NUMERO: **2461A**

ANNO: 2020

A P P U N T I

STUDENTE: Chiforeanu Loredana

MATERIA: Earthquake engineering - Part 1 - theory - Prof. Ceravolo

I Faculty of Engineering

Course of

EARTHQUAKE ENGINEERING

Teacher:

Prof. R. Ceravolo

References

- Dynamics of structures / Ray W. Clough, Joseph Penzien, 1993 (Central library; II Eng. Faculty, DISTR, DIASP);
- Dinamica strutturale : teoria e calcolo / Mario Paz, 1985 (Central library, DISTR, DIGET, Architecture Faculty);
-Dynamics of structures: theory and applications to earthquake engineering / Anil K. Chopra, 1995 (Central library, II Eng. Faculty, DISTR)
- Introduzione all' ingegneria antisismica / Carlo Gavarini (II Eng. Faculty);
- Theoretical and experimental modal analysis / ed. by Nuno Manuel Mendes Maia, Jùlio Martins Montalvao e Silva, 1997 (II Eng. Faculty, DISTR, DMEC);
- Fundamentals of earthquake engineering / Nathan M. Newmark, Emilio Rosenblueth, 1971 (Central library, DISTR);
- Reinforced concrete structures / R. Park, T. Paulay, 1990 (Central library, II Eng. Faculty, DISTR);

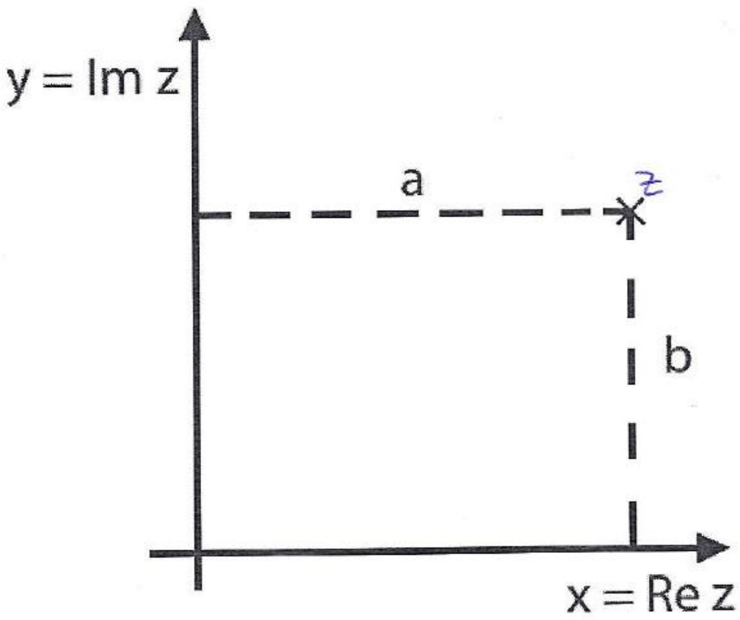
© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 3 di 262

12/3/18

BASICS ON COMPLEX NUMBERS

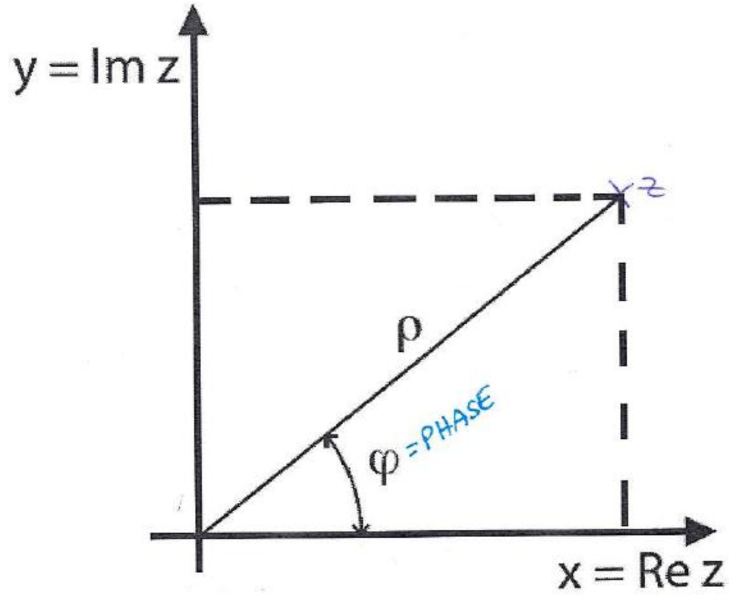
$$z = a + jb = \mathrm{Re}(z) + j\mathrm{Im}(z)$$

Cartesian representation



$$z = a + jb$$

Polar representation



$$z = \rho(\cos\varphi + j\sin\varphi) = \rho e^{j\varphi} = |z| e^{\angle z}$$

Phase can change by any multiple of 2π and still give the same angle. Hence the polydromy of $\sqrt{z}$, log z

$\varphi \in (-\pi, \pi]$

$\varphi = \mathrm{Arg}(z)$

$R = \rho = \text{module} = \sqrt{a^2 + b^2} = |z|$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 5 di 262

OPERATIONS WITH COMPLEX NUMBERS

Be z_1 e z_2 two complex numbers:

$$z_1 = a_1 + jb_1 = \rho_1\left(\cos\varphi_1 + j\sin\varphi_1\right) = \rho_1 e^{j\varphi_1}$$

$$z_2 = a_2 + jb_2 = \rho_2\left(\cos\varphi_2 + j\sin\varphi_2\right) = \rho_2 e^{j\varphi_2}$$

SUM $$z_1 + z_2 = \left(a_1 + a_2\right) + j\left(b_1 + b_2\right)$$

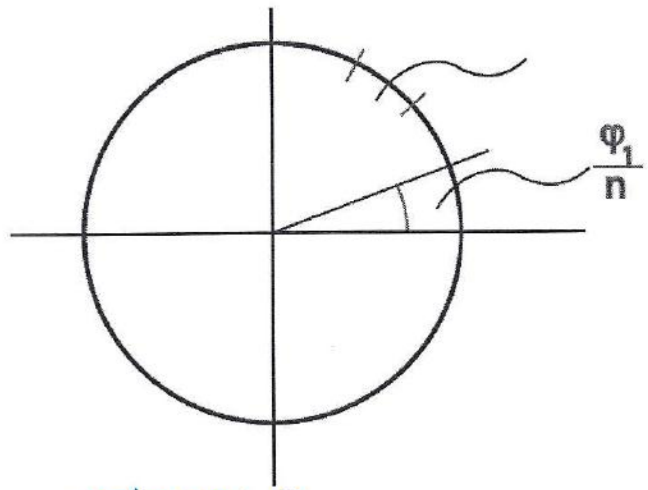
MULTIPLICATION $$z_1 z_2 = \rho_1\rho_2 e^{j\left(\varphi_1+\varphi_2\right)}$$

DIVISION $$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} e^{j\left(\varphi_1-\varphi_2\right)}$$

EXPONENTIATION $$z_1^n = \rho_1^n e^{jn\varphi_1}$$

ROOT

$$\sqrt[n]{z_1} = \sqrt[n]{\rho_1}\, e^{j\frac{\varphi_1+2k\pi}{n}} \qquad (k=0,1,2,\ldots,n-1)$$



ARGUMENT/PHASE: φ

COMPLEX CONJUGATE

$$z_1^* = a_1 - jb_1 = \rho_1 e^{-j\varphi_1}$$

MODULUS
MAGNITUDE
ABS VALUE

$$\left|z_1\right| = \square_1 = \sqrt{z_1 z_1^*} = \sqrt{a^2+b^2}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 7 di 262

CONJUGATE AND MODULUS

$$|z| = |z^*|$$

$$\mathrm{Re}(z) \le |z|$$

$$|z|^2 = 2z^*$$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

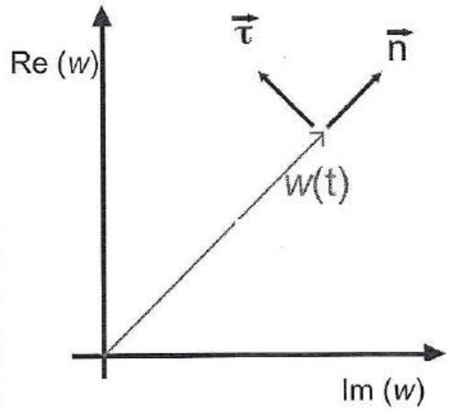
$$|z_1 \cdot z_2| = |z_1||z_2|$$

$$\big||z_1| + |z_2|\big| \ge |z_1 - z_2|$$

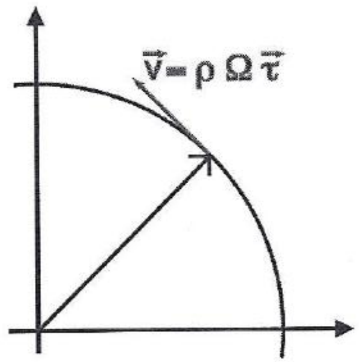
DERIVATIVE OF A COMPLEX FUNCTION

$$w(t) = \rho(t)e^{j\varphi(t)}$$

$$\frac{dw(t)}{dt} = \dot{\rho}(t)\underbrace{\boxed{e^{j\varphi(t)}}}_{\vec{n}} + \rho(t)\dot{\varphi}(t)\underbrace{\boxed{je^{j\varphi(t)}}}_{\vec{\tau} = \frac{de^{j\varphi(t)}}{dt}}$$



Example: uniform circular motion



$$\rho = \text{cost}$$

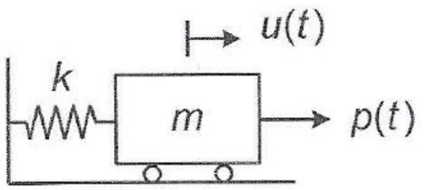
$$\frac{dw(t)}{dt} = \rho \cdot \Omega \cdot \vec{\tau} = \vec{v}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 9 di 262

12/3/18

SDF - INTRODUCTION

A single degree of freedom system consisting of a mass *m* and a spring with stiffness *k* is considered.

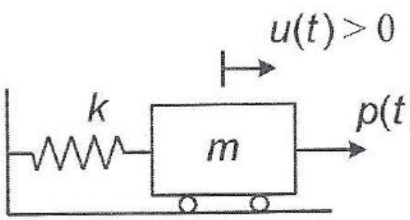


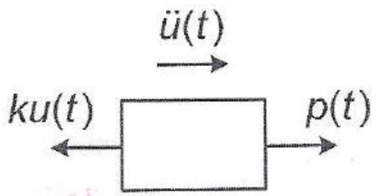
k (N/m)

The spring is undeformed for $u = 0$

The equation of motion can be derived in different ways. We will see the simplest ones.

Newton's second law



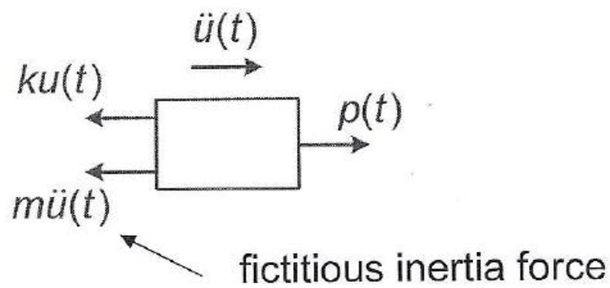


$$m\,\ddot{u}(t) = p(t) - k\,u(t) \quad \rightarrow \quad \boxed{m\,\ddot{u} + k\,u = p(t)}$$

D'Alembert's principle

The system is supposed in dynamic equilibrium. The principles of statics are applied by introducing a fictitious inertia force, a force equal to the product of mass times its acceleration and acting in a direction opposite to the acceleration.

Free-body diagram



Dynamic equilibrium

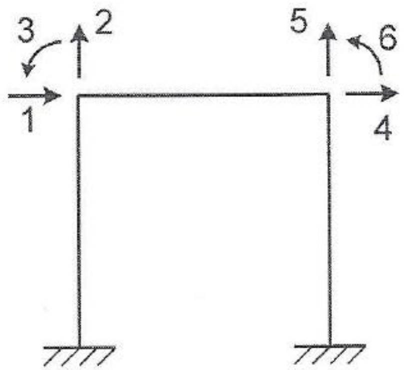
$$p(t) - k\,u(t) - m\,\ddot{u}(t) = 0$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 11 di 262

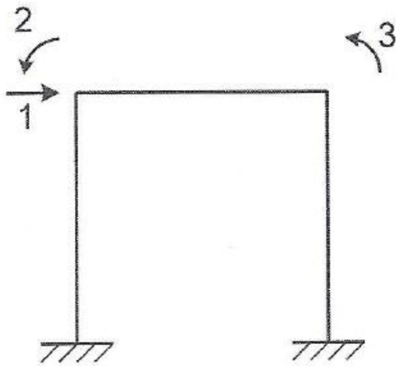
Some structures can be idealised as SDF

Example

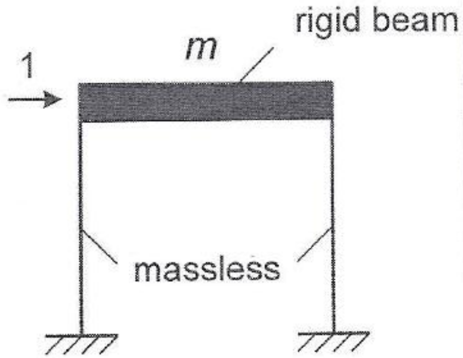
In statics, this frame has 6 active degrees of freedom.



By neglecting the axial deformations, 3 d.o.f. disappear.

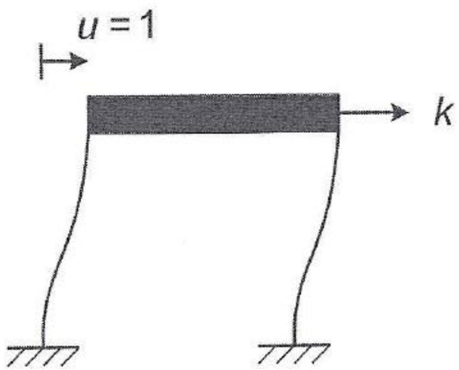


Only one d.o.f. is left if the frame is consisting of an heavy roof supported by light columns.



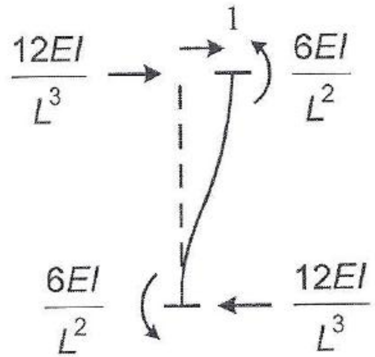
The mass of this SDF system is m, the mass of the roof.

The stiffness is determined in the classical way: the case of rigid beam and clamped columns is reported. The determination of k for other schemes (e.g. columns hinged at the base) are left to the student.



$$\boxed{k = \frac{24EI}{L^3}}$$

FOR CLAMPED COLUMNS

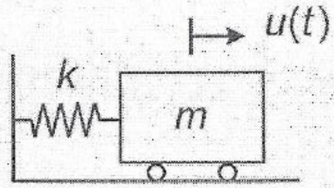


© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 13 di 262

SDF - FREE VIBRATION

UNDAMPED FREE VIBRATION

The structure is disturbed from its static equilibrium and then vibrates without any applied forces.



The equation of motion is: $\boxed{m\ddot{u}+ku=0}$

We search a solution in the form $e^{\alpha t}$

$$\alpha^2 m e^{\alpha t}+k\,e^{\alpha t}=0 \;\Rightarrow\; \alpha=\pm\sqrt{-k/m}=\pm j\,\omega_n$$

$\boxed{\alpha = \pm j \omega_n}$

$\boxed{\omega_n=\sqrt{k/m}}$ (rad/s) natural circular frequency

The solution is:

$$u(t)=G_1 e^{j\omega_n t}+G_2 e^{-j\omega_n t}$$

If we impose $G_1=G_2^*=|G|e^{-j\arg G}=\dfrac{|C|}{2}e^{-j\varphi}$, the
le costanti sono complesse e coniugate.
solution will be real:

$$u(t)=C\cos(\omega_n t-\varphi)=A\cos(\omega_n t)+B\sin(\omega_n t)$$

A and B are determined by the *initial conditions*

$$u_{t=0}=u_o \;\rightarrow\; u_o=A$$

$$\dot{u}_{t=0}=\dot{u}_o \;\rightarrow\; \dot{u}_o=B\omega_n$$

$$\boxed{u(t)=u_o\cos(\omega_n t)+\frac{\dot{u}_o}{\omega_n}\sin(\omega_n t)}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 15 di 262

Energy in undamped free vibration

At each instant of time, the total energy E is made of two parts, the kinetic energy E_k and the strain energy E_s.

$$\boxed{E_s(t) = \frac{1}{2} k\, u(t)^2} \qquad \boxed{E_k(t) = \frac{1}{2} m\, \dot{u}(t)^2}$$

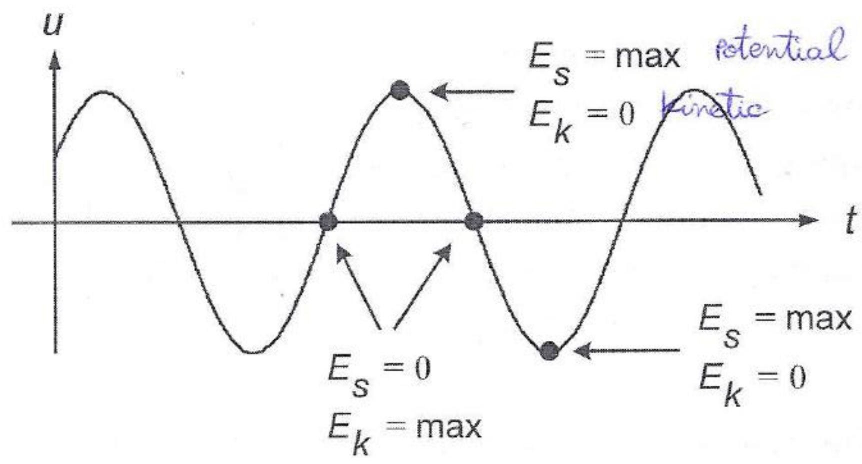
$$E(t) = E_k(t) + E_s(t)$$

$$= \frac{1}{2} m \left[C \omega_n \sin(\omega_n t - \varphi)\right]^2 + \frac{1}{2} k \left[C \cos(\omega_n t - \varphi)\right]^2$$

$$= \frac{1}{2} m C^2 \omega_n^2 \sin^2(\omega_n t - \varphi) + \frac{1}{2} k\, C^2 \cos^2(\omega_n t - \varphi)$$

$$= \frac{1}{2} k\, C^2 \qquad (k = m\omega_n^2)$$

> $E(t)$ is constant, which implies conservation of energy.



Remark : the conservation of energy can be used to derive the differential equation.

$$E(t) = \frac{1}{2} m\, \dot{u}^2 + \frac{1}{2} k\, u^2$$

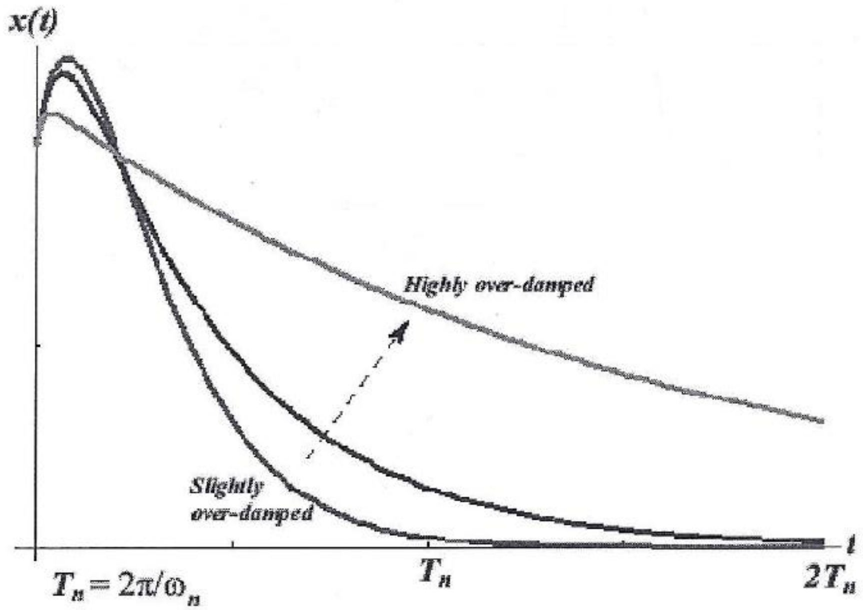
conservation of energy $\rightarrow \quad \dfrac{dE}{dt} = 0$

$$\frac{dE}{dt} = m\dot{u}\ddot{u} + ku\dot{u} = 0 \quad \rightarrow \quad m\ddot{u} + ku = 0$$

1° case: ζ>1 (overdamped systems)

α_1 and α_2 are real and the solution has the form:

$$\boxed{u(t) = K_1 e^{\alpha_1 t} + K_2 e^{\alpha_2 t}}$$



Overdamped systems have been recently adopted in seismic isolation

2° case: ζ<1 (underdamped systems)

α_1 and α_2 are a complex conjugate pair:

$$\alpha = -\zeta\omega_n \pm j\omega_D$$

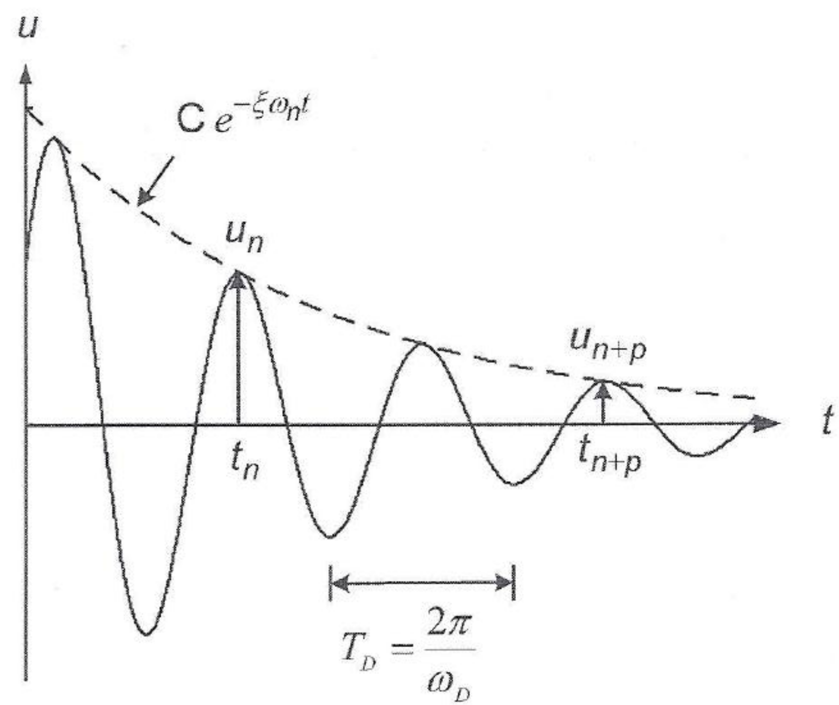
In most applications $\zeta \ll 1 \Rightarrow \omega_D \cong \omega_n$

$$u(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

By choosing for A_1 and A_2 a complex conjugate pair, we obtain a purely real solution: $A_1 = \overline{A_2}$

$$\boxed{\begin{aligned} u(t) &= e^{-\zeta\omega_n t} C\cos(\omega_D t - \varphi) \\ &= e^{-\zeta\omega_n t}\left[A\cos(\omega_D t) + B\sin(\omega_D t)\right] \end{aligned}}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 19 di 262



Decay of motion

A free vibration test (logarithmic decrement test) can be used to determine experimentally the natural frequency and the damping of a structure. ω_n, ζ

Measure the free vibrations on the structure: ex by applying a JACK (martinetto) → the structure is moved in some way but the structure can be damaged: in laboratory this method can be applied on beams/... but in CIVIL ENG. is not a used method.

p periods between two maximal points u_n and u_{n+p}

$$t_{n+p} = t_n + pT_D$$

$$\frac{u_n}{u_{n+p}} = \frac{C\,e^{-\zeta\omega_n t_n}\cos(\omega_D t_n - \varphi)}{C\,e^{-\zeta\omega_n(t_n+pT_D)}\cos(\omega_D(t_n+pT_D)-\varphi)} = e^{\zeta\omega_n p T_D}$$

$$\ln\frac{u_n}{u_{n+p}} = \zeta\,\omega_n\,p\,T_D = \zeta\,\omega_n\,p\,\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\zeta < 0.1 \;\rightarrow\; \sqrt{1-\zeta^2}\approx 1 \;\rightarrow\; \boxed{\zeta = \frac{1}{2\pi p}\ln\frac{u_n}{u_{n+p}}}$$

Nowadays curve fitting procedures are preferred to logarithmic decrement formulas.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 21 di 262

DAMPING

Different damping models can be adopted, a viscous proportional damping is the most used approach in dynamics.

There are two reasons for that:

- The equation which describes the motion is easy (linear).
- This model gives results which are often in good agreement with experiments.

A consequence is that the damping coefficient ζ can only be determined by experiments.

MULTIPLE DEGREES OF FREEDOM

SDF $m\ddot{u} + c\dot{u} + ku = p(t)$

MDF $[\mathbf{m}]\{\ddot{\mathbf{u}}\} + [\mathbf{c}]\{\dot{\mathbf{u}}\} + [\mathbf{k}]\{\mathbf{u}\} = \{\mathbf{p}(t)\}$

$[\mathbf{k}]$ stiffness matrix

$[\mathbf{c}]$ damping matrix

$[\mathbf{m}]$ mass matrix

Mass and stiffness matrices depend on the structure's discretisation and .on the choice of the degrees of freedom that are dynamically active, i.e. have their own inertia (dynamic reduction).

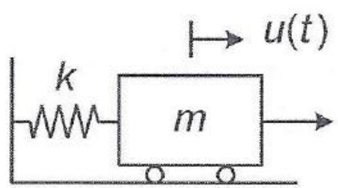
The damping matrix cannot be obtained directly by discretisation and further considerations are needed.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 23 di 262

19/3/18

SDF - HARMONIC EXCITATIONS

A harmonic load is applied to the structure.



↳ Induction of vibrations due to rotating machines

Without damping, the equation of motion is

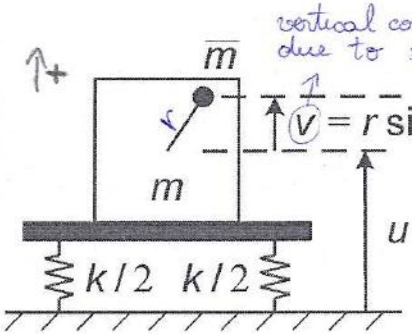
$$\boxed{m\ddot{u} + k\,u = p_o \sin(\omega t)}$$

Example 1

r = eccentricity

The system consisting of the mass m and the excentric mass $\overline{m}$ is considered for writing D'Alembert's equation.

ω_F = circular freq. of the excitation (machine)



INERTIAL FORCE OPPOSED TO MOTION

$$-(m-\overline{m})\ddot{u} - \overline{m}(\ddot{v}+\ddot{u}) - k\,u = 0$$

$$\boxed{m\ddot{u} + k\,u = \overline{m}\,r\,\omega^2 \sin(\omega_F t)}$$

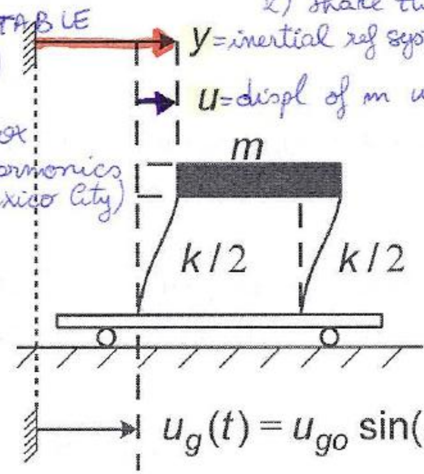
Amplitude of the vibration

WAYS:
1) Apply a force directly to the mass
2) Shake the support

Example 2 → SHAKE TABLE

(Response to earthquakes)
(non harmonics but there are some that can be approx to harmonics ex Mexico City)

$$y = u + u_g$$



inertial force is calculated WRT y ref syst of the fixed stars.

BALANCE EQUATION $-m\ddot{y} - k\,u = 0$

INERTIAL FORCE WRT ABSOL. REF. SYST. — RESTORING FORCE WRT LOCAL REF. SYSTEM. (RELATIVE)

If u is to be studied (e.g. earthquake) → means to study the relative displacement.

$$m(\ddot{u} + \ddot{u}_g) = -k\,u$$

A_u

$$\rightarrow \boxed{m\ddot{u} + k\,u = -m\ddot{u}_g = m\,u_{go}\,\omega^2 \sin(\omega t)}$$

EQ OF MOTION IN A CANONICAL FORM.

The damage of the structure depends on u.

If y is to be studied (e.g. floor isolation) → means to study the abs. displacement.

$$-m\ddot{y} - k(y - u_g) = 0$$

A_y

$$\rightarrow \boxed{m\ddot{y} + k\,y = k\,u_{go} \sin(\omega t)}$$

SAME EQ WRT y.

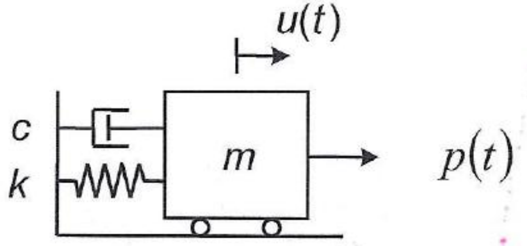
∃ some buildings with machines that cannot move from the absolute position

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 25 di 262

Every harmonic function can be written as a sum of trigonometric functions, which are equal to the exponential
$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$

If the phase is high (applying a high ...) ⇒ it's a phase opposition to a deformable structure ⇒ it's a phase opposite to the direction of the ⇒ the structure goes opposite to the excitation

WITH VISCOUS DAMPING



$$m\ddot{u} + c\dot{u} + k u = p_o e^{j\omega t}$$

The particular solution $u_p(t)$ (steady state response) is of the form:

$$u_p = \overline{U} e^{j\omega t}$$

By substituting it into the motion equation:

$$m j^2 \omega^2 \overline{U}\ e^{j\omega t} + c\, j\omega\, \overline{U}\ e^{j\omega t} + k\, \overline{U}\ e^{j\omega t} =$$

$$= p_o e^{j\omega t} \Rightarrow \overline{U} = \frac{p_o}{(k-\omega^2 m) + j\omega c}$$

it's better not to have imaginary values in the denominator.

If $\frac{\omega}{\omega_n}$ increase we can arrive to resonance up to a $\theta = 90°$ ⇒ which means excitation = cos ⇒ response = sin : $\frac{\omega}{\omega_n} = 1$

After some manipulations, it is obtained: MULTIPLY BY THE COMPLEX CONJUGATE TO FIND A REAL QUANTITY.

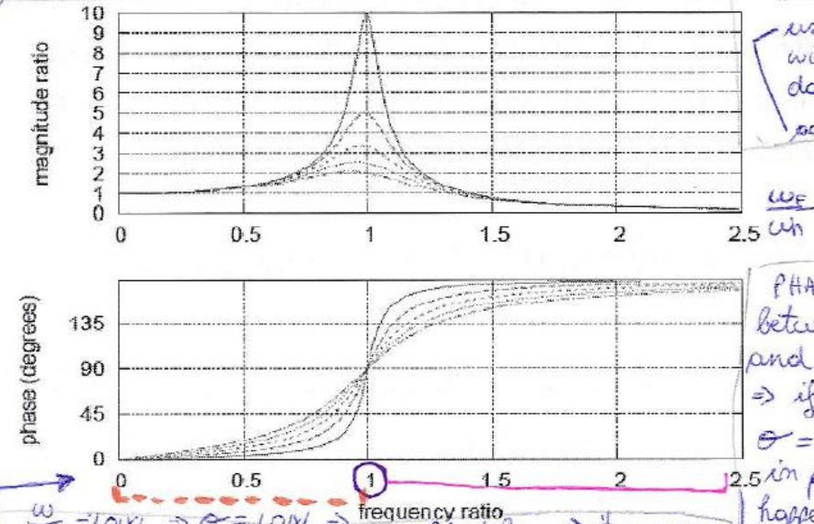
$$u_p(t) = \underbrace{\frac{p_o/k}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta(\omega/\omega_n)\right]^2}}}_{\text{MODULUS}}\, e^{j(\omega t - \vartheta)}$$

PHASE

The complex problem is shifted to the numerator which is better. It's possible to write them using modulus and phase.

$$\tan\vartheta = \frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2} \qquad 0 < \vartheta < 180°$$

The amplification is not ∞, but it depends on the damping coefficient c: a pedestrian bridges → ∃ big resonance ⇒ add more damping:
- use a material with higher damping
- add a damper.



PHASE = delay between excitation and the response ⇒ if c = 0 ⇒ θ = 0 ⇒ they're in phase so they happen at the same time.

If $\frac{\omega}{\omega_n}$ = low ⇒ θ = low ⇒ small delay ⇒ it means to excite very slowly a structure that is very stiff ⇒

Representation of $\overline{U}$ for different values of damping ζ

It's a QUASI STATIC EXCITATION, without an important delay.

23/4/18

Dynamic factor

After some while, the structure vibrates with the same frequency as the applied force. It is the steady state response $u_p(t)$.

The amplitude of these vibrations are now studied.

$$u_p(t) = \frac{p_o/k}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta(\omega/\omega_n)\right]^2}} e^{j(\omega t - \vartheta)}$$

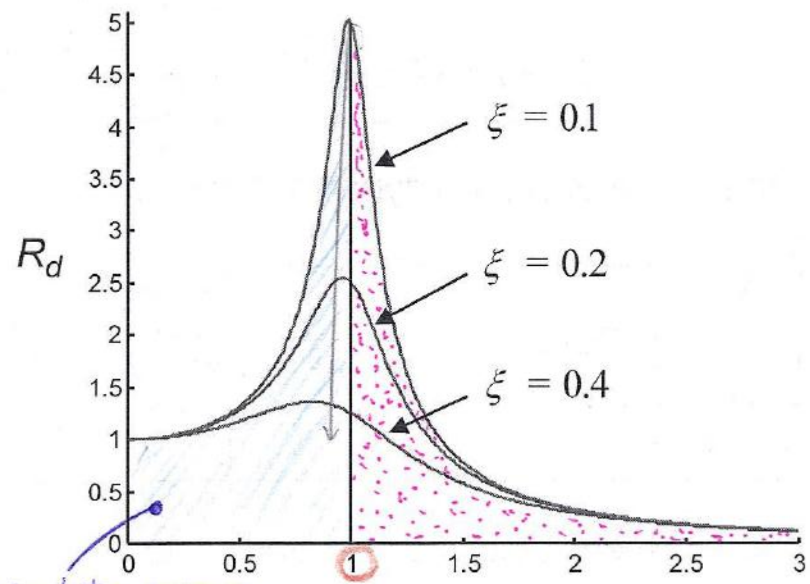
The static deformation due to a static load p_o is $\quad (u_{st})_o = \dfrac{p_o}{k}$

The amplitude of the vibration is equal to the product of the static deformation times a dimensionless dynamic factor R_d.

$$R_d(\zeta, \omega/\omega_n) = \frac{1}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta(\omega/\omega_n)\right]^2}}$$

R_d can be plotted as function of the ratio ω/ω_n for different values of the damping coefficient ζ. (ZETA)

The damping mitigates and shift the peaks towards left.



Quasi-static response governed by the restoring force ⇒ ω_F = small ⇒ neglected

(ω/ω_n) (= RELATIVE FREQUENCY / FREQ RATIO)

$\omega/\omega_n < 0.25 \;\rightarrow\; R_d \approx 1$ "quasi static" response

$\cancel{m\ddot{u}} + \cancel{c\dot{u}} + k\,u = p_o\, e^{j\omega t} \;\rightarrow\; u = p_o/k \;\; e^{j\omega t}$

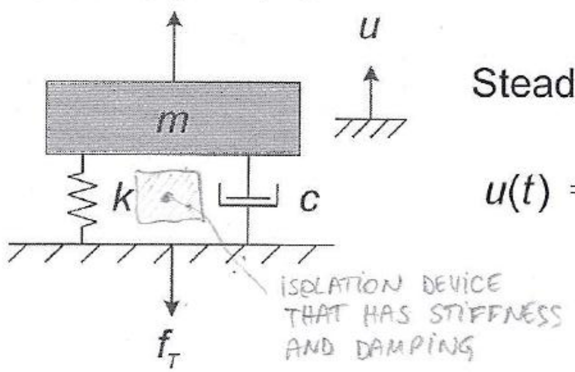
⇒ RESTORING FORCE ↑ = EXT. FORCE ↑ ⇒ easier solution.

$\omega \rightarrow \omega_n$ the amplitudes of vibrations become large : **Resonance**

Force transmission and vibration isolation

A harmonic load is applied to a structure. This structure is connected to the ground through a support modelled by a spring k and a damper c.

$$p(t) = p_o \sin(\omega t)$$



Steady state response

$$u(t) = \frac{p_o}{k} R_d \, e^{j(\omega t - \vartheta)}$$

The force transmitted to the ground is

$$f_T(t) = ku(t) + c\dot{u}(t)$$

$$= p_o R_d \, e^{j(\omega t-\vartheta)} + \frac{p_o c}{k} R_d \omega j e^{j(\omega t - \vartheta)}$$

IMAGINARY UNIT

If you multiply an harmonic function by $\boxed{j}$, the phase is delayed by $\frac{\pi}{2}$.

$$f_T(t) = p_o R_d \left[e^{j(\omega t-\vartheta)} + 2\zeta(\omega/\omega_n)\, j\, e^{j(\omega t-\vartheta)} \right]$$

dynamic loading factor

$$= \boxed{p_o} R_d \boxed{\sqrt{1+[2\zeta(\omega/\omega_n)]^2}} e^{j(\omega t-\vartheta+\alpha)}$$

amplitude of the excitation

MODULUS = represents the intensity of vibration.

The transmissibility TR is defined as the ratio between the amplitude of the transmitted force f_T and the amplitude of the force applied to the structure.

$$TR = \frac{f_{T\max}}{p_o} = \sqrt{\frac{1+[2\zeta(\omega/\omega_n)]^2}{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}$$

Effectiveness of possible isolation (device)

TR is dimensionless.

The objective is to choose the support (k, c) such that TR is as small as possible.

$\omega_n = \sqrt{\frac{k}{m}}$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 31 di 262

Concepts of seismic isolations

In seismic isolation we are more interested in acquired energy rather than in transmitted one. This calls for a different definition of transmissibility, *tr*, as ratio between the maximum displacement of the structure with respect to its foundation, and the displacement applied by the ground (see Example 2).

$$tr = \frac{u_{max}}{u_{go}} = \frac{(p_o/k)R_d}{u_{go}} = \frac{(mu_{go}\omega^2/k)R_d}{u_{go}} =$$

(relative)

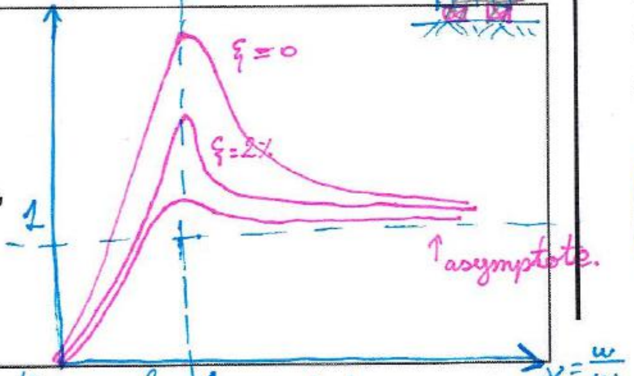
this refers to the acceleration in the example where Earthquake has harmonic law.

$$= \frac{(\omega/\omega_n)^2}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2+\left[2\zeta(\omega/\omega_n)\right]^2}}$$

$\omega/\omega_n \to 0$: $tr \to 0$,

peak at resonance,

$\omega/\omega_n \to \infty$: $tr \to 1$,



① Increase the stiffness: bunker, nuclear power plant BUT ε >>> 1 and cracks may happen so we can go towards resonance.
② Usually we stay at high $\nu = \frac{\omega}{\omega_n}$, by increasing damping ξ and deformability of isolators.

Usually seismic isolation is performed by ~~reducing~~ INCREASING stiffness. In fact $\omega/\omega_n \to 0$: $tr \to 0$, would call for very rigid, and expensive, structures.

On the contrary, when $\omega/\omega_n \to \infty$: $tr \to 1$, the maximum displacement does not depend on stiffness. Consequently, when stiffness is reduced, also force will reduce correspondingly. The better, deformation will concentrate on the isolation devices, thus reducing structural damage.

vibrations WRT the devices I put

Often isolation devices retain also dissipative properties (damping). Dampers are very effective in case of resonance (e.g. anomalous seismic motion).

But displacement tends to 1 always, while the transmitted force (shear) from ground to building is decreasing. All displ of the building are concentrated in the isolators, which are designed to support them, so ∄ much displ in the storeys.

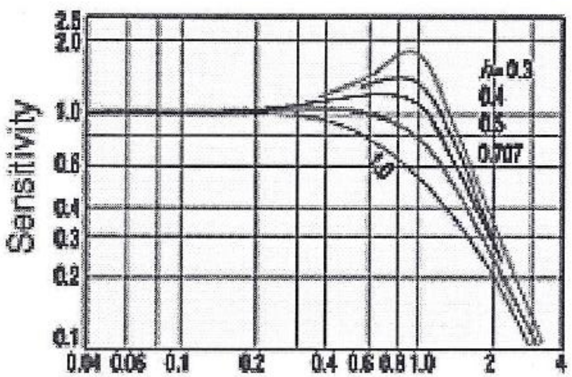
500€ 3000€ good but €€€ → special springs

Accelerometers

$\nu < 1 \quad TR \cong 1$ (quasi static response)

$u_g \cong y \quad \Rightarrow \ddot{u}_g \cong \ddot{y} \cong -k \cdot u/m = -\frac{1}{m} \cdot \overset{ku}{\text{Force}} = \text{acceler.}$

In order to reach small values for ν, high stiffness and small masses are employed. Accelerometers are constructively simple instruments and often the measured quantity often is force *ku*, rather than displacement.



Frequency ratio *U=f/fn*
(c) Acceleration seismometers
(measurement range < U(1))

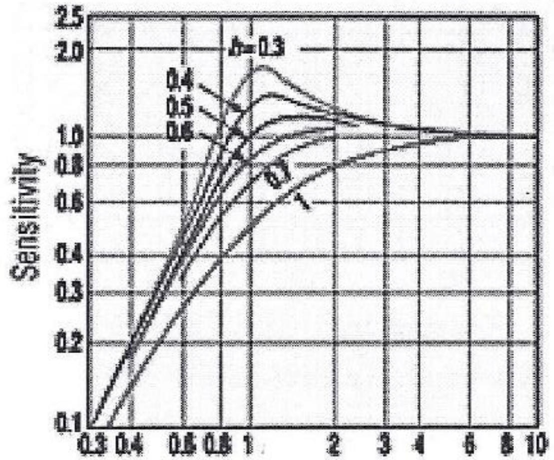
Displacement seismometers

$\nu > 1 \quad tr \cong 1$ (quasi - motionless mass)

$y = u_g + u \cong 0 \quad \Rightarrow u_g \cong -u$ HIGH MASS ⊕ SMALL STIFF

The elongation of the spring moves directly the displ. WRT ground.

In order to reach high values for ν, high masses are employed, this entailing large size instruments and a complex design. In fact, small stiffness make it difficult to obtain viscous damping: the higher the mass, the more accurate the seismometer.

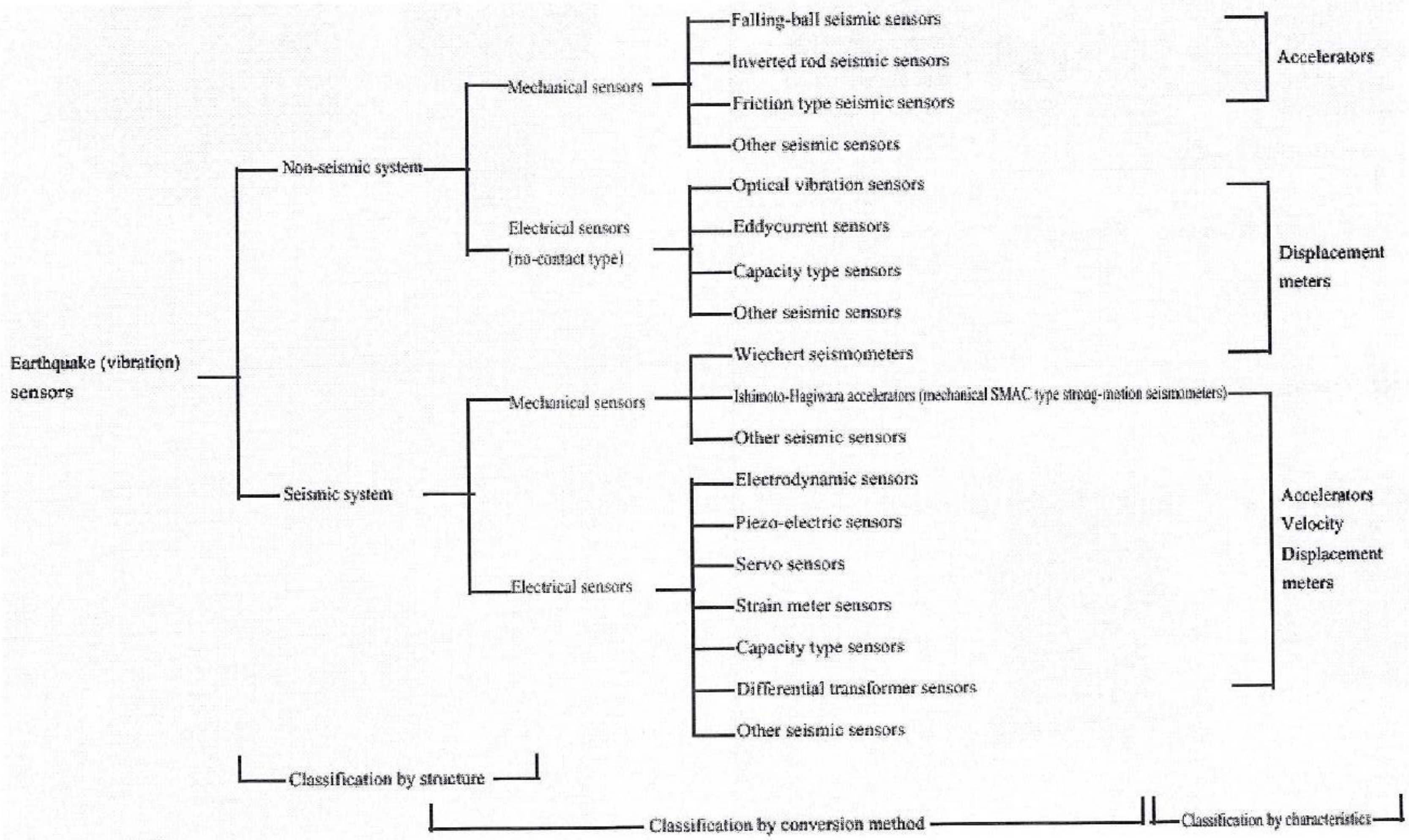


Frequency ratio *u=f/fn*
(a) Displacement seismometers
(measurement range>U(1))

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 35 di 262

Classification of earthquake (vibration) sensors

The following shows a classification of seismometers by seismic mass structure and signal conversion. Though an overwhelming number of vibration sensors are of piezo-electric type, seismometers mainly use electrodynamic sensors consisting of a coil and a magnet, or servo acceleration sensors.



© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 37 di 262

Classification by purpose of use

- Sensitivity seismometers for earthquake research observation
- Strong-motion seismometers for earthquake engineering - particularly for aseismic design
- Control-type seismometers for secondary disaster prevention (see table)

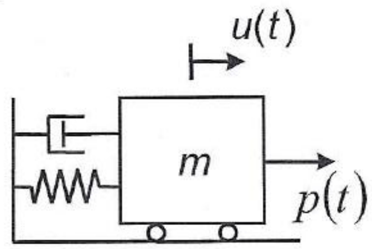
Shinkansen railroads and expressways	Shut down power in substations.
Nuclear power plants	Insert control rods.
Thermal power plants	Stop turbines.
General substations	Lock transformer protection relays.
Gas and fuel facilities	Close emergency shut-off valves.
Water facilities	Shut off top plugs.
Elevators	Shut down motor power.
General factories	Shut down receiving equipment relay.
Chemical plants	Shut off outlet and inlet valves.
Hospitals and department stores.	Interlocked with emergency broadcast system.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 39 di 262

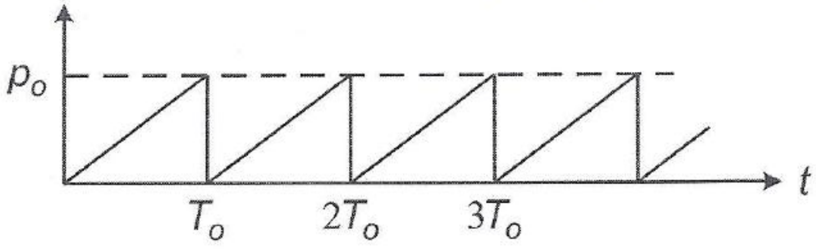
SDF - ARBITRARY EXCITATIONS

PERIODIC EXCITATION

A SDF system is excited by a periodic load, which is expanded in Fourier series.



$p(t)$ = PERIODIC EXCITATION (in between harmonic and general).



PERIODIC FUNCTION = SUM OF HARMONIC FUNCTIONS

$$A(t) = p(t)/m = \sum_{-\infty}^{+\infty} C_i e^{j2\pi \frac{i}{T_0} t}$$

$i = 0, 1, 2, 3, \dots -1, -2, \dots$

FOURIER CONSTANT

$$C_i = \frac{1}{T_o} \int_0^{T_0} A(t)\, e^{-j2\pi \frac{i}{T_0} t}\, dt$$

The equation of motion is LINEAR so I can apply the superposit. principle:

$$\ddot{u} + 2\zeta\omega_n \dot{u} + \omega_n^2 u = A(t) = \sum_{-\infty}^{+\infty} C_i e^{j2\pi \frac{i}{T_0} t}$$

The **steady state response** is calculated by using the superposition principle.

$$u_i = \frac{m C_i}{k} R_d(\omega_i/\omega_n, \zeta)\, e^{j(\omega_i t - \vartheta(\omega_i/\omega_n, \zeta))} =$$

$$\frac{|C_i|}{\omega_n^2} R_d(\omega_i/\omega_n, \zeta)\, e^{j(\omega_i t - \vartheta(\omega_i/\omega_n, \zeta) + \alpha_i)}$$

where: $\omega_i = 2\pi \frac{i}{T_0}$

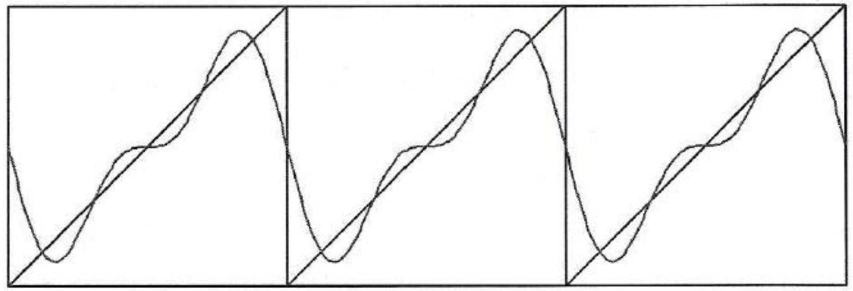
The total **steady state response** is then extended to the periodic excitation (same solution as for harmonics):

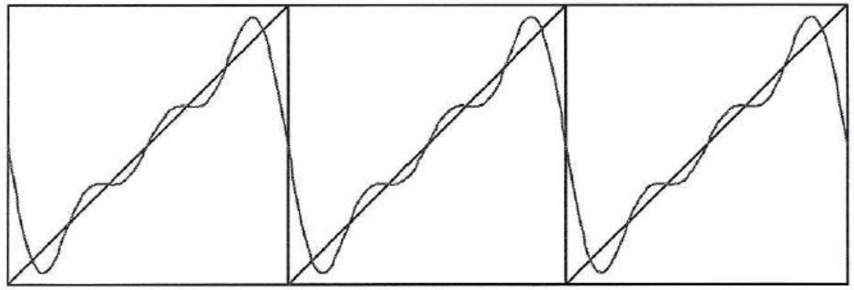
$$\boxed{u(t) = \sum_{-\infty}^{+\infty} \frac{|C_i|}{\omega_n^2} R_d(\omega_i/\omega_n, \zeta)\, e^{j(\omega_i t - \vartheta(\omega_i/\omega_n, \zeta) + \alpha_i)}}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 41 di 262

In this example, only the first three Fourier terms in $u(t)$ must be considered to get an error less than 2 %. The plot of the load approximations with 2 and 3 Fourier terms shows that an inaccurate approximation of the load gives an accurate approximation of the response. The reason is that, in this case, the higher frequencies in the load do not give any contributions to the response.

Not high quality because we used only 3 components of the series.
Good accuracy because the troncation has preserved the terms that produced resonance response, which is the most important one.





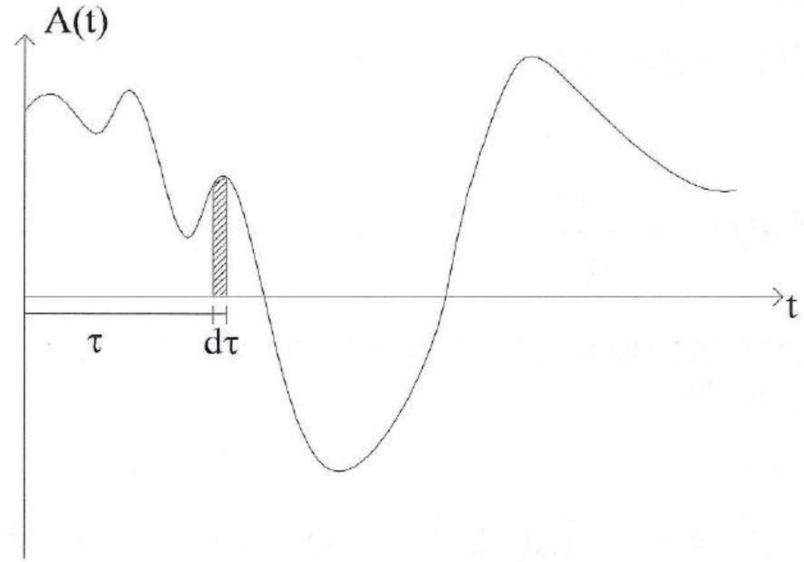
© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 43 di 262

$$H(f) = \frac{1}{\omega_n^{\ 2}} R_d\left(2\pi f / \omega_n, \zeta\right) e^{-j\vartheta\left(2\pi f / \omega_n, \zeta\right)}$$

H(f) is the system's *Frequency Response Function (FRF)*

NON PERIODIC EXCITATION: TIME DOMAIN APPROACH

In order to support an intuitive idea of this approach, let $A(t)$ be decomposed in a sequence of impulsive loads applied in an infinitesimal temporal support



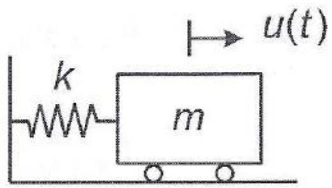
© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 45 di 262

RAYLEIGH'S METHOD

Rayleigh's method can be used to calculate approximately the lowest natural frequency of beams.

INTRODUCTION

Free vibrations of an undamped SDF system are considered.



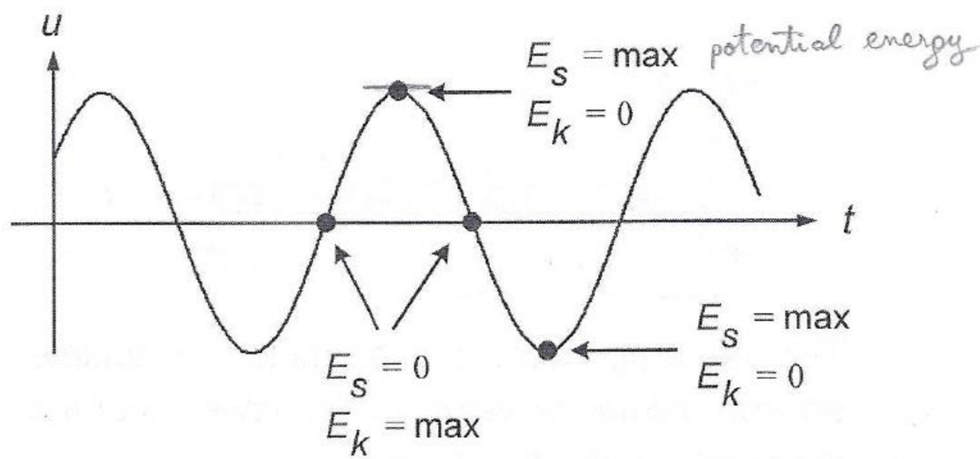
$$u(t) = u_o \sin(\omega_n t) \qquad \dot{u}(t) = u_o \omega_n \cos(\omega_n t)$$

strain energy

$$E_s(t) = \frac{1}{2} k\, u(t)^2 = \frac{1}{2} k\, u_o^2 \sin^2(\omega_n t)$$

kinetic energy

$$E_k(t) = \frac{1}{2} m\, \dot{u}(t)^2 = \frac{1}{2} m\, \omega_n^2\, u_o^2 \cos^2(\omega_n t)$$



$$E_{s\max} = \frac{1}{2} k\, u_o^2 \qquad E_{k\max} = \frac{1}{2} m\, \omega_n^2\, u_o^2$$

conservation of energy

$$E_{s\max} = E_{k\max} \quad \rightarrow \quad \frac{1}{2} k\, u_o^2 = \frac{1}{2} m\, \omega_n^2\, u_o^2$$

$$\rightarrow \quad \boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

kinetic energy

$$\dot{u}(x,t) = \frac{\partial u(x,t)}{\partial t} = Y\,\omega \sin\left(\frac{\pi x}{L}\right)\cos(\omega t)$$

$$E_k = E_{k\,M} + E_{k\,\mathrm{beam}}$$

$$E_{k\,M} = \frac{1}{2} M \left[\dot{u}(L/2,t)\right]^2 = \frac{1}{2} M\,Y^2 \omega^2 \cos^2(\omega t)$$

$$E_{k\,\mathrm{beam}} = \int_0^L \frac{1}{2} m \left[\dot{u}(x,t)\right]^2 \mathrm{d}x$$

$$= \frac{1}{2} m\,Y^2 \omega^2 \cos^2(\omega t) \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \mathrm{d}x$$

$$\boxed{E_{k\,\max} = \frac{1}{2}\,Y^2\,\omega^2 \left(M + \frac{mL}{2}\right)}$$

conservation of energy

$$E_{k\,\max} = E_{s\,\max} \;\rightarrow\; \omega^2 \left(M + m\frac{L}{2}\right) = EI\,\frac{\pi^4}{2L^3}$$

$$\rightarrow \quad \boxed{\omega = \pi^2 \sqrt{\frac{EI}{L^3(2M + mL)}}}$$

particular cases

$M = 0 \quad \omega = \pi^2 \sqrt{\frac{EI}{mL^4}}$ exact solution since the eigenmode is exact.

$$m = 0 \qquad \omega = \frac{\pi^2}{\sqrt{2}} \sqrt{\frac{EI}{ML^3}} = 6.98 \sqrt{\frac{EI}{ML^3}}$$

In that case, the beam is considered as a spring with $\quad k = \frac{48EI}{L^3}$

which gives as exact solution $\quad \omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{48EI}{ML^3}} = 6.93 \sqrt{\frac{EI}{ML^3}}$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 49 di 262

5/4/18

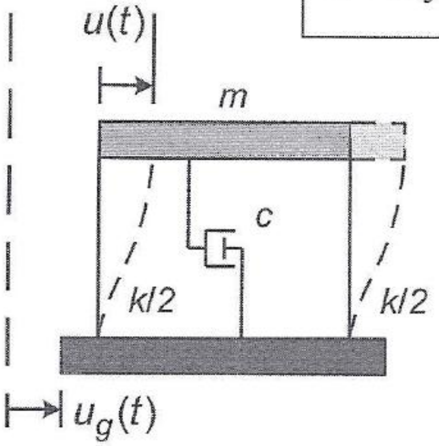
EARTHQUAKE ANALYSIS AND RESPONSE SPECTRA

An introduction to earthquake analysis is given, and the concept of elastic response spectra is introduced.

SDOF SYSTEMS

A SDOF system is subjected to a ground motion $u_g(t)$. The displacement response $u(t)$ is to be calculated.

$$m(\ddot{u}_g + \ddot{u}) + c\,\dot{u} + k\,u = 0$$
$$\ddot{u} + 2\zeta\omega_n\,\dot{u} + \omega_n^2 u = -\ddot{u}_g(t)$$



The ground acceleration can be registered using proper instruments (the figure shows a typical earthquake accelerogram).

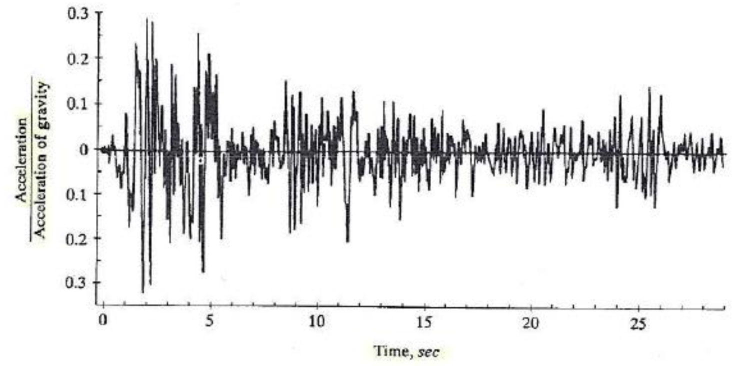
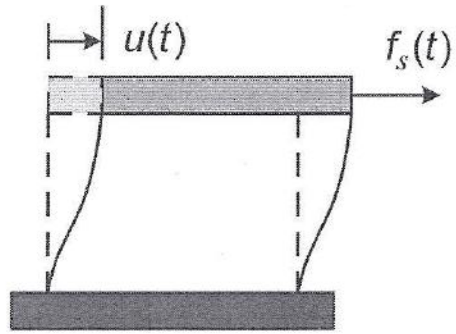


FIGURE 24-15
Accelerogram from El Centro earthquake, May 18, 1940 (*NS* component).

EQUIVALENT STATIC FORCE



$$f_s(t) = k\,u(t)$$
$$= m\,\omega_n^2\,u(t)$$

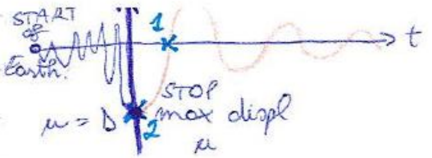
$\omega_n^2 u(t) \neq \ddot{u}(t)$:

Pseudo acceleration

(↑ work with this because displ is directly related to it

$f_s(t)$ is the force which must be applied statically in order to create a displacement $u(t)$.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 51 di 262



After end of earthquake ⇒ ∃ a free oscillation.
In point 1 the velocity is max.
$V_1 = \omega_n \cdot D$
and also accel in 2: $a_2 = \omega_n^2 D$

These 2 are pseudo ⇒ that we have if the Earthquake stops immediately after its max displacem.

Displacement, pseudo-velocity and pseudo-acceleration response spectra can be defined and plotted on the same graph

Pseudo-quantities are those a system would reach in a virtual free oscillation cycle next to the attainment of D, as if the earthquake would suddenly cease.

Spectral displacement	$D = \max\lvert u(t)\rvert$
Spectral pseudo - velocity	$V = \omega_n \cdot D$
Spectral pseudo - acceleration	$A = \omega_n^2 \cdot D$

$A = \omega_n V = \omega_n(\omega_n D)$

Pseudo-velocity is related to energy, while pseudo-acceleration is proportional to static load (directly usable for stress analysis).

BI-LOGARITHMIC REPRESENTATION:

$$\log V = \log(\underbrace{\frac{2\pi}{T_n}}_{=\omega_n} D) = \log(2\pi) - \log(T_n) + \log(D)$$

$$\log A = \log(\frac{2\pi}{T_n} V) = \log(2\pi) - \log(T_n) + \log(V)$$

COMBINED D-V-A SPECTRUM

- Spectrum = specific of each accelerogram
- ≠ spectra depend on ≠ values of damping

Linear relationships between logarithms make an easy interpretation of the bi-logarithmic graph:

4-WAY DIAGRAM: In 1 we can read
- period
- velocity
- displacement D
- pseudo-acceleration

In this case the accelerogram is El Centro.

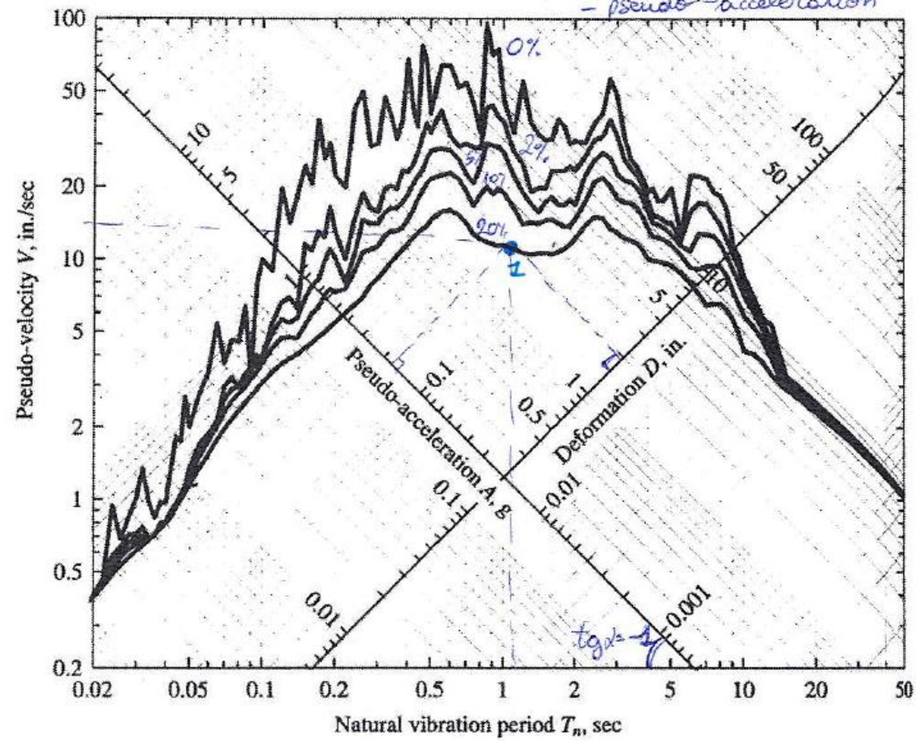


Figure 6.6.4 Combined D–V–A response spectrum for El Centro ground motion; ζ = 0, 2, 5, 10, and 20%.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 53 di 262

RESPONSE SPECTRUM CHARACTERISTICS

General characteristics can be derived from the analysis of response spectra.

PGA = Peak Ground Acceleration
PGV = Peak Ground Velocity
PGD = Peak Ground Displacement → All spectra start from PGA: in the origin $T_n = 0$ = period is zero ≈ very rigid structure: on the top the max accel is equal to that of the ground.

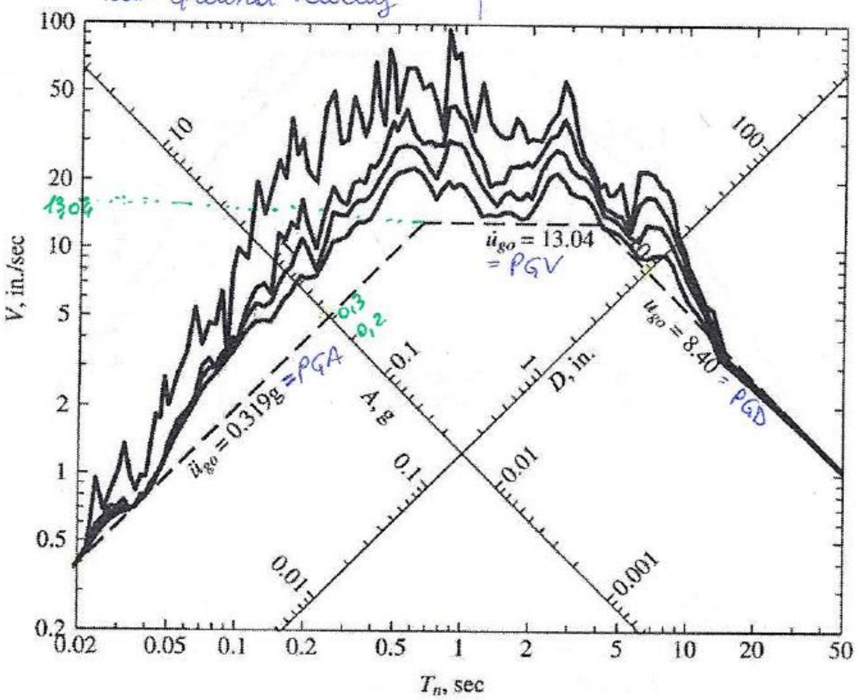


Figure 6.8.1 Response spectrum (ζ = 0, 2, 5, and 10%) and peak values of ground acceleration, ground velocity, and ground displacement for El Centro ground motion.

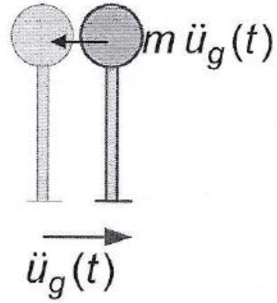
$$T_n = 2\pi\sqrt{m/k}$$

$T_n < 0.03$ s : rigid system

no deformation

$u(t) \approx 0 \rightarrow D \approx 0$

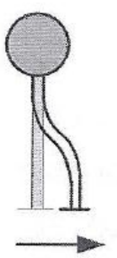
$A = \ddot{u}_{go}$ (PGA)



$T_n > 15$ s : flexible system

no total displacement

$u(t) = u_g(t) \rightarrow D = u_{go}$



The spectrum can be divided in 3 period ranges :

$T_n < 0.5$s : acceleration sensitive region

$0.5 < T_n < 3$s : velocity sensitive region

$T_n > 3$s : displacement sensitive region

Between $T_n \in (0; 0.2)$ ∃ an amplification of the spectra : ACCELERATION SESITIVE REGION, and then ∃ MAXIMUM AMPLIF REGION that is the most critical for buildings, in $T_n \in (0.2; 0.5)$.
⇒ 80% buildings are in this region.
$T_n \in (0.5; 3)$ ≡ VELOCITY SENSITIVE REGION : velocity ⇔ energy ⇔ damping
In the DISPL. SENS. REGION the structure stays still and the ground moves: if the struct is very SOFT ⇒ the max relative displ. of the struct is PGD.

If usually $T_n > 3$s, the structure is NATURALLY ISOLATED : ex skykreepers that suffer more for wind than for earthquakes.

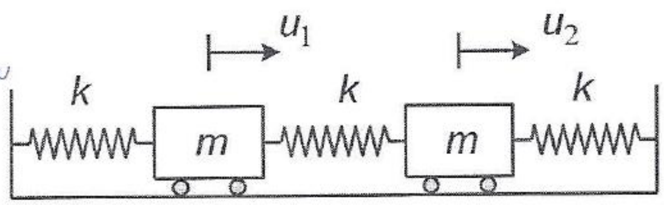
© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 55 di 262

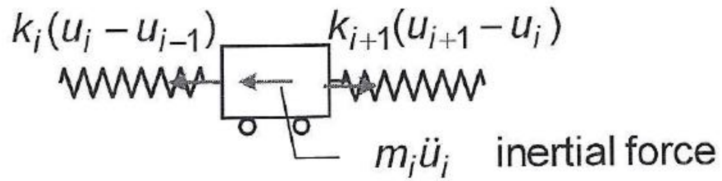
MDF – FREE VIBRATION

Free vibration of multi degree of freedom = nr of indipendent masses. discrete systems are studied.

EXAMPLE 1 - TRAIN

Hooke law of springs





$$-m_i\ddot{u}_i - k_i(u_i - u_{i-1}) + k_{i+1}(u_{i+1} - u_i) = 0$$

or $$m_i\ddot{u}_i - k_i u_{i-1} + (k_i + k_{i+1})u_i - k_{i+1}u_{i+1} = 0$$

If in the spring ∃ distributed mass ⇒ the [M] is not diagonal. [K] in linear elasticity is always symmetric due to Betty's theorem, and also [M] is symmetric always.

Differential equations

The equilibrium equation is written for each coach

$$i=1: \quad -m_1\ddot{u}_1 - k_1(u_1 - 0) + k_2(u_2 - u_1) = 0$$

$$i=2: \quad -m_2\ddot{u}_2 - k_2(u_2 - u_1) + k_3(0 - u_2) = 0$$

In this example k_1=k_2= k_3= k, thus

$$\begin{cases} m\ddot{u}_1 = (-2k)u_1 + (k)u_2 \\ m\ddot{u}_2 = (k)u_1 + (-2k)u_2 \end{cases}$$

$$\rightarrow \begin{cases} m\ddot{u}_1 + 2k\,u_1 - k\,u_2 = 0 \\ m\ddot{u}_2 - k\,u_1 + 2k\,u_2 = 0 \end{cases}$$

SYMMETRIC DIAGONAL because we assumed that the spring has no mass.

$$\rightarrow \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$[M]\{\ddot{u}\} + [K]\{u\} = \{0\}$ [2 x 2] SYMMETRIC

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 57 di 262

LUMPED MASS FORMULATION (easy) → concentrated mass : [M] = diagonal
CONSISTENT MASS " (difficult) → distributed mass

Eigenvalue problem

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The solution of this system is of the form

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underbrace{\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}}_{\text{spatial variability} \equiv \text{EIGENVECTOR}} \underbrace{e^{j\omega_k t}}_{\text{temporal variability}} \qquad \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} ? \qquad \omega_k ?$$

where $\{\phi\}$ is a spatial function and $e^{j\omega_k t}$ a temporal one.

Since ϕ_1, ϕ_2, ω_k are constants, derivation gives

$$\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = -\omega_k^2 \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} e^{j\omega_k t}$$

By introducing the two expressions above in the system, it is obtained

$$-\omega_n^2 [\mathbf{m}] \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} e^{j\omega_k t} + [\mathbf{k}] \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} e^{j\omega_k t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow \boxed{\left([\mathbf{k}] - \omega_k^2 [\mathbf{m}] \right) \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} e^{j\omega_k t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

This equation is valid for every t, which implies

$$\boxed{\left([\mathbf{k}] - \omega_k^2 [\mathbf{m}] \right) \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (1) \quad \text{eigenvalue problem}}$$

This system has one trivial solution ($\phi_1 = \phi_2 = 0$) which corresponds to equilibrium without motion.

Other solutions can be found if the following condition is respected:

$$\boxed{\det\left([\mathbf{k}] - \omega_k^2 [\mathbf{m}] \right) = 0 \quad (2)}$$

Conclusion : Two couples of complex conjugate solutions have been found

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \underbrace{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}_{\{\phi_1\}} e^{\pm j \overbrace{\sqrt{k/m}}^{\omega_1} \cdot t}$$

MODE 1 The 2 masses move in the same direction at the same time : in phase

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \underbrace{\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}_{\{\phi_2\}} e^{\pm j \overbrace{\sqrt{3k/m}}^{\omega_2} \cdot t}$$

MODE 2 The 2 masses are in phase opposition: they move in ≠ ways

$\omega_1 \quad \omega_2$	: natural circular frequencies
$\{\phi_1\} \ \{\phi_2\}$	: eigenmodes or modal shapes

The solution related to the first natural frequency is

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}^{(1)} = A_{1(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{j\omega_1 t} + A_{2(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-j\omega_1 t}$$

In order to obtain a real solution we choose:

$$A_{1(1)} = A^*_{2(1)} = C_{(1)}\, e^{-j\varphi_{(1)}}$$

Hence:

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \left(C_{(1)}\, e^{j(\omega_1 t - \varphi_{(1)})} + C_{(1)}\, e^{-j(\omega_1 t - \varphi_{(1)})} \right) =$$

$$= \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} C_{(1)} \cos(\omega_1 t - \varphi_{(1)})$$

MODE ≃ MODAL SHAPE ≈ EIGENVECTOR

A similar expression is found for the second natural frequency. Every linear combination of this two solutions is also a solution of the free vibration problem. The general solution can be written as

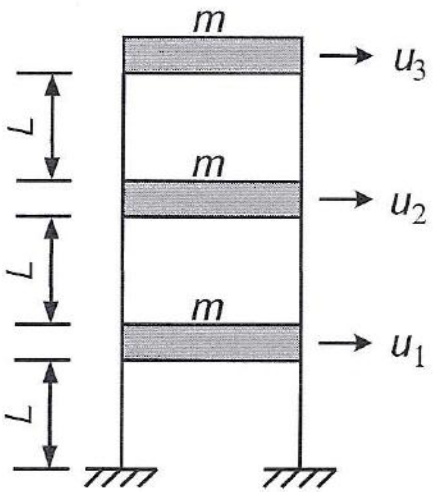
MODE 1 (SLOW) + MODE 2 (FAST)

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = C_{(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\omega_1 t - \varphi_{(1)}) + C_{(2)} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos(\omega_2 t - \varphi_{(2)})$$

SLOW or FAST depend on the frequency ω_i: $\omega_2 = 3\omega$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 61 di 262

Numerical example: 4 columns per storey, no damping.



$m = 2$ kg

$L = 200$ mm

columns

$b = 10$ mm

$h = 1$ mm

$E = 200$ GPa

$$k = \frac{48EI}{L^3} \qquad I = \frac{bh^3}{12}$$

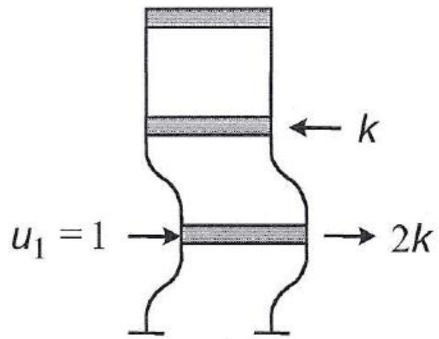
$$[\mathbf{m}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad [\mathbf{k}] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

(**[k]** is symmetric)

THE NUMBER OF DEGREES OF FREEDOM DEPENDS ON THE DISCRETIZATION OF THE STRUCTURE → so [K] can have ≠ dimensions, so it's not unique.

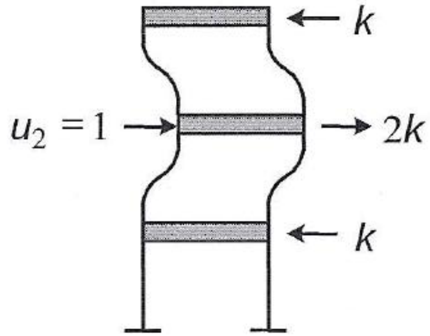
The mass matrix [M] has dimension depending on the nr of lumped mass (= concentrated mass).

Usually [K] is choosen to have the same dimension as [M]: reduction to the minimal nr of equation DYNAMIC COMPACTION.



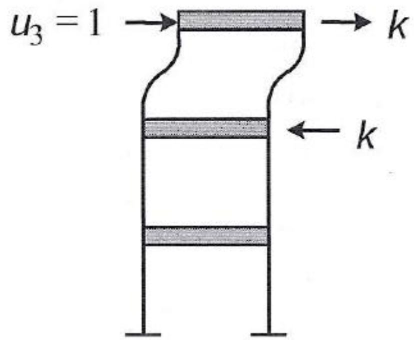
first column

$$\begin{Bmatrix} 2k \\ -k \\ 0 \end{Bmatrix}$$



second column

$$\begin{Bmatrix} -k \\ 2k \\ -k \end{Bmatrix}$$



third column

$$\begin{Bmatrix} 0 \\ -k \\ k \end{Bmatrix}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 63 di 262

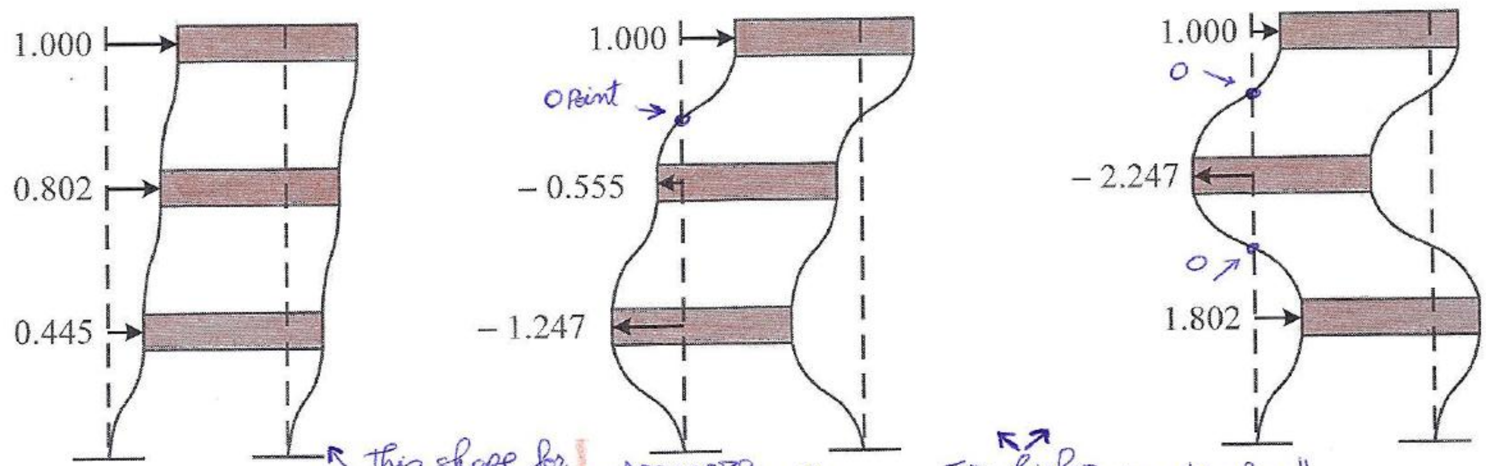
Results given by MATLAB

(normalized wrt top displacement) which is imposed = 1

$\omega_1 = 9.95$ rad/s $f_1 = 1.58$ Hz $\phi_1 = \begin{Bmatrix} 0.445 \\ 0.802 \\ 1.000 \end{Bmatrix}$

$\omega_2 = 27.9$ rad/s $f_2 = 4.44$ Hz $\phi_2 = \begin{Bmatrix} -1.247 \\ -0.555 \\ 1.000 \end{Bmatrix}$

$\omega_3 = 40.3$ rad/s $f_3 = 6.41$ Hz $\phi_3 = \begin{Bmatrix} 1.802 \\ -2.247 \\ 1.000 \end{Bmatrix}$



This shape for lower modes → Are more important ④

For higher modes "snake shape".

Physical interpretation of the eigenmodes

If free vibration is initiated by imposed displacements corresponding to the eigenmode k, the vibration of each story will be harmonic with frequency f_k and the structure will vibrate with a constant deflected shape corresponding to the eigenmode k. In the general case, the total vibration will result from the superposition of the vibration associated to each mode.

STRUCT
FOUND

④ Earthquakes usually have low frequencies, and usually the loads tend to sum up, so the force transmitted to the foundation is higher, while in the "snake shape", the forces tend to compensate STRUCT FOUND

9/4/18

MDOF – MODAL ANALYSIS

Exploits some properties of the modal shapes (eigenvectors) in order to decouple the eq. of motion, which is a non linear response with multiple degrees of freedom

The mode superposition approach is used in order to calculate the response of a linear MDOF system to an applied load vector.

First non damped systems are addressed. Finally also damping is introduced.

Mode orthogonality

The mode shapes vectors, being nothing but the eigenvectors satisfying the symmetric eingevalue problem, possess the important "orthogonality property".

Taking two particular modes *r* and *s*, we may write

$$\{[k]-\omega_r^2[m]\}\{\Phi\}_r = 0$$
$$\{[k]-\omega_s^2[m]\}\{\Phi\}_s = 0$$

Pre-multiplying the first equation by $\{\phi\}_s^T$

$$\mathrm{I} \qquad \{\Phi\}_s^T\{[k]-\omega_r^2[m]\}\{\Phi\}_r = 0$$

and transposing and post-multiplying the second one by $\{\phi\}_r$, we find

$$\{\Phi\}_s^T\{[k]^T-\omega_s^2[m]^T\}\{\Phi\}_r = 0$$

The last equation, due to the symmetry of the matrices (Betti's theorem), becomes

$$\mathrm{II} \qquad \{\Phi\}_s^T\{[k]-\omega_s^2[m]\}\{\Phi\}_r = 0$$

thus (II − I):

$$(\omega_r^2-\omega_s^2)\{\Phi\}_s^T[m]\{\Phi\}_r = 0$$

$\Leftrightarrow \begin{cases}(\omega_r^2-\omega_s^2)=0 \\ \{\Phi\}_s^T[m]\{\Phi\}_r=0 \quad \mathrm{III}\end{cases}$

$\Rightarrow$ If you put III into II, you fin IV

If the two modes are distinct, one has

$$\mathrm{III} \qquad \{\Phi\}_s^T[m]\{\Phi\}_r = 0$$

$$\mathrm{IV} \qquad \{\Phi\}_s^T[k]\{\Phi\}_r = 0$$

[M] and [K] are symmetric because of Betti's theorem, which is valid only in LINEAR ELASTICITY!

Modal decoupling

Let us consider the equation of a SDOF without damping

$$[m]\{\ddot{u}\}+[k]\{u\}=\{F(t)\}$$

The idea is to introduce modal coordinates {p} or principal coord

$$\boxed{\{u\}=[U]\{p\}}$$

GEOMETRIC COORD.

Derivation and multiplying on the left side by $[U]^T$ gives

$$\underbrace{[U]^T[m][U]}_{[I]}\{\ddot{p}\}+\underbrace{[U]^T[k][U]}_{[\Omega]}\{p\}=[U]^T\{F(t)\}$$

which can be rewritten as This is a generic uncoupled equation:

$$\boxed{[I]\{\ddot{p}\}+[\Omega]\{p\}=[U]^T\{F(t)\}}$$

and the corresponding uncoupled equation for the *k*-th mode has the form

$$\ddot{p}_k+\omega_k^2 p_k=\sum_i U_{ik}F_i(t)$$

It can be solved with the solution of an oscillator: ex: CONVOLUTION INTEGRAL or as for El Centro earthquake: RESPONSE SPECTRUM METHOD because the spectrum is known and this can be solved without solving the equa.

The squared natural circular frequencies are the eigenvalues which have been calculated by the eigenvalue problem.

Since equations are uncoupled they can be solved separately.

MODAL DECOUPLING + RESPONSE SPECTRUM.

Each uncoupled equation formally represents a SDF system. It can be solved independently to get {p(*t*)} and thus {u(*t*)}.

If a MDF structure is excited by a harmonic force whose frequency is one of the natural frequencies of the structure, then after some while (steady state response) the structure vibrates with the same frequency as the applied force and the deflected shape is the eigenmode associated to the natural frequency (**resonance**).

A MDF has many resonant frequencies (ex bridge).

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 69 di 262

$$\left[{}^{\backslash}m_{\backslash}\right]=[\Phi]^T[m][\Phi]=\begin{bmatrix}3.682 & 0 & 0\\ 0 & 5.726 & 0\\ 0 & 0 & 18.59\end{bmatrix}$$

$$\left[{}^{\backslash}k_{\backslash}\right]=[\Phi]^T[k][\Phi]=\begin{bmatrix}364.7 & 0 & 0\\ 0 & 4452 & 0\\ 0 & 0 & 30184\end{bmatrix}$$

$$[\Phi]^T\{F_0\}=\begin{Bmatrix}0.802\\ -0.555\\ -2.247\end{Bmatrix}$$

The uncoupled system is

$$3.682\ \ddot{p}_1+364.7\ p_1=0.802\sin(2\pi t)$$
$$5.726\ \ddot{p}_2+4452\ \ p_2=-0.555\sin(2\pi t)$$
$$18.59\ \ddot{p}_3+30184\ p_3=-2.247\sin(2\pi t)$$

We solve WRT the modal coord, that help us, then go back to the geometric ones.

Each SDF system is solved by only considering the steady state response, which directly gives

MODAL COORD

$$p_1(t)=\frac{0.802/364.7}{1-(2\pi/9.95)^2}\sin(2\pi t)=3.656\cdot10^{-3}\sin(2\pi t)$$

$$p_2(t)=\frac{-0.555/4452}{1-(2\pi/27.9)^2}\sin(2\pi t)=-1.313\cdot10^{-4}\sin(2\pi t)$$

$$p_3(t)=\frac{-2.247/30184}{\underbrace{1-(2\pi/40.3)^2}_{\text{amplification factor}}}\sin(2\pi t)=-7.629\cdot10^{-5}\sin(2\pi t)$$

same mode of oscillation but with ≠ amplitude

Transformation to geometric coordinates

GEOM COORD

$$\begin{Bmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\end{Bmatrix}=[\Phi]\begin{Bmatrix}p_1(t)\\ p_2(t)\\ p_3(t)\end{Bmatrix}=[\Phi]\begin{Bmatrix}3.656\ 10^{-3}\\ -1.313\ 10^{-4}\\ -7.629\ 10^{-5}\end{Bmatrix}\sin(2\pi t)$$

$$\boxed{\begin{Bmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\end{Bmatrix}=\begin{Bmatrix}3.449\ 10^{-3}\\ 3.177\ 10^{-3}\\ 1.653\ 10^{-3}\end{Bmatrix}\sin(2\pi t)}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 71 di 262

SYSTEMS WITH DAMPING

The equation system of MDOF systems with viscous damping is

$$\underbrace{[m]\{\ddot{u}\}}_{\text{INERTIA}}+\underbrace{[c]\{\dot{u}\}}_{\text{VISCOUS FORCE}}+\underbrace{[k]\{u\}}_{\text{RESTORING FORCE}}=\underbrace{\{F(t)\}}_{\text{FORCE}}$$

The transformation to modal coordinates can be performed as before, which gives

$$\{u\}=[U]\{p\}$$

$$\{\ddot{p}\}+[U]^T[c][U]\{\dot{p}\}+[\,\omega^2\,]\{p\}=[U]^T\{F(t)\}$$

Problem : in general $[U]^T$ $[\mathbf{c}]$ $[U]$ is a non diagonal matrix → means that I can't decouple the equations.

! ∃ some errors in forcing diagonalization, especially when ∃ a high damping ex. ∃ isolators. But usually, in ordinary situations, it's not.

Possible ways to decouple equations:

Method one 1) FORCED DIAGONALIZATION 2) ESTIMATE [c]

The terms out of the diagonal in $[U]^T$ $[\mathbf{c}]$ $[U]$ are neglected → I impose to 0 all those terms.

$$[U]^T\ [c][U]\cong[\,2\zeta\omega_i\,]=\begin{bmatrix}2\zeta_1\omega_1 & & & 0\\ & 2\zeta_2\omega_2 & & \\ & & \ddots & \\ 0 & & & 2\zeta_n\omega_n\end{bmatrix}$$

$$\rightarrow\quad [c]=\left([U]^T\right)^{-1}[\,2\zeta\omega_i\,][U]^{-1}$$ → FOLLOW THE INVERSE PATH TO CALCULATE [c]

This method supposes that all the modal damping ratios ζ_k are known, e.g. identified via experimental modal analysis procedures. When damping is low this technique supplies acceptable results.

In real practice ∄ methods to measure [c], but we measure the modal damping $2\xi_1$, $3\xi_1$, ... , so only the terms on the diagonal

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 73 di 262

The equation of motion becomes:

$$\begin{bmatrix} [C] & [M] \\ [M] & [0] \end{bmatrix}\{\dot{x}\}+\begin{bmatrix} [K] & [0] \\ [0] & -[M] \end{bmatrix}\{x\}=\begin{Bmatrix} \{F\} \\ \{0\} \end{Bmatrix}$$

having posed:

$$\{x\}=\begin{Bmatrix} \{u\} \\ \{\dot{u}\} \end{Bmatrix} \qquad \{\dot{x}\}=\begin{Bmatrix} \{\dot{u}\} \\ \{\ddot{u}\} \end{Bmatrix}$$

The eigenvalue problem is of the form:

$$\left\{ s\cdot\begin{bmatrix} [C] & [M] \\ [M] & [0] \end{bmatrix}+\begin{bmatrix} [K] & [0] \\ [0] & -[M] \end{bmatrix}\right\}\{\Phi'\}=\{0\}$$

where the n couples of eigenvalues

$$\boxed{s_k=-\zeta_k\omega_k\pm j\omega_k\sqrt{1-\zeta_k^2}}$$

correspond to n couples of complex conjugated eigenvectors, $\{\phi_k'\}$ and $\{\phi_k'\}^*$, such that:

$$\{\Phi'_{k}\}=\begin{Bmatrix} \{\Phi_k\} \\ \{\Phi_k\}s_k \end{Bmatrix}; \quad \{\Phi'_k\}^*=\begin{Bmatrix} \{\Phi_k\}^* \\ \{\Phi_k\}^* s_k^* \end{Bmatrix}$$

Free vibrations will be in the form:

$$\{x(t)\}=\sum_k\left(\{\Phi'\}A_{1(k)}e^{s_k t}+\{\Phi'\}^* A_{2(k)}^* e^{s_k^* t}\right)$$

By choosing complex conjugate values for the two constant, for a geometric coordinate:

Modal shapes become complex: having MODULUS & PHASE; complex valued shapes.

$\bar{A}_{1K}=\bar{A}_{2K}$

$$u_i(t)=\mathrm{Re}\left[\sum_k|\Phi'_{ik}|e^{-j\,\mathrm{phase}(\Phi'_{ik})}C_{(k)}e^{-j\varphi_{(k)}}e^{-\zeta_k\omega_k t+j\omega_{D(k)}t}\right]=$$

$$\sum_k|\Phi'_{ik}|C_{(k)}e^{-\zeta_k\omega_k t}\cos\left(\omega_{D(k)}t-\varphi_{(k)}-\mathrm{phase}(\Phi'_{ik})\right)$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 75 di 262

Other options without modal analysis: CONTROL ENGINEERING PROGRAMS:

In certain situations it may be convenient to solve directly the coupled linear differential equations, e.g. in a simpler form:

$$\{\dot{x}\} = [A]\{x\} + [B]\{F(t)\}$$

Having posed:

$$\{x\} = \begin{Bmatrix} \{u\} \\ \{\dot{u}\} \end{Bmatrix}; \quad [B] = \begin{bmatrix} [0] \\ [M^{-1}] \end{bmatrix};$$

$$[A] = \begin{bmatrix} [0] & [I] \\ -[M^{-1}][K] & -[M^{-1}][C] \end{bmatrix}$$

In analogy with the SDOF systems we search a solution in the form of matrix exponential:
solve the eq. in a closed form

$$\{x\} = [e^{At}]\{c(t)\} = \left(\sum_{m=0}^{\infty} \frac{[A]^m t^m}{m!} \right)\{c(t)\}$$

exp defined as a series

is an exp matrix that has the same properties of an exponential so that's why the notation

where $[\mathbf{e}^{At}]$ is the matrix exponential, a matrix function on square matrices analogous to the ordinary exponential function, and $\{\mathbf{c}(t)\}$ is an unknown vector function of time. From the product rule and by substituting :

$$[A][e^{At}]\{c(t)\} + [e^{At}]\{\dot{c}(t)\} =$$
$$[A][e^{At}]\{c(t)\} + [B]\{F(t)\} =$$

hence:

$$\{\dot{c}(t)\} = [e^{-At}][B]\{F(t)\}$$

Simple integration yields:

$$\{c(t)\} = \int_0^t [e^{-A\tau}][B]\{F(\tau)\}d\tau + \{c(0)\}$$

As $\{\mathbf{c}(0)\}=\{\mathbf{x}(0)\}$, finally we get a general MDF form for the Duhamel's integral :

refers to the initial condit. of the building usually $v_0 = 0$, $u_0 = 0$

$$\boxed{\{x(t)\} = \int_0^t [e^{A(t-\tau)}][B]\{F(\tau)\}d\tau + [e^{At}]\{x(0)\}}$$

CONVOLUTION IN MATRIX FORM

GENERAL DUHAMEL INTEGRAL

It's a matrix of functions

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 77 di 262

DISCRETE-TIME SYSTEMS

Some systems are modelled by a discrete-time formulations; ex control systems work with this formulation.

Continuous-time formulations may be easily converted into its discrete time version, which is the standard reference in case of active seismic and vibration control problems

Letting Δt be the sampling time and substituting $t_0=i\cdot\Delta t$ and $t=(i+1)\cdot\Delta t$, the general solution for consecutive times is

$$\{x[(i+1)\Delta t]\} = \int_{i\Delta t}^{(i+1)\Delta t} \left[e^{A((i+1)\Delta t-\tau)}\right][B]\{F(\tau)\}d\tau$$

$$+\left[e^{A\Delta t}\right]\{x(i\Delta t)\}$$ → this formulation is still continuous

or by changing the integration variable to $\sigma=(i+1)\Delta t-\tau$ in discrete form $-d\sigma = d\tau \Rightarrow$ extremes of the integration are swapped

$$\{x(i+1)\} = \int_0^{\Delta t}\left[e^{A\sigma}\right][B]\{F(i)\}d\sigma + \left[e^{A\Delta t}\right]\{x(i)\}$$

Thus, the discrete time version of the state space equation becomes

$$\boxed{\{x(i+1)\} = [A']\{x(i)\} + [B']\{F(i)\}}$$

It's a chain of equations.

having posed

$$[A'] = \left[e^{A\Delta t}\right];$$
$$[B'] = \int_0^{\Delta t}\left[e^{A\sigma}\right][B]d\sigma$$

Since in practical applications data are available in discrete time, usually with equidistant intervals between time points, the discrete version of state space equation is a standard use in experimental dynamics and control.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 79 di 262

and for any measurement time *i*:

$$\boxed{[Y(i)] = [D][A']^{i-1}[B']}$$

The matrices above are the system's Markov parameters. The Markov parameter sequence or chain for a state-space model is the discrete homologous to the IRF for the continuous-time formulation.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 81 di 262

16/04/18

SEISMIC ANALYSIS OF BUILDINGS

SHEAR TYPE FRAME

3 degrees of freedom

accel applied to the ground

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 + \boxed{\ddot{u}_g} \\ \ddot{u}_2 + \boxed{\ddot{u}_g} \\ \ddot{u}_2 + \boxed{\ddot{u}_g} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

↳ diagonal because assumption of lumped mass

$$[\mathbf{m}] \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + [\mathbf{c}] \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + [\mathbf{k}] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = -[\mathbf{m}] \begin{Bmatrix} \ddot{u}_g(t) \\ \ddot{u}_g(t) \\ \ddot{u}_g(t) \end{Bmatrix} = \overbrace{-[M] \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \ddot{u}_g(t)}^{\text{USED NOTATION}}$$

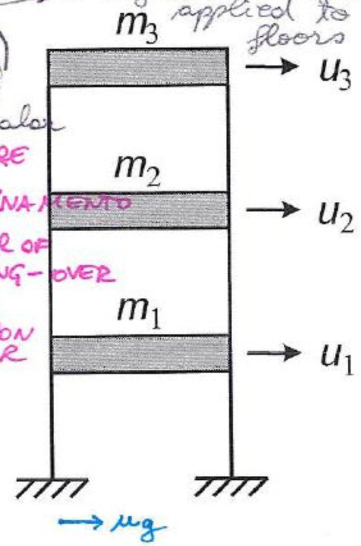
→ excitation acting on the structure at the support: it's the same for the 3 floors

→ to say that it's equally applied to all floors

$\ddot{u}_g(t)$ scalar

$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \{t\}$ = VETTORE DI TRASCINAMENTO = VECTOR OF CARRYING-OVER = FORCE LOCATION VECTOR

can be a matrix ⇒ CONNECTIVITY VECTOR OR MATRIX ≡ makes a connection between matrices with ≠ dimensions.



Eigenvalue problem

$$\omega_1 \; \omega_2 \; \omega_3 \qquad \boldsymbol{\phi}_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \qquad \boldsymbol{\phi}_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} \qquad \boldsymbol{\phi}_3 = \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} \qquad [\boldsymbol{\phi}] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}$$

Eigenvector normalisation

$$\begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} = \frac{1}{\sqrt{\{\phi\}^T [\mathbf{m}] \{\phi\}}} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \quad \text{etc.} \qquad [\mathbf{U}] = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 83 di 262

$\underbrace{[U]^T[M]}\{t\}$

$[\Gamma]\{t\} = \{\Gamma\}$

$K \perp h$ ← convolution

$h_k(t) * (\underbrace{-\Gamma_k}_{\text{participation factor}} \cdot \underbrace{\ddot{u}_g(t)}_{\text{accelerogram}}) = P_k(t)$; $h_k(t) = \frac{1}{\omega_{dk}} e^{-\xi_k \omega_k t} \cdot \sin(\omega_{dk} \cdot t) \rightarrow h_1, h_2, h_3$

$\omega_{dk} = \omega_{nk} \cdot \sqrt{1-\xi_k^2}$

RESPONSE HISTORY ANALYSIS ($\ddot{u}_g(t)$ known)

Solve numerically $\ddot{p}_k + 2\zeta\,\omega_k\,\dot{p}_k + \omega_k^2\,p_k = -\Gamma_k\,\ddot{u}_g(t) \rightarrow p_k(t) \qquad k = 1,2,3$

sufficient to know only this

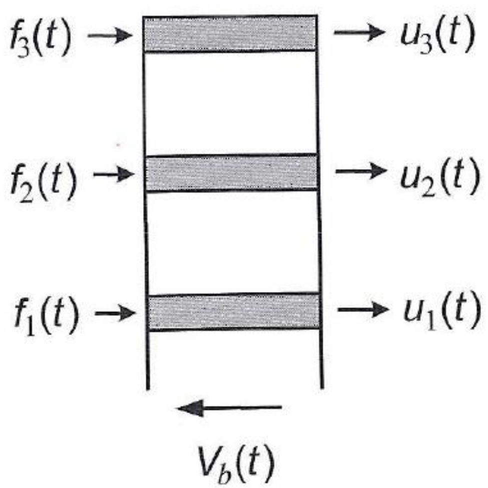
Summation of the modal contributions

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = [\mathbf{U}] \begin{Bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{Bmatrix} = \begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} p_1(t) + \begin{Bmatrix} U_{12} \\ U_{22} \\ U_{32} \end{Bmatrix} p_2(t) + \begin{Bmatrix} U_{13} \\ U_{23} \\ U_{33} \end{Bmatrix} p_3(t)$$

Equivalent static forces

$$\begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix} = [\mathbf{k}] \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = [\mathbf{k}][\mathbf{U}] \begin{Bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{Bmatrix}$$

Observation: in some instances, the contribution of the higher modes can be neglected.



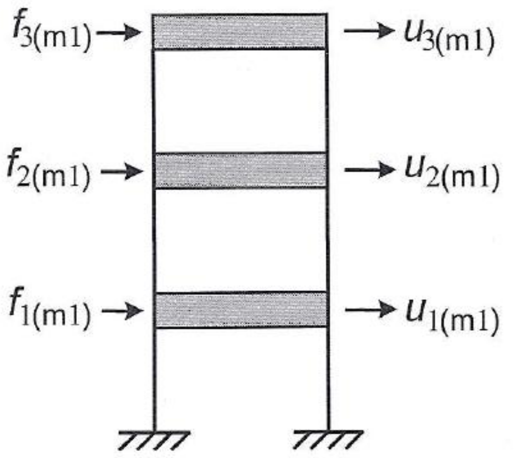
$$\begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix} = [\mathbf{k}] \begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} p_1(t) + [\mathbf{k}] \begin{Bmatrix} U_{12} \\ U_{22} \\ U_{32} \end{Bmatrix} p_2(t) + [\mathbf{k}] \begin{Bmatrix} U_{13} \\ U_{23} \\ U_{33} \end{Bmatrix} p_3(t)$$

The internal forces and stresses can now be determined using statics.

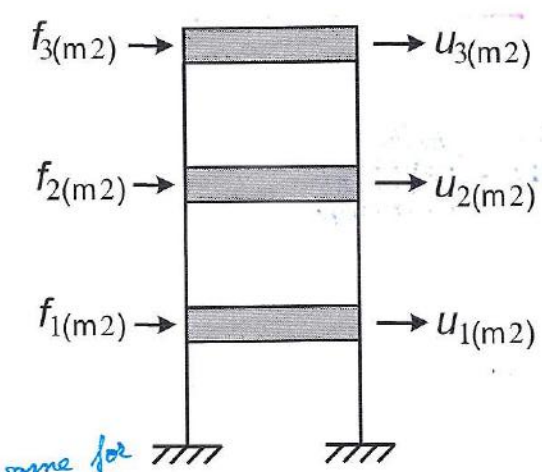
Example : base shear force

static $V_b(t) = f_1(t) + f_2(t) + f_3(t) \qquad V_{b\max} = \max|V_b(t)|$

The summation of modal contributions is performed in a different way.
The idea is to calculate first maximum displacements and equivalent static forces for each mode.



MODE 1



MODE 2

$$\begin{Bmatrix} u_{1(m1)} \\ u_{2(m1)} \\ u_{3(m1)} \end{Bmatrix} = [\mathbf{U}] \begin{Bmatrix} \Gamma_1 D_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} \Gamma_1 D_1$$

MAX OF SPECTRAL DISPL. OF 1ST MODE & MAX STATIC FORCES

$$\begin{Bmatrix} f_{1(m1)} \\ f_{2(m1)} \\ f_{3(m1)} \end{Bmatrix} = [\mathbf{k}][\mathbf{U}] \begin{Bmatrix} \Gamma_1 D_1 \\ 0 \\ 0 \end{Bmatrix} = [\mathbf{k}] \begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} \Gamma_1 D_1$$

The same for 2nd mode.

$$\begin{Bmatrix} u_{1(m2)} \\ u_{2(m2)} \\ u_{3(m2)} \end{Bmatrix} = [\mathbf{U}] \begin{Bmatrix} 0 \\ \Gamma_2 D_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} U_{12} \\ U_{22} \\ U_{32} \end{Bmatrix} \Gamma_2 D_2$$

$$\begin{Bmatrix} f_{1(m2)} \\ f_{2(m2)} \\ f_{3(m2)} \end{Bmatrix} = [\mathbf{k}][\mathbf{U}] \begin{Bmatrix} 0 \\ \Gamma_2 D_2 \\ 0 \end{Bmatrix} = [\mathbf{k}] \begin{Bmatrix} U_{12} \\ U_{22} \\ U_{32} \end{Bmatrix} \Gamma_2 D_2$$

Same calculations for the third mode

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 87 di 262

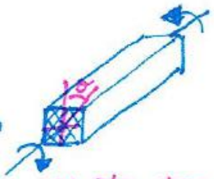
19/4/18

I HE 1 I IPE 2 □ 3 STEEL

2,1 : torsion not transmitted to much

3: More stiff → transmits torsion

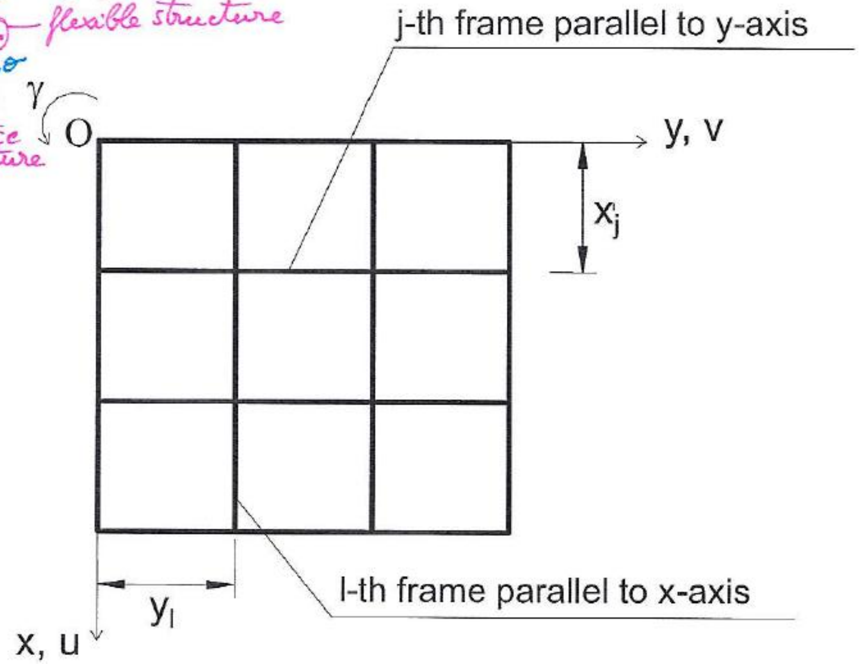
4 CONCRETE : elastic behaviour until cracks happen when the response become that of a flexible structure



3D BUILDING

In a multi-storey building the dynamic equilibrium is written under the following assumptions:
- the degrees of freedom at each storey are the two translations and the floor rotation (diaphragmatic behaviour at the storey level);
- the masses (m_i) are lumped at each floor;
- all the members are axially rigid;
- torsional stiffness of the elements is neglected (cylindrical hinges at the intersections between orthogonal frames).

M_t — stiff elastic structure



In free undamped vibration conditions, the equilibrium at the i-th floor in the x direction is expressed as follows:

$$\sum_{l} \{\mathbf{H_x}\}_i^l (\{\mathbf{u}\} - \{\boldsymbol{\gamma}\} y_l) + m_i (\ddot{u}_i - \ddot{\gamma}_i y_{G,i}) = 0$$

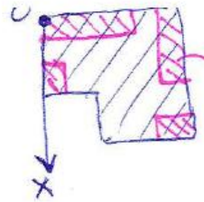
STIFFNESS

m_i : mass lumped at the i-th floor, whose centroid has coordinates ($x_{G,i}$, $y_{G,i}$)

$\{\mathbf{u}\}$, $\{\mathbf{v}\}$, $\{\boldsymbol{\gamma}\}$: vectors of the displacements (translation parallel to x-axis, translation parallel to y-axis, rotation with respect to the origin O, respectively)

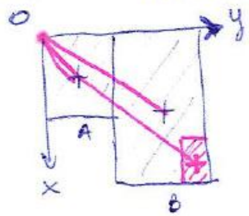
$\{\mathbf{H_x}\}_i^l$: *i*-th row of the *l*-th frame stiffness

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 89 di 262



≠ influence area because all mass partecipate to the inertia

In seismic POLAR MOMENTS are used.



centroid A | centroid of mas wall
centroid B |
⇒ transpose to the origin.

$$[\mathbf{M_{xx}}] = [\mathbf{M_{yy}}] = \begin{bmatrix} m_1 & 0 & & 0 \\ 0 & m_2 & & \\ & & \ldots & \\ & & & \ldots \\ 0 & 0 & & m_n \end{bmatrix}$$: mass matrix

$$[\mathbf{M_{xy}}] = \begin{bmatrix} -m_1 y_{G,1} & 0 & & 0 \\ 0 & -m_2 y_{G,2} & & \\ & & \ldots & \\ & & & \ldots \\ 0 & 0 & & -m_n y_{G,n} \end{bmatrix} ; \quad [\mathbf{M_{yy}}] = \begin{bmatrix} m_1 x_{G,1} & 0 & & 0 \\ 0 & m_2 x_{G,2} & & \\ & & \ldots & \\ & & & \ldots \\ 0 & 0 & & m_n x_{G,n} \end{bmatrix}$$

: matrices of static moments with respect to x and y axes

$$[\mathbf{M_{\gamma\gamma}}] = \begin{bmatrix} J_{0,1} & 0 & & 0 \\ 0 & J_{0,2} & & \\ & & \ldots & \\ & & & \ldots \\ 0 & 0 & & J_{0,n} \end{bmatrix}$$: matrix of polar moments with respect to the origin O

inertial moments WRT origin.

Difference between 2D and 3D : mass matrix is the same
[t] becomes n x 3
general def of the partecipation factor

Under earthquake loading the equation becomes:

$$[\mathbf{m}]\{\ddot{\mathbf{q}}\} + [\mathbf{k}]\{\mathbf{q}\} = -[\mathbf{m}][\mathbf{t}]\{\ddot{\mathbf{q}}_{\mathbf{g}}\}$$

$\{\ddot{\mathbf{q}}_{\mathbf{g}}\}^{\mathrm{T}} = \{\ddot{u}_g \quad \ddot{v}_g \quad \ddot{\gamma}_g\}$: vector of seismic accelerations (components parallel to x-axis and to y-axis and rotation)

$[\mathbf{t}]$: a *n*x3 matrix which properly applies the seismic input to the degrees of freedom.

This model is not good for vertical component of the acceleration $\ddot{z}_g$

If only one accelerogram shape is used along different axes, the previous equation may be written in a simpler way by introducing a nx1 vector **{t}**

$$[\mathbf{m}]\{\ddot{\mathbf{q}}\} + [\mathbf{k}]\{\mathbf{q}\} = -[\mathbf{m}]\{\mathbf{t}\}\,\ddot{u}_g(t)$$

and after the matrices' diagonalization MODAL COORD. SYSTEM.

$$[\mathbf{I}]\{\ddot{\mathbf{p}}\} + [\mathbf{\Omega}]\{\mathbf{p}\} = -\underbrace{[\mathbf{U}]^{\mathrm{T}}[\mathbf{m}]\{\mathbf{t}\}}_{[\Gamma]}\,\ddot{u}_g(t)$$

3D BUILDING: RESPONSE SPECTRUM ANALYSIS

As is usual in most structural problems, let us use the response spectrum for accelerations in order to evaluate the maximum acceleration resulting from the uncoupled equation:

S_a
→ pseudo-acceleration = $\ddot{p} = S_a \Gamma_k$

$$\boxed{\ddot{p}_{k_{max}} = \Gamma_k \cdot S_a(T_K)}$$

and consequently:

$$\boxed{p_{k_{max}} \cong \frac{\ddot{p}_{k_{max}}}{\omega_k^2} = \left(\frac{T_k}{2\pi}\right)^2 \Gamma_k\, S_a(T_k)}$$

gamma $_k$

modal displacement

At this stage, we need to work with each decoupled equation separately and, in particular, calculate local effects (internal loads and displacements) associated to each mode k. Contributions resulting from different modes will be then combined at the very end of the procedure.

When use the response spectrum, solve separately for each mode and at the very end combine them using the rules SRSS/CQC.

max global dispacements of the k-th mode

$$\{q\}_{k_{max}} = \begin{Bmatrix} \{u\}_{k_{max}} \\ \{v\}_{k_{max}} \\ \{\gamma\}_{k_{max}} \end{Bmatrix} = [U] \begin{Bmatrix} 0 \\ p_{k_{max}} \\ 0 \\ \dots \\ 0 \end{Bmatrix} = \{U\}_k \cdot p_{k_{max}} = \begin{Bmatrix} \{u\}_k \\ \{v\}_k \\ \{\gamma\}_k \end{Bmatrix} \cdot p_{k_{max}}$$

k-th mode: max forces acting on frames

$$\{f_x\}_{kmax}^{I} = [H_{xx}]^{I} \{S_x\}_{kmax}^{I} = [H_{xx}]^{I} \left(\{u\}_{k_{max}} - \{\gamma\}_{k_{max}}\, y_I\right)$$

$$\{f_y\}_{kmax}^{j} = [H_{yy}]^{j} \{S_y\}_{kmax}^{j} = [H_{yy}]^{j} \left(\{v\}_{k_{max}} + \{\gamma\}_{k_{max}}\, x_j\right)$$

MULTIMODAL ANALYSIS ACCORDING TO EUROCODE 8

- Seismic loading must be considered as acting simultaneously along the principal directions. In more detail, it is assumed that earthquake acts with 100% of its intensity in a single direction (x, y or z) and 30% in the other two. It must be noted that in framed building the vertical component can often be neglected.
- For each structural element the most onerous load configuration should be considered.

As an example, in the seismic analysis of a two storey spatial frame, the following configurations and corresponding {**t**} must be considered:

x y z → = 0 because can be neglected usually; ex Messina bridge z can't be neglected.

$$\{\mathbf{t}\}_1^T = \{1 \quad 1 \quad 0.3 \quad 0.3 \quad 0 \quad 0\}$$

$$\{\mathbf{t}\}_2^T = \{0.3 \quad 0.3 \quad 1 \quad 1 \quad 0 \quad 0\}$$

! not considered vertical axis

The two last zeros refer to the rotational component of the excitation at the base of the building, which is seismic problems is nil due to the great distance of the source (hypocenter).

! z is not considered because it's not so influent in seismic, the fail is due to horizontal excitations.
z should be taken into account if span is > 20 m