NUMERO: **2529A** ANNO: 2022

A P P U N T I

STUDENTE: Sobrero Giovanni

MATERIA: Mechanical System Dynamics - Prof. Bonisoli

Mechanical Systems Dynamics

English - Italian

Professor: Elvio Bonisoli

Giovanni Sobrero's Schemes

A.A. 2019 - 2020 (6 cfu)

Program

- vibrations of damped sdof systems [6 h]
    - *harmonic response (approach based on complex numbers)*
    - *examples: accelerometer, seismometer, ...*
    - *transmissibility*
    - *non periodic inputs (step, impulse, convolution)*
- vibrations of mdof systems with proportional viscous damping [12 h]
    - *equation of motion (matrix approach)*
    - *eigenvalues and eigenvectors*
    - *orthogonality of modes*
    - *modal analysis*
- elements of analytical dynamics [4.5 h]
    - *energy, work and equation of energy*
    - *principle of virtual work*
    - *Hamilton's principle*
    - *Lagrange's equations*
    - *examples*
- vibrations of continuous systems (distributed parameters) [9 h]
    - *wave equation: transverse oscillations of strings, axial and torsional oscillations of beams*
    - *bending vibrations of beams (Euler-Bernoulli)*
    - *approximate methods (Rayleigh)*
    - *examples*
- dynamics of rotors [7.5 h]
    - *Jeffcott's rotor*
    - *stability analysis (Campbell's diagrams)*
    - *influence of bearing flexibility*

Tutorial teaching

practical lectures are devoted to develop and complete the topics presented during the course

Exam

written test only

References

Meirovitch L., Fundamentals of Vibrations, Mc Graw Hill
Vigliani A., Lectures on Rotordynamics, Clut

Tutorial teaching

- sdof systems
    - *torsional oscillations of shafts with different diameters and 1 disk*
    - *beat phenomenon*
    - *response in resonance (convolution integral)*

- mdof systems
    - *shaft with 2 disks*
    - *shaft of different sections with 2 disks*
    - *2 shafts, 2 disks and gearing (gears with no inertia)*
    - *2 masses, 3 springs (response to initial excitation)*
    - *2 masses, 3 springs, 3 dampers (response to external excitation)*
    - *dynamic absorber (impedance, receptance and anti-resonance)*
    - *longitudinal vehicle model (2 dof – damped oscillations)*
    - *2 shafts, 2 disks and gearing with inertia (3 dof)*
    - *Matlab examples*

- analytical dynamics
    - *vehicle model (4 dof – Lagrange's equations)*
    - *mass and rigid ring (gyroscopic effects and circulatory forces)*

- distributed parameters systems

- rotordynamics

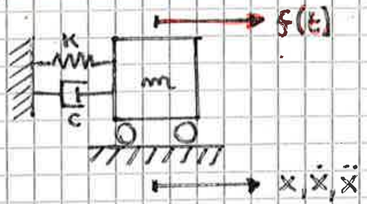
About the Exam:

Achieved learning outcomes will be assessed by means of a final exam. This is based on an analytical assessment of student achievement of the "expected learning outcomes" described above. In order to properly assess such achievement, the examination consists of a written test only, lasting 1 h and 30 min indicatively, with closed book and composed of three questions: the first is an **exercise similar to those proposed during the tutorial**; the second is focused on **one of the topics seen during the lectures**; with the third the students are required to apply the **methodologies explained in the subject to some new application or machine**. The exam aims at evaluating the ability of the students to model the dynamic behavior of mechanical systems, starting from the model definition and ending with the system analysis. In particular, the first part of the test assesses the ability to apply knowledge, while the second and third parts aim at assessing knowledge, communication skills and ability to use tools and method taught in the lectures for solving problems not directly proposed in the class hours.

Grading criteria:

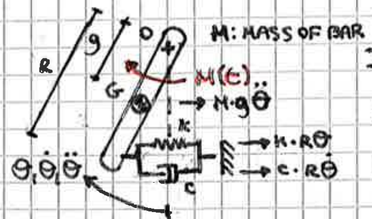
The maximum obtainable mark is 30/30 with merit (cum laude). Each answer to the three questions usually is evaluated from 0 to a maximum of 10 or 11 points, for a total of **32 available points.** During the semester, students are given an example of the final test, with discussion of the solution and hints on common errors and evaluation criteria. A few days after the written test, students are summoned for a review of the written output, in which examiners inform the students on grading criteria, and receive any student appeal supported by appropriate explanations. Computers, mobiles, electronic devices and any printed documentation are not allowed.

SDoF - SINGLE DEGREE OF FREEDOM [PREVIEW]



$$\begin{cases} x \to x(t) \\ \dot{x} = dx/dt \\ \ddot{x} = d^2x/dt^2 \end{cases}$$

- m [Kg] > 0 : MASS (→ KINEMATIC EN.)
- K [N/m] > 0 : ELASTICITY/RIGIDITY
- C [Ns/m] ≥ 0 : VISCOUS DISSIPATION /DAMPING



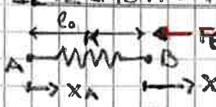
I_G [Kg·m²] : MASS MOMENT OF INERTIA

GRAVITATIONAL FIELD (ROLE OF K)

$\dot{\theta} = \omega$ [Rad/S] ; $\ddot{\theta} = \dot{\omega}$ [Rad/s²] $(x = R\theta ; \dot{x} = R\dot{\theta} ; \ddot{x} = R\ddot{\theta})$

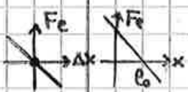
RESTORING FORCES (-)

1) ELASTIC FORCE (-)

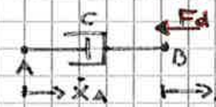


$\vec{\Delta x} = \vec{x_B} - \vec{x_A}$

$\vec{F_e} = -K\vec{\Delta x} = -K(\vec{x_B} - \vec{x_A})$



2) DAMPING FORCE (-)



$\vec{\Delta \dot{x}} = \vec{\dot{x}_B} - \vec{\dot{x}_A}$

$\vec{F_d} = -C\vec{\Delta \dot{x}} = -C(\vec{\dot{x}_B} - \vec{\dot{x}_A})$

(DAMPER)

3) COULOMB FRICTION F (-)

NON LINEAR

$\vec{F_c} = -\mu N \cdot \mathrm{SIGN}(\dot{x})$ (NON-LIN.)

4) INERTIAL FORCE (-)

$\vec{F_i} = -m\ddot{\vec{x}}_G$

$\Sigma \vec{F_E} = m\ddot{\vec{x}}_G$: NEWTON EQ.

$(\Sigma \vec{F_E} + \vec{F_i} = 0$: D'ALAMBERT$)$

5) INERTIAL MOMENT (-)

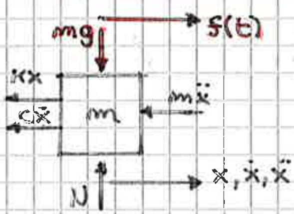
$\vec{M_i} = -I_G \ddot{\vec{\theta}}_G$

[(-) MEANS : OPPOSIT TO THE DIRECTION OF THE MOTION $(\vec{x})$]

FBD - FREE BODY DIAGRAM

- FREE SYSTEM ($\vec{f} = 0$) → FREE R.
- FORCED VIBRATION ($\vec{f} \neq 0$)

(f(t) OR F(t))



1) ISOLATE THE BODY
2) SET THE REFERENCE FRAME
3) ADD ALL THE GENERALISED FORCES

↑) $N = mg$ [REST. F = EXT. F]

←) $m\ddot{x} + c\dot{x} + Kx = f(t)$ [N] (ORD. DIFF. EQ.) (2nd ORDER)

↶) $\underbrace{I_G\ddot{\theta}}_{g \cdot Mg\theta} + \underbrace{R \cdot CR\dot{\theta}}_{\sim \dot{x}} + \underbrace{R \cdot KR\theta}_{\sim x} = M(t)$ [Nm] (ROTATIONAL EQ.)

(CANONICAL FORM)

←) $\ddot{x} + \frac{c}{m}\dot{x} + \frac{K}{m}x = \frac{1}{m}f(t) \Rightarrow \ddot{x} + 2\zeta\omega_m\dot{x} + \omega_m^2 x = \frac{\omega_m^2}{K}f(t)$

$$\omega_m = \sqrt{\frac{K}{m}} = \left[\sqrt{\frac{N}{m}}\sqrt{\frac{1}{Kg}}\right] = \left[\sqrt{\frac{Kg\,m}{m\,s^2}}\sqrt{\frac{1}{Kg}}\right] = \left[\frac{1}{S}\right] \to \left[\frac{Rad}{S}\right]$$

$\omega_m \left[\frac{Rad}{S}\right]$: NATURAL FREQUENCY | $f_m = \frac{\omega_m}{2\pi} = \ldots\left[\frac{Rad}{S}\right] \cdot \frac{1}{2\pi}\left[\frac{1}{Rad}\right] = \ldots\left[\frac{1}{S}\right]$

$$\zeta = \frac{c}{2m\omega_m} = \frac{c}{2m\sqrt{\frac{K}{m}}} = \frac{c}{2\sqrt{Km}} = \frac{c}{c_R}$$

: DAMPING FACTOR /RATIO (NON-DIM) | $T_m = \frac{1}{f_m}$ [S]

CONSIDERATIONS:

$m\ddot{x} + c\dot{x} + Kx = f(t)$ [N]

$$[Kg]\left[\frac{m}{s^2}\right] + \left[\frac{N \cdot S}{m}\right]\left[\frac{m}{S}\right] + \left[\frac{N}{m}\right][m] = [N]$$

$I_G\ddot{\theta} + C_t\dot{\theta} + K_t\theta = M(t)$ [Nm]

$$\left[\frac{Kg\,m^2}{Rad}\right]\left[\frac{Rad}{S^2}\right] + \left[\frac{Nm\,S}{Rad}\right]\left[\frac{Rad}{S}\right] + \left[\frac{Nm}{Rad}\right][Rad] = [Nm]$$

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DAMPING FACTOR (ζ) ($c\,[Ns/m]$ VISCOUS DAMPING / DISSIPATION)

→ IT IS NOT DIMENSIONAL

→ IT IS USED TO EVALUATE THE INTENSITY OF DAMPING

DEFINING:

$\omega_n = \sqrt{\frac{K}{m}}$: NATURAL FREQUENCY $\left[\sqrt{\frac{N}{m}}\cdot\sqrt{\frac{1}{Kg}}\right] = \left[\sqrt{\frac{Kg\,m}{m\,s^2}}\sqrt{\frac{1}{Kg}}\right] = \left[\sqrt{\frac{1}{s^2}}\right] = \left[\frac{1}{s}\right] = [Hz] \rightarrow \left[\frac{Rad}{s}\right]$

$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2m\sqrt{\frac{K}{m}}} = \frac{c}{2\sqrt{Km}} = \frac{c}{c_{CR}}$: DAMPING FACTOR/RATIO → [-] $\quad c_{CR} = 2\sqrt{Km}$: CRITICAL DAMPING

MOTION EQUATION ⟶ CANONICAL FORM (DIMENSIONLESS) → CHARACT (POLYNOMIAL) EQ.:

$$m\ddot{x} + c\dot{x} + Kx = 0 \rightarrow \ddot{x} + (c/m)\dot{x} + (K/m)x = 0 \rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

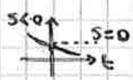
$$(ms^2 + cs + K)\,Ae^{st} = 0 \rightarrow ms^2 + cs + K = 0$$

SYSTEM POLES = EQ. ZERO ($s_{1,2}$)

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mK}}{2m} = \frac{-2m\zeta\omega_n \pm \sqrt{4m^2\zeta^2\omega_n^2 - 4m^2\omega_n^2}}{2m} = \frac{-2m\zeta\omega_n \pm 2m\omega_n\sqrt{\zeta^2-1}}{2m}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} = \dot{x}_{1,2}$$

TIME EVOLUTION DEPENDS ON THE SIGN OF ζ^2-1 (Δ): 3 CASES



- $\zeta > 1 \xrightarrow{\Delta>0}$ THE SYSTEM IS **OVERDAMPED** ⇒ $s_{1,2}$ ARE REAL AND NEGATIVE
- $\zeta = 1 \xrightarrow{\Delta=0}$ THE SYSTEM IS **CRITICALLY DAMPED** ⇒ $s_{1,2}$ ARE REAL, NEGATIVE AND COINCIDENT
- $\zeta < 1 \xrightarrow{\Delta<0}$ THE SYSTEM IS **UNDERDAMPED** ⇒ $s_{1,2}$ ARE COMPLEX AND CONJUGATE

(IF $\zeta = 0$: $s_{1,2} = \pm i\omega_n$: UNDAMPED (ZERO DAMPING)) [STARTING POINTS]

1) • CASE $\zeta > 1$: (IN THE PRACTICE THERE ARE FEW CASES WITH $\zeta > 1$)

$$x(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

$\begin{cases} s_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} \;;\; s_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \\ A_1, A_2 : \text{DETERMINABLE WITH INITIAL CONDITIONS: } \begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = V_0 \end{cases} \end{cases}$

2) • CASE $\zeta = 1$:

$x(t) = (A_1 + A_2t)e^{st}$ (BECAUSE $\Delta = 0$; $s_1 = s_2 = s$, THE SOLUTION TAKES THIS FORM)

$\begin{cases} s_1 = s_2 = s = -\omega_n \quad \text{(BECAUSE OF THAT } x(t) \text{ ASSUMES THIS DIFFERENT FORM)} \\ A_1, A_2 : \text{DETERMINABLE WITH INITIAL CONDITIONS: } \begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = V_0 \end{cases} \end{cases}$

3) • CASE $\zeta < 1$:

$x(t) = A_1e^{s_1t} + A_2e^{s_2t}$ $\quad$ ($s_{1,2} \in \mathbb{C}^*$ ← COMPLEX CONJUGATE)

$s_1 = \underbrace{-\zeta\omega_n}_{Re} - \underbrace{i\omega_n\sqrt{1-\zeta^2}}_{Im}$; $s_2 = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2}$, $i = \sqrt{-1}$ (COMPLEX NUMBER)

A_1, A_2 : DETERMINABLE WITH INITIAL CONDITIONS: $\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = V_0 \end{cases}$

WE WILL OBTAIN:

1) $x(t) = e^{-\zeta\omega_n t}\left[a\cosh(\omega_n\sqrt{\zeta^2-1}\,t) + b\sinh(\omega_n\sqrt{\zeta^2-1}\,t)\right]$, WITH: $a = x_0$; $b = \frac{V_0 + \zeta\omega_n x_0}{\omega_n\sqrt{\zeta^2-1}}$

2) $x(t) = \left[x_0 + (V_0 + \omega_n x_0)t\right]e^{-\omega_n t}$

3.1) $x(t) = e^{-\zeta\omega_n t}\left[a'\cos(\omega_d t) + b'\sin(\omega_d t)\right]$, WITH: $a' = a = x_0$; $b' = \frac{V_0 + \zeta\omega_n x_0}{\omega_d}$

3.2) $x(t) = A\sin(\omega_d t + \varphi)e^{-\zeta\omega_n t}$

3.3) $x(t) = C e^{-\zeta\omega_n t}\cdot\cos(\omega_d t - \varphi_+)$, WITH: $C\,(\text{or } A) = \sqrt{a'^2 + b'^2}$; $\varphi_+ = \arctan\left(\frac{b'}{a'}\right)$

- CASE $\zeta > 1$ (OVERDAMPING) ($s_{1,2}$: REAL AND NEGATIVE)

$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$: SOLUTION OF THE MOTION EQ. | $\dot{x}(t) = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$

$s_1 = -\zeta\omega_m - \omega_m\sqrt{\zeta^2-1}$; $s_2 = -\zeta\omega_m + \omega_m\sqrt{\zeta^2-1}$: ROOTS/SYSTEM POLES OF THE CHARACT. EQ.

IMPOSING THE INITIAL CONDITIONS : $\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = V_0 \end{cases}$ => WE FIND $A_1, A_2 = f(s_1, s_2)$

NON ZERO INITIAL CONDITIONS

$$\begin{cases} x(t=0) = x_0 = A_1 + A_2 \\ \dot{x}(t=0) = V_0 = s_1 A_1 + s_2 A_2 \end{cases}$$

(NOTE: n. OF I.C. = n. OF ORDER OF DIFF. EQ.)

THEREFORE:

$$\begin{cases} A_2 = x_0 - A_1 \Rightarrow A_2 = x_0 - \dfrac{V_0 - s_2 x_0}{s_1 - s_2} = \dfrac{s_1 x_0 - \cancel{s_1 x_0} - V_0 + \cancel{s_2 x_0}}{s_1 - s_2} = \dfrac{s_1 x_0 - V_0}{s_1 - s_2} \\ V_0 = s_1 A_1 + s_2 (x_0 - A_1) = s_1 A_1 + s_2 x_0 - s_2 A_1 \Rightarrow A_1 = \dfrac{V_0 - s_2 x_0}{s_1 - s_2} \end{cases}$$

WE OBTAIN THAT:

$$\begin{cases} A_1 = \dfrac{V_0 - s_2 x_0}{s_1 - s_2} \\ A_2 = \dfrac{s_1 x_0 - V_0}{s_1 - s_2} \end{cases}$$

CONSIDERING $s_1, s_2 \rightarrow A_1, A_2 \rightarrow$ SOLUTION $x(t)$ OF THE MOTION EQUATION

$$A_1 = \frac{V_0 - s_2 x_0}{s_1 - s_2} = \frac{V_0 + \zeta\omega_m x_0 - \omega_m x_0\sqrt{\zeta^2-1}}{-\cancel{\zeta\omega_m} - \omega_m\sqrt{\zeta^2-1} + \cancel{\zeta\omega_m} - \omega_m\sqrt{\zeta^2-1}} = \frac{V_0 + \zeta\omega_m x_0 - \omega_m x_0\sqrt{\zeta^2-1}}{-2\omega_m\sqrt{\zeta^2-1}} = \frac{-V_0 - \zeta\omega_m x_0 + \omega_m x_0\sqrt{\zeta^2-1}}{2\omega_m\sqrt{\zeta^2-1}}$$

$$= \frac{x_0}{2} - \frac{V_0 + \zeta\omega_m x_0}{2\omega_m\sqrt{\zeta^2-1}}$$

$$A_2 = \frac{s_1 x_0 - V_0}{s_1 - s_2} = \frac{-\zeta\omega_m x_0 - \omega_m x_0\sqrt{\zeta^2-1} - V_0}{-\cancel{\zeta\omega_m} - \omega_m\sqrt{\zeta^2-1} + \cancel{\zeta\omega_m} - \omega_m\sqrt{\zeta^2-1}} = \frac{-\zeta\omega_m x_0 - \omega_m x_0\sqrt{\zeta^2-1} - V_0}{-2\omega_m\sqrt{\zeta^2-1}} = \frac{+\zeta\omega_m x_0 + \omega_m x_0\sqrt{\zeta^2-1} + V_0}{2\omega_m\sqrt{\zeta^2-1}}$$

$$= \frac{x_0}{2} + \frac{V_0 + \zeta\omega_m x_0}{2\omega_m\sqrt{\zeta^2-1}}$$

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = \left(\frac{x_0}{2} - \frac{V_0 + \zeta\omega_m x_0}{2\omega_m\sqrt{\zeta^2-1}}\right) e^{(-\zeta\omega_m - \omega_m\sqrt{\zeta^2-1})t} + \left(\frac{x_0}{2} + \frac{V_0 + \zeta\omega_m x_0}{2\omega_m\sqrt{\zeta^2-1}}\right) e^{(-\zeta\omega_m + \omega_m\sqrt{\zeta^2-1})t} =$$

$$= \frac{e^{-\zeta\omega_m t}}{2\omega_m\sqrt{\zeta^2-1}} \left\{ \left[\omega_m x_0\sqrt{\zeta^2-1} - (V_0 + \zeta\omega_m x_0)\right] e^{-\omega_m\sqrt{\zeta^2-1}\,t} + \left[\omega_m x_0\sqrt{\zeta^2-1} + (V_0 + \zeta\omega_m x_0)\right] e^{+\omega_m\sqrt{\zeta^2-1}\,t} \right\} =$$

$$= \frac{e^{-\zeta\omega_m t}}{2\omega_m\sqrt{\zeta^2-1}} \left[(V_0 + \zeta\omega_m x_0)\left(e^{+\omega_m\sqrt{\zeta^2-1}\,t} - e^{-\omega_m\sqrt{\zeta^2-1}\,t}\right) + (\omega_m x_0\sqrt{\zeta^2-1})\left(e^{+\omega_m\sqrt{\zeta^2-1}\,t} + e^{-\omega_m\sqrt{\zeta^2-1}\,t}\right) \right]$$

CONSIDERING THAT:

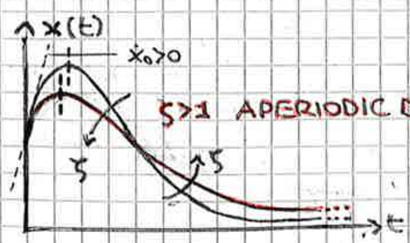
$e^{\alpha} - e^{-\alpha} = 2\sinh\alpha$; $e^{\alpha} + e^{-\alpha} = 2\cosh\alpha$ (FROM EULERO'S FORMULA)

$$x(t) = \frac{e^{-\zeta\omega_m t}}{\cancel{2}\omega_m\sqrt{\zeta^2-1}} \left[(V_0 + \zeta\omega_m x_0)\cancel{2}\sinh(\omega_m\sqrt{\zeta^2-1}\,t) + (\omega_m x_0\sqrt{\zeta^2-1})\cancel{2}\cosh(\omega_m\sqrt{\zeta^2-1}\,t) \right]$$

WE OBTAIN:

$$\boxed{x(t) = e^{-\zeta\omega_m t}\left[x_0 \cosh(\omega_m\sqrt{\zeta^2-1}\,t) + \frac{(V_0 + \zeta\omega_m x_0)}{\omega_m\sqrt{\zeta^2-1}} \sinh(\omega_m\sqrt{\zeta^2-1}\,t) \right]}$$

APERIODIC DECAY



$\zeta > 1$ APERIODIC DECAY

- CASE $\zeta < 1$ (UNDERDAMPING) ($S_{1,2}$ ARE COMPLEX AND CONJUGATED) ($0 \leq \zeta < 1$)

$x(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$: SOLUTION OF THE MOTION EQ. | $\dot{x}(t) = S_1 A_1 e^{S_1 t} + S_2 A_2 e^{S_2 t}$

$S_1 = -\zeta \omega_m - \omega_m \sqrt{\zeta^2 - 1} = -\zeta \omega_m - i \omega_m \sqrt{1 - \zeta^2}$: ROOTS/SYST. POLES OF THE CHAR. EQ.

$S_2 = -\zeta \omega_m + \omega_m \sqrt{\zeta^2 - 1} = -\zeta \omega_m + i \omega_m \sqrt{1 - \zeta^2}$, i OR j ($= \sqrt{-1}$)

DEFINING:

$\omega_d = \omega_m \sqrt{1 - \zeta^2}$: DAMPED FREQUENCY / FREQUENCY OF DAMPED SYSTEM (NATURAL)

THEREFORE: ($\omega_d \leq \omega_m$ [Rad/s] , $\omega = 2\pi f$, $T = \frac{1}{f}$)

$S_1 = -\zeta \omega_m - i \omega_d$ (ω_m IS A PARTICULAR CASE OF ω_d IN ABSENCE OF DAMPING ($\zeta = 0$))

$S_2 = -\zeta \omega_m + i \omega_d$

IMPOSING THE INITIAL CONDITIONS: $\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = V_0 \end{cases}$

$$\begin{cases} x(t=0) = x_0 = A_1 + A_2 \\ \dot{x}(t=0) = V_0 = S_1 A_1 + S_2 A_2 \end{cases}$$

THEREFORE:

$$\begin{cases} A_2 = x_0 - A_1 \Rightarrow A_2 = x_0 - \dfrac{V_0 - S_2 x_0}{S_1 - S_2} = \dfrac{S_1 x_0 - \cancel{S_2 x_0} - V_0 + \cancel{S_2 x_0}}{S_1 - S_2} = \dfrac{S_1 x_0 - V_0}{S_1 - S_2} \\ V_0 = S_1 A_1 + S_2 (x_0 - A_1) = S_1 A_1 + S_2 x_0 - S_2 A_1 \Rightarrow A_1 = \dfrac{V_0 - S_2 x_0}{S_1 - S_2} \end{cases}$$

WE OBTAIN THAT:

$$\begin{cases} A_1 = \dfrac{V_0 - S_2 x_0}{S_1 - S_2} \\ A_2 = \dfrac{S_1 x_0 - V_0}{S_1 - S_2} \end{cases}$$

CONSIDERING $S_1, S_2 \rightarrow A_1, A_2 \rightarrow$ SOLUTION $x(t)$ OF THE MOTION EQUATION

$$A_1 = \frac{V_0 + \zeta \omega_m x_0 - i \omega_d x_0}{-\cancel{\zeta \omega_m} - i \omega_d + \cancel{\zeta \omega_m} - i \omega_d} = \frac{V_0 + \zeta \omega_m x_0 - i \omega_d x_0}{-2 i \omega_d} = \frac{-V_0 - \zeta \omega_m x_0 + i \omega_d x_0}{2 i \omega_d} = \frac{x_0}{2} - \frac{V_0 + \zeta \omega_m x_0}{2 i \omega_d}$$

$$A_2 = \frac{-\zeta \omega_m x_0 - i \omega_d x_0 - V_0}{-\cancel{\zeta \omega_m} - i \omega_d + \cancel{\zeta \omega_m} - i \omega_d} = \frac{-\zeta \omega_m x_0 - i \omega_d x_0 - V_0}{-2 i \omega_d} = \frac{+\zeta \omega_m x_0 + i \omega_d x_0 + V_0}{2 i \omega_d} = \frac{x_0}{2} + \frac{V_0 + \zeta \omega_m x_0}{2 i \omega_d}$$

$$x(t) = \left(\frac{x_0}{2} - \frac{V_0 + \zeta \omega_m x_0}{2 i \omega_d} \right) e^{(-\zeta \omega_m - i \omega_d) t} + \left(\frac{x_0}{2} + \frac{V_0 + \zeta \omega_m x_0}{2 i \omega_d} \right) e^{(-\zeta \omega_m + i \omega_d) t} =$$

$$= \frac{e^{-\zeta \omega_m t}}{2 i \omega_d} \left\{ \left[i \omega_d x_0 - (V_0 + \zeta \omega_m x_0) \right] e^{-i \omega_d t} + \left[i \omega_d x_0 + (V_0 + \zeta \omega_m x_0) \right] e^{+i \omega_d t} \right\} =$$

$$= \frac{e^{-\zeta \omega_m t}}{2 i \omega_d} \left[(V_0 + \zeta \omega_m x_0) \left(e^{+i \omega_d t} - e^{-i \omega_d t} \right) + i \omega_d x_0 \left(e^{+i \omega_d t} + e^{-i \omega_d t} \right) \right]$$

CONSIDERING THAT:

$e^{j\alpha} - e^{-j\alpha} = 2 j \sin(\alpha)$; $e^{j\alpha} + e^{-j\alpha} = 2 \cos(\alpha)$

$$x(t) = \frac{e^{-\zeta \omega_m t}}{2 j \omega_d} \left[(V_0 + \zeta \omega_m x_0) \cdot 2 j \sin(\omega_d t) + i \omega_d x_0 \cdot 2 \cos(\omega_d t) \right]$$

WE OBTAIN:

$$\boxed{x(t) = e^{-\zeta \omega_m t} \left[\underbrace{x_0}_{a\,(Re)} \cos(\omega_d t) + \underbrace{\frac{(V_0 + \zeta \omega_m x_0)}{\omega_d}}_{b\,(Im)} \sin(\omega_d t) \right]}$$ PERIODIC DECAY

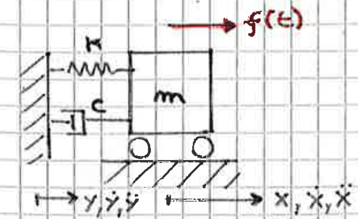
$x(t)$ CAN BE ALSO EXPRESSED AS:

$x(t) = C e^{-\zeta \omega_m t} \cdot \cos(\omega_d t - \varphi)$, WITH: $\begin{cases} C = \sqrt{a^2 + b^2} = \sqrt{x_0^2 + \left[\dfrac{(V_0 + \zeta \omega_m x_0)}{\omega_d} \right]^2} : \text{AMPLITUDE} \\ \varphi_+ = Tg^{-1}\left(\dfrac{b}{a}\right) ; \varphi_- = Tg^{-1}\left(-\dfrac{b}{a}\right) : \text{PHASE ANGLE} \end{cases}$

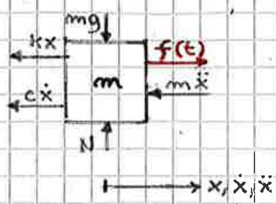
$x(t) = C e^{-\zeta \omega_m t} \cos(\omega_d t + \varphi_-)$

1.2 FORCED SYSTEM ($F \neq 0$) → FORCED VIBRATION $\left\{ \begin{array}{l} ^{1}\text{H.E.} \quad ^{2}\text{B.E.} \quad ^{3}\text{NON-PER. EXC.} \\ + \text{STABILITY} + \text{W.M.} + \text{DAMPING} \end{array} \right\}$

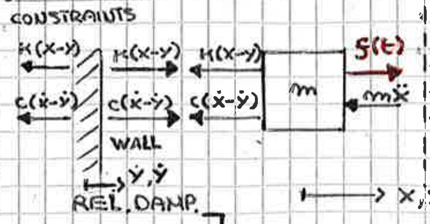
THE MODEL WITH CONCENTRATED PARAMETERS BECOMES:



FBD (STEADY WALL)



OR FBD (MOVING WALL)



WE'LL ANALYZE:
- HARMONIC EXCIT.
- BASE EXCIT.
- NON PERIODIC EX.
- STEP/IMPULSE EX.
- CONVOLUTION INT.

^{1} HARMONIC EXCITATION

$\leftarrow$) $m\ddot{x} + c(\dot{x}-\dot{y}) + K(x-y) = f(t)$ — REL. SPRING

$y = \dot{y} = \ddot{y} = 0 \qquad (e^{i\alpha} = \cos(\alpha) + i\sin(\alpha))$ (IF $\varphi = \pi \Rightarrow e^{i\pi} = -1$)

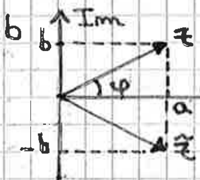
$f(t) = F_0 e^{i\omega t} \iff f(t) = F_0[\cos(\omega t) + i\sin(\omega t)]$: HARMONIC EXCITATION

WHERE: $F_0 \in \mathbb{R}e$, ω: FREQUENCY OF THE EXTERNAL FORCE (ω OR Ω)

$$\begin{cases} f(t) = F_0 \mathrm{Re}(e^{i\omega t}) \iff f(t) = F_0 \cos(\omega t) \Rightarrow \cos(\omega t) = \mathrm{Re}(e^{i\omega t}) \\ f(t) = F_0 \mathrm{Im}(e^{i\omega t}) \iff f(t) = F_0 \sin(\omega t) \Rightarrow \sin(\omega t) = \mathrm{Im}(e^{i\omega t}) \end{cases}$$

$F_0 = a + ib = z = \rho e^{i\varphi}$, $\varphi = \omega t$, $\bar{z} = a - ib$

$$\begin{cases} |z| = \sqrt{a^2+b^2} = \sqrt{\mathrm{Re}^2(z) + \mathrm{Im}^2(z)} = \sqrt{z \cdot \bar{z}} \\ \varphi = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left(\frac{b}{a}\right) \end{cases}$$



ARGAND-GAUSS

$\begin{cases} F_0 = z \\ \varphi = \omega t \end{cases}$

NOTE: $(a+ib)(a-ib) = (a^2+b^2)$ — TYPICALLY USED FOR $|z| = \sqrt{z \cdot \bar{z}}$

CONSIDERING THE MOTION EQUATION:

$\leftarrow$) $m\ddot{x} + c\dot{x} + Kx = f(t)$, CONSIDERING: $\omega_m = \sqrt{K/m}$; $\zeta = c/(2m\omega_m)$,

$\ddot{x} + 2\zeta\omega_m\dot{x} + \omega_m^2 x = \frac{1}{m} f(t) = \frac{1}{K}\omega_m^2 f(t) = \frac{F_0}{K}\omega_m^2 e^{i\omega t}$

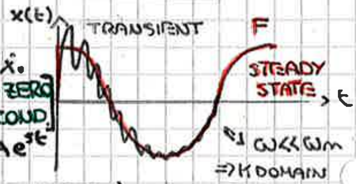
$\ddot{x} + 2\zeta\omega_m\dot{x} + \omega_m^2 x = \frac{F_0}{K}\omega_m^2 e^{i\omega t}$: MOTION EQUATION IN THE EXPONENTIAL FORM (CANONICAL)

$x(t) = \cancel{x_A(t)} + x_P(t)$: RESPONSE TO HARMONIC EXCITATION

AFTER SOME SEC. $x_A(t) = 0$

$$\begin{cases} x_A(t) : \text{HOMOGENEOUS (VANISHING) SOLUTION, TRANSIENT} \\ x_P(t) : \text{PARTICULAR SOLUTION/INTEGRAL, STEADY-STATE} \end{cases}$$

$x_0, \dot{x}_0$ $\left[\begin{array}{l} \text{NON ZERO} \\ \text{IN. COND.} \end{array} \right]$ $x(t) = Ae^{st}$



$x(t) = x_P(t)$, DEPENDING ON FORCING EXC. (FREQ. ω AND MODULUS F_0) AND ON SYSTEM CHARACTERISTICS (NAT. FREQ. ω_m AND DAMPING FACTOR ζ)

$x(t) = X_0 e^{i\omega t} \iff x(t) = x_0 \cos(\omega t - \underbrace{\varphi}_{\text{DELAY (CONST)}})$ OR $x(t) = x_0\cos(\omega t \oplus \varphi)$ — E. BONISOLI'S POINT OF VIEW (BETTER)

$X_0 = X_0(i\omega) \in \mathbb{C} \qquad x_0 \in \mathbb{R}e$

WITH φ [-] : DELAY
IN THIS CASE: $\varphi = Tg^{-1}\left(\frac{-b}{a}\right)$

$x_0 = |X_0(i\omega)|$ I THINK (TO CHECK)

$$\begin{cases} x(t) = X_0 e^{i\omega t} \\ \dot{x}(t) = i\omega X_0 e^{i\omega t} \\ \ddot{x}(t) = -\omega^2 X_0 e^{i\omega t} \end{cases}$$

$\cancel{\omega_m^2}\left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) - \left(\frac{\omega}{\omega_m}\right)^2\right] X_0 = \cancel{\omega_m^2}\frac{F_0}{K}$

$(-\omega^2 + 2i\zeta\omega_m\omega + \omega_m^2) X_0 \cancel{e^{i\omega t}} = \frac{F_0}{K}\omega_m^2 \cancel{e^{i\omega t}} \Rightarrow$

$X_0 = \frac{1}{\left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) - \left(\frac{\omega}{\omega_m}\right)^2\right]} \cdot \frac{F_0}{K} = G(i\omega) A$ WITH: $A = \frac{F_0}{K}$

DEFINING:

$z(i\omega) = 1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) - \left(\frac{\omega}{\omega_m}\right)^2$: IMPEDANCE FUNCTION $\left(\frac{1}{z(i\omega)} = G(i\omega)\right)$ A: STATIC DEFORM.

$G(i\omega) = \frac{X_0}{(F_0/K)} = \frac{1}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)}$: FREQUENCY RESPONSE (ADIMENSIONAL)
($G \leftrightarrow$ GAIN OF AMPLIFICATION) FRF: FREQ. RESP. FUNCT.

THEN:

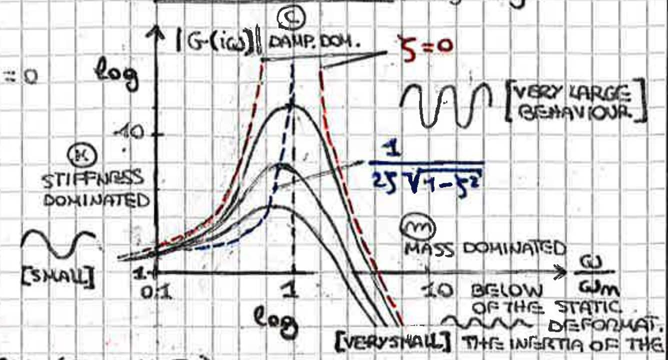
$x(t) = AG(i\omega)e^{i\omega t}$: HARMONIC RESPONSE (GENERAL FORM) , WITH $A = F_0/K$

RESONANCE DEMONSTRATION (FOR H.E.)

RESONANCE: AMPLIFICATION OF RESPONSE. THE SYSTEM IS CATCHING ENERGY FROM THE FORCE AND IT IS USING THE ENERGY TO IMPROVE AND ENLARGE THE OSCILLATING BEHAVIOUR

↓

BODE DIAGRAM (log-log)



IN ORDER TO OBTAIN THE VALUE AND LOCATION OF PEAKS: $\dfrac{d|G(i\omega)|}{d(\omega/\omega_m)}=0$

$$|G(i\omega)| = 1/\sqrt{[1-(\omega/\omega_m)^2]^2+[2\zeta(\omega/\omega_m)]^2}$$

$$\frac{d|G(i\omega)|}{d(\omega/\omega_m)} = -\frac{1}{2}\,\frac{2[1-(\omega/\omega_m)^2][-2(\omega/\omega_m)]+2[2\zeta(\omega/\omega_m)]2\zeta}{\{[1-(\omega/\omega_m)^2]^2+[2\zeta(\omega/\omega_m)]^2\}^{3/2}}=0$$

$$\cancel{2}[1-(\omega/\omega_m)^2][-2(\omega/\omega_m)]+\cancel{2}[2\zeta(\omega/\omega_m)]2\zeta=0$$

$$-\cancel{2}(\cancel{\omega/\omega_m})+\cancel{2}(\omega/\omega_m)^{\cancel{3}\,2}+\cancel{4}\,\zeta^2(\cancel{\omega/\omega_m})=0$$

$(\omega/\omega_m)^2=1-2\zeta^2$ THEREFORE $\omega/\omega_m=\sqrt{1-2\zeta^2}$

WE OBTAIN THAT: $(\omega_R=0$ IF $\zeta=1/\sqrt{2})$

$\omega_R=\omega=\omega_m\sqrt{1-2\zeta^2}$: **RESONANCE FREQUENCY (FOR H.E.)** $\quad \omega_R<\omega_d<\omega_m$

THE FREQUENCY RESPONSE $G(i\omega)$ HAS **NO PEAKS** ($\leftrightarrow$ NO MAX) **FOR** $\zeta>\frac{1}{\sqrt{2}}$ ($\cong$ 0.71) DUE TO THE E

$$|G(i\omega)|_{MAX}=\frac{1}{\sqrt{[1-(\sqrt{1-2\zeta^2})^2]^2+[2\zeta\sqrt{1-2\zeta^2}]^2}}=\frac{1}{\sqrt{[1-1+2\zeta^2]^2+[4\zeta^2(1-2\zeta^2)]}}=\frac{1}{\sqrt{4\zeta^4+4\zeta^2-8\zeta^4}}=$$

$$=\frac{1}{\sqrt{4\zeta^2-4\zeta^4}}=\frac{1}{\sqrt{4\zeta^2(1-\zeta^2)}}=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$: **MAXIMUM OF AMPLITUDE**

↳ $x_0=|X_0(i\omega)|=A|G(i\omega)|$

IN CASE OF LIGHT DAMPING:

$\zeta<0.05 \Rightarrow |G(i\omega)|_{MAX}=\frac{1}{2\zeta}=Q$: **QUALITY FACTOR** $(\leftrightarrow \omega=\omega_m)$ $\left(\text{EX: } Q=5 \Rightarrow \zeta=\frac{1}{2Q}=\frac{1}{10}=0.10\right)$

RESONANCE FREQUENCY (ω_R) DEF.

IT IS THE FREQUENCY FOR WHICH THE SYSTEM SHOWS THE MAXIMUM AMPLITUDE

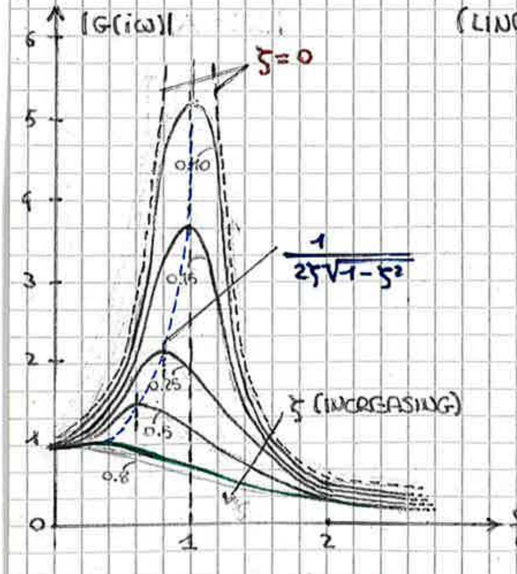
THEREFORE, IN CONCLUSION, WE HAVE **RESONANCE** ($\omega_R\neq0$) **ONLY IF** $\zeta<1/\sqrt{2}$

($\exists$ MAX IF $0<\zeta<\sqrt{2}/2$) (FOR HARMONIC EXCITATION)

($\omega_R=\omega_m\sqrt{1-2\zeta^2}$ → CONDITION: $1-2\zeta^2>0 \iff 2\zeta^2<1 \Rightarrow \zeta<1/\sqrt{2}$)

WE CAN CONSIDER ALSO ANOTHER DIAGRAM, PLOTTING $|X_0|=|G(i\omega)|A$, $A=\frac{F_0}{K}$

(LINEAR SCALE) ⇑ (NON log-log) (OR SIMPLY $|G(i\omega)|$)



NO PEAKS FOR $\zeta>\frac{1}{\sqrt{2}}$ ($\cong$0.71)

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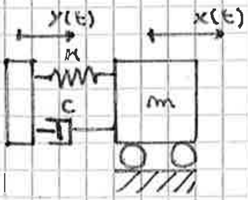
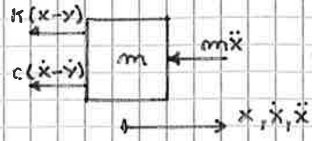
2 BASE EXCITATION

HARMONIC MOTION OF THE BASE ($y(t)$).



THE FBD BECOMES (N - mg NOT CONSIDERED)



[*] COMPUTING $X_0(i\omega)$ WITH THE LONG MATHEMATICAL PROCEDURE $X_0(i\omega) = X_0 \frac{\bar{G}(i\omega)}{\bar{G}(i\omega)}$ WE OBTAIN THAT THE **Im PART** IS **NEGATIVE (-)**. THEN THE **PHASE ANGLE**, $\varphi = Tg^{-1}\left(\frac{Im}{Re}\right)$, RESULTS **NEGATIVE (-)**. $\Rightarrow +\varphi_-$!

I INITIALLY USED AS CONVENTION, FOR FASTER EXECUTION, $-\varphi_+$ ... IT "WORKS" BUT IT'S MATHEMATICALLY AND PHYSIC. INCORRECT!

$f(t) = 0$

THE MOTION EQUATION BECOMES:

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \quad : \text{EQUATION OF EQUILIBRIUM}$$

BEING: [OR $x(t) = x_0\cos(\omega t + \varphi_-)$, $\varphi_- = Tg^{-1}(-b/a)$; $\varphi_+ = Tg^{-1}(b/a)$]

$$\begin{cases} x(t) = X_0 e^{i\omega t} \\ \dot{x}(t) = i\omega X_0 e^{i\omega t} \\ \ddot{x}(t) = -\omega^2 X_0 e^{i\omega t} \end{cases} \iff x(t) = x_0\cos(\omega t - \varphi_+) \quad X_0 \in \mathbb{C};\ x_0 \in Re \quad \text{WITH } x_0 = |X_0(i\omega)|$$

$$\begin{cases} y(t) = Y_0 e^{i\omega t} \\ \dot{y}(t) = i\omega Y_0 e^{i\omega t} \\ \ddot{y}(t) = -\omega^2 Y_0 e^{i\omega t} \end{cases} \iff y(t) = Y_0\cos(\omega t) \quad Y_0 \in Re$$

WITH $X_0 \in \mathbb{C}$ (CONST.) ; $Y_0 \in Re$, $X_0 = X_0(i\omega) = |X_0(i\omega)| e^{\mp i\varphi}$

THE EQUATION OF EQUILIBRIUM BECOMES:

$$\ddot{x} + 2\zeta\omega_m\dot{x} + \omega_m^2 x = 2\zeta\omega_m\dot{y} + \omega_m^2 y$$

$$\left[-\omega^2 + 2i\zeta\omega_m\omega + \omega_m^2\right] X_0 \cancel{e^{i\omega t}} = \left[2i\zeta\omega_m\omega + \omega_m^2\right] Y_0 \cancel{e^{i\omega t}}$$

$$\cancel{\omega_m^2}\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) + 1\right] X_0 = \cancel{\omega_m^2}\left[2i\zeta\left(\frac{\omega}{\omega_m}\right) + 1\right] Y_0 \qquad X_0(i\omega) = Y_0\left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] G(i\omega)$$

$$T_{RABS} = \frac{X_0}{Y_0} = \frac{1 + 2i\zeta(\omega/\omega_m)}{1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)} \quad : \underline{\text{ABSOLUTE TRANSMISSIBILITY}} \begin{cases} |T_{RABS}| = \frac{|X_0|}{Y_0} = \frac{|X_0(i\omega)|}{Y_0} \\ \angle \end{cases}$$

(1) $X_0(i\omega) = X_0 = Y_0 \frac{[1 + 2i\zeta(\omega/\omega_m)]}{[1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)]} = Y_0[1 + 2i\zeta(\omega/\omega_m)]\cdot G(i\omega)$

(2) $X_0(i\omega) = X_0 \frac{\bar{G}(i\omega)}{\bar{G}(i\omega)} = \frac{Y_0[1 + 2i\zeta(\omega/\omega_m)]}{[1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)]} \cdot \frac{[1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)]}{[1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)]} = \frac{Y_0[1 + 2i\zeta(\omega/\omega_m)]\cdot[1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)]}{[1 - (\omega/\omega_m)^2]^2 + [2\zeta(\omega/\omega_m)]^2} =$

$= Y_0|G(i\omega)|^2[1 + 2i\zeta(\omega/\omega_m)][1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)] = Y_0|G(i\omega)|^2[1 - (\omega/\omega_m)^2 - 2i\zeta\cancel{(\omega/\omega_m)} +$

$+ 2i\zeta\cancel{(\omega/\omega_m)} - 2i\zeta(\omega/\omega_m)^3 + 4\zeta^2(\omega/\omega_m)^2] = Y_0|G(i\omega)|^2[1 - (\omega/\omega_m)^2 + 4\zeta^2(\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)^3]$

FROM (1):

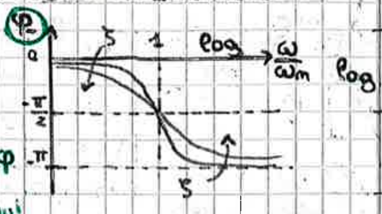
$$|X_0(i\omega)| = \sqrt{X_0(i\omega)\bar{X}_0(i\omega)} = Y_0\sqrt{[1 + 2i\zeta(\omega/\omega_m)]G(i\omega)\cdot[1 - 2i\zeta(\omega/\omega_m)]\bar{G}(i\omega)} = Y_0\sqrt{1 + [2\zeta(\omega/\omega_m)]^2}\,|G(i\omega)|$$

$$= Y_0\sqrt{\frac{1 + [2\zeta(\omega/\omega_m)]^2}{[1 - (\omega/\omega_m)^2]^2 + [2\zeta(\omega/\omega_m)]^2}} \Rightarrow |T_{RABS}| = \frac{|X_0(i\omega)|}{Y_0} = \sqrt{\frac{1 + [2\zeta(\omega/\omega_m)]^2}{[1 - (\omega/\omega_m)^2]^2 + [2\zeta(\omega/\omega_m)]^2}}$$

FROM (2):

$$\varphi_- = Tg^{-1}\left(\frac{Im}{Re}\right) = \frac{-2\zeta(\omega/\omega_m)^3}{1 - (\omega/\omega_m)^2 + 4\zeta^2(\omega/\omega_m)^2}$$

ABOUT THE PHASE:



THEREFORE:

$X_0(i\omega) = |X_0(i\omega)| e^{+i\varphi}$, $\varphi[-]$

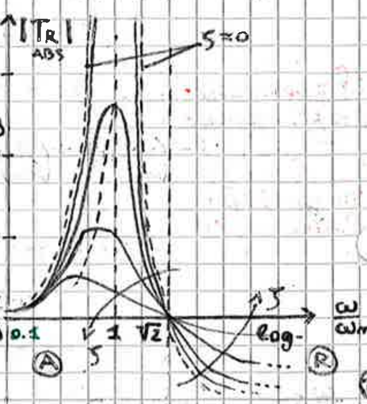
$x(t) = X_0 e^{i\omega t} = |X_0(i\omega)| e^{i\omega t} \cdot e^{+i\varphi_-}$

THERE IS A FASTER WAY TO COMPUTE φ. LOOK AT: 2nd ROUND SECTION!

WE OBTAIN THAT:

$x(t) = |X_0(i\omega)| e^{i(\omega t + \varphi_-)}$: MOTION RESPONSE

$[x(t) = X_0 e^{i\omega t} \iff x(t) = x_0\cos(\omega t + \varphi_-)] [x_0 = |X_0(i\omega)|]$



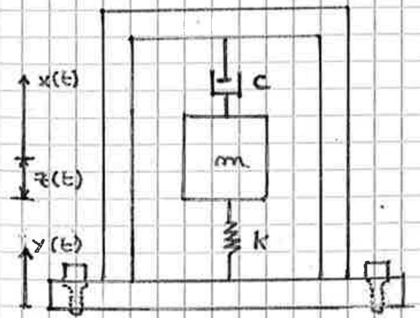
ABOUT TRANSM.:

$(\omega/\omega_m) < \sqrt{2}$: Ⓐ AMPLIFICATION

$(\omega/\omega_m) > \sqrt{2}$: Ⓡ EDUCTION

VIBRATION MEASURING INSTRUMENTS

ACCELEROMETER / SEISMOMETER

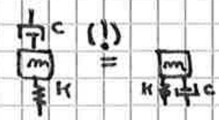


USED FOR MEASURING DISPLACEMENT AND ACCELERATION

y(t) : ABSOLUTE BASE DISPLACEMENT

x(t) : ABSOLUTE MASS DISPLACEMENT

z(t) : RELATIVE MASS DISPLACEMENT



THE TARGET IS TO DETERMINE THE MOTION y(t) OF THE BASE FROM THE MEASUREMENT OF THE RELATIVE DISP. z(t). $[z(t) = x(t) - y(t)]$

$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$

$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$; $y(t) = y_0 e^{i\omega t} \iff y(t) = y_0 \cos(\omega t)$; $\dot{y}(t) = i\omega y_0 e^{i\omega t}$; $\ddot{y}(t) = -\omega^2 y_0 e^{i\omega t}$

$m\ddot{z} + c\dot{z} + kz = m\omega^2 y_0 e^{i\omega t} \iff m\ddot{z} + c\dot{z} + kz = m\omega^2 y_0 \cos(\omega t)$, $y_0 \in \mathbb{R}e$

THE RESPONSE IS:

$z(t) = z_{10} e^{i\omega t} \iff z(t) = z_0 \cos(\omega t + \varphi)$, $z_{10} = z_{10}(i\omega) \in \mathbb{C}$

$z_{10} = z_{10}(i\omega) = |z_1(i\omega)| e^{+i\varphi}$ WITH: $\begin{cases} |z_1(i\omega)| = y_0 (\omega/\omega_m)^2 |G(i\omega)| \\ \varphi = Tg^{-1}\left\{\dfrac{-2\zeta(\omega/\omega_m)}{1-(\omega/\omega_m)^2}\right\} \end{cases}$

THEREFORE:

$z(t) = y_0 \left(\dfrac{\omega}{\omega_m}\right)^2 |G(i\omega)| e^{i(\omega t + \varphi)}$

RELATIVE TRANSMISSIB.:

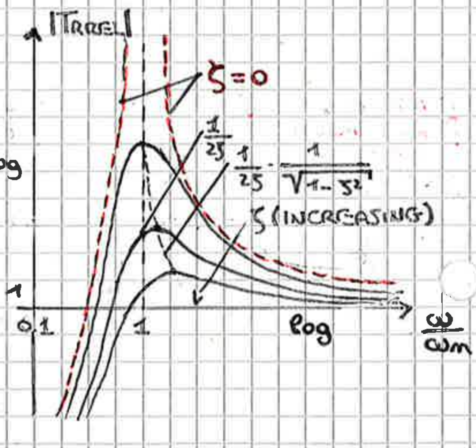
$T_{R_{REL}} = \dfrac{z_{10}}{y_0} = \dfrac{X_0 - Y_0}{Y_0} = T_{R_{ABS}} - 1$

THEN:

$|T_{R_{REL}}| = \dfrac{|z_{10}(i\omega)|}{y_0}$

RESONANCE (FOR B.E.)

IN ORDER TO FIND THE VALUE AND LOCATION OF PEAKS: $\dfrac{d|T_{R_{REL}}|}{d(\omega/\omega_m)} = 0$



$|T_{R_{REL}}| = \dfrac{|z_{10}(i\omega)|}{y_0} = \dfrac{y_0(\omega/\omega_m)^2 |G(i\omega)|}{y_0} = \dfrac{(\omega/\omega_m)^2}{\sqrt{[1-(\omega/\omega_m)^2]^2 + [2\zeta(\omega/\omega_m)]^2}}$

TO AVOID TO DO $F(x) = f(x)g(x) \rightarrow F'(x) = f'(x)g(x) + g'(x)f(x)$:

$|T_{R_{REL}}| = \dfrac{(\omega/\omega_m)^2}{\sqrt{1 - 2(\omega/\omega_m)^2 + (\omega/\omega_m)^4 + 4\zeta(\omega/\omega_m)^2}} =$

$= \dfrac{(\omega/\omega_m)^2}{(\omega/\omega_m)^2 \sqrt{\dfrac{1}{(\omega/\omega_m)^4}\left[1 - 2(\omega/\omega_m)^2 + (\omega/\omega_m)^4 + 4\zeta^2(\omega/\omega_m)^2\right]}} =$

$= \dfrac{1}{\sqrt{\dfrac{1}{(\omega/\omega_m)^4} - \dfrac{2}{(\omega/\omega_m)^2} + 1 + \dfrac{4\zeta^2}{(\omega/\omega_m)^2}}} = \dfrac{1}{\sqrt{(\omega/\omega_m)^{-4} - 2(\omega/\omega_m)^{-2} + 1 + 4\zeta^2(\omega/\omega_m)^{-2}}}$

THEN, WE DERIVE:

$\dfrac{d|T_{R_{REL}}|}{d(\omega/\omega_m)} = -\dfrac{1}{2} \dfrac{-4(\omega/\omega_m)^{-5} + 4(\omega/\omega_m)^{-3} - 8\zeta^2(\omega/\omega_m)^{-3}}{[(\omega/\omega_m)^{-4} - 2(\omega/\omega_m)^{-2} + 1 + 4\zeta^2(\omega/\omega_m)^{-2}]^{3/2}} = 0$

$-\cancel{4}(\omega/\omega_m)^{-5} + \cancel{4}(\omega/\omega_m)^{-3} - \cancel{8}^{2}\zeta^2(\omega/\omega_m)^{-3} = 0$

$-(\omega/\omega_m)^{\cancel{-5}\,-2} + \cancel{(\omega/\omega_m)^{-3}} - 2\zeta^2 \cancel{(\omega/\omega_m)^{-3}} = 0$

$-(\omega/\omega_m)^{-2} + 1 - 2\zeta^2 = 0 \Rightarrow (\omega/\omega_m)^{-2} = 1 - 2\zeta^2 \Rightarrow \dfrac{1}{(\omega/\omega_m)} = \sqrt{1-2\zeta^2} \iff (\omega/\omega_m) = \dfrac{1}{\sqrt{1-2\zeta^2}}$

THEREFORE:

$\omega_R = \omega = \omega_m \cdot \dfrac{1}{\sqrt{1-2\zeta^2}}$: RESONANCE FREQUENCY (FOR B.E) $(\zeta < 1/\sqrt{2})$

3 NON PERIODIC EXCITATIONS (GENERALITIES)

LINEAR SYSTEM → SUPERPOSITION OF EFFECTS (SUPERPOSITION PRINCIPLE)

$F(t) \rightarrow x(t)$ $\quad F(t) = F_1(t) + F_2(t) \rightarrow x(t) = x_1(t) + x_2(t)$ $\quad \begin{cases} \alpha F_1(t) \rightarrow \alpha x_1(t) \\ \beta F_2(t) \rightarrow \beta x_2(t) \end{cases}$

$\Rightarrow F(t) = \alpha F_1(t) + \beta F_2(t) = \alpha x_1(t) + \beta x_2(t) \rightarrow x(t) = \alpha x_1(t) + \beta x_2(t)$

CONSIDERATIONS:

- THE NON-PERIODIC EXCITATIONS ARE OFTEN REFERRED TO AS "TRANSIENT", IN SENSE THAT ARE NOT STATIONARY.
- USUALLY THEY START AT $t=0$.
- (TOTAL RESPONSE = RESPONSE OF LINEAR SYSTEM TO NON-PERIODIC EXCITATIONS + RESPONSE TO INITIAL EXCITATIONS : SUPERPOSITION PRINCIPLE)
- IN GENERAL, THE RESPONSE OF LINEAR SYSTEMS TO ARBITRARY EXCITATION CAN BE EXPRESSED AS A SUPERPOSITION OF IMPULSE RESPONSES BY MEANS OF THE CONVOLUTION INTEGRAL: $x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau$ [PREVIEW]

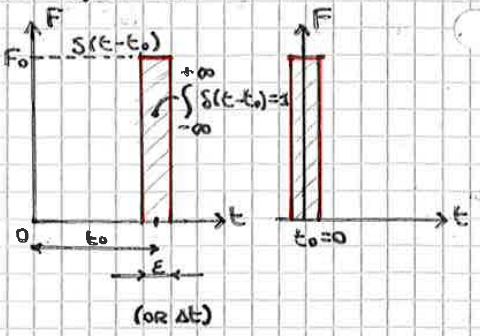
  IF THE I.C. ARE NOT NULL, IT IS NECESSARY TO ADD THE FREE RESPONSE TO THE CONVOLUTION INTEGRAL.

IMPULSE EXCITATION

WE CONSIDER IT AS A NOTO FUNCTION, BUT, REALLY, IT IS A DISTRIBUTION.

DEFINING : $\delta(t-t_0)$ DIRAC-DELTA $\lim_{\varepsilon\to 0} \delta(t-t_0) = \int_{-\infty}^{+\infty} \delta(t-t_0)\,dt = 1$

$$\begin{cases} \delta(t-t_0) \to \infty, & t = t_0 \\ \delta(t-t_0) = 0, & \forall t \neq t_0 \end{cases} \qquad hp_s: \begin{cases} F_0 = 1[N] \\ t_0 = 0[s] \end{cases}$$



THE IMPULSE CAN BE THOUGHT AS :

$I = \lim_{\substack{\varepsilon \to 0 \\ F_0 \to \infty}} F(t)$: IMPULSE OF A FORCE (FINITE VALUE)

$m\ddot{x} + c\dot{x} + kx = 0, \quad \forall t \neq t_0$

$$\begin{cases} x(t=0^-) = x(0^-) = 0 \\ \dot{x}(t=0^-) = \dot{x}(0^-) = 0 \end{cases} \quad \text{I.C. IN } t = 0^- \quad \left(\text{FROM } t \leq -\frac{\varepsilon}{2} \text{ OR } t \geq \frac{\varepsilon}{2}\right)$$

$m\ddot{x} + c\dot{x} + kx = F_0, \quad -\frac{\varepsilon}{2} \leq t \leq +\frac{\varepsilon}{2}$

WE HAVE TO TAKE THE LIMIT FOR $\varepsilon \to 0^+$ AND INTEGRATE :

$$\int_{-\varepsilon/2}^{+\varepsilon/2} (m\ddot{x} + c\dot{x} + kx)\,dt = \int_{-\varepsilon/2}^{+\varepsilon/2} F_0\,dt \Rightarrow m\dot{x}(0^+) + cx(0^+) + \lim_{\varepsilon\to 0} k \int_{-\varepsilon/2}^{+\varepsilon/2} x\,dt = \lim_{\varepsilon \to 0} \int_{-\varepsilon/2}^{+\varepsilon/2} F_0\,dt = I = 1 \quad [N \cdot s]$$

INTEGRATING AGAIN:

$$mx(0^+) + \lim_{\varepsilon\to 0} c \int_{-\varepsilon/2}^{+\varepsilon/2} x\,dt + \lim_{\varepsilon \to 0} k \int_{-\varepsilon/2}^{+\varepsilon/2} \int_{-\varepsilon/2}^{\tau} x\,d\tau\,dt = 0 \quad [N \cdot s^2]$$

(τ IS DUMMY VARIABLE OF INTEGR.) ($\tau = t - t_0$)

ASSUMING $x(t)$ LIMITED (NOT ∞) IN THE INTERVAL $[-\varepsilon/2, +\varepsilon/2]$, THEN ALL THE INT. ARE NULL.

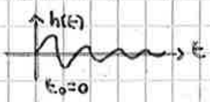
THEREFORE, WE OBTAIN:

$$\begin{cases} m\dot{x}(0^+) = I \\ mx(0^+) = 0 \end{cases} \Rightarrow \begin{cases} \dot{x}(0^+) = I/m \\ x(0^+) = 0 \end{cases} : \text{I.C. IN } t = 0^+$$

$$\begin{cases} x(t) = e^{-\zeta\omega_n t}[a\cos(\omega_d t) + b\sin(\omega_d t)] \\ \dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}[a\cos(\omega t) + b\sin(\omega_d t)] + e^{-\zeta\omega_n t}[-a\omega_d\sin(\omega_d t) + b\omega_d\cos(\omega_d t)] \end{cases}$$

$$\begin{cases} x(0^+) = a = 0 \\ \dot{x}(0^+) = a + b\omega_d = I/m \end{cases} \quad \begin{cases} a = 0 \\ b = I/(m\omega_d) = 1/m\omega_d \end{cases}$$

$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \cdot \sin(\omega_d t)$



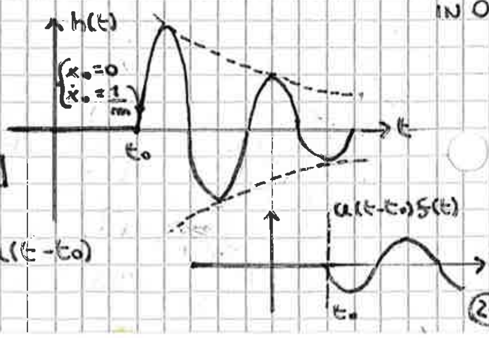
RELATION BETWEEN $\delta(t)$ AND $h(t)$:

$u(t-t_0) = \int_{-\infty}^{t_0} \delta(t - t_0)\,d\tau \iff s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$

$\delta(t-t_0) = \frac{d}{dt}[u(t-t_0)]$ — PROF. E. BONISOLI (PREVIOUSLY I THOUGHT $= \int_{-\infty}^{t_0} \delta(\tau - t_0)\,d\tau$ TO BE CLARIFIED!) (I STILL THINK IT IS LIKE THIS)

TOTAL RESPONSE TO THE IMPULSE (GENERAL FORM - IN CASE THE IMPULSE IS NOT CENTERED IN 0)

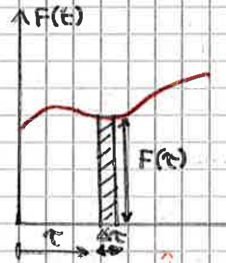
$x(t) = h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n(t-t_0)} \sin[\omega_d(t-t_0)] \cdot u(t-t_0)$



$$\begin{cases} u(t-t_0) = 0 & \text{FOR } t < t_0 \Rightarrow x(t) = 0 \\ u(t-t_0) = 1 & \text{FOR } t \geq t_0 \Rightarrow x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n(t-t_0)} \cdot \sin[\omega_d(t-t_0)] \end{cases}$$

- TAKE ANY FUNCTION $f(t)$ AND MULTIPLY BY STEP FUNCTION $u(t-t_0)$
- ANNIHILATE THE PART OF $f(t)$ BEFORE t_0

CONVOLUTION INTEGRAL



- $F(t)$: ARBITRARY EXCITATION
- FOCUS ON THE CONTRIBUTION TO THE RESPONSE OF AN IMPULSE ACTING OVER $\tau < t < \tau + \Delta\tau$
- hp: $\Delta\tau$ IS SUFFICIENTLY SMALL THAT $F(t)$ DOES NOT CHANGE MUCH OVER THE TIME INCREMENT.
- THE AREA CAN BE CONSIDERED AN IMPULSE OF MAGNITUDE : $F(\tau)\Delta\tau = \hat{F}(\tau)$

$$\hat{F}(\tau)\,\delta(t-\tau) = F(\tau)\Delta\tau\,\delta(t-\tau) \quad : \text{IMPULSE FORCE}$$

THE RESPONSE OF A LINEAR (TIME-INVARIANT) SYSTEM TO THE IMPULSE:

$$\Delta x(t,\tau) = F(\tau)\Delta\tau\, h(t-\tau)$$ (F(t) IS SEEN AS A SUPERPOSITION OF IMPULSIVE FORCES)

$$x(t) = \sum_{\tau} F(\tau)\Delta\tau\, h(t-\tau)$$

IN THE LIMIT, AS $\Delta\tau \to 0$, LET'S REPLACE THE SUMMATORY BY INTEGRATION, OBTAINING:

$$x(t) = \int_0^t F(\tau)\,h(t-\tau)\,d\tau$$: CONVOLUTION INTEGRAL OR SUPERPOSITION INTEGRAL

IT ALLOWS TO COMPUTE THE SYSTEM RESPONSE FOR ANY TIME t PROVIDED THAT THE INITIAL CONDITIONS ARE NULL AND THE IMPULSE RESPONSE IS KNOWN.

IF THE INITIAL CONDITIONS ARE NOT NULL, IT IS NECESSARY TO ADD THE FREE RESPONSE TO THE CONVOLUTION INTEGRAL.

SHIFTING AND FOLDING PROCESS ON $F(\tau)$:

LET'S CARRY OUT THE SHIFTING AND FOLDING PROCESS ON $F(\tau)$ INSTEAD OF $h(\tau)$.

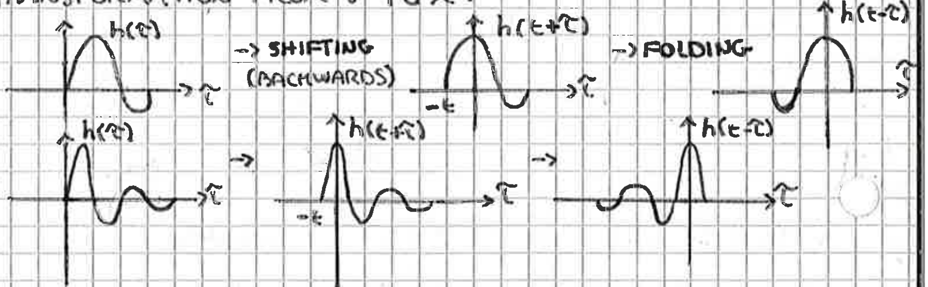
LET'S INTRODUCE A VARIABLE TRANSFORMATION FROM τ TO λ :

$\lambda = t - \tau$

$\tau = t - \lambda$

$d\tau = -d\lambda$

$$\begin{cases} \tau = 0 \rightarrow \lambda = t \\ \tau = t \rightarrow \lambda = 0 \end{cases}$$



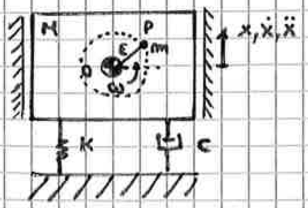
$$x(t) = \int_0^t F(\tau)\,h(t-\tau)\,d\tau = \int_t^0 F(t-\lambda)\,h(\lambda)(-d\lambda) = \int_0^t F(t-\lambda)\,h(\lambda)\,d\lambda$$

τ AND λ ARE DUMMY VARIABLES OF INTEGRATION.

THEREFORE:

$$x(t) = \int_0^t F(\tau)\,h(t-\tau)\,d\tau = \int_0^t F(t-\tau)\,h(\tau)\,d\tau$$

+ WASHING MACHINE



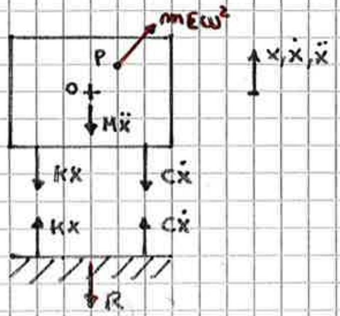
M: TOTAL MASS

m: ECCENTRIC MASS (STATIC UNBALANCE) (MASS NOT SET IN THE g-CENTRE)

$\varepsilon = \overline{OP}$: ECCENTRICITY

$\varepsilon\omega^2 \; (= \varepsilon \cdot \dot{\theta}^2)$: CENTRIFUGAL ACC.

FBD:



$$\begin{cases} \omega \ll \omega_m : \text{SUBCRITICAL BEHAVIOUR} \\ (\omega \sim \omega_m : \text{CRITICAL BEHAVIOUR}) \\ \omega \gg \omega_m : \text{OVERCRITICAL BEHAVIOUR (OR SUPERCRITICAL)} \end{cases}$$

$$M\ddot{x} + C\dot{x} + Kx = m\varepsilon\omega^2 \sin(\omega t) = m\varepsilon\omega^2 e^{i\omega t}$$

$$\ddot{x} + 2\xi\omega_m \dot{x} + \omega_m^2 x = \frac{m}{M}\varepsilon\omega^2 e^{i\omega t}$$

$$x(t) = X_0 e^{i\omega t} \; ; \; \dot{x}(t) = i\omega X_0 e^{i\omega t} \; ; \; \ddot{x}(t) = -\omega^2 X_0 e^{i\omega t}$$

$$\cancel{\omega_m^2}\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\xi\left(\frac{\omega}{\omega_m}\right) + 1\right] X_0 \cancel{e^{i\omega t}} = \frac{m\varepsilon}{M}\left(\frac{\omega}{\omega_m}\right)^2 \cancel{\omega_m^2} \cancel{e^{i\omega t}}$$

$$T_{R\,REL} = \frac{X_0}{\left(\frac{m\varepsilon}{M}\right)} = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega) \; : \; \text{RELATIVE TRANSMISSIBILITY} \; \Big\langle \; \begin{array}{l} |T_{R\,REL}| = \left(\frac{\omega}{\omega_m}\right)^2 \cdot |G(i\omega)| \\ \varphi_- = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left[\frac{-2\xi\left(\frac{\omega}{\omega_m}\right)}{1-\left(\frac{\omega}{\omega_m}\right)^2}\right] \end{array}$$

TRANSMITTED FORCE TO THE GROUND (R):

$$M\ddot{x} + C\dot{x} + Kx = m\varepsilon\omega^2 e^{i\omega t} \Rightarrow \omega_m^2\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\xi\left(\frac{\omega}{\omega_m}\right) + 1\right] X_0 e^{i\omega t} = \overbrace{\frac{m}{M}\varepsilon\omega_m^2}^{F_0/M}\left(\frac{\omega}{\omega_m}\right)^2 e^{i\omega t} = f, \; (F_0 = m\varepsilon\omega_m^2)$$

$$C\dot{x} + Kx = R_0 e^{i\omega t} \Rightarrow \omega_m^2\left[2i\xi\left(\frac{\omega}{\omega_m}\right) + 1\right] X_0 e^{i\omega t} = \frac{R_0}{M} e^{i\omega t} = f_{TR}$$

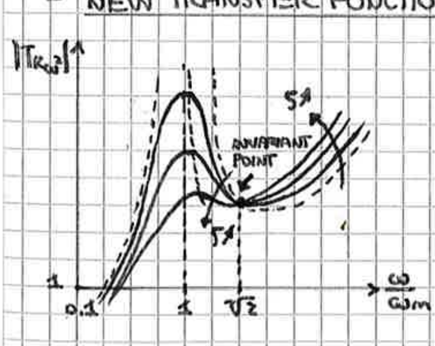
$$T_{R\,ABS} = \frac{f_{TR}}{f} = \frac{\left[1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right]}{\left[1-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right]} = \left[1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right] \cdot G(i\omega) \; \Big\langle \; \begin{array}{l} |T_{R\,ABS}| = \sqrt{1 + \left[2\xi\left(\frac{\omega}{\omega_m}\right)\right]^2} \cdot |G(i\omega)| \\ \varphi_- = Tg^{-1}\left[\frac{-2\xi\left(\frac{\omega}{\omega_m}\right)}{1-(\omega/\omega_m)^2}\right]_D - Tg^{-1}\left[-2\xi\left(\frac{\omega}{\omega_m}\right)\right]_N \end{array}$$

$$T_{R\omega^2} = \frac{R_0}{F_0} = \frac{\left[1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right] \cdot \left(\frac{\omega}{\omega_m}\right)^2}{\left[1-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right]} = \left[1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right] \cdot \left(\frac{\omega}{\omega_m}\right)^2 \cdot G(i\omega) = \begin{cases} \left[1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right] T_{R\,REL} \\ \left(\frac{\omega}{\omega_m}\right)^2 ; T_{R\,ABS} \end{cases}$$

$$|T_{R\omega^2}| = \frac{\sqrt{\left(\frac{\omega}{\omega_m}\right)^4 + \left[2\xi\left(\frac{\omega}{\omega_m}\right)^3\right]^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_m}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_m}\right)\right]^2}} = \left(\frac{\omega}{\omega_m}\right)^2 |T_{R\,ABS}|$$

$$\varphi_- = \varphi_-(T_{R\,ABS})$$

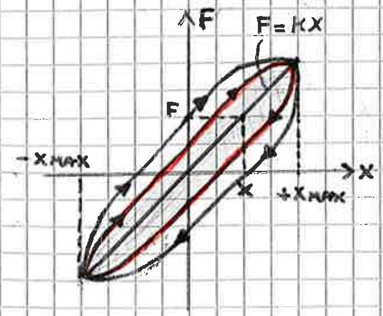
NEW TRANSFER FUNCTION



ENGINEERING COMPROMISE TO SET ξ:

- IF WE DON'T WANT TO DESTROY THE FLOOR WE WOULD LIKE TO HAVE A SMALLER DAMPING ($\xi\downarrow$) IN THE OVERCRITICAL REGIME $\left(\frac{\omega}{\omega_m} > \sqrt{2}\right)$
- INSTEAD, TO DO NOT HAVE THE WASHING MACHINE WITH VERY LARGE AMPLITUDE IN THE TRANSITORY BETWEEN SUB- AND OVER- CRITICAL, WE WOULD LIKE TO HAVE A LARGER DAMPING ($\xi\uparrow$) (FOR $\frac{\omega}{\omega_m} < \sqrt{2}$).

- HYSTERETIC / STRUCTURAL DAMPING

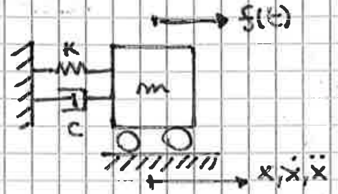


$F = Kx \Rightarrow x = \frac{F}{K}$ (STATIC) , $K \in \mathbb{R}e$

$F_{DYN} = K'x \Rightarrow x = \frac{F_{DYN}}{K'}$ (DYNAMIC), $K' \in \mathbb{C}$

HYSTERETIC AREA = LOST ENERGY ; $W = F \cdot x$ (W or L : WORK)

EQUIVALENT VISCOUS SYSTEM:



$m\ddot{x} + c\dot{x} + kx = f(t)$ (IN A CYCLE) / $m\ddot{x}$, kx ARE CONSERVATIVE $\Rightarrow$ AFTER A C. THEY ARE NE BLE

$L_c = \int_{CYCLE} c\dot{x}\,dx = \int_T c\dot{x}\frac{dx}{dt}dt = \int_T c\dot{x}^2 dt \geq 0$

$\begin{cases} L_c = \Delta E_{HYST} = \alpha \cdot x_0^2 & \text{: IN THE HYSTERETICAL MODEL} \\ L_c = \Delta E_V = \int_T C_{EQ}\dot{x}^2 dt & \text{: IN THE EQUIVALENT VISCOUS SYSTEM} \end{cases}$ $hp_s: \begin{cases} \text{INDEPENDENT FROM } \dot{x} \\ \text{STEADY STATE COND.} \end{cases}$

CONSIDERING $x(t) = x_0\cos(\omega t) \rightarrow \dot{x}(t) = \omega x_0 \sin(\omega t)\cdot(-1)$ (PERSONAL CONSIDERATION).

$L_c = \Delta E_V = \int_T C_{EQ}\dot{x}^2 dt = \int_{T=\frac{2\pi}{\omega}} C_{EQ}(\omega x_0)^2 \sin^2(\omega t)\,dt = C_{EQ}\pi\omega x_0^2$

$\begin{cases} L_c = \Delta E_{HYST} = \alpha \cdot x_0^2 \\ L_c = \Delta E_V = C_{EQ}\pi\omega x_0^2 \end{cases} \Rightarrow C_{EQ}\pi\omega \cancel{x_0^2} = \alpha \cancel{x_0^2} \Rightarrow C_{EQ} = \frac{\alpha}{\pi\omega} = \frac{h}{\omega}$, $h = \frac{\alpha}{\pi}$

THEREFORE:

$m\ddot{x} + C_{EQ}\dot{x} + kx = f(t) = F_0 e^{i\omega t}$ $(\dot{x} = i\omega x \leftrightarrow \dot{x} = i\omega X_0 e^{i\omega t} ; x = X_0 e^{i\omega t} \Rightarrow \dot{x} = i\omega x)$

$m\ddot{x} + \frac{h}{\cancel{\omega}} i\cancel{\omega} x + kx = f(t) = F_0 e^{i\omega t}$

$m\ddot{x} + (k + ih)x = F_0 e^{i\omega t}$ h: IMAGINARY PART OF THE COMPLEX STIFFNESS OF HYST. MODEL

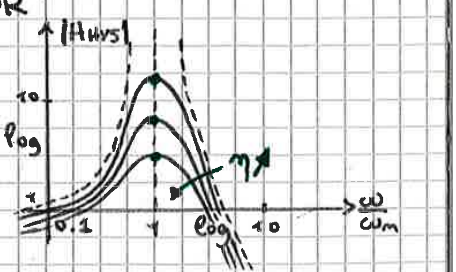
$\ddot{x} + \omega_m^2(1 + i\eta)x = \frac{F_0}{k}\omega_m^2 e^{i\omega t}$ $\eta = \frac{h}{k} \left(= \frac{\alpha}{\pi k}\right)$: LOSS FACTOR

$\cancel{\omega_m^2}\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 1 + i\eta\right] X_0 \cancel{e^{i\omega t}} = \frac{F_0}{k}\cancel{\omega_m^2}\cancel{e^{i\omega t}}$

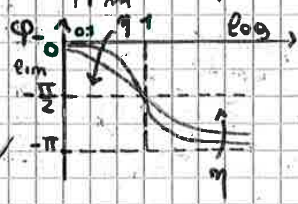
$H_{HYS} = \frac{X_0}{F_0/k} = \frac{1}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + i\eta}$

$|H_{HYS}| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]^2 + \eta^2}}$

$\varphi = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left[\frac{-\eta}{1 - \left(\frac{\omega}{\omega_m}\right)^2}\right]$ DELAY



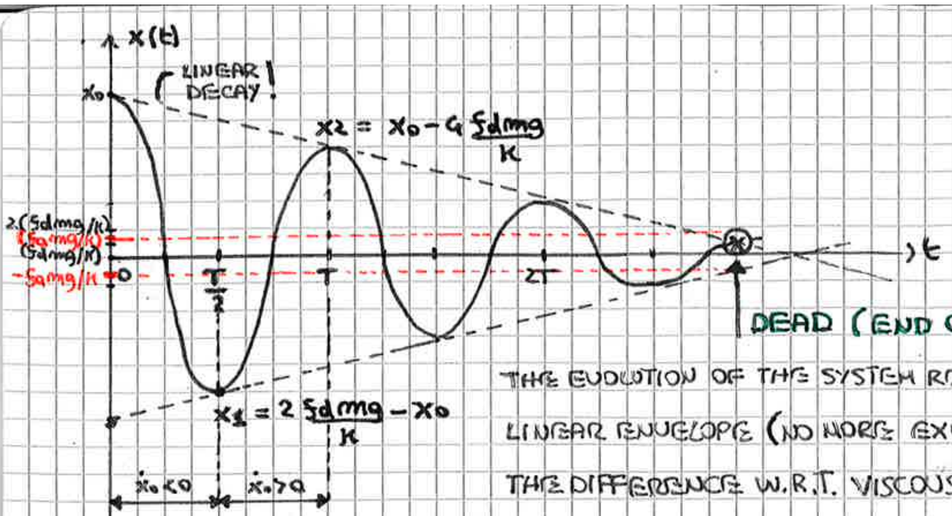
$C_{EQ} = \frac{\alpha}{\pi\omega} = \frac{h}{\omega} = \frac{\eta k}{\omega} \rightsquigarrow \omega = \omega_m$ $\left(\Rightarrow C_{EQ} = \eta \cdot \frac{k}{\omega_m} = \eta\frac{\omega_m^2 m}{\sqrt{\frac{k}{m}}} = \eta\,\omega_m m = \eta\sqrt{\frac{k}{m}}\,m = \eta\sqrt{km}\right)$



THEREFORE:

$C_{EQ} = \eta\sqrt{km}$

$\zeta_{EQ} = \frac{C_{EQ}}{2\sqrt{km}} = \frac{\eta\sqrt{km}}{2\sqrt{km}} = \frac{\eta}{2}$, $\eta = \frac{h}{k} = \frac{\alpha}{\pi k}$ COMPUTED ✓



THE EVOLUTION OF THE SYSTEM RESPONSE IS DESTROYED BY A LINEAR ENVELOPE (NO MORE EXPONENTIAL ONE). THIS IS THE DIFFERENCE W.R.T. VISCOUS SYSTEMS (EXP. ENVELOPE) !

[WITH FRICTION → LINEAR DECAY]

CONSIDERING m = NUMBER OF HALF CYCLES (T/2)

THE MOTION IS STOPPED (↔ ADHESION) WHEN:

$$\left| x_0 - 2m\frac{f_d mg}{K} \right| \leq \frac{f_a mg}{K}$$

$[M][\varphi]\{\ddot{\eta}\} + [K][\varphi]\{\eta\} = \{F\}$

PRE-MULTIPLY BOTH SIDES FOR $[\varphi]^T$

$\underbrace{[\varphi]^T[M][\varphi]}_{[M']}\{\ddot{\eta}\} + \underbrace{[\varphi]^T[K][\varphi]}_{[K']}\{\eta\} = \underbrace{[\varphi]^T\{F\}}_{\{F'\}}$, $[M'], [K']$ ARE SYMMETRIC BECAUSE $[M], [K]$ ARE SYMM. $\Rightarrow [M'] = [M']^T$, $[K'] = [K']^T$

$([M'])^T = ([\varphi]^T[M][\varphi])^T$

$[M'] = [\varphi]^T[M]^T[\varphi]$

NOTE: $[A]\{x\} = \{b\}$

$(m \times m)(m \times 1) = (m \times 1)$

$(\quad)^T = (\quad)^T$

$\{x\}^T[A]^T = \{b\}^T$

$[M']\{\ddot{\eta}\} + [K']\{\eta\} = \{F'\}$

CAN WE HAVE BOTH M' AND K' DIAGONAL? (IN ORDER TO HAVE UNCOUPLING EQ.)

$\Rightarrow$ YES!

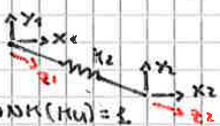
$m'_{R,R}\,\ddot{\eta}_R(t) + k'_{R,R}\,\eta(t) = f'_R(t)$ WITH $R = 1 \dots, m$, m INDEPENDENT EQUATIONS OF MOTION

$[\varphi]$ IS A "SPECIAL MATRIX" (UNIQUE) $\Rightarrow$ MODAL MATRIX / TRANSFORMATION MATRIX

η_R IS A NATURAL COORDINATE (PRINCIPAL)

$$[K] = \begin{bmatrix} k_1 & -k_1 & & \\ -k_1 & k_1+k_2 & -k_2 & \\ & -k_2 & k_2+k_3 & -k_3 \\ & & -k_3 & k_3 \end{bmatrix} \quad \dots \quad \begin{Bmatrix} \cancel{x}_{0,1} \\ x_1 \\ x_2 \\ \cancel{x}_{0,2} \end{Bmatrix}$$

$[K_u] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ RANK$(K_u) = 1$



$F_x = k_{2x}(x_1 - x_2)$
$F_y = k_{2y}(y_1 - y_2)$
$F = k_2(z_1 - z_2)$

ASSEMBLY TECHNIQUE

$[K]$ $\to$ SYMMETRY $\Leftrightarrow [K]^T = [K] \Leftrightarrow k_{j,h} = k_{h,j}$
$(m \times m)$
(MAIN DIAGONAL DOMINANT)

$[M]$ $([M'])$ IS A POSITIVE DEFINITE MATRIX

$[K]$ $([K'])$ IS A SEMI-POSITIVE DEFINITE MATRIX

$[C]$ $([C'])$ IS A SEMI-POSITIVE DEFINITE MATRIX

$\{z\}^T[M]\{z\} > 0$, $\forall \{z\} \in R^m$, $\{z\} \neq \{0\}$

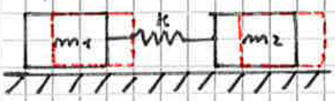
KINETIC ENERGY : $T = \sum_{R=1}^{m} \frac{1}{2} m_R \dot{x}_R^2 = \frac{1}{2}\{\dot{x}\}^T[M]\{\dot{x}\}$ (WHATEVER STATE OF SYSTEMS)

$\{z\}^T[K]\{z\} \geq 0$, $\forall \{z\} \in R^m$, $\{z\} \neq \{0\}$

POTENTIAL ENERGY : $V = \frac{1}{2}\{x\}^T[K]\{x\}$

EXAMPLE

$\omega_1 = 0$



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix}\{\ddot{x}\} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\{x\} = \{F\}$$

$[M] \rightarrow \det([M]) = m_1 \cdot m_2 > 0$ POSIT. DEF.

$[K] \rightarrow \det([K]) = k^2 - k^2 = 0$ SEMI-POSIT. DEF.

$\det([K]) = (k_1+k_2)(k_2+k_3) - k_2^2 > 0$

EIGENVALUE PROBLEM - SYNCHRONOUS SOLUTIONS

LTI SYSTEM n DOFS (LTI: LINEAR-TIME-INVARIANT)

$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$

LOOK FOR SYNCHRONOUS SOLUTIONS (1st THINK TO DO)

$\{x(t)\} = \{x_0\} f(t)$

— TIME SOLUTION $\Rightarrow$ NATURE OF MOTION (IMPOSED LAW)

— CONSTANT VECTOR $\Rightarrow$ MODE SHAPE

$\{\ddot{x}(t)\} = \{x_0\}\ddot{f}(t)$

$[M]\{x_0\}\ddot{f}(t) + [K]\{x_0\} f(t) = \{0\}$

PRE-MULTIPLYING BY $\{x_0\}^T$:

$$\underbrace{\{x_0\}^T[M]\{x_0\}}_{>0}\ddot{f}(t) + \underbrace{\{x_0\}^T[K]\{x_0\}}_{\geq 0} f(t) = \{0\}$$

EIGENVALUE

$$\frac{\ddot{f}(t)}{f(t)} = -\frac{\{x_0\}^T[K]\{x_0\}}{\{x_0\}^T[M]\{x_0\}} = -\omega^2 \leq 0$$

THEREFORE:

$\ddot{f}(t) + \omega^2 f(t) = 0$ PENDULUM

1) $\{\varphi_s\}^T[K]\{\varphi_R\} = \omega_R^2\{\varphi_s\}^T[M]\{\varphi_R\}$

2) $\{\varphi_s\}^T[K]\{\varphi_R\} = \omega_s^2\{\varphi_s\}^T[M]\{\varphi_R\}$

1) - 2) : $(\omega_R^2-\omega_s^2)\{\varphi_s\}^T[M]\{\varphi_R\} = 0$ M-ORTHOGONALITY

IF $R \neq S \Rightarrow \{\varphi_s\}^T[M]\{\varphi_R\} = 0$

IF $R = S \Rightarrow \{\varphi_s\}^T[M]\{\varphi_R\} = m_R \neq 0$ ([M] IS POSITIVE DEFINED)

1) $\frac{1}{\omega_R^2}\{\varphi_s\}^T[K]\{\varphi_R\} = \{\varphi_s\}^T[M]\{\varphi_R\}$

2) $\frac{1}{\omega_s^2}\{\varphi_s\}^T[K]\{\varphi_R\} = \{\varphi_s\}^T[M]\{\varphi_R\}$

1) - 2) : $\left(\frac{1}{\omega_R^2}-\frac{1}{\omega_s^2}\right)\{\varphi_s\}^T[K]\{\varphi_R\} = 0$ K-ORTHOGONALITY

IF $R \neq S \Rightarrow \{\varphi_s\}^T[K]\{\varphi_R\} = 0$

IF $R = S \Rightarrow \{\varphi_s\}^T[K]\{\varphi_R\} = K_R \neq 0$ ([K] IS STRICTLY POSITIVE DEFINED) (PAY ATT. TO RIGID BODY MOTION)

MASS AND STIFFNESS MODAL MATRICES:

$[\varphi]^T[M][\varphi] = [{}^{\nwarrow}m_{R\searrow}]$

$[\varphi]^T[K][\varphi] = [{}^{\nwarrow}K_{R\searrow}]$

$K_R = \omega_R^2 m_R \iff \omega_R = \sqrt{K_R/m_R}$

(2) THEOREM OF THE EXPANSION (MODAL SUPERPOSITION)

- EIGENVECTORS ARE A BASIS OF THE CONFIGURATION SPACE OF THE MDOF SYSTEM.

$\{x\} = [\varphi]\{\eta\}$ (DIRECT MODAL TRANSF. (DMT))

- LINEAR-TIME-INVARIANT (LTI) TRANSFORMATION BETWEEN PHYSICAL COORDINATES $\{x\}$ AND MODAL COORDINATES $\{\eta\}$
- ANY VECTOR CAN BE EXPRESSED AS A LINEAR COMBINATION OF MODE-SHAPES (EIGENVECTORS).
- EIGENVECTORS ARE LINEARLY INDEPENDENT.
- MODAL MATRIX $[\varphi]$ HAS FULL RANK.

PROOF BY CONTRADICTION:

hp: MODAL VECTORS ARE LINEARLY DEPENDENT, THEN:

$c_1\{\varphi_1\}+c_2\{\varphi_2\}+\ldots+c_m\{\varphi_m\} = \sum_{R=1}^{m} c_R\{\varphi_R\} = \{0\}$; WHERE c_R ARE NOT NULL

$\sum_{R=1}^{m} c_R\{\varphi_R\} = \{0\}$

PRE-MULTIPLYING FOR $\{\varphi_s\}^T[M]$

$\sum_{R=1}^{m} c_R \underbrace{\{\varphi_s\}^T[M]\{\varphi_R\}}_{\substack{\text{IF } R\neq S \ =0 \\ \text{IF } R=S \ \neq 0}} = \{0\}$

$c_s\{\varphi_s\}^T[M]\{\varphi_s\} = 0 \Rightarrow c_s = 0$

REPEATING THIS PROCEDURE TAKING ANY $\{\varphi_s\}^T$ IT FOLLOWS THAT $c_R = 0, \forall R$

WE OBTAIN THAT:

$[I]\{\ddot{\eta}\} + [{}^{\diagdown}\omega_R^2{}_{\diagdown}]\{\eta\} = \{\Gamma\}$

$\ddot{\eta}_R(t) + \omega_R^2 \eta_R(t) = \Gamma_R(t)$: R (1,..,m) UNCOUPLED PENDULUMS

($\{x(t)\} = [\Phi]\{\eta(t)\}$: EVOLUTION OF GEOMETRICAL COORD.)($\eta(t)$: MODAL COORDINATE)

(END OF UNDAMPED)

2.2 MODAL ANALYSIS FOR DAMPED SYSTEM

hp: NO EXTERNAL EXCITATION

$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\}$

MODAL ANALYSIS:

$\{x\} = [\varphi]\{\eta\}$

$[M][\varphi]\{\ddot{\eta}\} + [C][\varphi]\{\dot{\eta}\} + [K][\varphi]\{\eta\} = \{0\}$

PRE-MULTIPLYING FOR $[\varphi]^T$:

$[\varphi]^T[M][\varphi]\{\ddot{\eta}\} + [\varphi]^T[C][\varphi]\{\dot{\eta}\} + [\varphi]^T[K][\varphi]\{\eta\} = \{0\}$

$[{}^{\diagdown}m_R{}_{\diagdown}]\{\ddot{\eta}\} + \underbrace{[\varphi]^T[C][\varphi]}_{\text{NOT DIAGONAL}}\{\dot{\eta}\} + [{}^{\diagdown}K_R{}_{\diagdown}]\{\eta\} = \{0\}$

NOT DIAGONAL <=> DAMPING COUPLING, "NON PROPORTIONAL DAMPING"

"PROPORTIONAL DAMPING": $[C_P] = [\varphi]^T[C][\varphi] = [{}^{\diagdown}C_R{}_{\diagdown}]$, REASONABLE IF DAMPING IS SMALL.

CAUGHEY RELATIONS:

- PERMUTATION OF $[M]^{-1}$:

  $[C][M]^{-1}[K] = [K][M]^{-1}[C]$, CONSIDERING $[C] = [C_P]$ DIAGONAL

- CAUGHEY SERIES $[C_P]$

  $[C_P] = \sum_{i=0}^{m-1} \gamma_i [M]([M]^{-1}[K])^i$

- SIMPLEST CASE $\gamma_0, \gamma_1 \neq 0$:

  $[C_P] = \alpha[M] + \beta[K]$, CALLED "PROPORTIONAL DAMPING"

  WHERE: $\alpha \to [M]$: SKY-HOOK DAMPERS

  $\beta \to [K]$: MATERIAL DEFORMATIONS

$[\varphi]^T[C_P][\varphi] = [\varphi]^T(\alpha[M] + \beta[K])[\varphi] = \alpha[{}^{\diagdown}m_R{}_{\diagdown}] + \beta[{}^{\diagdown}K_R{}_{\diagdown}]$

MOTION EQUATION:

$m_R\ddot{\eta}_R + (\alpha m_R + \beta K_R)\dot{\eta}_R + K_R\eta_R = 0$, $R = 1,\dots,m$

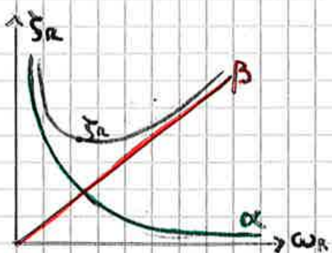
THIS CAN BE RE-WRITTEN AS:

$\ddot{\eta}_R + 2\zeta_R\omega_R\dot{\eta}_R + \omega_R^2\eta_R = 0$

WITH:

$\omega_R = \sqrt{\frac{K_R}{m_R}}$

$\zeta_R = \frac{\alpha}{2\cdot\omega_R} + \frac{\beta}{2}\cdot\omega_R$



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FREE RESPONSE MCK (PROP.)

$$[M]\{\ddot{x}\} + [C_P]\{\dot{x}\} + [K]\{x\} = \{0\}$$

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \longrightarrow \begin{cases} [\varphi] = [\{\varphi_1\};\{\varphi_2\};\dots;\{\varphi_m\}] & \text{: MODAL EIGENVECTOR MATRIX} \\ [\Lambda] = [{}^{\nwarrow}\omega_R^2{}_{\searrow}] = diag(\{\omega_R^2\}) & \text{: EIGENVALUE MATRIX} \end{cases}$$

$\{x\} = [\varphi]\{\eta\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_R$: SUPERPOSITION PRINCIPLE

$[C_P] = \alpha[M] + \beta[K]$: PROPORTIONAL DAMPING

$$[\varphi]^T[M][\varphi]\{\ddot{\eta}\} + [\varphi]^T(\alpha[M]+\beta[K])[\varphi]\{\dot{\eta}\} + [\varphi]^T[K][\varphi]\{\eta\} = [\varphi]^T\{0\} = \{0\}$$

$$[{}^{\nwarrow}m_R{}_{\searrow}]\{\ddot{\eta}\} + (\alpha[{}^{\nwarrow}m_R{}_{\searrow}] + \beta[{}^{\nwarrow}K_R{}_{\searrow}])\{\dot{\eta}\} + [{}^{\nwarrow}K_R{}_{\searrow}]\{\eta\} = \{0\}$$

$$m_R\ddot{\eta}_R + (\alpha m_R + \beta K_R)\dot{\eta}_R + K_R\eta_R = 0 \quad , \text{FOR } R = 1,\dots,m$$

$$\ddot{\eta}_R + 2\zeta_R\omega_R\dot{\eta}_R + \omega_R^2\eta_R = 0 \quad , \text{FOR } R=1,\dots,m \quad , \text{WITH } \omega_R = \sqrt{K_R/m_R}\ ,\ \zeta_R = \frac{\alpha}{2\cdot\omega_R} + \frac{\beta}{2}\cdot\omega_R$$

FREE RESPONSE:

$$\eta_R(t) = e^{-\zeta_R\omega_R t}\left[A_R\cos(\omega_R\sqrt{1-\zeta_R^2}\,t) + B_R\sin(\omega_R\sqrt{1-\zeta_R^2}\,t)\right]$$

LET'S IMPOSE THE INITIAL CONDITIONS (GEOMETRICAL COORD.):

$$\begin{cases}\{x(t=0)\} = \{x_0\} \\ \{\dot{x}(t=0)\} = \{\dot{x}_0\}\end{cases}$$

INVERSE MODAL TRANSFORMATION (→ MODAL COORD.):

$\{\eta\} = [\varphi]^{-1}\{x\}$ (NOTE: IF WE CONSIDER $[\Phi]^T[M][\Phi] = [I] \Rightarrow [\Phi]^{-1} = [\Phi]^T[M]$)

CONSIDERING THE SUPERPOSITION PRINCIPLE AND IMPOSING THE I.C.:

$$\{x\} = [\varphi]\{\eta\} = \sum_{R=1}^{m}(\{\varphi_R\}\eta_R)$$

$$\{x\} = \sum_{R=1}^{m}\{\varphi_R\}e^{-\zeta_R\omega_R t}[A_R\cos(\omega_{dR}t) + B_R\sin(\omega_{dR}t)]$$

$$\{\dot{x}\} = \sum_{R=1}^{m}\{\varphi_R\}\{-\zeta_R\omega_R e^{-\zeta_R\omega_R t}[A_R\cos(\omega_{dR}t)+B_R\sin(\omega_{dR}t)] + e^{-\zeta_R\omega_R t}[-A_R\omega_{dR}\sin(\omega_{dR}t) + B_R\omega_{dR}\cos(\omega_{dR}t)]\}$$

PRE-MULTIPLYING FOR $\{\varphi_S\}^T[M]$:

$$\{\varphi_S\}^T[M]\{x\} = \sum_{R=1}^{m}\{\varphi_S\}^T[M]\{\varphi_R\}e^{-\zeta_R\omega_R t}[A_R\cos(\omega_{dR}t) + B_R\sin(\omega_{dR}t)]$$

AT TIME $t=0$:

$$\{\varphi_S\}^T[M]\{x_0\} = \sum_{R=1}^{m}\{\varphi_S\}^T[M]\{\varphi_R\}A_R = m_S A_S$$

$$\{\varphi_S\}^T[M]\{\dot{x}_0\} = \sum_{R=1}^{m}\{\varphi_S\}[M]\{\varphi_R\}\{-\zeta_R\omega_R A_R + B_R\omega_{dR}\} = m_S(B_S\omega_{dS} - \zeta_S\omega_S A_S)$$

$$\begin{cases} A_R = \dfrac{\{\varphi_R\}^T[M]\{x_0\}}{m_R} \\ B_R = \dfrac{\{\varphi_R\}^T[M]}{m_R\,\omega_{dR}}\left[\{\dot{x}_0\} + \zeta_R\omega_R\{x_0\}\right]\end{cases}$$

2) MODAL APPROACH (ONLY WITH M C K PROP.)

$$[\varphi]^T[M][\varphi]\{\ddot\eta\} + [\varphi]^T[C][\varphi]\{\dot\eta\} + [\varphi]^T[K][\varphi]\{\eta\} = [\varphi]^T\{F\} = [\varphi]^T\{F_0\}e^{i\omega t} = \{\Gamma\}e^{i\omega t}$$

$$[{}^\diagdown m_R {}_\diagdown]\{\ddot\eta\} + [{}^\diagdown c_R {}_\diagdown]\{\dot\eta\} + [{}^\diagdown k_R {}_\diagdown]\{\eta\} = [\varphi]^T\{F_0\}e^{i\omega t} = \{\Gamma\}e^{i\omega t} \quad \Big| \quad \{\varphi_R\}^T\{F_0\}e^{i\omega t} = \Gamma_R e^{i\omega t}$$

$$m_R\ddot\eta_R + c_R\dot\eta_R + k_R\eta_R = \{\varphi_R\}^T\{F_0\}e^{i\omega t} = \Gamma_R e^{i\omega t} \qquad R = 1, \dots, m \quad (m \text{ INDIP. PENDULUMS})$$

CONSIDERING:

$$\{\eta\} = \{\eta_0\}e^{i\omega t}, \quad \{\dot\eta\} = i\omega\{\eta_0\}e^{i\omega t}, \quad \{\ddot\eta\} = -\omega^2\{\eta_0\}e^{i\omega t} \qquad \text{WITH } \eta_{R,0} \in \mathbb{C}$$

$$-m_R\omega^2\eta_{R,0}e^{i\omega t} + c_R i\omega\eta_{R,0}e^{i\omega t} + k_R\eta_{R,0}e^{i\omega t} = \{\varphi_R\}^T\{F_0\}e^{i\omega t}$$

$$(k_R + i\omega c_R - \omega^2 m_R)\eta_{R,0}e^{i\omega t} = \{\varphi_R\}^T\{F_0\}e^{i\omega t}$$

$$(k_R + i\omega c_R - \omega^2 m_R)\eta_{R,0} = \{\varphi_R\}^T\{F_0\}$$

$$\eta_{R,0} = \frac{\{\varphi_R\}^T\{F_0\}}{k_R + i\omega c_R - \omega^2 m_R} \qquad (\text{= SCALAR})$$

CONSIDERING:

$$\{x_0\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_{R,0} \quad \in \mathbb{C}$$

THEREFORE:

$$\{x_0\} = \sum_{R=1}^{m}\frac{\{\varphi_R\}^T\{F_0\}\{\varphi_R\}}{k_R + i\omega c_R - \omega^2 m_R} \qquad (\{x_0\} = [\alpha]\{F_0\})$$

FOR SINGLE FORCE EXCITATION:

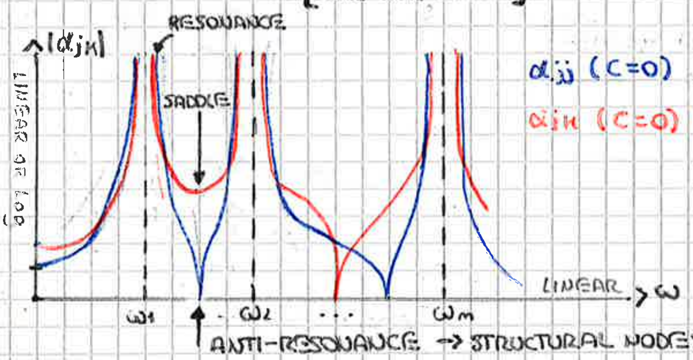
$$\alpha_{j,k}(\omega) = \frac{x_j}{F_k} = \sum_{R=1}^{m}\frac{\varphi_{j,R}\cdot\varphi_{k,R}}{k_R + i\omega c_R - \omega^2 m_R} = \alpha_{jk} = \alpha_{kj} \text{ : CROSS-RECEPTANCE (SYMMETRIC)}$$

$$\alpha_{j,j}(\omega) = \frac{x_j}{F_j} = \sum_{R=1}^{m}\frac{\varphi_{j,R}^2}{k_R + i\omega c_R - \omega^2 m_R} = \alpha_{jj} = \text{AUTO-RECEPTANCE (DIAGONAL)}$$

THEN:

$$\alpha_{jk}, \alpha_{jj} \rightarrow [\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mm} \end{bmatrix} = \text{RECEPTANCE MATRIX (SYMMETRIC)}$$

(CAUSE <=> EFFECT)

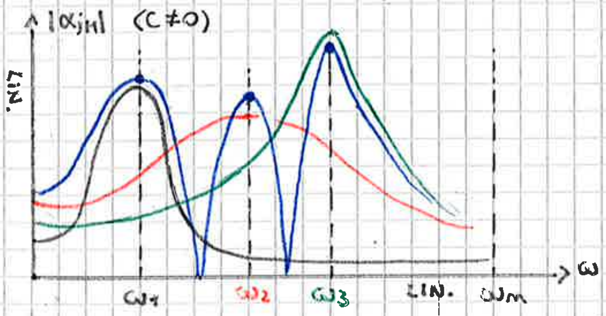


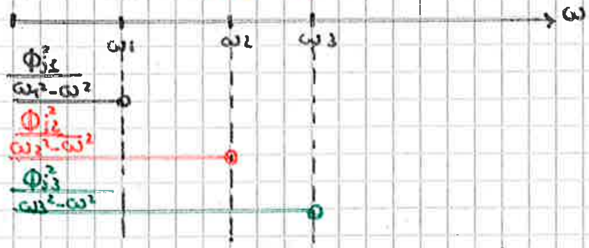
$$\begin{cases} \alpha_{jj} \rightarrow m \text{ RESONANCES}, \ m-1 \text{ ANTI-RESONANCES} \\ \alpha_{jk} \rightarrow m \text{ RESONANCES}, \ \text{ANTI-RESONANCES OR SADDLES} \end{cases}$$

IF USING UNITARY MODAL MASS NORMALISATION

$$\alpha_{jk}(\omega) = \frac{x_j}{F_k} = \sum_{R=1}^{m}\frac{\phi_{j,R}\phi_{k,R}}{\omega_R^2 + 2i\xi_R\omega_R\omega - \omega^2}, \text{ WITH: } \begin{cases} m_R = 1 \\ c_R = 2\xi_R\omega_R \\ k_R = \omega_R^2 \end{cases}$$

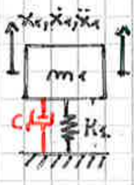
FOR AUTO-RECEPTANCE ($\alpha_{jj}(\omega)$), CONSIDERING $m_R = 1$, $C = 0$:





$$\alpha_{jj}(\omega) = \sum_{R=1}^{m} \frac{\Phi_{jR}^2}{\omega_R^2 + 2i\xi_R\omega_R\omega - \omega^2} = \sum_{R=1}^{m} \frac{\Phi_{jR}^2}{\omega_R^2 - \omega^2} = \frac{\Phi_{j1}^2}{\omega_1^2-\omega^2} + \frac{\Phi_{j2}^2}{\omega_2^2-\omega^2} + \frac{\Phi_{j3}^2}{\omega_3^2-\omega^2} + \dots + \frac{\Phi_{jm}^2}{\omega_m^2-\omega^2}$$

+ DYNAMIC ABSORBER

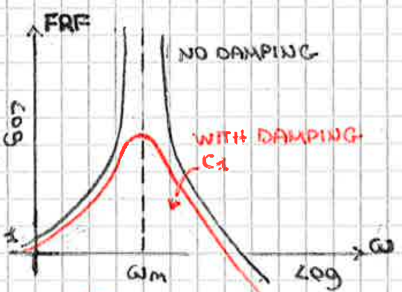


$$m_1\ddot{x}_1 + K_1x_1 = F(t) = F_0\cos(\omega t) = F_0 e^{i\omega t}$$

$$x_1 = X_{1,0}\cos(\omega t + \varphi) = X_{1,0}e^{i\omega t}$$

$$(-\omega^2 m_1 + K_1)X_{1,0}e^{i\omega t} = F_0 e^{i\omega t} \Rightarrow X_{1,0} = \frac{F_0}{K_1 - \omega^2 m_1} \quad ; \quad \omega_m = \omega_1 = \sqrt{\frac{K_1}{m_1}}$$

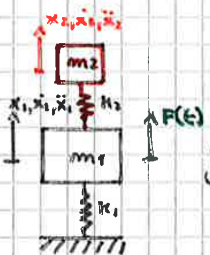
$$F_t = K_1x_1 \Rightarrow \frac{F_t}{F_0} = \frac{K_1X_{1,0}}{(K_1-\omega^2m_1)X_{1,0}} = \frac{K_1}{K_1-\omega^2m_1}$$

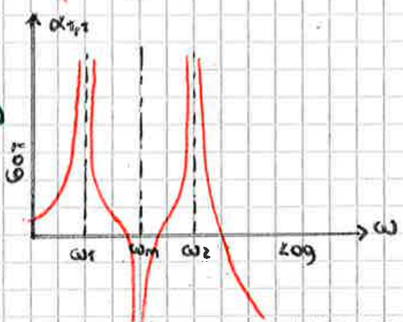


TO DESIGN DIFFERENTLY ?

⇒ WE ADD AN ADDITIONAL m-K SUB-SYSTEM

⇒ DYNAMIC ABSORBER





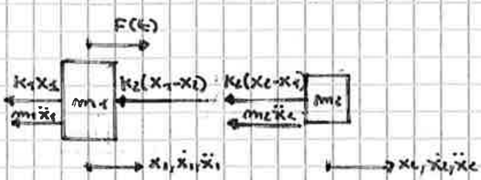
TUNED MASS DAMPER EFFECT

$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \cdot \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} F_0\cos(\omega t)$$

BECAUSE:

$\leftarrow$) $m_1\ddot{x}_1 + (K_1+K_2)x_1 - K_2x_2 = F_0\cos(\omega t)$

$\leftarrow$) $m_2\ddot{x}_2 - K_2x_1 + K_2x_2 = 0$



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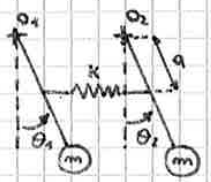
+ BEAT PHENOMENON

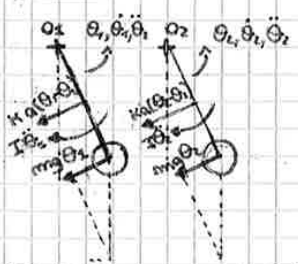
- 2 IDENTICAL SDOF CONNECTED BY A "SOFT SPRING" ⇒ WEAK COUPLING
- hp: SMALL OSCILLATIONS ⇒ LINEARISED SYSTEM



$$\begin{cases} x_1 \cong a\theta_1 \\ x_2 \cong a\theta_2 \end{cases}$$

$|F_{SPRING}| = k|x_1 - x_2| = ka|\theta_1 - \theta_2|$; $I = m\ell^2$ $[Kg\, m^2]$

FBD:



$\overset{\curvearrowright}{O_1}$ $I\ddot{\theta}_1 + a\cdot ka(\theta_1-\theta_2) + \ell\cdot mg\theta_1 = 0 \Rightarrow m\ell^2\ddot{\theta}_1 + (mg\ell + ka^2)\theta_1 - ka^2\theta_2 = 0$

$\overset{\curvearrowright}{O_2}$ $I\ddot{\theta}_2 + a\cdot ka(\theta_2-\theta_1) + \ell mg\theta_2 = 0 \Rightarrow m\ell^2\ddot{\theta}_2 + (mg\ell + ka^2)\theta_2 - ka^2\theta_1 = 0$

$$\begin{bmatrix} m\ell^2 & \\ & m\ell^2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} mg\ell + ka^2 & -ka^2 \\ -ka^2 & mg\ell + ka^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\det([K] - \omega^2[M]) = 0 \rightarrow \omega_1 = \sqrt{\frac{g}{\ell}}$, $\omega_2 = \sqrt{\frac{g}{\ell} + 2\frac{k}{m}\frac{a^2}{\ell^2}}$, BECAUSE:

$$\det([K] - \omega^2[M]) = \det\begin{pmatrix} mg\ell + ka^2 - \omega^2 m\ell^2 & -ka^2 \\ -ka^2 & mg\ell + ka^2 - \omega^2 m\ell^2 \end{pmatrix} = (mg\ell + ka^2 - \omega^2 m\ell^2)^2 - (-ka^2)^2 = 0$$

$(mg\ell + ka^2 - \omega^2 m\ell^2)^2 - k^2a^4 = 0 \Rightarrow mg\ell + ka^2 - \omega^2 m\ell^2 = ka^2 \Rightarrow \omega_1 = \sqrt{\frac{g}{\ell}}$

$(mg\ell + ka^2 - \omega^2 m\ell^2)^2 = (-ka^2)^2 \Rightarrow mg\ell + ka^2 - \omega^2 m\ell^2 = -ka^2 \Rightarrow \omega_2 = \sqrt{\frac{g}{\ell} + 2\frac{k}{m}\frac{a^2}{\ell^2}}$

$\{\varphi_1\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$; $\{\varphi_2\} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = C_1\cos(\omega_1 t - \alpha_1)\{\varphi_1\} + C_2\cos(\omega_2 t - \alpha_2)\{\varphi_2\}$

$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = -\omega_1 C_1 \sin(\omega_1 t - \alpha_1)\{\varphi_1\} - \omega_2 C_2 \sin(\omega_2 t - \alpha_2)\{\varphi_2\}$

I.C. $\begin{cases} \theta_1(t=0) = \theta_0 \\ \theta_2(t=0) = 0 \end{cases}$ $\begin{cases} \dot{\theta}_1(t=0) = 0 \\ \dot{\theta}_2(t=0) = 0 \end{cases}$

$$\begin{cases} \theta_0 = C_1\cos\alpha_1 + C_2\cos\alpha_2 \\ 0 = C_1\cos\alpha_1 - C_2\cos\alpha_2 \end{cases}$$

$$\begin{cases} 0 = -\omega_1 C_1 \sin(-\alpha_1) - \omega_2 C_2 \sin(-\alpha_2) \\ 0 = -\omega_1 C_1 \sin(-\alpha_1) + \omega_2 C_2 \sin(-\alpha_2) \end{cases}$$

$\Rightarrow \alpha_1 = \alpha_2 = 0$, $C_1 = C_2 = \frac{\theta_0}{2}$

$$\begin{cases} \theta_1 = \frac{1}{2}\theta_0\cos(\omega_1 t) + \frac{1}{2}\theta_0\cos(\omega_2 t) \\ \theta_2 = \frac{1}{2}\theta_0\cos(\omega_1 t) - \frac{1}{2}\theta_0\cos(\omega_2 t) \end{cases}$$

$\{\varphi_1\}$ $\{\varphi_2\}$

+ LAGRANGE'S EQUATIONS

$\begin{cases} T = \text{KINETIC ENERGY} \\ V = \text{POTENTIAL ENERGY} \end{cases}$ $\begin{cases} T = \frac{1}{2}\{\dot{x}\}^T[M]\{\dot{x}\} \\ V = \frac{1}{2}\{x\}^T[K]\{x\} \end{cases}$ IF ONLY ONE L. COORD. $q_1 = x_1 = x \Rightarrow \begin{cases} T = \frac{1}{2}M\dot{x}^2 \left(+ \frac{1}{2}I_0\dot{\theta}^2 > 0\right) \\ V = \frac{1}{2}Kx^2 \left(+ \frac{1}{2}K_t\theta^2 \geq 0\right) \end{cases}$

$L = T - V$ = LAGRANGIAN SCALAR

$L = L(q_i, \dot{q}_i)$; q_i = LAGRANGIAN COORDINATE

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$, $k = 1, \ldots, m$ (m DOFs) , Q_k = GENERALISED FORCES $\begin{cases} \text{- EXTERNAL FORCES} \\ \text{- DISSIPATIVE FORCES} \end{cases}$

$Q_k = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial q_k} = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \dot{\vec{R}}_i}{\partial \dot{q}_k}$; R_i = GENERIC COORDINATE POSITION

FOR GENERALISED FORCES:

$\vec{R}_i = \vec{R}_i(q_1; q_2; \ldots; q_m)$

$\delta\vec{R}_i = \delta\vec{R}_i(q_1; q_2; \ldots; q_m) = \frac{\partial \vec{R}_i}{\partial q_1}\partial q_1 + \frac{\partial \vec{R}_i}{\partial q_2}\partial q_2 + \ldots + \frac{\partial \vec{R}_i}{\partial q_m}\partial q_m$

$\dot{\vec{R}}_i = \frac{d\vec{R}_i}{dt} = \frac{\partial \vec{R}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{R}_i}{\partial q_2}\frac{dq_2}{dt} + \ldots + \frac{\partial \vec{R}_i}{\partial q_m} \cdot \frac{dq_m}{dt}$

NOT CONSIDERING THE EXTERNAL FORCES, IF DISSIPATIVE FORCES ARE ONLY VISCOUS:

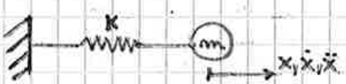
$\Rightarrow$ RAYLEIGHT DISSIPATIVE FUNCTION (D)

$Q_k = -\frac{\partial D}{\partial \dot{q}_k}$, $D = \frac{1}{2}\sum_{j,k=1}^{m} c_{jk}\dot{q}_k\dot{q}_j$, $D = \frac{1}{2}\{\dot{x}\}^T[C]\{\dot{x}\}$, IF $m = 1$: $q_1 = x_1 = x$, $D = \frac{1}{2}c\dot{x}^2$

IF THERE ARE AN EXTERNAL FORCE PLUS VISCOUS DISSIPATIVE FORCE:

$Q_k = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \dot{\vec{R}}_i}{\partial \dot{q}_k} = \vec{F}_1 \frac{\partial \dot{\vec{R}}_1}{\partial \dot{q}_k} - \frac{\partial D}{\partial \dot{q}_k} = \vec{F}_1 \frac{\partial \vec{R}_1}{\partial q_k} - \frac{\partial D}{\partial \dot{q}_k}$

EXAMPLE 1:



$\begin{cases} T = \frac{1}{2}M\dot{x}^2 \\ V = \frac{1}{2}Kx^2 \end{cases} \Rightarrow L = T - V = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}Kx^2$; $q_k = x$

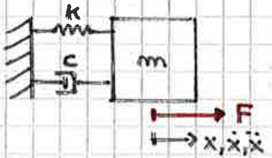
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$

$\begin{cases} \frac{\partial L}{\partial q_k} = \frac{\partial L}{\partial x} = -Kx \\ \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{x}} = M\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = M\ddot{x} \end{cases}$

$Q_k = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \dot{R}_i}{\partial \dot{q}_k} = 0$

$\Rightarrow M\ddot{x} + Kx = 0$

EXAMPLE 2:



$\begin{cases} T = \frac{1}{2}M\dot{x}^2 \\ V = \frac{1}{2}Kx^2 \\ D = \frac{1}{2}c\dot{x}^2 \end{cases}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \iff M\ddot{x} + Kx = Q_k$

$Q_k = \sum \vec{F}_i \frac{\partial \dot{R}_i}{\partial \dot{q}_k} = \sum \vec{F}_i \frac{\partial \vec{R}_i}{\partial q_k} = F \cdot \vec{1} \frac{\partial (x\vec{1})}{\partial x} - \frac{\partial D}{\partial \dot{x}} = F - c\dot{x}$

$\Rightarrow M\ddot{x} + c\dot{x} + Kx = F$

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BOUNDARY VALUE PROBLEM (PDE + I.C. + B.C.)

- II ORDER PARTIAL DERIVATIVE EQUATION (II ORDER PDE):

$$\frac{\partial}{\partial x}\left[T\frac{\partial y}{\partial x}\right] + f(x,t) = \rho\frac{\partial^2 y}{\partial t^2}$$

- INITIAL CONDITIONS (I.C.)

$$\begin{cases} y(x,0) = y_0(x) \\ \left.\frac{\partial y}{\partial x}\right|_{t=0} = \dot{y}_0(x) \end{cases}$$

- BOUNDARY CONDITIONS (B.C.) $\begin{cases} \text{GEOMETRIC (ESSENTIAL)} \\ \text{NATURAL (DYNAMIC)} \end{cases}$

$$\begin{cases} y(0,t) = 0 \\ y(L,t) = 0 \end{cases}$$

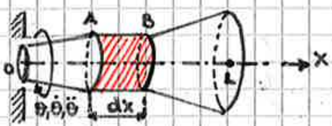
IN CASE OF FLEXIBLE CONSTRAINT:

$$\begin{cases} y(0,t) = 0 \\ y(L,t) \neq 0 \end{cases}$$

: FBD (FOR $y>0, \theta>0$)

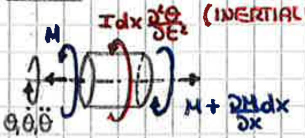
$\Rightarrow$ IN B: FORCE EQUILIBRIUM: $\uparrow)\ -T\left.\frac{\partial y}{\partial x}\right|_{x=L} - K\cdot y\Big| = 0$

3.3 SHAFT TORSIONAL VIBRATIONS (ONLY ROTATIONAL DEFORMATIONS)



$$\begin{cases} A \rightarrow \theta \\ B \rightarrow \theta + \frac{\partial \theta}{\partial x} dx \end{cases}$$

FBD:



$\circlearrowleft$) $-M + M + \frac{\partial M}{\partial x} dx - I dx \frac{d^2\theta}{dt^2} = 0$, WITH I : MASS MOMENT PER UNIT LENGTH $[Kg \cdot m^2/m] = [Kg \cdot m]$

$$\frac{\partial M}{\partial x} = I \frac{\partial^2 \theta}{\partial t^2}$$

FOR CIRCULAR SECTIONS : $M = GJ \, \partial\theta/\partial x$ $\begin{cases} G : \text{SHEAR MODULUS } [Pa] = [N/m^2] \\ J : \text{AREA MOMENT OF INERTIA } [m^4] \end{cases}$

$$\frac{\partial}{\partial x}\left[GJ \frac{\partial \theta}{\partial x}\right] = I \frac{\partial^2\theta}{\partial t^2}$$: PDE OF THE SHAFT

IF $GJ = \text{CONST}$ ($\partial(GJ)/\partial x = 0$):

$$GJ \frac{\partial^2\theta}{\partial x^2} = I \frac{\partial^2\theta}{\partial t^2}$$: WAVE EQUATION

$$\frac{\partial^2\theta}{\partial t^2} = \frac{GJ}{I}\frac{\partial^2\theta}{\partial x^2} = c^2 \frac{\partial^2\theta}{\partial x^2}$$, c : SOUND/PERTURBATION VELOCITY $[m/s]$

$$c^2\left[\frac{m^2}{s^2}\right] = \frac{GJ}{I} \; [Pa][m^4]\frac{1}{[Kg \cdot m]} = \left[\frac{N}{m^2}\right]\frac{[m^4]}{[Kg \cdot m]} = \left[\frac{Kg \cdot m}{s^2 \cdot m^2}\right]\frac{[m^4]}{[Kg \cdot m]} = \left[\frac{m^2}{s^2}\right]$$

- BOUNDARY CONDITIONS (B.C.)

$$\begin{cases} \theta(0,t) = 0 \\ M(L,t) = 0 \Rightarrow \left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0 \end{cases}$$

$EJ\frac{\partial^4 y}{\partial x^4} + \mu\frac{\partial^2 y}{\partial t^2} = 0$: EULER-BERNOULLI BEAM

$y(x,t) = \phi(x)\eta(t)$: SOLUTION

$EJ\frac{d^4\phi}{dx^4}\eta(t) + \mu\phi\frac{d^2\eta}{dt^2} = 0$

$EJ\phi^{IV}\eta + \mu\phi\ddot{\eta} = 0$

$EJ\frac{\phi^{IV}}{\phi} = -\mu\frac{\ddot{\eta}}{\eta}$: HARMOTIC MOTION, WITH: $\begin{cases}\eta(t) = \eta_0 \sin(\omega t + \alpha)\\ \ddot{\eta}(t) = -\omega^2\eta_0\sin(\omega t + \alpha)\end{cases}$

CONSIDERING: $\beta^4 = \frac{\omega^2\mu}{EJ}$, $\beta = \beta(\omega^2)$: SPATIAL EIGENVALUE, RELATED TO TIME EIGENVALUE (ω^2)

$\frac{d^4\phi}{dx^4} - \beta^4\phi = 0$

THIS PROVIDES THE EIGENFUNCTIONS:

$\phi(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x)$

$\begin{cases}\sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}\\ \cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}\end{cases}$ $\begin{cases}\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}\\ \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}\end{cases}$

A, B, C, D ARE DEPENDENT AND CONSTANT ON B.C.

- TYPICALLY USED BOUNDARY CONDITIONS (BC)

1) FREE END 

$\begin{cases}M(P,t) = EJ\frac{\partial^2 y}{\partial x^2}\Big|_{P,t} = 0\\ Q(P,t) = -EJ\frac{\partial^3 y}{\partial x^3}\Big|_{P,t} = 0\end{cases}$

IF: 

$\begin{cases}M(P,t) = EJ\frac{\partial^2 y}{\partial x^2}\Big|_{P,t} = 0\\ Q(P,t) = -EJ\frac{\partial^3 y}{\partial x^3}\Big|_{P,t} = -m\frac{\partial^2 y}{\partial t^2}\Big|_{x}\end{cases}$

IF:  (SPRING SUPPORTED END)

$M(P,t) = EJ\frac{\partial^2 y}{\partial x^2}\Big|_{P,t} = 0$

$Q(P,t) = -EJ\frac{\partial^3 y}{\partial x^3}\Big|_{P,x} = -ky(L,t)$

2) PINNED END  ($P=0$; $P=L$; ..)

$\begin{cases}y(P,t) = 0 \quad \text{(NULL DISPLACEMENT)}\\ M(P,t) = EJ\frac{\partial^2 y}{\partial x^2}\Big|_{P,t} = 0 \quad \text{(NULL MOMENT)}\end{cases}$

3) CLAMPED END  (FIXED) ($P=0$; $P=L$; ...)

$\begin{cases}y(P,t) = 0 \quad \text{(NULL DISPLACEMENT)}\\ \frac{\partial y}{\partial x}\Big|_{P,t} = 0 \quad \text{(NULL ROTATION)}\end{cases}$

SOLUTION OF WAVE EQUATION (FLEXURAL OSCILLATIONS)

CONSIDERING 3.4 (BEAM FLEXURAL V.), IN PARTICULAR THE EULER-BERNOULLI PDE:

$$EJ\frac{\partial^4 y}{\partial x^4} + \mu\frac{\partial^2 y}{\partial t^2} = 0$$

$$y(x,t) = \phi(x)\,\eta(t)$$

$$EJ\,\eta(t)\frac{\partial^4\phi}{\partial x^4} + \mu\,\phi(x)\frac{\partial^2\eta}{\partial t^2} = 0$$

$$EJ\,\eta\,\phi^{IV} + \mu\,\phi\,\ddot{\eta} = 0$$

$$EJ\frac{\phi^{IV}}{\phi} + \mu\frac{\ddot{\eta}}{\eta} = 0 \quad , \frac{\ddot{\eta}}{\eta} = -\omega^2 \quad , \beta^4 = \frac{\mu\omega^2}{EJ} \quad , \beta = \beta(\omega^2)$$: SPATIAL EIGENVALUE, RELATED TO TIME B.

$$\frac{\partial^4\phi}{\partial x^4} - \beta^4\phi = 0$$, THIS PROVIDES THE EIGENFUNCTIONS:

$$\phi(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x)$$: EIGENFUNCTION

EXAMPLE: BEAM (PINNED-PINNED BEAM) FLEXURAL OSCILLATIONS

CONSIDERING THE B.C.:

$$L:\begin{cases} y(0,t) = 0 \\ M(0,t) = EJ\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0} = 0 \end{cases} \qquad \begin{cases} \phi(0) = 0 \\ \phi''(0) = 0 \end{cases}$$

$$R:\begin{cases} y(L,t) = 0 \\ M(L,t) = EJ\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=L} = 0 \end{cases} \qquad \begin{cases} \phi(L) = 0 \\ \phi''(L) = 0 \end{cases}$$

$$\phi'(x) = -\beta A\sin(\beta x) + \beta B\cos(\beta x) + \beta C\sinh(\beta x) + \beta D\cosh(\beta x)$$
$$\phi''(x) = -\beta^2 A\cos(\beta x) - \beta^2 B\sin(\beta x) + \beta^2 C\cosh(\beta x) + \beta^2 D\sinh(\beta x)$$
$$\phi'''(x) = \beta^3 A\sin(\beta x) - \beta^3 B\cos(\beta x) + \beta^3 C\sinh(\beta x) + \beta^3 D\cosh(\beta x)$$
$$\phi^{IV}(x) = \beta^4 A\cos(\beta x) + \beta^4 B\cos(\beta x) + \beta^4 C\cosh(\beta x) + \beta^4 D\sinh(\beta x)$$

$$L:\begin{cases} \phi(0) = A + C = 0 & \Rightarrow A = -C \\ \phi''(0) = -\beta^2 A + \beta^2 C = 0 & \Rightarrow A = C \end{cases} \Rightarrow A = C = 0$$

$$\phi(x) = B\sin(\beta x) + D\sinh(\beta x)$$

$$R:\begin{cases} \phi(L) = B\sin(\beta L) + D\sinh(\beta L) = 0 \\ \phi''(L) = -\beta^2 B\sin(\beta L) + \beta^2 D\sinh(\beta L) = 0 \end{cases}$$ $\Leftrightarrow$ MATRIX FORM:

$$\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & +\beta^2\sinh(\beta L) \end{bmatrix}\begin{Bmatrix} B \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

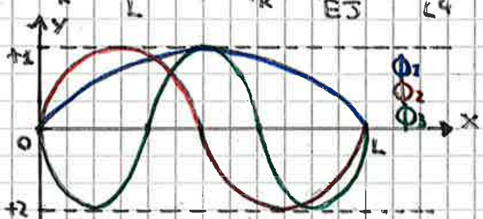
$\begin{Bmatrix} B \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ TRIVIAL SOLUTION; $\det\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & \beta^2\sinh(\beta L) \end{bmatrix} = \sin(\beta L)\beta^2\sinh(\beta L) + \sinh(\beta L)\beta^2\sin(\beta L) = 0$

$$2\beta^2\sin(\beta L)\sinh(\beta L) = 0 \quad \text{(CHARACTERISTIC EQ)}$$

SOLUTION FOR $D=0$, DUE TO $\beta \neq 0 \Rightarrow \sin(\beta L) = 0$: EIGENFUNCTION $(\sinh(\beta L) \neq 0)$

$$\beta_R L = R\cdot\pi \quad , \text{WITH } R = 1,2,\dots,\infty$$

$$\beta_R = \frac{R\cdot\pi}{L} \Rightarrow \beta_R^4 = \frac{\mu\omega_R^2}{EJ} = \frac{R^4\pi^4}{L^4} \Rightarrow \omega_R = R^2\pi^2\sqrt{\frac{EJ}{\mu L^4}}$$: NATURAL FREQUENCY



$$\omega_1 = \pi^2\sqrt{\frac{EJ}{ML^4}}$$
$$\omega_2 = 4\pi^2\sqrt{\frac{EJ}{ML^4}}$$
$$\omega_3 = 9\pi^2\sqrt{\frac{EJ}{ML^4}}$$

TIME RESPONSE (CONSIDERING THE I.C.):

$$y(x,t) = \sum_{R=1}^{\infty} M_R \sin\left(\frac{\pi R}{L}x\right)\sin(\omega_R t + \alpha)$$

UNIFIED APPROACH ($f(x,t)=0$)

$y(x,t) = \phi(x)\eta(t)$

$\mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0$: GENERAL FORM FOR PDE

THE BOUNDARY CONDITION IS:

$B_i[y] = 0$: B.C.

WITH $\mathcal{M}, K, B$: DIFFERENTIAL OPERATORS

3.1 STRING TRANSVERSAL OSCILLATIONS

$\rho \frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0$ $\begin{cases} y = y \\ \mathcal{M} = \rho \; ; \; K = -T\frac{\partial^2}{\partial x^2}(\cdot) \end{cases}$

3.2 ROD AXIAL OSCILLATIONS

$\mu \frac{\partial^2 u}{\partial t^2} - AE\frac{\partial^2 u}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0$ $\begin{cases} y = u \\ \mathcal{M} = \mu \; ; \; K = -AE\frac{\partial^2}{\partial x^2}(\cdot) \end{cases}$

3.3 SHAFT TORSIONAL OSCILLATIONS

$I\frac{\partial^2 \theta}{\partial t^2} - GJ\frac{\partial^2 \theta}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0$ $\begin{cases} y = \theta \\ \mathcal{M} = I \; ; \; K = -GJ\frac{\partial^2}{\partial x^2}(\cdot) \end{cases}$

3.4 BEAM FLEXURAL OSCILLATIONS

$\mu\frac{\partial^2 y}{\partial t} + EJ\frac{\partial^4 y}{\partial x^4} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0$ $\begin{cases} y = y \\ \mathcal{M} = \mu \; ; \; K = +EJ\frac{\partial^4}{\partial x^4}(\cdot) \end{cases}$

SUBSTITUTING SOLUTIONS: $y(x,t) = \phi(x)\cdot\eta(t)$, $\eta(t)$ IS HARMONIC

$\mathcal{M}[\phi]\cdot\ddot{\eta} + K[\phi]\cdot\eta = 0$

$\frac{\ddot{\eta}}{\eta} = -\frac{K[\phi]}{\mathcal{M}[\phi]} = -\omega^2$

DIFFERENTIAL EIGENPROBLEM:

$K[\phi] = \omega^2\mathcal{M}[\phi]$

SELF-ADJOINT PROPERTY

- ORTHOGONALITY PROPERTY WITH $\mathcal{M}, K$ DIFFERENT OPERATORS.
- DEF: GENERIC OPERATOR (L), TWO FUNCTIONS (u,v) THAT SATISFY B.C. ON DOMAIN (D).

$(u, L[v]) = \int_D u\cdot L[v]\,dD$

$\Rightarrow L$ IS SELF-ADJOINT $\overset{(IFF)}{\Longleftrightarrow}$ $(u, L[v]) = (v, L[u])$

EXAMPLE: CONSIDERING 3.1

$\mathcal{M} = \rho \; ; \; K = -T\frac{\partial^2}{\partial x^2}(\cdot) = -\frac{\partial}{\partial x}\left[T\cdot\frac{\partial}{\partial x}\cdot\right]$

$\mathcal{M}$ IS SELF-ADJUST ?

TAKE ϕ_i, ϕ_j EIGENFUNCTIONS THAT SATISFY B.C.

$(\phi_i, \mathcal{M}[\phi_j]) = \int_0^L \phi_i\rho\phi_j\,dx = \int_0^L \phi_j\rho\phi_i\,dx = (\phi_j, \mathcal{M}[\phi_i])$

$\Rightarrow \mathcal{M}$ IS SELF-ADJUST

UNIFIED APPROACH ($f(x,t) \neq 0$: FORCED OSCILLATIONS)

$Y(x,t) = \sum_{R=1}^{\infty} \phi_R(x)\eta(t)$

$M\left[\frac{\partial^2 Y}{\partial t^2}\right] + K[Y] = f(x,t)$

...

$m_R \ddot{\eta}_R + k_R \eta_R = \int_D \phi_R f(x,t) dD = N_R(t)$

APPROXIMATED METHODS (RAYLEIGH) (TO ESTIMATE THE 1st NATURAL FREQ.)

STARTING FROM MDOF:

$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \Rightarrow \omega_R \; ; \; \{\varphi_R\}$

$\{x\} \cong \{\varphi_R\} \Rightarrow \omega_R^2 \{\varphi_R\}^T[M]\{\varphi_R\} = \{\varphi_R\}^T[K]\{\varphi_R\} \iff \omega_R^2 = \frac{\{\varphi_R\}^T[K]\{\varphi_R\}}{\{\varphi_R\}^T[M]\{\varphi_R\}}$

$\begin{cases} T = \frac{1}{2}\{\dot{x}\}^T[M]\{\dot{x}\} \\ V = \frac{1}{2}\{x\}^T[K]\{x\} \end{cases}$

USING THE EXPANSION THEOREM:

$\{u\} = \sum_{R=1}^{m} c_R\{\varphi_R\}$

...

$R(\{u\}) = \omega_R^2 \left[1 + \frac{\frac{1}{\omega_R^2}\sum_{i\neq R}^{m} \varepsilon_i^2 \frac{m_i}{m_R}(\omega_i^2 - \omega_R^2)}{1 + \sum_{i\neq R}^{m} \varepsilon_i \frac{m_i}{m_R}} \right]$, WITH $\varepsilon_i = \frac{c_i}{c_R}$, $i \neq R$

...

IF $\{u\}$ IS "CLOSE" TO $\{\varphi_R\} \Rightarrow \varepsilon_i \ll 1$

R QUOTIENT IS "STATIONARY" IN THE NEIGHBOURHOOD OF THE "R" EIGENVALUE

IF $R=1 \Rightarrow R(\{u\}) \cong \omega_1^2 + \underbrace{\sum \dots}_{>0} > \omega_1^2$: OVERSTIMATED VALUE

THINKING IN PDE DOMAIN:

$T = \dots \qquad T_{MAX}$

$V = \dots \qquad V_{MAX}$

$R(\{x_0\}) = \frac{V_{MAX}}{T_{MAX}} = \frac{\{x_0\}^T[K]\{x_0\}}{\underbrace{\{x_0\}^T[M]\{x_0\}}_{\text{TRIAL FUNCTION}}} = \omega^2 \Rightarrow \omega_1^2$

PDE  (II ORDER IV ORDER)

$T(t) = \frac{1}{2}\int_0^L m(x)\left(\frac{\partial w}{\partial t}\right)^2 dx + \frac{1}{2}\sum_{i=1}^{m} m_i \left(\frac{\partial w}{\partial t}\right)^2_{x_i}$ WHERE $m(x)$: $\begin{cases} \rho(x) & \text{FOR STRING} \\ \mu(x) & \text{FOR ROD/EB BEAM} \\ I(x) & \text{FOR SHAFT} \end{cases}$

WHERE $w(x)$: $y(x)$; $u(x)$; $\gamma(x)$: GENERALISED DISPL. COORD.

$V(t) = \frac{1}{2}\int_0^L g(x)\left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{1}{2}\sum_{j=1}^{m} k_j w^2(x,t)$ WHERE $g(x)$: $\begin{cases} T(x) & \text{FOR STRING} \\ EA(x) \text{ FOR ROD} & \mid EJ(x) \text{ FOR EB BEAM} \\ GJ(x) & \text{FOR SHAFT} \end{cases}$

WHERE $(\partial w/\partial x) = (\partial^2 y/\partial x^2)$ FOR EB BEAM (CLAPEYRON Th.)

$w(x,t) = f(x)\cdot\eta(t)$ WITH TRIAL FUNCTION FOR $f(x)$

$T(t) = \frac{1}{2}\int_0^L m(x) f^2 dx\, \omega^2\eta^2(t) + \frac{1}{2}\sum_{i=1}^{m} m_i f^2(x_i)\omega^2\eta^2(t) \rightarrow \tilde{T}_{MAX} = \frac{T_{MAX}}{\omega^2} = \frac{1}{2}\int_0^L m(x) f^2 dx + \frac{1}{2}\sum_{i=1}^{m} m_i f^2(x_i)$

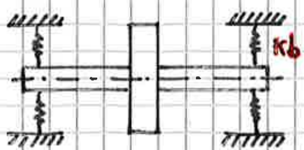
$V(t) = \frac{1}{2}\int_0^L g(x)\left(\frac{df}{dx}\right)^2 dx\, \eta^2(t) + \frac{1}{2}\sum_{j=1}^{m} k_j f^2(x_j)\eta^2(t) \rightarrow V_{MAX} = \frac{1}{2}\int_0^L g(x)\left(\frac{df}{dx}\right)^2 dx + \frac{1}{2}\sum_{j=1}^{m} k_j f^2(x_j)$

$R(f) = \frac{V_{MAX}}{\tilde{T}_{MAX}} = \omega^2 = \frac{\int \dots}{\int \dots}$

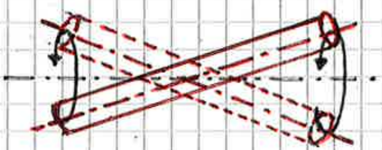
TYPICAL CAUSES OF INSTABILITY IN ROTORDYNAMICS:

- RIGID SHAFT (AND DISK) ON FLEXIBLE BEARINGS

1st MODE: CYLINDRICAL

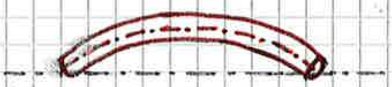


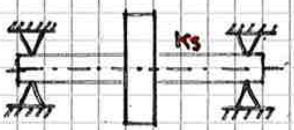
2nd MODE: CONICAL



- FLEXIBLE SHAFT ON RIGID BEARINGS (← JEFFCOTT MODEL)

1st MODE





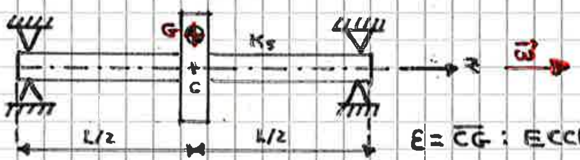
2nd MODE

JEFFCOTT MODEL

- RIGID DISK MOUNTED IN THE MIDDLE OF SHAFT L.
- FLEXIBLE AND MASSLESS SHAFT
- IDEAL RIGID BEARINGS
- DISK HAS MASS (m) AND MASS MOMENT OF OF INERTIA (I_G) (POLAR)

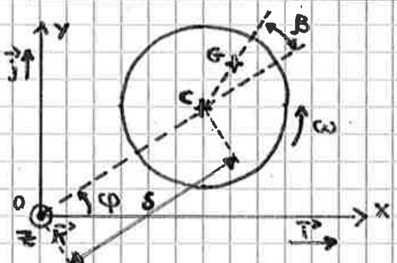
[PREVIEW]:

- ONLY 1 CRITICAL SPEED: $\omega_{CR} = \sqrt{K_s/m}$
- SYNCHRONOUS WHIRL : $\omega = \text{const} = \dot{\psi} \Rightarrow \ddot{\psi} = 0$
- SELF-CENTERING EFFECT : $\omega \gg \omega_{CR}$ $(\delta = \varepsilon)$
- (CANNOT EXPLAIN BACKWARD WHIRL)



$\varepsilon = \overline{CG}$: ECCENTRICITY

LET'S CONSIDER THE DISK SECTION



$\delta = \overline{OC}$: SHAFT DEFLECTION

$\vec{i}, \vec{j}, \vec{k}$: FIXED REFERENCE FRAME (x, y, z)

β : PHASE ANGLE | $\psi = \varphi + \beta = \omega t$

φ : WHIRL ANGLE | $\omega = \frac{\psi}{t} = \frac{\varphi}{t} + \frac{\beta}{t} = \dot{\psi} = \dot{\varphi} + \dot{\beta}$

$$\vec{\delta} = x_c \vec{i} + y_c \vec{j} = \delta e^{i\varphi} = x_c + i y_c = z_c$$

GIVEN A ROTOR WITH:

$\boxed{\omega = \text{CONST}} \Rightarrow \begin{cases} \delta = \text{CONST} \Rightarrow \text{CIRCLE} \\ \beta = \text{CONST} \Rightarrow \dot{\beta} = 0 \end{cases} \Rightarrow \omega = \dot{\psi} = \dot{\varphi} + \dot{\beta} = \dot{\varphi}$: SYNCHRONOUS (FORWARD) WHIRL MOTION

$\Rightarrow$ SPINNING SPEED $\equiv$ WHIRL SPEED

IN ORDER TO FIND THE VALUE AND LOCATION OF PEAKS:

PERSONAL CONSIDERATION

$\frac{d|\cdot|}{d(\omega/\omega_{cr})} = 0 \Rightarrow \omega_{MAX}$

$|\cdot|_{MAX} = |\cdot|(\omega_{MAX}/\omega_{cr})$

THEREFORE:

$$|\cdot| = \frac{(\omega/\omega_{cr})^2}{\sqrt{[1-(\omega/\omega_{cr})^2]^2 + [2\zeta(\omega/\omega_{cr})]^2}}$$

TO AVOID TO DO: $F(x) = f(x)g(x) \rightarrow F'(x) = f'(x)g(x) + g'(x)f(x)$:

$$|\cdot| = \frac{(\omega/\omega_{cr})^2}{\sqrt{1 - 2(\omega/\omega_{cr})^2 + (\omega/\omega_{cr})^4 + 4\zeta^2(\omega/\omega_{cr})^2}} = \frac{(\omega/\omega_{cr})^2}{(\omega/\omega_{cr})^2 \sqrt{\frac{1}{(\omega/\omega_{cr})^4}\left[1 - 2\left(\frac{\omega}{\omega_{cr}}\right)^2 + \left(\frac{\omega}{\omega_{cr}}\right)^4 + 4\zeta^2\left(\frac{\omega}{\omega_{cr}}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{(\omega/\omega_{cr})^{-4} - 2(\omega/\omega_{cr})^{-2} + 1 + 4\zeta^2(\omega/\omega_{cr})^{-2}}}$$

$$\frac{d|\cdot|}{d(\omega/\omega_{cr})} = -\frac{1}{2}\,\frac{-4(\omega/\omega_{cr})^{-5} + 4(\omega/\omega_{cr})^{-3} - 8\zeta^2(\omega/\omega_{cr})^{-3}}{[(\omega/\omega_{cr})^{-4} - 2(\omega/\omega_{cr})^{-2} + 1 + 4\zeta^2(\omega/\omega_{cr})^{-2}]^{3/2}} = 0 \iff -4(\omega/\omega_{cr})^{-5} + 4(\omega/\omega_{cr})^{-3} - 8\zeta^2(\omega/\omega_{cr})^{-3} = 0$$

$$-\cancel{4}(\omega/\omega_{cr})^{-5} + \cancel{4}(\omega/\cancel{\omega_{cr}})^{-3} - \cancel{8}\zeta^2(\omega/\cancel{\omega_{cr}})^{-3} = 0$$

$$-(\omega/\omega_{cr})^{-2} + 1 - 2\zeta^2 = 0 \Rightarrow (\omega/\omega_{cr})^{-2} = 1 - 2\zeta^2 \Rightarrow \frac{1}{(\omega/\omega_{cr})} = \sqrt{1-2\zeta^2} \Rightarrow \omega = \frac{\omega_{cr}}{\sqrt{1-2\zeta^2}}$$

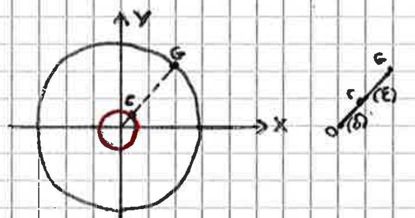
$$\omega_{MAX} = \omega_{cr}\frac{1}{\sqrt{1-2\zeta^2}}$$

WE OBTAINED:

$\omega_{MAX} = \frac{\omega_{cr}}{\sqrt{1-2\zeta^2}}$: MAX AMPLITUDE

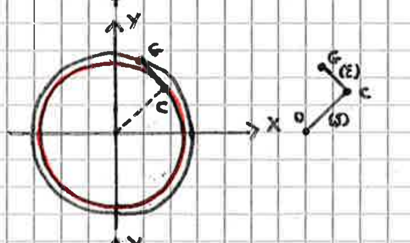
1) $\boxed{\omega \ll \omega_{cr}}$

$\begin{cases} \beta \cong 0 \\ R = \delta + \varepsilon \quad , \delta\,(\text{SMALL}) \end{cases}$



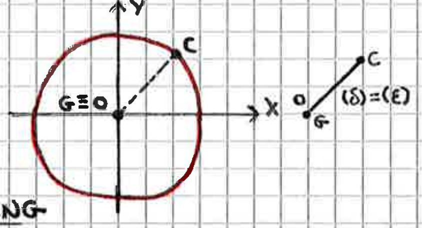
2) $\boxed{\omega \cong \omega_{cr}}$

$\begin{cases} \beta = -\frac{\pi}{2} \\ R = \sqrt{\delta^2 + \varepsilon^2} \quad , \delta\,(\text{LARGE}) \end{cases}$



3) $\boxed{\omega \gg \omega_{cr}}$

$\begin{cases} \beta = -\pi \\ R = \delta = \varepsilon \end{cases}$



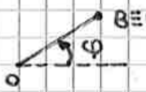
SUPERCRITICAL $\Rightarrow$ SELF-CENTERING

1) $0 < \omega < min(\omega_x, \omega_y)$

2) $min(\omega_x, \omega_y) < \omega < MAX(\omega_x, \omega_y)$

3) $MAX(\omega_x, \omega_y) < \omega < +\infty$

CONSIDERATION: IS IT ALWAYS TRUE THAT $\omega = \dot{\varphi}$? (NOT, ONLY IF SYNCHR-WHIRL :



$Tg\,\varphi = \frac{y}{x} = \frac{y_0 sin(\omega t)}{x_0 cos(\omega t)} = \frac{y_0}{x_0} Tg(\omega t)$

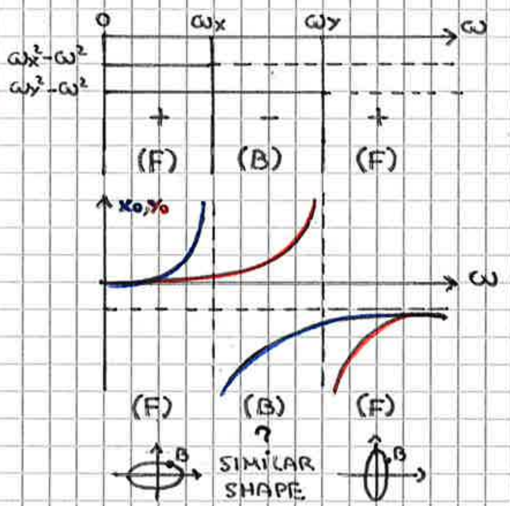
$\omega = CONST \Rightarrow \begin{cases} \delta = CONST \Rightarrow CIRCLE \\ \beta = CONST \Rightarrow \dot{\beta} = 0 \Rightarrow \omega = \dot{\varphi} = \dot{\varphi} \end{cases}$)

DERIVING: $\frac{1}{cos^2\varphi}\dot{\varphi} = \frac{y_0}{x_0}\frac{1}{cos^2(\omega t)} \cdot \omega \Rightarrow \frac{\dot{\varphi}}{\omega} = \frac{y_0}{x_0}\frac{cos^2\varphi}{cos^2(\omega t)} \neq 1$ NON-SYNCHRONOUS WHIRL

LET'S CONSIDER THE SIGN:

$SIGN\left(\frac{\dot{\varphi}}{\omega}\right) \propto SIGN\left(\frac{y_0}{x_0}\right) = SIGN\left(\frac{\omega_x^2 - \omega^2}{\omega_y^2 - \omega^2}\right)$ $\begin{cases} IF + \Rightarrow FORWARD\ WHIRL \\ IF - \Rightarrow BACKWARD\ WHIRL \end{cases}$

WITH $\omega_x < \omega_y$:



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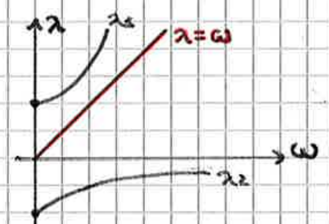
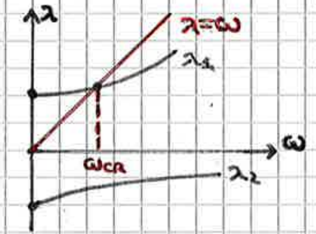
$s_{1,2} = \sigma + i\lambda$, WHERE $\begin{cases} \sigma = Re(s) \begin{cases} \leq 0 : \text{STABLE} \\ > 0 : \text{UNSTABLE} \end{cases} \\ \lambda = Im(s) \begin{cases} > 0 : \text{FORWARD WHIRL} \\ < 0 : \text{BACKWARD WHIRL} \end{cases} \end{cases}$

ABOUT THE WHIRL FREQUENCY (λ):



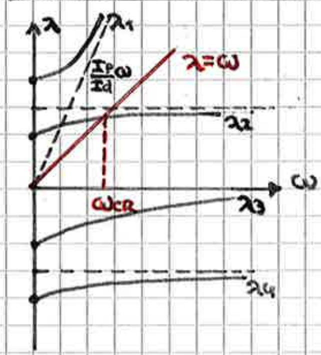
$\begin{cases} \lambda_1 > 0 \\ \lambda_2 < 0 \end{cases}$

$\Rightarrow$ CAMPBELL WITH NO CRITICAL SPEED : $\frac{I_p}{I_d} > 1$



$\Rightarrow$ ONE CRITICAL SPEED : $\frac{I_p}{I_d} < 1$; $\omega_{CR} = \sqrt{\frac{k}{I_d - I_p}}$

STATIC + DYNAMIC UMBALANCE



$$\begin{cases} m\ddot{z} + \alpha z - i\beta\theta = m\varepsilon\omega^2 e^{i\omega t} \\ I_d\ddot{\theta} - iI_p\omega\dot{\theta} + i\beta z + \delta\theta = (I_d - I_p)\, i\gamma\omega^2 e^{i(\omega t+\varphi)} \end{cases}$$

IN MATRIX FORM:

$$\begin{bmatrix} m & \\ & I_d \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & \\ & -iI_p\omega \end{bmatrix} \begin{Bmatrix} \dot{z} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} \alpha & -i\beta \\ +i\beta & \delta \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} m\varepsilon\omega^2 e^{i\omega t} \\ (I_d - I_p)\, i\gamma\omega^2 e^{i(\omega t+\varphi)} \end{Bmatrix}$$

MASS MATRIX $\in Re$ — GYROSCOPIC $\in Im$ — STIFFNESS — EXCITATIONS

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$\boxed{\text{MSD}}$ STUDY (2021 - 2022 GIOVANNI SOBRERO'S SCHEMES)
2nd ROUmd

- INTERNATIONAL SYSTEM OF UNITS (SI) -> TECHNICAL SYSTEM (TS) (27/12/21)

INDIPENDENT	DEPENDENT
[s]	[N]
[kg] -> [tonn]	[Nm] -> [Nmm]
[m] -> [mm]	$[Pa] = [N/m^2] \to [MPa] = [N/mm^2]$
[Rad]	$[kg/m^3] \to [tonn/mm^3]$
[K] or [°C]	[Hz] = [1/s]

- CONVERSIONS

$$\omega\left[\frac{Rad}{s}\right] = 2\pi f\,[Hz] \qquad V\left[\frac{m}{s}\right] = \frac{1000}{3600}\cdot V\left[\frac{Km}{h}\right]$$

$$\omega\left[\frac{Rad}{s}\right] = \frac{2\pi n}{60}\,[rpm]$$

- REFERENCE SYSTEM

GCS: GLOBAL COORDINATE SYSTEM

LCS: LOCAL COORDINATE SYSTEM

TOPICS:

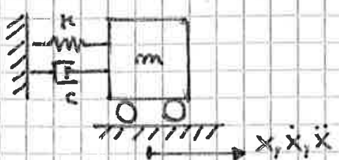
1. VIBRATIONS OF DAMPED SDOF SYSTEMS
   - 1.1 FREE SYSTEM ($f=0$) -> FREE RESPONSE
   - 1.2 FORCED SYSTEM ($f\neq0$) -> FORCED RESPONSE
     - 1 HARMONIC EXC.
     - 2 BASE EXC.
     - 3 NON-PER. EXC.
       - STEP
       - IMPULSE
       - ARBITRARY
     - + STABILITY
     - + WASHING M.
     - + DAMPING
       - VISCOUS
       - HYSTERETIC
       - COULOMB
2. VIBRATIONS OF DAMPED MDOF SYSTEMS
   - 2.1 MODAL ANALYSIS ($C=0$)
   - 2.2 MODAL ANALYSIS ($C\neq0$)
     - hps [M]: POS.DEF
       - [K][C]: SEMI-POS.D.
     - E.V.P.
     - PROPERTIES
       - (1) UNCOPLING EFFECT -> M+K D.
       - (2) THEOREM OF EXPANSION (MODAL SUPERP.)
       - (3) MODES NORMALIZATION
       - (4) DIRECT MODAL TRANSFORMATION
   - FREE RESPONSE
   - FORCED RESPONSE (TO H.E.)
   - + DYNAMIC ABSORBER
   - + BEAT PHENOMENON
   - + LAGRANGE'S EQUATIONS

1. VIBRATIONS OF DAMPED SDOF SYSTEMS [2nd Round]

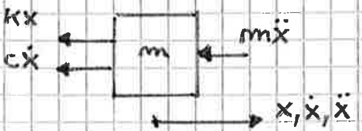
1.1 FREE SYSTEM ($f=0$) → FREE RESPONSE

1.2 FORCED SYSTEM ($f\neq 0$) → FORCED RESPONSE

1.1 FREE SYSTEM ($f=0$) → FREE RESPONSE



FREE BODY DIAGRAM (FBD) & MOTION EQUATION & SOLUTIONS



$\leftarrow)$ $m\ddot{x} + c\dot{x} + kx = 0$: MOTION EQ. - 2nd ORDER DIFF. EQ.

$x = Ae^{st}$, $\dot{x} = sAe^{st}$, $\ddot{x} = s^2Ae^{st}$: SOLUTION FORM (RESPONSE)

$(ms^2 + cs + k)Ae^{st} = 0$

$A = 0$: TRIVIAL SOLUTION

$ms^2 + cs + k = 0$: CHARACTERISTIC EQ.

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$\leftarrow)$ $\ddot{x} + 2\zeta\omega_m\dot{x} + \omega_m^2 x = 0$: MOTION EQ. - CANONICAL FORM

$$\omega_m = \sqrt{\frac{k}{m}} \quad \sqrt{\left[\frac{N}{m}\right]\left[\frac{1}{kg}\right]} = \sqrt{\left[\frac{kg\, m}{s^2\, m}\right]\left[\frac{1}{kg}\right]} = \left[\frac{1}{s}\right] \to \left[\frac{Rad}{s}\right] \Rightarrow f = \frac{\omega_m}{2\pi}\left[\frac{Rad}{s}\right]\left[\frac{1}{Rad}\right] = \left[\frac{1}{s}\right]$$

$$\zeta = \frac{c}{2\omega_m m} = \frac{c}{2\sqrt{km}} = \frac{c}{c_R} \text{ [ADIM.]} \leftrightarrow 2\zeta\omega_m = \frac{c}{m}$$

$(s^2 + 2\zeta\omega_m s + \omega_m^2)Ae^{st} = 0$

$A = 0$: TRIVIAL SOLUTION

$s^2 + 2\zeta\omega_m s + \omega_m^2 = 0$: CHARACTERISTIC EQ.

$$s_{1,2} = \frac{-2\zeta\omega_m \pm \sqrt{4\zeta^2\omega_m^2 - 4\omega_m^2}}{2} = -\zeta\omega_m \pm \omega_m\sqrt{\zeta^2 - 1}$$

$s_1 = \dot{x}_1 = -\zeta\omega_m - \omega_m\sqrt{\zeta^2-1}$; $s_2 = \dot{x}_2 = -\zeta\omega_m + \omega_m\sqrt{\zeta^2-1}$

THE TIME EVOLUTION DEPENDS ON THE SIGN OF $\zeta^2 - 1$ (Δ):

$\zeta > 1 \xrightarrow{\Delta > 0}$ OVERDAMPED SYSTEM : $s_{1,2} = \dot{x}_{1,2}$ ARE REAL AND NEGATIVE

$\zeta = 1 \xrightarrow{\Delta = 0}$ CRITICALLY DAMPED SYSTEM : $s_{1,2} = \dot{x}_{1,2}$ ARE REAL, NEGATIVE AND COINC.

$\zeta < 1 \xrightarrow{\Delta < 0}$ UNDERDAMPED SYSTEM : $s_{1,2} = \dot{x}_{1,2}$ ARE COMPLEX AND CONJUGATE

ON THE BASIS OF Δ THE SOLUTION TAKES DIFFERENT FORMS:

$\zeta > 1$ ($\Delta > 0$) : $x(t) = A_1e^{s_1t} + A_2e^{s_2t}$

$\zeta = 1$ ($\Delta = 0$) : $x(t) = (A_1 + A_2t)e^{st}$

$\zeta < 1$ ($\Delta < 0$) : $x(t) = A_1e^{s_1t} + A_2e^{s_2t}$

INITIAL CONDITIONS (I.C.):

$$\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = v_0 \end{cases}$$

A_1, A_2 DETERMINABLE FROM I.C.

3

CASE $\zeta = 1$ ($\Delta = 0$): CRITICAL DAMPING

1) SOLUTION FORM:

$x(t) = (A_1 + A_2 t)e^{st}$: SOLUTION ($\dot{x}(t) = sA_1 e^{st} + A_2 e^{st} + sA_2 t e^{st}$)

$s_1 = s_2 = \dot{x} = -\zeta\omega_m$: ROOTS/SYSTEM POLES

2) INITIAL CONDITIONS (I.C.):

$$\begin{cases} x(t=0) = x_0 = A_1 \\ \dot{x}(t=0) = v_0 = sA_1 + A_2 \end{cases}$$

3) THEREFORE (A_1, A_2):

$A_1 = x_0$

$A_2 = v_0 - sx_0$

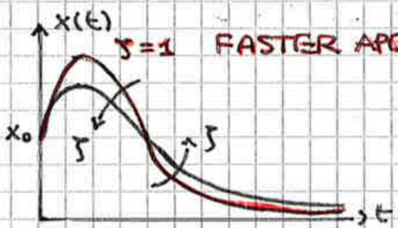
4) $s \rightarrow A_1, A_2$

$A_1 = x_0$

$A_2 = v_0 + \zeta\omega_m x_0 = v_0 + \omega_m x_0$

5) SOLUTION OF THE MOTION EQ.:

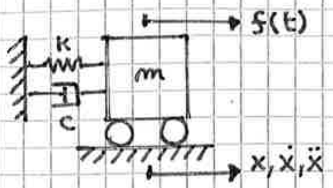
$x(t) = (A_1 + A_2 t)e^{st} = [x_0 + (v_0 + \omega_m x_0)t]\, e^{-\omega_m t}$: FASTER APERIODIC DECAY



FASTER APERIODIC DECAY (THE FASTEST!)

5

1.2 FORCED SYSTEM ($f \neq 0$) → FORCED RESPONSE



1. HARMONIC EXCITATION ($x(t) = x_p(t) = X_0 e^{i\omega t}$)
2. BASE EXCITATION ($x(t) = x_p(t) = X_0 e^{i\omega t}$)
3. NON-PERIODIC EXCITATION
   - → STEP EXC. ($x(t) = x_p(t) + x_A(t)$; $x_0 = 0, \dot{x}_0 = 0$)
   - → IMPULSE EXC. ($x(t) = x_A(t)$; $x_0 = 0, \dot{x}_0 \neq 0$)
   - → ARBITRARY EXC. → CONVOLUTION INTEGRAL

1. HARMONIC EXCITATION:

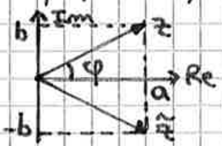
$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$; $(a+ib)(a-ib) = a^2 + b^2$

$f(t) = F_0 e^{i\omega t} \leftrightarrow f(t) = F_0 [\underbrace{\cos(\omega t)}_{Re} + i \underbrace{\sin(\omega t)}_{Im}]$: HARMONIC EXCITATION

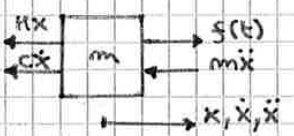
ω: FREQ. OF EXTERNAL FORCE

$z = a + ib = \rho e^{i\varphi}$; $z = F_0$, $\varphi = \omega t$; $\tilde{z} = a - ib$

$$\begin{cases} |z| = \sqrt{Re^2 + Im^2} = \sqrt{a^2+b^2} = \sqrt{z \cdot \tilde{z}} \\ \varphi_+ = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left(\frac{b}{a}\right) \\ \left(\varphi_- = Tg^{-1}\left(-\frac{b}{a}\right)\right) \end{cases}$$



FREE BODY DIAGRAM (FBD) & MOTION EQ. & RESPONSE TO HARM. EXC.



* ONLY IF THE SYSTEM IS LINEAR (IT IS)
⇒ SUPERPOSITION OF EFFECTS.

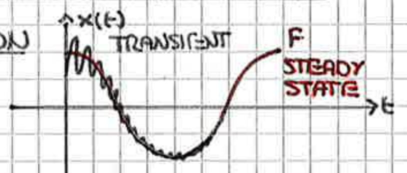
($x_A(t)$: HOMOG. EQ. WITH NON-ZERO x_0, V_0)

$\leftarrow m\ddot{x} + c\dot{x} + kx = f(t) = F_0 e^{i\omega t}$ MOTION EQ. – 2^{nd} ORDER DIFF. EQ.

$\leftarrow \ddot{x} + 2\zeta\omega_m \dot{x} + \omega_m^2 x = \frac{F_0}{k}\omega_m^2 e^{i\omega t}$ MOTION EQ. – CANONICAL EXPONENTIAL FORM

$x(t) = \cancel{x_A(t)} + x_p(t)$: RESPONSE TO HARMONIC EXCITATION

$$\left.\begin{cases} x_A(t) : \text{HOMOGENEOUS (VANISHING) SOLUTION (TRANSIENT)} \\ x_p(t) : \text{PARTICULAR (INTEGRAL) SOLUTION (STEADY STATE)} \end{cases}\right\} *$$



$x(t) = x_p(t) = X_0 e^{i\omega t} \leftrightarrow x(t) = x_0 \cos(\omega t - \varphi_+)$; $X_0 \in \mathbb{C}$; $x_0 = |X_0(\omega t)| \in Re$

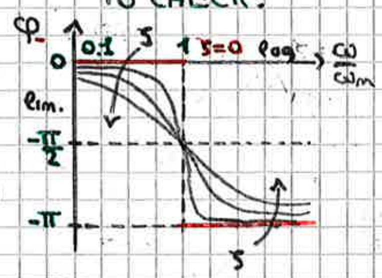
$x(t) = X_0 e^{i\omega t}$; $\dot{x}(t) = i\omega X_0 e^{i\omega t}$; $\ddot{x}(t) = -\omega^2 X_0 e^{i\omega t}$

(I THINK $= A \cdot |G(i\omega)|$ TO CHECK!)

$(-\omega^2 + 2i\zeta\omega_m\omega + \omega_m^2) X_0 \cancel{e^{i\omega t}} = \frac{F_0}{k}\omega_m^2 \cancel{e^{i\omega t}}$

$\cancel{\omega_m^2}\left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) - \left(\frac{\omega}{\omega_m}\right)^2\right] X_0 = \frac{F_0}{k}\cancel{\omega_m^2}$

$Z(i\omega) = 1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) - \left(\frac{\omega}{\omega_m}\right)^2$: IMPEDENCE FUNCTION



$G(i\omega) = \left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) - \left(\frac{\omega}{\omega_m}\right)^2\right]^{-1}$: FREQUENCY RESPONSE
($G \leftrightarrow$ GAIN OF AMPLIFICATION)
(ADIMENSIONAL) $G(i\omega) = \frac{1}{Z(i\omega)}$

$G(i\omega) = |G(i\omega)| e^{+i\varphi_-}$

$$\begin{cases} |G(i\omega)| = \sqrt{G(i\omega)\tilde{G}(i\omega)} = \sqrt{\frac{1}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} \cdot \frac{1}{1 - \left(\frac{\omega}{\omega_m}\right)^2 - 2i\zeta\left(\frac{\omega}{\omega_m}\right)}} = \frac{1}{\sqrt{\underbrace{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]^2}_{Re} + \underbrace{\left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2}_{Im}}} \\ \angle G(i\omega) = \varphi_- = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_m}\right)}{1 - \left(\frac{\omega}{\omega_m}\right)^2}\right] \end{cases}$$

$X_0 = A \cdot G(i\omega) = A \cdot |G(i\omega)| e^{+i\varphi_-}$, $A = \frac{F_0}{k}$

$x(t) = X_0 e^{i\omega t} = A \cdot G(i\omega) \cdot e^{i\omega t} = A \cdot |G(i\omega)| e^{+i\varphi_-} \cdot e^{i\omega t} = A \cdot |G(i\omega)| e^{i(\omega t + \varphi_-)}$: HARMONIC RESP.

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UNDAMPED SYSTEM ($\zeta = 0$)

$$\omega_R = \omega = \omega_m \sqrt{1 - 2\zeta^2} \Rightarrow \omega_R = \omega = \omega_m$$

$$|G(i\omega)| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2}} = \frac{1}{1 - \left(\frac{\omega}{\omega_m}\right)^2}$$

FOR $\omega \to \omega_m$: $|G(i\omega)| \to \infty$

IT IS NECESSARY TO STUDY THE TRANSIENT MOTION.

$$\ddot{x} + 2\zeta\omega_m \dot{x} + \omega_m^2 x = \frac{F_0}{K}\omega_m^2 e^{i\omega t}$$

CONSIDERING THE REAL PART OF $F_0 e^{i\omega t}$

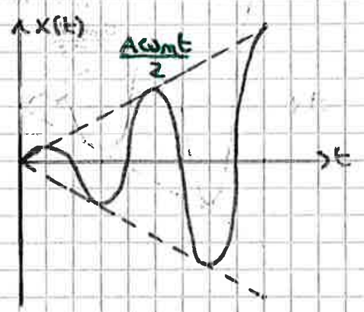
$$\ddot{x} + \omega_m^2 x = \frac{F_0}{K}\omega_m^2 \cos(\omega t)$$

PARTICULAR SOLUTION:

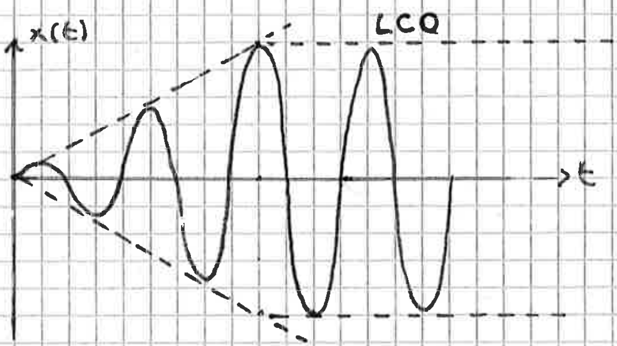
$$x_p(t) = X e^{i\omega t} \cdot t$$

$$x(t) = \frac{A\omega_m t}{2} \cdot \sin(\omega_m t)$$

IT'S AN OSCILLATORY RESPONSE WITH AN AMPLITUDE INCREASING LINEARLY WITH t

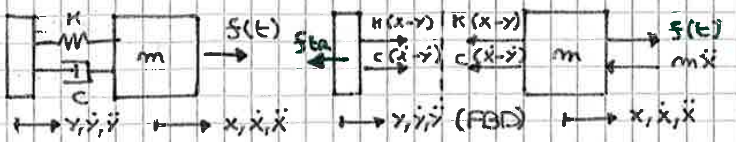


$\Rightarrow$



LCO: LIMITED CYCLE OSCILLATION

FORCE TRANSMISSIBILITY / VIBRATION ISOLATION



MOTION EQUATIONS:

$$f_{tr} = c(\dot x - \cancel{\dot y}) + k(x - \cancel{y}) = c\dot x + kx$$
$$f = m\ddot x + c\dot x + kx$$

* $\begin{cases} z(t) = z_{10} e^{i\omega t} \longleftrightarrow z(t) = z_0 \cos(\omega t + \varphi_-) \\ \dot z(t) = i\omega z_{10} e^{i\omega t} \qquad z_{10} \in \mathbb{C} \quad z_0 \in Re \\ \ddot z(t) = -\omega^2 z_{10} e^{i\omega t} \qquad z_0 = z_0(i\omega) = |z_0(i\omega)| e^{+i\varphi_-} \end{cases}$

CONSIDERING:

$$\frac{f_{tr}}{f} = T_{RABS} = \frac{X_0}{Y_0} = \frac{c\dot x + kx}{m\ddot x + c\dot x + kx} = \frac{1 + 2i\zeta(\omega/\omega_m)}{1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)} = \left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] G(i\omega)$$

RELATIVE TRANSMISSIBILITY:

$$T_{RREL} = \frac{z_{10}}{Y_0} = \frac{X_0 - Y_0}{Y_0} = T_{RABS} - 1 = \frac{1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} - 1 = \frac{\cancel{1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} - \cancel{1} + \left(\frac{\omega}{\omega_m}\right)^2 - \cancel{2i\zeta\left(\frac{\omega}{\omega_m}\right)}}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} =$$

NW!

$$= \frac{\left(\frac{\omega}{\omega_m}\right)^2}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega) \; : \; \text{RELATIVE TRANSMISSIBILITY} \; \begin{cases} |T_{RREL}| = \frac{|z_0(i\omega)|}{Y_0} = \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)| \\ \angle \end{cases}$$

RELATIVE MOTION EQUATION*:

$$m\ddot x + c(\dot x - \dot y) + k(x - y) = 0$$
$$m\ddot z + c\dot z + kz = -m\ddot y$$
$$\ddot z + 2\zeta\omega_m \dot z + \omega_m^2 z = -\ddot y$$
$$(-\omega^2 + 2i\zeta\omega_m\omega + \omega_m^2) z_{10} \cancel{e^{i\omega t}} = \omega^2 Y_0 \cancel{e^{i\omega t}}$$
$$\cancel{\omega_m^2}\left[1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] z_{10} = \omega_m^2 \left(\frac{\omega}{\omega_m}\right)^2 Y_0$$

(1) $z_{10}(i\omega) = z_{10} = \dfrac{Y_0(\omega/\omega_m)^2}{[1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)]} = Y_0\left(\dfrac{\omega}{\omega_m}\right)^2 G(i\omega) \quad (z_0 = Y_0 \cdot T_{RREL})$

(2) $z_{10}(i\omega) = z_{10}\dfrac{\tilde G(i\omega)}{\tilde G(i\omega)} = \dfrac{Y_0(\omega/\omega_m)^2}{[1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)]} \cdot \dfrac{[1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)]}{[1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)]} = \dfrac{Y_0 \cdot (\omega/\omega_m)^2 [1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)]}{[1 - (\omega/\omega_m)^2]^2 + [2\zeta(\omega/\omega_m)]^2}$

$$= Y_0 |G(i\omega)|^2 [(\omega/\omega_m)^2 - (\omega/\omega_m)^4 - 2i\zeta(\omega/\omega_m)^3]$$

SOLUTION:

$$z(t) = z_{10} e^{i\omega t}$$
$$z_{10} = |z_{10}(i\omega)| e^{+i\varphi_-}$$
$$z(t) = |z_0(i\omega)| e^{i(\omega t + \varphi_-)} \; : \; \text{RELATIVE MOTION RESPONSE}$$

WITH:

(1) $|z_0(i\omega)| = \sqrt{z(i\omega)\tilde z(i\omega)} = Y_0(\omega/\omega_m)^2\sqrt{G(i\omega)\tilde G(i\omega)} = Y_0\left(\dfrac{\omega}{\omega_m}\right)^2 |G(i\omega)| = Y_0 |T_{RREL}|$

(2) $\varphi_- = Tg^{-1}\left(\dfrac{Im}{Re}\right) = Tg^{-1}\left[\dfrac{-2\zeta(\omega/\omega_m)^3}{(\omega/\omega_m)^2 - (\omega/\omega_m)^4}\right] = Tg^{-1}\left[\dfrac{-2\zeta(\omega/\omega_m)}{1 - (\omega/\omega_m)^2}\right] \qquad \left(\varphi_+ = Tg^{-1}\left(\dfrac{+|Im|}{Re}\right)\right)$

SUNTO:

H.E. : $x(t) = X_0 e^{i\omega t} = |X_0(i\omega)| e^{i(\omega t + \varphi_-)} = A|G(i\omega)| e^{i(\omega t + \varphi_-)}$

B.E. : $x(t) = X_0 e^{i\omega t} = |X_0(i\omega)| e^{i(\omega t + \varphi_-)} = Y_0 \cdot |T_{RABS}| e^{i(\omega t + \varphi_-)} = Y_0 \cdot \sqrt{1 + [2\zeta(\omega/\omega_m)]^2}\,|G(i\omega)| e^{i(\omega t + \varphi_-)}$

R(B.E.) : $z(t) = z_{10} e^{i\omega t} = |z_0(i\omega)| e^{i(\omega t + \varphi_-)} = Y_0 \cdot |T_{RREL}| e^{i(\omega t + \varphi_-)} = Y_0 \cdot \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)| e^{i(\omega t + \varphi_-)}$

ω_{RES}^{BE-REL} – DEMONSTRATION:

$\frac{d|T_{R\,REL}|}{d(\omega/\omega_m)} = 0 \to$ PEAKS ; $T_{R\,REL} = \frac{Z_0}{Y_0} = \frac{X_0 - Y_0}{Y_0} = T_{R\,ABS} - 1 = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega)$; $|T_{R\,REL}| = \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)|$

$$|T_{R\,REL}| = \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)| = \frac{\left(\frac{\omega}{\omega_m}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_m}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2}} = \frac{\left(\frac{\omega}{\omega_m}\right)^2}{\sqrt{1-2\left(\frac{\omega}{\omega_m}\right)^2+\left(\frac{\omega}{\omega_m}\right)^4+4\zeta^2\left(\frac{\omega}{\omega_m}\right)^2}} =$$

$$= \frac{\cancel{\left(\frac{\omega}{\omega_m}\right)^2}}{\cancel{\left(\frac{\omega}{\omega_m}\right)^2}\sqrt{\frac{1}{\left(\frac{\omega}{\omega_m}\right)^4}\cdot\left[1-2\left(\frac{\omega}{\omega_m}\right)^2+\left(\frac{\omega}{\omega_m}\right)^4+4\zeta^2\left(\frac{\omega}{\omega_m}\right)^2\right]}} = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_m}\right)^{-4}-2\left(\frac{\omega}{\omega_m}\right)^{-2}+1+4\zeta^2\left(\frac{\omega}{\omega_m}\right)^{-2}}}$$

$$\frac{d|T_{R\,REL}|}{d(\omega/\omega_m)} = -\frac{1}{2}\,\frac{-4\left(\frac{\omega}{\omega_m}\right)^{-5}+4\left(\frac{\omega}{\omega_m}\right)^{-3}-8\zeta^2\left(\frac{\omega}{\omega_m}\right)^{-3}}{\left[\left(\frac{\omega}{\omega_m}\right)^{-4}-2\left(\frac{\omega}{\omega_m}\right)^{-2}+1+4\zeta^2\left(\frac{\omega}{\omega_m}\right)^{-2}\right]^{3/2}} = 0$$

IMPOSING $x=(\omega/\omega_m)$ AND CONSIDERING ONLY THE NUMERATOR $\left(N=0 \Rightarrow \frac{\omega}{\omega_m} = \frac{\omega_{RES}}{\omega_m}\right)$:

$N = -4x^{-5}+4x^{-3}-8\zeta^2x^{-3} = 0 \quad \cdot 1/x^{-3}$

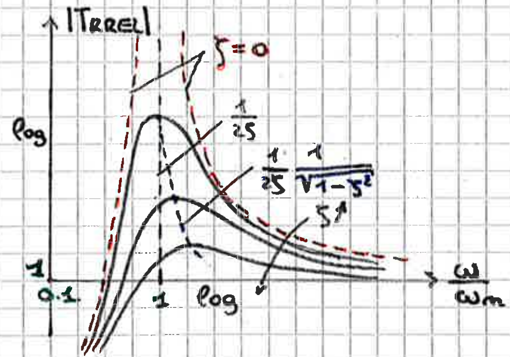
$N = -4x^{-2}+4-8\zeta^2 = 0$

$N = x^{-2} = 1-2\zeta^2 \Rightarrow x = (1-2\zeta^2)^{-1/2} = \frac{1}{\sqrt{1-2\zeta^2}} \Rightarrow \omega_{RES}^{BE-REL} = \omega = \omega_m \cdot \frac{1}{\sqrt{1-2\zeta^2}}$ ✓

$|T_{R\,REL}|_{MAX}$ – DEMONSTRATION:

$$|T_{R\,REL}|_{MAX} = \left|T_{R\,REL}\left(\frac{\omega}{\omega_m} = \frac{\omega_R}{\omega_m} = \frac{1}{\sqrt{1-2\zeta^2}}\right)\right| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_m}\right)^{-4}-2\left(\frac{\omega}{\omega_m}\right)^{-2}+1+4\zeta^2\left(\frac{\omega}{\omega_m}\right)^{-2}}} = \frac{1}{\sqrt{(1-2\zeta^2)^2-2(1-2\zeta^2)+1+4\zeta^2(1-2\zeta^2)}}$$

$$= \frac{1}{\sqrt{\cancel{1}-\underline{4\zeta^2}+\underline{4\zeta^4}-\cancel{2}+\underline{4\zeta^2}+\cancel{1}+4\zeta^2-\underline{8\zeta^4}}} = \frac{1}{\sqrt{4\zeta^2-4\zeta^4}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$ ✓



NOTE: $\frac{\omega}{\omega_m} > 0$: RESONANCE CONDITION H.E. B.E.(REL) : $\zeta < \frac{1}{\sqrt{2}}$ (≈ 0.71)

CONSIDERATION ABOUT: $T_{R\,REL} = \frac{Z_0}{Y_0} = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega) \to |T_{R\,REL}| = \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)|$:

IF ω (HARMONIC MOTION) $\ll \omega_m$ (INSTRUMENT) $\Rightarrow$ ACCELEROMETER (HIGH f INSTR.)

THEN, IF $\frac{\omega}{\omega_m} \ll 1 \Rightarrow \left\{\begin{array}{l}|G(i\omega)| \approx 1\\ \varphi = 0\end{array}\right\}_{ACC.} \Rightarrow z(t) = Z_0 e^{i\omega t} = |Z_0(i\omega)| e^{i(\omega t+\varphi)} = Y_0 \cdot |T_{R\,REL}| e^{i(\omega t+\varphi)} = Y_0 \cdot \left(\frac{\omega}{\omega_m}\right)^2 \underbrace{\cancel{|G(i\omega)|}}_{\approx 1} e^{i(\omega t+\cancel{\varphi})}$, $\varphi=0$ $\quad (\omega_m^A \approx 2\pi \cdot 10'000\,[Hz])$

IF ω (HARMONIC MOTION) $\gg \omega_m$ (INSTRUMENT) $\Rightarrow$ SEISMOMETER (LOW f INSTR.)

THEN, IF $\frac{\omega}{\omega_m} \gg 1 \Rightarrow \left\{\begin{array}{l}T_{R\,REL} \approx 1\\ \varphi = -\pi\end{array}\right\}_{SEISM.} \Rightarrow z(t) = Z_0 e^{i\omega t} = |Z_0(i\omega)| e^{i(\omega t+\varphi)} = Y_0 \cdot |T_{R\,REL}| e^{i(\omega t+\varphi)} = Y_0 \cdot \underbrace{\cancel{\left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)|}}_{\approx 1} e^{i(\omega t-\pi)}$ $\quad (\omega_m^S \approx 2\pi \cdot 0.001\,[Hz])$

$z(t) = Z_0 e^{i\omega t} = |Z_0(i\omega)| e^{i(\omega t+\varphi)} = Y_0 \cdot |T_{R\,REL}| e^{i(\omega t+\varphi)} = Y_0 \cdot \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)| e^{i(\omega t+\varphi)}$ <

- $(\omega_m \gg \omega)$ HIGH ω_m — ACCEL.: $z(t) = Y_0 \left(\frac{\omega}{\omega_m}\right)^2 e^{i\omega t}$
- LOW ω_m $(\omega \gg \omega_m)$ — SEISM: $z(t) = Y_0 \cdot e^{i(\omega t+\pi)}$

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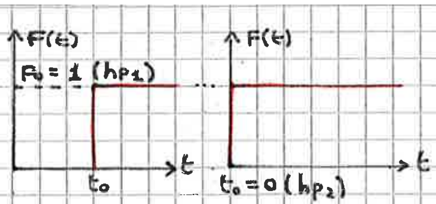
STEP EXCITATION

$$F(t): \begin{cases} \text{FOR } t<t_0: F(t)=0 \\ \text{FOR } t\geq t_0: F(t)=F_0 \end{cases} \qquad hp_s: \begin{cases} hp_1: F_0 = 1[N] \\ hp_2: t_0 = 0[s] \end{cases}$$



MOTION EQUATION $(t \geq t_0)$:

$$m\ddot{x} + c\dot{x} + kx = F_0 \;,\; t\geq t_0$$

SOLUTION CONSIDERING I.C. $\begin{cases} x(t=0) = x_0 = 0 \\ \dot{x}(t=0) = V_0 = 0 \end{cases}$

$$x(t) = x_P(t) + x_A(t)$$

$$\begin{cases} x_P(t) = F_0/k \quad (kx = F_0) \\ x_A(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right] \end{cases} \text{(UNDAMPED SYSTEM } \zeta<1)$$

THEREFORE:

$$x(t) = x_P(t) + x_A(t) = \frac{F_0}{k} + A_1 e^{s_1 t} + A_2 e^{s_2 t} = \frac{F_0}{k} + e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$\dot{x}(t) = \dot{x}_P(t) + \dot{x}_A(t) = 0 + s_1A_1e^{s_1t} + s_2A_2e^{s_2t} = 0 - \zeta\omega_n e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right] + e^{-\zeta\omega_n t}\left[-a\omega_d\sin(\omega_d t) + b\omega_d\cos(\omega_d t)\right]$$

IMPOSING THE I.C. (METHOD A = SLOWER):

$$\begin{cases} x(t=0) = \frac{F_0}{k} + A_1 + A_2 = 0 \\ \dot{x}(t=0) = s_1A_1 + s_2A_2 = 0 \end{cases} \Rightarrow \begin{cases} A_2 = -\frac{F_0}{k} - A_1 \\ s_1A_1 - s_2\frac{F_0}{k} - s_2A_1 = 0 \end{cases} \Rightarrow \begin{cases} A_2 = -\frac{F_0}{k} - \frac{s_2F_0}{k(s_1-s_2)} = \frac{F_0(s_2-s_1-s_2)}{k(s_1-s_2)} \\ A_1 = \frac{s_2F_0}{k(s_1-s_2)} \end{cases}$$

CONSIDERING: $s_1 = -\zeta\omega_n - i\omega_d \;;\; s_2 = -\zeta\omega_n + i\omega_d$

$$\begin{cases} A_1 = \frac{s_2F_0}{k(s_1-s_2)} = \frac{F_0(-\zeta\omega_n + i\omega_d)}{k(-2i\omega_d)} \\ A_2 = \frac{-s_1F_0}{k(s_1-s_2)} = \frac{F_0(+\zeta\omega_n + i\omega_d)}{k(-2i\omega_d)} \end{cases}$$

$$x(t) = \frac{F_0}{k} + A_1e^{s_1t} + A_2e^{s_2t} = \frac{F_0}{k}\left\{1 - \frac{1}{2i\omega_d}\left[(-\zeta\omega_n + i\omega_d)e^{(-\zeta\omega_n - i\omega_d)t} + (\zeta\omega_n + i\omega_d)e^{(-\zeta\omega_n + i\omega_d)t}\right]\right\} =$$

$$= \frac{F_0}{k}\left\{1 - \frac{e^{-\zeta\omega_n t}}{2i\omega_d}\left[(-\zeta\omega_n + i\omega_d)e^{-i\omega_d t} + (\zeta\omega_n + i\omega_d)e^{+i\omega_d t}\right]\right\}$$

BECAUSE: $e^{+i\alpha} - e^{-i\alpha} = 2i\sin\alpha \;;\; e^{+i\alpha} + e^{-i\alpha} = 2\cos\alpha$

$$x(t) = \frac{F_0}{k}\left\{1 - \frac{e^{-\zeta\omega_n t}}{2i\omega_d}\left[\zeta\omega_n \cdot 2i\sin(\omega_d t) + i\omega_d 2\cos(\omega_d t)\right]\right\} = \frac{F_0}{k}\left\{1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \zeta\left(\frac{\omega_n}{\omega_d}\right)\sin(\omega_d t)\right]\right\}$$

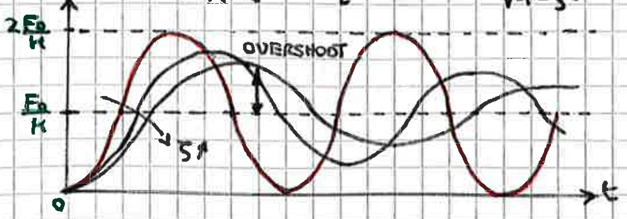
IMPOSING THE I.C. (METHOD B = FASTER!):

$$\begin{cases} x(t=0) = \frac{F_0}{k} + A_1 + A_2 = \frac{F_0}{k} + a = 0 \\ \dot{x}(t=0) = -\zeta\omega_n a + b\omega_d = 0 \end{cases} \Rightarrow \begin{cases} a = -\frac{F_0}{k} \\ b = \zeta\frac{\omega_n}{\omega_d}a = -\frac{F_0}{k}\zeta\left(\frac{\omega_n}{\omega_d}\right) \end{cases}$$

$$x(t) = \frac{F_0}{k} + A_1e^{s_1t} + A_2e^{s_2t} = \frac{F_0}{k} + e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right] = \frac{F_0}{k}\left\{1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \zeta\left(\frac{\omega_n}{\omega_d}\right)\sin(\omega_d t)\right]\right\}$$

THEREFORE:

$$s(t) = x(t) = \frac{F_0}{k}\left\{1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]\right\}$$: TOTAL STEP EXC. RESPONSE



$c\uparrow\ \zeta\uparrow\ \omega_d\downarrow \Rightarrow T_d\uparrow$

IF $\zeta = 0$: $x(t) = \frac{F_0}{k}\left\{1 - \cos(\omega_d t)\right\}$

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RELATION BETWEEN $s(t) - h(t)$ - DEMONSTRATION (NOT SEEN DURING LESSONS)

$$s(t) = u(t-t_0) = \int_{-\infty}^{t} h(\tau)\,d\tau = \int_{-\infty}^{t_0} \delta(\tau - t_0)\,d\tau$$

$$s(t) = \int_{-\infty}^{t} \frac{1}{m\omega_d} \cdot e^{-\zeta\omega_m \tau} \cdot \mathrm{sim}(\omega_d \tau) \cdot u(\tau)\,d\tau = \int_{0}^{t} \frac{1}{m\omega_d} \cdot e^{-\zeta\omega_m \tau}\, \mathrm{sim}(\omega_d \tau)\,d\tau$$

CONSIDERING: $e^{i\alpha} - e^{-i\alpha} = 2i\,\mathrm{sim}(\alpha)$; $e^{i\alpha} + e^{-i\alpha} = 2\cos(\alpha)$

$$s(t) = \int_{0}^{t} \frac{1}{m\omega_d} e^{-\zeta\omega_m \tau} \left(\frac{e^{i\omega_d \tau} - e^{-i\omega_d \tau}}{2i} \right) d\tau = \frac{1}{2im\omega_d} \int_{0}^{t} \left(e^{-(\zeta\omega_m - i\omega_d)\tau} - e^{-(\zeta\omega_m + i\omega_d)\tau} \right) d\tau =$$

$$= \frac{1}{2im\omega_d} \left[\frac{e^{-(\zeta\omega_m - i\omega_d)\tau}}{-(\zeta\omega_m - i\omega_d)} - \frac{e^{-(\zeta\omega_m + i\omega_d)\tau}}{-(\zeta\omega_m + i\omega_d)} \right]_0^t = \frac{1}{2im\omega_d} \left[\frac{e^{-(\zeta\omega_m - i\omega_d)\tau}}{-(\zeta\omega_m - i\omega_d)} \left(\frac{-(\zeta\omega_m + i\omega_d)}{-(\zeta\omega_m + i\omega_d)} \right) - \dots \right]_0^t =$$

$$= \frac{1}{2im\omega_d} \left\{ \left[\frac{-(\zeta\omega_m + i\omega_d) e^{-(\zeta\omega_m - i\omega_d)t}}{(\zeta\omega_m)^2 + \omega_d^2} + \frac{(\zeta\omega_m - i\omega_d) e^{-(\zeta\omega_m + i\omega_d)t}}{(\zeta\omega_m)^2 + \omega_d^2} \right] - \left[-\frac{(\zeta\omega_m + i\omega_d)}{(\zeta\omega_m)^2 + \omega_d^2} + \frac{(\zeta\omega_m - i\omega_d)}{(\zeta\omega_m)^2 + \omega_d^2} \right] \right\} =$$

$$= \frac{1}{2im\omega_d \omega_m^2} \left\{ -e^{-\zeta\omega_m t} \left[+(\zeta\omega_m + i\omega_d) e^{+i\omega_d t} - (\zeta\omega_m - i\omega_d) e^{-i\omega_d t} \right] - \left[-2i\omega_d \right] \right\} =$$

$$= \frac{-e^{-\zeta\omega_m t}}{\cancel{2i}m\omega_d \omega_m^2} \left[\cancel{2i}\omega_d \cos(\omega_d t) + \cancel{2i}\zeta\omega_m \,\mathrm{sim}(\omega_d t) \right] + \frac{\cancel{2i}\cancel{\omega_d}}{\cancel{2i}m\cancel{\omega_d}\omega_m^2} = \quad (m\omega_m^2 = k)$$

$$= \frac{-e^{-\zeta\omega_m t}}{k} \cdot \left[\frac{\omega_d}{\omega_d} \cos(\omega_d t) + \frac{\zeta\omega_m}{\omega_m \sqrt{1-\zeta^2}}\, \mathrm{sim}(\omega_d t) \right] + \frac{1}{k} = \frac{1}{k} \left\{ 1 - e^{-\zeta\omega_m t} \cdot \left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\, \mathrm{sim}(\omega_d t) \right] \right\} ✓$$

+ STABILITY

• GENERAL CASE

MOTION EQ: $\circlearrowright)\ m\ddot{y} = F(y;\dot{y})$

1^{st}: EQUILIBRIUM CONDITION:

hp: $y = y_e = \text{const} \neq 0 \Rightarrow \ddot{y} = \dot{y} = 0 \Rightarrow$ WE HAVE TO SOLVE $F(y_e;0)=0$

2^{nd}: PERTURBATION DEFINITION (SMALL):

$$\begin{cases} y(t) = y_e + x(t) & , x(t) = \text{SMALL PERTURBATION} \\ \dot{y}(t) = \dot{x}(t) \\ \ddot{y}(t) = \ddot{x}(t) \end{cases}$$

3^{rd}: TAYLOR EXPANSION THEOREM ($\to$ LINEARISATION):

$$F(y;\dot{y}) = F(y_e,0) + \frac{\partial F(y,\dot{y})}{\partial y}\Big|_{\substack{y=y_e\\ \dot{y}=0}} \cdot x + \frac{\partial F(\dot{y},\ddot{y})}{\partial \dot{y}}\Big|_{\substack{y=y_e\\ \dot{y}=0}} \cdot \dot{x} + O(x^2;\dot{x}^2)$$

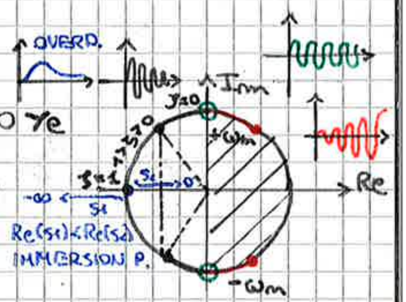
$$\frac{1}{m}\frac{\partial F}{\partial y}\Big|_{y_e,0} = -b$$

$$\frac{1}{m}\frac{\partial F}{\partial \dot{y}}\Big|_{y_e,0} = -a$$

$\ddot{x} + a\dot{x} + bx = 0$: LINEARIZED SYSTEM FOR EACH y_e

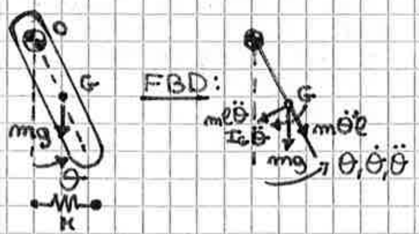
4^{rd}: VALUTATION OF THE CHARACTERISTIC EQUATION:

$$s^2 + as + b = 0 \to s_{1,2} = -\frac{a}{2} \mp \sqrt{\left(\frac{a}{2}\right)^2 - b}$$



$$\begin{cases} Re(s_1,s_2) < 0 \Rightarrow \text{ASYMPTOTIC STABLE: SYST. WILL RETURN TO } y_e \\ Re(s_1,s_2) = 0 \Rightarrow \text{STABLE: SYST. WILL OSCILLATE AROUND } y_e \\ Re(s_1,s_2) > 0 \Rightarrow \text{UNSTABLE: SYST. WILL GO AWAY} \end{cases}$$

• EXAMPLE: PENDULUM



$$\begin{cases} \ddot{\theta}\ell : \text{TANGENTIAL ACC} \\ \dot{\theta}^2\ell : \text{CENTRIFUGAL ACC} \\ I_O = I_G + m\ell^2 : \text{HUGENS-STEINER TH.} \end{cases}$$

$$O\circlearrowright)\ (I_G + m\ell^2)\ddot{\theta} + mg\ell \sin\theta = 0$$

$$I_O\ddot{\theta} = -mg\ell\sin\theta = F(\theta,\underset{=0}{\dot{\theta}})$$

1^{st}: $\ddot{\theta} = \dot{\theta} = 0 \Rightarrow \theta_e = \text{const} \Rightarrow F(\theta_e,0) = -mg\ell\sin\theta_e = 0 \Rightarrow \theta_e = \begin{cases} 0\ (,2\pi,\dots) \\ \pi\ (,3\pi,\dots) \end{cases}$ ($\leftrightarrow \infty\ \theta_e$) WHICH IS STABLE? $\to \dots 4^{th}$

2^{nd}: $\begin{cases} \theta(t) = \theta_e + \delta(t) \\ \dot{\theta}(t) = \dot{\delta}(t) \\ \ddot{\theta}(t) = \ddot{\delta}(t) \end{cases}$ $\begin{cases} \delta(t) = Ae^{st} \\ \dot{\delta}(t) = sAe^{st} \\ \ddot{\delta}(t) = s^2Ae^{st} \end{cases}$

3^{rd}: $F(\theta,\dot{\theta}) = F(\theta_e,0) + \frac{\partial F}{\partial \theta}(\theta,\dot{\theta})\Big|_{\theta_e}\cdot\delta + \cancel{\frac{\partial F(\dot{\theta},\dot{\theta})}{\partial \dot{\theta}}}\Big|_{\theta_e}\cdot\dot{\delta} + O(\delta^2,\dot{\delta}^2)$

$F(\theta_e,0) = -mg\ell\sin\theta_e$

$\frac{\partial F}{\partial \theta}\Big|_{\theta_e} = -mg\ell\cos\theta_e$

$\Rightarrow I_O\ddot{\delta} = \underbrace{-mg\ell\sin\theta_e}_{=0\ \forall\theta_e} - mg\ell\cos\theta_e\,\delta \Rightarrow I_O\ddot{\delta} + mg\ell\cos\theta_e\,\delta = 0$ LINEARIZED

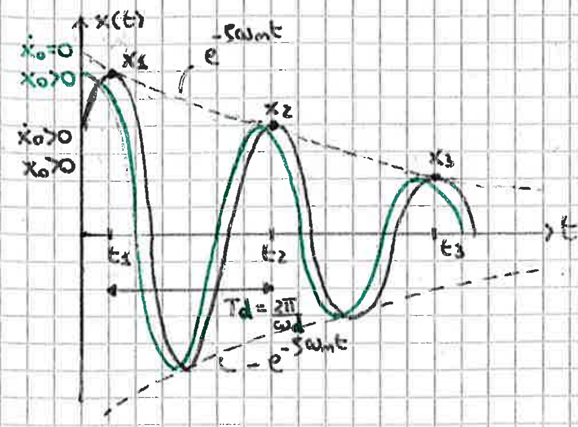
4^{th}: $(I_O s^2 + mg\ell\cos\theta_e)Ae^{st} = 0$

$I_O s^2 + mg\ell\cos\theta_e = 0 \to s_{1,2} = \pm\sqrt{-\frac{mg\ell\cos\theta_e}{I_O}}$ $\begin{cases} \theta_e = 0 : s_{1,2} = \pm i\sqrt{\frac{mg\ell}{I_O}} \Rightarrow \text{STABLE} \\ \theta_e = \pi : s_{1,2} = \oplus\sqrt{\frac{mg\ell}{I_O}} \Rightarrow \text{UNSTABLE} \end{cases}$

(IT'S ENOUGH ONLY ONE $+s$ TO DESTROY THE SYSTEM!)

+ DAMPING

- VISCOUS DAMPING (LINEAR) - LOGARITHMIC DECREMENT METHOD (LDM)



$$x(t) = x_0 e^{-\zeta\omega_n t} \cdot \cos(\omega t + \varphi_-)$$

$$\begin{Bmatrix} t_1 \\ x_1 \end{Bmatrix}; \begin{Bmatrix} t_2 \\ x_2 \end{Bmatrix}; \begin{Bmatrix} t_3 \\ x_3 \end{Bmatrix}$$

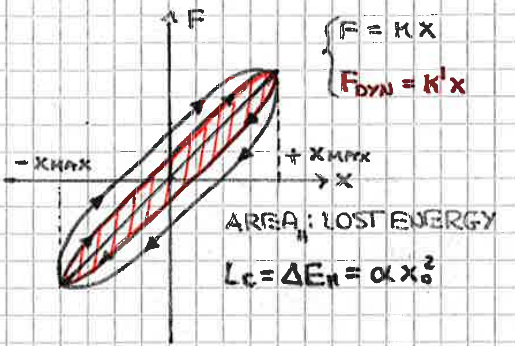
$$\frac{x_1(t_1)}{x_2(t_2)} = \frac{\cancel{x_0} e^{-\zeta\omega_n t_1} \cdot \cancel{\cos(\omega t + \varphi_-)}}{\cancel{x_0} e^{-\zeta\omega_n t_2} \cancel{\cos(\omega t + \varphi_-)}} = e^{\zeta\omega_n \underbrace{(t_2 - t_1)}_{T_d}}$$

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n T_d = \zeta\omega_n \frac{2\pi}{\omega_d} = 2\pi \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$\delta\sqrt{1-\zeta^2} = 2\pi\zeta \longleftrightarrow \delta^2 - \delta^2\zeta^2 = 4\pi^2\zeta^2 \longleftrightarrow \zeta^2(\delta^2 + 4\pi^2) = \delta^2$$

$$\zeta = \sqrt{\frac{\delta^2}{(4\pi^2 + \delta^2)}} \cong \frac{\delta}{2\pi}, \quad \delta = \ln\left(\frac{x_1}{x_2}\right) \quad ✓$$

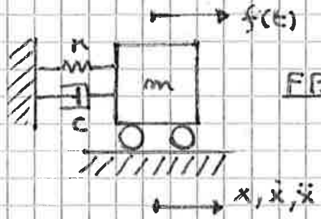
- HYSTERETIC/STRUCTURAL DAMPING | EQUIVALENT VISCOUS SYSTEM (QUASI-LIN)



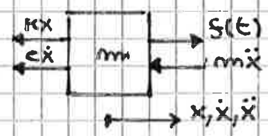
$$\begin{cases} F = kx \\ F_{DYN} = k'x \end{cases}$$

AREA$_H$: LOST ENERGY

$$L_c = \Delta E_H = \alpha x_0^2$$



FBD:



$$x(t) = x_0 \cos(\omega t)$$
$$\dot{x}(t) = -\omega x_0 \sin(\omega t)$$

$$L_c = \Delta E_V = \int_{cycle} c_{EQ}\dot{x}\, dx = \int_T c_{EQ}\dot{x}^2 dt = \int^T c_{EQ}(\omega x_0)^2 \sin^2(\omega t) dt$$
$$= c_{EQ}\pi\omega x_0^2$$

$$\alpha \cancel{x_0^2} = c_{EQ}\pi\omega \cancel{x_0^2} \Rightarrow c_{EQ} = \frac{\alpha}{\pi\omega} = \frac{h}{\omega}, \quad h = \frac{\alpha}{\pi}$$

$$m\ddot{x} + c_{EQ}\dot{x} + kx = f(t) = F_0 e^{i\omega t}$$

$$m\ddot{x} + c_{EQ}(i\omega x) + kx = F_0 e^{i\omega t}$$

$$m\ddot{x} + (k + c_{EQ} i\omega)x = F_0 e^{i\omega t}$$

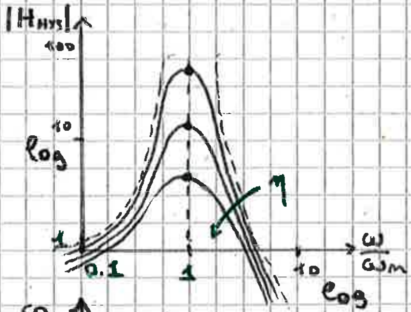
$$m\ddot{x} + (k + ih)x = F_0 e^{i\omega t}, \quad (k + ih) = \text{COMPLEX STIFFNESS}$$

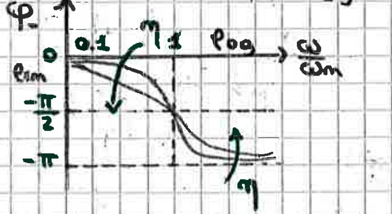
$$m\ddot{x} + (1 + i\eta)kx = F_0 e^{i\omega t}, \quad \eta = \frac{h}{k}$$

$$\ddot{x} + (1 + i\eta)\omega_n^2 x = \frac{F_0}{k}\omega_n^2 e^{i\omega t}$$

$$\cancel{\omega_n^2}\left[-\left(\frac{\omega}{\omega_n}\right)^2 + (1 + i\eta)\right] X_0 \cancel{e^{i\omega t}} = \frac{F_0}{k}\cancel{\omega_n^2}\cancel{e^{i\omega t}}$$

$$H_{HYS} = \frac{X_0}{(F_0/k)} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\eta} \begin{cases} |H_{HYS}| = \dfrac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \eta^2}} \\ \varphi_- = Tg^{-1}\left(\dfrac{Im}{Re}\right) = Tg^{-1}\left[\dfrac{-\eta}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \end{cases}$$





$$c_{EQ} = \frac{\alpha}{\pi\omega} = \frac{h}{\omega} = \frac{\eta k}{\omega} \rightsquigarrow \omega = \omega_n$$

$$c_{EQ} = \frac{\eta k}{\omega_n} = \eta\frac{\omega_n^{\cancel{2}} m}{\cancel{\omega_n}} = \eta\omega_n m = \eta\sqrt{\frac{k}{m}}\, m = \eta\sqrt{km}$$

$$\zeta = \frac{c_{EQ}}{2\sqrt{km}} = \frac{\eta\sqrt{km}}{2\sqrt{km}} = \frac{\eta}{2}, \quad \eta = \frac{h}{k} = \frac{\alpha}{\pi k} \quad ✓$$

2. VIBRATIONS OF DAMPED MDOF SYSTEMS [2^{nd} Round]

2.0 INTRO

2.1 MODAL ANALYSIS FOR UNDAMPED SYSTEM ($c=0$)

2.2 MODAL ANALYSIS FOR DAMPED SYSTEM ($c \neq 0$)

2.0 INTRO

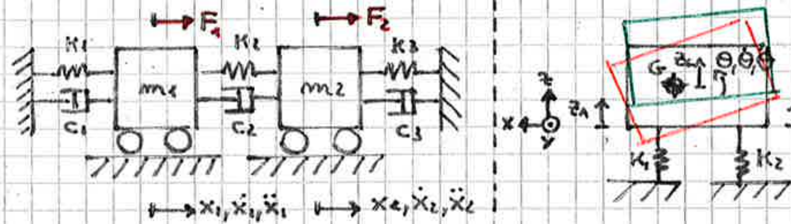
MDOF: MULTI DEGREES OF FREEDOM

m-DOF$_s$ → m-INDEPENDENT VARIABLES → m EIGENV. RIGID BODY MOTION

m-ODE$_s$ → m-NAT. FREQ., m-MODE-SHAPES ($\omega_0 \le \omega_1 \le \omega_2 \le \ldots \le \omega_m$)

hp$_s$: { LINEAR SYSTEMS → LINEAR IND. ODE ; m, c, k CONSTANT
NATURAL SYSTEMS → E_K EXPRESSED AS A QUADRATIC FORM OF $\dot{x}$ }

EXAMPLE: $m=2$ $[x_1, x_2]$; $[z_G, \theta]$

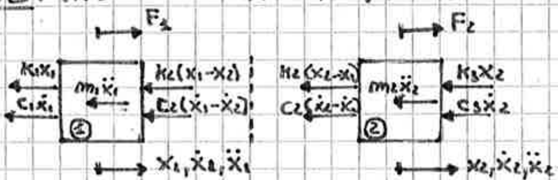


(QUITE VERTICAL ↓)

REBOUND M : ↕ + (↶) $\begin{cases} z_A \cong z_B \cong z_G \\ \theta \cong 0 \end{cases}$

PITCH M : ↷ + (↕) $\begin{cases} z_G \cong 0;\ z_A \neq z_B \\ \theta \neq 0 \end{cases}$

FBD: (NEWTONIAN APPROACH) (LCS)



*NOTE ON SYMMETRY:

$\begin{cases} a_{jk} = a_{kj} : \text{SYMMETRIC } A : [A]=[A]^T \\ a_{jk} = -a_{kj} : \text{SKEW-SYM. } A \end{cases}$

x_1 ①: $m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + c_2(\dot{x}_1-\dot{x}_2) + k_2(x_1-x_2) = F_1 \leftrightarrow m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2x_2 = F_1$

x_2 ②: $m_2\ddot{x}_2 + c_2(\dot{x}_2-\dot{x}_1) + k_2(x_2-x_1) + c_3\dot{x}_2 + k_3x_2 = F_2 \leftrightarrow m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2+c_3)\dot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = F_2$

MATRIX FORM:

$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$; $\{\dot{x}\} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}$; $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$; $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$ (m ROWS, $(m \times 1)$)

$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\{\ddot{x}\} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}\{\dot{x}\} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\{x\} = \{F\}$

$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\}$

UNCOUPLED (DIAGONAL) — COUPLED (NOT DIAGONAL) — ELASTIC/STIFFNESS AND DAMPING COUPLING

TO UNDERSTAND IF A MATRIX IS POSITIVE OR SEMI-POSITIVE DEFINITED A FASTER WAY (INSTEAD OF THE EVALUATION OF THE EIGENVALUES λ FROM $\det(K-\lambda I)=0$) IS THE EVALUATION OF THE DET. OF THE MATRIX. → IF $\det[\,] \neq 0$

IF $\det[K] > 0$ ⇒ POSITIVE DEF. ($\forall \lambda > 0$); FULL RANK!

IF $\det[K] = 0$ ⇒ SEMI-POSITIVE DEF. (SOME $\lambda = 0$)

(IF THE ROWS OF THE MATRIX ARE LINEAR COMB. OF EACH OTHER ↔ LINEAR DEPENDENT ROWS ⇒ NO FULL RANK)

NOTES:

$[M],[C],[K]$ ARE:

- SQUARE MATRIX $(m \times m)$
- * REAL SYMMETRIC (Re), $[\ldots] = [\ldots]^T$ ($a_{mm} \to a_{mm}$)
- $[M]$: POSIT. DEF. $\underset{1\times m}{\{z\}^T}\underset{m\times m}{[M]}\underset{m\times 1}{\{z\}} > 0$
- $[C],[K]$: SEMI-P. DEF. $\{z\}^T[K]\{z\} \ge 0$

RELATED TO EIGENVALUE K, $\det(K-\lambda I)=0$

$K = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $K-\lambda I = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$

$\det(K-\lambda I) = (2-\lambda)(3-\lambda) - 0\cdot 0 = 6-2\lambda-3\lambda+\lambda^2 = 0$

$\lambda^2 - 5\lambda + 6 = 0 \to \lambda_{1,2} = \frac{5 \pm \sqrt{25-4\cdot 6}}{2} \begin{cases} + & 3 \\ - & 2 \end{cases}$

$K = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ $K-\lambda I = \begin{bmatrix} 3-\lambda & -3 \\ -3 & 3-\lambda \end{bmatrix}$

$\det(K-\lambda I) = (3-\lambda)(3-\lambda) - (-3)(-3) = (3-\lambda)^2 - (-3)^2$

$= \cancel{9} - 6\lambda + \lambda^2 - \cancel{9} = 0 \to \lambda(\lambda-6) = 0 \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 6 \end{cases}$

FLEXIBILITY INFLUENCE COEFF. (a_{jk}) → FLEXIBILITY MATRIX [A] [IMPORTANT PROPERTIES]

$[a_{jk}] = [A]$ a_{jk} IS THE DISP. OF POINT x_j DUE TO F_k (UNIT FORCE) APPLIED IN x_k.

$x_j = \sum_{k=1}^{m} a_{jk} \cdot F_k$: SUPERPOSITION PRINCIPLE $\Rightarrow \{x\} = [A]\{F\}$

STIFFNESS INFLUENCE COEFF. (k_{jk}) → STIFFNESS MATRIX [K] (WHEN ALL OTHER DISPL. ARE NULL)

$[k_{jk}] = [K]$ k_{jk} IS WHEN ACTING A FORCE AT POINT x_j TO PRODUCE A (UNIT DISPL.) OF x_k

$F_j = \sum_{k=1}^{m} k_{jk} x_k$: SUPERPOSITION PRINCIPLE $\Rightarrow \{F\} = [K]\{x\}$

THEREFORE:

$\{x\} = [A]\{F\} = [A][K]\{x\} = [I]\{x\}$, $[I] = [A][K]$: IDENTITY MATRIX : $[A] = [K]^{-1}$

VALUTATION OF THE $\exists$ OF $[K]^{-1}$:

$[K]\{x\} = \{F\} \leftrightarrow [K]^{-1}[K]\{x\} = [K]^{-1}\{F\} \rightarrow [I]\{x\} = [K]^{-1}\{F\} \rightarrow \{x\} = [A]\{F\}$ FULL RANK RANK = min(m,m)

IF $\det[K] > 0 \Rightarrow [K]$ IS POSITIVE DEF. ($\{z\}^T[K]\{z\} > 0$) ($\forall \lambda > 0$ FROM $\det(K - \lambda I) = 0$) $\longrightarrow \exists [K]^{-1}$

IF $\det[K] = 0 \Rightarrow [K]$ IS SEMI-POSITIVE DEF. ($\{z\}^T[K]\{z\} \geq 0$) (SOME $\lambda = 0$) $\xrightarrow[\text{RANK < min(m,m)}]{\text{NO FULL RANK}} \nexists [K]^{-1}$

EXAMPLE: ‐

$[K] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$: $\det[K] = (k_1+k_2)\cdot(k_2+k_3) - (-k_2)^2 = k_1k_2 + k_1k_3 + \cancel{k_2^2} + k_2k_3 - \cancel{k_2^2} > 0$: FULL RANK $\Rightarrow \exists [K]^{-1}$

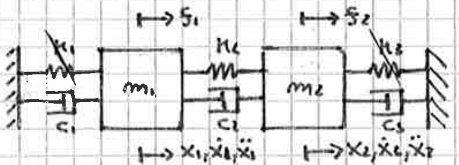
RANK = 2 (FULL!) $\Rightarrow$ THE FLEXIBILITY MATRIX $[A] = [K]^{-1}$ EXISTS! RANK = min(m,m)

PHYSICAL MEANING: WE CAN TRANSPOSE THE SYSTEM AND THE SYSTEM WILL COME BACK TO THE SAME I.C.

$[K] = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$: $\det[K] = k_2^2 - (-k_2^2) = 0$; NO FULL RANK $\Rightarrow \nexists [K]^{-1}$

RANK = 1 (0 ≤) RANK < min(m,m)

PHYSICAL MEANING: WE CAN HAVE RIGID BODY MOTION: IF WE DELETE k_1 AND k_3, IF WE MOVE THE SYSTEM, THE SYSTEM REMAINS WHERE IT HAS BEEN PUT (THERE ARE NOT RESTORING FORCES THAT TO COME BACK THE SYSTEM TO THE ORIGINAL CONFIGURATION!).



(m = 2)

‐ ‐

RANK CALCULATION (NOT SEEN DURING LESSONS)

$[A] = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \\ 3 & 2 & 6 \end{bmatrix}$ $\begin{matrix} -1 & 2 \\ 3 & 4 \\ 2 & 6 \end{matrix}\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \\ 3 & 2 & 6 \end{bmatrix}\begin{matrix} 1 & -1 \\ 2 & 3 \\ 3 & 2 \end{matrix}$

$j_1 = \min(m,m) = 3$ (RANK = 3 IF $\det(A) > 0$)

(MAJOR DIAG. FROM SX→DX) − (ANTI-DIAG. FROM DX→SX)

$\det(A) = (1\cdot3\cdot6) + (-1\cdot4\cdot3) + (2\cdot2\cdot2) - [(2\cdot3\cdot3) + (-1\cdot2\cdot6) + (1\cdot4\cdot2)] = 0$

$\det(A) = 0 \Rightarrow$ WE HAVE TO EVALUATE A'

$[A'] = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$j_2 = j_1 - 1 = 2$ (RANK = 2 IF $\det(A') > 0$)

$\det(A') = 1\cdot3 - (-1\cdot2) = 6 > 0 \Rightarrow$ RANK = 2

$[A] = \begin{bmatrix} 1 & -5 & 3 & 0 \\ 0 & 3 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

$j_1 = \min(m,m) = 3$ (RANK = 3 IF ONE OF det OF THE 4 SUB-M IS > 0)

(DELETING 1 m)

$\begin{bmatrix} -5 & 3 & 0 \\ 3 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & -5 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & -5 & 3 \\ 0 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$\det(A_1) = 10 + 3 + 0 - [0 + 18 - 5] = 0$
$\det(A_2) = -2 + 3 + 0 - [0 + 0 + 1] = 0$
$\det(A_3) = 6 - 5 + 0 - [0 + 0 + 1] = 0$
$\det(A_4) = 3 + 5 + 0 - [9 + 0 - 1] = 0$

$\det(A) = 0 \Rightarrow$ WE HAVE TO EVALUATE A'

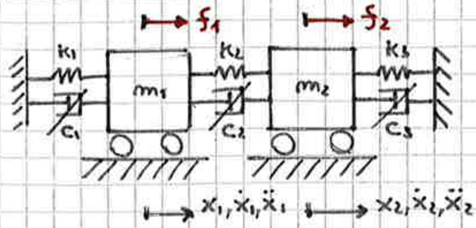
$[A'] = \begin{bmatrix} 1 & -5 \\ 0 & 3 \end{bmatrix}$ $j_2 = j_1 - 1 = 2$

$\det(A') = 3 - 0 = 3 > 0 \Rightarrow$ RANK = 2

‐ ‐

LET A BE SYMM. OF m. A PRINCIPAL MINOR OF ORDER k IS A MINOR (DETERMINANT OF THE MATRIX) OBTAINED BY DELETING m−k ROWS AND COLUMNS (SAME ROW AND COLUMN POSITION). THE COLLECTORS OF ALL PRINCIPAL MINORS OF ORDER k WILL BE DENOTED A_k.

CONSIDERING:



2 dofs $\rightarrow$ 2 λ (λ_1, λ_2) $\rightarrow$ 2 ω_m $(\omega_{m1}, \omega_{m2})$ $\rightarrow$ 2 MODES

CASE A) $k_2 = 0$ $\Rightarrow$ NO COUPLING $\Rightarrow$ 2 INDEPENDENT SYSTEMS ($\leftrightarrow$ [M], [K] = DIAG.)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \omega_1 = \sqrt{\frac{k_1}{m_1}} \; ; \; \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

(NAT. FREQUENCIES)

(WE DON'T NEED MODAL ANALYSIS)

CASE B) $k_2 \neq 0$ $\Rightarrow$ COUPLING

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$\det(K) = (k_1+k_2)(k_2+k_3) - (-k_2)^2 = k_1k_2 + k_1k_3 + \cancel{k_2^2} + k_2k_3 - \cancel{k_2^2} > 0$ $\Rightarrow$ FULL RANK $\Rightarrow$ POS. DEF.

$\Rightarrow \exists [K]^{-1}$ $\Rightarrow$ GAUSS REDUCTION $\Rightarrow$ 2 INDEPENDENT SYSTEMS $\Rightarrow$ ω_1, ω_2

NOTE: A LITTLE CHANGE IN k_2 CAN LEAD TO A VERY STRONG CHANGE IN THE SYSTEM (LEISSA SENTENCE).

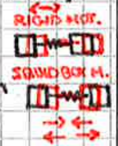
CASE C) $k_1 = k_2 = k_3 = k$; $m_1 = m_2 = m$ $\Rightarrow$ COUPLING ($\pm$ 2 DEPENDENT SYST. EQs)

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \pm$$

$\det(K) = 4k^2 - (-k)^2 = 3k^2 > 0$ $\Rightarrow$ FULL RANK $\Rightarrow$ POS. DEF. $\Rightarrow \exists [K]^{-1} \Rightarrow$ GAUSS RED.

$y = x_1 + x_2 \Rightarrow m(\ddot{x}_1 + \ddot{x}_2) + k(x_1 + x_2) = F_1 + F_2 \Rightarrow m\ddot{y} + ky = F_1 + F_2 \qquad \omega_1 = \sqrt{\frac{k}{m}}$

$z = x_1 - x_2 \Rightarrow m(\ddot{x}_1 - \ddot{x}_2) + 3k(x_1 - x_2) = F_1 - F_2 \Rightarrow m\ddot{z} + 3kz = F_1 - F_2 \qquad \omega_2 = \sqrt{\frac{3k}{m}}$



CASE D) $k_1 = k_3 = 0$; $k_2 \neq 0$ $\Rightarrow$ COUPLING (RIGID BODY MOTION)

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$\det(K) = k_2^2 - (-k_2)^2 = 0$ $\Rightarrow$ NOT FULL RANK $\Rightarrow$ RANK = 1 $\Rightarrow$ RIGID BODY MOTION

PROPERTIES: (1), (2), (3), (4)

(1) UNCOUPLING EFFECT ⇒ M-K ORTHOGONALITY DEMONSTRATION

1) CONSIDERING 2 MODES (R, S) WITH [M], [K] SYMMETRIC $\begin{cases}[M]^T=[M]\\ [K]^T=[K]\end{cases}$ hp!

$([K]-\omega_R^2[M])\{\varphi_R\}=\{0\}$

$([K]-\omega_S^2[M])\{\varphi_S\}=\{0\}$

2) PREMULTIPLYING BOTH BY $\{\varphi_S\}^T$ AND $\{\varphi_R\}^T$ RESPECTIVELY:

$\{\varphi_S\}^T([K]-\omega_R^2[M])\{\varphi_R\}=\{\varphi_S\}^T\{0\}=0$ (SCALAR) $\leftrightarrow \begin{cases}(1): \{\varphi_S\}^T[K]\{\varphi_R\}=\omega_R^2\cdot\{\varphi_S\}^T[M]\{\varphi_R\}\\ (1)': \frac{1}{\omega_R^2}\{\varphi_S\}^T[K]\{\varphi_R\}=\{\varphi_S\}^T[M]\{\varphi_R\}\end{cases}$

$\{\varphi_R\}^T([K]-\omega_S^2[M])\{\varphi_S\}=\{\varphi_R\}^T\{0\}=0$

3) MAKING THE TRANSPOSE OF THE 2nd EQ:

$[\{\varphi_R\}^T([K]-\omega_S^2[M])\{\varphi_S\}]^T=[\{\varphi_R\}^T\{0\}]^T=[0]^T=0$

$\{\varphi_S\}^T([K]-\omega_S^2[M])^T\{\varphi_R\}=0$

$\{\varphi_S\}^T([K]-\omega_S^2[M])\{\varphi_R\}=0 \leftrightarrow \begin{cases}(2): \{\varphi_S\}^T[K]\{\varphi_R\}=\omega_S^2\cdot\{\varphi_S\}^T[M]\{\varphi_R\}\\ (2)': \frac{1}{\omega_S^2}\{\varphi_S\}^T[K]\{\varphi_R\}=\{\varphi_S\}^T[M]\{\varphi_R\}\end{cases}$

4) DOING: (2)-(1) WE GET:

$(\omega_R^2-\omega_S^2)\{\varphi_S\}^T[M]\{\varphi_R\}=0$

$\begin{cases}\text{IF } R\neq S \Rightarrow \{\varphi_S\}^T[M]\{\varphi_R\}=0 & \text{M-ORTHOGONALITY (ORTHOGONALITY IF } \{a\}^T\{b\}=0)\\ \text{IF } R=S \Rightarrow \{\varphi_S\}^T[M]\{\varphi_R\}=m_R>0, & m_R: \text{MODAL MASS } (>0 \text{ BECAUSE } [M] \text{ IS POS.DEF.})\end{cases}$

5) DOING: (2)'-(1)' WE GET:

$\left(\frac{1}{\omega_S^2}-\frac{1}{\omega_R^2}\right)\{\varphi_S\}^T[K]\{\varphi_R\}=0$

$\begin{cases}\text{IF } R\neq S \Rightarrow \{\varphi_S\}^T[K]\{\varphi_R\}=0 & \text{K-ORTHOGONALITY}\\ \text{IF } R=S \Rightarrow \{\varphi_S\}^T[K]\{\varphi_R\}=K_R\geq 0, & K_R: \text{MODAL STIFFNESS } (\geq 0 \text{ BECAUSE } [K] \text{ IS SEMI-P.})\end{cases}$

6) THEREFORE:

FOR MODE R $(\omega_R, \{\varphi_R\})$ ⇒ WE GET m_R, K_R: $\omega_R=\sqrt{\frac{K_R}{m_R}}$

$\begin{cases}[\varphi]^T[M][\varphi]=[{}^{\diagdown}m_{R\diagdown}] & \text{: MASS MODAL MATRIX}\\ [\varphi]^T[K][\varphi]=[{}^{\diagdown}K_{R\diagdown}] & \text{: STIFFNESS MODAL MATRIX}\end{cases}$

MEANING: WE HAVE UNCOUPLED THE EQUATIONS; WE CAN SOLVE ALL MODE SHAPES LIKE UNCOUPLED PENDOLUMS WITH THE SDOF TECHNIQUES.

NOTE: WE CAN USE MODAL TECHNIQUES FOR UNCOUPLING 2 MATRICES; BUT WHEN WE HAVE A THIRD (DAMPING ONE) IT COULD BE NOT SO EASY... BAD PROBLEM.

m_R, K_R : PARAMETER OF EACH INDEPENDENT PENDULUM

(2) THEOREM OF EXPANSION (MODAL SUPERPOSITION) → DEMONSTRATION OF:

1. THE MODAL MATRIX $[\varphi]$ IS A "BASE".
2. LTI TRANSFORMATION BETWEEN $\{x\}$ AND $\{\eta\}$: $\{x\} = [\varphi]\{\eta\}$
3. ANY VECTOR CAN BE EXPRESSED AS A LINEAR COMBINATION OF MODE-SHAPES (EIGENVE)
   - EIGENVECTORS $\{\varphi_1\}, \{\varphi_2\}, \dots$ ARE LINEARLY INDEPENDENT.
   - MODAL MATRIX $[\varphi]$ HAS FULL RANK.

PROOF BY CONTRADICTION:

hp: MODAL VECTORS $\{\varphi_1\}, \{\varphi_2\}, \dots$ ARE **LINEARLY DEPENDENT**,

THEN $c_1\{\varphi_1\} + c_2\{\varphi_2\} + \dots c_R\{\varphi_R\} + \dots c_m\{\varphi_m\} = \sum_{R=1}^{m} c_R\{\varphi_R\} = \{0\}$

WHERE c_R ARE NOT NULL.

PRE-MULTIPLYING BY $\{\varphi_S\}^T[M]$:

$$\sum_{R=1}^{m} c_R \underbrace{\{\varphi_S\}^T[M]\{\varphi_R\}}_{\substack{\text{IF } R\neq S \ = 0 \\ \text{IF } R=S \ = m_R \neq 0}} = 0$$

$c_S\{\varphi_S\}^T[M]\{\varphi_S\} = 0 \Rightarrow c_S = 0$

REPEATING THIS PROCEDURE TAKING ANY $\{\varphi_S\}^T$, IT FOLLOWS THAT $c_R = 0, \forall R$ (CONTRAD.)

⇒ HENCE, EIGENVECTORS ARE LINEARLY INDEPENDENT (THEY REPRESENT A BASE)

⇒ HENCE, ANY VECTOR CAN BE EXPRESSED AS A LINEAR COMB. OF EIGENVECTORS,

$\{V\} = \sum_{R=1}^{m} c_R\{\varphi_R\}$

PRE-MULTIPLYING BY $\{\varphi_S\}^T[M]$:

$\{\varphi_S\}^T[M]\{V\} = \sum_{R=1}^{m} c_R\{\varphi_S\}^T[M]\{\varphi_R\} = c_S m_S$

$c_R = \dfrac{\{\varphi_S\}^T[M]\{V\}}{m_R}$: SCALAR VALUE FOR EACH MODE (MODAL PARTECIPATION FACTOR)

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2.2 MODAL ANALYSIS FOR DAMPED SYSTEM ($C \neq 0$)

• **MODAL ANALYSIS WITH NO EXTERNAL FORCE**

$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\}$, $\{x\} = [\varphi]\{\eta\} = \sum_{R=1}^{m} \{\varphi_R\}\eta_R$: LINEAR T. (SUPERP. PRINC.)

$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \rightarrow [\Lambda], [\varphi]$ $\{x\} = \{x_0\}e^{i\omega t}$; $\{\dot{x}\} = i\omega\{x_0\}e^{i\omega t}$; $\{\ddot{x}\} = -\omega^2\{x_0\}e^{i\omega t}$

$([K] - \omega^2[M])\{x_0\} = \{0\}$

$\begin{cases} \{x_0\} = 0 \quad \text{TRIVIAL SOLUTION} \\ \det([K] - \omega^2[M]) = 0 \rightarrow \omega_R^2 \rightarrow [\Lambda] = [\diagdown \omega_R^2 \diagdown] , \ R = 1, \dots, m \end{cases}$

$([K] - \omega_R^2[M])\{\varphi_R\} = \{0\} \rightarrow \{\varphi_R\} \rightarrow [\varphi] = [\{\varphi_1\}; \{\varphi_2\}; \dots]$

CHARACTERIZATION OF DAMPING: **CAUGHEY RELATIONS**:

• PERMUTATION OF $[M]^{-1}$:

$[C][M]^{-1}[K] = [K][M]^{-1}[C]$, $[C] = [C_P] = \text{DIAG}$

• CAUGHEY SERIES $[C_P]$:

$$[C_P] = \sum_{i=0}^{m-1} \gamma_i [M]([M]^{-1}[K])^i$$

• SIMPLEST CASE $\gamma_0, \gamma_1 \neq 0$:

$[C_P] = \alpha[M] + \beta[K]$

$\alpha \rightarrow [M]$: SKY-HOOK DAMPERS

$\beta \rightarrow [K]$: MATERIAL DEFORMATIONS

• GENERALLY:

$[C] = [C_P] + [C_{NP}]$

IF $[C_{NP}] \neq 0 \Rightarrow$ COUPLING IN C !

MODAL TRANSFORMATION (ONLY WITH MCK PROP.)

$[M][\varphi]\{\ddot{\eta}\} + [C_P][\varphi]\{\dot{\eta}\} + [K][\varphi]\{\eta\} = \{0\}$

$[\varphi]^T[M][\varphi]\{\ddot{\eta}\} + [\varphi]^T[C_P][\varphi]\{\dot{\eta}\} + [\varphi]^T[K][\varphi]\{\eta\} = [\varphi]^T\{0\} = 0$

$[\varphi]^T[M][\varphi]\{\ddot{\eta}\} + [\varphi]^T(\alpha[M] + \beta[K])[\varphi]\{\dot{\eta}\} + [\varphi]^T[K][\varphi]\{\eta\} = 0$

$[\diagdown m_R \diagdown]\{\ddot{\eta}\} + (\alpha[\diagdown m_R \diagdown] + \beta[\diagdown K_R \diagdown])\{\dot{\eta}\} + [\diagdown K_R \diagdown]\{\eta\} = 0$

$[\diagdown m_R \diagdown]\{\ddot{\eta}\} + [\diagdown C_R \diagdown]\{\dot{\eta}\} + [\diagdown K_R \diagdown]\{\eta\} = 0$

$m_R \ddot{\eta}_R + (\alpha m_R + \beta K_R)\dot{\eta}_R + K_R \eta_R = 0$, $R = 1, \dots, m$ (m INDEP. PENDULUMS)

$m_R \ddot{\eta}_R + C_R \dot{\eta}_R + K_R \eta_R = 0$, $R = 1, \dots, m$ $\omega_R = \sqrt{K_R/m_R}$; $\zeta_R = C_R/(2 m_R \omega_R)$

IF USING UNITARY MODAL MASS NORMALIZATION ($m_R = 1$, $[\varphi] \rightarrow [\phi]$, $\{\phi_R\} = \frac{1}{\sqrt{m_R}}\{\varphi_R\}$):

$[\phi]^T[M][\phi]\{\ddot{\eta}\} + [\phi]^T[C_P][\phi]\{\dot{\eta}\} + [\phi]^T[K][\phi]\{\eta\} = \{0\}$

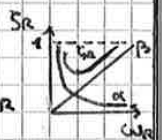
$[\phi]^T[M][\phi]\{\ddot{\eta}\} + [\phi]^T(\alpha[M] + \beta[K])[\phi]\{\dot{\eta}\} + [\phi]^T[K][\phi]\{\eta\} = \{0\}$

$[I]\{\ddot{\eta}\} + (\alpha[I] + \beta[\Lambda])\{\dot{\eta}\} + [\Lambda]\{\eta\} = \{0\}$

$[I]\{\ddot{\eta}\} + [\diagdown 2\zeta_R\omega_R \diagdown]\{\dot{\eta}\} + [\diagdown \omega_R^2 \diagdown]\{\eta\} = \{0\}$

$\ddot{\eta}_R + (\alpha + \beta\omega_R^2)\dot{\eta}_R + \omega_R^2 \eta_R = 0$

$\ddot{\eta}_R + 2\zeta_R\omega_R\dot{\eta}_R + \omega_R^2\eta_R = 0$, $\zeta_R = \frac{1}{2\omega_R m_R}(\alpha m_R + \beta K_R) = \frac{(\alpha + \beta\omega_R^2)}{2\omega_R} = \underbrace{\frac{\alpha}{2\omega_R}}_{\text{IPERBOLIC}} + \underbrace{\frac{\beta\omega_R}{2}}_{\text{LINEAR}}$



FREE RESPONSE:

$\eta_R(t) = \eta_{R\,H.S.}(t) = e^{-\zeta_R\omega_R t}\cdot[A_R\cos(\omega_R\sqrt{1-\zeta_R^2}\,t) + B_R\sin(\omega_R\sqrt{1-\zeta_R^2}\,t)]$

$\dot{\eta}_R(t) = \dot{\eta}_{R\,H.S.}(t) = -\zeta_R\omega_R e^{-\zeta_R\omega_R t}[A_R\cos(\omega_R\sqrt{1-\zeta_R^2}\,t) + B_R\sin(\omega_R\sqrt{1-\zeta_R^2}\,t)] + e^{-\zeta_R\omega_R t}\cdot[-A_R\omega_R\sqrt{1-\zeta_R^2}\sin(\omega_R\sqrt{1-\zeta_R^2}\,t)$
$+ B_R\omega_R\sqrt{1-\zeta_R^2}\cos(\omega_R\sqrt{1-\zeta_R^2}\,t)]$

• FORCED RESPONSE (HARMONIC EXCITATION)

$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F(t)\} = \{F_0\}\cos(\omega t) = \{F_0\}e^{i\omega t}$

CONSIDERATION ABOUT PRE-MULTIPLICATION OF $[\varphi]^T$ [PREVIEW]

$\{\Gamma\} = [\varphi]^T\{F_0\} \Rightarrow \{\Gamma\}e^{i\omega t} = [\varphi]^T\{F_0\}e^{i\omega t}$ | $\{X\} = \{X_0\}e^{i\omega t}$; $\{\eta\} = \{\eta_0\}e^{i\omega t}$; $\{X\} = [\varphi]\{\eta\}$

$\Gamma_R = \{\varphi_R\}^T\{F_0\} \Rightarrow \Gamma_R e^{i\omega t} = \{\varphi_R\}^T\{F_0\}e^{i\omega t}$ | $\{X\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_R$; $\{X_0\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_{R,0} \in \mathbb{C}$

TWO STRATEGIES:

1) INVERSION OF THE DYNAMIC STIFFNESS MATRIX (DUNCAN REDUCTION) $[C] = [C_P] + [C_{NP}]$

2) MODAL APPROACH (ONLY WITH MCK PROPORTIONALITY) $[C] = [C_P] \leftrightarrow [C]$ = DIAGONALIZABLE ($[\varphi]^T[C_P][\varphi]$ = DIAG)

1) INVERSION OF THE DYNAMIC STIFFNESS MATRIX (DUNCAN RED.) → COMPLEX EIGENVECT (NOT SYNCHR. MOTIONS)

$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F\}$ $[C] = [C_P] + [C_{NP}]$

CONSIDERING: $\{X\} = \{X_0\}e^{i\omega t}$; $\{\dot{X}\} = i\omega\{X_0\}e^{i\omega t}$; $\{\ddot{X}\} = -\omega^2\{X_0\}e^{i\omega t}$; $\{F\} = \{F_0\}e^{i\omega t}$

$([K] - \omega^2[M] + i\omega[C])\{X_0\}\cancel{e^{i\omega t}} = \{F_0\}\cancel{e^{i\omega t}}$

$[K_{DYN}]\{X_0\} = \{F_0\}$, $[K_{DYN}] = [Z(i\omega)]$

$\{X_0\} = [K_{DYN}]^{-1}\{F_0\} = [Z(i\omega)]^{-1}\{F_0\} = [\alpha]\{F_0\}$, $[\alpha] = [K_{DYN}]^{-1}$: RECEPTANCE MATRIX [m/N]

$\alpha_{jj} = \frac{x_j}{F_j}$; $\alpha_{jk} = \frac{x_j}{F_k}$

↳ TO $\exists [K_{DYN}]^{-1}$ HAS TO BE FULL RANK $\leftrightarrow \det[K_{DYN}] \neq 0$!

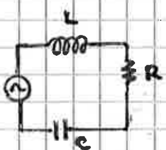
$\frac{x_j}{F_k}$ $\begin{cases} j \neq k: \text{CROSS-RECEPTANCE} \in \mathbb{C} \ (\alpha_{jk}) \\ j = k: \text{AUTO-RECEPTANCE} \ (\alpha_{jj}) \end{cases}$ RECEPTANCE [m/N]

$\frac{\dot{x}_j}{F_k}$ $\begin{cases} j \neq k: \text{CROSS-MOBILITY} \in \mathbb{C} \\ j = k: \text{AUTO-MOBILITY} \end{cases}$ MOBILITY [(m/s)/N] $Y_{jj} = \frac{\dot{x}_j}{F_j} = \frac{i\omega x_j}{F_j} = i\omega\alpha_{jj}$

$\frac{\ddot{x}_j}{F_k}$ $\begin{cases} j \neq k: \text{CROSS-INERTANCE} \in \mathbb{C} \\ j = k: \text{AUTO-INERTANCE} \end{cases}$ INERTANCE [(m/s²)/N] $A_{jj} = \frac{\ddot{x}_j}{F_j} = \frac{-\omega^2 x_j}{F_j} = -\omega^2\alpha_{jj}$

NOTE: SIMILITUDE WITH:

$\begin{cases} M: \text{MASS} \\ C: \text{DAMPING} \\ K: \text{STIFFNESS} \end{cases} \Leftrightarrow \begin{cases} L: \text{INDUCTANCE} \\ R: \text{RESISTOR} \\ 1/C: \text{CAPACITOR} \end{cases}$



($\Delta V = Ri \leftrightarrow c\dot{x} = F_{DAMPING}$)

VOLTAGE KIRCHOFF LAWS $\begin{cases} 1^{st} \text{ LAW}: L\frac{d i^2}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{de(t)}{dt} \\ \ldots \end{cases}$

IF USING UNITARY MODAL MASS NORMALIZATION $\left(m_R = 1,\ [\phi] \to [\Phi],\ \{\Phi_R\} = \frac{1}{\sqrt{m_R}}\{\varphi_R\}\right)$

$$[\Phi]^T[M][\Phi]\{\ddot{\eta}\} + [\Phi]^T[C_P][\Phi]\{\dot{\eta}\} + [\Phi]^T[K][\Phi]\{\eta\} = [\Phi]^T\{F_0\}e^{i\omega t} = \{\Gamma\}e^{i\omega t}$$

$$[\Phi]^T[M][\Phi]\{\ddot{\eta}\} + [\Phi]^T(\alpha[M]+\beta[K])[\Phi]\{\dot{\eta}\} + [\Phi]^T[K][\Phi]\{\eta\} = [\Phi]^T\{F_0\}e^{i\omega t} = \{\Gamma\}e^{i\omega t}$$

$$[I]\{\ddot{\eta}\} + (\alpha[I] + \beta[\Lambda])\{\dot{\eta}\} + [\Lambda]\{\eta\} = [\Phi]^T\{F_0\}e^{i\omega t} = \{\Gamma\}e^{i\omega t}$$

$$[I]\{\ddot{\eta}\} + [\diagdown 2\zeta_R\omega_R \diagdown]\{\dot{\eta}\} + [\diagdown \omega_R^2 \diagdown]\{\eta\} = [\Phi]^T\{F_0\}e^{i\omega t} = \{\Gamma\}e^{i\omega t}$$

$$\ddot{\eta}_R + (\alpha + \beta\omega_R^2)\dot{\eta}_R + \omega_R^2\eta_R = \{\Phi_R\}^T\{F_0\}e^{i\omega t} = \Gamma_R \cdot e^{i\omega t}$$

$$\ddot{\eta}_R + 2\zeta_R\omega_R\dot{\eta}_R + \omega_R^2\eta_R = \{\Phi_R\}^T\{F_0\}e^{i\omega t} = \Gamma_R \cdot e^{i\omega t}$$

CONSIDERING: $\{\eta\} = \{\eta_0\}e^{i\omega t}$; $\{\dot{\eta}\} = i\omega\{\eta_0\}e^{i\omega t}$; $\{\ddot{\eta}\} = -\omega^2\{\eta_0\}e^{i\omega t}$

$$(\omega_R^2 - \omega^2 + 2i\zeta_R\omega\omega_R)\eta_{R,0}\cancel{e^{i\omega t}} = \{\Phi_R\}^T\{F_0\}\cancel{e^{i\omega t}}$$

$$\eta_{R,0} = \frac{\{\Phi_R\}^T\{F_0\}}{(\omega_R^2 - \omega^2 + 2i\zeta_R\omega\omega_R)}$$

CONSIDERING: $\{x\} = [\Phi]\{\eta\} = \sum_{R=1}^{m}\{\Phi_R\}\eta_R$; $\{x_0\} = \sum_{R=1}^{m}\{\Phi_R\}\eta_{R,0}$

$$\{x_0\} = \sum_{R=1}^{m}\frac{\{\Phi_R\}^T\{F_0\}\{\Phi_R\}}{(\omega_R^2 - \omega^2 + 2i\zeta_R\omega\omega_R)}$$

FOR SINGLE FORCE EXCITATION:

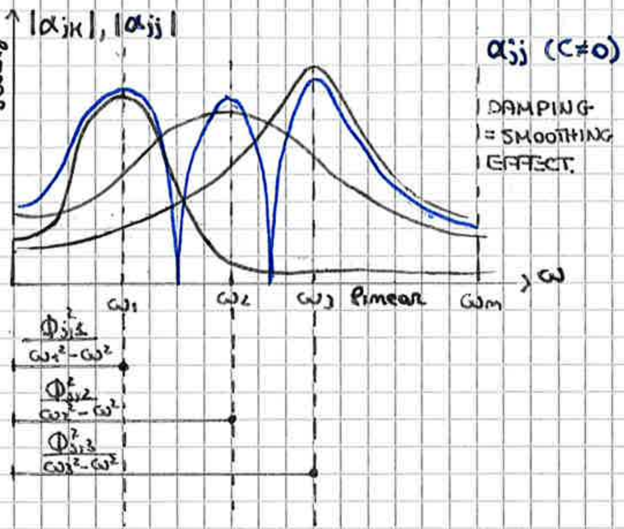
$\{x_0\} = [\alpha]\{F_0\}$, $[\alpha]$ = RECEPTANCE MATRIX

$$\alpha_{jj} = \frac{x_j}{F_j} = \sum_{R=1}^{m}\frac{\Phi_{j,R}^2}{(\omega_R^2 - \omega^2 + 2i\zeta_R\omega\omega_R)}$$: AUTO-R.

$$\alpha_{jk} = \frac{x_j}{F_k} = \sum_{R=1}^{m}\frac{\Phi_{j,R}\,\Phi_{k,R}}{(\omega_R^2 - \omega^2 + 2i\zeta_R\omega\omega_R)}$$: CROSS-R.

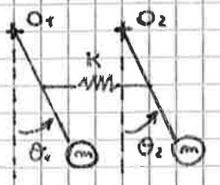
CONSIDERING $C=0$:

$$\alpha_{jj} = \sum_{R=1}^{m}\frac{\Phi_{jR}^2}{\omega_R^2 - \omega^2} = \frac{\Phi_{j,1}^2}{\omega_1^2 - \omega^2} + \frac{\Phi_{j,2}^2}{\omega_2^2 - \omega^2} + \dots$$

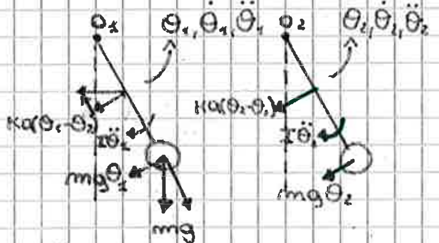


+ BEAT PHENOMENON

- 2 IDENTICAL SDOF CONNECTED BY A "SOFT SPRING" => WEAK COUPLING
- hp: **SMALL OSCILLATIONS** => LINEARIZED SYSTEM



FBD:



I.C. $\begin{cases} \{\theta(t=0)\} = \begin{Bmatrix} \theta_0 \\ 0 \end{Bmatrix} \\ \{\dot\theta(t=0)\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \end{cases}$

$$\text{a)}\ I\ddot\theta_1 + a^2k(\theta_1-\theta_2) + mgl\theta_1 = 0 \rightarrow ml^2\ddot\theta_1 + (mgl + ka^2)\theta_1 - ka^2\theta_2 = 0$$

$$\text{b)}\ I\ddot\theta_2 + a^2k(\theta_2-\theta_1) + mgl\theta_2 = 0 \rightarrow ml^2\ddot\theta_2 - ka^2\theta_1 + (mgl + ka^2)\theta_2 = 0$$

$$\begin{bmatrix} ml^2 & \\ & ml^2 \end{bmatrix}\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{Bmatrix} + \begin{bmatrix} mgl+ka^2 & -ka^2 \\ -ka^2 & mgl+ka^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[M]\{\ddot\theta\} + [K]\{\theta\} = \{0\}, \quad \{\theta\} = \{\theta_0\}e^{i\omega t}$$

$$([K] - \omega^2[M])\{\theta_0\} = \{0\}, \quad \{\theta_0\} = 0 \text{ TRIVIAL S.}$$

$$\det([K] - \omega^2[M]) = 0 \rightarrow \omega^2$$

$$\det\begin{bmatrix} mgl+ka^2-\omega^2ml^2 & -ka^2 \\ -ka^2 & mgl+ka^2-\omega^2ml^2 \end{bmatrix} = (mgl + ka^2 - \omega^2ml^2)^2 - (-ka^2)^2 = 0$$

$$mgl + ka^2 - \omega^2ml^2 + ka^2 = 0 \Rightarrow \omega_2 = \sqrt{\frac{mgl+2ka^2}{ml^2}} = \sqrt{\frac{g}{l} + \frac{2ka^2}{ml^2}}$$

$$mgl + \cancel{ka^2} - \omega^2ml^2 - \cancel{ka^2} = 0 \Rightarrow \omega_1 = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

MODE #1

$$\begin{bmatrix} \cancel{mgl}+ka^2-\cancel{mgl} & -ka^2 \\ -ka^2 & \cancel{mgl}+ka^2-\cancel{mgl} \end{bmatrix}\begin{Bmatrix} 1 \\ \varphi_{2,1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-ka^2 + ka^2\varphi_{2,1} = 0 \Rightarrow \varphi_{2,1} = 1$$

MODE #2

$$\begin{bmatrix} \cancel{mgl}+ka^2-\cancel{mgl}-2ka^2 & -ka^2 \\ -ka^2 & \cancel{mgl}+ka^2-\cancel{mgl}-2ka^2 \end{bmatrix}\begin{Bmatrix} 1 \\ \varphi_{2,2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-ka^2 - ka^2\varphi_{2,2} = 0 \Rightarrow \varphi_{2,2} = -1$$

MODAL MATRIX:

$$[\varphi] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\{x\} = [\varphi]\{\eta\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_R, \quad \eta_R = C_R\cos(\omega_R t - \alpha_R)$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} C_1\cos(\omega_1 t - \alpha_1) \\ C_2\cos(\omega_2 t - \alpha_2) \end{Bmatrix} = \{\varphi_1\}\eta_1 + \{\varphi_2\}\eta_2 = \{\varphi_1\}C_1\cos(\omega_1 t - \alpha_1) + \{\varphi_2\}C_2\cos$$

$$\begin{Bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} \dot\eta_1 \\ \dot\eta_2 \end{Bmatrix} = \{\varphi_1\}\dot\eta_1 + \{\varphi_2\}\dot\eta_2 = \{\varphi_1\}(-\omega_1C_1)\sin(\omega_1 t - \alpha_1) + \{\varphi_2\}(-\omega_2C_2)\sin(\omega_2 t - \alpha_2)$$

$$\theta_1 = C_1\cos(\omega_1 t - \alpha_1) + C_2\cos(\omega_2 t - \alpha_2)$$

$$\theta_2 = C_1\cos(\omega_1 t - \alpha_1) - C_2\cos(\omega_2 t - \alpha_2)$$

$$\dot\theta_1 = -\omega_1C_1\sin(\omega_1 t - \alpha_1) - \omega_2C_2\sin(\omega_2 t - \alpha_2)$$

$$\dot\theta_2 = -\omega_1C_1\sin(\omega_1 t - \alpha_1) + \omega_2C_2\sin(\omega_2 t - \alpha_2)$$

+ LAGRANGE'S EQUATIONS

$T = \frac{1}{2}\{\dot{x}\}^T[M]\{\dot{x}\}$: KINETIC ENERGY (>0) $\left(T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_0\dot{\theta}^2 > 0 \right)$

$V = \frac{1}{2}\{x\}^T[K]\{x\}$: POTENTIAL ENERGY (≥ 0) $\left(V = \frac{1}{2}kx^2 + \frac{1}{2}k_t\theta^2 \geq 0 \right)$

$D = \frac{1}{2}\{\dot{x}\}^T[C]\{\dot{x}\}$: DISSIPATION (≥ 0)

$L = T - V$: LAGRANGIAN SCALAR

$L = L(q_i, \dot{q}_i)$, q_i : LAGRANGIAN COORDINATE

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$, $k = 1, \dots, m$ (m DOFs), Q_k = GENERALISED F $\begin{cases} \text{EXTERNAL FORCES} \\ \text{DISSIPATIVE FORCES} \end{cases}$

$Q_k = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial \dot{q}_k} = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial q_k}$, R_i : GENERIC COORDINATE POSITION

FOR GENERALISED FORCES:

$\vec{R}_i = \vec{R}_i(q_1; q_2; \dots; q_m)$

$\delta\vec{R}_i = \delta\vec{R}_i(q_1; q_2; \dots; q_m) = \frac{\partial \vec{R}_i}{\partial q_1}\partial q_1 + \frac{\partial \vec{R}_i}{\partial q_2}\partial q_2 + \dots + \frac{\partial \vec{R}_i}{\partial q_m}\partial q_m$

$\dot{\vec{R}}_i = \frac{d\vec{R}_i}{dt} = \frac{\partial \vec{R}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{R}_i}{\partial q_2}\frac{dq_2}{dt} + \dots + \frac{\partial \vec{R}_i}{\partial q_m}\frac{dq_m}{dt}$

NOT CONSIDERING THE EXTERNAL FORCES, IF **DISSIPATIVE FORCES** ARE ONLY **VISCOUS**:

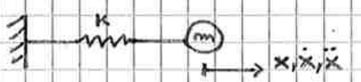
$\Rightarrow$ RAYLEIGHT DISSIPATIVE FUNCTION (D)

$Q_k = -\frac{\partial D}{\partial \dot{q}_k}$, $D = \frac{1}{2}\sum_{j,k=1}^{m} c_{jk}\dot{q}_k\dot{q}_j$ $\left(\text{IF } m=1 : q_1 = x_1 = x \Rightarrow D = \frac{1}{2}c\dot{x}^2 \right)$

IF THERE ARE 1 EXTERNAL F. + 1 VISCOUS DISSIPATIVE FORCE:

$Q_k = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial \dot{q}_k} = \vec{F}_1 \frac{\partial \vec{R}_1}{\partial \dot{q}_k} - \frac{\partial D}{\partial \dot{q}_k} = \vec{F}_1 \frac{\partial \vec{R}_1}{\partial q_k} - \frac{\partial D}{\partial \dot{q}_k}$

EXAMPLE 1:



$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

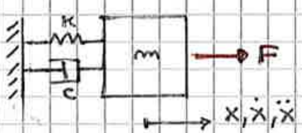
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$ $\quad k = 1$, $q_1 = x$

$\begin{cases} \frac{\partial L}{\partial q_k} = \frac{\partial L}{\partial x} = -kx \\ \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = m\ddot{x} \end{cases}$

$Q_k = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial \dot{q}_k} = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial q_k} = 0$

$\Rightarrow m\ddot{x} + kx = 0$

EXAMPLE 2:



$\begin{cases} T = \frac{1}{2}m\dot{x}^2 \\ V = \frac{1}{2}kx^2 \\ D = \frac{1}{2}c\dot{x}^2 \end{cases}$

$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$

$\begin{cases} \frac{\partial L}{\partial q_k} = \frac{\partial L}{\partial x} = -kx \\ \frac{\partial L}{\partial \dot{q}_k} = m\dot{x} ; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = m\ddot{x} \end{cases}$

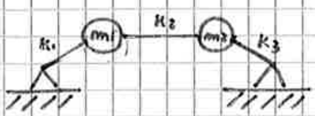
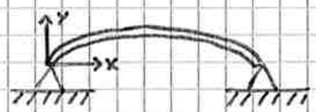
(LOOK AT PAGE L28)

$Q_k = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial \dot{q}_k} = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial q_k} = F \cdot \overbrace{1}^{\frac{\partial (x_1)}{\partial x}} - \frac{\partial D}{\partial \dot{q}_k} =$

$= F - c\dot{x}$

$\Rightarrow m\ddot{x} + kx = F - c\dot{x} \Leftrightarrow m\ddot{x} + c\dot{x} + kx = F$

3. DISTRIBUTED PARAMETER SYSTEMS

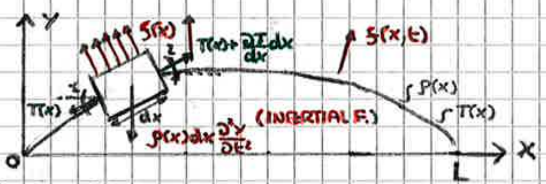
DISCRETE MODEL	CONTINUOUS MODEL (←)
	
• m ODEs	• 1 PDE
• m DOFs, ω_m, φ_m, FINITE m OF MODES	• ∞ DOFs, ω_i, ∞ MODES
• I.C.	• IC + BC (BOUNDARY VALUE PROBLEM : PDE + IC + BC)
• DISPLACEMENT = f(TIME) : $x = x(t)$	• DISPLACEMENTS = f(SPACE, TIME) : $y = y(x,t)$
• EIGENVECTORS	• EIGENFUNCTIONS

DOMAIN
$$\begin{cases} \text{SPATIAL} : 0 \leq x \leq L \\ \text{TIME} : 0 \leq t \leq +\infty \end{cases}$$

WE EXAMINATE ONLY 1D-PROBLEMS:

3.1 STRING TRANSVERSAL VIBRATIONS	II ORDER PDE	y	ρ	T (TENSION)
3.2 ROD AXIAL VIBRATIONS	II ORDER PDE	u	μ	$N = EA\dfrac{\partial y}{\partial x}$
3.3 SHAFT TORSIONAL VIBRATIONS	II ORDER PDE	θ	I	$M = GJ\dfrac{\partial \theta}{\partial x}$
3.4 BEAM FLEXURAL VIBRATIONS	IV ORDER PDE	y, θ	μ	$\begin{cases} M = EJ\dfrac{\partial \theta}{\partial x} = EJ\dfrac{\partial^2 y}{\partial x^2} \\ Q = -\dfrac{\partial M}{\partial x} = -EJ\dfrac{\partial^3 y}{\partial x^3} \end{cases}$

3.1 STRING TRANSVERSAL VIBRATIONS



$$\begin{cases} \rho(x)\ [\text{Kg/m}] : \text{DISTRIBUTED MASS PER UNIT LENGHT} \\ T(x)\ [\text{N}] : \text{TENSION} \\ f(x,t)\ [\text{N/m}] : \text{FORCE PER UNIT LENGTH} \end{cases}$$

hp: SMALL OSCILLATIONS

$$(\uparrow)\quad T_2\theta_2 - T_1\theta_1 + f(x,t)dx - \rho(x)dx\frac{\partial^2 y}{\partial t^2} = 0$$

$$\begin{cases} T_1 = T(x),\ \theta_1 = \dfrac{\partial y}{\partial x} \\ T_2 = T(x) + \dfrac{\partial T}{\partial x}dx,\ \theta_2 = \dfrac{\partial y}{\partial x} + \dfrac{\partial^2 y}{\partial x^2}dx \end{cases}$$

$$\left[T(x) + \frac{\partial T}{\partial x}dx\right]\left[\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}dx\right] - \left[T(x)\right]\left[\frac{\partial y}{\partial x}\right] + f(x)dx - \rho(x)dx\frac{\partial^2 y}{\partial t^2} = 0$$

NEGLECTING HOT (HIGH ORDER TERMS) (d^2x):

$$T\frac{\partial^2 y}{\partial x^2} + \frac{\partial T}{\partial x}\frac{\partial y}{\partial x} + f(x,t) = \rho\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left[T\frac{\partial y}{\partial x}\right] + f(x,t) = \rho\frac{\partial^2 y}{\partial t^2} \quad : \text{PDE OF THE STRING}$$

IF $T(x) = \text{CONST}$ ($\partial T/\partial x = 0$), NO EXTERNAL FORCES ($f(x,t) = 0$):

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \quad : \text{WAVE EQUATION}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial x^2}, \quad c : \text{SOUND/PERTURBATION VELOCITY [m/s]}$$

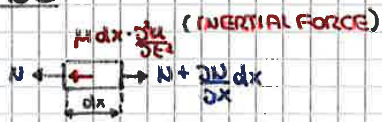
3.2 ROD AXIAL VIBRATIONS

CONVENTIONS:

$$\begin{cases} \text{NORMAL FORCE: } N>0 \Rightarrow \text{TRACTION} \\ \text{SHEAR: } T>0 \Rightarrow \text{CLOCKWISE} \\ \text{BENDING MOM.: } M>0 \Rightarrow \text{TRACTION OF LOWER FIBERS} \end{cases}$$

hps:
- LINEAR ELASTIC MATERIAL
- ISOTROPIC MATERIAL
- NEGLECTED TRANSVERSE DEFORMATIONS

FBD:



$\Rightarrow N + \frac{\partial N}{\partial x}dx - N - \mu dx \frac{\partial^2 u}{\partial t^2} = 0$, μ [kg/m] : MASS PER UNIT LENGHT

$$\frac{\partial N}{\partial x} = \mu \frac{\partial^2 u}{\partial t^2}$$

FROM BEAM THEORY: $\sigma = N/A = E\varepsilon = E\,\partial u/\partial x \Rightarrow N = \sigma A = E\varepsilon A = AE\,\partial u/\partial x$

$$\frac{\partial}{\partial x}\left[AE\frac{\partial u}{\partial x}\right] = \mu\frac{\partial^2 u}{\partial t^2}$$: PDE OF THE ROD

IF AE = CONST:

$$AE\frac{\partial^2 u}{\partial x^2} = \mu\frac{\partial^2 u}{\partial t^2}$$: WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = \frac{AE}{\mu}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

- BOUNDARY CONDITIONS (BC)

$$\begin{cases} u(x=0,t)=0 \\ N(x=L,t)=0 \leftrightarrow \left(N = AE\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0\right) \end{cases}$$

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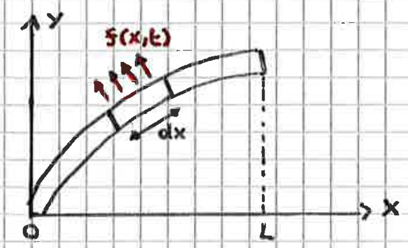
3.4 BEAM FLEXURAL VIBRATIONS

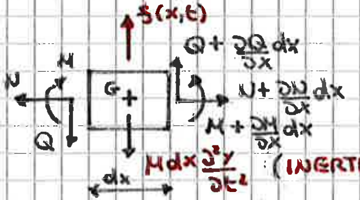
- IV ORDER PDE
- EACH POINT: 2 DOFs: $y, \frac{\partial y}{\partial x}$ (DISPLACEMENT, ROTATION)
- 4 B.C. (2 FOR EACH END)

- EULER-BERNOULLI MODEL → M
- TIMOSHENKO MODEL → M, T (MORE FLEXIBLE)



μ [Kg/m]: MASS PER UNIT LENGTH
M [Nm]: BENDING MOMENT
Q [N]: SHEAR FORCE

FBD:



$$\uparrow)\ \cancel{Q} + \frac{\partial Q}{\partial x}dx - \cancel{Q} + f(x,t)dx - \mu\, dx \frac{\partial^2 y}{\partial t^2} = 0 \qquad (0 \leq x \leq L)$$

$$G)\ \cancel{M} + \frac{\partial M}{\partial x}dx - \cancel{M} + \left(Q + \frac{\partial Q}{\partial x}dx\right)\frac{dx}{2} + Q\frac{dx}{2} = 0$$

NEGLECTING HOT (dx^2):

$$\uparrow)\ \frac{\partial Q}{\partial x} + f(x,t) - \mu\frac{\partial^2 y}{\partial t^2} = 0 \quad \Rightarrow \quad \frac{\partial Q}{\partial x} + f(x,t) = \mu\frac{\partial^2 y}{\partial t^2}$$

$$G)\ \frac{\partial M}{\partial x} + Q = 0 \quad \Rightarrow \quad Q = -\frac{\partial M}{\partial x}$$

THEREFORE:

$$-\frac{\partial^2 M}{\partial x^2} + f(x,t) = \mu\frac{\partial^2 y}{\partial t^2}$$

FROM THE BEAM THEORY: $M = EJ\,\partial\theta/\partial x = EJ\,\partial^2 y/\partial x^2$

$$-\frac{\partial^2}{\partial x^2}\left[EJ\frac{\partial^2 y}{\partial x^2}\right] + f(x,y) = \mu\frac{\partial^2 y}{\partial t^2} \ : \ \text{PDE OF THE BEAM}, \quad Q = -\frac{\partial M}{\partial x} = -EJ\frac{\partial^3 y}{\partial x^3}$$

IF EJ = CONST, NO EXT. FORCES:

$$-EJ\frac{\partial^4 y}{\partial x^4} = \mu\frac{\partial^2 y}{\partial t^2} \ : \ \text{WAVE EQUATION}$$

$$\left(\frac{\partial^2 y}{\partial t^2} = -\frac{EJ}{\mu}\frac{\partial^4 y}{\partial x^4} = -c^2\frac{\partial^4 y}{\partial x^4}\right)$$

$$EJ\frac{\partial^4 y}{\partial x^4} + \mu\frac{\partial^2 y}{\partial t^2} = 0 \ : \ \text{EULER-BERNOULLI BEAM EQ.}$$

- BOUNDARY CONDITIONS (B.C.):

1) FREE END

$$\begin{cases} M(P,t) = EJ\left.\frac{\partial^2 y}{\partial x^2}\right|_{P,t} = 0 \\ Q(P,t) = -EJ\left.\frac{\partial^3 y}{\partial x^3}\right|_{P,t} = 0 \end{cases}$$

IF (m) (LEFT) $\downarrow$ $\downarrow m\ddot{y}$ $\uparrow y, \dot{y}, \ddot{y}$

$$\begin{cases} M(P,t) = EJ\left.\frac{\partial^2 y}{\partial x^2}\right|_{P,t} = 0 \\ Q(P,t) = -EJ\left.\frac{\partial^3 y}{\partial x^3}\right|_{P,t} = -m\left.\frac{\partial^2 y}{\partial t^2}\right|_{x} \end{cases}$$

IF k (LEFT) $\downarrow$ $\downarrow ky$ $\uparrow y, \dot{y}, \ddot{y}$

$$\begin{cases} M(P,t) = EJ\left.\frac{\partial^2 y}{\partial x^2}\right|_{P,t} = 0 \\ Q(P,t) = -EJ\left.\frac{\partial^3 y}{\partial x^3}\right|_{P,t} = -ky(L,t) \end{cases}$$

2) PINNED END

(P = 0; P = L; ...)

$$\begin{cases} y(P,t) = 0 \\ M(P,t) = EJ\left.\frac{\partial^2 y}{\partial x^2}\right|_{P,t} = 0 \end{cases}$$

3) CLAMPED END

$$\begin{cases} y(P,t) = 0 \\ \partial y/\partial x\,|_{P,t} = 0 \end{cases}$$

SOLUTIONS OF WAVE EQS. (TRANSVERSAL/AXIAL/TORSIONAL OSCILLATIONS) II PDEs

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial x^2} \; ; \; \frac{\partial^2 u}{\partial t^2} = \frac{EA}{\mu}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2} \; ; \; \frac{\partial^2 \theta}{\partial t^2} = \frac{GJ}{I}\frac{\partial^2 \theta}{\partial x^2} = c^2\frac{\partial^2 \theta}{\partial x^2}$$

SOLUTION FORM:

$y(x,t) = \phi(x)\eta(t)$, $\phi(x), \eta(t)$ EIGENFUNCTIONS (SPACE, TIME)

$$\phi(x)\frac{\partial^2 \eta}{\partial t^2} = c^2\eta(t)\frac{\partial^2 \phi}{\partial x^2}$$

$$\phi\ddot{\eta} = c^2\eta\phi^{II} \leftrightarrow \phi^{II} - \frac{\ddot{\eta}}{\eta}\frac{1}{c^2}\phi = 0 \; , \; \frac{\ddot{\eta}}{\eta} = -\omega^2$$

$$\phi^{II} + \frac{\omega^2}{c^2}\phi = 0$$

$$\phi^{II} + \beta^2\phi = 0 \; , \; \beta = \frac{\omega}{c} \; , \; c_y = \sqrt{\frac{T}{\rho}} \; ; \; c_u = \sqrt{\frac{AE}{\mu}} \; ; \; c_\theta = \sqrt{\frac{GJ}{I}}$$

$\Rightarrow$ SYSTEM OF 2ODEs:

$$\begin{cases} \ddot{\eta} + \omega^2\eta = 0 \\ \phi^{II} + \beta^2\phi = 0 \end{cases}$$: HARMONIC FUNCTIONS

$$\begin{cases} \eta(t) = A\cos(\omega t) + B\sin(\omega t) \\ \phi(x) = C\cos(\beta x) + D\sin(\beta x) \end{cases}$$: EIGENFUNCTIONS $\begin{cases} A, B & \text{DEPENDING ON I.C.} \\ C, D & \text{DEPENDING ON B.C.} \end{cases}$

SOLUTION:

$$y(x,t) = \phi(x)\eta(t) = \sum_{R=1}^{\infty}\left[A_R\cos(\omega_R t) + B_R\sin(\omega_R t)\right]\left[C_R\cos(\beta_R x) + D_R\sin(\beta_R x)\right]$$

SOLUTION OF WAVE EQ. (FLEXURAL OSCILLATIONS) IV ORDER PDE

$$\frac{\partial^2 y}{\partial t^2} = -\frac{EJ}{\mu}\frac{\partial^4 y}{\partial x^4} = -c^2\frac{\partial^4 y}{\partial x^4}$$

SOLUTION FORM:

$y(x,t) = \phi(x)\eta(t)$

$$\phi(x)\frac{\partial^2 \eta}{\partial t^2} = -c^2\eta(t)\frac{\partial^4 \phi}{\partial x^4}$$

$$\phi\ddot{\eta} = -c^2\eta\phi^{IV} \leftrightarrow \phi^{IV} + \frac{\ddot{\eta}}{\eta}\frac{1}{c^2}\phi = 0$$

$$\phi^{IV} - \frac{\omega^2}{c^2}\phi = 0$$

$$\phi^{IV} - \beta^4\phi = 0 \; , \; \beta^2 = \frac{\omega}{c} \; , \; c = \sqrt{\frac{EJ}{\mu}}$$

$\Rightarrow$ SYSTEM OF 2ODEs:

$$\begin{cases} \ddot{\eta} + \omega^2\eta = 0 \\ \phi^{IV} - \beta^4\phi = 0 \end{cases}$$

$$\begin{cases} \eta(t) = A\cos(\omega t) + B\sin(\omega t) \\ \phi(x) = C\cos(\beta x) + D\sin(\beta x) + E\cosh(\beta x) + F\sinh(\beta x) \end{cases}$$

EXAMPLE: BEAM (PINNED-PINNED) FLEXURAL OSCILLATIONS

$$\frac{\partial^2 y}{\partial t^2} = -\frac{EJ}{M}\frac{\partial^4 y}{\partial x^4} = -c^2\frac{\partial^4 y}{\partial x^4}$$

$$y(t) = \phi(x)\eta(t)$$

$$\phi(x)\frac{\partial^2\eta}{\partial t^2} = -c^2\eta(t)\frac{\partial^4\phi}{\partial x^4}$$

$$\phi\ddot{\eta} = -c^2\eta\phi^{IV} \iff -\phi^{IV} - \frac{\ddot{\eta}}{\eta}\frac{1}{c^2}\phi = 0$$

$$\phi^{IV} - \frac{\omega^2}{c^2}\phi = 0$$

$$\phi^{IV} - \beta^4\phi = 0, \quad \beta^2 = \frac{\omega}{c}, \quad c = \sqrt{\frac{EJ}{M}}$$

$$\begin{cases} \ddot{\eta} + \omega^2\eta = 0 \\ \phi^{IV} + \beta^4\phi = 0 \end{cases}$$

$$\begin{cases} \eta(t) = A\cos(\omega t) + B\,\mathrm{sim}(\omega t) \\ \phi(x) = C\cos(\beta x) + D\,\mathrm{sim}(\beta x) + E\cosh(\beta x) + F\,\mathrm{simh}(\beta x) \end{cases}$$

IMPOSING THE B.C. (PINNED-PINNED): 

$$L\begin{cases} y(x=0,t) = 0 = \phi(0)\eta(t) \\ M(x=0,t) = EJ\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0} = 0 \end{cases} \Rightarrow \begin{cases} \phi(0) = 0 \\ \phi''(0) = 0 \end{cases}$$

$$R\begin{cases} y(x=L,t) = 0 = \phi(L)\eta(t) \\ M(x=L,t) = EJ\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=L} = 0 \end{cases} \Rightarrow \begin{cases} \phi(L) = 0 \\ \phi''(L) = 0 \end{cases}$$

$$\phi'(x) = \beta[-C\,\mathrm{sim}(\beta x) + D\cos(\beta x) + E\,\mathrm{simh}(\beta x) + F\cosh(\beta x)]$$

$$\phi''(x) = \beta^2[-C\cos(\beta x) - D\,\mathrm{sim}(\beta x) + E\cosh(\beta x) + F\,\mathrm{simh}(\beta x)]$$

$$\phi'''(x) = \beta^3[+C\,\mathrm{sim}(\beta x) - D\cos(\beta x) + E\,\mathrm{simh}(\beta x) + F\cosh(\beta x)]$$

$$L\begin{cases} \phi(0) = 0 = C + E \\ \phi''(0) = 0 = \beta^2[-C + E] \end{cases} \Rightarrow \begin{cases} C = -E \\ C = E \end{cases} \Rightarrow C = E = 0$$

$$\phi(x) = D\,\mathrm{sim}(\beta x) + F\,\mathrm{simh}(\beta x)$$

$$R\begin{cases} \phi(L) = 0 = D\,\mathrm{sim}(\beta L) + F\,\mathrm{simh}(\beta L) \\ \phi''(L) = 0 = \beta^2[-D\,\mathrm{sim}(\beta L) + F\,\mathrm{simh}(\beta L)] \end{cases} \iff \begin{bmatrix} \mathrm{sim}(\beta L) & \mathrm{simh}(\beta L) \\ -\beta^2\mathrm{sim}(\beta L) & \beta^2\mathrm{simh}(\beta L) \end{bmatrix}\begin{Bmatrix} D \\ F \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\begin{Bmatrix} D \\ F \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ TRIVIAL SOLUTION; $\det[\ \ ] = \mathrm{sim}(\beta L)\beta^2\mathrm{simh}(\beta L) + \beta^2\mathrm{sim}(\beta L)\mathrm{simh}(\beta L) = 0$

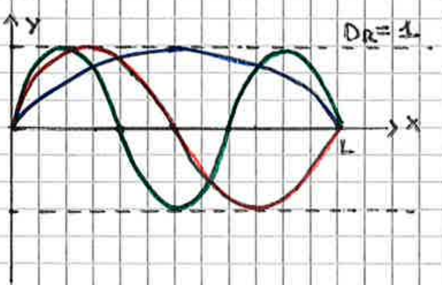
$2\beta^2\mathrm{sim}(\beta L)\mathrm{simh}(\beta L) = 0$ CHARACTERISTIC EQ.

$\beta \neq 0, \mathrm{simh}(\beta L) \neq 0 \Rightarrow \mathrm{sim}(\beta L) = 0 \iff \beta L = \pi$

$\beta_R L = R\cdot\pi \iff \sqrt{\frac{\omega_R}{c}}L = R\cdot\pi \rightarrow \omega_R = \frac{R^2\cdot\pi^2}{L^2}c = R^2\pi^2\sqrt{\frac{EJ}{ML^4}}$: NATURAL FREQUENCIES

$$\begin{cases} \omega_1 = \pi^2\sqrt{\frac{EJ}{ML^4}} : \text{FUNDAMENTAL FREQ.} \\ \omega_2 < \omega_3 < \ldots < \omega_m : \text{OVERTONES} \end{cases}$$

$\phi_R(x) = D_R\,\mathrm{sim}(\beta_R x) + F\,\mathrm{simh}(\beta_R x) = D_R\,\mathrm{sim}\left(\frac{R\pi}{L}x\right) + F\,\mathrm{simh}\left(\frac{R\pi}{L}x\right)$; hp$_s$: $F = 0$, $D_R = 1\ \forall R$



$R = 1$, ϕ_1, $\omega_1 = \pi^2\sqrt{\frac{EJ}{ML^4}}$

$R = 2$, ϕ_2, $\omega_2 = 4\pi^2\sqrt{\frac{EJ}{ML^4}}$

$R = 3$, ϕ_3, $\omega_3 = 9\pi^2\sqrt{\frac{EJ}{ML^4}}$

→ UNIFIED APPROACH

- CASE $f(x,t)=0$

$$\mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + \mathcal{K}[y] = 0$$

$$B_i[y] = 0 \quad (\text{B.C.})$$

3.1 STRING TRANSVERSAL VIBRATIONS

$$\rho\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + \mathcal{K}[y] = 0 \quad \begin{cases} y = y \\ \mathcal{M} = \rho \end{cases} \quad \mathcal{K} = -T\frac{\partial^2}{\partial x^2}(\cdot)$$

3.2 ROD AXIAL VIBRATIONS

$$\mu\frac{\partial^2 u}{\partial t^2} - AE\frac{\partial^2 u}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + \mathcal{K}[y] = 0 \quad \begin{cases} y = u \\ \mathcal{M} = \mu \end{cases} \quad \mathcal{K} = -AE\frac{\partial^2}{\partial x^2}(\cdot)$$

3.3 SHAFT TORSIONAL VIBRATIONS

$$I\frac{\partial^2 \theta}{\partial t^2} - GJ\frac{\partial^2 \theta}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + \mathcal{K}[y] = 0 \quad \begin{cases} y = \theta \\ \mathcal{M} = I \end{cases} \quad \mathcal{K} = -GJ\frac{\partial^2}{\partial x^2}(\cdot)$$

3.4 BEAM FLEXURAL VIBRATIONS

$$\mu\frac{\partial^2 y}{\partial t^2} + EJ\frac{\partial^4 y}{\partial x^4} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + \mathcal{K}[y] = 0 \quad \begin{cases} y = y \\ \mathcal{M} = \mu \end{cases} \quad \mathcal{K} = +EJ\frac{\partial^4}{\partial x^4}(\cdot)$$

DIFFERENTIAL EIGENPROBLEM

$$y(x,t) = \phi(x)\eta(t)$$

$$\mathcal{M}[\phi]\ddot{\eta} + \mathcal{K}[\phi]\eta = 0$$

$$\frac{\ddot{\eta}}{\eta} = -\frac{\mathcal{K}[\phi]}{\mathcal{M}[\phi]} = -\omega^2 \leq 0$$

$$\mathcal{K}[\phi] = \omega^2\mathcal{M}[\phi]$$

SELF-ADJOINT PROPERTY → $\mathcal{M}$-$\mathcal{K}$ ORTHOGONALITY

- GENERIC OPERATOR (L), TWO FUNCTIONS (u,v) THAT SATISFY THE B.C. ON THE DOMAIN (D).

$(u, L[v]) = \int_D u\,L[v]\,dD$; L IS SELF-ADJOINT IF $(u, L[v]) = (v, L[u])$

* WE TAKE TWO EIGENFUNCTIONS (ϕ_i, ϕ_j) AND WE DEMON. THAT $\mathcal{M}, \mathcal{K}$ ARE SELF-ADJOINT

- TWO EIGENFUNCTIONS $(\phi_i; \phi_j) \longleftrightarrow$ TWO EIGENVALUES $(\omega_i; \omega_j)$

$$\begin{cases} \mathcal{K}[\phi_i] = \omega_i^2\mathcal{M}[\phi_i] \\ \mathcal{K}[\phi_j] = \omega_j^2\mathcal{M}[\phi_j] \end{cases}$$

$$\int_D \phi_j\,\mathcal{K}[\phi_i]\,dD = \int_D \omega_i^2\mathcal{M}[\phi_i]\,dD$$

$$\int_D \phi_i\,\mathcal{K}[\phi_j]\,dD = \int_D \omega_j^2\mathcal{M}[\phi_j]\,dD$$

$$(\omega_i^2 - \omega_j^2)\int_D \phi_i\,\mathcal{M}[\phi_j]\,dD = 0$$

$$\begin{cases} \mathcal{M}[\phi_i] = \frac{1}{\omega_i^2}\mathcal{K}[\phi_i] \\ \mathcal{M}[\phi_j] = \frac{1}{\omega_j^2}\mathcal{K}[\phi_j] \end{cases}$$

$$\int_D \phi_j\,\mathcal{M}[\phi_i]\,dD = \int_D \frac{1}{\omega_i^2}\mathcal{K}[\phi_i]\,dD$$

$$\int_D \phi_i\,\mathcal{M}[\phi_j]\,dD = \int_D \frac{1}{\omega_j}\mathcal{K}[\phi_j]\,dD$$

$$\left(\frac{1}{\omega_i^2} - \frac{1}{\omega_j^2}\right)\int_D \phi_i\,\mathcal{K}[\phi_j]\,dD = 0$$

$$\begin{cases} \omega_i \neq \omega_j \Rightarrow \int_D \phi_i\,\mathcal{M}[\phi_j]\,dD = 0 \quad : \text{ $\mathcal{M}$-ORTHOGONALITY} \\ \omega_i = \omega_j \Rightarrow \int_D \phi_i\,\mathcal{M}[\phi_i]\,dD = m_i \quad : \text{ MODAL MASS} \end{cases}$$

$$\begin{cases} \Rightarrow \int_D \phi_i\,\mathcal{K}[\phi_j]\,dD = 0 \quad : \text{ $\mathcal{K}$-ORTHOGONALITY} \\ \Rightarrow \int_D \phi_i\,\mathcal{K}[\phi_i]\,dD = k_i \quad : \text{ MODAL STIFFNESS} \end{cases}$$

CONSIDERATION (MDOF):

IF $[M] = \alpha[I] \rightarrow [\varphi]^T[M][\varphi] \longleftrightarrow [\varphi]^T[\varphi] \rightarrow \{\varphi\}^T[M]\{\varphi\} \longleftrightarrow \{\varphi\}^T\{\varphi\}$

$$\begin{cases} R \neq S \Rightarrow \{\varphi\}_S^T\{\varphi\}_R = 0 \\ R = S \Rightarrow \{\varphi_R\}^T\{\varphi_R\} = 1 \end{cases}$$

6. ROTORDYNAMICS

INTRODUCTION

CAMPBELL DIAGRAM

TYPICAL CAUSES OF INSTABILITY

JEFFCOTT MODEL (FLEXIBLE SHAFT ON RIGID BEARINGS)

RIGID SHAFT ON FLEXIBLE BEARINGS

DYNAMIC UMBALANCE

STATIC + DYNAMIC UMBALANCE

INTRODUCTION

ROTOR: BODY SUSPENDED THROUGH BEARINGS WHICH ALLOW IT TO ROTATE FREELY AROUND AN AXIS. BEHAVIOUR'S
- **SPIN** (ABOUT ITS OWN AXIS)
- **WHIRL** (PRECESSION SPEED OF DEF. AXIS LINE

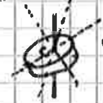
G = GRAVITY CENTER

C = GEOMETRICAL CENTER (**ORBIT** δ)

ω_{CR} : IT CAUSES THE MAXIMUM DISPLACEMENT OF THE ORBIT ($\delta = \delta_{MAX}$)

- RESONANCE ⇒ "**SLOW EFFECT**" : GROWTH OF THE ORBIT IS LINEAR
- IT'S DUE TO UNBALANCE : **STATIC** $\varepsilon = \overline{CG}$

; **DYNAMIC** $\varepsilon = 0$



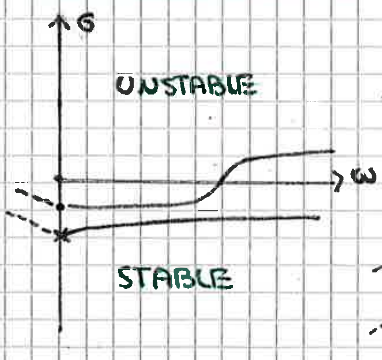
C≡G

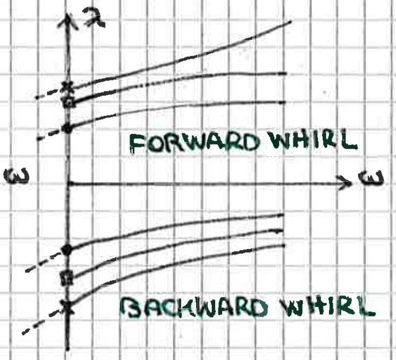
- INTERSECTION OF $\lambda = \omega$

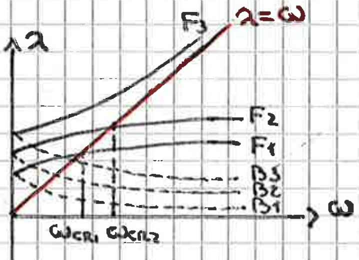
$\delta = \delta(t) = \delta_0 e^{st}$, $s = \sigma + i\lambda$, $\sigma = Re(s)$, $\lambda = Im(s)$

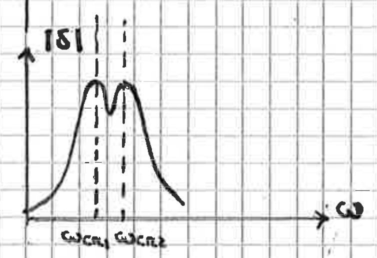
- λ : **WHIRL FREQUENCY**
- σ : **DECAY RATE**
  - $\sigma \leq 0$: **STABILITY**
  - $\sigma > 0$: **INSTABILITY** ⇒
    - DAMPING: INTERNAL / EXTERNAL
    - "**FAST EFFECT**" : SUDDEN GROWTH OF THE ORBIT : EXPONENTIAL

CAMPBELL DIAGRAMS









FROM GEOMETRY:

$$\begin{cases} x_G = x_C + \varepsilon\cos\psi \\ y_G = y_C + \varepsilon\sin\psi \end{cases} \Rightarrow \begin{cases} \dot{x}_G = \dot{x}_C - \varepsilon\dot{\psi}\sin\psi \\ \dot{y}_G = \dot{y}_C + \varepsilon\dot{\psi}\cos\psi \end{cases} \Rightarrow \begin{cases} \ddot{x}_G = \ddot{x}_C - \varepsilon\ddot{\psi}\sin\psi - \varepsilon\dot{\psi}^2\cos\psi \\ \ddot{y}_G = \ddot{y}_C + \varepsilon\ddot{\psi}\cos\psi - \varepsilon\dot{\psi}^2\sin\psi \end{cases}$$

THEREFORE:

$$\begin{cases} m\ddot{x}_C + C_S\dot{x}_C + K_S x_C = m\varepsilon(\ddot{\psi}\sin\psi + \dot{\psi}^2\cos\psi) \\ m\ddot{y}_C + C_S\dot{y}_C + K_S y_C = m\varepsilon(-\ddot{\psi}\cos\psi + \dot{\psi}^2\sin\psi) - mg \\ I_G\ddot{\psi} + (C_S\dot{x}_C + K_S x_C)\varepsilon\sin\psi - (C_S\dot{y}_C + K_S y_C)\varepsilon\cos\psi = M \end{cases}$$

CONSIDERING:

- STEADY-STATE MOTION $\begin{cases} \ddot{\psi} = 0 \\ \omega = \dot{\psi} = \text{CONST} \end{cases}$
- VERTICAL ROTOR ($mg = 0$)
- COMPLEX VARIABLE: $z_C = x_C + i y_C$

$$m(\ddot{x}_C + i\ddot{y}_C) + C_S(\dot{x}_C + i\dot{y}_C) + K_S(x_C + i y_C) = m\varepsilon\omega^2(\cos\psi + i\sin\psi) = m\varepsilon\omega^2 e^{i\omega t}$$

$$m\ddot{z}_C + C_S\dot{z}_C + K_S x_C = m\varepsilon\omega^2 e^{i\omega t}$$

$$z_C = z_0 e^{i\omega t}, \quad \dot{z}_C = i\omega z_0 e^{i\omega t}, \quad \ddot{z}_C = -\omega^2 e^{i\omega t}$$

$$\omega_{CR}^2\left[1 - \left(\frac{\omega}{\omega_{CR}}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_{CR}}\right)\right] z_0 \cancel{e^{i\omega t}} = \varepsilon\omega^2 \cancel{e^{i\omega t}}, \quad \omega_{CR} = \sqrt{\frac{K_S}{m}} \text{ : CRITICAL SPEED}$$

FREQUENCY RESPONSE FUNCTION (FRF):

$$\frac{z_0}{\varepsilon} = \left(\frac{\omega}{\omega_{CR}}\right)^2 \frac{1}{\left[1 - \left(\frac{\omega}{\omega_{CR}}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_{CR}}\right)\right]}$$

$$|\cdot| = \frac{|z_0|}{\varepsilon} = \frac{|\delta|}{\varepsilon} = \left(\frac{\omega}{\omega_{CR}}\right)^2 \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_{CR}}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_{CR}}\right)\right]^2}}$$

$$\angle = Tg_0^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left(\frac{2\zeta(\omega/\omega_{CR})}{1 - (\omega/\omega_{CR})^2}\right)$$

GIVEN A ROTOR WITH:

$$\omega = \text{CONST} \begin{cases} \delta = \text{CONST} \Rightarrow \text{CIRCLE} \\ \beta = \text{CONST} \Rightarrow \dot{\beta} = 0 \end{cases} \Rightarrow \omega = \dot{\psi} = \dot{\varphi}\ ; \quad \ddot{\psi} = \ddot{\varphi} = 0$$

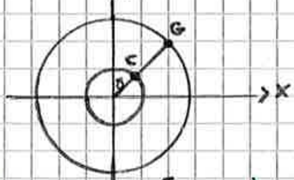
SYNCHRONOUS (FORWARD) WHIRL MOTION

MAXIMUM AMPLITUDE:

$$\omega_{MAX} = \omega_{CR}\frac{1}{\sqrt{1 - 2\zeta^2}}$$

1) $\boxed{\omega \ll \omega_{CR}}$

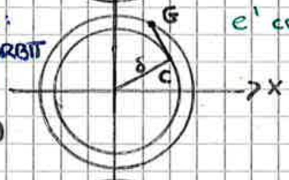
$$\begin{cases} \beta \cong 0 \\ R = \delta + \varepsilon, \ \delta(\text{SMALL}) \end{cases}$$



2) $\boxed{\omega \cong \omega_{CR}}$ RESONANCE: GROWTH OF ORBIT

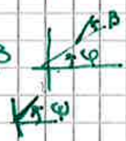
$$\begin{cases} \beta = -\pi/2 \\ R = \sqrt{\delta^2 + \varepsilon^2}, \ \delta(\text{LARGE}) \end{cases}$$

secondo me è + ...



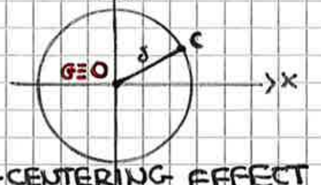
e' come se desse la spinta (G)

$\psi = \varphi + \beta$



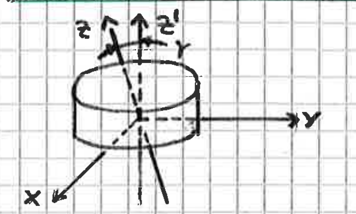
3) $\boxed{\omega \gg \omega_{CR}}$

$$\begin{cases} \beta = -\pi \\ R = \delta = \varepsilon \end{cases}$$



SUPERCRITICAL → SELF-CENTERING EFFECT

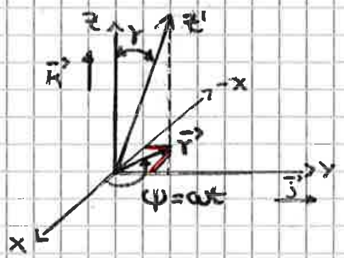
DYNAMIC UMBALANCE



- z' : PRINCIPAL AXIS OF INERTIA
- γ : DYNAMIC UMBALANCE ($\gamma = CONST$)
- $\varepsilon = 0$ ($C \equiv G$)

$\gamma \Rightarrow$ GYROSCOPIC MOMENT (M_g) $\Rightarrow$ SHAFT ANGULAR DEFLECTIONS ($\theta_{x,y}$) $\Rightarrow$ SHAFT ELASTIC TORQUE (M_e)

EQUILIBRIUM BETWEEN M_g AND M_e.

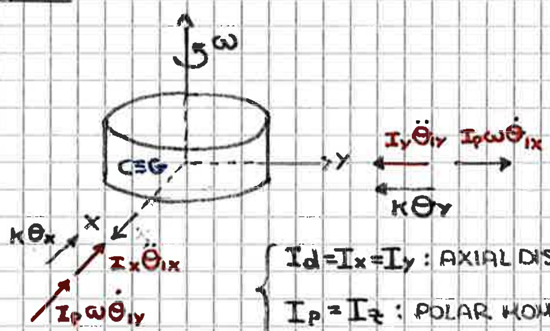


- $\vec{\omega} = \omega \vec{k}$
- $\vec{\gamma} \perp zz'$ PLANE, PROJECTED ON PLANE XY
- $\begin{cases} \gamma_x = -\gamma \sin(\omega t) \\ \gamma_y = +\gamma \cos(\omega t) \end{cases}$

SHAFT UNDERGOES A TOTAL ANGULAR DEFORMATION DUE TO γ AND ELASTIC ROT θ

$$\begin{cases} \theta_{1x} = \theta_x + \gamma_x = \theta_x - \gamma \sin(\omega t) \\ \theta_{1y} = \theta_y + \gamma_y = \theta_y + \gamma \cos(\omega t) \end{cases} \quad \begin{cases} \dot\theta_{1x} = \dot\theta_x - \gamma\omega \cos(\omega t) \\ \dot\theta_{1y} = \dot\theta_y - \gamma\omega \sin(\omega t) \end{cases} \quad \begin{cases} \ddot\theta_{1x} = \ddot\theta_x + \gamma\omega^2 \sin(\omega t) \\ \ddot\theta_{1y} = \ddot\theta_y - \gamma\omega^2 \cos(\omega t) \end{cases}$$

FBD:



$\begin{cases} I_d = I_x = I_y : \text{AXIAL DISK } I \\ I_p = I_z : \text{POLAR MOM. OF INERTIA} \end{cases}$

- INERTIAL TORQUES (M'): $-I_x\ddot\theta_{1x}$; $-I_y\ddot\theta_{1y}$
- GYROSCOPIC TORQUES (M_g): $-I_p\omega\dot\theta_{1y}$; $+I_p\omega\dot\theta_{1x}$
- ELASTIC TORQUES (M_e): $-k\theta_x$; $-k\theta_y$

$$\begin{cases} I_x\ddot\theta_{1x} + I_p\omega\dot\theta_{1y} + k\theta_x = 0 \\ I_y\ddot\theta_{1y} - I_p\omega\dot\theta_{1x} + k\theta_y = 0 \end{cases}$$

$$\begin{cases} I_d\ddot\theta_x + I_p\omega\dot\theta_y + k\theta_x = -I_d\gamma\omega^2\sin(\omega t) + I_p\gamma\omega^2\sin(\omega t) = \gamma\omega^2(I_p - I_d)\sin(\omega t) \\ I_d\ddot\theta_y - I_p\omega\dot\theta_x + k\theta_y = +I_d\gamma\omega^2\cos(\omega t) - I_p\gamma\omega^2\cos(\omega t) = -\gamma\omega^2(I_p - I_d)\cos(\omega t) \end{cases}$$

$\theta = \theta_x + i\theta_y$: COMPLEX ANGLE

$I_d\ddot\theta - iI_p\omega\dot\theta + k\theta = i\gamma\omega^2(I_p - I_d)e^{i\omega t}$ $\qquad \omega_{CR} = \sqrt{\dfrac{K}{I_d - I_p}}$ $\qquad$ IF $I_p > I_d \Rightarrow$ NO ω_{CR}

$\theta = \theta_0 e^{st}$; $\dot\theta = s\theta_0 e^{st}$; $\ddot\theta = s^2\theta_0 e^{st}$

$(I_d s^2 - iI_p\omega s + K)\theta_0 \cancel{e^{st}} = 0$: CHARACTERISTIC EQ.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{iI_p\omega \pm \sqrt{-I_p^2\omega^2 - 4I_dK}}{2I_d} = \sigma + i\lambda$$

$\sigma = Re(s) \begin{cases} \sigma < 0 : \text{STABLE} \\ \sigma > 0 : \text{UNSTABLE} \end{cases}$ $\qquad \lambda = Im(s) \begin{cases} \lambda > 0 \quad \text{FORWARD WHIRL} \\ \lambda < 0 \quad \text{BACWARD WHIRL} \end{cases}$

M.
S.
D.

THEORY SUMMARIES

MECHANICAL SYSTEM DYNAMICS

THEORY - SUMMARIES

1.2 FORCED SYSTEM ($f(t) \neq 0$) -> FORCED RESPONSE

1 HARMONIC EXCITATION ($f(t) = F_0 e^{i\omega t} = F_0 \cdot [\cos(\omega t) + i \sin(\omega t)]$)

$\ddot{x} + 2\zeta\omega_m \dot{x} + \omega_m^2 x = \frac{F_0}{m} e^{i\omega t} = \frac{F_0}{K}\omega_m^2 e^{i\omega t}$: CANONICAL FORM OF THE MOTION EQ

$x(t) = x_p(t) = X_0 e^{i\omega t}$: PARTICULAR (INTEGRAL) SOLUTION FORM ; $x(t) = x_0 \cos(\omega t - \varphi)$

$(\dot{x}(t) = i\omega X_0 e^{i\omega t} ; \ddot{x}(t) = -\omega^2 X_0 e^{i\omega t})$ WITH $X_0 = |X_0(i\omega)| e^{+i\varphi} = A \cdot |G(i\omega)| e^{+i\varphi} \in \mathbb{C}$, $x_0 = |X_0(i\omega)| \in \mathbb{R}e$

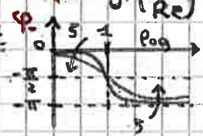
$\cancel{\omega_m^2}\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) + 1\right] X_0 \cancel{e^{i\omega t}} = \frac{F_0}{K}\cancel{\omega_m^2}\cancel{e^{i\omega t}}$; $\left[1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] = Z(i\omega)$: IMPEDENCE

$\Rightarrow Z(i\omega) X_0 = \frac{F_0}{K} \Rightarrow X_0 = A \cdot \frac{1}{Z(i\omega)} = A \cdot G(i\omega) = A \cdot |G(i\omega)| e^{+i\varphi}$

$G(i\omega) = \frac{1}{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right]}$: FREQUENCY RESPONCE FUNCTION (FRF) : $G(i\omega) = |G(i\omega)| e^{+i\varphi}$

$|G(i\omega)| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_m}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2}}$

$\varphi = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_m}\right)}{1 - \left(\frac{\omega}{\omega_m}\right)^2}\right]$



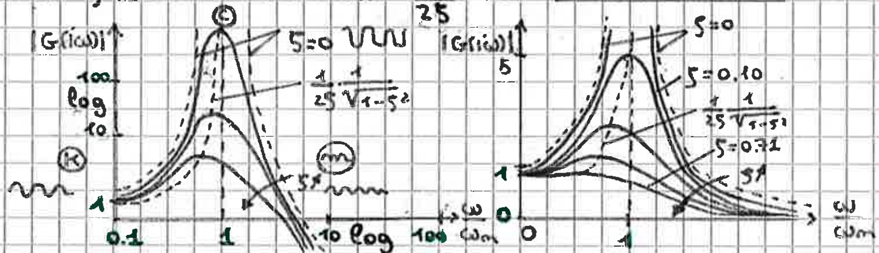
$x(t) = X_0 e^{i\omega t} = |X_0(i\omega)| e^{i(\omega t + \varphi)} = A \cdot |G(i\omega)| e^{i(\omega t + \varphi)}$: HARMONIC RESPONSE

$\frac{d\,|G(i\omega)|}{d(\omega/\omega_m)} = 0$ -> PEAKS ; $N\left(\frac{d\,|G(i\omega)|}{d(\omega/\omega_m)}\right) = 0$ -> ω_{RES} ($<\omega_d < \omega_m$)

$\omega_{RES} = \omega = \omega_m\sqrt{1 - 2\zeta^2}$: RESONANCE f. $\zeta < \frac{1}{\sqrt{2}}$ ($\cong 0.71$) (FOR $\zeta > \frac{1}{\sqrt{2}}$ -> NO PEAKS.)

$|G(i\omega)|_{MAX} = \left|G(i\omega)\left(\frac{\omega}{\omega_m} = \frac{\omega_R}{\omega_m} = \sqrt{1-2\zeta^2}\right)\right| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$: MAXIMUM F.R.F.

FOR $\zeta < 0.05$: $|G(i\omega)|_{MAX} = \frac{1}{2\zeta} = Q$: QUALITY FACTOR



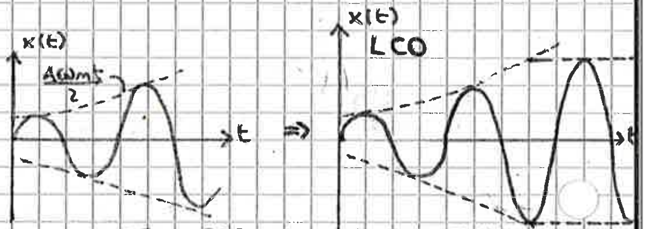
IN CASE OF UNDAMPED SYSTEM ($\zeta = 0$):

$\ddot{x} + \omega_m^2 x = \frac{F_0}{K}\omega_m^2 \cos(\omega t)$: CANONICAL FORM

$x(t) = x_p(t) = X_0 e^{i\omega t} \cdot t$: PARTICULAR SOLUTION

$x(t) = \frac{A\omega_m t}{2} \cdot \sin(\omega_m t)$: OSCILLATORY RESPONSE

$|G(i\omega)| = \frac{1}{1 - \left(\frac{\omega}{\omega_m}\right)^2}$ => FOR $\omega \to \omega_m$: $|G(i\omega)| \to \infty$



2 BASE EXCITATION

$m\ddot{x} + c(\dot{x} - \dot{y}) + K(x - y) = 0$ { ABS: $m\ddot{x} + c\dot{x} + Kx = c\dot{y} + Ky$; REL: $m\ddot{z} + c\dot{z} + Kz = -m\ddot{y}$ ($z = x - y$)

$\ddot{x} + 2\zeta\omega_m \dot{x} + \omega_m^2 x = 2\zeta\omega_m \dot{y} + \omega_m^2 y$: CANONICAL F.

$x(t) = x_p(t) = X_0 e^{i\omega t} \leftrightarrow x(t) = x_0 \cos(\omega t + \varphi_1)$; $X_0 = |X_0(i\omega)| e^{-i\varphi_1} \in \mathbb{C}$, $x_0 \in \mathbb{R}e$

$y(t) = Y_0 e^{i\omega t} \leftrightarrow y(t) = Y_0 \cos(\omega t)$, $Y_0 \in \mathbb{R}e$

$\cancel{\omega_m^2}\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) + 1\right] X_0 \cancel{e^{i\omega t}} = \cancel{\omega_m^2}\left[2i\zeta\left(\frac{\omega}{\omega_m}\right) + 1\right] Y_0 \cancel{e^{i\omega t}}$

(FORCE TRANSM.)

$Tr_{ABS} = \frac{X_0}{Y_0} = \frac{1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} = \left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] \cdot G(i\omega)$: ABSOLUTE TRANSMISS.

($= f_{TR}/f$) -> TO BASE

$|Tr_{ABS}| = \frac{|X_0(i\omega)|}{Y_0} = \sqrt{Tr_{ABS} \cdot \overline{Tr_{ABS}}} = \sqrt{1 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2} \cdot |G(i\omega)|$

$\left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] G(i\omega) \cdot \frac{\overline{Z}(i\omega)}{\overline{Z}(i\omega)} = |G(i\omega)|\left[1 - \left(\frac{\omega}{\omega_m}\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_m}\right)^2 - 2i\zeta\left(\frac{\omega}{\omega_m}\right)^3\right]$

$Tr_{REL} = \frac{Z_{10}}{Y_0} = \frac{X_0 - Y_0}{Y_0} = Tr_{ABS} - 1 = \left(\frac{\omega}{\omega_m}\right)^2 \cdot G(i\omega)$: RELATIVE TRANSMISS , $|Tr_{REL}| = \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)|$

$\Rightarrow \varphi_1 = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left(\frac{+2\zeta\left(\frac{\omega}{\omega_m}\right)^3}{1 - \left(\frac{\omega}{\omega_m}\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_m}\right)^2}\right)$

$x(t) = |X_0(i\omega)| e^{i(\omega t + \varphi_1)} = Y_0 \cdot |Tr_{ABS}| e^{i(\omega t + \varphi_1)} = Y_0 \cdot \sqrt{1 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2} |G(i\omega)| e^{i(\omega t + \varphi_1)}$

$\varphi_2 = Tg^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_m}\right)}{1 - \left(\frac{\omega}{\omega_m}\right)^2}\right]$

OR $\varphi = Tg^{-1}\left(\frac{-|Im|_D}{Re_D}\right) - Tg^{-1}\left(\frac{-|Im|_N}{Re_N}\right) =$

$z(t) = |Z_0(i\omega)| e^{i(\omega t + \varphi_2)} = Y_0 \cdot |Tr_{REL}| e^{i(\omega t + \varphi_2)} = Y_0 \cdot \left(\frac{\omega}{\omega_m}\right)^2 \cdot |G(i\omega)| e^{i(\omega t + \varphi_2)}$

(FASTER WAY TO COMPUTE) $= Tg^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_m}\right)}{1 - \left(\frac{\omega}{\omega_m}\right)^2}\right]_D - Tg^{-1}\left[-2\zeta\left(\frac{\omega}{\omega_m}\right)\right]_N$ VERIFIED ✓

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IN $t=0^+$:

$$\int_{-\varepsilon/2}^{+\varepsilon/2}(m\ddot{x}+c\dot{x}+kx)\,dt=\int_{-\varepsilon/2}^{+\varepsilon/2}F_0\,dt$$

$$m\dot{x}(0^+)+c\,x(0^+)+\lim_{\varepsilon\to 0}k\int_{-\varepsilon/2}^{+\varepsilon/2}x\,dt=\lim_{\varepsilon\to 0}\int_{-\varepsilon/2}^{+\varepsilon/2}F_0\,dt=I=1 \Rightarrow \text{I.C.}\begin{cases}\dot{x}(0^+)=I/m\\ x(0^+)=0\end{cases}$$

$$m\,x(0^+)+\lim_{\varepsilon\to 0}c\int_{-\varepsilon/2}^{\varepsilon/2}\dot{x}\,dt+\lim_{\varepsilon\to 0}k\int_{-\varepsilon/2}^{\varepsilon/2}\int_{-\varepsilon/2}^{t}\dot{x}\,d\tau\,dt=0$$

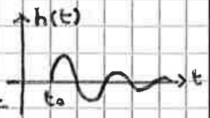
- $x(t)=x_A(t)=A_1e^{s_1t}+A_2e^{s_2t}=e^{-\zeta\omega_n t}\cdot[a\cos(\omega_d t)+b\sin(\omega_d t)]$

$$\dot{x}(t)=\dot{x}_A(t)=s_1A_1e^{s_1t}+s_2A_2e^{s_2t}=-\zeta\omega_n e^{-\zeta\omega_n t}\cdot[a\cos(\omega_d t)+b\sin(\omega_d t)]+e^{-\zeta\omega_n t}\cdot[-\omega_d a\sin(\omega_d t)+\omega_d b\cos(\omega_d t)]$$

$$\begin{cases}x(0^+)=a=0\\ \dot{x}(0^+)=-\zeta\omega_n a+\omega_d b=I/m\end{cases}\Rightarrow\begin{cases}a=0\\ b=I/(\omega_d m)=1/(\omega_d m)\end{cases}\quad \text{I.C. }(0^+)$$

$x(t)=\frac{1}{m\omega_d}\cdot e^{-\zeta\omega_n t}\cdot\sin(\omega_d t)$: RESPONSE

$h(t)=\frac{1}{m\omega_d}\cdot e^{-\zeta\omega_n(t-t_0)}\cdot\sin[\omega_d(t-t_0)]\cdot u(t-t_0)$, $\begin{cases}u(t-t_0)=1, & t>t_0\\ u(t-t_0)=0 & t<t_0\end{cases}$ (GENERAL FORM) : TOTAL RESP. TO THE IMPULSE



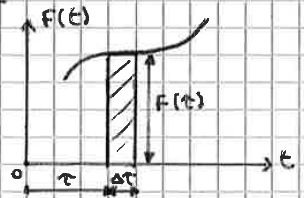
RELATION BETWEEN $s(t)$ + $h(t)$:

$$s(t)=u(t-t_0)=\int_{-\infty}^{t}h(\tau)\,d\tau=\int_{-\infty}^{t_0}\delta(\tau-t_0)\,d\tau$$

$$\delta(t-t_0)=\frac{d}{dt}[u(t-t_0)]$$

ARBITRARY EXCITATION → CONVOLUTION INTEGRAL

$F(t)$: ARBITRARY EXCITATION



$\hat{F}(\tau)=F(\tau)\cdot\Delta\tau$: AREA = IMPULSE MAGNITUDE [N/S]

$\hat{F}(\tau)\,\delta(t-\tau)=F(\tau)\Delta\tau\,\delta(t-\tau)$: IMPULSE FORCE [N]

$x(t)=\sum_\tau \Delta x(t,\tau)=\sum_\tau F(\tau)\Delta\tau\, h(t-\tau)$

$\Delta\tau\to 0$: $x(t)=\int_0^t F(\tau)h(t-\tau)\,d\tau$: CONVOLUTION INTEGRAL (IF I.C. $\neq 0$ ⇒ WE ADD THE FREE RESP)

SHIFTING AND FOLDING PROCESSES:

$\lambda=t-\tau$

$\tau=t-\lambda$

$d\tau=-d\lambda$

$$\begin{cases}\tau=t & \lambda=0\\ \tau=0 & \lambda=t\end{cases}$$

$$x(t)=\int_0^t F(\tau)h(t-\tau)\,d\tau=\int_t^0 F(t-\lambda)h(\lambda)(-d\lambda)=\int_0^t F(t-\lambda)h(\lambda)\,d\lambda$$

τ,λ ARE DUMMY VARIABLES, THEREFORE:

$$x(t)=\int_0^t F(\tau)h(t-\tau)\,d\tau=\int_0^t F(t-\tau)h(\tau)\,d\tau$$

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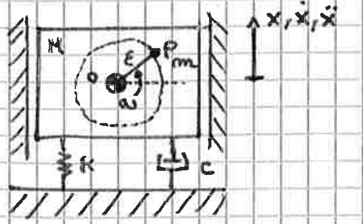
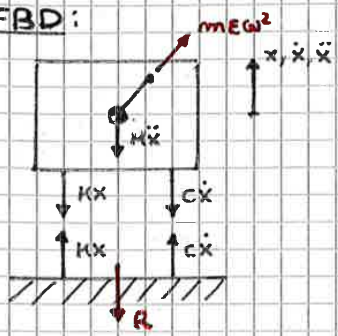
+ WASHING MACHINE



FBD:



$\downarrow)$ $M\ddot{x} + c\dot{x} + Kx = m\varepsilon\omega^2 \sin(\omega t) = m\varepsilon\omega^2 e^{i\omega t} = \left(\frac{\omega}{\omega_m}\right)^2 \overbrace{\underbrace{m\varepsilon\omega_m^2}_{F_0}\, e^{i\omega t}}^{F}$

$$\omega_m^2\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) + 1\right] X_0 e^{i\omega t} = \underbrace{m\varepsilon\omega_m^2}_{F_0}\left(\frac{\omega}{\omega_m}\right)^2 \frac{1}{M} e^{i\omega t} = f$$

$$T_{RREL} = \frac{X_0}{\left(\frac{m\varepsilon}{M}\right)} = \frac{\left(\frac{\omega}{\omega_m}\right)^2}{1-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega) \;\Big\langle\; \begin{array}{l} |T_{RREL}| = \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)| \\ \varphi = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_m}\right)}{1-\left(\frac{\omega}{\omega_m}\right)^2}\right] \end{array}$$

$\downarrow)$ $c\dot{x} + Kx = R = R_0 e^{i\omega t}$

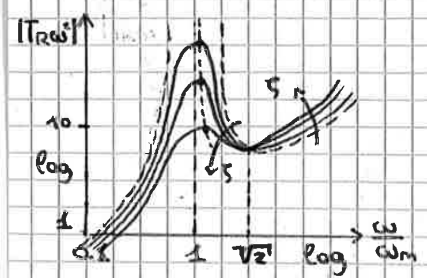
$$\omega_m^2\left[2i\zeta\left(\frac{\omega}{\omega_m}\right) + 1\right] X_0 e^{i\omega t} = R_0 \frac{1}{M} e^{i\omega t} = f_{TR}$$

$$T_{RABS} = \frac{f_{TR}}{f} = \frac{\left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right]}{1-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} = \left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] \cdot G(i\omega) \;\Big\langle\; \begin{array}{l} |T_{RABS}| = \sqrt{1 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2} \cdot |G(i\omega)| \\ \varphi = Tg^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_m}\right)}{1-\left(\frac{\omega}{\omega_m}\right)^2}\right]_D - Tg^{-1}\left[-2\zeta\left(\frac{\omega}{\omega_m}\right)\right]_N \end{array}$$

$$T_{R\omega^2} = \frac{R_0}{F_0} = \frac{\left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right]\left(\frac{\omega}{\omega_m}\right)^2}{1-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)} = \left\{ \begin{array}{l} \left[1+2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] \cdot T_{RREL} \\ T_{RABS} \cdot \left(\frac{\omega}{\omega_m}\right)^2 \end{array} \right.$$

$$= \left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right]\left(\frac{\omega}{\omega_m}\right)^2 \cdot G(i\omega) \;\Big\langle\; \begin{array}{l} |T_{R\omega^2}| = \sqrt{\left(\frac{\omega}{\omega_m}\right)^4 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)^3\right]^2} \cdot |G(i\omega)| \\ \varphi_{T_{R\omega^2}} = \varphi_{T_{RABS}} \end{array}$$

└ NEW TRANSFER FUNCTION



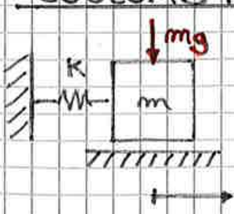
(ENGINEERING COMPROMISE TO SET ζ:)

$\omega \ll \omega_m$: SUBCRITICAL BEHAVIOUR: FOR $\frac{\omega}{\omega_m} < \sqrt{2} \Rightarrow$ HIGHER ζ (↑)

$\omega \gg \omega_m$: OVERCRITICAL BEHAVIOUR: FOR $\frac{\omega}{\omega_m} > \sqrt{2} \Rightarrow$ LOWER ζ (↓)

(IS BETTER)

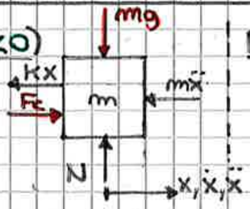
- COULOMB FRICTION (NON-LINEAR) - PIECEWISE LINEAR SYSTEM



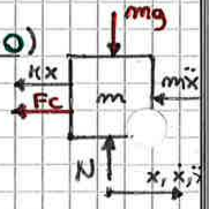
$F_c = -\mu N \cdot \mathrm{sign}(\dot{x})$

$\mu = \begin{cases} f_a : \text{ADHESION } (> f_d) \\ f_d : \text{DYNAMIC} \end{cases}$

FBD ($\dot{x}<0$)



FBD ($\dot{x}>0$)



$\Rightarrow m\ddot{x} + Kx = \pm F_c = \pm f_d N$: NON LINEAR → "PIECEWISE LINEAR SYSTEM"

$x(t) = x_{P.I}(t) + x_{H.S}(t) + \text{I.C.}(x_0, \dot{x}_0 = 0)$; LET'S INTEGRATE SEPARATING TIME:

$$\boxed{\begin{cases} 0 \le t \le \frac{T}{2} \Rightarrow \dot{x} < 0 \quad (1) \\ \frac{T}{2} \le t \le T \Rightarrow \dot{x} > 0 \quad (2) \end{cases}}$$

1) $m\ddot{x} + Kx = +f_d N = +f_d mg$

$\begin{cases} x_{P.I.}(t) = +\frac{f_d mg}{K} \quad (\ddot{x} = \dot{x} = 0) \\ x_{H.S.}(t) = A_0 \cos(\omega_m t) + B_0 \sin(\omega_m t) \end{cases}$

$x(t) = x_{P.I}(t) + x_{H.S}(t) = +\frac{f_d mg}{K} + A_0 \cos(\omega_m t) + B_0 \sin(\omega_m t)$

$\dot{x}(t) = -\omega_m A_0 \sin(\omega_m t) + \omega_m B_0 \cos(\omega_m t)$

$\begin{cases} x(t=0) = x_0 = +\frac{f_d mg}{K} + A_0 \\ \dot{x}(t=0) = 0 = \omega_m B_0 \end{cases} \Rightarrow \begin{cases} A_0 = x_0 - \frac{f_d mg}{K} \\ B_0 = 0 \end{cases}$ I.C. $(t=0)$

$x(t) = +\frac{f_d mg}{K} + \left(x_0 - \frac{f_d mg}{K}\right) \cos(\omega_m t)$

WHEN $t = \frac{T}{2} = \frac{2\pi}{2\omega_m} = \frac{\pi}{\omega_m} \Rightarrow x\left(t = \frac{T}{2}\right) = +\frac{f_d mg}{K} - x_0 + \frac{f_d mg}{K} = \frac{2 f_d mg}{K} - x_0 = x_1$

2) $m\ddot{x} + Kx = -f_d N = -f_d mg$

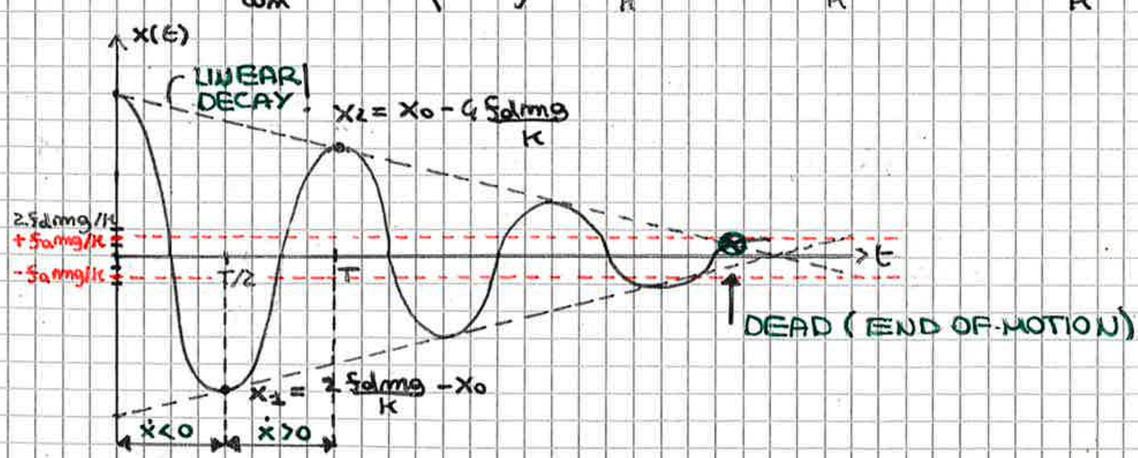
$x(t) = x_{P.I}(t) + x_{H.S.}(t) = -\frac{f_d mg}{K} + A_0' \cos(\omega_m t) + B_0' \sin(\omega_m t)$

$\dot{x}(t) = -\omega_m A_0' \sin(\omega_m t) + \omega_m B_0' \cos(\omega_m t)$

$\begin{cases} x\left(t = \frac{T}{2}\right) = 2\frac{f_d mg}{K} - x_0 = -\frac{f_d mg}{K} - A_0' \\ \dot{x}\left(t = \frac{T}{2}\right) = 0 = -\omega_m B_0' \end{cases} \Rightarrow \begin{cases} A_0' = x_0 - 3\frac{f_d mg}{K} \\ B_0' = 0 \end{cases}$ I.C. $(t = T/2)$

$x(t) = -\frac{f_d mg}{K} + \left(x_0 - 3\frac{f_d mg}{K}\right) \cos(\omega_m t)$

WHEN $t = T = \frac{2\pi}{\omega_m} \Rightarrow x(t)(t = T) = -\frac{f_d mg}{K} + x_0 - 3\frac{f_d mg}{K} = x_0 - 4\frac{f_d mg}{K} = x_2$



THE MOTION IS STOPPED WHEN $\left| x_0 - 2m \frac{f_d mg}{K} \right| \le \frac{f_a mg}{K}$, m = m. OF HALF-CYCLES

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+ DYNAMIC ABSORBER

IN ORDER TO "REDUCE" $F_{TR} \to$ TMD : $\omega_m^2 = \sqrt{\frac{k_2}{m_2}}$: $\omega \to \omega_m = \omega_2 \Rightarrow X_{10} \to 0 \Rightarrow F_{TR} \to 0$

$$\frac{F_{TR0}}{F_0} = \frac{k_1 X_{10}}{(k_1 - m_1\omega^2)}$$

+ BEAT PHENOMENON

MODE #1 $\omega_1 = \sqrt{g/l}$

MODE #2 $\omega_2 = \sqrt{\frac{g}{l} + \frac{2ka^2}{ml^2}}$

$\{x\} = [\phi]\{\eta\}$, $\eta_R = C_R \cos(\omega_R t - \alpha_R)$

$$IC \begin{cases} \{x_0\} = \begin{Bmatrix} \theta_0 \\ 0 \end{Bmatrix} \\ \{\dot{x}_0\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \end{cases} \Rightarrow \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} (\theta_0/2)\cos(\omega_1 t) \\ (\theta_0/2)\cos(\omega_2 t) \end{Bmatrix} \Rightarrow \begin{cases} \theta_1 = \\ \theta_2 = \end{cases}$$

$$\begin{cases} \alpha = \frac{\omega_1 - \omega_2}{2} t \\ \beta = \frac{\omega_1 + \omega_2}{2} t \end{cases} \Rightarrow \begin{cases} \alpha + \beta = \omega_1 t \\ \alpha - \beta = \omega_2 t \end{cases} \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\begin{cases} \omega_A = \frac{\omega_1 + \omega_2}{2} & : \text{MODULATED AMPLITUDE} \\ \omega_B = \frac{\omega_2 - \omega_1}{2} & : \text{CARRIER BAS BAND} \end{cases}$$

"SOFT SPRING" $k \to 0$ $\omega_B \to 0 \Leftrightarrow \omega_2 \to \omega_1$

+ LAGRANGE'S EQUATIONS

$L = T - V$

$L = L(q_i, \dot{q}_i)$

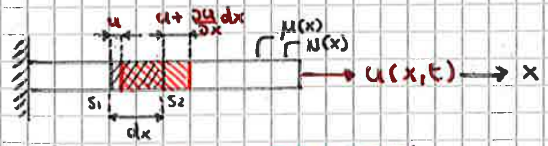
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$$

$$\begin{cases} T = \frac{1}{2}\{\dot{x}\}[M]\{\dot{x}\} \\ V = \frac{1}{2}\{x\}[K]\{x\} \\ D = \frac{1}{2}\{\dot{x}\}[C]\{\dot{x}\} \end{cases}$$

$$Q_k = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{\dot{R}}_i}{\partial \dot{q}_k} = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial q_k}$$

$$Q_k = -\frac{\partial D}{\partial \dot{q}_k}, \quad D = \frac{1}{2}\sum_{j=1}^{m} c_{jk}\, \dot{q}_k \cdot \dot{q}_j$$

3.2 ROD AXIAL VIBRATIONS ($u(x,t)$, μ, $N = AE\,\partial u/\partial x$) II ORDER PDE, 2 B.C.



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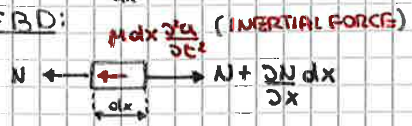


$$\rightarrow)\quad N + \frac{\partial N}{\partial x}dx - N - \mu dx \frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial N}{\partial x} = \mu \frac{\partial^2 u}{\partial t^2}$$

FROM BEAM THEORY: $\sigma = N/A = E\varepsilon = E\,\partial u/\partial x \Rightarrow N = A\sigma = AE\,\partial u/\partial x$

$$\frac{\partial}{\partial x}\left[AE\frac{\partial u}{\partial x}\right] = \mu \frac{\partial^2 u}{\partial t^2} \quad : \text{PDE OF THE ROD}$$

IF $AE = CONST$:

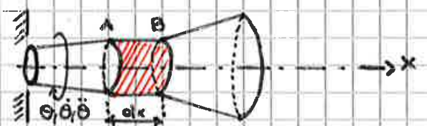
$$AE\frac{\partial^2 u}{\partial x^2} = \mu \frac{\partial^2 u}{\partial t^2} \quad : \text{WAVE EQUATION}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{AE}{\mu}\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- BOUNDARY CONDITIONS (B.C.):

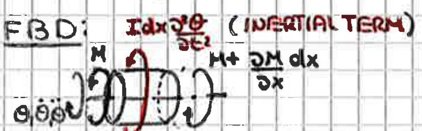
$$\begin{cases} u(x=0,t) = 0 \\ N(x=L,t) = 0 \quad (\Rightarrow \partial u/\partial x = 0) \end{cases}$$

3.3 SHAFT TORSIONAL VIBRATIONS ($\theta(x,t)$, I, $M = GJ\,\partial\theta/\partial x$) II ORDER PDE, 2 B.C.



$$\begin{cases} A \rightarrow \theta \\ B \rightarrow \theta + \frac{\partial \theta}{\partial x}dx \end{cases}$$

FBD:



$$M + \frac{\partial M}{\partial x}dx - M - I dx \frac{\partial^2 \theta}{\partial t^2} = 0$$

$$\frac{\partial M}{\partial x} = I \frac{\partial^2 \theta}{\partial t^2}$$

FOR CIRCULAR SECTION: $M = GJ\,\partial\theta/\partial x$

$$\frac{\partial}{\partial x}\left[GJ\frac{\partial \theta}{\partial x}\right] = I \frac{\partial^2 \theta}{\partial t^2} \quad : \text{PDE OF THE SHAFT}$$

IF $GJ = CONST$:

$$GJ\frac{\partial^2 \theta}{\partial x^2} = I \frac{\partial^2 \theta}{\partial t^2} \quad : \text{WAVE EQUATION}$$

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{GJ}{I}\frac{\partial^2 \theta}{\partial x^2} = c^2 \frac{\partial^2 \theta}{\partial x^2}$$

- BOUNDARY CONDITIONS (B.C.):

$$\begin{cases} \theta(x=0,t) = 0 \\ M(x=L,t) = 0 \quad (\Rightarrow \partial\theta/\partial x = 0) \end{cases}$$

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SOLUTIONS OF WAVE EQS. (TRANSVERSAL/AXIAL/TORSIONAL OSCILLATIONS) II ORDER PDEs

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial x^2} \; ; \quad \frac{\partial^2 u}{\partial t^2} = \frac{AE}{\mu}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2} \; ; \quad \frac{\partial^2 \theta}{\partial t^2} = \frac{GJ}{I}\frac{\partial^2 \theta}{\partial x^2} = c^2\frac{\partial^2 \theta}{\partial x^2}$$

- SOLUTION FORM:

$y(x,t) = \phi(x)\eta(t)$, $\phi(x), \eta(t)$: EIGENFUNCTIONS (SPATIAL, TIME)

$$\phi(x)\frac{\partial^2 \eta}{\partial t^2} = c^2\eta(t)\frac{\partial^2 \phi}{\partial x^2}$$

$$\phi\ddot{\eta} = c^2\eta\,\phi^{II} \longleftrightarrow \phi^{II} - \frac{\ddot{\eta}}{\eta}\frac{1}{c^2}\phi = 0 \qquad \begin{cases} \eta(t) = \eta_0 \sin(\omega t + \alpha_-) \\ \ddot{\eta}(t) - \omega^2\eta_0\sin(\omega t + \alpha_-) \end{cases} \Rightarrow \frac{\ddot{\eta}(t)}{\eta(t)} = -\omega^2$$

$$\phi^{II} + \frac{\omega^2}{c^2}\phi = 0$$

$$\phi^{II} + \beta^2\phi = 0 \; , \quad \beta = \frac{\omega}{c} \; , \quad c = \sqrt{\frac{T}{\rho}} \; ; \quad c = \sqrt{\frac{EA}{\mu}} \; ; \quad c = \sqrt{\frac{GJ}{I}}$$

- $\Rightarrow$ SYSTEM OF 2 ODEs:

$$\begin{cases} \ddot{\eta} + \omega^2\eta = 0 \\ \phi^{II} + \beta^2\phi = 0 \end{cases} : \text{HARMONIC FUNCTIONS}$$

$$\begin{cases} \eta(t) = A\cos(\omega t) + B\sin(\omega t) \\ \phi(x) = C\cos(\beta x) + D\sin(\beta t) \end{cases} : \text{EIGENFUNCTIONS} \begin{cases} A, B \text{ DEPENDING ON I.C.} \\ C, D \text{ DEPENDING ON B.C.} \end{cases}$$

- SOLUTION

$$y(x,t) = \phi(x)\eta(t) = \sum_{R=1}^{\infty}\left[A_R\cos(\omega_R t) + B_R\sin(\omega_R t)\right]\left[C_R\cos(\beta_R x) + D_R\sin(\beta_R t)\right]$$

SOLUTION OF WAVE EQ. (FLEXURAL OSCILLATIONS) IV ORDER PDE.

$$\frac{\partial^2 y}{\partial t^2} = -\frac{EJ}{\mu}\frac{\partial^4 y}{\partial x^4} = -c^2\frac{\partial^4 y}{\partial x^4}$$

- SOLUTION FORM:

$y(x,t) = \phi(x)\eta(t)$

$$\phi(x)\frac{\partial^2 \eta}{\partial t^2} = -c^2\eta(t)\frac{\partial^4 \phi}{\partial x^4}$$

$$\phi\ddot{\eta} = -c^2\eta\,\phi^{IV} \longleftrightarrow \phi^{IV} + \frac{\ddot{\eta}}{\eta}\frac{1}{c^2}\phi = 0 \; , \quad \frac{\ddot{\eta}}{\eta} = -\omega^2$$

$$\phi^{IV} - \frac{\omega^2}{c^2}\phi = 0$$

$$\phi^{IV} - \beta^4\phi = 0 \; , \quad \beta^2 = \frac{\omega}{c} \; , \quad c = \sqrt{\frac{EJ}{\mu}}$$

- $\Rightarrow$ SYSTEM OF 2 ODEs:

$$\begin{cases} \ddot{\eta} + \omega^2\eta = 0 \\ \phi^{IV} - \beta^4\phi = 0 \end{cases}$$

$$\begin{cases} \eta(t) = A\cos(\omega t) + B\sin(\omega t) \\ \phi(x) = C\cos(\beta x) + D\sin(\beta x) + E\cosh(\beta x) + F\sinh(\beta x) \end{cases}$$

- USING THE SEPARATION OF VARIABLES:

$w(x,t) = \xi(x)\eta(t)$, $\xi(x)$ = TRIAL FUNCTION

$$\begin{cases} T(t) = \frac{1}{2}\int_0^L m(x)\,\xi^2\,dx\,\omega^2\eta^2(t) + \frac{1}{2}\sum_{i=1}^{m} m_i\,\xi^2(x_i)\,\omega^2\eta^2(t) \\ V(t) = \frac{1}{2}\int_0^L g(x)\left(\frac{d\xi}{dx}\right)^2 dx\,\eta^2(t) + \frac{1}{2}\sum_{j=1}^{m} K_j\,\xi^2(x_j)\,\eta^2(t) \end{cases}$$

$\Rightarrow T_{MAX}, V_{MAX}$; $\tilde{T}_{MAX} = \frac{T_{MAX}}{\omega^2}$

$R(\xi) = \frac{V_{MAX}}{\tilde{T}_{MAX}} = \frac{\int\ldots}{\int\ldots}$: RAYLEIGH QUOTIENT

CONSIDERING:

- STEADY STATE $\begin{cases} \omega = \dot{\psi} = \text{CONST} \\ \ddot{\psi} = 0 \quad (\psi = \varphi + \beta = \omega t) \end{cases}$
- VERTICAL ROTOR
- COMPLEX VARIABLE $z_c = x_c + i y_c$

$$m(\ddot{x}_c + i\ddot{y}_c) + c_s(\dot{x}_c + i\dot{y}_c) + k_s(x_c + i y_c) = m\varepsilon\omega^2(\cos(\omega t) + i\sin(\omega t)) = m\varepsilon\omega^2 e^{i\omega t}$$

$$m\ddot{z}_c + c_s\dot{z}_c + k_s z_c = m\varepsilon\omega^2 e^{i\omega t}$$

$$z_c = z_0 e^{i\omega t} \; ; \; \dot{z}_c = i\omega z_0 e^{i\omega t} \; ; \; \ddot{z}_c = -\omega^2 z_0 e^{i\omega t}$$

$$\omega_{cr}^2\left[1 - \left(\frac{\omega}{\omega_{cr}}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_{cr}}\right)\right] z_0 \cancel{e^{i\omega t}} = \varepsilon\omega^2 \cancel{e^{i\omega t}} \quad , \quad \omega_{cr} = \sqrt{\frac{k_s}{m}} \; : \text{CRITICAL SPEED}$$

F.R.F.:

$$\frac{z_0}{\varepsilon} = \left(\frac{\omega}{\omega_{cr}}\right)^2 \frac{1}{\left[1 - \left(\frac{\omega}{\omega_{cr}}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_{cr}}\right)\right]} \quad \begin{cases} |\cdot| = \dfrac{\left(\frac{\omega}{\omega_{cr}}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_{cr}}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_{cr}}\right)\right]^2}} \\ \angle = Tg^{-1}\left(\dfrac{2\zeta\left(\frac{\omega}{\omega_{cr}}\right)}{1 - \left(\frac{\omega}{\omega_{cr}}\right)^2}\right) \end{cases}$$

GIVEN A ROTOR WITH:

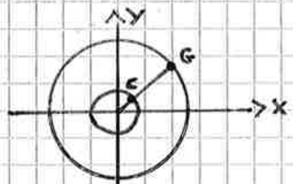
$\omega = \text{CONST} \begin{cases} \delta = \text{CONST} \Rightarrow \text{CIRCLE} \\ \beta = \text{CONST} \Rightarrow \dot{\beta} = 0 \end{cases} \quad \omega = \dot{\psi} = \dot{\varphi} = \text{CONST} \Rightarrow \ddot{\psi} = 0$: SYNCHRONOUS (FORWARD) WHIRL MOTION

MAXIMUM AMPLITUDE:

$$\omega = \omega_{cr} \frac{1}{\sqrt{1 - 2\zeta^2}}$$

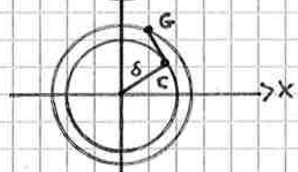
1) $\boxed{\omega \ll \omega_{cr}}$

$\begin{cases} \beta = 0 \\ R = \delta + \varepsilon \; , \; \delta \text{(SMALL)} \end{cases}$



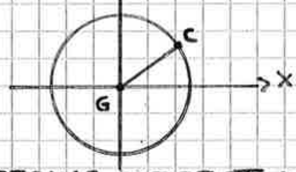
2) $\boxed{\omega \cong \omega_{cr}}$

$\begin{cases} \beta = -\pi/2 \\ R = \sqrt{\delta^2 + \varepsilon^2} \; , \; \delta \text{(LARGE)} \end{cases}$



3) $\boxed{\omega \gg \omega_{cr}}$

$\begin{cases} \beta = -\pi \\ R = \delta = \varepsilon \end{cases}$



SUPERCRITICAL → SELF CENTERING EFFECT : $\delta = \varepsilon$, $G \equiv O$

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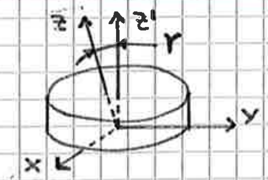
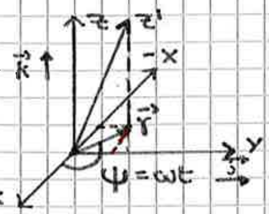
DYNAMIC UMBALANCE



- z': PRINCIPAL AXIS OF INERTIA
- γ = CONST : DYNAMIC UMBALANCE
- $C \equiv G \leftrightarrow \varepsilon = 0$

$\gamma \rightarrow$ GYROSCOPIC MOMENT (M_g) $\rightarrow$ SHAFT ANGULAR DEFLECTION ($\theta_{x,y}$) $\rightarrow$ SHAFT ELASTIC TORQUE (M_e)

EQUILIBRUM BETWEEN M_g AND M_e



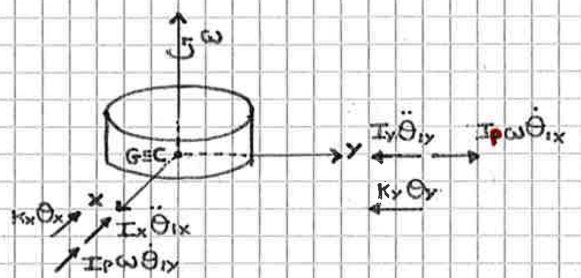
- $\vec{\omega} = \omega \vec{k}$
- $\gamma \perp zz'$ PLANE, PROJECTED ON xy PLANE

$$\begin{cases} \gamma_x = -\gamma \sin(\omega t) \\ \gamma_y = \gamma \cos(\omega t) \end{cases}$$

SHAFT UNDERGOES A TOTAL ANGULAR DEFORMATION DUE TO γ AND ELASTIC ROT. θ

$$\begin{cases} \theta_{1x} = \theta_x + \gamma_x = \theta_x - \gamma \sin(\omega t) \\ \theta_{1y} = \theta_y + \gamma_y = \theta_y + \gamma \cos(\omega t) \end{cases} \quad \begin{cases} \dot\theta_{1x} = \dot\theta_x - \gamma\omega\cos(\omega t) \\ \dot\theta_{1y} = \dot\theta_y - \gamma\omega\sin(\omega t) \end{cases} \quad \begin{cases} \ddot\theta_{1x} = \ddot\theta_x + \gamma\omega^2\sin(\omega t) \\ \ddot\theta_{1y} = \ddot\theta_y - \gamma\omega^2\cos(\omega t) \end{cases}$$

FBD:



- INERTIAL TORQUES: $-I_x\ddot\theta_{1x}$, $-I_y\ddot\theta_{1y}$
- GYROSCOPIC TORQUES: $-I_p\omega\dot\theta_{1y}$, $+I_p\omega\dot\theta_{1x}$
- ELASTIC TORQUES: $-K_x\theta_x$, $-K_y\theta_y$

$$\begin{cases} I_d = I_x = I_y : \text{DISK AXIAL MOM. OF INERTIA} \\ I_p = I_z : \text{POLAR MOM. OF INERTIA} \end{cases}$$

$$\begin{cases} I_x\ddot\theta_{1x} + I_p\omega\dot\theta_{1y} + K_x\theta_x = 0 \\ I_y\ddot\theta_{1y} - I_p\omega\dot\theta_{1x} + K_y\theta_y = 0 \end{cases}$$

$$\begin{cases} I_x\ddot\theta_x + I_p\omega\dot\theta_y + K_x\theta_x = -\gamma\omega^2 I_d\sin(\omega t) + \gamma\omega^2 I_p\sin(\omega t) = \gamma\omega^2(I_p - I_d)\sin(\omega t) \\ I_y\ddot\theta_y - I_p\omega\dot\theta_x + K_y\theta_y = +\gamma\omega^2 I_d\cos(\omega t) - \gamma\omega^2 I_p\cos(\omega t) = -\gamma\omega^2(I_p - I_d)\cos(\omega t) \end{cases}$$

$\theta = \theta_x + i\theta_y$

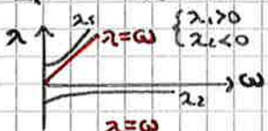
$I_d\ddot\theta - iI_p\omega\dot\theta + K\theta = i\gamma\omega^2(I_p - I_d)e^{i\omega t}$, $\quad \omega_{CR} = \sqrt{\dfrac{K}{I_d - I_p}}$ IF $\dfrac{I_p}{I_d} > 1 \Rightarrow$ NO ω_{CR}

$\theta = \theta_0 e^{st}$; $\dot\theta = s\theta_0 e^{st}$; $\ddot\theta = s^2\theta_0 e^{st}$

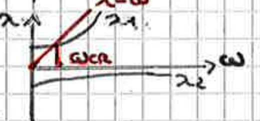
$(I_d s^2 - iI_p\omega s + K)\cancel{\theta_0}\cancel{e^{st}} = 0$ CHARACT. EQ.

$s_{1,2} = \dfrac{-b \pm\sqrt{b^2 - 4ac}}{2a} = \dfrac{iI_p\omega \pm\sqrt{-I_p^2\omega^2 - 4I_dK}}{2I_d} = \sigma + i\lambda$

$\sigma = Re(s)\begin{cases} \leq 0 : \text{STABLE} \\ > 0 : \text{UNSTABLE} \end{cases} \qquad \lambda = Im(s)\begin{cases} > 0 \text{ FORWARD WHIRL} \\ < 0 \text{ BACKWARD WHIRL} \end{cases}$



$\Rightarrow$ NO ω_{CR} $\left(\dfrac{I_p}{I_d} > 1\right)$

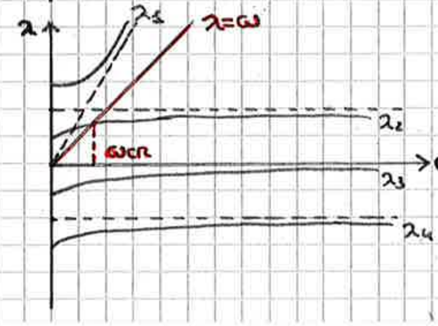


$\Rightarrow \omega_{CR} = \sqrt{\dfrac{K}{I_d - I_p}}$, $\dfrac{I_p}{I_d} < 1$

STATIC + DYNAMIC UMBALANCE

$$\begin{cases} m\ddot z + \alpha z - i\beta\theta = m\varepsilon\omega^2 e^{i\omega t} \\ I_d\ddot\theta - iI_p\omega\dot\theta + i\beta z + \delta\theta = i\gamma\omega^2(I_d - I_p)e^{i(\omega t + \varphi)} \end{cases}$$

$$\begin{bmatrix} m & \\ & I_d \end{bmatrix}\begin{Bmatrix} \ddot z \\ \ddot\theta \end{Bmatrix} + \begin{bmatrix} 0 & \\ & -iI_p\omega \end{bmatrix}\begin{Bmatrix} \dot z \\ \dot\theta \end{Bmatrix} + \begin{bmatrix} \alpha & -i\beta \\ +i\beta & \delta \end{bmatrix}\begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} m\varepsilon\omega^2 e^{i\omega t} \\ i\gamma\omega^2(I_d - I_p)e^{i(\omega t + \varphi)} \end{Bmatrix}$$



- CASE $\zeta < 1$ ($\Delta < 0$) UNDERDAMPING (2 COMPLEX CONJUGATE SOLUTIONS)

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (\dot{x}(t) = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}), \quad s_{1,2} = -\zeta\omega_m \mp i\omega_m\sqrt{1-\zeta^2} = -\zeta\omega_m \mp i\omega_d$$

$$\text{I.C.} \begin{cases} x(t=0) = x_0 = A_1 + A_2 \\ \dot{x}(t=0) = \dot{x}_0 = V_0 = s_1 A_1 + s_2 A_2 \end{cases} \Rightarrow \begin{cases} A_2 = x_0 - A_1 \\ V_0 = s_1 A_1 + s_2 x_0 - s_2 A_1 \end{cases} \Rightarrow \begin{cases} A_2 = \dfrac{s_1 x_0 - \cancel{s_2 x_0} - V_0 + \cancel{s_2 x_0}}{s_1 - s_2} \\ A_1 = \dfrac{V_0 - s_2 x_0}{s_1 - s_2} \end{cases}$$

$$A_1 = \frac{V_0 - s_2 x_0}{s_1 - s_2} = \frac{V_0 + \zeta\omega_m x_0 - i\omega_d x_0}{-2i\omega_d} = \frac{x_0}{2} - \frac{V_0 + \zeta\omega_m x_0}{2i\omega_d}$$

$$A_2 = \frac{s_1 x_0 - V_0}{s_1 - s_2} = \frac{-\zeta\omega_m x_0 - i\omega_d x_0 - V_0}{-2i\omega_d} = \frac{x_0}{2} + \frac{V_0 + \zeta\omega_m x_0}{2i\omega_d}$$

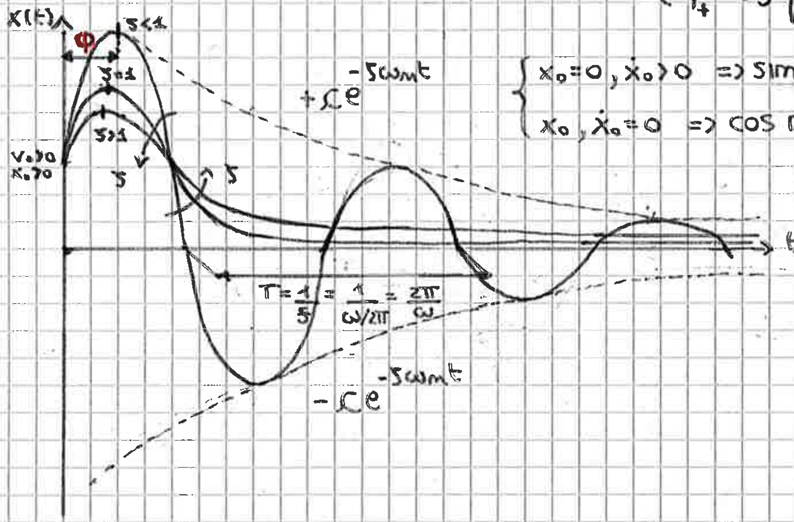
$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = \left[\frac{x_0}{2} - \frac{V_0 + \zeta\omega_m x_0}{2i\omega_d}\right] e^{(-\zeta\omega_m - i\omega_d)t} + \left[\frac{x_0}{2} + \frac{V_0 + \zeta\omega_m x_0}{2i\omega_d}\right] e^{(-\zeta\omega_m + i\omega_d)t}$$

CONSIDERING THE EULERO'S FORMULAS:

$\dfrac{e^{\alpha} - e^{-\alpha}}{2} = \sinh\alpha$	$\dfrac{e^{i\alpha} - e^{-i\alpha}}{2} = i\sin\alpha$	$e^{i\alpha} = \cos\alpha + i\sin\alpha$
$\dfrac{e^{\alpha} + e^{-\alpha}}{2} = \cosh\alpha$	$\dfrac{e^{i\alpha} + e^{-i\alpha}}{2} = \cos\alpha$	$e^{-i\alpha} = \cos(-\alpha) + i\sin(-\alpha) = \cos\alpha - i\sin\alpha$

$$x(t) = e^{-\zeta\omega_m t}\Big[\underbrace{x_0}_{a}\cos(\omega_d t) + \underbrace{\frac{V_0 + \zeta\omega_m x_0}{\omega_d}}_{b}\sin(\omega_d t)\Big] \quad : \text{PERIODIC DECAY}$$

$$x(t) = \mathcal{C} e^{-\zeta\omega_m t}\cdot\cos(\omega_d t - \varphi_+), \quad \text{WITH} \begin{cases} \mathcal{C} = \sqrt{a^2+b^2} = \sqrt{x_0^2 + \left(\dfrac{V_0 + \zeta\omega_m x_0}{\omega_d}\right)^2} : \text{AMPLITUDE} \\ \varphi_+ = Tg^{-1}\left(\dfrac{Im}{Re}\right) = Tg^{-1}\left(\dfrac{b}{a}\right) = Tg^{-1}\left(\dfrac{\frac{V_0 + \zeta\omega_m x_0}{\omega_d}}{x_0}\right) : \text{PHASE ANGLE} \end{cases}$$



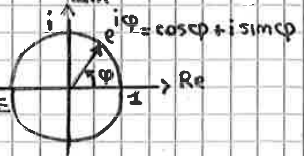
$$\begin{cases} x_0 = 0, \dot{x}_0 > 0 \Rightarrow \sin \text{ EV.} \\ x_0, \dot{x}_0 = 0 \Rightarrow \cos \text{ EV.} \end{cases}$$

RESUMING:

- CASE $\zeta > 1$: $x(t) = e^{-\zeta\omega_m t}\cdot\left[x_0\cosh(\omega_m\sqrt{\zeta^2-1}\,t) + \dfrac{V_0 + \zeta\omega_m x_0}{\omega_m\sqrt{\zeta^2-1}}\sinh(\omega_m\sqrt{\zeta^2-1}\,t)\right]$
- CASE $\zeta = 1$: $x(t) = e^{-\omega_m t}\cdot\left[x_0 + (V_0 + x_0\omega_m)t\right]$
- CASE $\zeta < 1$: $x(t) = e^{-\zeta\omega_m t}\cdot\left[x_0\cos(\omega_d t) + \dfrac{V_0 + \zeta\omega_m x_0}{\omega_d}\sin(\omega_d t)\right]$; $x(t) = \mathcal{C}e^{-\zeta\omega_m t}\cdot\cos(\omega_d t - \varphi_+)$

EULERO'S FORMULAS:

$\dfrac{e^{\alpha} - e^{-\alpha}}{2} = \sinh\alpha$	$\dfrac{e^{i\alpha} - e^{-i\alpha}}{2} = i\sin\alpha$	$e^{i\alpha} = \cos\alpha + i\sin\alpha$	$(a+ib)(a-ib) = (a^2+b^2)$
$\dfrac{e^{\alpha} + e^{-\alpha}}{2} = \cosh\alpha$	$\dfrac{e^{i\alpha} + e^{-i\alpha}}{2} = \cos\alpha$	$e^{-i\alpha} = \cos\alpha - i\sin\alpha$	ARGAND AND GAUSS $\mathbb{C}$ PLANE



WERNER'S FORMULAS [PREVIEW]:

$\sin\alpha\cos\beta = \dfrac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$	$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos\alpha\cos\beta = \dfrac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$	$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin\alpha\sin\beta = \dfrac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$	

L2

IN CASE OF UNDAMPED SYSTEM ($\zeta = 0$)

$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} = F_0 (\cos(\omega t) + i\sin(\omega t)) \rightarrow m\ddot{x} + kx = F_0 \cos(\omega t)$

$\omega_{RES} = \omega = \omega_m \sqrt{1 - 2\zeta^2} \rightarrow \omega_{RES} = \omega = \omega_m$

$|G(i\omega)| = \dfrac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2}} \rightarrow |G(i\omega)| = \dfrac{1}{1 - \left(\frac{\omega}{\omega_m}\right)^2} \Rightarrow \boxed{\omega \rightarrow \omega_m : |G(i\omega)| \rightarrow \infty}$

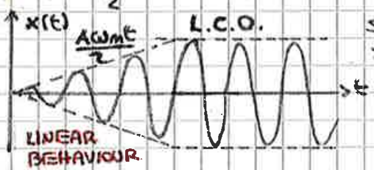
$m\ddot{x} + kx = F_0 \cos(\omega t)$

$x(t) = x_p(t) = X_0 \cos(\omega t)$

$\cancel{\omega_m^2} \left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right] X_0 \cancel{\cos(\omega t)} = \frac{F_0}{k} \cancel{\omega_m^2} \cancel{\cos(\omega t)} \Rightarrow X_0 = \frac{F_0}{k} \frac{1}{1 - (\omega/\omega_m)^2} \Rightarrow x(t) = \frac{F_0}{k} \frac{\cos(\omega t)}{1 - (\omega/\omega_m)^2}$

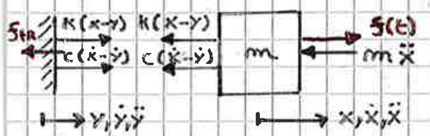
$\boxed{\lim_{\omega \to \omega_m} x(t) = \lim_{\omega \to \omega_m} \frac{dN/d\omega}{dD/d\omega}} = \lim_{\omega \to \omega_m} \dfrac{-F_0 \sin(\omega t)\, t}{-\frac{k\,2}{\omega_m}\left(\frac{\omega}{\omega_m}\right)} = \frac{F_0 \omega_m t}{2k} \sin(\omega t) = \frac{A \omega_m t}{2} \sin(\omega_m t)$

$x(t) = \frac{A\omega_m t}{2} \sin(\omega_m t)$, $A = \frac{F_0}{k}$



SOMETHING (NATURE OF SYST./FAILURE) STOPS THE AMPLITUDE INCREASING

2 BASE EXCITATION



$\begin{cases} m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = \mathfrak{f} \\ c(\dot{x} - \dot{y}) + k(x - y) = \mathfrak{f}_{TR} \end{cases}$: MOTION EQ.s

CONSIDERING $\mathfrak{f}(t) = 0$

$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

$x_p(t) = X_0 e^{i\omega t}$, $X_0 \in \mathbb{C}$; $y_p(t) = Y_0 e^{i\omega t}$, $Y_0 \in \mathbb{R}e$

$\cancel{\omega_m^2}\left[1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] X_0 \cancel{e^{i\omega t}} = \cancel{\omega_m^2}\left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] Y_0 \cancel{e^{i\omega t}}$

$T_{RABS} = \frac{\mathfrak{f}_{TR}}{\mathfrak{f}} = \frac{X_0}{Y_0} = \left[1 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] G(i\omega) \begin{cases} |T_{RABS}| = \sqrt{1 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2} \cdot |G(i\omega)| \quad \left(|T_{RABS}| = \sqrt{T_{RABS} \cdot \hat{T}_{RABS}}\right) \\ \varphi_- = Tg^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_m}\right)}{1 - (\omega/\omega_m)^2}\right]_D - Tg^{-1}\left[-2\zeta\left(\frac{\omega}{\omega_m}\right)\right]_N = Tg^{-1}\left[\frac{+2\zeta(\omega/\omega_m)^3}{1 - (\omega/\omega_m)^2 + 4\zeta^2(\omega/\omega_m)^2}\right] \end{cases}$

$\left(\begin{array}{l} T_{RABS} \\ \rightarrow \varphi_- \end{array} = T_{RABS} \frac{\hat{G}(i\omega)}{\hat{G}(i\omega)} = \frac{[1 + 2i\zeta(\omega/\omega_m)]\,[1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)]}{[1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)]\,[1 - (\omega/\omega_m)^2 - 2i\zeta(\omega/\omega_m)]} = |G(i\omega)|^2 [1 - R^2 - \cancel{2i\zeta R} + \cancel{2i\zeta R} - 2i\zeta R^3 + 4\zeta^2 R^2]\right)$

$T_{RREL} = T_{RABS} - 1 = \frac{X_0 - Y_0}{Y_0} = \frac{Z_0}{Y_0} = \frac{[1 + 2i\zeta(\omega/\omega_m)] - [1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)]}{[1 - (\omega/\omega_m)^2 + 2i\zeta(\omega/\omega_m)]} = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega) \begin{cases} |T_{RREL}| = \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)| \\ \varphi_- = Tg^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_m}\right)}{1 - (\omega/\omega_m)^2}\right]_D \end{cases}$

$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \rightarrow m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$: RELATIVE MOTION EQ.

$z_p(t) = Z_{10} e^{i\omega t}$, $Z_{10} \in \mathbb{C}$; $y_p(t) = Y_0 e^{i\omega t}$, $Y_0 \in \mathbb{R}e$; $z = x - y \Rightarrow m\ddot{x} = m\ddot{z} + m\ddot{y}$

$\cancel{\omega_m^2}\left[1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right)\right] Z_{10} \cancel{e^{i\omega t}} = + \cancel{\omega_m^2}\left(\frac{\omega}{\omega_m}\right)^2 Y_0 \cancel{e^{i\omega t}} \rightarrow \frac{Z_{10}}{Y_0} = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega)$

(H.E.: $x_p(t) = X_0 e^{i\omega t} = A G(i\omega) e^{i\omega t} = A|G(i\omega)| e^{i(\omega t + \varphi_-)}$, $A = F_0/k$)

B.E.: $x_p(t) = X_0 e^{i\omega t} = Y_0 T_{RABS} e^{i\omega t} = Y_0 |T_{RABS}| e^{i(\omega t + \varphi_-)} = Y_0 \sqrt{1 + [2\zeta(\omega/\omega_m)]^2}\, |G(i\omega)| e^{i(\omega t + \varphi_-)}$

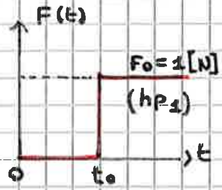
R.B.E.: $z_p(t) = Z_{10} e^{i\omega t} = Y_0 T_{RREL} e^{i\omega t} = Y_0 |T_{RREL}| e^{i(\omega t + \varphi_-)} = Y_0 \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)| e^{i(\omega t + \varphi_-)}$

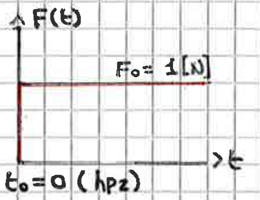
$\frac{d|T_{RABS}|}{d(\omega/\omega_m)} = 0 \rightarrow$ PEAKS ; $N\left(\frac{d|T_{RABS}|}{d(\omega/\omega_m)}\right) = 0 \rightarrow \omega_{RES}^{BE-TRABS} = \omega_m \sqrt{\frac{2}{1 + \sqrt{1 + 8\zeta^2}}}$; $|T_{RABS}|_{MAX} = \left|T_{RABS}\left(\frac{\omega}{\omega_m} = \frac{\omega_{RES}}{\omega_m}\right)\right|$

$\frac{d|T_{RREL}|}{d(\omega/\omega_m)} = 0 \rightarrow$ PEAKS ; $N\left(\frac{d|T_{RABS}|}{d(\omega/\omega_m)}\right) = 0 \rightarrow \omega_{RES}^{BE-TRREL} = \omega_m \cdot \frac{1}{\sqrt{1 - 2\zeta^2}}$; $|T_{RREL}|_{MAX} = \frac{1}{2\zeta} \frac{1}{\sqrt{1 - \zeta^2}}$

STEP EXCITATION

$$F(t) = \begin{cases} 0 & t < t_0 \\ F_0 & t \geq t_0 \end{cases} \qquad hps: \begin{cases} F_0 = 1[N] \\ t_0 = 0[s] \end{cases}$$





MOTION EQUATION $(t \geq t_0)$:

$$m\ddot{x} + c\dot{x} + kx = F_0$$

SOLUTION FORM CONSIDERING I.C. $\begin{cases} x(t=0) = 0 \\ \dot{x}(t=0) = 0 \end{cases}$

$$x(t) = x_P(t) + x_A(t)$$

$$\begin{cases} x_P(t) = \dfrac{F_0}{k} \\ x_A(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = e^{-\zeta\omega_m t} \cdot [a\cos(\omega_d t) + b\sin(\omega_d t)] \end{cases} \quad (\text{UNDERDAMPING } \zeta < 1)$$

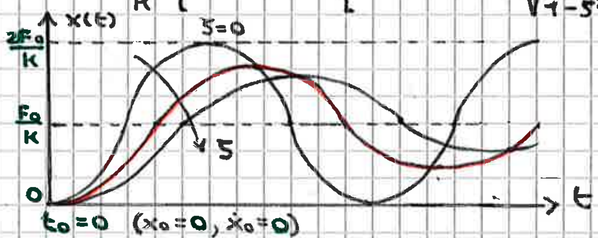
$$x(t) = \frac{F_0}{k} + e^{-\zeta\omega_m t} [a\cos(\omega_d t) + b\sin(\omega_d t)]$$

$$\dot{x}(t) = -\zeta\omega_m e^{-\zeta\omega_m t}[a\cos(\omega_d t) + b\sin(\omega_d t)] + e^{-\zeta\omega_m t}[-\omega_d a\sin(\omega_d t) + \omega_d b\cos(\omega_d t)]$$

IMPOSING THE I.C.

$$\begin{cases} x(t=0) = 0 = \dfrac{F_0}{k} + a \\ \dot{x}(t=0) = 0 = -\zeta\omega_m a + \omega_d b \end{cases} \Rightarrow \begin{cases} a = -\dfrac{F_0}{k} \\ b = \zeta\dfrac{\omega_m}{\omega_d} a = -\dfrac{F_0}{k}\dfrac{\zeta}{\sqrt{1-\zeta^2}} \end{cases}$$

$$S(t) = \frac{F_0}{k}\left\{1 - e^{-\zeta\omega_m t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]\right\}$$: STEP EXCITATION RESPONSE



IF $\zeta \nearrow$ $\omega_d = \omega_m\sqrt{1-\zeta^2} \searrow$ $\Rightarrow T_d = \dfrac{2\pi}{\omega_d} \nearrow$

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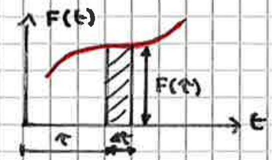
ARBITRARY EXCITATION → CONVOLUTION INTEGRAL



$F(t)$: ARBITRARY EXCITATION

$\hat{F}(\tau) = F(\tau)\Delta\tau$: AREA = IMPULSE MAGNITUDE [N/S]

$\hat{F}(\tau)\delta(t-\tau) = F(\tau)\Delta\tau\,\delta(t-\tau)$: IMPULSE FORCE

$x(t) = \sum_\tau \Delta x(t,\tau) = \sum_\tau F(\tau)\Delta\tau\, h(t-\tau)$: RESPONSE

$\Delta\tau \to 0 \quad x(t) = \int_0^t F(\tau)h(t-\tau)d\tau$: CONVOLUTION INTEGRAL

NW: IT IS VALID FOR NULL I.C.; IF THEY ARE NOT NULL, WE HAVE TO ADD THE FREE RESPONSE.

SHIFTING AND FOLDING PROCESS:

$\lambda = t-\tau$

$\tau = t-\lambda$

$d\tau = -d\lambda$

$$\begin{cases} \tau = 0 \to \lambda = t \\ \tau = t \to \lambda = 0 \end{cases}$$

$$x(t) = \int_0^t F(\tau)h(t-\tau)d\tau = \int_t^0 F(t-\lambda)h(\lambda)(-d\lambda) = \int_0^t F(t-\lambda)h(\lambda)d\lambda$$

τ, λ ARE DUMMY VARIABLES OF INTEGRATION, THEREFORE:

$$x(t) = \int_0^t F(\tau)h(t-\tau)d\tau = \int_0^t F(t-\tau)h(\tau)d\tau = F(t) * h(t)$$

IN PRACTICE:

PROBLEM GIVES US: $F(t) = F_0\cos(\omega_m t)u(t)$ / $F(t) = F_0\sin(\omega_m t)u(t)$ / $F(t) = F_0\sin(\omega t)u(t)$

NULL I.C. AND $\zeta = 0$ (UNDAMPED SDOF SYSTEMS).

1) CONVOLUTION INTEGRAL

$x(t) = \int_0^t F(\tau)h(t-\tau)d\tau \quad , \quad h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_m(t-t_0)} \cdot \sin[\omega_d(t-t_0)]\,u(t-t_0) \quad , \quad \zeta = 0$

$F(\tau) = F_0\cos(\omega_m\tau)u(\tau)$

$h(t-\tau) = \frac{1}{m\omega_m}\sin[\omega_m(t-\tau)]u(t-\tau)$

$x(t) = \int_0^t \frac{F_0}{m\omega_m}\cos(\omega_m\tau)u(\tau)\sin[\omega_m(t-\tau)]u(t-\tau)d\tau$

CONSIDERING $t \geq t_0$:

$x(t) = \int_0^t \frac{F_0}{m\omega_m}\cos(\omega_m\tau)\sin[\omega_m(t-\tau)]d\tau$

WERNER'S FORMULAS:

$$\boxed{\begin{aligned} \sin(\alpha)\cos(\beta) &= \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \\ \cos(\alpha)\cos(\beta) &= \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \\ \sin(\alpha)\sin(\beta) &= \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} \end{aligned}}$$

2) ALTERNATIVE APPROACH:

$F(t) = F_0\cos(\omega_m t)u(t) \xrightarrow{*} F(t) = F_0\cos(\omega t)$

$m\ddot{x} + kx = F_0\cos(\omega t)$, $x_p(t) = X_0\cos(\omega t)$

$\omega_m^2\left[1-\left(\frac{\omega}{\omega_m}\right)^2\right]X_0\cancel{\cos(\omega t)} = \frac{F_0}{k}\omega_m^2\cancel{\cos(\omega t)} \to X_0 = \frac{F_0}{k}\frac{1}{1-(\omega/\omega_m)^2}$

$x(t) = x_p(t) + x_A(t)$, THEN WE IMPOSE THE I.C. ! NW: THE I.C. HAVE TO BE IMPOSED AFTER SOL. FORM DEF. !

$$\begin{cases} x_p(t) = X_0\cos(\omega t) = \frac{F_0}{k}\frac{\cos(\omega t)}{1-(\omega/\omega_m)^2} \\ x_A(t) = A_1e^{s_1t} + A_2e^{s_2t} = e^{-\zeta\omega_m t}[a\cos(\omega_d t) + b\sin(\omega_d t)] \end{cases} \Rightarrow \begin{cases} x(t) = \frac{F_0}{k}\frac{\cos(\omega t)}{1-(\omega/\omega_m)^2} + a\cos(\omega_m t) + b\sin(\omega_m t) \\ \dot{x}(t) = -\frac{F_0}{k}\frac{\omega\sin(\omega t)}{1-(\omega/\omega_m)^2} - \omega_m a\sin(\omega_m t) + \omega_m b\cos(\omega_m t) \end{cases}$$

$$\text{I.C.}\begin{cases} x(t=0) = 0 = \frac{F_0}{k}\frac{1}{1-(\omega/\omega_m)^2} + a \\ \dot{x}(t=0) = 0 = \omega_m b \end{cases} \Rightarrow \begin{cases} a = -\frac{F_0}{k}\frac{1}{1-(\omega/\omega_m)^2} \\ b = 0 \end{cases}$$

$\Rightarrow x(t) = \frac{F_0}{k}\frac{1}{1-(\omega/\omega_m)^2}\left[\cos(\omega t) - \cos(\omega_m t)\right]$

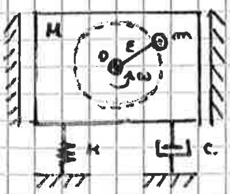
$\xrightarrow{*} x(t) = \lim_{\omega\to\omega_m} x(t) = \lim_{\omega\to\omega_m}\frac{dN/d\omega}{dD/d\omega}$

*WE DO THIS IF $F(t) = \delta(\omega_m)$

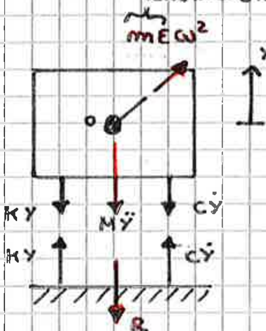
+ WASHING MACHINE

ENGINE STATIC UMBALANCE × $G = \varepsilon \omega$

$\varepsilon = \frac{G}{\omega} \cdot 10^3 \, [\mu m]$



FBD:



$\left(f = \left(\frac{\omega}{\omega_m}\right)^2 F \right)$

MOTION EQs:

$\uparrow) \; -M\ddot{y} - c\dot{y} - Ky + m\varepsilon\omega^2 \sin(\omega t) = 0$

$\downarrow) \; M\ddot{y} + c\dot{y} + Ky = m\varepsilon\omega^2 \sin(\omega t) = f = \left(\frac{\omega}{\omega_m}\right)^2 \overbrace{\underbrace{m\varepsilon\omega_m^2}_{F_0}}^{F} e^{i\omega t}$

$\uparrow) \; c\dot{y} + Ky - R = 0$

$\downarrow) \; R = c\dot{y} + Ky \Leftrightarrow c\dot{y} + Ky = R = f_{tR} = R_0 e^{i\omega t}$

$y(t) = Y_0 e^{i\omega t}$

$$\cancel{\omega_m^2}\left[1 - \left(\frac{\omega}{\omega_m}\right)^2 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right] Y_0 \cancel{e^{i\omega t}} = \frac{m\varepsilon\omega^2}{M} \cancel{e^{i\omega t}} = \left(\frac{\omega}{\omega_m}\right)^2 \overbrace{\frac{m\varepsilon\omega_m^2}{M}}^{F_0/M} \cancel{e^{i\omega t}} \quad F/M$$

$$\omega_m^2\left[1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right] Y_0 \cancel{e^{i\omega t}} = \frac{R_0}{M} \cancel{e^{i\omega t}}$$

- $TR_{REL} = \frac{Y_0}{(m\varepsilon/M)} = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega)$

$$|TR_{REL}| = \left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)|$$

$$\varphi = Tg^{-1}\left[\frac{-2\xi(\omega/\omega_m)}{1 - (\omega/\omega_m)^2}\right] = \varphi_{TR_{REL}} = \varphi_{G(i\omega)}$$

- $TR_{ABS} = \frac{f_{tR}}{f} = \frac{R}{f} = \frac{c\dot{y} + Ky}{M\ddot{y} + c\dot{y} + Ky} = \left[1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right] \cdot G(i\omega)$

$$|TR_{ABS}| = \sqrt{1 + \left[2\xi\left(\frac{\omega}{\omega_m}\right)\right]^2} \, |G(i\omega)|$$

$$\varphi = Tg^{-1}\left[\frac{-2\xi(\omega/\omega_m)^3}{1 - (\omega/\omega_m)^2 + 4\xi^2(\omega/\omega_m)^2}\right]$$

$$\left(\begin{matrix} TR_{ABS} \\ \to \varphi_t \end{matrix} \; \frac{G(i\omega)}{\bar{G}(i\omega)} = [1 + 2i\xi R][1 - R^2 - 2i\xi R] |G(i\omega)| = |G(i\omega)| (1 - R^2 - \cancel{2i\xi R} + \cancel{2i\xi R} - 2i\xi R^3 + 4\xi^2 R^2) \right)$$

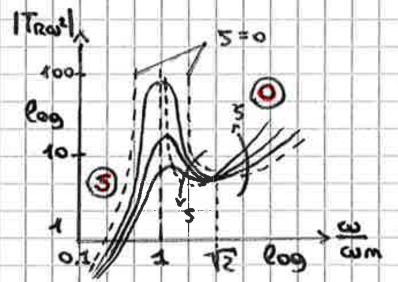
- $TR\omega^2 = \frac{R_0}{F_0} = \frac{\cancel{M}\cancel{\omega_m^2}\left[1 + 2i\xi(\omega/\omega_m)\right] Y_0}{\cancel{M}\cancel{\omega_m^2}\left(\frac{\omega_m}{\omega}\right)^2 \left[1 - (\omega/\omega_m)^2 + 2i\xi(\omega/\omega_m)\right] Y_0} = \left(\frac{\omega}{\omega_m}\right)^2 \left[1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)\right] G(i\omega) = \begin{cases} \left(\frac{\omega}{\omega_m}\right)^2 TR_{ABS} \\ \sqrt{1 + 2i\xi\left(\frac{\omega}{\omega_m}\right)}^{-1} TR_{REL} \end{cases}$

$TR\omega^2$:

$$|TR\omega^2| = \left(\frac{\omega}{\omega_m}\right)^2 \sqrt{1 + [2\xi(\omega/\omega_m)]^2} \, |G(i\omega)|$$

$$\varphi = Tg^{-1}\left[\frac{-2\xi(\omega/\omega_m)^3}{1 - (\omega/\omega_m)^2 + 4\xi^2(\omega/\omega_m^2)}\right] = \varphi_{TR\omega^2} = \varphi_{TR_{ABS}}$$

NEW TRANSFER FUNCTION



ENGINEERING COMPROMISE TO SET ξ:

$\omega \ll \omega_m$: SUBCRITICAL BEHAVIOUR (S)

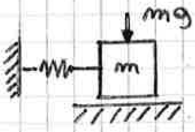
$\omega \gg \omega_m$: OVERCRITICAL BEHAVIOUR (O)

IF $\frac{\omega}{\omega_m} < \sqrt{2}$ IT'S MORE CONVENIENT TO SELECT AN HIGHER ξ (↑)

IF $\frac{\omega}{\omega_m} > \sqrt{2}$ IT'S MORE CONVENIENT TO SELECT A LOWER ξ (↓)

IN ORDER TO REDUCE THE AMPLIFICATION.

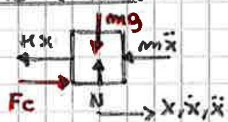
- COULOMB FRICTION (NON-LINEAR)



$F_c = -\mu N \,\mathrm{sigm}(\dot{x})$

$\mu = \begin{cases} f_a : \text{ADHESION} \ (> f_d) \\ f_d : \text{DYNAMIC} \end{cases}$

FBD ($\dot{x}<0$):



FBD ($\dot{x}>0$)

$m\ddot{x} + c\dot{x} + Kx = \pm F_c = \pm f_d N$

$x(t) = x_P(t) + x_A(t)$ + I.C. ($x_0, \dot{x}_0 = 0$)

$$\begin{cases} 0 \leq t \leq \frac{T}{2} : \dot{x} < 0 : F_c = f_d mg & (1) \\ \frac{T}{2} \leq t \leq T : \dot{x} > 0 : F_c = -f_d mg & (2) \end{cases}$$

1) $m\ddot{x} + Kx = f_d mg$

$x(t) = x_P(t) + x_A(t) = \frac{f_d mg}{K} + A_0 \cos(\omega_m t) + B_0 \,\mathrm{sim}(\omega_m t)$

$\dot{x}(t) = -\omega_m A_0 \,\mathrm{sim}(\omega_m t) + \omega_m B_0 \cos(\omega_m t)$

$$\begin{cases} x(t=0) = x_0 = \frac{f_d mg}{K} + A_0 \\ \dot{x}(t=0) = 0 = \omega_m B_0 \end{cases} \Rightarrow \begin{cases} A_0 = x_0 - \frac{f_d mg}{K} \\ B_0 = 0 \end{cases}$$

$x(t) = \frac{f_d mg}{K} + \left(x_0 - \frac{f_d mg}{K}\right) \cos(\omega_m t)$

WHEN $t = \frac{T}{2} = \frac{2\pi}{2\omega_m} = \frac{\pi}{\omega_m} \rightarrow x\left(t = \frac{T}{2}\right) = \frac{f_d mg}{K} - \left(x_0 - \frac{f_d mg}{K}\right) = 2\frac{f_d mg}{K} - x_0 = x_1$

2) $m\ddot{x} + Kx = -f_d mg$

$x(t) = x_P(t) + x_A(t) = -\frac{f_d mg}{K} + A_0' \cos(\omega_m t) + B_0' \,\mathrm{sim}(\omega_m t)$

$\dot{x}(t) = -\omega_m A_0' \,\mathrm{sim}(\omega_m t) + \omega_m B_0' \cos(\omega_m t)$

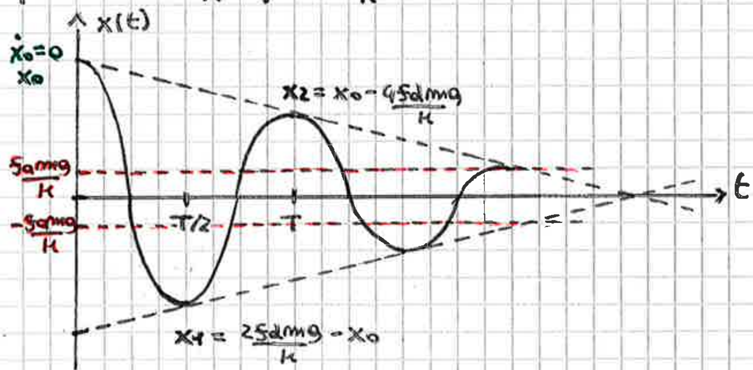
$$\begin{cases} x\left(t = \frac{T}{2}\right) = 2\frac{f_d mg}{K} - x_0 = -\frac{f_d mg}{K} - A_0' \\ \dot{x}\left(t = \frac{T}{2}\right) = 0 = -\omega_m B_0' \end{cases} \Rightarrow \begin{cases} A_0' = x_0 - 3\frac{f_d mg}{K} \\ B_0' = 0 \end{cases}$$

$x(t) = -\frac{f_d mg}{K} + \left(x_0 - 3\frac{f_d mg}{K}\right) \cos(\omega_m t)$

WHEN $t = T = \frac{2\pi}{\omega_m} \rightarrow x(t = T) = -\frac{f_d mg}{K} + x_0 - 3\frac{f_d mg}{K} = x_0 - 4\frac{f_d mg}{K} = x_2$

THE MOTION IS STOPPED WHEN:

$\left| x_0 - 2m \frac{f_d mg}{K} \right| \leq \frac{f_a mg}{K}$, m = m. OF HALF CYCLES



RANK AND "SOLUTION" COMPUTATION METHODS

- GAUSS REDUCTION (TO EVALUATE THE RANK AND TO SOLVE THE PROBLEM)

THE AIM IS TO OBTAIN SOMETHING SIMILAR TO $[\triangleright\!\!\!\triangle]$ OR $[\nabla]$ BY MULT. PER COEFFs AND SUBTR. ROWS

- IF $[\triangle]$ -> FULL RANK -> $\exists [K]^{-1}$ WE CAN DO $\{x\} = [K]^{-1}\{F\}$
- IF ONE ROW RESULTS FULL 0 -> NOT FULL RANK.

EXAMPLE:

$$[A] = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} - \\ \cdot 2 \end{matrix} \to \begin{bmatrix} 2 & 2 & 1 \\ 0 & 5 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} - \\ \\ \cdot 2 \end{matrix} \to \begin{bmatrix} 0 & 2 & -1 \\ 0 & 5 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \cdot 5 - \\ \cdot 2 \end{matrix} \to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 5 & -2 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \text{FULL RANK}$$

NOTE: THE SUBTRACTED ROW REMAINS THE SAME.

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 5 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \to \begin{cases} -x_3 = F_1 \\ 5x_2 - 2x_3 = F_2 \\ x_1 + x_3 = F_3 \end{cases} \Rightarrow \begin{cases} x_3 = -F_1 \\ x_2 = (F_2 - 2F_1)/5 \\ x_1 = F_3 + F_1 \end{cases}$$

- SYLVESTER CRITERION (TO EVALUATE IF A MATRIX IS POSITIVE-DEF)

IT IS NECESSARY AND SUFF. CONDITION TO DETERMINE WHETHER A MATRIX IS **POSIT.-DEF.**

PRINCIPAL MINOR OF ORDER K DEF: IT IS A MINOR (DETERMINANT OF THE MATRIX) OBTAINED BY DELETING $m-k$ ROWS AND COLUMNS (SAME POSITION). THE COLLECTORS OF ALL PRINCIPAL MINORS OF ORDER K WILL BE DENOTED A_k^*.

- IF THE PRINCIPAL MINORS (= DETERMINANT OF PRINCIPAL SUB-MATRICES = A_3^*, A_2^*, A_1^*) CONSIDERING ONLY THE LEFT UPPER CORNER $\left([A], \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, [a_{11}]\right)$ ARE >0 => **POSITIVE DEF.**

RANK = $j_1 = \min(m, m)$ **(=> FULL RANK (BECAUSE $\neq 0$))**

NOTE: IF $[A] = [m \times m]$, $m > m$ WE HAVE TO EVALUATE THE $\det()$ OF EACH SQUARE-MATRIX (DELETING ONE m PER TIME): IF ONE $\det() > 0$ => $\det(A) > 0$

EXAMPLE:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} ; \; A_3^* = \det[A] \; ; \; A_2^* = \left\{ \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}; \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \right\}; \; A_1^* = \{a_{11}\; a_{22}\; a_{33}\}$$

$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$: **SYLVESTER METHOD**

$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32})$: **OTHER METHOD**

- IF $\det(A) = 0$ **(=> NOT FULL RANK) => SEMI-POSIT. DEF.**

NOTE: TO UNDERSTAND WHICH IS THE RANK: $j_2 = j_1 - 1$; IF THE $\det()$ OF THE LEFT UPPER CORNER SUBMATRICES $\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, [a_{11}]\right)$ ARE >0 => RANK = j_2. IF $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0$ AND $a_{11} > 0$ => RANK = $j_3 = j_1 - 2$.

EIGENVALUE PROBLEM (E.V.P.)

LTI SYSTEM, n DOFs (LTI: LINEAR-TIME-INVARIANT: NO EXTERNAL F., NO DISSIPATION)

$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$

LOOKING FOR SYNCHRONOUS MOTION:

$$\begin{cases} \{x(t)\} = \{x_0\}\, g(t) \\ \{\ddot{x}(t)\} = \{x_0\}\, \ddot{g}(t) \end{cases}$$

$g(t)$: TIME SOLUTION ⇒ NATURE OF MOTION (IMPOSED LAW)

$\{x_0\}$: CONSTANT VECTOR ⇒ MODE SHAPE

$[M]\{x_0\}\ddot{g}(t) + [K]\{x_0\}g(t) = \{0\}$

PRE-MULTIPLYING BY $\{x_0\}^T$:

$\{x_0\}^T[M]\{x_0\}\ddot{g}(t) + \{x_0\}^T[K]\{x_0\}g(t) = \{0\}$

$$\frac{\ddot{g}(t)}{g(t)} = -\frac{\{x_0\}^T[K]\{x_0\}}{\{x_0\}^T[M]\{x_0\}} = -\omega^2 \le 0 \quad ; \quad \omega^2 : \text{EIGENVALUE}$$

$\ddot{g}(t) + \omega^2 g(t) = 0$: PENDULUM

THEREFORE, IN PRACTICE:

$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$

$$\begin{cases} \{x\} = \{x_0\}e^{i\omega t} \\ \{\dot{x}\} = i\omega\{x_0\}e^{i\omega t} \\ \{\ddot{x}\} = -\omega^2\{x_0\}e^{i\omega t} \end{cases}$$

$([K] - \omega^2[M])\{x_0\} = 0$

$\{x_0\} = 0$ TRIVIAL SOL. ⇒ $([K]-\omega^2[M])$ HAS TO BE NOT FULL RANK.

$\det([K]-\omega^2[M]) = 0 \rightarrow \lambda_R = \omega_R^2$: EIGENVALUES, $R = 1,..,n$ $(\omega_1^2 \le \omega_2^2 \le \omega_n^2)$

$[K]-\omega^2[M]$ = DYN. STIFFNESS MATRIX, $\omega_R = \sqrt{\omega_R^2}$: NATURAL FREQUENCIES

⇒ ∞ POSSIBLE SOLUTIONS (ROUCHE-CAPELLI THEOREM) (SAME SHAPE $\{x_0\}$, DIFF. AMPL. COEFF)

WE CALL $\{x_0\} \to \{\varphi\}$: $\omega_1 \le \omega_2 \le \omega_3 \dots \le \omega_n$

$\{\varphi_1\}\ \{\varphi_2\}\ \{\varphi_3\}\ \{\varphi_n\}$ DEFINED A PART FROM AN ARBITRARY CONST.

COLLECTORS OF EIGENVALUES AND EIGENVECTORS:

$[\Lambda] = [\diagdown \omega_R^2 \diagdown]$: EIGENVALUE MATRIX

$[\varphi] = [\{\varphi_1\};\{\varphi_2\};\dots;\{\varphi_n\}]$: EIGENVECTOR/MODAL MATRIX (UNIQUE)(CONSTANT)(FULL RANK)

$\{x\} = [\varphi]\{\eta\}$: LINEAR MODAL TRANSFORMATION

(2) THEOREM OF EXPANSION (MODAL SUPERPOSITION) → DEMONSTRATION OF:

1. MODAL MATRIX $[\varphi]$ IS A BASE;
2. LINEAR TRANSF. BETWEEN $\{x\}$ AND $\{\eta\}$: $\{x\} = [\varphi]\{\eta\}$
3. ANY VECTOR CAN BE EXPRESSED AS A LINEAR COMBINATION OF MODE SHAPES (EIGENVECTORS)

$\left\{ \begin{array}{l} \text{EIGENVECTORS } \{\varphi_1\}, \{\varphi_2\}, \dots, \{\varphi_m\} \text{ ARE LINEARLY INDEPENDENT.} \\ [\varphi] \text{ HAS FULL RANK.} \end{array} \right.$

PROOF BY CONTRADICTION:

hp: EIGENVECTORS $\{\varphi_1\}, \{\varphi_2\}, \dots, \{\varphi_m\}$ ARE LINEARLY DEPENDENT.

THEN $c_1\{\varphi_1\} + c_2\{\varphi_2\} + \dots + c_m\{\varphi_m\} = \sum_{R=1}^{m} c_R\{\varphi_R\} = \{0\}$ WITH $c_R \neq 0$, $\forall R$.

PRE-MULTIPLYING BY $\{\varphi_S\}^T[M]$:

$$\sum_{R=1}^{m} c_R \{\varphi_S\}^T [M] \{\varphi_R\} = 0$$

$$\begin{cases} \text{IF } R \neq S \rightarrow \{\varphi_S\}^T[M]\{\varphi_R\} = 0 \\ \text{IF } R = S \rightarrow \{\varphi_S\}^T[M]\{\varphi_R\} = m_R \Rightarrow c_S = 0 \end{cases}$$

REAPITING THE PROCEDURE TAKING ANY $\{\varphi_S\}^T$, IT FOLLOWS THAT $c_R = 0$, $\forall R$ (CONTRADICTION)

⇒ HENCE, EIGENVECTORS ARE LINEARLY INDEPENDENT.

⇒ HENCE, ANY VECTOR CAN BE EXPRESSED AS A LINEAR COMB. OF EIGENVECTORS.

$$\{V\} = \sum_{R=1}^{m} c_R\{\varphi_R\}$$

PRE-MULTIPLYING BY $\{\varphi_S\}^T[M]$:

$$\{\varphi_S\}^T[M]\{V\} = \sum_{R=1}^{m} c_R \{\varphi_S\}^T[M]\{\varphi_R\} = c_S m_S$$

$c_R = \dfrac{\{\varphi_S\}^T[M]\{V\}}{m_R}$: SCALAR VALUE FOR EACH MODE (MODAL PARTECIPATION FACTOR)

2.2 MODAL ANALYSIS FOR DAMPED SYSTEM ($C \neq 0$)

- **MODAL ANALYSIS WITH NO EXTERNAL FORCE**

$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{0\}$, $\{x\}=[\varphi]\{\eta\}=\sum_{R=1}^{m}\{\varphi_R\}\eta_R$: LINEAR TRANSF. (SUPERP. PRINC.)

$[M]\{\ddot{x}\}+[K]\{x\}=\{0\} \rightarrow [\Lambda],[\varphi]$, $\{x\}=\{x_0\}e^{i\omega t}$, $\{\ddot{x}\}=-\omega^2\{x_0\}e^{i\omega t}$

$([K]-\omega^2[M])\{x_0\}=\{0\}$

$\begin{cases}\{x_0\}=0 : \text{TRIVIAL S.} \\ \det([K]-\omega_R^2[M])=0 \rightarrow \omega_R^2 \rightarrow [\Lambda]=[\diagdown \omega_R^2 \diagdown]\end{cases}$

$([K]-\omega_R^2[M])\{\varphi_R\}=\{0\} \rightarrow \{\varphi_R\} \rightarrow [\varphi]=[\{\varphi_1\};\{\varphi_2\};..;\{\varphi_m\}]$

CHARACTERIZATION OF DAMPING : **CAUGHEY RELATIONS**

- PERMUTATION OF $[M]^{-1}$:

$[C][M]^{-1}[K]=[K][M]^{-1}[C]$, $[C]=[C_P]=\text{DIAG}$

- GENERALLY:

$[C]=[C_P]+[C_{NP}]$, IF $[C_{NP}]\neq 0 \Rightarrow$ COUPLING IN $[C]$!

- CAUGHEY SERIES:

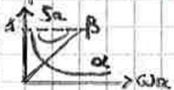
$[C]=\sum_{i=0}^{m-1}\gamma_i[M]([M]^{-1}[K])^i$

$\alpha \rightarrow [M]$: SKY HOOK DAMPERS

$\beta \rightarrow [K]$: MATERIAL DEFORMATIONS

- SIMPLEST CASE $\gamma_0,\gamma_1 \neq 0$:

$[C_P]=\alpha[M]+\beta[K]$



NW: IF $[C]=[C_P]$ THE MODE SHAPES ARE THE SAME OF THE UNDAMPED SYSTEM.

MODAL ANALYSIS (WITH MC_PK PROP.):

$[M][\varphi]\{\ddot{\eta}\}+[C_P][\varphi]\{\dot{\eta}\}+[K][\varphi]\{\eta\}=\{0\}$

$[\varphi]^T[M][\varphi]\{\ddot{\eta}\}+[\varphi]^T[C_P][\varphi]\{\dot{\eta}\}+[\varphi]^T[K][\varphi]\{\eta\}=[\varphi]^T\{0\}=0$

$[\varphi]^T[M][\varphi]\{\ddot{\eta}\}+[\varphi]^T(\alpha[M]+\beta[K])[\varphi]\{\dot{\eta}\}+[\varphi]^T[K][\varphi]\{\eta\}=0$

$[\diagdown m_R \diagdown]\{\ddot{\eta}\}+(\alpha[\diagdown m_R \diagdown]+\beta[\diagdown k_R \diagdown])\{\dot{\eta}\}+[\diagdown k_R \diagdown]\{\eta\}=0$

$m_R\ddot{\eta}_R+(\alpha m_R+\beta k_R)\dot{\eta}_R+k_R\eta_R=0$, $R=1,\dots,m$, $m=m.\text{dofs}$

$m_R\ddot{\eta}_R+c_R\dot{\eta}_R+k_R\eta_R=0$, $R=1,\dots,m$ $\quad \omega_R=\sqrt{\dfrac{k_R}{m_R}}$; $\dfrac{c_R}{m_R}=2\zeta_R\omega_R \Rightarrow \zeta_R=\dfrac{c_R}{2m_R\omega_R}=\dfrac{(\alpha m_R+\beta k_R)}{2m_R\omega_R}$

IF USING UNIT MODAL MASS NORMALIZATION:

$[\phi]^T[M][\phi]\{\ddot{\eta}\}+[\phi]^T(\alpha[M]+\beta[K])[\phi]\{\dot{\eta}\}+[\phi]^T[K][\phi]\{\eta\}=[\phi]^T\{0\}=0$, $\{\phi_R\}=\dfrac{1}{\sqrt{m_R}}\{\varphi_R\}$

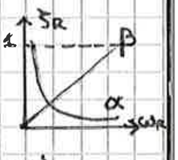
$[I]\{\ddot{\eta}\}+(\alpha[I]+\beta[\Lambda])\{\dot{\eta}\}+[\Lambda]\{\eta\}=0$

$[I]\{\ddot{\eta}\}+[\diagdown 2\zeta_R\omega_R \diagdown]\{\dot{\eta}\}+[\diagdown \omega_R^2 \diagdown]\{\eta\}=0$

$\ddot{\eta}_R+(\alpha+\beta\omega_R^2)\dot{\eta}_R+\omega_R^2\eta_R=0$, $R=1,\dots,m$

$\ddot{\eta}_R+2\zeta_R\omega_R\dot{\eta}_R+\omega_R^2\eta_R=0$, $R=1,\dots,m$ $\quad \omega_R=\sqrt{\dfrac{k_R}{m_R}}$, $\zeta_R=\dfrac{(\alpha+\beta\omega_R^2)}{2\omega_R}=\dfrac{\alpha}{2\omega_R}+\dfrac{\beta\omega_R}{2}$

(IPERBOLIC)(LINEAR)



FREE RESPONSE:

$\eta_R(t)=\eta_{R\,H.S.}(t)=e^{-\zeta_R\omega_R t}\cdot[A_R\cos(\omega_{dR}t)+B_R\sin(\omega_{dR}t)]$, $\omega_{dR}=\omega_R\sqrt{1-\zeta^2}$, $R=1,\dots,m$

$\dot{\eta}_R(t)=\dot{\eta}_{R\,H.S}(t)=-\zeta_R\omega_R e^{-\zeta_R\omega_R t}\cdot[A_R\cos(\omega_{dR}t)+B_R\sin(\omega_{dR}t)]+e^{-\zeta_R\omega_R t}\cdot[-\omega_{dR}A_R\sin(\omega_{dR}t)+\omega_{dR}B_R\cos(\omega_{dR}t)]$

IMPOSING THE **I.C.** [PREVIEW]:

$\begin{cases}A_R=\dfrac{\{\varphi_R\}^T[M]\{x_0\}}{m_R} \\ B_R=\dfrac{\{\varphi_R\}^T[M]\{\dot{x}_0\}}{m_R\omega_{dR}}+\dfrac{\zeta_R\omega_R A_R}{\omega_{dR}}=\dfrac{\{\varphi_R\}^T[M]}{m_R\omega_{dR}}(\{\dot{x}_0\}+\zeta_R\omega_R\{x_0\})\end{cases}$

SOLUTION [PREVIEW]:

$\{x\}=[\varphi]\{\eta\}=\sum_{R=1}^{m}\{\varphi_R\}\eta_R$

• FORCED RESPONSE (HARMONIC EXCITATION)

$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} = \{F_0\} e^{i\omega t}$

CONSIDERATION ABOUT THE PRE-MULTIPLICATION OF $[\varphi]^T$:

$\{\Gamma\} = [\varphi]^T\{F_0\} \rightarrow \{\Gamma\}e^{i\omega t} = [\varphi]^T\{F_0\}e^{i\omega t}$ | $\{x\} = [\varphi]\{\eta\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_R$

$\Gamma_R = \{\varphi_R\}^T\{F_0\} \rightarrow \Gamma_R e^{i\omega t} = \{\varphi_R\}^T\{F_0\}e^{i\omega t}$ | $\{x_0\} = [\varphi]\{\eta_0\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_{R,0}$

TWO STRATEGIES:

1) IF $[C] = [C_P] + [C_{NP}]$ -> INVERSION OF THE DYNAMIC STIFFNESS MATRIX

2) IF $[C] = [C_P]$ -> MODAL APPROACH IF $[\varphi]^T[C][\varphi]$ (OR $[\phi]^T[C][\phi]$) = DIAG => $[C] = [C_P]$

DETERMINED NOT CONSIDERING DAMP.

1) INVERSION OF THE DYNAMIC STIFFNESS MATRIX

$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} = \{F_0\}e^{i\omega t}$

$\{x\} = \{x_0\}e^{i\omega t}$; $\{\dot{x}\} = i\omega\{x_0\}e^{i\omega t}$; $\{\ddot{x}\} = -\omega^2\{x_0\}e^{i\omega t}$

$([K] - \omega^2[M] + i\omega[C])\{x_0\}\cancel{e^{i\omega t}} = \{F_0\}\cancel{e^{i\omega t}}$

$[K_{DYN}]\{x_0\} = \{F_0\}$

$\{x_0\} = [K_{DYN}]^{-1}\{F_0\} = [\alpha]\{F_0\}$; $[\alpha]$ = RECEPTANCE MATRIX $\quad [K_{DYN}]\overbrace{[K_{DYN}]^{-1}}^{=[\alpha]} \overset{*}{=} [I]$

$\alpha_{jj} = \frac{x_j}{F_j}$; $\alpha_{jk} = \frac{x_j}{F_k}$

*TO BE INVERTIBLE: $\det[K_{DYN}] \neq 0$ (70)

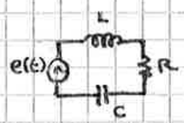
$\frac{x_j}{F_k}$: { $j \neq k$: CROSS-RECEPTANCE $\in \mathbb{C}$; $j = k$: AUTO-RECEPTANCE } RECEPTANCE [m/N] $\quad \alpha_{jj} = \frac{x_j}{F_j}$; $\alpha_{jk} = \frac{x_j}{F_k}$

$\frac{\dot{x}_j}{F_k}$: { $j \neq k$: CROSS-MOBILITY $\in \mathbb{C}$; $j = k$: AUTO-MOBILITY } MOBILITY [(m/s)/N] $\quad Y_{jj} = \frac{\dot{x}_j}{F_j} = i\omega\frac{x_j}{F_j} = i\omega\alpha_{jj}$

$\frac{\ddot{x}_j}{F_k}$: { $j \neq k$: CROSS-INERTANCE $\in \mathbb{C}$; $j = k$: AUTO-INERTANCE } INERTANCE [(m/s²)/N] $\quad A_{jj} = \frac{\ddot{x}_j}{F_j} = -\omega^2\frac{x_j}{F_j} = -\omega^2\alpha_{jj}$

SIMILITUDE WITH:

- m <-> L : INDUCTANCE
- c <-> R : RESISTOR
- k <-> $1/C$: CAPACITOR



($\Delta V = Ri$ <-> $F_c = c\dot{x}$)

1st VOLTAGE KIRCHOFF LAW:

$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{de(t)}{dt}$

NOT-SEEN DURING LESSONS

*INVERSE OF A MATRIX (GAUSS ALGORYTHM) OR (COFACTOR METHOD : PAG. 124 : FASTER!)

$[A] = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ IS INVERTIBLE SINCE $\det(A) \neq 0$ $\quad \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ $R_2 = R_2 - R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_3 = R_3 + \frac{1}{2}R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1/2 & -1/2 & 1/2 & 1 \end{array}\right]$ $\begin{matrix} R_2 = -(1/2)R_2 \\ R_3 = +(2)R_3 \end{matrix} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & +1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$

$R_2 = R_2 - \frac{R_3}{2} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$ $R_1 = R_1 - 2R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$

$[A]^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$ CHECK: $[A]\cdot[A]^{-1} = [I]$ $\quad \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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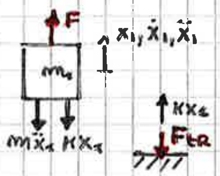
+ DYNAMIC ABSORBER (WE CAN DO THE SAME CONSIDERING ALSO C)

FBD:



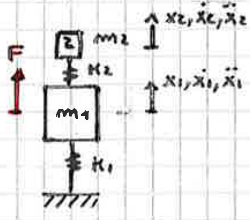
MOTION EQs:

$$\begin{cases} m\ddot{x}_1 + k_1 x_1 = F(t) \\ k_1 x_1 = F_{tR}(t) \end{cases} \rightarrow \begin{cases} (-m\omega^2 + k_1) x_{1,0} e^{i\omega t} = F_0 e^{i\omega t} \\ k_1 x_{1,0} e^{i\omega t} = F_{tR,0} e^{i\omega t} \end{cases}$$

$$\frac{F_{tR,0}}{F_0} = \frac{k_1 x_{1,0}}{(k_1 - m_1\omega^2)}$$

IN ORDER TO "REDUCE" (DELETE, PROPERLY) THE F_{tR} → TMD $\omega_m = \sqrt{\frac{k_2}{m_2}}$

[PREVIEW: WHEN $\omega = \omega_m = \omega_2$ (ANTI-R. ∮) ⇒ $x_{1,0} = 0$ ⇒ $F_{tR,0} = 0$]



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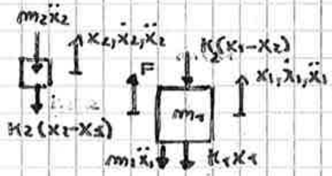


MOTION EQs:

$$\begin{cases} m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \\ m_1\ddot{x}_1 + k_1 x_1 + k_2 x_1 - k_2 x_2 = F \end{cases}$$

MATRIX FORM:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F\}$$

DUE TO THE FACT THERE IS NOT C WE CAN USE THE METHOD: INVERSE OF $[K_{DYN}]$:

$$\{x\} = \{x_0\} e^{i\omega t}$$

$$([K] - \omega^2[M])\{x_0\} e^{i\omega t} = \{F_0\} e^{i\omega t}$$

$$\{x_0\} = [K_{DYN}]^{-1}\{F_0\} = [\alpha]\{F_0\}$$

$$[K_{DYN}] = ([K] - \omega^2[M]) = \begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix}$$

INVERSE OF A MATRIX (COFACTOR METHOD) (FASTER)

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow [A]^{-1} = \frac{1}{\det[A]} \begin{bmatrix} \text{COF}(a_{11}) & \text{COF}(a_{12}) \\ \text{COF}(a_{21}) & \text{COF}(a_{22}) \end{bmatrix}^T \; ; \; \text{COF}(a_{jk}) = (-1)^{j+k} \cdot C_{jk} \; , \; C_{jk} = \det[A_{jk}]$$

A_{jk}: WE DELETE ROW j AND COLUMN k ; $A_{22} = a_{11}$, $C_{11} = \det(A_{22}) = a_{11}$

$$\det[K_{dyn}] = (k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - (-k_2)^2 = k_1 k_2 - \omega^2 k_1 m_2 + \cancel{k_2^2} - \omega^2 k_2 m_2 - \omega^2 m_1 k_2 + \omega^4 m_1 m_2 - \cancel{k_2^2}$$

$$k_1 = \omega_1^2 m_1 \; ; \; k_2 = \omega_2^2 m_2 \Rightarrow \det[K_{dyn}] = \omega_1^2\omega_2^2 m_1 m_2 - \omega^2\omega_1^2 m_1 m_2 - \omega^2\omega_2^2 m_2^2 - \omega^2\omega_2^2 m_1 m_2 + \ldots$$

$$\text{COF}(a_{11}) = (-1)^{1+1}(k_2 - \omega^2 m_2) = k_2 - \omega^2 m_2 = m_2(\omega_2^2 - \omega^2)$$

$$\text{COF}(a_{12}) = (-1)^{1+2}(-k_2) = k_2 = m_2\omega_2^2$$

$$\text{COF}(a_{21}) = (-1)^{2+1}(-k_2) = k_2 = m_2\omega_2^2$$

$$\text{COF}(a_{22}) = (-1)^{2+2}(k_1 + k_2 - \omega^2 m_1) = (m_1\omega_1^2 + m_2\omega_2^2 - \omega^2 m_1) = m_1(\omega_1^2 - \omega^2) + m_2\omega_2^2$$

$$[K_{DYN}]^{-1} = \frac{1}{\det[K_{dyn}]} \begin{bmatrix} m_2(\omega_2^2 - \omega^2) & m_2\omega_2^2 \\ m_2\omega_2^2 & m_1(\omega_1^2 - \omega^2) + m_2\omega_2^2 \end{bmatrix}$$

$$\{x_0\} = \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} = [K_{DYN}]^{-1}\{F_0\} = \frac{1}{\det[K_{DYN}]} \begin{bmatrix} m_2(\omega_2^2 - \omega^2) & m_2\omega_2^2 \\ m_2\omega_2^2 & m_1(\omega_1^2 - \omega^2) + m_2\omega_2^2 \end{bmatrix} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$x_{10} = \frac{m_2(\omega_2^2 - \omega^2) F_0}{\det[K_{DYN}]} \Rightarrow \text{WHEN } \omega = \omega_2 \sqrt{\frac{k_2}{m_2}} \Rightarrow x_{10} = 0 \Rightarrow F_{tR0} = 0 \; ! \; (\text{GREAT!})$$

• WAY 2 : WE CONSIDER $\eta_R = e^{-\zeta\omega_R t}\left[A_R\cos(\omega_{dR}t)+B_R\sin(\omega_{dR}t)\right] \rightarrow \eta_R = A_R\cos(\omega_R t)+B_R\sin(\omega_R t)$

$$\begin{Bmatrix}\theta_1\\\theta_2\end{Bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{Bmatrix}A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t)\\A_2\cos(\omega_2 t)+B_2\sin(\omega_2 t)\end{Bmatrix}$$

$$\theta_1 = A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t)+A_2\cos(\omega_2 t)+B_2\sin(\omega_2 t)$$

$$\theta_2 = A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t)-A_2\cos(\omega_2 t)-B_2\sin(\omega_2 t)$$

$$\dot\theta_1 = -\omega_1A_1\sin(\omega_1 t)+\omega_1B_1\cos(\omega_1 t)-\omega_2A_2\sin(\omega_2 t)+\omega_2B_2\cos(\omega_2 t)$$

$$\dot\theta_2 = -\omega_1A_1\sin(\omega_1 t)+\omega_1B_1\cos(\omega_1 t)+\omega_2A_2\sin(\omega_2 t)-\omega_2B_2\cos(\omega_2 t)$$

$$\text{I.C.}\begin{cases}\theta_1(t=0)=\theta_0=A_1+A_2\\ \theta_2(t=0)=0=A_1-A_2\\ \dot\theta_1(t=0)=0=\omega_1B_1+\omega_2B_2\\ \dot\theta_1(t=0)=0=\omega_1B_1-\omega_2B_2\end{cases}\Rightarrow\begin{cases}\theta_0=2A_1\\A_2=A_1\\ \omega_2B_2=-\omega_1B_1\\ \omega_2B_2=\omega_1B_1\end{cases}\begin{cases}A_1=\frac{\theta_0}{2}\\A_2=\frac{\theta_0}{2}\\ \cancel{\omega_2}\frac{\omega_1}{\cancel{\omega_2}}B_1=-\omega_1B_1\Rightarrow B_1=0\\ B_2=\frac{\omega_1}{\omega_2}B_1\Rightarrow B_2=0\end{cases}$$

$$\Rightarrow\begin{Bmatrix}\theta_1\\\theta_2\end{Bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{Bmatrix}(\theta_0/2)\cos(\omega_1 t)\\(\theta_0/2)\cos(\omega_2 t)\end{Bmatrix}\Rightarrow\begin{cases}\theta_1=(\theta_0/2)\cos(\omega_1 t)+(\theta_0/2)\cos(\omega_2 t)\\\theta_2=(\theta_0/2)\cos(\omega_1 t)-(\theta_0/2)\cos(\omega_2 t)\end{cases}$$ OK ✓

• WAY 3 : 1^{st} GENERAL APPROACH THAT WE HAVE SEEN :

$$\{x\}=[\varphi]\{\eta\}=\sum_{R=1}^{m}\{\varphi_R\}\eta_R=\sum_{R=1}^{m}\{\varphi_R\}e^{-\zeta\omega_R t}\left[A_R\cos(\omega_{dR}t)+B_R\sin(\omega_{dR}t)\right]$$

$$\{\dot x\}=[\varphi]\{\dot\eta\}=\sum_{R=1}^{m}\{\varphi_R\}\dot\eta_R=\sum_{R=1}^{m}\{\varphi_R\}\{-\zeta\omega_R e^{-\zeta\omega_R t}\left[A_R\cos(\omega_{dR}t)+B_R\sin(\omega_{dR}t)\right]+e^{-\zeta\omega_R t}\left[-\omega_{dR}A_R\sin(\omega_{dR}t)+\omega_{dR}\dots\right.$$

$$\begin{cases}\{x(t=0)\}=\{x_0\}=\sum_{R=1}^{m}\{\varphi_R\}A_R\\ \{\dot x(t=0)\}=\{\dot x_0\}=\sum_{R=1}^{m}\{\varphi_R\}\omega_R B_R\end{cases}$$

PRE-MULTIPLYING BY $\{\varphi_S\}^T[M]$:

$$\begin{cases}\{\varphi_S\}^T[M]\{x_0\}=\sum_{R=1}^{m}\{\varphi_S\}^T[M]\{\varphi_R\}A_R=m_SA_S\\ \{\varphi_S\}^T[M]\{\dot x_0\}=\sum_{R=1}^{m}\{\varphi_S\}^T[M]\{\varphi_R\}\omega_RB_R=m_S\omega_SB_S\end{cases}$$

NW! TO COMPUTE m_R : $m_1^!, m_2^!$ ← DIAG

$$m_R=\{\varphi_R\}^T[M]\{\varphi_R\}$$

$$m_1^!=\{1\ 1\}\begin{bmatrix}me^2&0\\0&me^2\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}=\{me^2\ me^2\}\begin{Bmatrix}1\\1\end{Bmatrix}=2me^2$$

$$m_2^!=\{1\ -1\}\begin{bmatrix}me^2&0\\0&me^2\end{bmatrix}\begin{Bmatrix}1\\-1\end{Bmatrix}=\{me^2\ -me^2\}\begin{Bmatrix}1\\-1\end{Bmatrix}=2me^2$$

$$\begin{cases}A_R=\dfrac{\{\varphi_R\}^T[M]\{x_0\}}{m_R}\\ B_R=\dfrac{\{\varphi_R\}^T[M]\{\dot x_0\}}{m_R\omega_R}\end{cases}\Rightarrow\begin{cases}A_1=\dfrac{1}{2me^2}\{1\ 1\}\begin{bmatrix}me^2&0\\0&me^2\end{bmatrix}\begin{Bmatrix}\theta_0\\0\end{Bmatrix}=\dfrac{1}{2me^2}\{me^2\ me^2\}\begin{Bmatrix}\theta_0\\0\end{Bmatrix}=\dfrac{1}{2me^2}me^2\theta_0=\dfrac{\theta_0}{2}\\ B_1=\dfrac{1}{2me^2\omega_1}\{1\ 1\}\begin{bmatrix}me^2&0\\0&me^2\end{bmatrix}\begin{Bmatrix}0\\0\end{Bmatrix}=\dfrac{1}{2me^2\omega_1}\{me^2\ me^2\}\begin{Bmatrix}0\\0\end{Bmatrix}=0\end{cases}$$

$$\begin{cases}A_2=\dfrac{1}{2me^2}\{1\ -1\}\begin{bmatrix}me^2&0\\0&me^2\end{bmatrix}\begin{Bmatrix}\theta_0\\0\end{Bmatrix}=\dfrac{1}{2me^2}\{me^2\ -me^2\}\begin{Bmatrix}\theta_0\\0\end{Bmatrix}=\dfrac{1}{2me^2}me^2\theta_0=\dfrac{\theta_0}{2}\\ B_2=0\end{cases}$$

$$\eta_1=A_1\cos(\omega_1 t)=\theta_0\cos(\omega_1 t)$$

$$\eta_2=A_2\cos(\omega_2 t)=\theta_0\cos(\omega_2 t)$$

$$\Rightarrow\begin{Bmatrix}\theta_1\\\theta_2\end{Bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{Bmatrix}(\theta_0/2)\cos(\omega_1 t)\\(\theta_0/2)\cos(\omega_2 t)\end{Bmatrix}\Rightarrow\begin{cases}\theta_1=(\theta_0/2)\cos(\omega_1 t)+(\theta_0/2)\cos(\omega_2 t)\\\theta_2=(\theta_0/2)\cos(\omega_1 t)-(\theta_0/2)\cos(\omega_2 t)\end{cases}$$

IN ANYCASE, THE STARTING POINT FOR THE NEXT STEPS IS:

$$\begin{Bmatrix}\theta_1\\\theta_2\end{Bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{Bmatrix}(\theta_0/2)\cos(\omega_1 t)\\(\theta_0/2)\cos(\omega_2 t)\end{Bmatrix}\Rightarrow\begin{cases}\theta_1=(\theta_0/2)\cos(\omega_1 t)+(\theta_0/2)\cos(\omega_2 t)\\\theta_2=(\theta_0/2)\cos(\omega_1 t)-(\theta_0/2)\cos(\omega_2 t)\end{cases}$$

+ LAGRANGE'S EQUATIONS

$T = \frac{1}{2}\{\dot{x}\}[M]\{\dot{x}\}$: KINETIC E. >0 $\quad \left(T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I_0 \dot{\theta}^2 > 0 \right)$

$V = \frac{1}{2}\{x\}[K]\{x\}$: POTENTIAL E. ≥ 0 $\quad \left(V = \frac{1}{2} kx^2 + \frac{1}{2} k_t \theta^2 \geq 0 \right)$

$D = \frac{1}{2}\{\dot{x}\}[C]\{\dot{x}\}$: DISSIPATION ≥ 0

$L = T - V$: LAGRANGE SCALAR

$L = L(q_i, \dot{q}_i)$, q_i : LAGRANGE COORD.

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$: LAGRANGE EQ. , $k = 1, \dots, m$ (m dofs) , Q_k : GENERALIZED FORCE

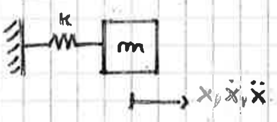
EXTERNAL FORCES :

$$Q_k = \sum_{i=1}^{m} \vec{F}_i \cdot \frac{\partial \dot{\vec{R}}_i}{\partial \dot{q}_k} = \vec{F}_i \cdot \frac{\partial \vec{R}_i}{\partial q_k}$$

DISSIPATIVE FORCES:

$$Q_k = -\frac{\partial D}{\partial \dot{q}_k} \quad , \quad D = \frac{1}{2} \sum_{i,k=1}^{m} c_{ik} \dot{q}_k \dot{q}_j$$: RAYLEIGHT DISSIPATIVE FUNCT.

EXAMPLE 1



$$\begin{cases} T = \frac{1}{2} m\dot{x}^2 \\ V = \frac{1}{2} kx^2 \\ D = 0, F = 0 \Rightarrow Q = 0 \end{cases}$$

$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

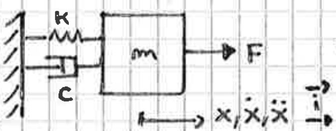
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k = 0 \quad (k = 1, q_k = x)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$$\begin{cases} \frac{\partial L}{\partial x} = -kx \\ \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \end{cases}$$

$m\ddot{x} + kx = 0$

EXAMPLE 2



$$\begin{cases} T = \frac{1}{2} m\dot{x}^2 \quad D = \frac{1}{2} c\dot{x}^2 \\ V = \frac{1}{2} kx^2 \\ Q_F = \vec{F} \cdot \frac{\partial \vec{R}}{\partial x} = F \cdot \vec{i} \cdot \underbrace{\frac{\partial \vec{x}}{\partial x}}_{=1} = F \\ Q_c = -\frac{\partial D}{\partial \dot{q}_k} = -\frac{\partial D}{\partial \dot{x}} = -c\dot{x} \end{cases}$$

$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \quad (k = 1, q_k = x)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q = F - c\dot{x}$

$$\begin{cases} \frac{\partial L}{\partial x} = -kx \\ \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \end{cases}$$

$m\ddot{x} + kx = F - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = F$

PERSONAL CONSIDERATION (NOT 100% SURE):

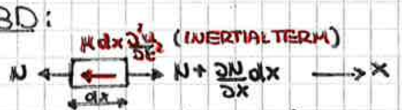
$$Q_k = \sum_{i=1}^{m} \vec{F}_i \cdot \frac{\partial \dot{\vec{R}}_i}{\partial \dot{q}_k} = \sum_{i=1}^{m} \vec{F}_i \cdot \frac{\partial \vec{R}_i}{\partial q_k}$$

$$Q_k = \vec{F} \cdot \frac{\partial \vec{R}}{\partial x} = F \cdot \vec{i} \cdot \frac{\partial \vec{x}}{\partial x} = F \cdot \vec{i} \cdot \frac{\partial (x\vec{i})}{\partial x} = F \cdot \vec{i} \cdot \left(\frac{\partial x}{\partial x} \vec{i} + \underbrace{\frac{\partial \vec{i}}{\partial x}}_{=0} x \right) = F \cdot \underbrace{\vec{i} \cdot \vec{i}}_{=1}$$

3.2 ROD AXIAL VIBRATIONS

$u = u(x,t)$

FBD:



$\rightarrow)\ \cancel{N} + \frac{\partial N}{\partial x}dx - \cancel{N} - \mu\, dx \frac{\partial^2 u}{\partial t^2} = 0$

$$\frac{\partial N}{\partial x} = \mu \frac{\partial^2 u}{\partial t^2}$$

FROM BEAM THEORY: $\sigma = N/A = E\varepsilon \Rightarrow N = EA\, \partial u/\partial x$

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right] = \mu \frac{\partial^2 u}{\partial t^2}$$: PDE OF THE ROD

IF AE = CONST:

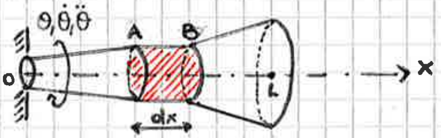
$$EA\frac{\partial^2 u}{\partial x^2} = \mu \frac{\partial^2 u}{\partial t^2}$$: WAVE EQ.

$$\frac{\partial^2 u}{\partial t^2} = \frac{EA}{\mu}\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c = \sqrt{\frac{EA}{\mu}}\ [m/s]$$

- BOUNDARY CONDITIONS (B.C.):

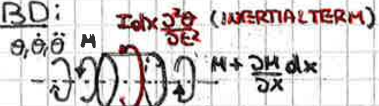
$$\begin{cases} y(x=0,t) = 0 \\ N(x=L,t) = EA\dfrac{\partial u}{\partial x} = 0 \end{cases}$$

3.3 SHAFT TORSIONAL VIBRATIONS



$$\begin{cases} A \rightarrow \theta \\ B \rightarrow \theta + \dfrac{\partial \theta}{\partial x}dx \end{cases} \qquad \theta = \theta(x,t)$$

FBD:



$\circlearrowleft)\ \cancel{M} + \frac{\partial M}{\partial x}dx - \cancel{M} - I\, dx \frac{\partial^2 \theta}{\partial t^2} = 0$

$$\frac{\partial M}{\partial x} = I \frac{\partial^2 \theta}{\partial t^2}$$

FOR CIRCULAR SECTION: $M = GJ\, \partial\theta/\partial x$

$$\frac{\partial}{\partial x}\left[GJ\frac{\partial \theta}{\partial x}\right] = I \frac{\partial^2 \theta}{\partial t^2}$$: PDE OF THE SHAFT

IF GJ = CONST:

$$GJ\frac{\partial^2 \theta}{\partial x^2} = I \frac{\partial^2 \theta}{\partial t^2}$$: WAVE EQ.

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{GJ}{I}\frac{\partial^2 \theta}{\partial x^2} = c^2 \frac{\partial^2 \theta}{\partial x^2}, \quad c = \sqrt{\frac{GJ}{I}}\ [m/s]$$

- BOUNDARY CONDITIONS (B.C.):

$$\begin{cases} \theta(x=0,t) = 0 \\ M(x=L,t) = GJ\left.\dfrac{\partial \theta}{\partial x}\right|_{x=L} = 0 \end{cases}$$

WAVE EQs SOLUTIONs (TRANSVERSAL / AXIAL / TORSIONAL) II ORDER PDEs (2 B.C.)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad c = \sqrt{\frac{T}{\rho}} \quad ; \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c = \sqrt{\frac{AE}{\mu}} \quad ; \quad \frac{\partial^2 \theta}{\partial t^2} = c^2 \frac{\partial^2 \theta}{\partial x^2}, \quad c = \sqrt{\frac{GJ}{I}}$$

SOLUTION FORM:

$$y(x,t) = \Phi(x)\eta(t)$$

$$\Phi(x)\frac{\partial^2 \eta}{\partial t^2} = c^2 \eta(t) \frac{\partial^2 \Phi}{\partial x^2}$$

$$\Phi \ddot{\eta} = c^2 \eta \Phi^{II} \longleftrightarrow \Phi^{II} - \frac{\ddot{\eta}}{\eta}\frac{1}{c^2}\Phi = 0 \longleftrightarrow \Phi^{II} + \frac{\omega^2}{c^2}\Phi = 0, \quad \beta = \frac{\omega}{c}, \quad \frac{\ddot{\eta}(t)}{\eta(t)} = -\omega^2$$

$$\Phi^{II} + \beta^2 \Phi = 0, \quad \beta = \frac{\omega}{c}$$

$\Rightarrow$ SYSTEM OF 2 ODEs:

$$\begin{cases} \ddot{\eta} + \omega^2 \eta = 0 \\ \Phi^{II} + \beta^2 \Phi = 0 \end{cases} \quad : \text{HARMONIC MOTIONS}$$

$$\begin{cases} \eta(t) = A\cos(\omega t) + B\sin(\omega t) \\ \Phi(x) = C\cos(\beta x) + D\sin(\beta x) \end{cases} \quad : \text{EIGENFUNCTIONS} \quad \begin{cases} \text{I.C.} \rightarrow A, B \\ \text{B.C.} \rightarrow C, D \end{cases}$$

SOLUTION:

$$y(x,t) = \Phi(x)\eta(t) = \sum_{k=1}^{\infty} \Phi_k \cdot \eta_k = \sum_{k=1}^{\infty} [A_k \cos(\omega_k t) + B_k \sin(\omega_k t)][C_k \cos(\beta_k x) + D_k \sin(\beta_k x)]$$

WAVE EQ. SOLUTION (BEAM FLEXURAL OSCILLATIONS) IV ORDER PDE (4 B.C.)

$$\frac{\partial^2 y}{\partial t^2} = -c^2 \frac{\partial^4 y}{\partial x^4}, \quad c = \sqrt{\frac{EJ}{\mu}}$$

SOLUTION FORM:

$$y(x,t) = \Phi(x)\eta(t)$$

$$\Phi(x)\frac{\partial^2 \eta}{\partial t^2} = -c^2 \eta(t) \frac{\partial^4 \Phi}{\partial x^4}$$

$$\Phi \ddot{\eta} = -c^2 \eta \Phi^{IV} \longleftrightarrow \Phi^{IV} + \frac{\ddot{\eta}}{\eta}\frac{1}{c^2}\Phi = 0 \longleftrightarrow \Phi^{IV} - \frac{\omega^2}{c^2}\Phi = 0, \quad \beta^2 = \frac{\omega}{c}, \quad \frac{\ddot{\eta}(t)}{\eta(t)} = -\omega^2$$

$$\Phi^{IV} - \beta^4 \Phi = 0, \quad \beta = \sqrt{\frac{\omega}{c}} \longleftrightarrow \beta^2 = \frac{\omega}{c}$$

$\Rightarrow$ SYSTEM OF 2 ODEs:

$$\begin{cases} \ddot{\eta} + \omega^2 \eta = 0 \\ \Phi^{IV} - \beta^4 \Phi = 0 \end{cases}$$

$$\begin{cases} \eta(t) = A\cos(\omega t) + B\sin(\omega t) \\ \Phi(x) = C\cos(\beta x) + D\sin(\beta x) + E\cosh(\beta x) + F\sinh(\beta x) \end{cases}$$

SOLUTION:

$$y(x,t) = \Phi(x)\eta(t) = \sum_{k=1}^{\infty} \Phi_k \cdot \eta_k$$

EXAMPLE : BEAM (PINNED-PINNED) FLEXURAL OSCILLATIONS

$$\frac{\partial^2 y}{\partial t^2} = -\frac{EJ}{\mu}\frac{\partial^4 y}{\partial x^4} = -c^2\frac{\partial^4 y}{\partial x^2}$$

$$y(x,t) = \Phi(x)\eta(t)$$

$$\Phi(x)\frac{\partial^2\eta}{\partial t^2} = -c^2\eta(t)\frac{\partial^4\Phi}{\partial x^4}$$

$$\Phi\ddot{\eta} = -c^2\eta\Phi^{IV} \quad\leftrightarrow\quad \Phi^{IV} + \frac{\ddot{\eta}}{\eta}\frac{1}{c^2}\Phi = 0 \quad\leftrightarrow\quad \Phi^{IV} - \frac{\omega^2}{c^2}\Phi = 0 \quad , \quad \beta^2 = \frac{\omega}{c}$$

$$\Phi^{IV} - \beta^4\Phi = 0$$

$$\begin{cases}\ddot{\eta} + \omega^2\eta = 0 \\ \Phi^{IV} - \beta^4\Phi = 0\end{cases} \quad : \text{HARMONIC FUNCTIONS}$$

$$\begin{cases}\eta(t) = A\cos(\omega t) + B\sin(\omega t) \\ \Phi(x) = C\cos(\beta x) + D\sin(\beta x) + E\cosh(\beta x) + F\sinh(\beta x)\end{cases} \quad : \text{EIGENFUNCTIONS}$$

$$y(x,t) = \Phi(x)\eta(t) = \sum_{R=1}^{\infty}\Phi_R(x)\eta_R(t)$$

IMPOSING THE B.C. (PINNED-PINNED):

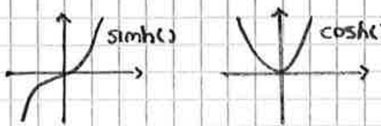
$$L\begin{cases}y(x=0,t) = 0 = \Phi(0)\eta(t) \\ M(x=0,t) = EJ\frac{\partial^2 y}{\partial x^2} = 0\end{cases} \Rightarrow \begin{cases}\Phi(0) = 0 \\ \Phi''(0) = 0\end{cases}$$

$$R\begin{cases}y(x=L,t) = 0 = \Phi(L)\eta(t) \\ M(x=L,t) = EJ\frac{\partial^2 y}{\partial x^2} = 0\end{cases} \Rightarrow \begin{cases}\Phi(L) = 0 \\ \Phi''(L) = 0\end{cases}$$

$$\Phi'(x) = \beta[-C\sin(\beta x) + D\cos(\beta x) + E\sinh(\beta x) + F\cosh(\beta x)]$$

$$\Phi''(x) = \beta^2[-C\cos(\beta x) - D\sin(\beta x) + E\cosh(\beta x) + F\sinh(\beta x)]$$

$$\Phi'''(x) = \beta^3[C\sin(\beta x) - D\cos(\beta x) + E\sinh(\beta x) + F\cosh(\beta x)]$$



$$L\begin{cases}\Phi(0) = 0 = C + E \\ \Phi''(0) = 0 = -\beta^2 C + \beta^2 E\end{cases} \Rightarrow \begin{cases}C = -E \\ C = E\end{cases} \Rightarrow C = E = 0$$

$$R\begin{cases}\Phi(L) = 0 = D\sin(\beta L) + F\sinh(\beta L) \\ \Phi''(L) = 0 = \beta^2[-D\sin(\beta L) + F\sinh(\beta L)]\end{cases}$$

$$\begin{bmatrix}\sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & \beta^2\sinh(\beta L)\end{bmatrix}\begin{Bmatrix}D \\ F\end{Bmatrix} = \begin{Bmatrix}0 \\ 0\end{Bmatrix} , \quad \begin{Bmatrix}D \\ F\end{Bmatrix} = \begin{Bmatrix}0 \\ 0\end{Bmatrix} \ \text{TRIV. SOL.}$$

$$\det[\ ] = \beta^2\sin(\beta L)\sinh(\beta L) + \beta^2\sin(\beta L)\sinh(\beta L) = 0$$

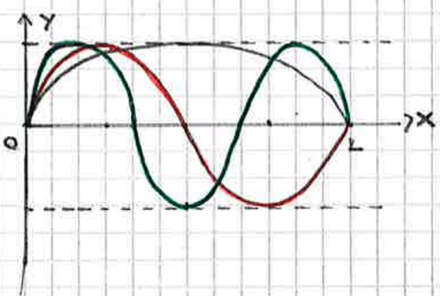
$$2\beta^2\sin(\beta L)\sinh(\beta L) = 0 \quad : \text{CHARACT. EQ.}$$

$$\text{IF } \beta \neq 0 \quad \sinh(\beta L) \neq 0 \Rightarrow \sin(\beta L) = 0 \Rightarrow \beta L = \pi \Rightarrow \beta_R L = R\pi \Rightarrow \sqrt{\frac{\omega_R}{c_R}} = R\frac{\pi}{L}$$

$$\omega_R = R^2\frac{\pi^2}{L^2}c_R = \frac{R^2\pi^2}{L^2}\sqrt{\frac{EJ}{\mu}} = R^2\pi^2\sqrt{\frac{EJ}{\mu L^4}}$$

$$\begin{cases}\omega_1 = \pi^2\sqrt{\frac{EJ}{\mu L^4}} & : \text{FUNDAMENTAL FREQ.} \\ \omega_2 < \omega_3 < \ldots < \omega_\infty & : \text{OVERTONES}\end{cases}$$

$$\Phi_R(x) = D_R\sin(\beta_R x) + F\sinh(\beta_R x) = D_R\sin\left(\frac{R\pi}{L}x\right) + F\sinh\left(\frac{R\pi}{L}x\right) \ ; \ hp: F = 0 \ ; \ D_R = 1, \forall R$$



$R = 1 \quad \omega_1 = \pi^2\sqrt{\frac{EJ}{\mu L^4}}$ — 0 STRUCTURAL NODES

$R = 2 \quad \omega_2 = 4\pi^2\sqrt{\frac{EJ}{\mu L^4}}$ — 1 STRUCTURAL NODE

$R = 3 \quad \omega_2 = 9\pi^2\sqrt{\frac{EJ}{\mu L^4}}$ — 2 STURCTURAL NODES

UNIFIED APPROACH

- CASE $f(x,t)=0$

$\mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0$: GENERAL FORM (FOR PDE)

$B_i[y] = 0$: B.C.

3.1 STRING TRANSVERSAL VIBRATIONS

$\rho \frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0 \quad \begin{cases} y = y \\ \mathcal{M} = \rho \end{cases} \quad K = -T\frac{\partial^2}{\partial x^2}(\cdot)$

3.2 ROD AXIAL VIBRATIONS

$\mu \frac{\partial^2 u}{\partial t^2} - AE\frac{\partial^2 u}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0 \quad \begin{cases} y = u \\ \mathcal{M} = \mu \end{cases} \quad K = -AE\frac{\partial^2}{\partial x^2}(\cdot)$

3.3 SHAFT TORSIONAL VIBRATIONS

$I \frac{\partial^2 \theta}{\partial t^2} - GJ\frac{\partial^2 \theta}{\partial x^2} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0 \quad \begin{cases} y = \theta \\ \mathcal{M} = I \end{cases} \quad K = -GJ\frac{\partial^2}{\partial x^2}(\cdot)$

3.4 BEAM FLEXURAL VIBRATIONS

$\mu \frac{\partial^2 y}{\partial t^2} + EJ\frac{\partial^4 y}{\partial x^4} = 0 \rightarrow \mathcal{M}\left[\frac{\partial^2 y}{\partial t^2}\right] + K[y] = 0 \quad \begin{cases} y = y \\ \mathcal{M} = \mu \end{cases} \quad K = +EJ\frac{\partial^4}{\partial x^4}(\cdot)$

SUBSTITUTING $y(x,t) = \Phi(x)\eta(t)$:

$\mathcal{M}[\Phi]\ddot{\eta} + K[\Phi]\eta = 0$

$\frac{\ddot{\eta}}{\eta} = -\frac{K[\Phi]}{\mathcal{M}[\Phi]} = -\omega^2 \leq 0$

DIFFERENTIAL EIGENPROBLEM

$K[\Phi] = \omega^2 \mathcal{M}[\Phi]$

SELF-ADJOINT PROPERTY -> $\mathcal{M}$-K ORTHOGONALITY PROPERTY

- ORTHOGONALITY PROPERTY WITH $\mathcal{M}$ AND K.
- GENERIC OPERATOR (L), TWO FUNCTIONS (u,v) THAT SATISFY B.C. ON DOMAIN (D).

$(u, L[v]) = \int_D u\, L[v]\, dD$

$\Rightarrow$ L IS SELF ADJOINT IF $(u, L[v]) = (v, L[u])$

- WE TAKE $(u,v) = (\phi_i, \phi_j)$
- WE DEMONSTRATE THAT $\mathcal{M}$ AND K ARE SELF-ADJOINT (LOOK THE EXAMPLE)
- TWO EIGENFUNCTIONS (ϕ_i, ϕ_j) <-> EIGENVALUES (ω_i, ω_j)

$\begin{cases} K[\phi_i] = \omega_i^2 \mathcal{M}[\phi_i] \\ K[\phi_j] = \omega_j^2 \mathcal{M}[\phi_j] \end{cases} \qquad \begin{cases} \mathcal{M}[\phi_i] = \frac{1}{\omega_i^2} K[\phi_i] \\ \mathcal{M}[\phi_j] = \frac{1}{\omega_j^2} K[\phi_j] \end{cases}$

$\int_D \phi_j K[\phi_i]\, dD = \int_D \omega_i^2 \mathcal{M}[\phi_i]\, dD \qquad \int_D \phi_j \mathcal{M}[\phi_i]\, dD = \int_D \frac{1}{\omega_i^2} K[\phi_i]\, dD$

$\int_D \phi_i K[\phi_j]\, dD = \int_D \omega_j^2 \mathcal{M}[\phi_j]\, dD \qquad \int_D \phi_i \mathcal{M}[\phi_j]\, dD = \int_D \frac{1}{\omega_j^2} K[\phi_j]\, dD$

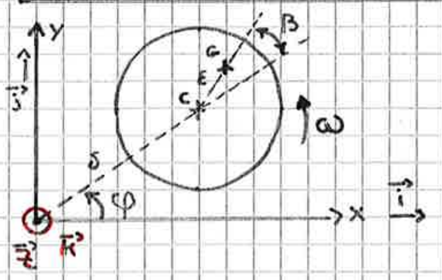
$(\omega_i^2 - \omega_j^2)\int_D \phi_i \mathcal{M}[\phi_j]\, dD = 0 \qquad \left(\frac{1}{\omega_i^2} - \frac{1}{\omega_j^2}\right)\int_D \phi_i K[\phi_j]\, dD$

$\begin{cases} \text{IF } \omega_i \neq \omega_j \Rightarrow \int_D \phi_i \mathcal{M}[\phi_j]\, dD = 0 : \mathcal{M}\text{-ORTHOGONALITY} \\ \text{IF } \omega_i = \omega_j \Rightarrow \int_D \phi_i \mathcal{M}[\phi_i]\, dD = m_i : \text{MODAL MASS} \end{cases} \qquad \begin{cases} \Rightarrow \int_D \phi_i K[\phi_j]\, dD = 0 : K\text{-ORTHOGONALITY} \\ \Rightarrow \int_D \phi_i K[\phi_i]\, dD = K_i : \text{MODAL STIFFNESS} \end{cases}$

4. ROTORDYNAMICS

JEFFCOTT MODEL (RIGID BEARINGS - FLEXIBLE SHAFT)

LET'S CONSIDER THE DISK SECTION:



$\varepsilon = \overline{CG}$: ECCENTRICITY

$\delta = \overline{OC}$: SHAFT DEFLECT. (ORBIT)

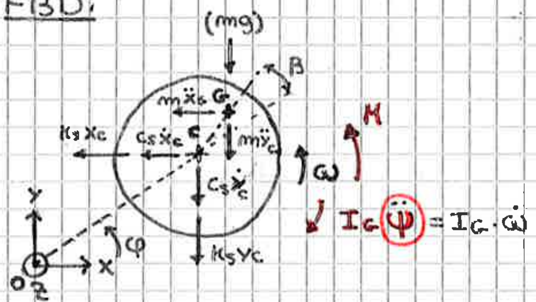
φ: WHIRL ANGLE ; β = PHASE ANGLE

$\omega = \frac{\psi}{t} = \frac{\varphi}{t} + \frac{\beta}{t} = \dot{\varphi} + \dot{\beta} = \dot{\psi}$

$\vec{i}, \vec{j}, \vec{k}$: FIXED REF. FRAME

$\delta = x_C\vec{i} + y_C\vec{j} = \delta_0 e^{i\omega t} = \delta_0 e^{i\varphi} = x_C + i y_C = z_C$

FBD:



MOTION EQS:

$$\begin{cases} \downarrow)\ m\ddot{y}_G + C_s\dot{y}_C + K_s y_C + mg = 0 \\ \leftarrow)\ m\ddot{x}_G + C_s\dot{x}_C + K_s x_C = 0 \\ \circlearrowleft G)\ M - I_G\ddot{\psi} + \varepsilon\cos\psi\,(C_s\dot{y}_C + K_s y_C) - \varepsilon\sin\psi\,(C_s\dot{x}_C + K_s x_C) \end{cases}$$

FROM GEOMETRY:

$$\begin{cases} x_G = x_C + \varepsilon\cos\psi \\ y_G = y_C + \varepsilon\sin\psi \end{cases} \quad \begin{cases} \dot{x}_G = \dot{x}_C - \dot{\psi}\varepsilon\sin\psi \\ \dot{y}_G = \dot{y}_C + \dot{\psi}\varepsilon\cos\psi \end{cases} \quad \begin{cases} \ddot{x}_G = \ddot{x}_C - \dot{\psi}^2\varepsilon\cos\psi - \ddot{\psi}\varepsilon\sin\psi \\ \ddot{y}_G = \ddot{y}_C - \dot{\psi}^2\varepsilon\sin\psi + \ddot{\psi}\varepsilon\cos\psi \end{cases}$$

THEREFORE:

$$\begin{cases} m\ddot{x}_C + C_s\dot{x}_C + K_s x_C = m\varepsilon\,(\dot{\psi}^2\cos\psi + \ddot{\psi}\sin\psi) \\ m\ddot{y}_C + C_s\dot{y}_C + K_s y_C = m\varepsilon\,(\dot{\psi}^2\sin\psi - \ddot{\psi}\cos\psi) - mg \end{cases}$$

CONSIDERING (hps):

- STEADY STATE MOTION $\begin{cases} \dot{\omega} = \ddot{\psi} = 0 \\ \omega = \dot{\psi} = \text{CONST} \end{cases}$
- VERTICAL ROTOR (mg=0)
- COMPLEX VARIABLE : $z_C = x_C + i y_C$

SYNCHR (FORWARD) WHIRL

$\omega = \text{CONST} \begin{cases} \delta = \text{CONST} \Rightarrow \text{CIRCLE} \\ \beta = \text{CONST} \Rightarrow \dot{\beta} = 0 \end{cases} \Rightarrow \omega = \dot{\psi} = \dot{\varphi} \Rightarrow \ddot{\psi} = 0$

$m(\ddot{x}_C + i\ddot{y}_C) + C_s(\dot{x}_C + i\dot{y}_C) + K_s(x_C + iy_C) = m\varepsilon\,[\dot{\psi}^2(\cos\psi + i\sin\psi)] = m\varepsilon\omega^2 e^{i\omega t}$

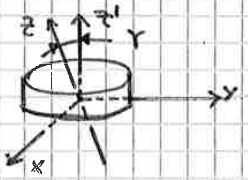
$m\ddot{z}_C + C_s\dot{z}_C + K_s z_C = m\varepsilon\omega^2 e^{i\omega t}$, $z(t) = z_0 e^{i\omega t}$

$\omega_{CR}^2\left[1 - \left(\frac{\omega}{\omega_{CR}}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_{CR}}\right)\right] z_0 \cancel{e^{i\omega t}} = \varepsilon\omega^2 \cancel{e^{i\omega t}}$, $\omega_{CR} = \sqrt{\frac{K_s}{m}}$ (ONLY ONE)

$\frac{z_0}{\varepsilon} = \left(\frac{\omega}{\omega_{CR}}\right)^2 G(i\omega)$ $\left\langle \begin{array}{l} \left|\frac{z_0}{\varepsilon}\right| = \frac{|\delta|}{\varepsilon} = \left(\frac{\omega}{\omega_{CR}}\right)^2 |G(i\omega)| \\ \varphi_+ = \text{Tg}^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_{CR}}\right)}{1 - (\omega/\omega_{CR})^2}\right] \end{array} \right.$

F.R.F. $\omega_{MAX} = \omega_{CR}\frac{1}{\sqrt{1 - 2\zeta^2}}$

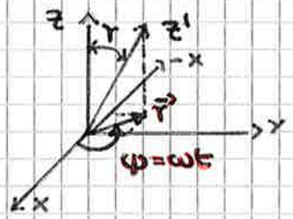
DYNAMIC UMBALANCE



- z' : PRINCIPAL AXIS OF INERTIA
- γ : DYNAMIC UMBALANCE ($\gamma = \text{const}$)
- $\varepsilon = 0$ ($C \equiv G$) – COMPLEX ANGLE $\theta = \theta_x + i\theta_y$

$\gamma \Rightarrow$ GYROSCOPIC MOMENT (M_g) $\Rightarrow$ SHAFT ANGULAR DEFLECTION ($\theta_{x,y}$) $\Rightarrow$ SHAFT ELASTIC TORQUE (M_e)

EQUILIBRIUM BETWEEN M_g AND M_e.

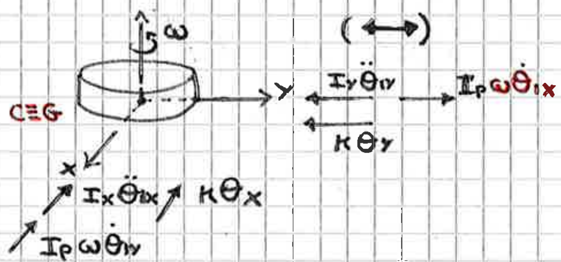


- $\vec{\omega} = \omega \vec{k}$
- $\vec{\gamma} \perp zz'$ PLANE, PROJECTED ON $\overline{XY}$ PLANE
- $\begin{cases} \gamma_x = -\gamma \sin(\omega t) \\ \gamma_y = \gamma \cos(\omega t) \end{cases}$

SHAFT UNDERGOES A TOTAL ANGULAR DEFORMATION DUE TO γ AND ELASTIC R. θ

$$\begin{cases} \theta_{1x} = \theta_x + \gamma_x = \theta_x - \gamma \sin(\omega t) \\ \theta_{1y} = \theta_y + \gamma_y = \theta_y + \gamma \cos(\omega t) \end{cases} \quad \begin{cases} \dot\theta_{1x} = \dot\theta_x - \gamma\omega\cos(\omega t) \\ \dot\theta_{1y} = \dot\theta_y - \gamma\omega\sin(\omega t) \end{cases} \quad \begin{cases} \ddot\theta_{1x} = \ddot\theta_x + \gamma\omega^2\sin(\omega t) \\ \ddot\theta_{1y} = \ddot\theta_y - \gamma\omega^2\cos(\omega t) \end{cases}$$

FBD:



INERTIAL TORQUE (M') $\begin{cases} x: -I_x\ddot\theta_{1x} \\ y: -I_y\ddot\theta_{1y} \end{cases}$

GYROSCOPIC TORQUE (M_g) $\begin{cases} x: -I_p\omega\dot\theta_{1y} \\ y: +I_p\omega\dot\theta_{1x} \end{cases}$

ELASTIC TORQUE (M_e) $\begin{cases} -k\theta_x \\ -k\theta_y \end{cases}$

MOTION EQS:

$$\begin{cases} I_x\ddot\theta_{1x} + I_p\omega\dot\theta_{1y} + k\theta_x = 0 \\ I_y\ddot\theta_{1y} - I_p\omega\dot\theta_{1x} + k\theta_y = 0 \end{cases}$$

CONSIDERING:

$\begin{cases} I_d = I_x = I_y : \text{AXIAL DISK } I \\ I_p = I_z : \text{POLAR MOM. OF INERTIA} \end{cases}$

$$\begin{cases} I_d\ddot\theta_x + I_p\omega\dot\theta_y + k\theta_x = -I_d\gamma\omega^2\sin(\omega t) + I_p\gamma\omega^2\sin(\omega t) = \gamma\omega^2(I_p - I_d)\sin(\omega t) \\ I_d\ddot\theta_y - I_p\omega\dot\theta_x + k\theta_y = +I_d\gamma\omega^2\cos(\omega t) - I_p\gamma\omega^2\cos(\omega t) = -\gamma\omega^2(I_p - I_d)\cos(\omega t) \end{cases}$$

$\theta = \theta_x + i\theta_y$: COMPLEX ANGLE

$$I_d\ddot\theta - iI_p\omega\dot\theta + k\theta = i\gamma\omega^2(I_p - I_d)e^{i\omega t} \qquad \omega_{CR} = \sqrt{\frac{k}{I_d - I_p}} \qquad \text{IF } I_p > I_d \Rightarrow \text{NO } \omega_{CR}$$

$$\theta = \theta_0 e^{st}; \quad \dot\theta = s\theta_0 e^{st}; \quad \ddot\theta = s^2\theta_0 e^{st}$$

$$(I_d s^2 - iI_p\omega s + k)\cancel{\theta_0 e^{st}} = 0 \quad : \text{CHARACT. EQ.}$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{iI_p\omega \pm \sqrt{-I_p^2\omega^2 - 4I_d k}}{2I_d} = \sigma + i\lambda$$

$\sigma = Re(s) \begin{cases} \sigma \le 0 : \text{STABLE} \\ \sigma > 0 : \text{UNSTABLE} \end{cases} \qquad \lambda = Im(s) \begin{cases} > 0 \text{ FORWARD WHIRL} \\ < 0 \text{ BACKWARD WHIRL} \end{cases}$

MECHANICAL SYSTEM DYNAMICS

APPLIED LECTURES → EXERCISES

M. S. D.

APPLIED LECTURES

EXERCISES

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- CONVOLUTION INTEGRAL

PROBLEM GIVES: $\begin{cases} F(t) = F_0 \cos(\omega_m t) \cdot u(t) \\ \text{NULL I.C.} \end{cases}$ $\rightarrow F(\tau) = F_0 \cos(\omega_m \tau)\, u(\tau)$

$x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau$

$h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_m(t-t_0)} \cdot \sin[\omega_d(t-t_0)] \cdot u(t-t_0)$

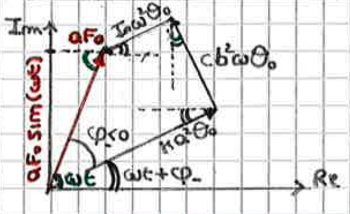
$h(t-\tau) = \frac{1}{m\omega_m} \cdot \sin[\omega_m(t-\tau)] \cdot u(t-\tau)$, WITH $\zeta = 0$

$t \geq t_0 \Rightarrow u(t-\tau) = 1$, $u(\tau) = 1$

hp: $I_0\ddot{\theta} + cb^2\dot{\theta} + ka^2\theta = F_0 \sin(\omega t) = F_0\, \mathrm{Im}(e^{i\omega t})$

$\theta = \theta_0 e^{i\omega t} = |\theta_0| e^{i(\omega t - \varphi_+)} = |\theta_0| \sin(\omega t \mp \varphi_{\pm})$

ARGAND & GAUSS COMPLEX PLANE:



FORMULARY' (FASTER CONSULTATION)

$\boxed{I_R = m\rho^2\ [\mathrm{Kg\,m^2}]}$ $\boxed{k_t = \frac{G I_P}{L}\ [\mathrm{Nm}]}$ $\boxed{I_P = \frac{\pi D^4}{32}\ [\mathrm{m^4}]}$ $\boxed{K_{EQ}^{//} = \sum_{i=1}^{m} K_i = K_1 + K_2 + \dots}$

$\boxed{K_{EQ}^{SERIE} = \left(\sum_{i=1}^{n} \frac{1}{K_i}\right)^{-1} \leftrightarrow \frac{1}{K_{EQ}^{SERIE}} = \frac{1}{K_1} + \frac{1}{K_2} + \dots}$ $\boxed{\ddot{\theta}\,\overline{OG}: \text{TANG. ACC.}}$ $\boxed{\dot{\theta}^2\,\overline{OG}: \text{CENTRIF. ACC.}}$

$\boxed{I_0 = I_G + m(\overline{OG})^2}$ $\boxed{I_G(\text{BAR}) = \frac{m\ell^2}{12}}$ $\boxed{I_G(\text{PULLEY}) = \frac{mR^2}{2}}$ $\boxed{I_0 = \int_0^{\ell} \mu y^2\, dy}$ (*)

$\boxed{\text{IN } O: R_V, R_h\ (\leftrightarrow N, T)}$ $\boxed{\theta_{TOT} = \theta_{ST} + \theta,\ \theta_{ST} = \theta(\ddot{\theta} = \dot{\theta} = 0)}$ $\boxed{x_A \simeq \overline{OA}\,\theta}$ $\boxed{\text{IF } \omega > \omega_m \rightarrow \varphi_+ < 0 \Rightarrow \varphi_+ = \underset{(<0)}{\varphi_+'} + \pi}$

STABILITY: 1^{st}: EQ. COND.: $\ddot{\theta} = \dot{\theta} = 0 \rightarrow \theta_E \Rightarrow I_0\ddot{\theta} = F(\theta, \dot{\theta}) \rightarrow F(\theta_E, 0) = 0$

2^{nd}: SMALL PERT.: $\theta(t) = \theta_E + \delta(t)$; $\dot{\theta}(t) = \dot{\delta}(t)$; $\ddot{\theta}(t) = \ddot{\delta}(t)$

3^{rd}: TAYL. EXP. TH: $F(\theta, \dot{\theta}) = F(\theta_E, 0) + \frac{\partial F}{\partial \theta}(\theta, \dot{\theta})\Big|_{\theta_E, 0} \cdot \delta + \frac{\partial F}{\partial \dot{\theta}}(\dot{\theta}, \ddot{\theta})\Big|_{\theta_E, 0} \cdot \dot{\delta} + O(\delta^2 + \dot{\delta}^2)$

4^{th}: VALUT. OF CHAR. EQ: $I_0\ddot{\delta} - F(\theta, \dot{\theta}) = 0$ WITH $\delta = Ae^{st}$; $\dot{\delta} = sAe^{st}$; $\ddot{\delta} = s^2Ae^{st}$

$s^2 + as + b = 0$ $\quad s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; IF $\mathrm{Re}(s_1, s_2) = 0 \Rightarrow \theta_E$ = STABLE | IF $\mathrm{Re}(s_1/s_2) > 0 \Rightarrow$ UNS.
$< 0 \Rightarrow \theta_E$ = ASYMPTOTIC STABLE

CONVOLUTION INTEGRAL: $x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau$, $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n(t-t_0)} \cdot \sin[\omega_d(t-t_0)]\, u(t-t_0)$

$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$ $\quad | \ u(\tau) = 1;\ u(t-\tau) = 1,\ t \geq t_0$

$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$

CLASSICAL APPROACH: $F(t) = F_0\cos(\omega_m t) \rightarrow F(t) = F_0 \cos(\omega t) \Rightarrow x_p(t) = X_0 \cos(\omega t)$

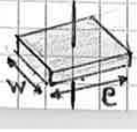
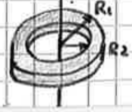
$\lim_{\omega \to \omega_m} \left[x(t) = x_p(t) + x_A(t) \right] = \lim_{\omega \to \omega_m} \frac{dN/d\omega}{dD/d\omega}$; THEN $\omega = \omega_m \rightarrow x_p(t)$ $\Big|$ $x_A(t) = e^{-\zeta\omega_m t} \cdot [a\cos(\omega_d t) + b\sin(\omega_d t)]$

$\left.\begin{array}{l} x(t) = x_p(t) + x_A(t) \\ \dot{x}(t) = \dot{x}_p(t) + \dot{x}_A(t) \end{array}\right. \rightarrow$ I.C. $\begin{cases} x(t=0) = 0 \\ \dot{x}(t=0) = 0 \end{cases} \Rightarrow a, b \rightarrow x(t) = x(t)_{\text{CONV.INT.}}$

ACCURACY $= \frac{|Z_A| - |Z_0|}{|Z_A|} = \frac{Y_0R^2 - Y_0R^2|G(i\omega)|}{Y_0R^2} = 1 - |G(i\omega)|$, $R = \frac{\omega}{\omega_m}$ (ACCELEROMETER)

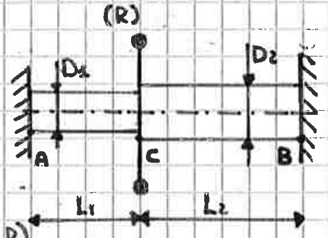
(*) INERTIAL MOMENTS (I_G) (IF $O \neq G \Rightarrow I_0 = I_G + m(\overline{OG})^2$ [$\mathrm{Kg\,m^2}$])

 mR^2  $\frac{mR^2}{2}$  $\frac{m\ell^2}{12}$ ( $\frac{m\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}m\ell^2$)

 $\frac{1}{12}m(\ell^2 + w^2)$  $\frac{1}{2}m(R_1^2 + R_2^2)$  $\frac{2}{5}mR^2$

APPLIED LECTURE 1 : SDOF VIBRATIONS   1st round

EXERCISE 1: FREE TORSIONAL VIBRATIONS



ROTOR (R) IS MOUNTED ON SHAFT AB, WHICH IS CLAMPED AT BOTH ENDS.

$L_1 = 0.9\,[m]$; $L_2 = 1.5\,[m]$; $D_1 = 10\,[mm]$; $D_2 = 12\,[mm]$

$m = 200\,[Kg]$, $\rho = 650\,[mm]$ (MASS AND INERTIA RADIUS OF R)

$G = 80000\,[N/mm^2]$ (TANGENTIAL ELASTICITY MODULUS)

T (PERIOD OF OSCILLATIONS) = ?

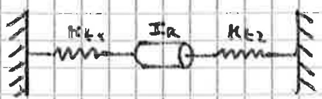
PARALLEL SHAFTS:

$K_{EQ} = \sum_{i=1}^{m} K_i \iff K_{EQ} = K_1 + K_2 + \ldots + K_m$

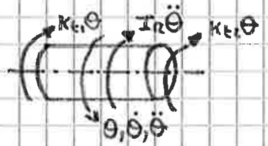
EXECUTION

$\boxed{T = \frac{1}{f_m} = \frac{1}{\frac{\omega_m}{2\pi}} = \frac{2\pi}{\omega_m}, \ (\omega_m = 2\pi f_m)}$, $\boxed{I_R = m\rho^2}$, $\boxed{I_P = \frac{\pi D^4}{32}}$, $\boxed{K_t = \frac{G \cdot I_P}{L}\,[Nm]}$, $\boxed{\omega_m = \sqrt{\frac{K_{tEQ}}{I}}}$

1. SIMPLIFIED SYSTEM:



2. ISOLATING THE MASS: FBD



3. MOTION EQUATION:

$$I_R\ddot{\theta} + K_{t1}\theta + K_{t2}\theta = 0$$

$$I_R\ddot{\theta} + (K_{t1} + K_{t2})\theta = 0$$

$$I_R = m \cdot \rho^2 = 200\,[Kg] \cdot (0.65\,[m])^2 = 60.5\ [Kg\,m^2]$$

$$I_{P1} = \frac{\pi D_1^4}{32} = \frac{3.14\,(0.01\,[m])^4}{32} = 9.812 \cdot 10^{-10}\ [m^4]$$

$$I_{P2} = \frac{\pi D_2^4}{32} = \frac{3.14\,(0.012\,[m])^4}{32} = 2 \cdot 10^{-9}\,[m^4]$$

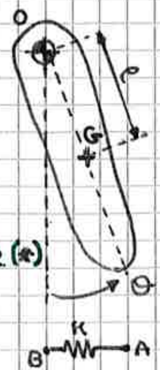
$$K_{t1} = \frac{G \cdot I_{P1}}{L_1} = 80000 \cdot 10^6 \left[\frac{N}{m^2}\right] \cdot 9.812 \cdot 10^{-10}\,[m^4] \cdot \frac{1}{0.9\,[m]} = 87.22\ [Nm]$$

$$K_{t2} = \frac{G \cdot I_{P2}}{L_2} = 80000 \cdot 10^6 \left[\frac{N}{m^2}\right] \cdot 2 \cdot 10^{-9}\,[m^4] \cdot \frac{1}{1.5\,[m]} = 106.67\,[Nm]$$

$$\omega_m = \sqrt{\frac{K_{t1} + K_{t2}}{I_R}} = \sqrt{\frac{1}{60.5\,[Kg\,m^2]}(87.22 + 106.67)\,[Nm]} = 2.2\ \left[\frac{1}{s}\right] \rightarrow \left[\frac{Rad}{s}\right]$$

$$T = \frac{2\pi}{\omega_m} = \frac{2 \cdot 3.14}{2.2\left[\frac{1}{s}\right]} = 2.855\,[s] \quad ✓$$

EXERCISE 3: COMPOUND PENDULUM



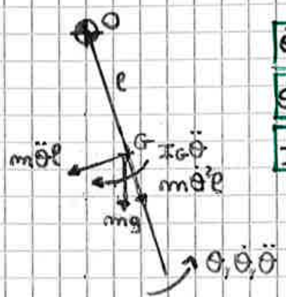
$m, I_G, \ell = \overline{OG}$

1) DRAW THE PENDULUM FBD
2) MOTION EQUATION (θ)
3) EQUILIBRIUM POSITION AND DISCUSS STABILITY
4) MOTION EQUATION LINEARIZATION **NEAR STABLE θ_e (*)**
5) NATURAL ANGULAR FREQUENCY = ?

GRAVITY & SPRING

EXECUTION:

1) FBD:



$\boxed{\ddot\theta\ell = \text{TANGENTIAL ACC.}}$

$\boxed{\dot\theta^2\ell = \text{CENTRIFUGAL ACC.}}$

$\boxed{I_0 = I_G + m\ell^2 \quad \text{Th. HUGENS-STEINER}}$ $(\ell = \overline{OG})$

2) MOTION EQUATION:

$\circlearrowright\ I_G\ddot\theta + m\ell^2\ddot\theta + mg\sin\theta\,\ell = 0 \iff I_0\ddot\theta + mg\sin\theta\,\ell = 0$ ✓

3) EQUILIBRIUM POSITION:

$\boxed{\text{CONDITION OF EQUILIBRIUM}: \ddot\theta = \dot\theta = 0}$

$mg\sin\theta\,\ell = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta_e = \begin{cases} 0° \\ 180° \end{cases}$ ✓ $\left\{\begin{matrix} 0, 2\pi, \dots \\ \pi, 3\pi, \dots \end{matrix}\ \infty\ \text{EQ.}\theta\right\}$

4) MOTION EQUATION LINEARIZATION:

THE LINEARIZATION SHOULD BE DONE ONLY IN THE EQUILIBRIUM POINT:

$I_0\ddot\theta = F(\theta, \dot\theta)$, $\dot\theta = 0$

CONSIDERING THE TAYLOR SERIES EXPRESSION **(METHOD A)**

$F(\theta) = F(\theta_e) + \left.\dfrac{dF}{d\theta}\right|_{\theta=\theta_e}(\theta - \theta_e)$, e: EQUILIBRIUM

CALLING:

$(\theta - \theta_e) = \delta$

$\theta(t) = \theta_e + \delta(t)$

$\dot\theta(t) = \dot\delta(t)$

$\ddot\theta(t) = \ddot\delta(t)$

THEREFORE:

$I_0\ddot\delta + mg\ell\sin(\theta_e + \delta) = 0$

CONSIDERING **(METHOD B)**

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$\sin(\theta_e + \delta) = \sin\theta_e\cos\delta + \sin\delta\cos\theta_e$
$= 0\cos\delta + \sin\delta\cos\theta_e =$
$\cong \delta\cos\theta_e$

THEREFORE: $\Rightarrow I_0\ddot\delta + mg\ell\,\delta\cos\theta_e = 0$ (*)

$F(\theta) = I_0\ddot\theta = -mg\ell\sin\theta$

$F(\theta_e) = I_0\ddot\theta_e = -mg\ell\sin\theta_e$

$\left.\dfrac{dF}{d\theta}\right|_{\theta=\theta_e} = \dfrac{dF(\theta_e)}{dt} = -mg\ell\cos\theta_e$

$F(\theta_e) + \left.\dfrac{dF}{d\theta}\right|_{\theta=\theta_e}(\theta - \theta_e) = -mg\ell\sin\theta_e + (-mg\ell\cos\theta_e)(\theta - \theta_e)$
$= 0 - mg\ell\cdot\delta\cos\theta_e$

THEREFORE: $I_0\ddot\theta = I_0\ddot\delta = -mg\ell\,\delta\cos\theta_e \Rightarrow I_0\ddot\delta + mg\ell\cdot\delta\cos\theta_e = 0$

$I_0\ddot\delta + mg\ell\,\delta\cos\theta_e = 0 \qquad \delta = Ae^{st},\ \dot\delta = sAe^{st},\ \ddot\delta = s^2Ae^{st}$

$I_0 s^2 \cancel{Ae^{st}} + mg\ell\cos\theta_e \cancel{Ae^{st}} = 0 \Rightarrow s = \pm\sqrt{-\dfrac{mg\ell\cos\theta_e}{I_0}}$

WITH $\theta_e = 0, 2\pi, 4\pi, \dots$: $s = \pm i\sqrt{\dfrac{mg\ell}{I_0}}$ (2 im. POLES CONJUGATE) $\Rightarrow Re(s_1, s_2) = 0 \Rightarrow I_0\ddot\delta + mg\ell\,\delta = 0$ ✓

WITH $\theta_e = \pi, 3\pi, 5\pi, \dots$: $s = \pm\sqrt{\dfrac{mg\ell}{I_0}}$ (2 REAL POLES) $\Rightarrow Re(s_1) > 0$: UNSTABLE θ_e (ONE + Re S. IS ENOUGH TO DESTROY THE SYSTEM!)

5) $\omega_m = ?$

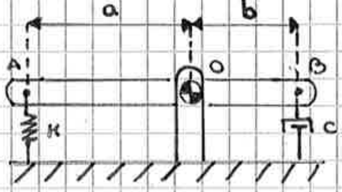
$\omega_m = +\sqrt{\dfrac{mg\ell}{I_0}}$ ✓

(*) STABLE θ_e ($\Rightarrow \cos\theta_e = 1$)

EXERCISE 5 : DAMPED FREE VIBRATIONS

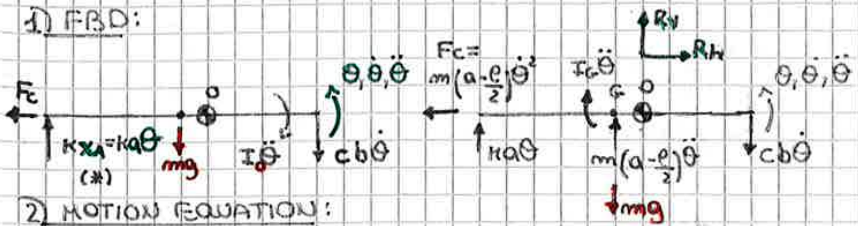
$a = 1.2\,[m]$; $b = 0.8\,[m]$; $m = 80\,[kg]$; $k = 50\,[kN/m]$; (UNIFORM BAR)

1) FBD
2) MOTION EQUATION
3) DAMPING COEFF. $c(\zeta = 0.5) = ?$



EXECUTION

1) FBD:



→ $R_h = F_c$

↑ $R_v = \dots$

2) MOTION EQUATION:

(STATIC FORCE)

$$\circlearrowleft O)\quad I_O\ddot{\theta} + cb^2\dot{\theta} + ka^2\theta = mgR \iff I_G\ddot{\theta} + mR^2\ddot{\theta} + \underbrace{cb^2}_{c_{eq}}\dot{\theta} + \underbrace{ka^2}_{k_{eq}}\theta = mgR \qquad R = a - \frac{\ell}{2},\ \ell = a + b$$

CONSIDERING AN OFFSET OF GCS → WE CAN NEGLECT THE STATIC FORCE mg

$$\theta_{STATIC}(\ddot{\theta} = \dot{\theta} = 0) = \frac{mgR}{ka^2} = \text{CONST}$$

$$\theta = \theta_{st} + \theta_{NEW}$$

$$I_G\ddot{\theta}_{NEW} + mR^2\ddot{\theta}_{NEW} + cb^2\dot{\theta}_{NEW} + ka^2\theta_{NEW} = 0$$

$$(\theta_N = \theta_{NEW})$$

3) DAMPING COEFF. (c):

$$\ddot{\theta}_N + 2\zeta\omega_m\dot{\theta}_N + \omega_m^2\theta_N = 0 \iff \ddot{\theta}_N + \frac{cb^2}{I_O}\dot{\theta}_N + \frac{ka^2}{I_O}\theta_N = 0$$

$$\omega_m = \sqrt{\frac{ka^2}{I_O}} \qquad \frac{cb^2}{I_O} = 2\zeta\omega_m \Rightarrow c = \frac{2I_O\zeta\omega_m}{b^2}$$

$$I_O = I_G + m\left(a - \frac{\ell}{2}\right)^2 = m_{EQ}$$

$$\ell = a + b = 1.2 + 0.8 = 2\,[m]$$

$$I_G\,(BAR) = \frac{m\ell^2}{12} = \frac{80 \cdot 2^2}{12} = 26.67\ [kg\,m^2]$$

$$I_O = I_G + m\left(a - \frac{\ell}{2}\right)^2 = 26.67 + 80\left(1.2 - \frac{2}{2}\right)^2 = 29.87\ [kg\,m^2]$$

$$\omega_m = \sqrt{\frac{ka^2}{I_O}} = \sqrt{\frac{50 \cdot 10^3 \cdot 1.2^2}{29.87}} = 49.1\ \left[\frac{Rad}{s}\right]$$

$$c = \frac{2I_O\zeta\omega_m}{b^2} = \frac{2 \cdot 29.87 \cdot 0.5 \cdot 49.1}{0.8^2} = 2291\ \left[\frac{Ns}{m}\right]\ ✓$$

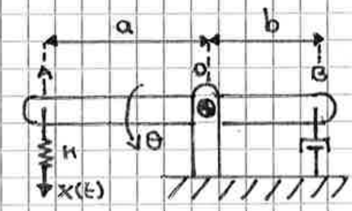
(*) $x_A = a\sin\theta \simeq a\theta$ (FOR LINEARIZATION!)

EXERCISE 7: FORCED VIBRATIONS

UNIFORM BAR: $x(t) = X_0 \sin(\Omega t)$, $X_0 = 10\,[mm]$; $f = 7\,[Hz]$

$a = 1.2\,[m]$; $b = 0.8\,[m]$; $m = 80\,[kg]$; $k = 50\,[kN/m]$

$c = 2291\,[Ns/m]$;



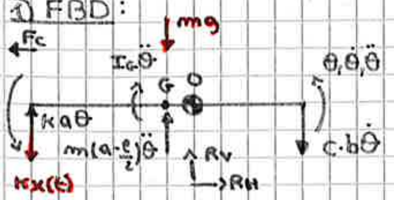
1) FBD

2) MOTION EQUATION

3) STEADY STATE RESPONSE ($\theta_0 = ?$)($\varphi = ?$)

EXECUTION

1) FBD:



2) MOTION EQUATION:

$$O) \quad I_O\ddot{\theta} + c \cdot b^2\dot{\theta} + ka^2\theta = a k x(t) = a k X_0 \sin(\omega t) \quad ✓ \quad \left(\theta_{TOT} = \theta_{ST} + \theta,\ \theta_{ST} = \frac{mgR}{ka^2}\right)$$

$$I_O = I_G + m\left(a - \frac{\ell}{2}\right)^2 = \frac{m\ell^2}{12} + m\left(a - \frac{\ell}{2}\right)^2 = 28.87\,[kg\,m^2]$$

3) STEADY STATE RESPONSE

$$\ddot{\theta} + \frac{cb^2}{I_O}\dot{\theta} + \frac{ka^2}{I_O}\theta = \frac{ak}{I_O}x(t), \quad \omega_n = \sqrt{\frac{ka^2}{I_O}} = 49.1\left[\frac{rad}{s}\right], \quad \omega = 2\pi f = 43.96\left[\frac{rad}{s}\right]$$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{ak}{I_O}x(t) = \frac{\cancel{a}k}{a^2k}\omega_n^2 x(t) = \frac{\omega_n^2}{a}x(t)$$

$$(-\omega^2 + 2i\zeta\omega_n\omega + \omega_n^2)\,\Theta_0(i\omega)\,e^{i\omega t} = \frac{\omega_n^2}{a}X_0 e^{i\omega t}$$

$$[1 - (\omega/\omega_n)^2 + 2i\zeta(\omega/\omega_n)]\,\Theta_0(i\omega) = \frac{X_0}{a}$$

$$\Theta_0(i\omega) = G(i\omega)\frac{X_0}{a} = |G(i\omega)|\frac{X_0}{a}e^{\mp i\varphi} = \theta_0\cos(\mp\varphi)$$

$$\Theta(i\omega) = \Theta_0(i\omega)e^{i\omega t} = |G(i\omega)|\frac{X_0}{a}e^{i(\omega t \mp \varphi)} = |\Theta(i\omega)|e^{i(\omega t \mp \varphi)} = \theta_0\cos(\omega t \mp \varphi)$$

$$\theta_0 = |\Theta_0(i\omega)| = |G(i\omega)|\frac{X_0}{a} = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}\frac{X_0}{a} = 0.00909\,[rad] \hat{=} 0.0091\,[rad] \quad ✓$$

$$\Theta_0(i\omega) = G(i\omega)\frac{\bar{G}(i\omega)}{\bar{G}(i\omega)}\frac{X_0}{a} = |G(i\omega)|^2[1 - (\omega/\omega_n)^2 - 2i\zeta(\omega/\omega_n)]\frac{X_0}{a}; \quad Re = 1 - (\omega/\omega_n)^2, \ Im = -2\zeta(\omega/\omega_n)$$

$$\varphi = Tg^{-1}\left(\frac{Im}{Re}\right) = Tg^{-1}\left(-\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right) = -77.5° \quad ✓$$

$$x(t) = x_p(t) + x_h(t) = \frac{F_0}{K}\frac{1}{1-\left(\frac{\omega}{\omega_m}\right)^2}\cos(\omega t) + a\cos(\omega_m t) + b\sin(\omega_m t)$$

$$\dot{x}(t) = -\frac{F_0}{K}\omega\frac{1}{1-\left(\frac{\omega}{\omega_m}\right)^2}\sin(\omega t) - \omega_m a\sin(\omega_m t) + \omega_m b\cos(\omega_m t)$$

$$\begin{cases} x(t=0) = 0 = \frac{F_0}{K}\frac{1}{1-(\omega/\omega_m)^2} + a \\ \dot{x}(t=0) = 0 = \omega_m b \end{cases} \Rightarrow \begin{cases} a = -\frac{F_0}{K}\frac{1}{1-(\omega/\omega_m)^2} \\ b = 0 \end{cases}$$

$$x(t) = \frac{F_0}{K}\frac{1}{1-\left(\frac{\omega}{\omega_m}\right)^2}\left(\cos(\omega t) - \cos(\omega_m t)\right)$$

$$\boxed{\lim_{\omega\to\omega_m} x(t) = \lim_{\omega\to\omega_m}\frac{dN/d\omega}{dD/d\omega}} = \lim_{\omega\to\omega_m}\frac{F_0}{K}\frac{\left[-t\cdot\sin(\omega t)\right]}{\left[-\frac{2}{\omega_m}\left(\frac{\omega}{\omega_m}\right)\right]}$$

$\omega = \omega_m$

$$x(t) = +\frac{F_0}{2K}\omega_m t\sin(\omega_m t) = +\frac{F_0}{2m\omega_m^2}\omega_m t\sin(\omega_m t) = \frac{F_0\cdot t}{2m\omega_m}\sin(\omega_m t)$$

$$x(t) = \frac{F_0 t}{2m\omega_m}\sin(\omega_m t)\cdot u(t)$$ ✓

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2) OTHER APPROACH

$f(t) = F_0 \sin(\omega_m t) \rightarrow f(t) = F_0 \sin(\omega t) \rightarrow x(t) = x_p(t) = x_0 \sin(\omega t)$

↑ NW!

$$\begin{cases} x_p(t) = x_0 \sin(\omega t) \\ \dot{x}_p(t) = \omega x_0 \cos(\omega t) \\ \ddot{x}_p(t) = -\omega^2 x_0 \sin(\omega t) \end{cases}$$

$m\ddot{x} + Kx = F_0 \sin(\omega t)$

$\cancel{\omega_m^2}\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 1\right] x_0 \cancel{\sin(\omega t)} = \frac{F_0}{K} \cancel{\omega_m^2} \cancel{\sin(\omega t)} \Rightarrow x_0 = \frac{F_0}{K} \cdot \frac{1}{1-\left(\frac{\omega}{\omega_m}\right)^2}$

$x(t) = \frac{F_0}{K} \frac{1}{1-(\omega/\omega_m)^2} \sin(\omega t)$

$$\boxed{\lim_{\omega \to \omega_m} x(t) = \lim_{\omega \to \omega_m} \frac{dN/d\omega}{dD/d\omega}} = \lim_{\omega \to \omega_m} \frac{F_0}{K} \frac{t\cos(\omega t)}{-2\left(\frac{\omega}{\omega_m}\right)\frac{1}{\omega_m}}$$

$\omega = \omega_m$

$x_p(t) = -\frac{F_0 \omega_m t}{2K} \cos(\omega_m t) = -\frac{F_0 t}{2m\omega_m} \cos(\omega_m t)$

$x(t) = x_p(t) + x_A(t) = -\frac{F_0 t}{2m\omega_m} \cos(\omega_m t) + a\cos(\omega_m t) + b\sin(\omega_m t)$

$\dot{x}(t) = -\frac{F_0 t \cancel{\omega_m}}{2m\cancel{\omega_m}} \sin(\omega_m t) - \frac{F_0}{2m\omega_m} \cos(\omega_m t) - \omega_m a \sin(\omega_m t) + \omega_m b \cos(\omega_m t)$

$$\begin{cases} x(t=0) = 0 = a \\ \dot{x}(t=0) = 0 = -\frac{F_0}{2m\omega_m} + \omega_m b \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = \frac{F_0}{2m\omega_m^2} \end{cases}$$

$x(t) = -\frac{F_0 t}{2m\omega_m} \cos(\omega_m t) + \frac{F_0}{2m\omega_m^2} \sin(\omega_m t) = \frac{F_0}{2m\omega_m} \cdot \left[\frac{\sin(\omega_m t)}{\omega_m} - t \cdot \cos(\omega_m t)\right]$

$x(t) = \frac{F_0}{2m\omega_m} \left[\frac{1}{\omega_m} \sin(\omega_m t) - t\cos(\omega_m t)\right] u(t)$ ✓

EXERCISE 11: DYNAMIC TEST OF AN AIRPLANE ELEVATOR

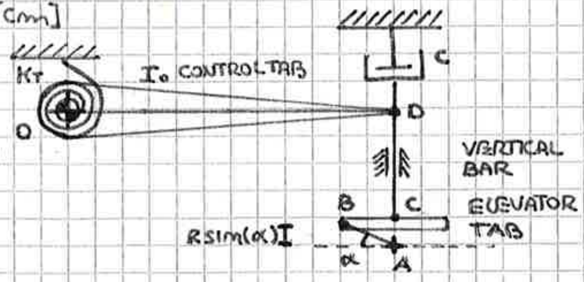
CONTROL TAB OF AN AIRPLANE ELEVATOR

I_0: CONTROL TAB MASS MOMENT OF INERTIA ; $\omega_m = \sqrt{K_T/I_0}$

$\mu = \mu_0(1 - y/L)$: CONTROL TAB MASS PER UNIT LENGTH

$\mu_0 = 100\,[Kg/m]$; $\overline{OD} = L = 50\,[cm]$; $\overline{AB} = R = 5\,[cm]$

$K = 40\,[Kg/mm]$; $\zeta = 60\%$; $\theta_{MAX} = 5°$



1) MOTION EQUATION OF THE ELEVATOR TAB (CONTROL TAB...)

2) STIFFNESS (K_t)

3) NATURAL FREQUENCY (ω_m)

4) RESONANCE FREQUENCY (ω_{RES})

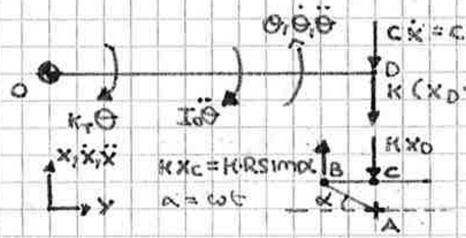
5) PHASE (φ) OF THE RESPONSE WITH RESPECT TO EXCITATION IN RESONANCE CONDITION

6) MODULUS OF FORCE (F_0) APPLIED ON THE ELEVATOR TAB BY THE SPRING.

EXECUTION

1) MOTION EQUATION OF THE ELEVATOR TAB

FBD



$K = 40\left[\frac{Kg}{mm}\right] = 40\cdot10^3\left[\frac{Kg}{m}\right]$

$K = 40\cdot10^3\left[\frac{Kg}{m}\right]\cdot 9.8\left[\frac{m}{s^2}\right] = 392\cdot10^3\left[\frac{N}{m}\right]$

✓

$$I_0\ddot\theta + L^2c\dot\theta + \underbrace{L\cdot K(L\theta - R\sin\alpha)}_{F \text{ APPLIED BY THE SPRING}} + K_t\theta = 0 \;\longleftrightarrow\; \underbrace{I_0}_{m_{EQ}}\ddot\theta + \underbrace{L^2c}_{c_{EQ}}\dot\theta + \underbrace{(L^2K + K_t)}_{K_{EQ}}\theta = LKR\sin\overset{\omega t}{\alpha}$$

$Kx_D = K\cdot R\sin\alpha$

2) STIFFNESS (K_t)

$$\ddot\theta + \frac{L^2c}{I_0}\dot\theta + \frac{(L^2K + K_t)}{I_0}\theta = \frac{LKR}{I_0}e^{i\omega t} \qquad \omega_m^2 = \frac{(L^2K + K_t)}{I_0} \;;\; 2\zeta\omega_m = \frac{L^2c}{I_0}$$

$$\cancel{\omega_m^2}\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\frac{\omega}{\omega_m}\right) + 1\right]\theta_0\cancel{e^{i\omega t}} = \frac{LKR}{(L^2K + K_t)}\cancel{\omega_m^2}\cancel{e^{i\omega t}}$$

$= 0.0872\,[Rad]$

$$\theta_0 = \frac{LKR}{(L^2K + K_t)}\cdot G(i\omega) \rightarrow |\theta_0| = \frac{LKR}{(L^2K + K_t)}\cdot|G(i\omega)| \rightarrow |\theta_{MAX}| = \frac{LKR}{(L^2K + K_t)}|G(i\omega)|_{MAX} = \frac{5\pi}{180}$$

$$|\theta_{MAX}| = \frac{LKR}{(L^2K + K_t)}\cdot\frac{1}{2\zeta\sqrt{1-\zeta^2}} \rightarrow (L^2K + K_t) = \frac{LKR}{|\theta_{MAX}|}\cdot|G(i\omega)|_{MAX} \;;\; |G(i\omega)|_{MAX} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.042$$

$$K_t = \frac{LKR}{|\theta_{MAX}|}\cdot\frac{1}{2\zeta\sqrt{1-\zeta^2}} - L^2K = LK\left[\frac{R\cdot|G(i\omega)|_{MAX}}{|\theta_{MAX}|} - L\right] = 0.5\cdot(40\cdot10^3\cdot9.8)\left[\frac{0.05\cdot1.042}{0.0872} - 0.5\right] = 19105.5\,[Nm/Rad]$$

✓

3) NATURAL FREQUENCY (ω_m)

$$\omega_m = \sqrt{\frac{L^2K + K_t}{I_0}} \;;\; \boxed{I_0 = \int_0^L \mu y^2\,dy} = \int_0^L \mu_0\left(1 - \frac{y}{L}\right)y^2\,dy = \int_0^L \mu_0\left(y^2 - \frac{y^3}{L}\right)dy = \mu_0\left[\frac{y^3}{3} - \frac{y^4}{4L}\right]_0^L =$$

$$= \mu_0\left[\frac{L^3}{3} - \frac{L^4}{4L}\right] = \mu_0L^3\left(\frac{1}{3} - \frac{1}{4}\right) = \mu_0L^3\frac{4-3}{12} = \frac{\mu_0L^3}{12} = \frac{100\cdot0.5^3}{12} = 1.042\,[Kg\,m^2]$$

$$\omega_m = \sqrt{\frac{L^2K + K_t}{I_0}} = \sqrt{\frac{0.5^2\cdot392\cdot10^3 + 19105.5}{1.042}} = 335.24\left[\frac{Rad}{s}\right]$$ ✓

EXERCISE 12: ACCELEROMETER

DATA

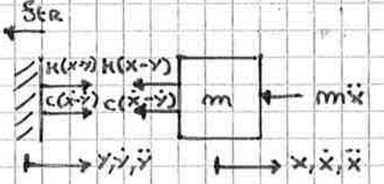
$m = 0.08$ [Kg]

$\xi = 70\%$

ACCURACY : 0.4%

$K = ?$ $c = ?$

FBD:



EXECUTION

$m\ddot{x} + c(\dot{x}-\dot{y}) + K(x-y) = 0$, $m(\ddot{z}) = m(\ddot{x}-\ddot{y}) \Rightarrow m\ddot{x} = m(\ddot{z}+\ddot{y})$

$m\ddot{z} + c\dot{z} + Kz = -m\ddot{y}$

$\cancel{\omega_m^2}\left[-\left(\frac{\omega}{\omega_m}\right)^2 + 2i\xi\left(\frac{\omega}{\omega_m}\right) + 1\right] z_0 \cancel{e^{i\omega t}} = +\omega^2 y_0 \cancel{e^{i\omega t}} \frac{\cancel{\omega_m}}{\omega_m^2}$

$T_{RREL} = \frac{z_0}{y_0} = \left(\frac{\omega}{\omega_m}\right)^2 G(i\omega)$; $z(t) = z_0 e^{i\omega t} = y_0 |T_{RREL}| e^{i(\omega t+\varphi_-)} = \underbrace{y_0\left(\frac{\omega}{\omega_m}\right)^2 |G(i\omega)|}_{|z_0|} e^{i(\omega t+\varphi_-)}$

IF $\omega_m \gg \omega \Rightarrow \begin{cases} |G(i\omega)| = 1 \\ \varphi_- = 0 \end{cases}$: ACCELEROMETER $\Rightarrow z(t) = \underbrace{y_0\left(\frac{\omega}{\omega_m}\right)^2}_{|z_A|} e^{i\omega t}$

~ . ~ . ~ . ~ . ~

$0 \le f \le 50$ [Hz] $\Rightarrow 0 \le \omega \le 314$ [Rad/s] $\Rightarrow \omega_{CR} = 314$ [Rad/s]

a = ACCURACY ; $R = \omega/\omega_m$

$$\boxed{a = \frac{y_0R^2 - |z_0|}{y_0R^2} = \frac{y_0R^2 - y_0R^2|G(i\omega)|}{y_0R^2} = 1 - |G(i\omega)| = \frac{0.4}{100} \quad \left(a = \frac{|z_A| - |z_0|}{|z_A|}\right)}$$

$a = 1 - |G(i\omega)| = 1 - \frac{1}{\sqrt{(1-R^2)^2 + 4\xi^2R^2}} \iff 1 - a = \frac{1}{\sqrt{(1-R^2)^2+4\xi^2R^2}}$

$(1-R^2)^2 + 4\xi^2R^2 = \frac{1}{(1-a)^2} = \frac{1}{\left(1-\frac{0.4}{100}\right)^2} = 1.008$

$1 - 2R^2 + R^4 + 4\xi^2R^2 = 1.008$

$R^4 + (4\xi^2-2)R^2 + (1-1.008) = 0$

$R^4 + (4\cdot 0.7^2 - 2)R^2 - 0.008 = 0$

$R^4 - 0.04R^2 - 0.008 = 0$

$R^2_{1,2} = \frac{0.04 \pm \sqrt{(0.04)^2 + 4\cdot 0.008}}{2} \begin{cases} + \; 0.11165 \rightarrow R = \sqrt{0.1116} = 0.334 = \frac{\omega}{\omega_m} \\ - \; -0.072 \end{cases}$

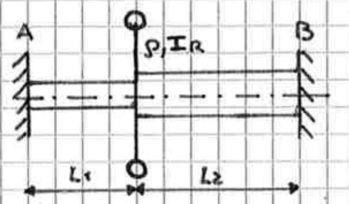
$\omega_m = \frac{\omega}{R} = \frac{314}{0.334} = 940.12 \left[\frac{Rad}{s}\right]$

$\omega_m = \sqrt{\frac{K}{m}} \rightarrow K = \omega_m^2 \cdot m = 940.12 \cdot 0.08 = 70706 \left[\frac{N}{m}\right]$ ✓

$\frac{c}{m} = 2\xi\omega_m \rightarrow c = 2m\xi\omega_m = 2\cdot 0.08\cdot 0.7\cdot 940.12 = 105.3 \left[\frac{Ns}{m}\right]$ ✓

APPLIED LECTURE 1 : SDOF VIBRATIONS   2nd Round

EXERCISE 1: FREE TORSIONAL VIBRATIONS



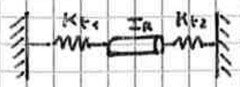
DATA

$L_1 = 0.9\,[m]$; $L_2 = 1.5\,[m]$; $D_1 = 10\,[mm]$; $D_2 = 12\,[mm]$

$m = 200\,[Kg]$, $\rho = 450\,[mm]$; $G = 80\,000\,[N/mm^2]$

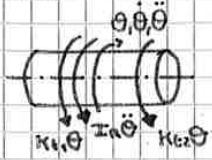
$T = ?$

EXECUTION



PARALLEL SHAFTS: $K_{EQ} = \sum_{i=1}^{m} K_i \Rightarrow K_{tEQ} = K_{t1} + K_{t2}$, $K_t = \frac{G I_P}{L}$, $I_P = \frac{\pi D^4}{32}$

FBD:



MOTION EQUATION:

$I_R\ddot\theta + K_{tEQ}\theta = 0$, $\omega_m = \sqrt{\frac{K_{tEQ}}{I_R}} = 2\pi f_m$; $T_m = \frac{1}{f_m} = \frac{2\pi}{\omega_m}$

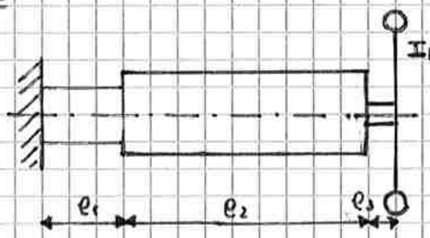
$I_{P1} = \frac{\pi D_1^4}{32} = \frac{3.14 \cdot (0.01)^4}{32} = 9.8125 \cdot 10^{-10}\,[m^4]$; $I_{P2} = \frac{\pi D_2^4}{32} = 2 \cdot 10^{-9}\,[m^4]$

$K_{t1} = \frac{G I_{P1}}{L_1} = 80\,000 \cdot 10^6 \left[\frac{N}{m^2}\right] \cdot 9.8125 \cdot 10^{-10}\,[m^4]\, \frac{1}{0.9}\left[\frac{1}{m}\right] = 87.22\,[Nm]$

$K_{t2} = 106.67\,[Nm]$; $K_{tEQ} = K_{t1} + K_{t2} = 193.89\,[Nm]$; $I_R = m\rho^2 = 200 \cdot 0.45^2 = 40.5\,[Kg\,m^2]$

$\omega_m = \sqrt{\frac{K_{tEQ}}{I_R}} = \sqrt{\frac{193.89}{40.5}} = 2.19\left[\frac{Rad}{s}\right]$; $T = \frac{2\pi}{\omega_m} = 2.87\,[s]$ ✓

EXERCISE 2: FREE TORSIONAL VIBRATIONS



DATA

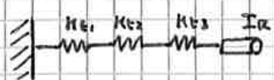
$\ell_1 = 25\,[cm]$; $\ell_2 = 75\,[cm]$; $\ell_3 = 10\,[cm]$;

$d_1 = 2\,[cm]$; $d_2 = 3\,[cm]$; $d_3 = 1\,[cm]$;

$I_R = 2.5\,[Kg\,m^2]$; $G = 82 \cdot 10^9\,[N/m^2]$

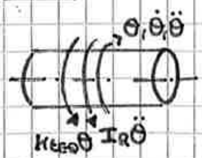
$\omega_m = ?$ ($f_m = ?$)

EXECUTION



SHAFTS IN SERIES: $K_{EQ} = \left(\sum_{i=1}^{m} \frac{1}{K_i}\right)^{-1}$, $\frac{1}{K_{tEQ}} = \frac{1}{K_{t1}} + \frac{1}{K_{t2}} + \frac{1}{K_{t3}}$, $K_t = \frac{G I_P}{L}$, $I_P = \frac{\pi D^4}{32}$

FBD:



MOTION EQUATION:

$I_R\ddot\theta + K_{tEQ}\theta = 0$, $\omega_m = \sqrt{\frac{K_{tEQ}}{I_R}}$

$I_{P1} = \frac{\pi D_1^4}{32} = \frac{3.14 \cdot (0.02)^4}{32} = 1.5 \cdot 10^{-8}\,[m^4]$; $I_{P2} = 7.9 \cdot 10^{-8}\,[m^4]$; $I_{P3} = 9.8125 \cdot 10^{-10}\,[m^4]$

$K_{t1} = \frac{G I_P}{\ell_1} = 82 \cdot 10^9 \left[\frac{N}{m^2}\right] \cdot 1.5 \cdot 10^{-8}\,[m^4]\, \frac{1}{0.25}\left[\frac{1}{m}\right] = 4920\,[Nm]$

$K_{t2} = 8637.33\,[Nm]$; $K_{t3} = 804.625\,[Nm]$; $K_{tEQ} = \left(\frac{1}{K_{t1}} + \frac{1}{K_{t2}} + \frac{1}{K_{t3}}\right)^{-1} = 640.27\,[Nm]$

$\omega_m = \sqrt{\frac{K_{tEQ}}{I_R}} = \sqrt{640.27\,[Nm]\, \frac{1}{2.5}\left[\frac{1}{Kg\,m^2}\right]} = 16\left[\frac{Rad}{s}\right]$

$f_m = \frac{\omega_m}{2\pi} = \frac{16}{2 \cdot 3.14} = 2.55\,[Hz]$ ✓

EXERCISE 4: FORCED VIBRATIONS

DATA

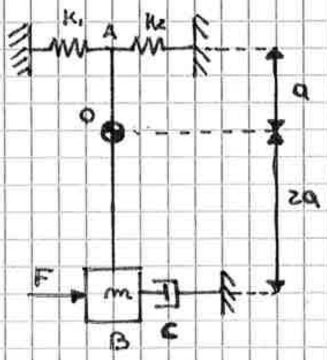
$F(t) = F_0 \sin(\Omega t)$ $k_1 = 100\,[N/m]$; $k_2 = 300\,[N/m]$

1) FBD $a = 0.1\,[m]$; $m = 1\,[kg]$

2) $\omega_m = ?$ $F_0 = 1\,[N]$; $\Omega = 2\,[rad/s]$

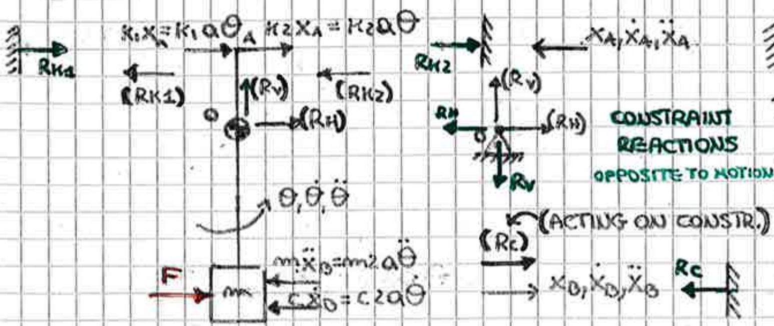
3) $c\ (\Rightarrow \zeta = 0.5)$

4) $|\theta_0|, \varphi = ?$



EXECUTION:

1) FBD:



$\leftarrow)_A\ R_{K1} + R_{K2} - (k_1+k_2)a\theta = 0$

$\rightarrow)_O\ R_H - (k_1+k_2)a\theta - m2a\ddot\theta - c2a\dot\theta + F = 0$

$\rightarrow)_B\ R_C - m2a\ddot\theta - c2a\dot\theta + F = 0$

2) $\omega_m = ?$

$\circlearrowleft_O)\ -m4a^2\ddot\theta - c4a^2\dot\theta - (k_1+k_2)a^2\theta + 2aF = 0$

$m4a^2\ddot\theta + c4a^2\dot\theta + (k_1+k_2)a^2\theta = +2aF$

$\omega_m = \sqrt{\dfrac{(k_1+k_2)}{4m}} = \sqrt{\dfrac{400}{4}} = 10\left[\dfrac{rad}{s}\right]$ ✓ $\omega_m^2 = \dfrac{(k_1+k_2)a^2}{m4a^2}$

3) $\dfrac{\cancel{4a^2}c}{\cancel{4a^2}m} = 2\zeta\omega_m \Rightarrow c = 2\zeta\omega_m m = 2\cdot\frac{1}{2}\cdot 10\cdot 1 = 10\left[\dfrac{Ns}{m}\right]$ ✓

4) $|\theta_0|, \varphi = ?$

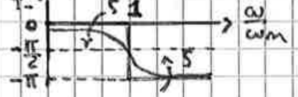
$\theta(t) = \theta_0 e^{i\omega t}$

$\omega_m^2\left[1 - \left(\dfrac{\omega}{\omega_m}\right)^2 + 2i\zeta\left(\dfrac{\omega}{\omega_m}\right)\right]\theta_0 e^{i\omega t} = \dfrac{+2aF_0}{(k_1+k_2)a^2}\,\omega_m^2\, \cancel{e^{i\omega t}}$

$\theta_0 = \dfrac{+2F}{a(k_1+k_2)}\,G(i\omega) \rightarrow |\theta_0| = \dfrac{+2F_0}{a(k_1+k_2)}\,|G(i\omega)| = \dfrac{+2F_0}{a(k_1+k_2)}\,\dfrac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_m}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_m}\right)\right]^2}} =$

$\dfrac{\omega}{\omega_m} = \dfrac{\Omega}{\omega_m} = \dfrac{2}{10} = 0.2$ $= \dfrac{+2\cdot 1}{40}\,\dfrac{1}{\sqrt{(1-0.04)^2 + 4\frac{1}{4}0.04}} = 0.05099\,[rad]$

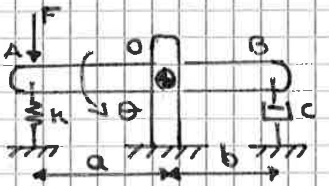
$\varphi_- = Tg^{-1}\left(\dfrac{Im}{Re}\right) = Tg^{-1}\left[-\dfrac{2\zeta\left(\frac{\omega}{\omega_m}\right)}{1-(\omega/\omega_m)^2}\right] = Tg^{-1}\left[-\dfrac{2\cdot\frac{1}{2}\cdot 0.2}{1-0.04}\right] = -11.77°$ $\left(\dfrac{\omega}{\omega_m} < 1\right)$ ✓



$\theta(t) = \theta_0 e^{i\omega t} = |\theta_0| e^{i(\omega t + \varphi_-)}$

EXERCISE 6: FORCED VIBRATIONS

DATA



$F(t) = F_0 \sin(\Omega t)$

$a = 1.2\,[m]$; $b = 0.8\,[m]$; $m = 80\,[Kg]$; $K = 50\,[KN/m]$

$C = 2291\,[Ns/m]$; $F_0 = 200\,[N]$; $f = 13\,[Hz]$

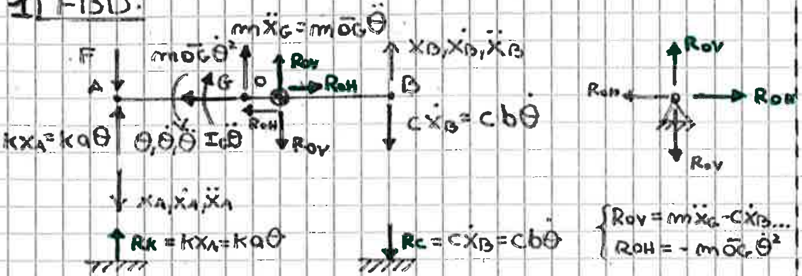
1) FBD

2) MOTION EQ.

3) STEADY-STATE RESPONSE (SMALL OSCILLATIONS) $|\theta_0|, \varphi = ?$

EXECUTION

1) FBD:



$\begin{cases} R_{OV} = m\ddot{x}_G - C\dot{x}_B \ldots \\ R_{OH} = -m\overline{OG}\dot{\theta}^2 \end{cases}$

IF WE CONSIDER ALSO **mg**

$\overset{\curvearrowleft}{O})\ -I_O\ddot{\theta}_T - cb^2\dot{\theta}_T - Ka^2\theta_T + aF + mg\overline{OG} = 0$

$\theta_T = \theta + \theta_{ST}$, $\theta_{ST} = \dfrac{mg\,\overline{OG}}{Ka^2}$

WE CAN ACCOUNT FOR mg IN THE PRE-LOAD FORCE OF THE SPRING

$aF_{K\,PRE-L} = \overline{OG}\,mg$

2) MOTION EQ.:

$\overset{\curvearrowleft}{O})\ -(I_G + m\overline{OG}^2)\ddot{\theta} - cb^2\dot{\theta} - Ka^2\theta + Fa = 0$, $\overline{OG} = a - \dfrac{(a+b)}{2} = \dfrac{a-b}{2}$ ✓

$(I_G + m\overline{OG}^2)\ddot{\theta} + cb^2\dot{\theta} + Ka^2\theta = Fa$, $I_G = \dfrac{m(a+b)^2}{12}$; $I_O = \dfrac{m(a+b)^2}{12} + \dfrac{m(a-b)^2}{4} = 29.87\,[Kg\,m^2]$

3) STEADY-STATE RESPONSE:

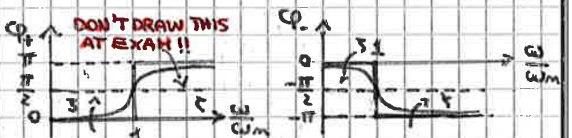
$\omega_n^2 = \dfrac{Ka^2}{I_O} \Rightarrow \dfrac{1}{I_O} = \omega_n^2 \dfrac{1}{Ka^2}$; $\dfrac{cb^2}{I_O} = 2\zeta\omega_n \Rightarrow \zeta = \dfrac{cb^2}{2I_O\omega_n} = 0.5$

$\theta = \theta_0 e^{i\omega t}$

$\omega_n^2\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + 2i\zeta\left(\dfrac{\omega}{\omega_n}\right)\right]\theta_0 \cancel{e^{i\omega t}} = \dfrac{F_0\omega_n^2 \cancel{e^{i\omega t}}}{Ka}$, $\omega = 2\pi f = 81.64\left[\dfrac{Rad}{s}\right]$; $\omega_n = 49.1\left[\dfrac{Rad}{s}\right]$

$\theta_0 = \dfrac{F_0}{Ka}G(i\omega)$; $|\theta_0| = \dfrac{F_0}{Ka}|G(i\omega)| = \dfrac{F_0}{Ka}\dfrac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} = 0.00137\,[Rad]$ ✓

$\varphi_-' = Tg^{-1}\left(\dfrac{I_m}{Re}\right) = Tg^{-1}\left[-\dfrac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] = Tg^{-1}(+0.942) = 43.3$



IF $\omega < \omega_n \Rightarrow \varphi_- = Tg^{-1}\left(\dfrac{I_m}{Re}\right)$; $\varphi_+ = Tg^{-1}\left(\dfrac{|I_m|}{Re}\right)$

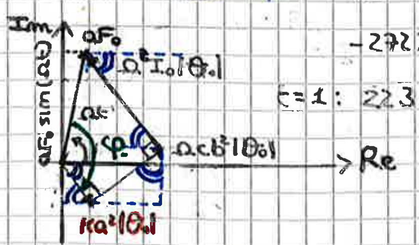
IF $\omega > \omega_n \Rightarrow \varphi_- = \varphi_-'(>0) - \pi$; $\varphi_+ = \varphi_+'(<0) + \pi$

$\Rightarrow \varphi_- = 43.3 - 180 = -136.7$ ✓; $\theta(t) = \theta_0 e^{i\omega t} = |\theta_0| e^{i(\omega t + \varphi_-)}$ $\wedge$ $\theta(t) = |\theta_0| e^{i(\omega t - \varphi_+)}$

ARGAND & GAUSS COMPLEX PLANE:

$F(t) = F_0 \sin(\Omega t) \rightarrow \theta(t) = \theta_0 \sin(\Omega t)$; $\dot{\theta}(t) = \Omega\theta_0\cos(\Omega t)$; $\ddot{\theta}(t) = -\Omega^2\theta_0\sin(\Omega t)$

$-\Omega^2 I_O|\theta_0|\sin(\Omega t + \varphi_-) + \Omega cb^2|\theta_0|\cos(\Omega t + \varphi_-) + Ka^2|\theta_0|\sin(\Omega t + \varphi_-) = aF_0\sin(\Omega t)$



$-272.75\sin(81.64t - 136.7) + 163.99\cos(\ ) + 98.64\sin(\ ) = 240\sin(81.64t)$

$t = 1$: $223.59 + 93.92 - 80.86 \cong 237.45\,[Nm]$

EXERCISE 8: CONVOLUTION INTEGRAL

UNDAMPED SDOF SYSTEM

$f(t) = F_0 \cos(\omega_m t)\, u(t)$; $\omega_m = \sqrt{\frac{K}{m}}$, $u(t)$: UNIT STEP FUNCTION

NULL I.C.

1) DERIVE THE RESPONSE ($x(t)$): CONVOLUTION INTEGRAL

2) COMPARE IT WITH THE RESULT OBTAINED WITH AN ALTERNATIVE APPROACH

EXECUTION

1) CONVOLUTION INTEGRAL

$$x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau \quad , \quad h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_m(t-t_0)} \cdot \mathrm{sim}[\omega_d(t-t_0)]\, u(t-t_0) \quad , \quad \tau = t - t_0$$

$F(\tau) = F_0 \cos(\omega_m \tau)\, u(\tau)$ $\qquad (\zeta = 0)$

$$h(t-\tau) = \frac{1}{m\omega_m} \mathrm{sim}[\omega_m(t-\tau)]\, u(t-\tau)$$

$$x(t) = \int_0^t F_0 \cos(\omega_m \tau)\, u(\tau) \frac{1}{m\omega_m} \mathrm{sim}[\omega_m(t-\tau)]\, u(t-\tau)\, d\tau$$

CONSIDERING $t \geq t_0$:

$$x(t) = \int_0^t \frac{F_0}{m\omega_m} \cos(\omega_m \tau)\, \mathrm{sim}[\omega_m(t-\tau)]\, d\tau \quad , \quad u(\tau) = 1 \ , \ u(t-\tau) = 1$$

CONSIDERING THAT:

$$\boxed{\begin{aligned} \mathrm{sim}\,\alpha \cos\beta &= \frac{\mathrm{sim}(\alpha+\beta) + \mathrm{sim}(\alpha-\beta)}{2} \\ \mathrm{sim}\,\alpha\, \mathrm{sim}\,\beta &= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \end{aligned}}$$

WE OBTAIN:

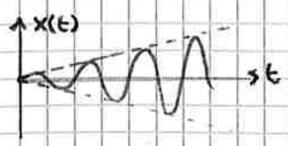
$$x(t) = \frac{F_0}{2m\omega_m} \int_0^t \left[\mathrm{sim}(\omega_m t - \cancel{\omega_m \tau} + \cancel{\omega_m \tau}) + \mathrm{sim}(\omega_m t - \omega_m \tau - \omega_m \tau) \right] d\tau$$

$$x(t) = \frac{F_0}{2m\omega_m} \int_0^t \left[\mathrm{sim}(\omega_m t) + \mathrm{sim}(\omega_m t - 2\omega_m \tau) \right] d\tau$$

$$x(t) = \frac{F_0}{2m\omega_m} \left[\mathrm{sim}(\omega_m t)\tau - \frac{\cos(\omega_m t - 2\omega_m \tau)}{-2\omega_m} \right]_0^t = \frac{F_0}{2m\omega_m} \left[\mathrm{sim}(\omega_m t)\, t + \frac{\cos(-\cancel{\omega_m t})}{\cancel{2\omega_m}} - \frac{\cancel{\cos(\omega_m t)}}{\cancel{2\omega_m}} \right]$$

$$x(t) = \frac{F_0 \cdot t}{2m\omega_m} \mathrm{sim}(\omega_m t) \cdot u(t)$$

$t \to \infty$: $x_0 \to \infty$ ✓



2) →

EXERCISE 9: CONVOLUTION INTEGRAL

UNDAMPED SDOF SYSTEM

$F(t) = F_0 \sin(\omega_m t)\, u(t)$, $u(t)$: UNIT STEP FUNCTION ; $\omega_m = \sqrt{\frac{K}{m}}$

NULL I.C.

1) DERIVE THE RESPONSE ($x(t)$) : CONVOLUTION INTEGRAL

2) COMPARE IT WITH THE RESULT OBTAINED WITH AN ALTERNATIVE APPROACH.

EXECUTION

1) CONVOLUTION INTEGRAL

$$x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau \quad , \quad h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_m (t-t_0)} \sin[\omega_d (t-t_0)]\, u(t-t_0) \; ; \; \tau = t - t_0$$

$$F(\tau) = F_0 \sin(\omega_m \tau)\, u(\tau) \qquad (\zeta = 0)$$

$$h(t-\tau) = \frac{1}{m\omega_m} \sin[\omega_m (t-\tau)]\, u(t-\tau)$$

$$x(t) = \int_0^t \frac{F_0}{m\omega_m} \sin(\omega_m \tau)\, u(\tau) \sin[\omega_m (t-\tau)]\, u(t-\tau)\, d\tau$$

CONSIDERING $t \geq t_0$ ($u(\tau) = 1$, $u(t-\tau) = 1$) :

$$x(t) = \int_0^t \frac{F_0}{m\omega_m} \sin(\omega_m \tau) \sin(\omega_m t - \omega_m \tau)\, d\tau$$

CONSIDERING THAT:

$$\boxed{\begin{aligned} \sin(\alpha)\cos(\beta) &= \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2} \\ \sin(\alpha)\sin(\beta) &= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \end{aligned}}$$

WE OBTAIN:

$$x(t) = \frac{F_0}{2m\omega_m} \int_0^t [\cos(\omega_m \tau - \omega_m t + \omega_m \tau) - \cos(\cancel{\omega_m \tau} + \omega_m t - \cancel{\omega_m \tau})]\, d\tau$$

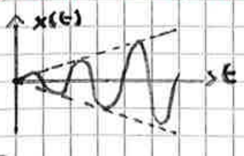
$$x(t) = \frac{F_0}{2m\omega_m} \int_0^t [\cos(2\omega_m \tau - \omega_m t) - \cos(\omega_m t)]\, d\tau$$

$$x(t) = \frac{F_0}{2m\omega_m} \cdot \left[\frac{\sin(2\omega_m \tau - \omega_m t)}{2\omega_m} - \cos(\omega_m t)\tau \right]_0^t = \frac{F_0}{2m\omega_m} \left[\frac{\sin(\omega_m t)}{2\omega_m} - \cos(\omega_m t) t - \right.$$

$$\left. - \frac{\sin(-\omega_m t)}{2\omega_m} \right] = \frac{F_0}{2m\omega_m} \left[\frac{\sin(\omega_m t)}{\omega_m} - \cos(\omega_m t)\, t \right]$$

$$x(t) = \frac{F_0}{2m\omega_m} \left[\frac{\sin(\omega_m t)}{\omega_m} - t \cdot \cos(\omega_m t) \right] u(t) \quad ✓$$

$t \to \infty$: $x(t) \to \infty$



2) →

EXERCISE 10: CONVOLUTION INTEGRAL

UNDAMPED SDOF SYSTEM

$F(t) = F_0 \sin(\omega t)\, u(t)$, $u(t)$: UNIT STEP FUNCTION

NULL I.C.

1) DEFINE THE RESPONSE ($x(t)$): CONVOLUTION INTEGRAL

2) COMPARE IT WITH THE RESULT OBTAINED WITH AN ALTERNATIVE APPROACH.

EXECUTION

1) CONVOLUTION INTEGRAL

$$x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau \quad , \quad h(t) = \frac{1}{\omega_d m} e^{-\zeta\omega_n (t-t_0)} \sin[\omega_d (t-t_0)]\, u(t-t_0) \; , \; \tau = t - t_0$$

$$F(\tau) = F_0 \sin(\omega \tau)\, u(\tau) \qquad (\zeta = 0)$$

$$h(t-\tau) = \frac{1}{\omega_n m} \sin[\omega_n (t-\tau)]\, u(t-\tau)$$

$$x(t) = \int_0^t \frac{F_0}{m\omega_n} \sin(\omega\tau) u(\tau) \sin(\omega_n t - \omega_n \tau)\, u(t-\tau)\, d\tau$$

CONSIDERING $t \geq t_0$ ($u(\tau) = 1$, $u(t-\tau) = 1$):

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin(\omega\tau) \sin(\omega_n t - \omega_n\tau)\, d\tau$$

CONSIDERING THAT:

$$\boxed{\begin{aligned} \sin\alpha \cos\beta &= \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2} \\ \sin\alpha \sin\beta &= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \end{aligned}}$$

WE OBTAIN:

$$x(t) = \frac{F_0}{2m\omega_n} \int_0^t [\cos(\omega\tau - \omega_n t + \omega_n \tau) - \cos(\omega\tau + \omega_n t - \omega_n \tau)]\, d\tau$$

$$x(t) = \frac{F_0}{2m\omega_n} \left\{ + \frac{\sin(\omega\tau - \omega_n t + \omega_n\tau)}{(\omega_n + \omega)} - \frac{\sin(\omega\tau + \omega_n t - \omega_n\tau)}{(\omega - \omega_n)} \right\}_0^t$$

$$x(t) = \frac{F_0}{2m\omega_n} \left\{ \frac{\sin(\omega t)}{\omega + \omega_n} - \frac{\sin(\omega t)}{\omega - \omega_n} - \frac{\sin(-\omega_n t)}{\omega + \omega_n} + \frac{\sin(\omega_n t)}{\omega - \omega_n} \right\} \quad + \omega \sin(\omega_n t)\, \cancel{+ \omega_n \sin(\omega_n t)}$$

$$x(t) = \frac{F_0}{2m\omega_n} \left\{ \frac{\cancel{\omega \sin(\omega t)} - \omega_n \sin(\omega t) - \cancel{\omega \sin(\omega t)} - \omega_n \sin(\omega t) - \omega \sin(-\omega_n t) + \cancel{\omega_n \sin(-\omega_n t)}}{(\omega + \omega_n)(\omega - \omega_n)} \right\}$$

$$x(t) = \frac{F_0}{2m\omega_n} \cdot \frac{1}{(\omega^2 - \omega_n^2)} \cdot [2\omega \sin(\omega_n t) - 2\omega_n \sin(\omega t)]$$

$$x(t) = \frac{F_0}{m \cancel{\omega_n} \omega_n^2 \left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]} \cancel{\omega_n} \left[\sin(\omega t) - \left(\frac{\omega}{\omega_n}\right) \sin(\omega_n t)\right]$$

$$x(t) = \frac{F_0}{K} \cdot \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \left[\sin(\omega t) - \left(\frac{\omega}{\omega_n}\right) \sin(\omega_n t)\right] u(t) \quad ✓$$

2) ALTERNATIVE APPROACH

$$m\ddot{x} + Kx = F_0 \sin(\omega t) \; ; \; x(t) = x_P(t) + x_A(t)$$

$$\begin{cases} x_P(t) = x_0 \sin(\omega t) \Rightarrow \omega_n^2 [1 - (\omega/\omega_n)^2]\, x_0 \cancel{\sin(\omega t)} = \frac{F_0}{K} \omega_n^2 \cancel{\sin(\omega t)} : x_0 = \frac{F_0}{K} \frac{1}{1 - (\omega/\omega_n)^2} \\ x_A(t) = a \cos(\omega_n t) + b \sin(\omega_n t) \end{cases}$$

IMPOSING THE I.C.

$$\begin{cases} x(t=0) = 0 = a \\ \dot{x}(t=0) = 0 = \frac{F_0}{K} \frac{\omega}{1 - (\omega/\omega_n)^2} + \omega_n b \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = -\frac{F_0}{K\omega_n} \cdot \frac{\omega}{1 - (\omega/\omega_n)^2} \end{cases} \Rightarrow x(t) = \frac{F_0}{K[1 - (\omega/\omega_n)^2]} \left[\sin(\omega t) - \frac{\sin(\omega_n t)}{\omega_n}\right]$$

$$x(t) = \frac{F_0}{K} \cdot \frac{1}{1 - (\omega/\omega_n)^2} \left[\sin(\omega t) - \left(\frac{\omega}{\omega_n}\right) \sin(\omega_n t)\right] \quad ✓$$

5) PHASE (φ), $\omega = \omega_{RES}$:

$$\varphi_- = Tg^{-1}\left(-\frac{Im}{Re}\right) = Tg^{-1}\left[-\frac{2\xi\left(\frac{\omega}{\omega_m}\right)}{1-\left(\frac{\omega}{\omega_m}\right)^2}\right] = -41,41^\circ \;;\; |\theta_0| = \frac{40\cdot 9.8\cdot 10^3\cdot 0.05\cdot 0.5}{(40\cdot 9.8\cdot 10^3\cdot 0.5^2+19038.22)\sqrt{[1-0.28]^2+0.4032}}$$

$$= 0.0872\,[Rad]$$

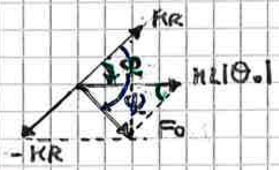
✓

6) $|F_0| = ?$:

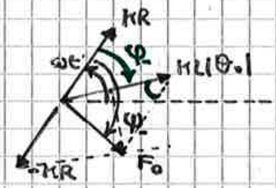
$$F = K(L\theta - R\sin\alpha) = K(L\theta - R\sin(\omega t))$$

$$\theta = \theta_0 e^{i\omega t} = |\theta_0| e^{i(\omega t+\varphi_-)} = |\theta_0|\sin(\omega t+\varphi_-)$$

$$F = KL|\theta_0|\sin(\omega t+\varphi_-) - KR\sin(\omega t) = F_0\sin(\omega t+\psi_-)$$



PROPERLY:



$$F_0^2 = (KL|\theta_0|)^2 + (KR)^2 - 2K^2L|\theta_0|R\cos(\varphi_+) = (40\cdot 9.8\cdot 10^3\cdot 0.5\cdot$$

$$F_0 = \sqrt{(40\cdot 9.8\cdot 10^3\cdot 0.5\cdot 0.0872)^2 + (40\cdot 9.8\cdot 10^3\cdot 0.05)^2 - 2\cdot(40\cdot 9.8\cdot 10^3)^2\cdot 0.5\cdot 0.0872\cdot 0.05\cdot\cos(41.41)} = 13'183\,[N]$$

$$KR\sin\varphi_+ = F_0\sin(\psi_+ - \varphi_+) \Rightarrow \psi_+ = \sin^{-1}\left(\frac{KR\sin\varphi_+}{F_0}\right) + \varphi_+ = \sin^{-1}\left(\frac{40\cdot 9.8\cdot 10^3\cdot 0.05\sin(41.41)}{13183}\right) + 41.41$$

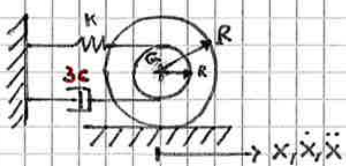
$$\psi_- = -120.96^\circ$$

✓

+ EXERCISE : PULLEY SYSTEM (EA04_2021-10-29_16:00)

DATA

TWO HOMOGENEOUS COAXIAL PULLEYS



$h_1 = h_2 = h = 2\,[cm]$: THICKNESS

$R = 2R$, $R = 10\,[cm]$; $C_{EQ} = 3C$

$m = 10\,[kg]$: TOTAL MASS

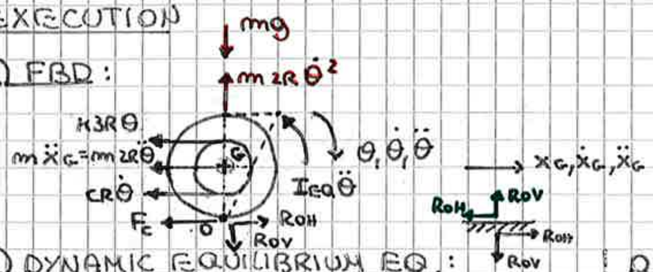
$C = 130\,[Ns/m]$; $K = 1500\,[N/m]$

1) FBD

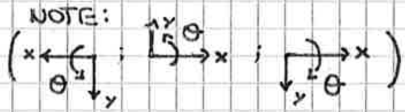
2) DYNAMIC EQUILIBRIUM EQ. $(\omega_n = ?\ ,\ \zeta = ?)$

EXECUTION

1) FBD :



NOTE:



2) DYNAMIC EQUILIBRIUM EQ. :

$O\circlearrowright \; -I_{EQ}\ddot\theta - m4R^2\ddot\theta - K9R^2\theta - 3CR^2\dot\theta = 0$

$(I_{EQ} + m4R^2)\ddot\theta + 3CR^2\dot\theta + K9R^2\theta = 0$ ✓

$I_{EQ} = I_1 + I_2 = \frac{mR^2}{2} + \frac{m(2R)^2}{2} = \frac{5}{2}mR^2$ ✗

BECAUSE $= \frac{m_1R_1^2}{2} + \frac{m_2R_2^2}{2}$, $m_1 \neq m_2$

$I_{EQ} = \frac{\rho V_1 R^2}{2} + \frac{\rho V_2 4R^2}{2} = \frac{\rho R^2}{2}(V_1 + 4V_2) = \frac{\rho R^2}{2}\pi R^2 h\,(1+16) = \frac{m\pi}{\pi R^2 h 5}\frac{R^4 h\,17}{2} = \frac{m R^2\,17}{10} = 0.17\,[kg\,m^2]$

$V_1 = \pi R_1^2 \cdot h_1 = \pi R^2 h$

$V_2 = \pi 4R^2 h$

$\rho = \frac{m}{V_1+V_2} = \frac{m}{\pi R^2 h 5}$

$I_O = I_{EQ} + 4mR^2 = 0.17 + 4\cdot 10\cdot 0.1^2 = 0.57\,[kg\,m^2]$

REACTIONS:

$\left\{ \begin{array}{l} (\rightarrow) \; +R_{OH} = F_c + CR\dot\theta + m2R\ddot\theta + K3R\theta = R_{OH}\,(\leftarrow) \\ (\downarrow) \; +R_{OV} = m2R\dot\theta^2 - mg = R_{OV}\,(\uparrow) \end{array} \right.$

$\theta_{TOT} = \theta_{ST} + \theta$, $\theta_{ST} = \frac{4RF_c}{K9R^2} = \frac{4 f_d mg}{9KR}$

NOT SURE !

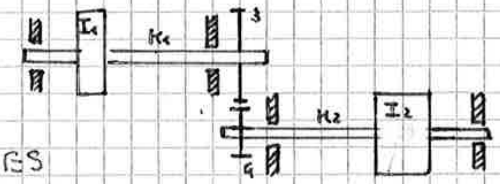
$\omega_n = \sqrt{\frac{9KR^2}{I_O}} = \sqrt{\frac{9\cdot 1500\cdot 0.1^2}{0.57}} = 15.39\,\left[\frac{Rad}{s}\right]$

$\frac{CR^2}{I_O} = 2\zeta\omega_n \Rightarrow \zeta = \frac{3CR^2}{2I_O\omega_n} = \frac{3\cdot 130\cdot 0.1^2}{2\cdot 0.57\cdot 15.39} = 0.222\,\left[\frac{Ns}{m}\right]$ ✓

APPLIED LECTURE 2: MDOF SYSTEMS 1st Round

EXERCISE 1: ELASTIC SHAFTS TORSIONAL OSCILLATIONS

DATA



NEGLECTING I_3, I_4, DISSIPATION

$\tau = \theta_4/\theta_3 \quad \left(\tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{\theta_4}{\theta_3} = \frac{R_{IN}}{R_{OUT}} = \frac{R_3}{R_4}\right)$

1) COMPUTE THE TORSIONAL NATURAL FREQUENCIES

2) COMPUTE THE NUMERICAL VALUES WITH: $I_1 = 0.3\,[kg\,m^2]$; $I_2 = 135\,[kg\,m^2]$

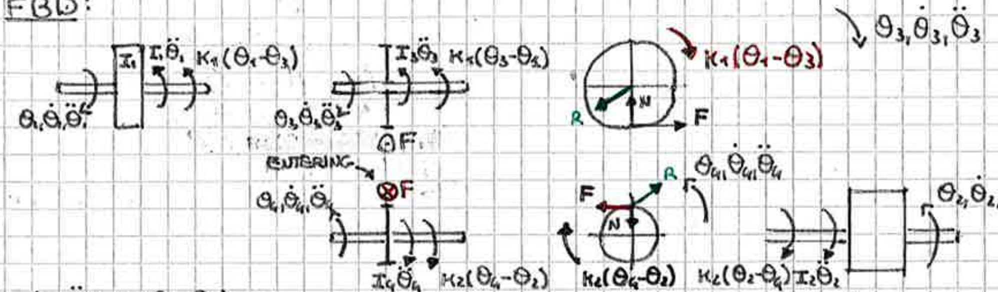
$k_1 = 600\,[N/m]$; $k_2 = 8000\,[N/m]$; $\tau = [0.07 \;\; 0.13 \;\; 0.20 \;\; 0.27 \;\; 0.38 \;\; 0.45]$

3) EVALUATE THE EFFECT OF τ ON THE NAT. FREQS. PLOT THE NAT FREQS $f(\tau)$.

EXECUTION

1) TORSIONAL NAT. FREQUENCIES:

FBD:



$$\begin{cases} I_1\ddot{\theta}_1 + k_1(\theta_1-\theta_3) = 0 \\ \cancel{I_3\ddot{\theta}_3} + k_1(\theta_3-\theta_1) + FR_3 = 0 \quad \longleftrightarrow \quad I_3\ddot{\theta}_3 + FR_3 = k_1(\theta_1-\theta_3) \\ \cancel{I_4\ddot{\theta}_4} + k_2(\theta_4-\theta_2) - FR_4 = 0 \quad \longleftrightarrow \quad I_4\ddot{\theta}_4 + k_2(\theta_4-\theta_2) = FR_4 \\ I_2\ddot{\theta}_2 + k_2(\theta_2-\theta_4) = 0 \end{cases}$$

KNOWING THAT: $\tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{\theta_{OUT}}{\theta_{IN}} = \frac{R_{IN}}{R_{OUT}} = \frac{\theta_4}{\theta_3} = \frac{R_3}{R_4}$

$k_1(\theta_3-\theta_1) + k_2\tau(\tau\theta_3-\theta_2) = 0$

$F = \frac{k_2}{R_4}(\theta_4-\theta_2) = \frac{k_2}{R_4}(\tau\theta_3-\theta_2)$

$$\begin{cases} \theta_3 = \frac{k_1}{k_1+k_2\tau^2}\theta_1 + \frac{k_2\tau}{k_1+k_2\tau^2}\theta_2 \\ \theta_4 = \tau\theta_3 = \frac{k_1\tau}{k_1+k_2\tau^2}\theta_1 + \frac{k_2\tau^2}{k_1+k_2\tau^2}\theta_2 \end{cases}$$

$$\begin{cases} I_1\ddot{\theta}_1 + k_1\theta_1 - k_1\theta_3 = I_1\ddot{\theta}_1 + k_1\theta_1 - \frac{k_1^2}{k_1+k_2\tau^2}\theta_1 - \frac{k_1k_2\tau}{k_1+k_2\tau^2}\theta_2 = I_1\ddot{\theta}_1 + \frac{\cancel{k_1^2}+k_1k_2\tau^2-\cancel{k_1^2}}{k_{EQ}}\theta_1 - \frac{k_1k_2\tau}{k_{EQ}}\theta_2 = 0 \\ I_2\ddot{\theta}_2 + k_2\theta_2 - k_2\theta_4 = I_2\ddot{\theta}_2 + k_2\theta_2 - \frac{k_1k_2\tau}{k_1+k_2\tau^2}\theta_1 - \frac{k_2^2\tau^2}{k_1+k_2\tau^2}\theta_2 = I_2\ddot{\theta}_2 + \frac{k_1k_2+\cancel{k_2^2\tau^2}-\cancel{k_2^2\tau^2}}{k_{EQ}}\theta_2 - \frac{k_1k_2\tau}{k_{EQ}}\theta_1 = 0 \end{cases}$$

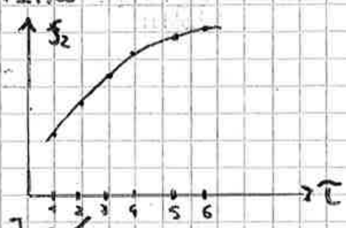
$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \frac{k_1k_2\tau^2}{k_{EQ}} & -\frac{k_1k_2\tau}{k_{EQ}} \\ -\frac{k_1k_2\tau}{k_{EQ}} & \frac{k_1k_2}{k_{EQ}} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad k_{EQ} = k_1 + k_2\tau^2$$

$$\det([K]-\omega^2[M]) = \det\begin{bmatrix} k_1k_2\tau^2/k_{EQ} - \omega^2 I_1 & -k_1k_2\tau/k_{EQ} \\ -k_1k_2\tau/k_{EQ} & k_1k_2/k_{EQ} - \omega^2 I_2 \end{bmatrix} = \left(\frac{k_1k_2\tau^2}{k_{EQ}} - \omega^2 I_1\right)\left(\frac{k_1k_2}{k_{EQ}} - \omega^2 I_2\right) - \left(-\frac{k_1k_2\tau}{k_{EQ}}\right)^2 = 0$$

$$\frac{k_1^2k_2^2\tau^2}{k_{EQ}^2} - \frac{I_2k_1k_2\tau^2}{k_{EQ}}\omega^2 - \frac{I_1k_1k_2}{k_{EQ}}\omega^2 + I_1I_2\omega^4 - \frac{k_1^2k_2^2\tau^2}{k_{EQ}^2} = 0 \rightarrow I_1I_2\omega^4 - \frac{k_1k_2}{k_{EQ}}(I_2\tau^2+I_1)\omega^2 = 0$$

$$\omega^2\left[I_1I_2\omega^2 - \frac{k_1k_2}{k_{EQ}}(I_2\tau^2+I_1)\right] = 0 \longrightarrow \omega_1 = 0\,[Rad/s]$$

$$\omega_2 = \sqrt{\frac{k_1k_2(I_2\tau^2+I_1)}{k_{EQ}I_1I_2}}\left[\frac{Rad}{s}\right] ✓$$

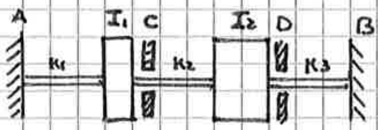


2) NUMERICAL VALUES: $\tau = [0.07 \;\; 0.13 \;\; 0.20 \;\; 0.27 \;\; 0.38 \;\; 0.45]$ (1 2 3 4 5 6)

$\left(f = \frac{2\pi}{\omega}\right)$ $\quad f_2 = [2.13 \;\; 3.25 \;\; 4.31 \;\; 5.07 \;\; 5.82 \;\; 6.11]\,[Hz]$ ✓

EXERCISE 3: ELASTIC SHAFTS TORSIONAL OSCILLATIONS

DATA



DERIVE THE EQS OF MOTION CONSIDERING:

1) $K_1 = K_2 = K_3 = K_t$; $I_2 = 2I_1$; $[\Lambda]$; $[\varphi]$ = ?

2) HARMONIC TORQUE ON DISK 1, f_{AR} = ?

3) NO CONSTRAINTS IN A, B ; $\dot{\theta}_0$ = CONST ; SUDDENLY THE SHAFT IS BLOCKED IN A AND B; DERIVE THE CONSEQUENT SYSTEM RESPONSE.

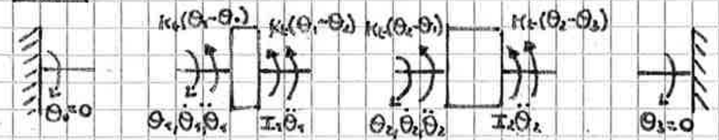
4) COMPUTE THE NUMERICALLY VALUES OF THE NAT FREQS ; $I_1 = 40.5\ [kg m^2]$

$d = 10\ [mm]$; $\ell_1 = \ell_2 = \ell_3 = 900\ [mm]$; $G = 80000\ [N/mm^2]$

EXECUTION

1) EQS OF M. AND $[\Lambda]$, $[\varphi]$:

FBD:



1) $I_1\ddot{\theta}_1 + 2K_t\theta_1 - K_t\theta_2 = 0$

2) $I_2\ddot{\theta}_2 + 2K_t\theta_2 - K_t\theta_1 = 0$

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{Bmatrix} + \begin{bmatrix} 2K_t & -K_t \\ -K_t & 2K_t \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$

$([K] - \omega^2[M])\{X_0\} = \{0\}$, $\{X_0\} = \{0\}$ TRIVIAL

$\det([K] - \omega^2[M]) = 0 \rightarrow \omega^2$

$$\det\begin{bmatrix} 2K_t - \omega^2 I_1 & -K_t \\ -K_t & 2K_t - \omega^2 2I_1 \end{bmatrix} = (2K_t - \omega^2 I_1)(2K_t - \omega^2 2I_1) - (-K_t)^2 = 4K_t^2 - 4I_1K_t\omega^2 - 2I_1K_t\omega^2 + 2I_1^2\omega^4 - K_t^2 = 0$$

$$2I_1^2\omega^4 - 6I_1K_t\omega^2 + 3K_t^2 = 0 \qquad \omega^2 = \frac{1}{4I_1^2}\left(6I_1K_t \pm \sqrt{36I_1^2K_t^2 - 24I_1^2K_t^2}\right) = \begin{cases} + & \omega_2^2 \\ - & \omega_1^2 \end{cases}$$

$$\omega_1^2 = \frac{1}{4I_1^2}(6I_1K_t - \sqrt{12}I_1K_t) = \frac{K_t}{I_1}\cdot\frac{6-\sqrt{12}}{4} = 0.634\frac{K_t}{I_1}$$

$$\omega_2^2 = \frac{K_t}{I_1}\,\frac{6+\sqrt{12}}{4} = 2.366\frac{K_t}{I_1}$$

$$[\Lambda] = [\,\diagdown\ \omega_R^2\ \diagdown\,] = \begin{bmatrix} 0.634 & 0 \\ 0 & 2.366 \end{bmatrix}\frac{K_t}{I_1} \quad ✓$$

MODE # 1

$$\begin{bmatrix} 2K_t - 0.634K_t & -K_t \\ -K_t & 2K_t - 2\cdot 0.634 \end{bmatrix}\begin{Bmatrix} 1 \\ \varphi_{2,1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$(2 - 0.634)K_t - K_t\varphi_{2,1} = 0 \Rightarrow \varphi_{2,1} = 1.366$

MODE # 2

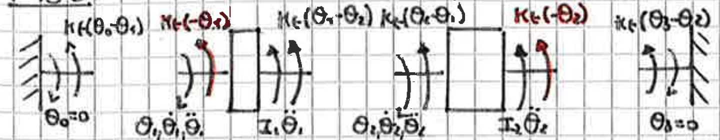
$$\begin{bmatrix} 2K_t - 2.366K_t & -K_t \\ -K_t & 2K_t - 2\cdot 2.366 \end{bmatrix}\begin{Bmatrix} 1 \\ \varphi_{2,2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\varphi_{2,2} = (2 - 2.366) = -0.366$

$$[\varphi] = \begin{bmatrix} 1 & 1 \\ 1.366 & -0.366 \end{bmatrix} \quad ✓$$

3) INITIALLY $\dot{\theta}_0 = \text{CONST}$, NO CONSTRAINT IN A, B.
THEN THE SHAFT IS BLOCKED IN A AND B
WHICH IS THE SYSTEM RESPONSE ?

FBD:



1) $I_1\ddot\theta_1 + k_t(\theta_1-\theta_2) = -k_t\theta_1 \quad \longleftrightarrow \quad I_1\ddot\theta_1 + 2k_t\theta_1 - k_t\theta_2 = 0$

2) $I_2\ddot\theta_2 + k_t(\theta_2-\theta_1) = -k_t\theta_2 \quad\quad I_2\ddot\theta_2 + 2k_t\theta_2 - k_t\theta_1 = 0$

$$\begin{bmatrix} I_1 & \\ & I_2 \end{bmatrix}\begin{Bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{Bmatrix} + \begin{bmatrix} 2k_t & -k_t \\ -k_t & 2k_t\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

$\{x\} = [\varphi]\{\eta\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_R \rightarrow \{x_0\} = \sum_{R=1}^{m}\{\varphi_R\}\eta_{R,0}$

$\eta_R(t) = \eta_{R\,HS}(t) = A_R\cos(\omega_R t) + B_R \sin(\omega_R t)$

$\{x\} = \sum_{R=1}^{m}\{\varphi_R\}[A_R\cos(\omega_R t) + B_R\sin(\omega_R t)]$

$\{\dot x\} = \sum_{R=1}^{m}\{\varphi_R\}[-\omega_R A_R \sin(\omega_R t) + \omega_R B_R\cos(\omega_R t)]$

I.C.:

$$\begin{cases}\{x(t=0)\} = \{x_0\} = \begin{Bmatrix}0\\0\end{Bmatrix}\\ \{\dot x(t=0)\} = \{\dot x_0\} = \begin{Bmatrix}\dot\theta_{1,0}\\ \dot\theta_{2,0}\end{Bmatrix} = \begin{Bmatrix}1\\1\end{Bmatrix}\dot\theta_0\end{cases}$$

PRE-MULTIPLYING BY $\{\varphi_S\}^T[M]$:

$\{\varphi_S\}^T[M]\{x\} = \sum_{R=1}^{m}\{\varphi_S\}^T[M]\{\varphi_R\}[A_R\cos(\omega_R t) + B_R\sin(\omega_R t)]$

$\{\varphi_S\}^T[M]\{\dot x\} = \sum_{R=1}^{m}\{\varphi_S\}^T[M]\{\varphi_R\}[-\omega_R A_R\sin(\omega_R t) + \omega_R B_R\cos(\omega_R t)]$

AT $t=0$:

$\{\varphi_S\}^T[M]\{x_0\} = \sum_{R=1}^{m}\{\varphi_S\}^T[M]\{\varphi_R\}\cdot A_R = m_S A_S$

$\{\varphi_S\}^T[M]\{\dot x_0\} = \sum_{R=1}^{m}\{\varphi_S\}^T[M]\{\varphi_R\}\cdot \omega_R B_R = m_S\omega_S B_S$

THEREFORE:

$A_R = \dfrac{\{\varphi_R\}^T[M]\{x_0\}}{m_R}$

$B_R = \dfrac{\{\varphi_R\}^T[M]\{\dot x_0\}}{m_R\omega_R}$

$A_1 = \dfrac{\{\varphi_1\}^T[M]\{x_0\}}{m_1} = \{1 \;\; 1.366\}\begin{bmatrix} I_1 & 0\\ 0 & 2I_1\end{bmatrix}\begin{Bmatrix}0\\0\end{Bmatrix}\cdot\dfrac{1}{m_1} = 0$

$A_2 = 0$

$B_1 = \{1 \;\; 1.366\}\begin{bmatrix} I_1 & 0\\ 0 & 2I_1\end{bmatrix}\begin{Bmatrix}\dot\theta_{1,0}\\ \dot\theta_{2,0}\end{Bmatrix}\dfrac{1}{m_1\omega_1} = \{I_1 \;\; 2.732 I_1\}\begin{Bmatrix}1\\1\end{Bmatrix}\dfrac{\dot\theta_0}{m_1\omega_1} = \dfrac{3.732\cancel{I_1}\,\dot\theta_0}{4.732\cancel{I_1}\cdot\sqrt{0.634\frac{k_t}{I_1}}} = \sqrt{\dfrac{I_1}{k_t}}\,\dot\theta_0$

$B_2 = \{1 \;\; -0.366\}\begin{bmatrix} I_1 & 0\\ 0 & 2I_1\end{bmatrix}\begin{Bmatrix}\dot\theta_{1,0}\\ \dot\theta_{2,0}\end{Bmatrix}\dfrac{1}{m_2\omega_2} = \{I_1 \;\; -0.732 I_1\}\begin{Bmatrix}1\\1\end{Bmatrix}\dfrac{\dot\theta_0}{m_2\omega_2} = \dfrac{0.268\,I_1\dot\theta_0}{1.268\,I_1\sqrt{2.366\frac{k_t}{I_1}}} = 0.137\sqrt{\dfrac{I_1}{k_t}}\,\dot\theta_0$

$\eta_1 = B_1\sin(\omega_1 t) = \sqrt{\dfrac{I_1}{k_t}}\,\dot\theta_0\sin(\omega_1 t)$

$\eta_2 = B_2\sin(\omega_2 t) = 0.137\sqrt{\dfrac{I_1}{k_t}}\,\dot\theta_0\sin(\omega_2 t)$

$\{x\} = [\varphi]\{\eta\}$

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{bmatrix}1 & 1\\ 1.366 & -0.366\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\end{Bmatrix} = \begin{Bmatrix}\eta_1+\eta_2\\ 1.366\,\eta_1 - 0.366\,\eta_2\end{Bmatrix} = \begin{Bmatrix}\sin(\omega_1 t) + 0.137\sin(\omega_2 t)\\ 1.366\sin(\omega_1 t) - 0.05\sin(\omega_2 t)\end{Bmatrix}\sqrt{\dfrac{I_1}{k_t}}\,\dot\theta_0 \quad ✓$$

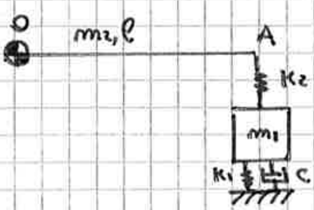
EXERCISE 4: BEAM AND MASS OSCILLATIONS

DATA

$\overline{OA} = \ell = 0.5\ [m]$

$m_1 = 5\ [kg]$ $m_2 = 10\ [kg]$

$k_1 = 2\ [N/m]$; $k_2 = 4\ [N/m]$



BEING THE MASSES DEFINED AS $m_2 (\to \theta)$ AND $m_1 (\to y)$ IT WOULD BE BETTER TO CONSIDER

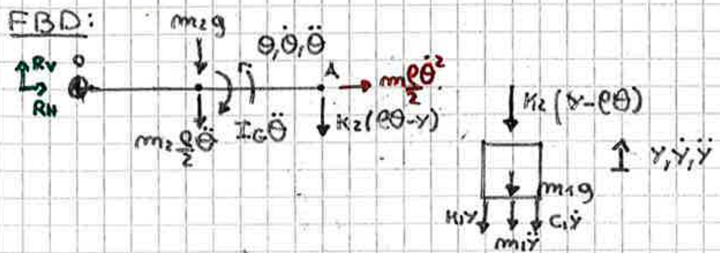
$[M]\begin{Bmatrix}\ddot{y}\\ \ddot{\theta}\end{Bmatrix} + [C]\begin{Bmatrix}\dot{y}\\ \dot{\theta}\end{Bmatrix} + [K]\begin{Bmatrix}y\\ \theta\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

1) DERIVE THE MOTION EQS, ASSUMING SMALL OSCILLATIONS.

2) COMPUTE THE SYSTEM NATURAL FREQUENCIES AND MODAL MATRIX ($C_1 = 0$).

EXECUTION

FBD:



$\circlearrowleft O)\ \left(I_G + \frac{m_2\ell^2}{4}\right)\ddot{\theta}_T + k_2\ell^2\theta_T - k_2\ell y_T = m_2 g \frac{\ell}{2}$

$\uparrow)\ m_1\ddot{y}_T + C_1\dot{y}_T + (k_1 + k_2)y_T - k_2\ell\theta_T = m_1 g$

$\theta_{TOT} = \theta_{st} + \theta$, $\theta_{st} = \left(m_2 g \frac{\ell}{2} + k_2 \ell y_{st}\right)\frac{1}{k_2\ell^2}$

$y_{TOT} = y_{st} + y$, $y_{st} = \left(m_1 g + k_2\ell\theta_{st}\right)\frac{1}{k_1 + k_2}$

1) MOTION EQUATIONS, CONSIDERING ONLY SMALL OSCILLATIONS

$\circlearrowleft O)\ \left(I_G + \frac{m_2\ell^2}{4}\right)\ddot{\theta} + k_2\ell^2\theta - k_2\ell y = 0$

$\uparrow)\ m_1\ddot{y} + C_1\dot{y} + (k_1 + k_2)y - k_2\ell\theta = 0$

$\begin{bmatrix}m_1 & 0\\ 0 & I_G + \frac{m_2\ell^2}{4}\end{bmatrix}\begin{Bmatrix}\ddot{y}\\ \ddot{\theta}\end{Bmatrix} + \begin{bmatrix}C_1 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{y}\\ \dot{\theta}\end{Bmatrix} + \begin{bmatrix}(k_1+k_2) & -k_2\ell\\ -k_2\ell & k_2\ell^2\end{bmatrix}\begin{Bmatrix}y\\ \theta\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

$$\begin{bmatrix}I_G + \frac{m_2\ell^2}{4} & 0\\ 0 & m_1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}\\ \ddot{y}\end{Bmatrix} + \begin{bmatrix}0 & 0\\ 0 & C_1\end{bmatrix}\begin{Bmatrix}\dot{\theta}\\ \dot{y}\end{Bmatrix} + \begin{bmatrix}k_2\ell^2 & -k_2\ell\\ -k_2\ell & (k_1+k_2)\end{bmatrix}\begin{Bmatrix}\theta\\ y\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \iff [M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

✓

2) NATURAL FREQUENCIES AND MODAL MATRIX ($C_1 = 0$)

$([K] - \omega_R^2[M])\{\varphi_R\} = \{0\}$ $R = 1, 2$

$\{\varphi_R\} = 0$ TRIVIAL

$\det([K] - \omega_R^2[M]) = 0 \Rightarrow \omega_R^2$

$I_0 = I_G + \frac{m_2\ell^2}{4} = \frac{m_2\ell^2}{12} + \frac{m_2\ell^2}{4} = \frac{m_2\ell^2}{3} = \frac{10 \cdot 0.5^2}{3} = 0.833\ [kg\,m^2]$

$k_2\ell^2 = 4 \cdot 0.5^2 = 1$; $k_1 + k_2 = 2 + 4 = 6\ [N/m]$

$-k_2\ell = -4 \cdot 0.5 = -2$

$\det\begin{bmatrix}1 - \omega_R^2 \cdot 0.833 & -2\\ -2 & 6 - \omega_R^2 \cdot 5\end{bmatrix} = (1 - \omega_R^2\, 0.833)(6 - \omega_R^2 \cdot 5) - 4 = 0 \iff 6 - \omega_R^2 5 - 6 \cdot 0.833\,\omega_R^2 + 0.833 \cdot 5 \cdot \omega_R^4 - 4 = 0$

$4.165\,\omega_R^4 - 9.998\,\omega_R^2 + 2 = 0 \Rightarrow \omega_R^2 = \frac{9.998 \pm \sqrt{9.998^2 - 4 \cdot 4.165 \cdot 2}}{2 \cdot 4.165} = \frac{9.998 \pm 8.163}{8.33}$

$\omega_R^2 \begin{cases} + & 2.18 \to \omega_2 = 1.476\ [Rad/s] = \omega_2 \\ - & 0.22 \to \omega_1 = 0.469\ [Rad/s] = \omega_1 \end{cases}$ ✓

MODE #1 $\omega_1 = 0.469\ [Rad/s]$

$\begin{bmatrix}0.817 & -2\\ -2 & 4.9\end{bmatrix}\begin{Bmatrix}1\\ \varphi_{2,1}\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

$0.817 - 2\varphi_{2,1} = 0 \Rightarrow \varphi_{2,1} = \frac{0.817}{2} = 0.408$

$\{\varphi_1\} = \begin{Bmatrix}1\\ 0.408\end{Bmatrix}$

IF WE CONSIDER $\begin{Bmatrix}\ddot{y}\\ \ddot{\theta}\end{Bmatrix}$ WE OBTAIN: $\begin{Bmatrix}1\\ 2.45\end{Bmatrix}$ ✓

MODE #2 $\omega_2 = 1.476\ [Rad/s]$

$\begin{bmatrix}-0.816 & -2\\ -2 & -4.9\end{bmatrix}\begin{Bmatrix}1\\ \varphi_{2,2}\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

$-0.816 - 2\varphi_{2,2} = 0 \Rightarrow \varphi_{2,2} = -\frac{0.816}{2} = -0.408$

$\{\varphi_2\} = \begin{Bmatrix}1\\ -0.408\end{Bmatrix}$

IF $\begin{Bmatrix}\ddot{y}\\ \ddot{\theta}\end{Bmatrix} \to \begin{Bmatrix}1\\ -2.45\end{Bmatrix}$ ✓

$[\varphi] = [\{\varphi_1\}\{\varphi_2\}] = \begin{bmatrix}1 & 1\\ 0.408 & -0.408\end{bmatrix}$

$\begin{bmatrix}1 & 1\\ 2.45 & -2.45\end{bmatrix}$

2-b) NATURAL FREQUENCIES AND NODES

$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$

$\{x\} = \{x_0\}e^{i\omega t}$; $\{\dot{x}\} = i\omega\{x_0\}e^{i\omega t}$; $\{\ddot{x}\} = -\omega^2\{x_0\}e^{i\omega t}$

$([K] - \omega^2[M])\{x_0\} = \{0\}$

$\begin{cases} \{x_0\} = 0 \text{ TRIVIAL S.} \\ \det([K]-\omega^2[M]) = 0 \rightarrow \omega^2 \end{cases}$

$\det\begin{bmatrix} K_1+K_2-\omega^2 m & K_2\ell_2-K_1\ell_1 \\ K_2\ell_2-K_1\ell_1 & K_1\ell_1^2+K_2\ell_2^2-\omega^2 I_G \end{bmatrix} = \det\begin{bmatrix} 65'673-\omega^2 1461 & 31\,150.9 \\ 31\,150.9 & 165\,423.35-\omega^2 2034 \end{bmatrix} =$

$= 10863847660 - 133578882\,\omega^2 - 241683514.4\,\omega^2 + 2971674\,\omega^4 - 970378570.8 = 0$

$= 2971674\,\omega^4 - 375'262'396.4\,\omega^2 + 9'893'469\,089 = 0$

$\omega^2_{1,2} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \begin{cases} + \rightarrow 88.78 \rightarrow \omega_2 = 9.42\ [\text{Rad/s}] \\ - \rightarrow 37.5 \rightarrow \omega_1 = 6.12\ [\text{Rad/s}] \end{cases}$ ✓

MODE #1 $\omega_1 = 6.12$ [Rad/s]

$([K] - \omega_1^2[M])\{\varphi_1\} = \{0\}$

$\begin{bmatrix} 10\,952.12 & 31\,150.9 \\ 31\,150.9 & 89'241.1 \end{bmatrix}\begin{Bmatrix} \varphi_{1,1} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ WE PUT $\varphi_{2,1} = 1$ BECAUSE IT CAN BE EASIER TO SEE A QUANTITY IN DISPLACEMENT ([m]) RATHER THAN ANG. D. ([Rad])

$10\,952.12\,\varphi_{1,1} + 31\,150.9 = 0 \Rightarrow \varphi_{1,1} = -2.84 \Rightarrow \varphi_1 = \begin{Bmatrix} -2.84 \\ 1 \end{Bmatrix}$ OR

$\left(10\,952.12 + 31\,150.9\,\varphi'_{2,1} = 0 \Rightarrow \varphi'_{2,1} = -0.35 \Rightarrow \varphi'_1 = \begin{Bmatrix} 1 \\ -0.35 \end{Bmatrix}\right)$ (BOTH CORRECT)

MODE #2 $\omega_2 = 9.42$ [Rad/s]

$([K] - \omega_2^2[M])\{\varphi_2\} = \{0\}$

$\begin{bmatrix} -63\,970.9 & 31\,150.9 \\ 31\,150.9 & -15\,066.5 \end{bmatrix}\begin{Bmatrix} \varphi_{1,2} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$\Rightarrow [\varphi] = \begin{bmatrix} -2.84 & 0.487 \\ 1 & 1 \end{bmatrix}$ OR $\Rightarrow [\varphi'] = \begin{bmatrix} 1 & 1 \\ -0.35 & 2.054 \end{bmatrix}$ ✓

$-63\,970.9\,\varphi_{1,2} + 31\,150.9 = 0 \Rightarrow \varphi_{1,2} = 0.487 \Rightarrow \varphi_2 = \begin{Bmatrix} 0.487 \\ 1 \end{Bmatrix}$ OR

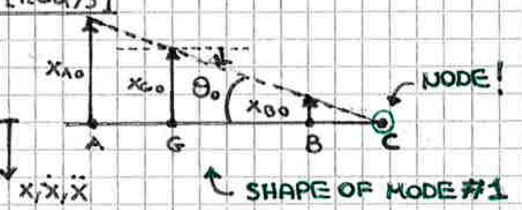
$\left(-63\,970.9 + 31\,150.9\,\varphi'_{2,2} = 0 \Rightarrow \varphi'_{2,2} = 2.054 \Rightarrow \varphi'_2 = \begin{Bmatrix} 1 \\ 2.054 \end{Bmatrix}\right)$

NODE C1 AND SHAPE OF MODE #1 $\omega_1 = 6.12$ [Rad/s]

IF $\theta_0 = 1$ [Rad] $\Rightarrow x_{G0} = -2.84$ [m]

$\Rightarrow x_{A0} = x_G - \ell_1\theta_0 = -2.84 - 1.219\cdot 1 = -4.06$ [m]

$\Rightarrow x_{B0} = x_G + \ell_2\theta_0 = -2.84 + 1.829\cdot 1 = -1.01$ [m]



$x_{B0} = \overline{BC}\,\mathrm{Tg}\,\theta_0 \cong \overline{BC}\,\theta_0 \Rightarrow BC = \dfrac{|x_{B0}|}{\theta_0} = 1.01$ [m]

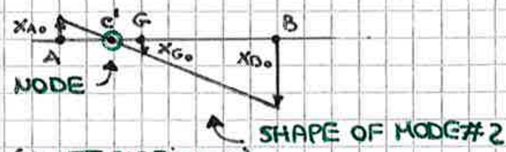
$\overline{GC} = \ell_2 + \overline{BC} = 1.829 + 1.01 = 2.84$ [m] ✓ **(RIGHT SIDE OF G)**

NODE C2 AND SHAPE OF MODE #2 $\omega_2 = 9.42$ [Rad/S]

IF $\theta_0 = 1$ [Rad] $\Rightarrow x_{G0} = 0.487$ [m]

$\Rightarrow x_{A0} = x_G - \ell_1\theta_0 = 0.487 - 1.219\cdot 1 = -0.732$ [m]

$\Rightarrow x_{B0} = x_G + \ell_2\theta_0 = 0.487 + 1.829\cdot 1 = 2.316$ [m]



$x_{G0} = \overline{GC'}\,\mathrm{Tg}\,\theta_0 \cong \overline{GC'}\,\theta_0 \Rightarrow \overline{GC'} = \dfrac{x_{G0}}{\theta_0} = 0.487$ [m] **(LEFT SIDE OF G)**

NOTE: MODE #1 AND MODE # ARE SYNCHRONOUS ; THE SOLUTIONS REPRESENTED BY THESE 2 MODES ARE BOTH POSSIBLE SYNCHR. SOLUTION; SINCE THE S. IS LINEAR → SUPERPOS. OF EFFECTS!

→ GO BACK TO "REAL WORLD":

$\{x(t)\} = [\varphi]\{\eta(t)\}$ DIRECT MODAL TRANSFORMATION (DMT)

$$\begin{Bmatrix} x_G(t) \\ \theta(t) \end{Bmatrix} = [\varphi]\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = \begin{bmatrix} -2.84 & 0.487 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix}$$

$$\begin{cases} x_G(t) = -2.84\,\eta_1(t) + 0.487\,\eta_2(t) = -0.852\,\Delta\cos(6.12t) + 0.146\,\Delta\cos(9.42t) \\ \theta(t) = \eta_1(t) + \eta_2(t) = 0.30\,\Delta\left[\cos(6.12t) + \cos(9.42t)\right] \end{cases}$$

✓

FREE RESPONSE OF THE SYSTEM.

4-b) FINALLY, COMPUTE THE DAMPING FACTORS DUE TO THE PRESENCE OF TWO VISCOUS DAMPERS, ASSUMING A PROPORTIONAL DAMPING DISTRIBUTION:

$[C] = [C_P] = \alpha[M] + \beta[K]$, WITH $\alpha = \beta = 0.2$

$$m_R\ddot{\eta}_R + (\alpha m_R + \beta k_R)\dot{\eta}_R + k_R\eta_R = 0 \quad R = 1, 2$$

$$\ddot{\eta}_R + (\alpha + \beta\omega_R^2)\dot{\eta}_R + \omega_R^2\eta_R = 0 \quad \longleftrightarrow \quad \ddot{\eta}_R + 2\zeta_R\omega_R\dot{\eta}_R + \omega_R^2\eta_R = 0$$

$$2\zeta_R\omega_R = \alpha + \beta\omega_R^2 \Rightarrow \zeta_R = \frac{\alpha}{2\omega_R} + \frac{\beta\omega_R}{2}$$

$$\zeta_1 = \frac{\alpha}{2\omega_1} + \frac{\beta\omega_1}{2} = \frac{0.2}{2\cdot 6.12} + \frac{0.2\cdot 6.12}{2} = 0.628 = 62.8\,\%$$

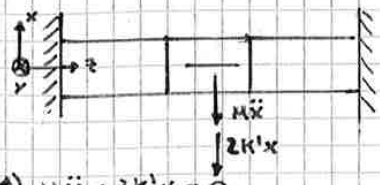
$$\zeta_2 = \frac{\alpha}{2\omega_2} + \frac{\beta\omega_2}{2} = \frac{0.2}{2\cdot 9.42} + \frac{0.2\cdot 9.42}{2} = 0.953 = 95.3\,\%$$

5) TRY WITH DIFFERENT SETS OF COORDINATES ...

5-a) WE CAN USE THE LAGRANGE'S EQUATIONS

...

2-b) NAT. FREQ. IN THE HORIZONTAL PLANE (ω_H)

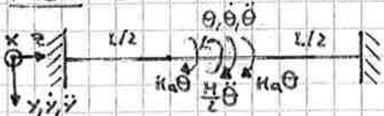


$\downarrow)\ M\ddot{x} + 2K'x = 0$

$$\omega_H = \sqrt{\frac{2K'}{M}} = \sqrt{\frac{2\cdot 192\cdot E\cdot I_y}{M L^3}} = \sqrt{\frac{2\cdot 192\cdot 2.1\cdot 10^{11}\cdot 3.64\cdot 10^{-8}}{100\cdot 3^3}} = 32.97\ \left[\frac{Rad}{s}\right]\ ✓$$

2-C) NAT. FREQ. OF AXIAL OSCILLATIONS (ω_a) ✗

FBD: (CONSIDERING ONLY ONE BEAM)



$\boxed{K_a = \frac{EA}{L/2}}$ $\quad A = 5.64\ [cm^2] = 5.64\cdot 10^{-4}\ [m^2]$

$\circlearrowleft)\ \frac{M}{2}\ddot{\theta} + 2K_a\theta = 0$ TOTALLY WRONG! LOOK 2nd ROUND VERSION.

$$\omega_a = \sqrt{\frac{2K_a}{M/2}} = \sqrt{\frac{4K_a}{M}} = \sqrt{\frac{4EA}{M(L/2)}} = \sqrt{\frac{8EA}{ML}} = \sqrt{\frac{8\cdot 2.1\cdot 10^{11}\cdot 5.64\cdot 10^{-4}}{100\cdot 3}} = 1777.2\ \left[\frac{Rad}{s}\right]$$

2-d) ENGINE SPEED (ω) - FLEXURAL RESONANCE

$\downarrow)\ M\ddot{y} + 2K_0 y = m\varepsilon\omega^2 \sin(\omega t) \iff \ddot{y} + \omega_n^2 y = \frac{m\varepsilon\omega^2}{M} e^{i\omega t}$, $y(t) = Y_0 e^{i\omega t}$

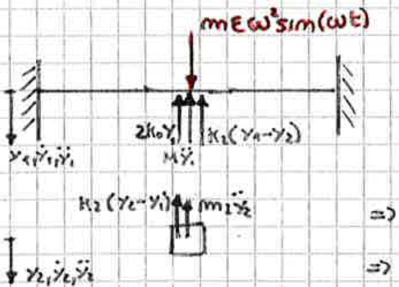
$\omega_R = \omega = \omega_y = 106.67\ \left[\frac{Rad}{s}\right]$; $\omega = \frac{2\pi n}{60}\ [Rpm]$

$n = \frac{\omega\cdot 60}{2\cdot\pi} = \frac{106.67\cdot 60}{2\cdot 3.14} = 1019.1\ [Rpm]$ ✓

3) $m_2 = ?\ K_2 = ?\ (\Rightarrow X_{2,0} < 1\,[cm])$

FBD:

(NOTE: $m\varepsilon = m_0 R = 5\ [Kg\,mm] = 5\cdot 10^{-3}\ [Kg\,m]$)



METHOD (1): INVERSION OF THE DYNAMIC STIFF. MATRIX

$$\{X_0\} = \frac{\{F_0\}}{([K] - \omega^2[M])} = [K_{DYN}]^{-1}\{F_0\} = [\alpha]\{F_0\}$$

$$\begin{Bmatrix} X_{1,0} \\ X_{2,0} \end{Bmatrix} = \frac{1}{\det[K_{DYN}]} \begin{bmatrix} COF(a_{11}) & COF(a_{12}) \\ COF(a_{21}) & COF(a_{22}) \end{bmatrix}^T \begin{Bmatrix} m\varepsilon\omega^2 \\ 0 \end{Bmatrix}$$

$\Rightarrow X_{2,0} = \frac{1}{\det[K_{DYN}]}\cdot COF(a_{21})\cdot m\varepsilon\omega^2 < 1\,[cm] = 10^{-2}\,[m]$

$\Rightarrow X_{1,0} = \frac{1}{\det[K_{DYN}]}\cdot COF(a_{11})\cdot m\varepsilon\omega^2 \boxed{= 0}$

$\downarrow)\ M\ddot{y}_1 + (2K_0 + K_2)y_1 - K_2 y_2 = m\varepsilon\omega^2 \sin(\omega t)$

$\downarrow)\ m_2\ddot{y}_2 - K_2 y_1 + K_2 y_2 = 0$

$$\iff \begin{bmatrix} M & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 2K_0 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} 2K_0 + K_2 - \omega^2 M & -K_2 \\ -K_2 & K_2 - \omega^2 m_2 \end{bmatrix} \quad [A]^{-1} = \frac{1}{\det[A]}\begin{bmatrix} COF(a_{11}) & COF(a_{12}) \\ COF(a_{21}) & COF(a_{22}) \end{bmatrix}^T, \quad COF(a_{jk}) = (-1)^{j+k} C_{jk}, \quad C_{jk} = \det(..)$$

$$\det[A] = (2K_0 + K_2 - \omega^2 M)(K_2 - \omega^2 m_2) - (-K_2)^2 = 2K_0K_2 - 2K_0\omega^2 m_2 + \cancel{K_2^2} - K_2\omega^2 m_2 - K_2\omega^2 M + \omega^4 M m_2 - \cancel{K_2^2} =$$

$$= \omega^4 M m_2 - (2K_0 m_2 + K_2 m_2 + K_2 M)\omega^2 + 2K_0 K_2$$

$COF(a_{12}) = (-1)^{1+2}\cdot C_{jk} = -C_{jk} = -\det(-K_2) = K_2$; $COF(a_{11}) = (-1)^{1+1}\cdot C_{jk} = C_{jk} = K_2 - \omega^2 m_2$

$X_{1,0} = \frac{(K_2 - \omega^2 m_2)\, m\varepsilon\omega^2}{\det[K_{DYN}]} = 0 \iff K_2 - \omega^2 m_2 = 0 \rightarrow \boxed{K_2 = \omega^2 m_2\ \text{TUNED MASS DAMPER}}$

$$X_{2,0} = \frac{+K_2 m\varepsilon\omega^2}{\det[K_{DYN}]} < 10^{-2} \rightarrow \frac{+\omega^4 m_2 m\varepsilon}{\omega^4 M m_2 - (2K_0 m_2 + \omega^2 m_2^2 + \omega^2 m_2 M)\omega^2 + 2K_0\omega^2 m_2} < 10^{-2}$$

$+\omega^2 m\varepsilon < 10^{-2}[\cancel{\omega^2 M} - \cancel{2K_0} - \omega^2 m_2 - \cancel{\omega^2 M} + \cancel{2K_0}] \rightarrow +m\varepsilon < -10^{-2} m_2 \rightarrow m_2 > \frac{m\varepsilon}{10^{-2}} = \frac{5\cdot 10^{-3}}{10^{-2}} = 0.5\,[Kg]$

$\Rightarrow K_2 = \omega^2 m_2 = 106.67^2\cdot 0.5 = 5689.24\ [N/m]$ ✓ ✓

CASE $K = 2 \Rightarrow q_k = q_2 = \theta$

$$\frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} = 0 - \left(-a_1 k_{s1}(z - a_1\theta - z_1) + a_2 k_{s2}(z + a_2\theta - z_2)\right)$$

$$\frac{\partial L}{\partial \dot\theta} = \frac{\partial T}{\partial \dot\theta} - \frac{\partial V}{\partial \dot\theta} = I_y\dot\theta - 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = I_y\ddot\theta$$

$$Q_2 = -\frac{\partial D}{\partial \dot\theta} = -\left(-a_1 c_{s1}(\dot z - a_1\dot\theta - \dot z_1) + a_2 c_{s2}(\dot z + a_2\dot\theta - \dot z_2)\right)$$

$$I_y\ddot\theta - a_1 k_{s1}(z - a_1\theta - z_1) + a_2 k_{s2}(z + a_2\theta - z_2) - a_1 c_{s1}(\dot z - a_1\dot\theta - \dot z_1) + a_2 c_{s2}(\dot z + a_2\dot\theta - \dot z_2) = 0$$

(2) $I_y\ddot\theta + (a_2 c_{s2} - a_1 c_{s1})\dot z + (a_1^2 c_{s1} + a_2^2 c_{s2})\dot\theta + a_1 c_{s1}\dot z_1 - a_2 c_{s2}\dot z_2 + (a_2 k_{s2} - a_1 k_{s1})z +$

$+ (a_1^2 k_{s1} + a_2^2 k_{s2})\theta + a_1 k_{s1} z_1 - a_2 k_{s2} z_2 = 0$

CASE $K = 3 \Rightarrow q_k = q_3 = z_1$

$$\frac{\partial L}{\partial z_1} = \frac{\partial T}{\partial z_1} - \frac{\partial V}{\partial z_1} = 0 - \left[k_1(z_1 - u_1) - k_{s1}(z - a_1\theta - z_1)\right]$$

$$\frac{\partial L}{\partial \dot z_1} = \frac{\partial T}{\partial \dot z_1} - \frac{\partial V}{\partial \dot z_1} = m_1\dot z_1 - 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot z_1}\right) = m_1\ddot z_1$$

$$Q_3 = -\frac{\partial D}{\partial \dot z_1} = -\left[c_1(\dot z_1 - \dot u_1) - c_{s1}(\dot z - a_1\dot\theta - \dot z_1)\right]$$

$$m_1\ddot z_1 + k_1(z_1 - u_1) - k_{s1}(z - a_1\theta - z_1) + c_1(\dot z_1 - \dot u_1) - c_{s1}(\dot z - a_1\dot\theta - \dot z_1) = 0$$

(3) $m_1\ddot z_1 - c_{s1}\dot z + c_{s1}a_1\dot\theta + (c_1 + c_{s1})\dot z_1 - k_{s1}z + k_{s1}a_1\theta + (k_1 + k_{s1})z_1 = c_1\dot u_1 + k_1 u_1$

CASE $K = 4 \Rightarrow q_k = q_4 = z_2$

$$\frac{\partial L}{\partial z_2} = \frac{\partial T}{\partial z_2} - \frac{\partial V}{\partial z_2} = 0 - \left[k_2(z_2 - u_2) - k_{s2}(z + a_2\theta - z_2)\right]$$

$$\frac{\partial L}{\partial \dot z_2} = \frac{\partial T}{\partial \dot z_2} - \frac{\partial V}{\partial \dot z_2} = m_2\dot z_2 - 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot z_2}\right) = m_2\ddot z_2$$

$$Q_4 = -\frac{\partial D}{\partial \dot z_2} = -\left[c_2(\dot z_2 - \dot u_2) - c_{s2}(\dot z + a_2\dot\theta - \dot z_2)\right]$$

$$m_2\ddot z_2 + k_2(z_2 - u_2) - k_{s2}(z + a_2\theta - z_2) + c_2(\dot z_2 - \dot u_2) - c_{s2}(\dot z + a_2\dot\theta - \dot z_2) = 0$$

(4) $m_2\ddot z_2 - c_{s2}\dot z - c_{s2}a_2\dot\theta + (c_2 + c_{s2})\dot z_2 - k_{s2}z - k_{s2}a_2\theta + (k_2 + k_{s2})z_2 = c_2\dot u_2 + k_2 u_2$

CASE IN MATRIX :

$$\begin{bmatrix} m & & & \\ & I_y & & \\ & & m_1 & \\ & & & m_2 \end{bmatrix}\begin{Bmatrix} \ddot z \\ \ddot\theta \\ \ddot z_1 \\ \ddot z_2 \end{Bmatrix} + \begin{bmatrix} c_{s1}+c_{s2} & c_{s2}a_2 - c_{s1}a_1 & -c_{s1} & -c_{s2} \\ a_2c_{s2} - a_1c_{s1} & a_1^2c_{s1} + a_2^2c_{s2} & a_1c_{s1} & -a_2c_{s2} \\ -c_{s1} & a_1c_{s1} & c_1+c_{s1} & 0 \\ -c_{s2} & -a_2c_{s2} & 0 & c_2+c_{s2} \end{bmatrix}\begin{Bmatrix} \dot z \\ \dot\theta \\ \dot z_1 \\ \dot z_2 \end{Bmatrix} +$$

$$\begin{bmatrix} k_{s1}+k_{s2} & k_{s2}a_2 - k_{s1}a_1 & -k_{s1} & -k_{s2} \\ a_2k_{s2} - a_1k_{s1} & a_1^2k_{s1} + a_2^2k_{s2} & a_1k_{s1} & -a_2k_{s2} \\ -k_{s1} & k_{s1}a_1 & k_1+k_{s1} & 0 \\ -k_{s2} & -a_2k_{s2} & 0 & k_2+k_{s2} \end{bmatrix}\begin{Bmatrix} z \\ \theta \\ z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ c_1\dot u_1 + k_1u_1 \\ c_2\dot u_2 + k_2u_2 \end{Bmatrix}$$

2) $u_2 = u_1 = 0$; $F(t)$ APPLIED IN B ; DERIVE THE GENERALIZED FORCES VECTOR ($\Rightarrow Q$)

CASE $K = 1 \Rightarrow q_k = q_1 = z$ $\left[\vec F(t) = F(t)\vec k \Rightarrow \vec R_i = z_B\vec k\right]$

$$Q_1 = \sum_{i=1}^{m} \vec F_i \cdot \frac{\partial \vec R_i}{\partial z} = \vec F_1 \cdot \frac{\partial (z_B\vec k)}{\partial z} = F(t)\vec k \cdot \frac{\partial (z_B)}{\partial z}\vec k = F(t)$$

CASE $K = 2 \Rightarrow q_k = q_2 = \theta$ [NOTE: $\vec k \cdot \vec k = 1$]

$$Q_2 = \vec F_1 \cdot \frac{\partial (z_B\vec k)}{\partial \theta} = F(t)\vec k\,\frac{\partial (z_B)}{\partial \theta}\vec k = a_2 F(t)$$

CASE $3 \Rightarrow q_k = q_3 = z_1$

$$Q_3 = \vec F_1\,\frac{\partial (z_B\vec k)}{\partial z_1} = F(t)\vec k\,\frac{\partial (z_B)}{\partial z_1}\vec k = 0$$

CASE $4 \Rightarrow q_k = q_4 = z_2$

$$Q_4 = \vec F_1\,\frac{\partial (z_B\vec k)}{\partial z_2} = F(t)\vec k\,\frac{\partial (z_B)}{\partial z_2}\vec k = 0$$

(DUE ONLY TO EXT. F) $\Rightarrow Q = \begin{Bmatrix} F(t) \\ F(t)a_2 \\ 0 \\ 0 \end{Bmatrix} \leftrightarrow Q = \{F(t)\ \ F(t)a_2\ \ 0\ \ 0\}^T$ ✓

ASKED BY POINT 3

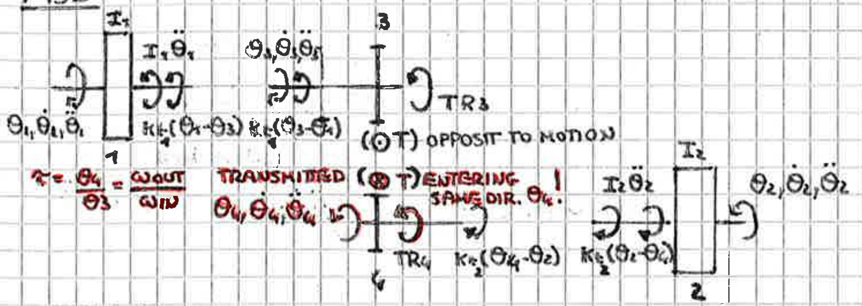
APPLIED LECTURE 2 : MDOF SYSTEMS 2nd round (ONLY EXECUTION)

EXERCISE 1 : ELASTIC SHAFTS TORSIONAL OSCILLATIONS

EXECUTION:

1) NEGLECTING I_3, I_4 ; $\tau = \theta_4/\theta_3$, NO DISSIPATION ; $\omega_1 = ?$ $\omega_2 = ?$

FBD:



MOTION EQs:

$$I_1\ddot\theta_1 + k_{t_1}(\theta_1-\theta_3) = 0$$

$$\begin{cases} k_{t_1}(\theta_3-\theta_1) + TR_3 = 0 \Rightarrow k_{t_1}\theta_3 - k_{t_1}\theta_1 + k_{t_2}\tau\theta_3\tau - k_{t_2}\tau\theta_2 \Rightarrow \theta_3 = \frac{1}{k_{t_1}+k_{t_2}\tau^2}\left(k_{t_1}\theta_1 + k_{t_2}\tau\theta_2\right) \\ k_{t_2}(\theta_4-\theta_2) - TR_4 = 0 \Rightarrow k_{t_2}(\tau\theta_3-\theta_2) - \frac{TR_3}{\tau} = 0 \Rightarrow T = \frac{k_{t_2}(\tau\theta_3-\theta_2)\tau}{R_3} \end{cases}$$

$$I_2\ddot\theta_2 + k_{t_2}(\theta_2-\theta_4) = 0 \quad ; \quad \tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{\theta_{OUT}}{\theta_{IN}} = \frac{\theta_4}{\theta_3} = \frac{R_3}{R_4} \Rightarrow \begin{cases} R_4 = R_3/\tau \\ \theta_4 = \tau\theta_3 \end{cases}$$

$$\begin{cases} \theta_3 = \frac{k_{t_1}\theta_1}{k_{t_1}+k_{t_2}\tau^2} + \frac{k_{t_2}\tau\theta_2}{k_{t_1}+k_{t_2}\tau^2} \\ \theta_4 = \tau\theta_3 \end{cases}$$

$$\begin{cases} I_1\ddot\theta_1 + k_{t_1}\theta_1 - \frac{k_{t_1}^2\theta_1}{k_{t_1}+k_{t_2}\tau^2} - \frac{k_{t_1}k_{t_2}\tau\theta_2}{k_{t_1}+k_{t_2}\tau^2} = 0 \\ I_2\ddot\theta_2 + k_{t_2}\theta_2 - \frac{k_{t_1}k_{t_2}\tau\theta_1}{k_{t_1}+k_{t_2}\tau^2} - \frac{k_{t_2}^2\tau^2\theta_2}{k_{t_1}+k_{t_2}\tau^2} = 0 \end{cases} \quad \begin{cases} I_1\ddot\theta_1 + \left(\frac{\cancel{k_{t_1}^2} + k_{t_1}k_{t_2}\tau^2 - \cancel{k_{t_1}^2}}{k_{t_1}+k_{t_2}\tau^2}\right)\theta_1 - \frac{k_{t_1}k_{t_2}\tau\,\theta_2}{k_{t_1}+k_{t_2}\tau^2} = 0 \\ I_2\ddot\theta_2 - \frac{k_{t_1}k_{t_2}\tau\theta_1}{k_{t_1}+k_{t_2}\tau^2} + \left(\frac{k_{t_1}k_{t_2} + \cancel{k_{t_2}^2\tau^2} - \cancel{k_{t_2}^2\tau^2}}{k_{t_1}+k_{t_2}\tau^2}\right)\theta_2 = 0 \end{cases}$$

MATRIX FORM, CONSIDERING $k_{EQ} = k_{t_1}+k_{t_2}\tau^2$, $k_1 = k_{t_1}$, $k_2 = k_{t_2}$:

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{Bmatrix} + \begin{bmatrix} \frac{k_1k_2\tau^2}{k_{EQ}} & -\frac{k_1k_2\tau}{k_{EQ}} \\ -\frac{k_1k_2\tau}{k_{EQ}} & \frac{k_1k_2}{k_{EQ}} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$[M]\{\ddot x\} + [K]\{x\} = \{0\}$ $\quad \{x\} = \{x_0\}e^{i\omega t}$

$([K] - \omega^2[M])\{x_0\} = \{0\}$, $\{x_0\} = 0$: TRIVIALS.

$\det([K] - \omega^2[M]) = 0 \rightarrow [\Lambda], [\phi]$

$$\det\begin{bmatrix} \frac{k_1k_2\tau^2}{k_{EQ}} - \omega^2 I_1 & -\frac{k_1k_2\tau}{k_{EQ}} \\ -\frac{k_1k_2\tau}{k_{EQ}} & \frac{k_1k_2}{k_{EQ}} - \omega^2 I_2 \end{bmatrix} = \left(\frac{k_1k_2\tau^2}{k_{EQ}} - \omega^2 I_1\right)\left(\frac{k_1k_2}{k_{EQ}} - \omega^2 I_2\right) - \left(-\frac{k_1k_2\tau}{k_{EQ}}\right)^2 = \cancel{\frac{k_1^2k_2^2\tau^2}{k_{EQ}^2}} - \frac{k_1k_2\tau^2\omega^2 I_2}{k_{EQ}} - \frac{k_1k_2\omega^2 I_1}{k_{EQ}} + \omega^4 I_1I_2 - \cancel{\frac{k_1^2k_2^2\tau^2}{k_{EQ}^2}} = 0$$

$$I_1I_2\omega^4 - \left(\frac{k_1k_2\tau^2 I_2}{k_{EQ}} + \frac{k_1k_2 I_1}{k_{EQ}}\right)\omega^2 = 0 \quad \left[I_1I_2\omega^2 - \frac{k_1k_2}{k_{EQ}}(I_1 + \tau^2 I_2)\right]\omega^2 = 0$$

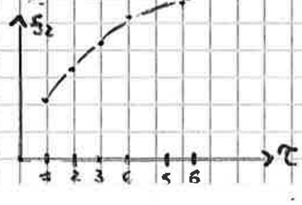
$\omega_1 = 0$ [Rad/S] ; $\omega_2 = \sqrt{\frac{k_1k_2(I_1+\tau^2 I_2)}{k_{EQ}I_1I_2}}$ $\left[\frac{Rad}{S}\right]$ ✓

2) $I_1 = 0.3\,[kg\,m^2]$, $I_2 = 135\,[kg\,m^2]$; $k_1 = 600\,[N/m]$ $k_2 = 8000\,[N/m]$

	1	2	3	4	5	6	
τ = [	0.07	0.13	0.20	0.27	0.38	0.45	] $f(\tau) = ?$
f_2 = [	2.13	3.25	4.31	5.07	5.82	6.11	] [Hz] ✓

$\left(f_2 = \frac{\omega_2}{2\pi}\right)$

3) EFFECT OF τ ON f_m ? PLOT $f_m(\tau)$



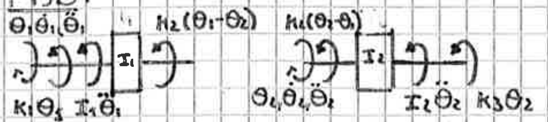
✓

EXERCISE 3: ELASTIC SHAFTS TORSIONAL OSCILLATIONS

EXECUTION

0) $k_1 \neq k_2 \neq k_3$ MOTION EQs $(\theta_1, \theta_2) = ?$

FBD:



$$\begin{cases} I_1\ddot\theta_1 + (k_1+k_2)\theta_1 - k_2\theta_2 = 0 \\ I_2\ddot\theta_2 - k_2\theta_1 + (k_2+k_3)\theta_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad ✓$$

1) $k_1 = k_2 = k_3 = k_t,\ I_2 = 2I_1$ $[\Lambda] = ?$ $[\Phi] = ?$ (I CONSIDER $k_t = k$)

$$\begin{bmatrix} I_1 & 0 \\ 0 & 2I_1 \end{bmatrix}\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det([K]-\omega^2[M]) = \det\begin{bmatrix} 2k-\omega^2 I_1 & -k \\ -k & 2k-2\omega^2 I_1 \end{bmatrix} = (2k-\omega^2 I_1)(2k-2\omega^2 I_1) - (-k)^2 = 0$$

$$4k^2 - 4k\omega^2 I_1 - 2k\omega^2 I_1 + 2\omega^4 I_1^2 - k^2 = 2I_1^2\omega^4 - (4kI_1 + 2kI_1)\omega^2 + 3k^2 = 0$$

$$2I_1^2\omega^4 - 6kI_1\omega^2 + 3k^2 = 0 \Rightarrow \omega_{1/2}^2 = \frac{-b \mp \sqrt{b^2-4ac}}{2a} = \frac{1}{4I_1^2}\left(6kI_1 \mp \sqrt{36k^2I_1^2 - 4\cdot 2\cdot 3\cdot I_1^2k^2}\right) =$$

$$= \frac{1}{4I_1^2}\left(6kI_1 \mp \sqrt{12}\,kI_1\right) = \frac{k}{I_1}\left(\frac{6 \mp \sqrt{12}}{4}\right)$$

$$\omega_1^2 = \frac{6-\sqrt{12}}{4}\frac{k}{I_1} = 0.634\frac{k}{I_1} \qquad \omega_2^2 = \frac{6+\sqrt{12}}{4}\frac{k}{I_1} = 2.366\frac{k}{I_1}$$

$$\left(\Rightarrow \begin{matrix} \omega_1 = 0.796\sqrt{\frac{k}{I_1}}\left[\frac{Rad}{s}\right] \\ \omega_2 = 1.538\sqrt{\frac{k}{I_1}}\left[\frac{Rad}{s}\right] \end{matrix}\right) \Rightarrow [\Lambda] = \begin{bmatrix} 0.634\frac{k}{I_1} & 0 \\ 0 & 2.366\frac{k}{I_1} \end{bmatrix} ✓$$

$$\begin{bmatrix} 2k-\omega_R^2 I_1 & -k \\ -k & 2k-2\omega_R^2 I_1 \end{bmatrix}\begin{Bmatrix} \varphi_{1,R} \\ \varphi_{2,R} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \text{ SETTING } \varphi_{1,1} = 1,\ \varphi_{1,2} = 1$$

$$-k\varphi_{1,R} + (2k - 2\omega_R^2 I_1)\varphi_{2,R} = 0 \Rightarrow \varphi_{2,R} = \frac{k\cdot 1}{2(k-\omega_R^2 I_1)} \Rightarrow \begin{cases} \varphi_{2,1} = \dfrac{k}{2(k-0.634k)} = 1.366 \\ \varphi_{2,2} = \dfrac{k}{2(k-2.366k)} = -0.366 \end{cases}$$

$$\Rightarrow [\Phi] = [\{\varphi_1\};\{\varphi_2\}] = \begin{bmatrix} 1 & 1 \\ 1.366 & -0.366 \end{bmatrix} ✓$$

2) HARMONIC TORQUE ON DISK 1, f_{AR} (ANTI-RES. f) = ?

$$\begin{bmatrix} I_1 & 0 \\ 0 & 2I_1 \end{bmatrix}\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} M_0 e^{i\omega t} \\ 0 \end{Bmatrix}$$
, **CONSIDERING** $x_1 = \theta_1;\ x_2 = \theta_2$

$[M]\{\ddot x\} + [K]\{x\} = \{F\}$, $\{x\} = \{x_0\}e^{i\omega t}$ (METHOD OF $[K_{DYN}]^{-1}$) (FASTER !)

$([K] - \omega^2[M])\{x_0\}e^{i\omega t} = \{F_0\}e^{i\omega t}$

$\{x_0\} = [K_{DYN}]^{-1}\{F_0\} = [\alpha]\{F_0\}$

$[K_{DYN}] = \begin{bmatrix} 2k-\omega^2 I_1 & -k \\ -k & 2k-2\omega^2 I_1 \end{bmatrix}$ $[K_{DYN}]^{-1} = \dfrac{1}{\det[K_{DYN}]_1}\begin{bmatrix} 2k-2\omega^2 I_1 & +k \\ +k & 2k-\omega^2 I_1 \end{bmatrix}$

NOTE: A MATRIX TO BE INVERTIBLE MUST HAVE FULL RANK

$\alpha_{jj} = \dfrac{x_j}{F_j}$; $\alpha_{jk} = \dfrac{x_j}{F_k}$

$$\begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} = \frac{1}{\det[K_{DYN}]_1}\begin{bmatrix} 2k-2\omega^2 I_1 & +k \\ +k & 2k-\omega^2 I_1 \end{bmatrix}\begin{Bmatrix} M_0 \\ 0 \end{Bmatrix}$$

$\{x\} = \{x_0\}e^{i\omega t}$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix}e^{i\omega t} \Rightarrow \begin{cases} x_1(t) = x_{10}e^{i\omega t} = \dfrac{(2k-2\omega^2 I_1)M_0}{\det[K_{DYN}]}e^{i\omega t} = 0 \iff \omega = \sqrt{\dfrac{k}{I_1}} \rightarrow f_{AR} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{I_1}} ✓ \\ x_2(t) = x_{20}e^{i\omega t} = \dfrac{kM_0}{\det[K_{DYN}]}e^{i\omega t} = -\dfrac{M_0}{k}e^{i\omega t} \end{cases}$$

$x_j = \alpha_{jj}F_j$	$x_j = \alpha_{jk}F_k$	
$\begin{cases} x_1 = \alpha_{11}F_1 \\ x_2 = \alpha_{22}F_2 = 0 \end{cases}$	$\begin{cases} x_1 = \alpha_{1,2}F_2 = 0 \\ x_2 = \alpha_{2,1}F_1 \end{cases}$	$x_2(t)$ IS DUE TO THE H.M. APPLIED ON I_1.

3) INITIAL SPEED $\dot{\theta}_0$; THEN THE SYST. IS BLOCKED; DERIVE THE RESPONSE

$\{x\} = [\varphi]\{\eta\}$

$\eta_R = A_R \cos(\omega_R t) + B_R \sin(\omega_R t)$

$$\begin{cases} A_R = \dfrac{\{\varphi_R\}^T[M]\{x_0\}}{m_R} = 0 \\ B_R = \dfrac{\{\varphi_R\}^T[M]\{\dot{x}_0\}}{m_R \omega_R} \end{cases}$$

$$B_1 = \frac{\{\varphi_1\}^T[M]\{\dot{x}_0\}}{m_1' \omega_1} = \frac{\{1 \quad 1.366\}}{4.732 I_1 \omega_1}\begin{bmatrix} I_1 & \\ & 2I_1 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_0 \\ \dot{\theta}_0 \end{Bmatrix} = \frac{\{I_1 \quad 2\cdot 1.366 I_1\}}{4.732 I_1 \cdot 0.796\sqrt{\frac{K}{I_1}}}\begin{Bmatrix} \dot{\theta}_0 \\ \dot{\theta}_0 \end{Bmatrix} = \frac{\dot{\theta}_0 \sqrt{I_1}\cdot 3.732}{\sqrt{K}\ 3.7667} =$$

$$= 0.9908\, \dot{\theta}_0 \sqrt{\frac{I_1}{K}}$$

$$B_2 = \frac{\{1 \quad -0.366\}}{m_2' \omega_2}\begin{bmatrix} I_1 & \\ & 2I_1 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_0 \\ \dot{\theta}_0 \end{Bmatrix} = \frac{\{I_1 \quad -2\cdot 0.366 I_1\}}{1.268 I_1 \cdot 1.538\sqrt{\frac{K}{I_1}}}\begin{Bmatrix} \dot{\theta}_0 \\ \dot{\theta}_0 \end{Bmatrix} = \frac{\dot{\theta}_0 \sqrt{I_1}}{\sqrt{K}}\,\frac{0.268}{1.950} = 0.1374\, \dot{\theta}_0 \sqrt{\frac{I_1}{K}}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.366 & -0.366 \end{bmatrix}\begin{Bmatrix} 0.9908 \sin(\omega_1 t) \\ 0.1374 \sin(\omega_2 t) \end{Bmatrix} \dot{\theta}_0 \sqrt{\frac{I_1}{K}}$$

$$\Rightarrow \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.9908 \sin(\omega_1 t) + 0.1374 \sin(\omega_2 t) \\ 1.3534 \sin(\omega_1 t) - 0.0503 \sin(\omega_2 t) \end{Bmatrix} \dot{\theta}_0 \sqrt{\frac{I_1}{K}} \quad ✓$$

4) $I_1 = 40.5$ [kg m²], $d = 10$ [mm], $\ell_1 = \ell_2 = 900$ [mm], $G = 80\,000$ [N/mm²]; $\omega_1 = ?$ $\omega_2 = ?$

$$\omega_1 = 0.796\sqrt{\frac{K}{I_1}} = 1.168 \text{ [Rad/s]} \quad ✓$$

$$\omega_2 = 1.538\sqrt{\frac{K}{I_1}} = 2.257 \text{ [Rad/s]}$$

$$K = \frac{G I_P}{L} = \frac{80\,000\cdot 10^6 \cdot 9.8125\cdot 10^{-10}}{0.9} = 87.22 \text{ [Nm]}$$

$$I_P = \frac{\pi D^4}{32} = \frac{3.14\cdot 0.01^4}{32} = 9.8125\cdot 10^{-10} \text{ [m}^4\text{]}$$

EXERCISE 5: SIMPLIFIED VEHICLE MODEL

EXECUTION

1) MOTION EQs FOR THE FOLLOWING SETS OF COORDINATES:

a) VERTICAL DISPL. OF POINT A AND BEAM ROTATION ($\Rightarrow x_A, \theta$)

b) VERTICAL DISPL. OF POINT G AND BEAM ROTATION ($\Rightarrow x_G, \theta$)

DATA:

$m = 1461$ [Kg]

$I_G = 2034$ [Kg m^2]

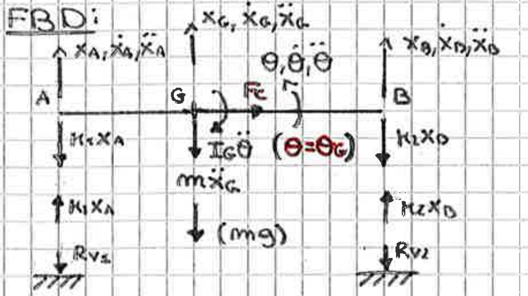
$k_1 = 29188$ [N/m]

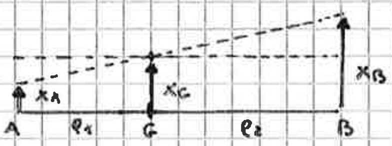
$k_2 = 36485$ [N/m]

$l_1 = 1.219$ [m]

$l_2 = 1.829$ [m]

FBD:





$$\begin{cases} x_G = x_A + l_1\theta \iff x_A = x_G - l_1\theta \\ x_B = x_A + (l_1+l_2)\theta = x_G + l_2\theta \end{cases}$$

1-a) MOTION EQs:

$$\uparrow)\ -m\ddot{x}_G - k_1 x_A - k_2 x_B = 0 \qquad \Rightarrow \begin{cases} m(\ddot{x}_A + l_1\ddot{\theta}) + k_1 x_A + k_2[x_A + (l_1+l_2)\theta] = 0 \\ I_G\ddot{\theta} + k_2 l_2 [x_A + (l_1+l_2)\theta] - k_1 l_1 x_A = 0 \end{cases}$$

$$G)\ -I_G\ddot{\theta} - k_2 l_2 x_B + k_1 l_1 x_A = 0$$

$$\begin{cases} m\ddot{x}_A + m l_1 \ddot{\theta} + (k_1+k_2)x_A + k_2(l_1+l_2)\theta = 0 \\ I_G\ddot{\theta} + (k_2 l_2 - k_1 l_1)x_A + k_2 l_2 (l_1+l_2)\theta = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} m & m l_1 \\ 0 & I_G \end{bmatrix}}_{\text{NOT SYMM.}} \begin{Bmatrix} \ddot{x}_A \\ \ddot{\theta} \end{Bmatrix} + \underbrace{\begin{bmatrix} (k_1+k_2) & k_2(l_1+l_2) \\ k_2 l_2 - k_1 l_1 & k_2 l_2 (l_1+l_2) \end{bmatrix}}_{\text{NOT SYMM.}} \begin{Bmatrix} x_A \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(WE CANNOT PERFORM MODAL ANALYSIS)

1-b) MOTION EQs

$$\begin{cases} m\ddot{x}_G + k_1(x_G - l_1\theta) + k_2(x_G + l_2\theta) = 0 \\ I_G\ddot{\theta} + k_2 l_2 (x_G + l_2\theta) - k_1 l_1 (x_G - l_1\theta) = 0 \end{cases}$$

$$\begin{cases} m\ddot{x}_G + (k_1+k_2)x_G + (k_2 l_2 - k_1 l_1)\theta = 0 \\ I_G\ddot{\theta} + (k_2 l_2 - k_1 l_1)x_G + (k_2 l_2^2 + k_1 l_1^2)\theta = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & I_G \end{bmatrix}}_{\text{SYMM.}} \begin{Bmatrix} \ddot{x}_G \\ \ddot{\theta} \end{Bmatrix} + \underbrace{\begin{bmatrix} k_1+k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_2 l_2^2 + k_1 l_1^2 \end{bmatrix}}_{\text{SYMM.}} \begin{Bmatrix} x_G \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(WE CAN PERFORM MODAL ANALYSIS)

2) $\omega_1 = ?$ $\omega_2 = ?$ MODES $= ?$ $\rightarrow$ 2-b

$$\det([K] - \omega^2[M]) = \det\begin{bmatrix} k_1+k_2 - m\omega^2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_2 l_2^2 + k_1 l_1^2 - I_G\omega^2 \end{bmatrix} = \det\begin{bmatrix} 65\,673 - 1461\omega^2 & 31'150.9 \\ 31'150.9 & 165'423.3 - 2034\omega^2 \end{bmatrix} =$$

$$= (65\,673 - 1461\omega^2)(165'423.3 - 2034\omega^2) - (31'150.9)^2 = 9'893'465'810 - 133'578'882\omega^2 - 241'683'441.3\omega^2$$

$$+ 2'971'674\omega^4 = 2'971'674\omega^4 - 375'262'323.3\omega^2 + 9'893'465'810 = 0$$

$$\omega_{1,2} = \sqrt{\frac{1}{2a}\left(-b \mp \sqrt{b^2 - 4ac}\right)} \qquad \omega_1 = 6.124 \left[\frac{Rad}{s}\right];\ \omega_2 = 9.422 \left[\frac{Rad}{s}\right] \ \checkmark$$

SETTING $\varphi_{1,R} = 1\ \forall R$

$$31'150.9 + (165'423.3 - 2034 \cdot 6.124^2)\varphi_{2,1} = 0 \Rightarrow \varphi_{2,1} = -0.3494$$

$$31'150.9 + (165'423.3 - 2034 \cdot 9.422^2)\varphi_{2,2} = 0 \Rightarrow \varphi_{2,2} = 2.0571$$

SETTING $\varphi_{2,R} = 1\ \forall R$:

$\varphi_{1,1} = -2.862$

$\varphi_{1,2} = 0.4861$

EXERCISE 6: DYNAMIC ABSORBER

EXECUTION

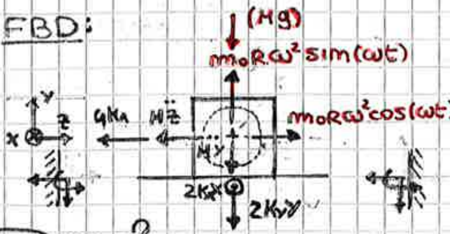
1) CHOOSE THE SECTION SO THAT $y_{ST} < \delta_{MAX} = 1\,[mm]$; $k_0 = 192\dfrac{EI}{L^3}$; $M = 100\,[kg]$; $L = 3\,[m]$; $E = 2.1\cdot10^{11}\,[N/m^2]$

$$k_0 y_{st} = \frac{M}{2}g \Rightarrow y_{ST} = \frac{Mg}{2k_0} = \frac{Mg\,L^3}{2\cdot192EI} < \delta_{MAX} \Leftrightarrow I > \frac{MgL^3}{2\cdot192\cdot E\cdot\delta_{MAX}}$$

$$I > \frac{100\cdot9.8\cdot3^3}{2\cdot192\cdot2{,}1\cdot10^{11}\cdot0.001} = 0.000\,000\,328\,[m^4] = 32.8\,[cm^4] \Rightarrow \text{WE CHOOSE } [80\times20\times3]$$

$(b \times a \times s)$ ✓

FBD:



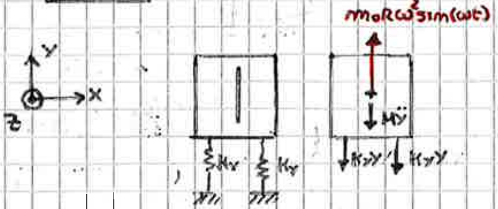
CONSIDERING THAT:

$$\boxed{\begin{aligned} k_y &= 192\frac{EI_x}{L} \\ k_x &= 192\frac{EI_y}{L} \\ k_A &= \frac{EA}{L}\end{aligned}}$$

$$k_y = \frac{192\cdot2.1\cdot10^{11}\cdot38.1\cdot10^{-8}}{3^3} = 568\,960\,[N/m]$$

$$k_x = \frac{192\cdot2.1\cdot10^{11}\cdot3.64\cdot10^{-8}}{3^3} = 54\,357.3\,[N/m]$$

2-a) $\omega_V = ?$

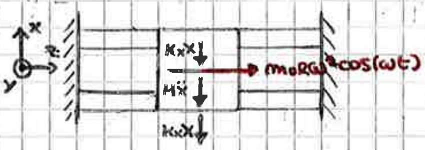


↑) $-M\ddot{y} - 2k_y y + m_oR\omega^2\sin(\omega t) = 0$

$M\ddot{y} + 2k_y y = m_oR\omega^2\sin(\omega t)$

$$\omega_V = \omega_y = \sqrt{\frac{2k_y}{M}} = \sqrt{\frac{2\cdot568\,960}{100}} = 106.67\left[\frac{Rad}{s}\right] ✓$$

2-b) $\omega_H = ?$

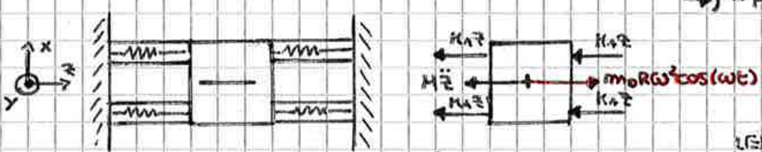


↑) $-M\ddot{x} - 2k_x x = 0$

$M\ddot{x} + 2k_x x = 0$

$$\omega_H = \omega_x = \sqrt{\frac{2k_x}{M}} = 32.97\left[\frac{Rad}{s}\right] ✓$$

2-c) $\omega_A = ?$



→) $-M\ddot{z} - 4k_A z + m_oR\omega^2\cos(\omega t) = 0$

$M\ddot{z} + 4k_A z = m_oR\omega^2\cos(\omega t)$

$$k_A = \frac{EA}{L} = \frac{EA}{L/2} = \frac{2.1\cdot10^{11}\cdot5.64\cdot10^{-4}}{1.5} = 78\,960\,000\,[N/m]$$

LENGTH OF THE PIECE → L

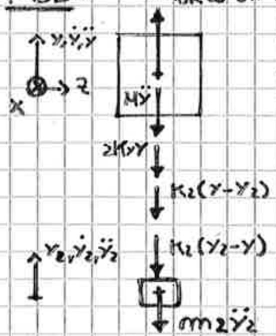
$$\omega_A = \omega_z = \sqrt{\frac{4k_A}{M}} = \sqrt{\frac{4\cdot78\,960\,000}{100}} = 1777.2\left[\frac{Rad}{s}\right] ✓$$

2-d) ω (→ FLEXURAL RESONANCE) = ?

$$\omega = \omega_R = \omega_V = 106.67\left[\frac{Rad}{s}\right] \;;\; \omega = \frac{2\pi n}{60} \Rightarrow n_e = \frac{\omega\cdot60}{2\pi} = \frac{106.67\cdot60}{2\cdot3.14} = 1019.16\,[RPM] ✓$$

3) $n = n_e = CONST$; TMD ; $m_oR = 5\,[kg\,mm]$, $m_2 = ?$ $k_2 = ?$ ⇒ $y_2 < 1\,[cm]$

FBD:



MOTION EQs:

$$F^* = \underbrace{\left(\frac{\omega}{\omega_m}\right)^2}_{=1} m_oR\omega_m^2\sin(\omega t)$$

$$\begin{cases} ↑)\; M\ddot{y} + (2k_y + k_2)y - k_2 y_2 = m_oR\omega^2\sin(\omega t) \\ ↑)\; m_2\ddot{y}_2 - k_2 y + k_2 y_2 = 0 \end{cases}$$

$$\begin{bmatrix} M & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix}\ddot{y}\\ \ddot{y}_2\end{Bmatrix} + \begin{bmatrix} 2k_y+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} y \\ y_2\end{Bmatrix} = \begin{Bmatrix} m \\ 0 \end{Bmatrix}$$

$$\{X_0\} = [K_{DYN}]^{-1}\{F_0\} \;;\; [K_{DYN}] = \begin{bmatrix} 2k_y+k_2-\omega^2M & -k_2 \\ -k_2 & k_2-\omega^2m_2\end{bmatrix} \rightarrow [K_{DYN}]^{-1} = \frac{1}{\det[K_{DYN}]}\begin{bmatrix} k_2-\omega^2m_2 & k_2 \\ k_2 & 2k_y+k_2-\omega^2M\end{bmatrix}$$

$$\begin{Bmatrix} x_{10} \\ x_{20}\end{Bmatrix} = \frac{1}{\det[K_d]}\begin{bmatrix} k_2-\omega^2m_2 & k_2 \\ k_2 & 2k_y+k_2-\omega^2M\end{bmatrix}\begin{Bmatrix} m_oR\omega_v^2 \\ 0\end{Bmatrix}$$

$$\det[K_{DYN}] = (2k_y+k_2-\omega^2M)(k_2-\omega^2m_2) - k_2^2 = 2k_yk_2 - 2k_y\omega^2m_2 - k_2\omega^2m_2 - k_2\omega^2M + m_2M\omega^4$$

$$x_{20} = \frac{k_2m_oR\omega^2}{\det[K_{DYN}]} = \left|-\frac{m_oR}{m_2}\right| < 0.01\,[m] \Rightarrow \begin{cases} m_2 > \dfrac{m_oR}{10^{-2}} = 5\cdot10^{-3}\cdot10^2 = 0.5\,[kg] \\ k_2 = \omega_V^2 m_2 = 5689.2\,[N/m]\end{cases} ✓$$

CONSIDERING $k=3$, $q_3 = z_1$:

$$\frac{\partial L}{\partial z_1} = \frac{\partial T}{\partial z_1} - \frac{\partial V}{\partial z_1} = -\left(-K_{s1}(z - a_1\theta - z_1) + K_1(z_1 - u_1)\right)$$

$$\frac{\partial L}{\partial \dot z_1} = \frac{\partial T}{\partial \dot z_1} - \frac{\partial V}{\partial \dot z_1} = m_1 \dot z_1$$

$$Q_3 = -\frac{\partial D}{\partial \dot z_1} = -\left(-C_{s1}(\dot z - a_1\dot\theta - \dot z_1) + C_1(\dot z_1 - \dot u_1)\right)$$

$$m_1\ddot z_1 - K_{s1}(z - a\theta - z_1) + K_1(z_1 - u_1) - C_{s1}(\dot z - a_1\dot\theta - \dot z_1) + C_1(\dot z_1 - \dot u_1) = 0$$

(3) $m_1\ddot z_1 - C_{s1}\dot z + a_1 C_{s1}\dot\theta + (C_1 + C_{s1})\dot z_1 - K_{s1} z + a_1 K_{s1}\theta + (K_{s1} + K_1) z_1 = K_1 u_1 + C_1 \dot u_1$

CONSIDERING $k=4$, $q_4 = z_2$:

$$\frac{\partial L}{\partial z_2} = \frac{\partial T}{\partial z_2} - \frac{\partial V}{\partial z_2} = -\left(-K_{s2}(z + a_2\theta - z_2) + K_2(z_2 - u_2)\right)$$

$$\frac{\partial L}{\partial \dot z_2} = \frac{\partial T}{\partial \dot z_2} - \frac{\partial V}{\partial \dot z_2} = m_2 \dot z_2$$

$$Q_4 = -\frac{\partial D}{\partial \dot z_2} = -\left(-C_{s2}(\dot z + a_2\dot\theta - \dot z_2) + C_2(\dot z_2 - \dot u_2)\right)$$

$$m_2\ddot z_2 - K_{s2}(z_B - z_2) + K_2(z_2 - u_2) - C_{s2}(\dot z + a_2\dot\theta - \dot z_2) + C_2(\dot z_2 - \dot u_2) = 0$$

(4) $m_2\ddot z_2 - C_{s2}\dot z - a_2 C_{s2}\dot\theta + (C_{s2} + C_2)\dot z_2 - K_{s2} z - K_{s2} a_2\theta + (K_{s2} + K_2) z_2 = K_2 u_2 + C_2 \dot u_2 = 0$

MATRIX FORM:

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_y & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot z \\ \ddot\theta \\ \ddot z_1 \\ \ddot z_2 \end{Bmatrix} + \begin{bmatrix} C_{s1} + C_{s2} & a_2 C_{s2} - a_1 C_{s1} & -C_{s1} & -C_{s2} \\ a_2 C_{s2} - a_1 C_{s1} & a_2^2 C_{s2} + a_1^2 C_{s1} & a C_{s1} & -a_2 C_{s2} \\ -C_{s1} & a_1 C_{s1} & C_1 + C_{s1} & 0 \\ -C_{s2} & -a_2 C_{s2} & 0 & C_2 + C_{s2} \end{bmatrix}\begin{Bmatrix} \dot z \\ \dot\theta \\ \dot z_1 \\ \dot z_2 \end{Bmatrix} +$$

$$+\begin{bmatrix} K_{s1} + K_{s2} & a_2 K_{s2} - a_1 K_{s1} & -K_{s1} & -K_{s2} \\ a_2 K_{s2} - a_1 K_{s1} & a_1^2 K_{s1} + a_2^2 K_{s2} & a_1 K_{s1} & -a_2 K_{s2} \\ -K_{s1} & a_1 K_{s1} & K_{s1} + K_1 & 0 \\ -K_{s2} & -a_2 K_{s2} & 0 & K_{s2} + K_2 \end{bmatrix}\begin{Bmatrix} z \\ \theta \\ z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ K_1 u_1 + C_1 \dot u_1 \\ K_2 u_2 + C_2 \dot u_2 \end{Bmatrix}$$ ✓

2) hp: $u_1 = u_2 = 0$ $F(t)$ APPLIED IN PONT B ; $\{Q\} = ?$

$$Q_k = \sum_{i=1}^{m} \vec F_i \frac{\partial \vec R_i}{\partial q_k} = \sum_{i=1}^{m} \vec F_i \frac{\partial \vec R_i}{\partial q_k} \rightarrow Q_k = \vec F \frac{\partial \vec R}{\partial q_k}$$

CONSIDERING $k=1$, $q_1 = z$:

$$Q_1 = \vec F \frac{\partial \vec R}{\partial z} = F(t)\vec k \frac{\partial \vec z_B}{\partial z} = F(t)\vec k \frac{\partial (z_B \vec k)}{\partial z} = F(t)\frac{\partial z_B}{\partial z}\vec k \cdot \vec k = F(t)$$

CONSIDERING $k=2$, $q_2 = \theta$:

$$Q_2 = \vec F \frac{\partial \vec R}{\partial \theta} = F(t)\vec k \frac{\partial \vec z_B}{\partial \theta} = F(t)\vec k \frac{\partial (z_B \vec k)}{\partial \theta} = F(t)\frac{\partial z_B}{\partial \theta}\vec k \cdot \vec k = F(t) a_2$$

CONSIDERING $k=3$, $q_3 = z_1$:

$$Q_3 = \vec F \frac{\partial \vec R}{\partial z_1} = F(t)\vec k \frac{\partial \vec z_B}{\partial z_1} = F(t)\vec k \frac{\partial (z_B \vec k)}{\partial z_1} = F(t)\frac{\partial z_B}{\partial z_1}\vec k \cdot \vec k = 0$$

CONSIDERING $k=4$, $q_4 = z_2$:

$$Q_4 = \vec F \frac{\partial \vec R}{\partial z_2} = F(t)\vec k \frac{\partial \vec z_B}{\partial z_2} = F(t)\vec k \frac{\partial (z_B \vec k)}{\partial z_2} = F(t)\frac{\partial z_B}{\partial z_2}\vec k \cdot \vec k = 0$$

$\{Q(t)\} = \{ F(t) \;\; a_2 F(t) \;\; 0 \;\; 0 \}^T$ ✓

EXERCISE 2: STRING VIBRATIONS

DATA

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2) CHARCTERISTIC EQ.

IMPOSING THE B.C. WE GET C, D:

L: $u(x=0,t) = \phi(0)\eta(t) = 0 = C \Rightarrow C = 0$

$\Rightarrow \phi(x) = D\sin(\beta x)$; $\phi'(x) = \beta D\cos(\beta x)$

R: $AE\phi'(L) + (k-\omega^2 m)\phi(L) = 0$

$\Rightarrow AE\beta D\cos(\beta L) + (k-\omega^2 m)D\sin(\beta L) = 0$

$D\left(AE\beta\cos(\beta L) + (k-\omega^2 m)\sin(\beta L)\right) = 0$, D=0 TRIVIAL S. (UNDETERMINED AMPL.)

$\boxed{AE\beta\cos(\beta L) + (k-\omega^2 m)\sin(\beta L) = 0}$: CHARACT. EQ. TO EVALUE.

THE PROBLEM GIVES US : $\gamma^2 = \frac{\mu\omega^2}{EA} = \frac{\omega^2}{c^2} = \beta^2 \Rightarrow \gamma = \beta$

$AE\beta + (k-\omega^2 m)\frac{\sin(\beta L)}{\cos(\beta L)} = 0$

$AE\beta + (k-\omega^2 m)Tg(\beta L) = 0$

SETTING $k_0 = \frac{AE}{L}$; $\mu = \frac{m_0}{L}$

$k_0(\beta L) + (k-\omega^2 m)Tg(\beta L) = 0$

$Tg(\beta L) + \frac{k_0(\beta L)}{k-\omega^2 m} = 0 \Leftrightarrow Tg(\beta L) + \frac{(\beta L)}{\frac{k}{k_0} - \frac{\omega^2 m}{k_0}} = 0 \rightarrow$

CHARACT. EQ.

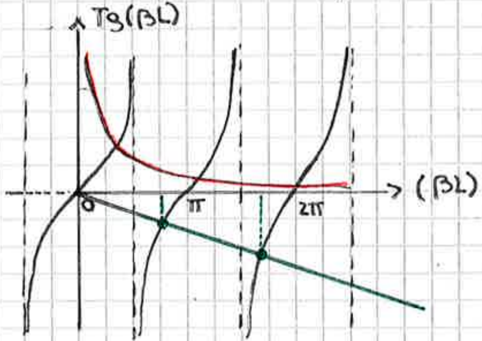
$$\boxed{Tg(\beta L) + \frac{(\beta L)}{\frac{k}{k_0} - \frac{m}{m_0}(\beta L)^2} = 0}$$ ✓

$\frac{\omega^2 m}{k_0} = \frac{\omega^2 m}{\frac{AE}{L}} = \frac{\beta^2 c^2 m L}{AE} = \frac{\beta^2 AE\, mL}{AE\,\mu} = \beta^2 L\frac{m}{\mu} = \beta^2 L\frac{m}{\frac{m_0}{L}} = \frac{m}{m_0}(\beta L)^2$

2') DISCUSSION OF CASES $\begin{cases} k=0 \\ m=0 \end{cases}$

$\boxed{k=0} \Rightarrow Tg(\beta L) = \frac{m_0}{m(\beta L)}$ ✓

$\boxed{m=0} \Rightarrow Tg(\beta L) = -\frac{k_0(\beta L)}{k}$



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$$\begin{bmatrix} I_1\omega^2 & GJ\beta \\ -GJ\beta\sin(\beta\ell)-I_2\omega^2\cos(\beta\ell) & GJ\beta\cos(\beta\ell)-I_2\omega^2\sin(\beta\ell) \end{bmatrix}\begin{Bmatrix} C \\ D \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad , \begin{Bmatrix} C \\ D \end{Bmatrix}=0 \text{ TRIVIAL S.}$$

$$\det[\,]=I_1\omega^2\left(GJ\beta\cos(\beta\ell)-I_2\omega^2\sin(\beta\ell)\right)-GJ\beta\left(-GJ\beta\sin(\beta\ell)-I_2\omega^2\cos(\beta\ell)\right)=0$$

$$I_1\omega^2 GJ\beta\cos(\beta\ell)-I_1I_2\omega^4\sin(\beta\ell)+G^2J^2\beta^2\sin(\beta\ell)+GJ\beta I_2\omega^2\cos(\beta\ell)=0, \quad \beta=\frac{\omega}{c}$$

$$I_1I_2\omega^4\sin(\beta\ell)-I_1\frac{\omega^3}{c}GJ\cos(\beta\ell)-\frac{\omega^2(GJ)^2}{c^2}\sin(\beta\ell)-I_2GJ\frac{\omega^3}{c}\cos(\beta\ell)=0$$

$$\omega^2\left\{I_1I_2\sin(\beta\ell)\omega^2-(I_1+I_2)\frac{GJ}{c}\cos(\beta\ell)\omega-\frac{(GJ)^2}{c^2}\sin(\beta\ell)\right\}=0$$

$\omega_0=0$

$$\sin(\beta\ell)\left[I_1I_2\omega^2-\left(\frac{GJ}{c}\right)^2\right]=\cos(\beta\ell)\left[\frac{GJ}{c}(I_1+I_2)\omega\right]$$

$$\left(c=\sqrt{\frac{GJ}{I_u}}\Rightarrow c^2=\frac{GJ}{I_u}\Rightarrow GJ=c^2I_u\right)$$

$$Tg(\beta\ell)=\frac{(I_1+I_2)GJ\omega/c}{I_1I_2\omega^2-\left(\frac{GJ}{c}\right)^2}=\frac{(I_1+I_2)}{\frac{c}{GJ\omega}\left[I_1I_2\omega^2-\frac{G^2J^2}{c^2}\right]}=\frac{I_1+I_2}{\frac{c\omega}{GJ}I_1I_2-\frac{GJ}{c\omega}}=\frac{I_1+I_2}{\frac{I_1I_2}{I_u}\left(\frac{\omega}{c}\right)-I_u\left(\frac{c}{\omega}\right)}$$

PUTTING $I_u\cdot\ell=I$:

$$Tg\left(\frac{\omega}{c}\ell\right)=\frac{I_1+I_2}{\frac{I_1I_2}{I}\left(\frac{\omega\ell}{c}\right)-I\left(\frac{c}{\omega\ell}\right)} \quad \Longleftrightarrow \quad \text{CHARACT. EQ.} \quad \boxed{Tg(\beta\ell)=\frac{I_1+I_2}{\frac{I_1I_2}{I}(\beta\ell)-\frac{I}{(\beta\ell)}}}$$

SOLVING NUMERICALLY (...) :

$$\frac{\omega L}{c}=0.697 \Rightarrow \omega_2=7786\left[\frac{Rad}{s}\right] \checkmark$$

1-C) RAYLEIGH APPROXIMATED METHOD

$$T_{MAX}=\tilde{T}_{MAX}\,\omega^2=V_{MAX}$$

$$\begin{cases} T(t)=\frac{1}{2}\int_0^L m(x)\left(\frac{\partial w}{\partial t}\right)^2dx+\frac{1}{2}\sum_{i=1}^{m}m_i\left(\frac{\partial w}{\partial t}\right)^2_{x_i} \\ V(t)=\frac{1}{2}\int_0^L g(x)\left(\frac{\partial w}{\partial x}\right)^2dx+\frac{1}{2}\sum_{j=1}^{\ell}k_j\,w^2(x_j,t) \end{cases}$$

$$\Theta(x,t)=\Phi(x)\eta(t)$$

$$\frac{\partial\Theta}{\partial t}=\Phi(x)\dot{\eta}(t)$$

$$\begin{cases} T(t)=\frac{1}{2}\int_0^L I_u\left(\frac{\partial\Theta}{\partial t}\right)^2dx+\frac{1}{2}I_1\left(\frac{\partial\Theta}{\partial t}\right)^2_{x=0}+\frac{1}{2}I_2\left(\frac{\partial\Theta}{\partial t}\right)^2_{x=L} \\ V(t)=\frac{1}{2}\int_0^L GJ\left(\frac{\partial\Theta}{\partial x}\right)^2dx \end{cases}$$

CONSIDERING:

$$\Theta(x,t)=\Phi(x)\eta(t)\begin{cases}\Phi(x) \text{ TRIAL FUNCTION} \\ \eta(t)=A\sin(\omega t+\varphi) \Rightarrow \dot{\eta}(t)=A\omega\cos(\omega t+\varphi)\end{cases}$$

$$\begin{cases} T(t)=\frac{1}{2}\int_0^L I_u\Phi^2(x)\cdot\dot{\eta}^2(t)dx+\frac{1}{2}I_1\Phi^2(0)\dot{\eta}^2(t)+\frac{1}{2}I_2\Phi^2(L)\dot{\eta}^2(t) \\ V(t)=\frac{1}{2}\int_0^L GJ\,\Phi'^2(x)\eta^2(t)dx \end{cases}$$

$$\begin{cases} T(t)=\frac{1}{2}\int_0^L I_u\Phi^2(x)(+\omega^2A^2\cos^2(\omega t+\varphi))dx+\frac{1}{2}I_1\Phi^2(0)(+\omega^2A^2\cos^2(\omega t+\varphi))+\frac{1}{2}I_2\Phi^2(L)(+\omega^2A^2\cos^2(\,)) \\ V(t)=\frac{1}{2}\int_0^L GJ\,\Phi'^2(x)\,A^2\sin^2(\omega t+\varphi)\,dx \end{cases}$$

$$\begin{cases} T_{MAX}=\frac{1}{2}\int_0^L I_u\omega^2A^2\Phi^2(x)dx+\frac{1}{2}I_1\omega^2A^2\Phi^2(0)+\frac{1}{2}I_2\omega^2A^2\Phi^2(L), \quad \tilde{T}_{MAX}=\frac{T_{MAX}}{\omega^2} \\ V_{MAX}=\frac{1}{2}\int_0^L GJA^2\Phi'^2(x)dx \end{cases}$$

$$R=\frac{V_{MAX}}{\tilde{T}_{MAX}}=\omega^2=\frac{GJ\cancel{A^2}\int_0^L\Phi'^2(x)dx}{I_u\cancel{A^2}\int_0^L\Phi^2(x)dx+I_1\cancel{A^2}\,\Phi^2(0)+I_2\cancel{A^2}\,\Phi^2(L)}$$

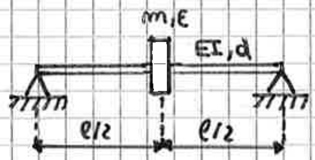
APPLIED LECTURE 4 : CRITICAL SPEEDS AND ROTORDYNAMICS   1st Round

EXERCISE 1 : FLEXURAL CRITICAL VELOCITY OF A TURBO-COMPRESSOR

ROTOR – FLEXIBLE SHAFT ON RIGID BEARINGS (JEFFCOTT MODEL)

$m = 9\,[kg]$; $\ell = 40\,[cm]$; $d = 2.5\,[cm]$; $G = 3.15$

$E = 2.1 \cdot 10^{11}\,[N/m^2]$; $k = 48\,EI/\ell^3$, I : S. ARGAM.I.



1) $\varepsilon = ?$ ($n_1 = 3200\,[rpm]$, $n_2 = 7000\,[rpm]$)

2) $n_{CR} = ?$ (FLEXURAL CRITICAL SPEED)

3) $\delta_1(n_1) = ?$ $\delta_2(n_2) = ?$ | $mg = 0$ | S. DAMPING NEGLIGIBLE ($\Rightarrow c = 0$)

EXECUTION

1) THE PROBLEM GIVES:

$G = \varepsilon\omega$: UMBALANCE GRADE

$\varepsilon = \frac{G}{\omega} 10^3\,[\mu m]$

$\varepsilon_{MAX} = min(\varepsilon_1, \varepsilon_2)$

$\omega_1 = \frac{2\pi n_1}{60} = \frac{2 \cdot 3.14 \cdot 3200}{60} = 334.93\,[Rad/s]$

$\omega_2 = \frac{2\pi n_2}{60} = \frac{2 \cdot 3.14 \cdot 7000}{60} = 732.67\,[Rad/s]$

$\varepsilon_1 = \frac{G}{\omega_1} 10^3 = \frac{3.15}{334.93} \cdot 10^3 = 9.4\,[\mu m]$; $\varepsilon_2 = \frac{3.15 \cdot 10^3}{732.67} = 4.3\,[\mu m]$

$\varepsilon_{MAX} = min(\varepsilon_1, \varepsilon_2) = 4.3\,[\mu m]$ ✓

2) FLEXURAL CRITICAL SPEED (n_{CR})

$\omega_{CR} = \sqrt{\frac{k_S}{m}}$, $k = 48\frac{EI}{\ell^3}$, $I = \frac{\pi d^4}{64}$ $\left(\neq I_P = \frac{\pi d^4}{32}\right)$, $\pi = 3.1416$

$I = \frac{\pi d^4}{64} = \frac{3.1416 \cdot (2.5)^4}{64} = 1.9175\,[cm^4] = 1.9175 \cdot 10^{-8}\,[m^4]$ | $k = \frac{EI}{\ell^3} = \left[\frac{N}{m^2}\right]\frac{[m^4]}{[m^3]} = \left[\frac{N}{m}\right]$

$k_S = 48\frac{EI}{\ell^3} = \frac{48 \cdot 2.1 \cdot 10^{11} \cdot 1.9175 \cdot 10^{-8}}{(0.4)^3} = 3'020'062.5\,\left[\frac{N}{m}\right] \cong 3020.1 \cdot 10^3\,\left[\frac{N}{m}\right]$

$\omega_{CR} = \sqrt{\frac{k_S}{m}} = \sqrt{\frac{3020.1 \cdot 10^3}{9}} = 579.28\,\left[\frac{Rad}{s}\right]$

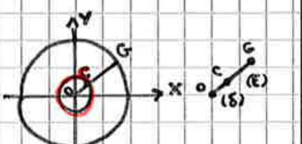
$n_{CR} = \frac{\omega_{CR}\,60}{2\pi} = \frac{579.28 \cdot 60}{2 \cdot 3.1416} = 5531.7\,[rpm]$ ✓

3) AMPLITUDES (δ_1, δ_2) OF THE TRAJECTORY FOR THE TWO SPINNING SPEEDS (n_1, n_2)

$\left|\frac{Z_0}{\varepsilon}\right| = \frac{|\delta|}{\varepsilon} = \frac{(\omega/\omega_{CR})^2}{\sqrt{[1-(\omega/\omega_{CR})^2]^2 + [2\zeta(\omega/\omega_{CR})]^2}}$, $\zeta = \frac{c_S}{2m\,\omega_{CR}} = 0$

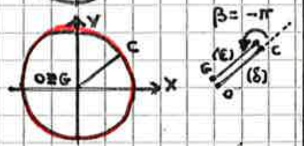
$\delta_1 = \frac{\varepsilon\,(\omega_1/\omega_{CR})^2}{\sqrt{[1-(\omega_1/\omega_{CR})^2]^2}} = \frac{4.3\,(334.93/579.28)^2}{\sqrt{[1-(334.93/579.28)^2]^2}} = 2.16\,[\mu m]$ ✓

$\omega < \omega_{CR}$ $\begin{cases} \beta \cong 0 \\ R = \delta + \varepsilon \end{cases}$



$\delta_2 = \frac{\varepsilon\,(\omega_2/\omega_{CR})^2}{\sqrt{[1-(\omega_2/\omega_{CR})^2]^2}} = \frac{4.3\,(732.67/579.28)^2}{\sqrt{[1-(732.67/579.28)^2]^2}} = +11.47\,[\mu m]$ ✓

$\omega > \omega_{CR}$ $\begin{cases} \beta = -\pi \\ R = \delta - \varepsilon \end{cases}$



REMIND:

$I_P = \frac{\pi D^4}{32}\,[m^4]$

$I_A = \frac{\pi D^4}{64}\,[m^4]$

$I_m = m\rho^2\,[kg\,m^2]$

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2) NATURAL FREQUENCIES :

$$[M]\{\ddot{\theta}\} + [K]\{\theta\} = \{0\}$$

$$\{\theta\} = \{\theta_0\} e^{i\omega t} \; ; \; \{\dot{\theta}\} = i\omega\{\theta_0\} e^{i\omega t} \; ; \; \{\ddot{\theta}\} = -\omega^2\{\theta_0\} e^{i\omega t}$$

$$(-\omega^2[M] + [K])\{\theta_0\} \cancel{e^{i\omega t}} = \{0\} \; , \; \{\theta_0\} = 0 \text{ : TRIVIAL SOLUTION}$$

$$\det([K] - \omega^2[M]) = 0$$

$$\det([K] - \omega^2[M]) = \det \begin{bmatrix} k_T - \omega^2 I_E & -k_T/\tau \\ -k_T/\tau & k_T/\tau^2 - \omega^2 I_R \end{bmatrix} = (k_T - \omega^2 I_E)\cdot\left(\frac{k_T}{\tau^2} - \omega^2 I_R\right) - \left(\frac{-k_T}{\tau}\right)^2 = 0$$

$$\cancel{\frac{k_T^2}{\tau^2}} - k_T\omega^2 I_R - \omega^2 I_E \frac{k_T}{\tau^2} + \omega^4 I_E I_R - \cancel{\frac{k_T^2}{\tau^2}} = 0$$

$$\omega^2\left(-k_T I_R - I_E \frac{k_T}{\tau^2} + \omega^2 I_E I_R\right) = 0$$

$\omega_1 = 0 \; (= \omega_{m1})$

$$\omega_2 = \sqrt{\frac{k_T I_R + (k_T/\tau^2) I_E}{I_E I_R}} = \sqrt{\frac{500\cdot 2 + [500/(1/6)^2]\cdot 3.1}{(2\cdot 3.1)}} = 95.71 \text{ [Rad/s]} \checkmark \; (= \omega_{m2} = \omega_m)$$

3) TORSIONAL CRITICAL SPEEDS (CONSIDERING ONLY THE FIRST TWO HARMONICS)

THE PROBLEM SAYS : T_E SUPPLIED BY A RECIPROCATING FOUR-STROKE ($m_S = 4$) SINGLE CYLINDER ($m_C = 1$) ICE ...

$$\begin{cases} 4 \text{ STROKES} \rightarrow m_S = 4 \\ 1 \text{ CYLINDER} \rightarrow m_C = 1 \end{cases}$$

$$T_E = T_0 + \sum_{k=1}^{m} [A_k \cos(k\Omega_S t) + B_k \sin(k\Omega_S t)] = T_0 + \sum_{k=1}^{m} B_k \sin(k\Omega_S t)$$

$$= T_0 + T_1 \sin(\underbrace{1\cdot\Omega_S}_{\Omega_1} t) + T_2 \sin(\underbrace{2\cdot\Omega_S}_{\Omega_2} t)$$

BY DEFINITION:

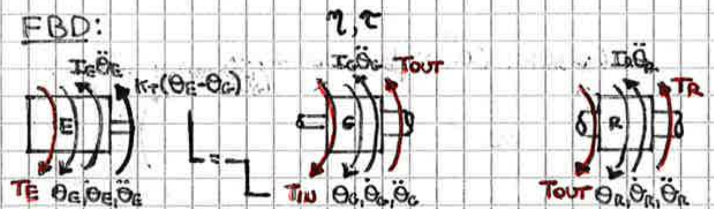
$$\Omega_S = 2\frac{m_C}{m_S}\cdot\omega_{CR} = 2\cdot\frac{1}{4}\omega_{CR} = \frac{1}{2}\omega_{CR} \quad \Big| \quad n_{CR} = \frac{\omega_{CR}\cdot 60}{2\pi}$$

$$\Omega_{S1} = \Omega_1 = \frac{1}{2}\omega_{CR1} \Rightarrow \omega_{CR1} = 2\Omega_1 = 2\omega_m = 2\cdot 95.71 = 191.42 \text{ [Rad/s]} \; ; \; n_{CR1} = 1827.9 \text{ [Rpm]}$$

$$\Omega_{S2} = \frac{\Omega_2}{2} = \frac{1}{2}\omega_{CR2} \Rightarrow \omega_{CR2} = \Omega_2 = \omega_m = 95.71 \text{ [Rad/s]} \; ; \; n_{CR2} = 913.96 \text{ [Rpm]}$$

CONSIDERING ANOTHER APPROACH:

FBD:



$$\eta = \frac{P_{OUT}}{P_{IN}} \; ; \; \tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{\dot{\theta}_{OUT}}{\dot{\theta}_{IN}} = \frac{\theta_R}{\theta_G}$$

$$\eta = \frac{P_{OUT}}{P_{IN}} = \frac{T_{OUT}\cdot\dot{\theta}_{OUT}}{k_T(\theta_E - \theta_G)\dot{\theta}_{IN}} = \frac{T_{OUT}}{k_T(\theta_E - \theta_G)}\cdot\tau = 1 \Rightarrow T_{OUT} = \frac{k_T(\theta_E - \theta_G)}{\tau} = \frac{k_T}{\tau}\left(\theta_E - \frac{\theta_R}{\tau}\right)$$

THEREFORE :

$$\begin{cases} I_E\ddot{\theta}_E + k_T(\theta_E - \theta_G) - T_E = 0 \\ I_R\ddot{\theta}_R - T_{OUT} + T_R = 0 \end{cases} \qquad \begin{cases} I_E\ddot{\theta}_E + k_T\theta_E - \frac{k_T}{\tau}\theta_R = T_E \\ I_R\ddot{\theta}_R + \frac{k_T}{\tau^2}\theta_R - \frac{k_T}{\tau}\theta_E = -T_R \end{cases} \checkmark$$

WE CAN RESUME REMEMBERING:

$$\eta = \frac{P_{OUT}}{P_{IN}} = \frac{T_{OUT}\,\dot{\theta}_{OUT}}{T_{IN}\,\dot{\theta}_{IN}} = \frac{T_{OUT}}{T_{IN}}\tau$$

$$\tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{\dot{\theta}_{OUT}}{\dot{\theta}_{IN}} = \frac{T_{IN}}{T_{OUT}}\eta$$

2) NATURAL FREQUENCIES (ω_1, ω_2) AND NATURAL MODES (θ_{10}/θ_{20})

$$[I]\{\ddot\theta\} + [K]\{\theta\} = \{0\}$$

$$\{\theta\} = \{\theta_0\}e^{i\omega t} \; ; \; \{\dot\theta\} = i\omega\{\theta_0\}e^{i\omega t} \; ; \; \{\ddot\theta\} = -\omega^2\{\theta_0\}e^{i\omega t}$$

$$(-\omega^2[I] + [K])\{\theta_0\}\cancel{e^{i\omega t}} = \{0\} \quad , \; \{\theta_0\} = \{0\} : \text{TRIVIAL SOLUTION}$$

$$\det([K] - \omega^2[I]) = 0$$

$$\det([K] - \omega^2[I]) = \det\begin{vmatrix} K_T - \omega^2 I_1 & -K_T \\ -K_T & K_T - \omega^2(I_2 + \tau^2 I_3) \end{vmatrix} = (K_T - \omega^2 I_1)(K_T - \omega^2(I_2 + \tau^2 I_3)) - (-K_T)^2 = 0$$

$$\cancel{K_T^2} - \omega^2 K_T(I_2 + \tau^2 I_3) - K_T\omega^2 I_1 + \omega^4 I_1(I_2 + \tau^2 I_3) - \cancel{K_T^2} = 0$$

$$\omega^2(-K_T(I_2 + \tau^2 I_3) - K_T I_1 + \omega^2 I_1(I_2 + \tau^2 I_3)) = 0$$

$$\begin{cases} \omega_1 = 0 = \omega_{m1} \\ \omega_2 = \sqrt{\dfrac{K_T(I_1 + I_2 + \tau^2 I_3)}{I_1(I_2 + \tau^2 I_3)}} = \sqrt{\dfrac{602100(2.45 + 3.92 + 0.46^2 \cdot 9.81)}{2.45(3.92 + 0.46^2 \cdot 9.81)}} = 480.82 \left[\dfrac{Rad}{s}\right] \end{cases}$$

NATURAL FREQUENCIES ✓

$$\begin{cases} I_1\ddot\theta_1 + K_T\theta_1 - K_T\theta_2 = 0 \\ (I_2 + I_3\tau^2)\ddot\theta_2 + K_T\theta_2 - K_T\theta_1 = 0 \end{cases}$$

$$\begin{cases} (-\omega_m^2 I_1 + K_T)\,\theta_{10}\cancel{e^{i\omega t}} = K_T\theta_{20}\cancel{e^{i\omega t}} \\ [-\omega_m^2(I_2 + I_3\tau^2) + K_T]\,\theta_{20}e^{i\omega t} = K_T\theta_{10}e^{i\omega t} \end{cases}$$

$$\frac{\theta_{10}}{\theta_{20}} = \frac{K_T}{K_T - \omega_m^2 I_1} = \frac{602100}{602100 - 480.82^2 \cdot 2.45} = -2.45$$

NATURAL MODES ✓

3) TORSIONAL CRITICAL SPEEDS (n_{CR1}, n_{CR2})

$$T = T_0 + \sum_{K=1}^{m}[A_K\cos(K\Omega_S t) + B_K\sin(K\Omega_S t)] = T_0 + \sum_{K=1}^{m} B_K\sin(K\Omega_S t) = T_0 + B_1\sin(\underbrace{1\cdot\Omega_S t}_{\Omega_1}) + B_2\sin(\underbrace{2\cdot\Omega_S t}_{\Omega_2})$$

$$\Omega_S = 2\frac{m_C}{m_S}\omega_{CR} \Rightarrow \omega_{CR} = \frac{1}{2}\frac{m_S}{m_C}\Omega_S \quad ; \quad n_{CR} = \frac{\omega\, 60}{2\pi}$$

$$\begin{cases} \Omega_1 = 1\cdot\Omega_S = \omega_m \Rightarrow \omega_{CR1} = \dfrac{1}{2}\dfrac{m_S}{m_C}\omega_m = \dfrac{1}{2}\dfrac{4}{6}\cdot 480.82 = 160.27 \left[\dfrac{Rad}{s}\right]; \; n_{CR1} = 1530.5\,[Rpm] \\ \Omega_2 = 2\cdot\Omega_S = \omega_m \Rightarrow \omega_{CR2} = \dfrac{1}{2}\dfrac{m_S}{m_C}\dfrac{\omega_m}{2} = \dfrac{\omega_{CR1}}{2} = 80.14 \left[\dfrac{Rad}{s}\right]; \; n_{CR2} = 765.2\,[Rpm] \end{cases}$$

✓

2) NATURAL FREQUENCIES (ω_{m1}, ω_{m2})

$[I]\{\ddot{\theta}\} + [K_c]\{\theta\} = \{0\}$

$\{\theta\} = \{\theta_0\}e^{i\omega t}$; $\{\dot{\theta}\} = i\omega\{\theta_0\}e^{i\omega t}$; $\{\ddot{\theta}\} = -\omega^2\{\theta_0\}e^{i\omega t}$

$(-\omega^2[I] + [K_c])\{\theta_0\}\cancel{e^{i\omega t}} = \{0\}$, $\{\theta_0\} = \{0\}$: TRIVIAL SOLUTION

$\det([K_c] - \omega^2[I]) = 0$

$$\det([K_c] - \omega^2[I]) = \det\begin{bmatrix} 2K_cR_1^2 - \omega^2(I_e+I_1) & -2K_cR_1R_2 \\ -2K_cR_1R_2 & 2K_cR_2^2 - \omega^2(I_R+I_2) \end{bmatrix} = 0$$

$[2K_cR_1^2 - \omega^2(I_e+I_1)][2K_cR_2^2 - \omega^2(I_R+I_2)] - (-2K_cR_1R_2)^2 = 0$

$\cancel{4K_c^2R_1^2R_2^2} - 2K_cR_1^2\omega^2(I_R+I_2) - 2K_cR_2^2\omega^2(I_e+I_1) + \omega^4(I_e+I_1)(I_R+I_2) - \cancel{4K_c^2R_1^2R_2^2} = 0$

$\omega^2[-2K_cR_1^2(I_R+I_2) - 2K_cR_2^2(I_e+I_1) + \omega^2(I_e+I_1)(I_R+I_2)] = 0$

$$\begin{cases} \omega_1 = 0 = \omega_{m1} \\ \omega_2 = \sqrt{\dfrac{2K_c[R_1^2(I_R+I_2) + R_2^2(I_e+I_1)]}{(I_e+I_1)(I_R+I_2)}} = \sqrt{\dfrac{2\cdot 60\cdot 10^5[0.2^2(1.5+0.03) + 0.25^2(0.5+0.03)]}{(0.5+0.03)(1.5+0.03)}} = 118.15 \ [Rad/s] \end{cases}$$

✓

3) TORSIONAL CRITICAL SPEEDS (n_{CR1}, n_{CR2})

$T = T_0 + \sum_{K=1}^{m}(A_K\cos(K\cdot\Omega_s t) + B_K\sin(K\cdot\Omega_s t)) = T_0 + \sum_{K=1}^{m} B_K\sin(K\cdot\Omega_s t) =$

$= T_0 + B_1\sin(\underbrace{1\cdot\Omega_s}_{\Omega_1} t) + B_2\sin(\underbrace{2\cdot\Omega_s}_{\Omega_2} t)$

$\Omega_s = 2\dfrac{m_c}{m_s}\omega_{CR} = 2\cdot\dfrac{1}{4}\omega_{CR} = \dfrac{1}{2}\omega_{CR} \Rightarrow \omega_{CR} = 2\Omega_s$ | $n_{CR} = \dfrac{\omega_{CR}\cdot 60}{2\pi}$

$\Omega_1 = 1\cdot\Omega_s = \omega_m \Rightarrow \omega_{CR1} = 2\Omega_s = 2\omega_m = 2\cdot 118.15 = 236.3\left[\dfrac{Rad}{s}\right]$; $n_{CR1} = 2256.5$ [Rpm]

$\Omega_2 = 2\cdot\Omega_s = \omega_m \Rightarrow \omega_{CR2} = 2\Omega_s = \dfrac{2}{2}\omega_m = \omega_m = 118.15\left[\dfrac{Rad}{s}\right]$; $n_{CR2} = 1128.2$ [Rpm]

✓

EXERCISE 2: TORSIONAL CRITICAL SPEEDS OF A TRANSMISSION WITH AN ELASTIC JOINT

DATA

RIGID SHAFTS (INFINITELY RIGID → $c=0$) → 2 ω_{CR} (TORSIONAL)

$I_E = 3.1\ [kg\,m^2]$ $\tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{1}{6}$; $\eta = 1$; $I_R = 2\ [kg\,m^2]$; $K_T = 500\ [Nm/Rad]$

1) DERIVE THE MOTION EQs (θ_E, θ_R)

2) COMPUTE THE NATURAL FREQUENCIES.

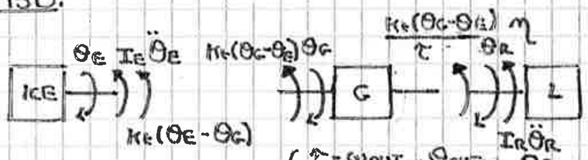
3) COMPUTE THE TORSIONAL CRITICAL SPEEDS (ONLY FIRST TWO HARMONICS) T_E, T_R = CONST

EXECUTION

$\boxed{\eta = \frac{P_{OUT}}{P_{IN}} = \frac{T_{OUT}\cdot\dot\theta_{OUT}}{T_{IN}\cdot\dot\theta_{IN}}}$; $\boxed{\tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{\theta_{OUT}}{\theta_{IN}} = \frac{R_{IN}}{R_{OUT}} = \frac{Z_{IN}}{Z_{OUT}} = \frac{T_{IN}}{T_{OUT}}\eta}$ (<1)

1) MOTION EQs:

FBD:



$\tau \begin{cases} \tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{\theta_{OUT}}{\theta_{IN}} = \frac{\theta_R}{\theta_E} \\ \tau = \frac{T_{IN}}{T_{OUT}}\cdot\eta \end{cases}$

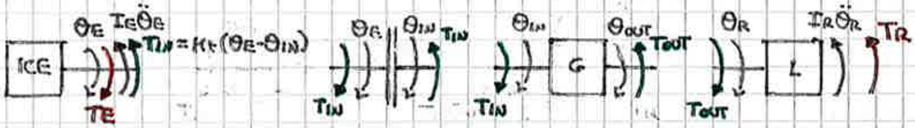
MOTION EQs:

$$\begin{cases} I_E\ddot\theta_G + K_t\left(\theta_E - \frac{\theta_R}{\tau}\right) = 0 \\ I_R\ddot\theta_R + \frac{K_t}{\tau}\left(\frac{\theta_R}{\tau} - \theta_E\right)\eta = 0 \end{cases} \quad \begin{cases} I_E\ddot\theta_E + K_t\theta_E - \frac{K_t}{\tau}\theta_R = 0 \\ I_R\ddot\theta_R - \frac{K_t}{\tau}\eta\theta_E + \frac{K_t}{\tau^2}\eta\theta_R = 0 \end{cases}$$

$$\begin{bmatrix} I_E\cdot\eta & 0 \\ 0 & I_R \end{bmatrix}\begin{Bmatrix}\ddot\theta_E \\ \ddot\theta_R\end{Bmatrix} + \begin{bmatrix} K_t\cdot\eta & -\frac{K_t}{\tau}\cdot\eta \\ -\frac{K_t}{\tau}\eta & \frac{K_t}{\tau^2}\eta \end{bmatrix}\begin{Bmatrix}\theta_E \\ \theta_R\end{Bmatrix} = \begin{Bmatrix} 0\cdot\eta \\ 0 \end{Bmatrix}$$ ✓

(BETTER AND) COMPLETE FBD:

SINCE [K] IS NOT SYMM → WE MULTIPLY THE 1st EQ $\cdot\eta$



$$\begin{cases} -I_E\ddot\theta_E - T_{IN} + T_E = 0 \\ -I_R\ddot\theta_R + T_{OUT} - T_R = 0 \end{cases} \Rightarrow \begin{cases} I_E\ddot\theta_E + T_{IN} = T_E \\ I_R\ddot\theta_R + T_{OUT} = -T_R \end{cases}$$

$T_{IN} = K_t(\theta_E - \theta_{IN}) = K_t\left(\theta_E - \frac{\theta_R}{\tau}\right)$

$T_{OUT} = \frac{T_{IN}}{\tau}\eta$; $\theta_{IN} = \frac{\theta_{OUT}}{\tau} = \frac{\theta_R}{\tau}$

$$\Rightarrow \begin{cases} I_E\ddot\theta_E + K_t\left(\theta_E - \frac{\theta_R}{\tau}\right) = T_E \\ I_R\ddot\theta_R - \frac{\eta}{\tau}K_t\left(\theta_E - \frac{\theta_R}{\tau}\right) = -T_R \end{cases} \Rightarrow \begin{bmatrix} I_E\cdot\eta & 0 \\ 0 & I_R \end{bmatrix}\begin{Bmatrix}\ddot\theta_E \\ \ddot\theta_R\end{Bmatrix} + \begin{bmatrix} K_t\cdot\eta & -\frac{K_t}{\tau}\cdot\eta \\ -\frac{K_t}{\tau}\eta & +\frac{K_t}{\tau^2}\eta \end{bmatrix}\begin{Bmatrix}\theta_E \\ \theta_R\end{Bmatrix} = \begin{Bmatrix} T_E\cdot\eta \\ -T_R \end{Bmatrix}$$ ✓

IF USING THE LAGRANGIAN APPROACH:

$$\begin{cases} T = \frac{1}{2}I_E\dot\theta_E^2 + \frac{1}{2}I_R\dot\theta_R \qquad \left(D = \frac{1}{2}c_T\dot\theta^2 = 0\right) \\ V = \frac{1}{2}K_t(\theta_E - \theta_{IN})^2 = \frac{1}{2}K_t\left(\theta_E - \frac{\theta_R}{\tau}\right)^2 \end{cases}$$

$L = T - V$; $L = L(q_i, \dot q_i)$; $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$, $Q_k = \sum_{i=1}^{m} \vec F_i \frac{\partial \vec R_i}{\partial q_k} = \sum_{i=1}^{m} \vec F_i \frac{\partial \vec R_i}{\partial q_k}$, $Q_k = -\frac{\partial D}{\partial \dot q_k}$, $D = \frac{1}{2}\sum_{j,k=1}^{m} c_{jk}\dot q_k \dot q_j$

$k=1$ $q_1 = \theta_E$:

$\frac{\partial L}{\partial \theta_E} = \frac{\partial T}{\partial \theta_E} - \frac{\partial V}{\partial \theta_E} = -K_t(\theta_E - \theta_{IN})$

$\frac{\partial L}{\partial \dot\theta_E} = \frac{\partial T}{\partial \dot\theta_E} - \frac{\partial V}{\partial \dot\theta_E} = I_E\dot\theta_E \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_E}\right) = I_E\ddot\theta_E$

$Q_1 = \sum_{i=1}^{m} \vec F_i \frac{\partial \vec R_i}{\partial \theta_E} = T_E \vec k \cdot \vec k = T_E$

$k=2$, $q_2 = \theta_R$:

$\frac{\partial L}{\partial \theta_R} = \frac{\partial T}{\partial \theta_R} - \frac{\partial V}{\partial \theta_R} = +\frac{1}{\tau}K_t\left(\theta_E - \frac{\theta_R}{\tau}\right)$

$\frac{\partial L}{\partial \dot\theta_R} = I_R\dot\theta_R \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_R}\right) = I_R\ddot\theta_R$

$Q_2 = \vec F \frac{\partial \vec R}{\partial \theta_R} = F\vec k(-\vec k) = -T_R$

$$\begin{cases} I_E\ddot\theta_E + K_t\left(\theta_E - \frac{\theta_R}{\tau}\right) = T_E \\ I_R\ddot\theta_R - \frac{K_t}{\tau}\left(\theta_E - \frac{\theta_R}{\tau}\right) = -T_R \end{cases} \Rightarrow \begin{cases} I_E\ddot\theta_E + K_t\theta_E - \frac{K_t}{\tau}\theta_R = T_E \\ I_R\ddot\theta_R - \frac{K_t}{\tau}\theta_E + \frac{K}{\tau^2}\theta_R = -T_R \end{cases}$$

2) NATURAL FREQ. (* IN PROF. EX. I'VE NEVER SEEN ω_0)

$$\det([K] - \omega^2[M]) = \begin{vmatrix} K_t - \omega^2 I_E & -\frac{K_t}{\tau} \\ -\frac{K_t}{\tau} & \frac{K_t}{\tau^2} - \omega^2 I_R \end{vmatrix} = (K_t - \omega^2 I_E)\left(\frac{K_t}{\tau^2} - \omega^2 I_R\right) - \left(\frac{K_t}{\tau}\right)^2$$

$\cancel{\frac{K_t^2}{\tau^2}} - K_t\omega^2 I_R - \frac{K_t}{\tau^2}I_E\omega^2 + I_E I_R\omega^4 - \cancel{\frac{K_t^2}{\tau^2}} = 0$

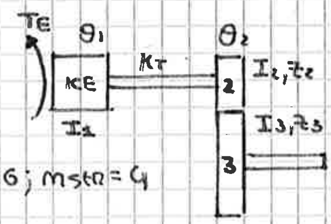
$\omega^2\left[I_E I_R\omega^2 - \left(K_t I_R + \frac{K_t}{\tau^2}I_E\right)\right] = 0$, $\omega_1^* = 0$ [Rad/s]

$\omega_2 = \sqrt{\frac{K_t}{I_E I_R}\left(I_R + \frac{I_E}{\tau^2}\right)}$

$\omega_2 = \sqrt{\frac{500}{3.1\cdot 2}\left(2 + 3.1\cdot 6^2\right)} = 95.715\left[\frac{Rad}{s}\right] \cong 95.7\left[\frac{Rad}{s}\right]$ ✓

EXERCISE 3 : TORSIONAL CRITICAL SPEEDS OF A TRANSMISSION WITH GEARBOX

DATA



$I_1 = 2.45\,[kg\,m^2]$; $I_2 = 3.92\,[kg\,m^2]$; $I_3 = 9.81\,[kg\,m^2]$

$\tau = \frac{\omega_{OUT}}{\omega_{IN}}$; $\eta = \frac{P_{OUT}}{P_{IN}} = 1$; $z_2 = 23$; $z_3 = 50$;

$L = 0.8\,[m]$; $d = 0.08\,[m]$; $G = 80000\,[N/mm^2]$; $m_{cyl} = 6$; $m_{str} = 4$

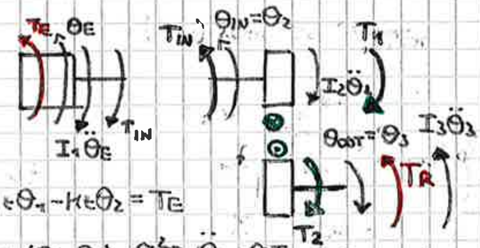
1) MOTION EQs (LAGRANGIAN APPROACH)

2) NATURAL FREQs AND NATURAL MODES

3) TORSIONAL CRITICAL SPEEDS (FIRST 2 HARMONICS)

EXECUTION

FBD:



0) CLASSICAL APPROACH

$$\begin{cases} I_1\ddot{\theta}_1 + T_{IN} = T_E \\ I_2\ddot{\theta}_2 - T_{IN} + T_4 = 0 \\ I_3\ddot{\theta}_3 + T_R - T_2 = 0 \end{cases} \Rightarrow \begin{cases} I_1\ddot{\theta}_1 + K_t(\theta_1 - \theta_2) = T_E \\ I_2\ddot{\theta}_2 - K_t(\theta_1 - \theta_2) + \tau T_2 = 0 \\ T_2 = \tau I_3 \ddot{\theta}_2 + T_R = 0 \end{cases} \quad \begin{cases} I_1\ddot{\theta}_1 + K_t\theta_1 - K_t\theta_2 = T_E \\ I_2\ddot{\theta}_2 - K_t(\theta_1-\theta_2) + \tau^2 I_3\ddot{\theta}_2 + \tau T_R = 0 \end{cases}$$

$$\tau = \frac{\omega_{OUT}}{\omega_{IN}} = \frac{\theta_{OUT}}{\theta_{IN}} = \frac{\theta_3}{\theta_2} = \frac{R_2}{R_3} = \frac{z_2}{z_3} = \frac{T_4}{T_2}\eta \Rightarrow T_{OUT} = \frac{T_{IN}}{\tau}; \quad \theta_3 = \tau\theta_2$$

$$\begin{cases} I_1\ddot{\theta}_1 + K_t\theta_1 - K_t\theta_2 = T_E \\ (I_2 + \tau^2 I_3)\ddot{\theta}_2 - K_t\theta_1 + K_t\theta_2 = -\tau T_R \end{cases} \Rightarrow \begin{bmatrix} I_1 & \\ & I_2+\tau^2 I_3 \end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{Bmatrix} + \begin{bmatrix} K_t & -K_t \\ -K_t & K_t \end{bmatrix}\begin{Bmatrix}\theta_1 \\ \theta_2\end{Bmatrix} = \begin{Bmatrix} T_E \\ -\tau T_R \end{Bmatrix} \checkmark$$

1) LAGRANGIAN APPROACH

$$\begin{cases} T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2 = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3(\tau\dot{\theta}_2)^2 \quad \text{(HERE WE HAVE TO SUBSTITUTE)} \\ V = \frac{1}{2}K_t(\theta_1 - \theta_2)^2 \end{cases}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_K}\right) - \frac{\partial L}{\partial q_K} = Q_K, \quad Q_K = \sum_{i=1}^{m} \vec{F}_i \frac{\partial \vec{R}_i}{\partial q_K}$$

$K=1, q_1 = \theta_1$:

$$\frac{\partial L}{\partial \theta_1} = \frac{\partial T}{\partial \theta_1} - \frac{\partial V}{\partial \theta_1} = -K_t(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = I_1\dot{\theta}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = I_1\ddot{\theta}_1$$

$$Q_1 = \vec{F}_E \frac{\partial \vec{R}_E}{\partial \theta_1} = T_E \cdot \vec{k} \cdot \vec{k} = T_E$$

$$I_1\ddot{\theta}_1 + K_t(\theta_1 - \theta_2) = T_E$$

(1) $I_1\ddot{\theta}_1 + K_t\theta_1 - K_t\theta_2 = T_E$

$K=2; q_2 = \theta_2$

$$\frac{\partial L}{\partial \theta_2} = \frac{\partial T}{\partial \theta_2} - \frac{\partial V}{\partial \theta_2} = +K_t(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = I_2\dot{\theta}_2 + I_3\tau^2\dot{\theta}_2 \Rightarrow \frac{d}{dt}() = I_2\ddot{\theta}_2 + I_3\tau^2\ddot{\theta}_2$$

$$Q_2 = T_R(-\vec{k})\vec{k} = -T_R$$

$$I_2\ddot{\theta}_2 + I_3\tau^2\ddot{\theta}_2 - K_t(\theta_1 - \theta_2) = -T_R$$

(2) $(I_2 + I_3\tau^2)\ddot{\theta}_2 - K_t\theta_1 + K_t\theta_2 = -T_R$

$$\begin{bmatrix} I_1 & \\ & I_2+I_3\tau^2 \end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{Bmatrix} + \begin{bmatrix} K_t & -K_t \\ -K_t & K_t \end{bmatrix}\begin{Bmatrix}\theta_1 \\ \theta_2\end{Bmatrix} = \begin{Bmatrix} T_E \\ -T_R \end{Bmatrix} \checkmark$$

2) NATURAL FREQs AND NATURAL MODES (⇒ NO MOMENTS T_E, T_R)

$K_t = \frac{G I_P}{L}$; $I_P = \frac{\pi d^4}{32}$

$K_t = 401\,920 \left[\frac{Nm}{rad}\right]$

$$\det\begin{bmatrix} K_t - \omega^2 I_1 & -K_t \\ -K_t & K_t - \omega^2(I_2 + I_3\tau^2) \end{bmatrix} = (K_t - \omega^2 I_1)(K_t - \omega^2(I_2 + I_3\tau^2)) - K_t^2 = 0$$

$$\cancel{K_t^2} - \omega^2 K_t(I_2 + I_3\tau^2) - K_t\omega^2 I_1 + I_1(I_2 + I_3\tau^2)\omega^4 - \cancel{K_t^2} = 0$$

$$\omega^2[I_1(I_2 + I_3\tau^2)\omega^2 - K_t(I_1 + I_2 + I_3\tau^2)] = 0 \Rightarrow \omega_1 = 0\,[rad/s]; \quad \omega_2 = \sqrt{\frac{K_t(I_1 + I_2 + I_3\tau^2)}{I_1(I_2 + I_3\tau^2)}} = 480.71\left[\frac{rad}{s}\right]$$

#1 : $\omega_1 = 0$, $\varphi_{1,1} = 1 \Rightarrow -K_t\varphi_{1,1} + K_t\varphi_{2,1} = 0 \Rightarrow \varphi_{2,1} = 1$

#2 : $\omega_2 = 480.71\left[\frac{rad}{s}\right]$, $\varphi_{1,2} = 1 \Rightarrow K_t - \omega^2 I_1 - K_t\varphi_{2,2} = 0 \Rightarrow \varphi_{2,2} = \frac{K_t - \omega^2 I_1}{K_t} = 1 - \frac{\omega^2 I_1}{K_t} = -0.4086 \cong -0.41$

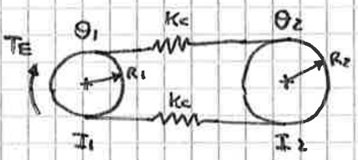
EXERCISE 4: ELASTIC SHAFTS TORSIONAL OSCILLATIONS

DATA

$I_E = 0.5\ [kg\,m^2]$; $I_R = 1.5\ [kg\,m^2]$, $I_1 = I_2 = 0.03\ [kg\,m^2]$

$\tau = \omega_{OUT}/\omega_{IN}$ $\quad \eta = P_{OUT}/P_{IN} = 1$; $R_1 = 0.2\ [m]$; $R_2 = 0.25\ [m]$

$K_c(BELT) = 60\ [kN/m]$; $m_{st} = 4$; $m_{cyl} = 1$



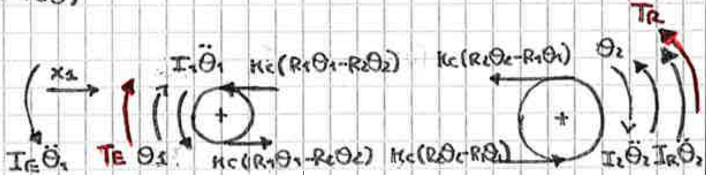
1) MOTION EQS (LAGRANGIAN APPROACH)

2) NATURAL FREQS.

3) TORSIONAL CRITICAL SPEEDS (TWO HARMONICS)

EXECUTION

0) CLASSICAL APPROACH



$$\begin{cases}(I_E + I_1)\ddot\theta_1 + 2K_c(R_1\theta_1 - R_2\theta_2)R_1 = T_E \\ (I_R + I_2)\ddot\theta_2 + 2K_c(R_2\theta_2 - R_1\theta_1)R_2 = -T_R\end{cases}$$

$$\begin{bmatrix} I_E + I_1 & \\ & I_R + I_2 \end{bmatrix}\begin{Bmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{Bmatrix} + \begin{bmatrix} 2K_cR_1^2 & -2K_cR_2R_1 \\ -2K_cR_1R_2 & +2K_cR_2^2\end{bmatrix}\begin{Bmatrix}\theta_1 \\ \theta_2\end{Bmatrix} = \begin{Bmatrix} T_E \\ -T_R\end{Bmatrix} \quad ✓$$

1) LAGRANGIAN APPROACH

$L = T - V$; $L = L(q_i; \dot q_i)$ $\quad q_i = \theta_1, \theta_2$; $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} = Q_k$, $\quad Q_k = \sum_{i=1}^{n} \vec F_i \frac{\partial \vec R_i}{\partial q_k}$, $\quad k = 1, 2$

$$T = \frac{1}{2}(I_1 + I_E)\dot\theta_1^2 + \frac{1}{2}(I_2 + I_R)\dot\theta_2^2$$

$$V = \frac{1}{2}K_c(R_1\theta_1 - R_2\theta_2)^2 + \frac{1}{2}K_c(R_1\theta_1 - R_2\theta_2)^2 = \frac{1}{2}(2K_c)(R_1\theta_1 - R_2\theta_2)^2$$

$k = 1$, $q_1 = \theta_1$:	$k = 2$, $q_2 = \theta_2$:
$\frac{\partial L}{\partial \theta_1} = \frac{\partial T}{\partial \theta_1} - \frac{\partial V}{\partial \theta_1} = -R_1(2K_c)(R_1\theta_1 - R_2\theta_2)$	$\frac{\partial L}{\partial \theta_2} = \frac{\partial T}{\partial \theta_2} - \frac{\partial V}{\partial \theta_2} = +R_2(2K_c)(R_1\theta_1 - R_2\theta_2)$
$\frac{\partial L}{\partial \dot\theta_1} = (I_1 + I_E)\dot\theta_1 \Rightarrow \frac{d()}{dt} = (I_1 + I_E)\ddot\theta_1$	$\frac{\partial L}{\partial \dot\theta_2} = (I_2 + I_R)\dot\theta_2 \Rightarrow \frac{d()}{dt} = (I_2 + I_R)\ddot\theta_2$
$Q_1 = T_E\,\vec k \cdot \vec k = T_E$	$Q_2 = T_R(-\vec k)(\vec k) = -T_R$
$(I_1 + I_E)\ddot\theta_1 + 2R_1K_c(R_1\theta_1 - R_2\theta_2) = T_E$	$(I_2 + I_R)\ddot\theta_2 - R_2(2K_c)(R_1\theta_1 - R_2\theta_2) = -T_R$
① $(I_1 + I_E)\ddot\theta_1 + 2K_cR_1^2\theta_1 - 2K_cR_1R_2\theta_2 = T_E$	② $(I_2 + I_R)\ddot\theta_2 - 2K_cR_1R_2\theta_1 + 2K_cR_2^2\theta_2 = -T_R$

$$\begin{bmatrix} I_1 + I_E & \\ & I_2 + I_R \end{bmatrix}\begin{Bmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{Bmatrix} + \begin{bmatrix} 2K_cR_1^2 & -2K_cR_1R_2 \\ -2K_cR_1R_2 & 2K_cR_2^2\end{bmatrix}\begin{Bmatrix}\theta_1 \\ \theta_2\end{Bmatrix} = \begin{Bmatrix} T_E \\ -T_R\end{Bmatrix} \quad ✓$$

NOTE: $Q_1 = \sum_{i=1}^{2} \vec F_i \frac{\partial \vec R_i}{\partial \theta_1} = T_E \vec k \frac{\partial \theta_1 \vec k}{\partial \theta_1} + T_2(-\vec k)\underbrace{\frac{\partial \theta_2}{\partial \theta_1}}_{=0}$

2) NATURAL FREQUENCIES:

$$\det([K] - \omega^2[M]) = \det\begin{bmatrix} 2K_cR_1^2 - \omega^2(I_1 + I_E) & -2K_cR_1R_2 \\ -2K_cR_1R_2 & 2K_cR_2^2 - \omega^2(I_2 + I_R)\end{bmatrix} = (2K_cR_1^2 - \omega^2(I_1 + I_E))(2K_cR_2^2 - \omega^2(I_2 + I_R)) - 4K_c^2R_1^2R_2^2$$

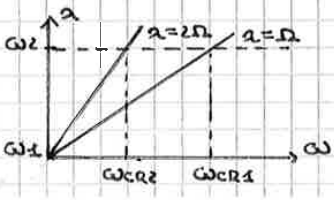
$$\cancel{4K_c^2R_1^2R_2^2} - 2K_cR_1^2\omega^2(I_2 + I_R) - 2K_cR_2^2\omega^2(I_1 + I_E) + \omega^4(I_1 + I_E)(I_2 + I_R) - \cancel{4K_c^2R_1^2R_2^2} = 0$$

$$\omega^2\left\{(I_1 + I_E)(I_2 + I_R)\omega^2 - 2K_c[R_1^2(I_2 + I_R) + R_2^2(I_1 + I_E)]\right\} = 0$$

$\omega_1 = 0\ [rad/s]$ ✓

$$\omega_2 = \sqrt{\frac{2K_c[R_1^2(I_2 + I_R) + R_2^2(I_1 + I_E)]}{(I_1 + I_E)(I_2 + I_R)}} = 118.15\ \left[\frac{rad}{s}\right]$$

3) TORSIONAL CRITICAL SPEEDS (2 HARMONICS)



$\Omega_{ENG} = k\left(2 \cdot \frac{m_{cyl}}{m_{st}}\right)\omega_{cr}$, $\quad \omega = \frac{2\pi M}{60}$ $\quad (\pi = 3.1416)$

$$n_{cr1} = \frac{\omega_{cr1}\,60}{2\pi} = \frac{60}{2\pi}\frac{\Omega_{ENG}}{k}\left(\frac{1}{2}\frac{m_{st}}{m_{cyl}}\right) = \frac{60}{2\pi}\frac{\omega_2}{1}\left(\frac{1}{2}\frac{4}{1}\right) = 2256.5\ [RPM]$$

$$n_{cr2} = \frac{60}{2\pi}\frac{\omega_2}{2}(2) = 1128.25\ [RPM] \quad ✓$$

5) 2 DOFs SYSTEM, $f_1 = 9\,[Hz]$, $f_2 = 15\,[Hz]$, WHICH ARE THE 2 CRITICAL SPEEDS $n_{CR}\,[rpm]$?

$C_m = C_0 + \sum_{k=1}^{n} A_k \cos(k\Omega t) + B_k \sin(k\Omega t)$: ENGINE TORQUE, Ω : ANG. SPEED OF ICE.

$\Omega = 2\frac{n_c}{n_s} \cdot \omega$: HARMONICS ; $n = \frac{60}{2\pi}\omega$

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5) ASSUMING THAT THE SPRING HAS A HYSTERETIC BEHAVIOUR (COMPLEX STIFFNESS: $(1+i\eta)k$, $\eta=\sqrt{3}$: LOSS FACTOR), DERIVE THE STEADY-STATE RESPONSE (θ_0, φ) FOR $\omega=\omega_m$.

$L_c = \Delta E_H = \alpha \cdot x_0^2$ | $L_c = \Delta E_V = \int_{CYCLE} C_{EQ} \cdot \dot{x}\, dx = \int_T C_{EQ}\dot{x}\frac{dx}{dt}dt = \int_T C_{EQ}\dot{x}^2 dt = \int^T C_{EQ}\omega^2 x_0^2 \sin^2(\omega t)dt$

$\Delta E_H = \Delta E_V$ | $x(t) = x_0\cos(\omega t)$, $\dot{x}(t) = -\omega x_0 \sin(\omega t)$ $= C_{EQ}\pi\omega x_0^2$

$\alpha \cdot \cancel{x_0^2} = C_{EQ}\pi\omega \cancel{x_0^2} \Rightarrow C_{EQ} = \frac{\alpha}{\pi\omega} = \frac{h}{\omega} \Rightarrow C_{EQ}\omega = h$, $\eta = \frac{h}{k}$

$(M+m)L^2\ddot{\theta} + \frac{3}{4}L^2k\theta = -\sqrt{3}LF_0e^{i\omega t}$

$k \to k(1+i\eta)$

$(M+m)L^2\ddot{\theta} + \frac{3}{4}L^2k(1+i\eta)\theta = -\sqrt{3}LF_0e^{i\omega t}$

$\to \ddot{\theta} + \omega_m^2(1+i\eta)\theta = -\frac{\omega_m^2 4\sqrt{3}}{3L^2k}LF_0e^{i\omega t}$

$\cancel{\omega_m^2}\left[1+i\eta-\left(\frac{\omega}{\omega_m}\right)^2\right]\theta_0\cancel{e^{i\omega t}} = -\frac{4\sqrt{3}}{3Lk}\cancel{\omega_m^2}F_0\cancel{e^{i\omega t}} \to \theta_0 = -\frac{4\sqrt{3}}{3Lk}\frac{1}{1-\left(\frac{\omega}{\omega_m}\right)^2+i\eta} = |\theta_0|e^{i\varphi}$

$|\theta_0| = -\frac{4\sqrt{3}}{3kL}\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_m}\right)^2\right]^2+\eta^2}}$; $\varphi = Tg^{-1}\left(\frac{-Im}{Re}\right) = Tg^{-1}\left[-\frac{\eta}{1-\left(\frac{\omega}{\omega_m}\right)^2}\right]$

CONSIDERING $\omega=\omega_m$:

$|\theta_0| = -\frac{4\sqrt{3}}{3kL\eta}$ [Rad] $\left(Tg\varphi = -\frac{0}{0}\right)$ $\varphi = -\frac{\pi}{2}$ [Rad] (QUADRATURE)

$$\frac{5}{3} m^2 \omega^4 - \frac{31}{3} m K_2 \omega_2^2 + \frac{7}{4} K_2^2 = 0$$

$$\omega_{1,2}^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3}{10m^2}\left(\frac{31}{3} m K_2 \pm \sqrt{\left(\frac{31}{3}\right)^2 m^2 K_2^2 - \frac{35}{3} m^2 K_2^2}\right) =$$

$$= \frac{3}{10m^2}\left(\frac{31}{3} m K_2 \pm 9.7525\, m K_2\right) = 3.1 \frac{K_2}{m} \pm 2.926 \frac{K_2}{m}$$

$(-)$ $\omega_1^2 = 0.174 \frac{K_2}{m}$

$(+)$ $\omega_2^2 = 6.026 \frac{K_2}{m}$

$$\omega_1 = 0.417 \sqrt{\frac{K_2}{m_2}} \left[\frac{Rad}{s}\right]$$

$$\omega_2 = 2.455 \sqrt{\frac{K_2}{m_2}} \left[\frac{Rad}{s}\right]$$

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MULTIPLE CHOICE (2 points)

Which of the following answers represents the unique correct definition of a generic eigenfunction $\phi_r(x)$ of a torsional problem with generic boundary conditions?

a) $\phi_r(x) = A\sin\left(\omega_r\sqrt{\frac{\rho}{G}}x\right) + B\cos\left(\omega_r\sqrt{\frac{\rho}{G}}x\right)$

b) $\phi_r(x) = A\sin\left(\omega_r\sqrt{\frac{\rho}{E}}x\right) + B\cos\left(\omega_r\sqrt{\frac{\rho}{E}}x\right)$

c) $\phi_r(x) = A\sin\left(\omega_r\sqrt{\frac{\rho}{E}}x\right) + B\cos\left(\omega_r\sqrt{\frac{\rho}{E}}x\right) + C\sinh\left(\omega_r\sqrt{\frac{\rho}{E}}x\right) + D\cosh\left(\omega_r\sqrt{\frac{\rho}{E}}x\right)$

d) $\phi_r(x) = A\sin\left(\omega_r\sqrt{\frac{\rho}{G}}x\right) + B\cos\left(\omega_r\sqrt{\frac{\rho}{G}}x\right) + C\sinh\left(\omega_r\sqrt{\frac{\rho}{G}}x\right) + D\cosh\left(\omega_r\sqrt{\frac{\rho}{G}}x\right)$

from theory : torsional continuous systems
$\Rightarrow$
- PDE II order
- only sin and cos terms
- G is the shear modulus [Pa] $\Rightarrow \tau \Rightarrow$ torsion behaviour
- E is the Young modulus [Pa] $\Rightarrow \sigma \Rightarrow$ axial behaviour
- in case, FBD of elementary piece, wave equation, EoM, ...

MULTIPLE CHOICE (3 points)

The natural frequencies of two DOFs torsional system are 9 Hz and 15 Hz. The driving torque is supplied by a reciprocating four-strokes three-cylinders internal combustion engine. Considering only the first two harmonics, which are the two highest critical speeds n_{cr} [rpm] of the system?

a) 1350 and 1620

b) 300 and 600

c) 180 and 300

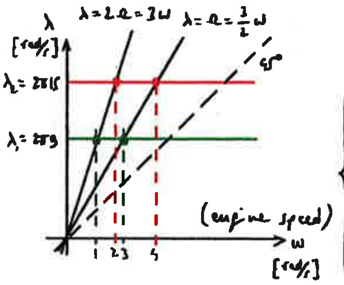
d) 360 and 600

from theory

Engine torque: (Fourier decomp.) $\boxed{C_m = C_0 + \sum_{k=1}^{n} A_k\cos(k\Omega t) + B_k\sin(k\Omega t)}$ angular speed of ICE

$\boxed{\Omega = 2\frac{n_c}{n_s}\cdot\omega}$ harmonics

with $n = \frac{60}{2\pi}\omega$

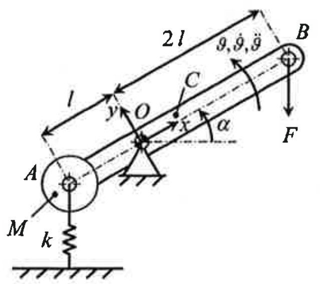


$\Omega = 2\frac{3}{4}\omega = \frac{3}{2}\omega \Rightarrow \omega = \frac{2}{3}\Omega$

$$\begin{cases} \omega_{cr,3} = \frac{2}{3}\lambda_1 \Rightarrow n_{cr,3} \doteq 60\frac{2}{3}9 = 360 \text{ rpm} \\ \omega_{cr,1} = \frac{1}{3}\lambda_1 \Rightarrow n_{cr,1} \doteq 60\frac{1}{3}9 = 180 \text{ rpm} \\ \omega_{cr,4} = \frac{2}{3}\lambda_2 \Rightarrow n_{cr,4} \doteq 60\frac{2}{3}15 = 600 \text{ rpm} \\ \omega_{cr,2} = \frac{1}{3}\lambda_2 \Rightarrow n_{cr,2} \doteq 60\frac{1}{3}15 = 300 \text{ rpm} \end{cases}$$

SINGLE DEGREE OF FREEDOM (11 points)

The SDOF system sketched in the figure is composed of a homogeneous crane bar C (overall length $L = 3l$, mass m, and barycentric mass moment of inertia $I_G = mL^2/12$) and a lumped mass M (negligible mass moment of inertia). The system is hinged in O and linked to the ground at A by a spring with stiffness k. A harmonic force $F = F_0 e^{i\omega t}$ is applied in B in the shown direction. The system is placed on a horizontal plane (weight does not have to be considered).
The absolute reference system x, y, ϑ describes the motion and the rotation of the bar C with respect to the shown configuration with $\alpha = 30°$.



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5) Assuming that the spring has a hysteretic behaviour, hence its stiffness is now complex with a value $k(1+i\eta)$ where $\eta=\sqrt{3}$ is the loss factor, derive the steady-state response (modulus ϑ_0 and phase φ) for $\omega=\omega_n$.

↳ note: underdamped case for $0 \le \eta < 2$

EoM: $(M+m)\ell^2\ddot{\vartheta} + \frac{3}{4}k(1+i\eta)\ell^2\vartheta = -\sqrt{3}\,\ell F_0 e^{i\omega t}$

$$\ddot{\vartheta} + \omega_n^2(1+i\eta)\vartheta = \frac{-\sqrt{3}F_0}{(M+m)\ell}e^{i\omega t}$$

$\vartheta = \vartheta_0 e^{i\omega t}$ in EoM $\left(-\omega^2+\omega_n^2(1+i\eta)\right)\vartheta_0 \cancel{e^{i\omega t}} = \frac{-\sqrt{3}F_0}{(M+m)\ell}\cancel{e^{i\omega t}}$

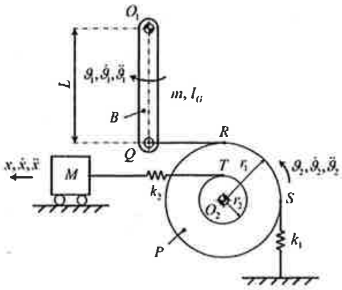
$$|\vartheta_0| = \frac{\sqrt{3}F_0}{(M+m)\ell\omega_n^2}\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\eta^2}} \; ; \quad \tan\varphi = -\frac{\eta}{1-\frac{\omega^2}{\omega_n^2}} < 0 \text{ per } \eta>0$$

if $\frac{\omega}{\omega_n}=1$; $\eta=\sqrt{3} \Rightarrow |\vartheta_0| = \frac{F_0}{(M+m)\ell\omega_n^2}$ rad ; $\varphi = -\frac{\pi}{2}$ rad

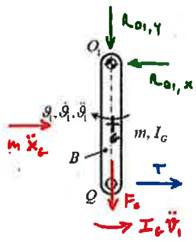
quadrature

MULTI DEGREE OF FREEDOM (11 points)

The system in figure is made up of: a rigid bar B, hinged in O_1, having length L, mass m and mass moment of inertia I_G; a massless pulley + drum device P, hinged in O_2; a lumped mass M, which slides without friction on a horizontal surface.
A rigid rope is wrapped around the pulley of radius r_1 and connects the rigid bar to the fixed frame, through a spring having stiffness k_1. No slippage occurs between the rope and the pulley.
A rope of stiffness k_2 connects the drum of radius r_2 with the lumped mass M.
The absolute reference systems x, ϑ_1 and ϑ_2 describe the horizontal displacement of the mass M, the rotation of the rigid bar B, and the rotation of the pulley + drum device P, respectively.
The system is placed on a horizontal plane (weight does not have to be considered).



1) Draw the free-body diagrams of: the bar B; the pulley + drum device P; the mass M. Consider small vibrations about the shown configuration.



small vibrations:

$\begin{cases} \sin\vartheta_1 \cong \vartheta_1 \\ \cos\vartheta_1 \cong 0 \end{cases}$ $\begin{cases} \sin\vartheta_2 \cong \vartheta_2 \\ \cos\vartheta_2 \cong 0 \end{cases}$

$x_G \cong \frac{L}{2}\vartheta_1$; $I_G = \frac{mL^2}{12}$