NUMERO: 2436A ANNO: 2019

A P P U N T I

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MATERIA: Esercizi di Analisi Matematica II + Schemi - Prof. Serra

Analisi Matematica II

Raccolta di esercizi

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CAMBI DI COORDINATE

<u>POLARI</u> $x = \rho\cos\vartheta$, $y = \rho\sin\vartheta$, Jacobiana $= \rho$. Per pezzi di circonferenze

<u>ELLITTICHE</u> $x = a\rho\cos\vartheta$, $y = b\rho\sin\vartheta$, Jacobiano $ab\rho$

$$\left|\iint_D f\,dx\,dy\right| \le \int_D |f|\,dx\,dy$$

(MATRICE JACOBIANA $J\Phi$ $\begin{bmatrix} -\nabla F_1- \\ -\nabla f_2- \end{bmatrix}$)

• attenzione che se ρ varia devo indicarlo! $D = \{(x,y) \in \mathbb{R}^2 \mid \frac{x}{\sqrt3} \le y \le \sqrt3 x,$
$1 \le xy \le 4\}$ $\qquad 1 \le \rho^2\cos\vartheta\sin\vartheta \le 4 \to$ ricavi ρ

per $\vartheta \to \quad tg^{-1}\left(\frac{1}{\sqrt3}\right) \le \vartheta \le tg^{-1}(\sqrt3)$

INTEGRALI TRIPLI

RIDUZIONE PER FILI $\to$ (CONVESSO su z) $\to$ $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D \wedge \alpha(x,y) \le z \le \beta(x,y)\}$

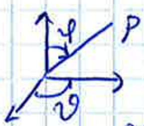
$$\int_\Omega f(x,y,z)\,dx\,dy\,dz = \int_D \int_\alpha^\beta f\,dz\,dx\,dy$$

OPPURE PER STRATI $\to$ $\Omega = \{z \in [c,d],\ (x,y) \in D_z\}$

CAMBI DI CORDINATE

<u>CILINDRICHE</u> ρ, ϑ, z . $x = \rho\cos\vartheta$, $y = \rho\sin\vartheta$, $z = z$; $|J\Phi| = \rho$

<u>SFERICHE</u>



$0 \le \varphi \le \pi \to$ limiti.

$x = \rho\cos\vartheta\sin\varphi$, $y = \rho\sin\vartheta\sin\varphi$, $z = \rho\cos\varphi$; $|J\Phi| = \rho^2\sin\varphi$

BARICENTRO (CENTRO DI MASSA)

$\sigma =$ densità $= \sigma(x,y)$ $\qquad x_G = \dfrac{\int \sigma(x,y)\,x\,dx\,dy}{\int \sigma(x,y)\,dx\,dy}$

σ costante $=$ se corpo omogeneo $\to$ $x_G = \dfrac{\int x\,dx\,dy}{\int dx\,dy}$

INTEGRALI CURVILINEI 1° SPECIE

curva parametrica in $R^3 \to$ funzione $[a,b] \to \mathbb{R}^3$, $\gamma(t) = (x(t), y(t), z(t))$

cioè 3 funzioni in una unica variabile. Regolare $\to C^1$ e $\gamma'(t) \neq 0\ \forall t$.

Semplice $\to$ senza autointersezioni.

INTEGRALE DI SUPERFICIE DI I^a SPECIE

Σ = superficie cartesiana - grafico di una f in due variabili $g(x,y)$

$\doteq \{(x,y,g(x,y)) \mid (x,y) \in \mathbb{R}^2\}$

$$\int_\Sigma F \cdot d\sigma = \int_D F(x,y,g(x,y))\sqrt{1+\|\nabla g\|^2}\, dy\, dx$$

proietta Σ su xy

- se f costante $= 1 \to \int_D \sqrt{1+\|\nabla g\|^2}\, dx\, dy$ = area Σ
- ricostruire masse da densità

INTEGRALE DI SUPERFICIE DI 2^a SPECIE

- integrale di flusso $\to \int_\Sigma F \hat{u}_N d\Sigma = \int_D F \hat{n} d\sigma$

$\hat{n} = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$ se $\hat{n}$ verso l'alto, $F(x,y,g(x,y))$

TEOREMA DI GAUSS o DELLA DIVERGENZA

$$\int_D F \hat{n} d\sigma = \int_\Omega \operatorname{div} F\, dx\, dy\, dz$$ $\to$ flusso di f uscente da Ω

TEOREMA DI STOKES o FLUSSO DEL ROTORE

$$\oint_\gamma F \cdot dP = \int_\Sigma \nabla \wedge F\, n\, d\Sigma$$

$\partial\Sigma$ antiorario $= (\gamma_1(t), \gamma_2(t), g(\gamma_1, \gamma_2))$

$\gamma = (\gamma_1, \gamma_2)$ in t $\to$ dominio

CALCOLO DIFFERENZIALE

Foglio 1: nozioni di base

1. Trovare e disegnare il dominio delle seguenti funzioni di due variabili

   1.1. $f(x,y) = \dfrac{x}{y}$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ y \neq 0\}]$

   1.2. $f(x,y) = \dfrac{y}{x-3}$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ x \neq 3\}]$

   1.3. $f(x,y) = \dfrac{2x}{x+y}$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ y \neq -x\}]$

   1.4. $f(x,y) = \dfrac{x+y}{x-y+1}$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ y \neq x+1\}]$

   1.5. $f(x,y) = \sqrt{x+y+2}$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ y \geq -x-2\}]$

   1.6. $f(x,y) = \dfrac{\sqrt{x-y}}{x-1}$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ x \geq y,\ x \neq 1\}]$

   1.7. $f(x,y) = \dfrac{x+y}{x^2+y^2-1}$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ x^2+y^2 \neq 1\}]$

   1.8. $f(x,y) = \log(4-x^2-y^2)$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ x^2+y^2 < 4\}]$

   1.9. $f(x,y) = \sqrt{x^2-1-y}$ $\quad [\{(x,y) \in \mathbb{R}^2 :\ y \leq x^2-1\}]$

2. Identificare il dominio delle seguenti funzioni di tre variabili

   2.1. $f(x,y,z) = \dfrac{x+y+z}{z-1}$ $\quad$ [$\mathbb{R}^3$ meno il piano $z=1$]

   2.2. $f(x,y,z) = \dfrac{x^2+y^2+3}{x^2+y^2+z^2}$ $\quad [\mathbb{R}^3 \setminus \{0,0,0\}]$

   2.3. $f(x,y,z) = \dfrac{z+2}{xyz}$ $\quad$ [$\mathbb{R}^3$ meno i piani coordinati]

   2.4. $f(x,y,z) = \sqrt{x+y+z-1}$ $\quad$ [il semispazio $x+y+z \geq 1$]

   2.5. $f(x,y,z) = \log(1-x^2-y^2-z^2)$ $\quad$ [l'interno della sfera di raggio 1 e centro l'origine]

   2.6. $f(x,y,z) = \sqrt{x^2+y^2+z^2-4}$ $\quad$ [$\mathbb{R}^3$ meno la sfera aperta di raggio 2 e centro l'origine]

   2.7. $f(x,y,z) = \log(z-x^2-y^2)$ $\quad$ [l'interno del paraboloide $z=x^2+y^2$]

   2.8. $f(x,y,z) = \dfrac{\sqrt{1-z}}{\sqrt{z-x^2-y^2}}$ $\quad$ [l'interno del paraboloide $z=x^2+y^2$ sotto al piano $z=1$]

3. Disegnare alcune curve di livello delle seguenti funzioni:

   a) $f(x,y) = \dfrac{x-y}{x+y}$ $\quad$ b) $f(x,y) = \sin(x^2+y^2)$ $\quad$ c) $f(x,y) = x^2+5y^2$ $\quad$ d) $f(x,y) = e^{x^2-y^2}$

4. Calcolare le derivate parziali $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ delle seguenti funzioni

   a) $f(x,y) = 2x+3y-1$ $\quad$ b) $f(x,y) = x^2-y^2$ $\quad$ c) $f(x,y) = y^2-x^2$

   d) $f(x,y) = 3x^2-4y^3$ $\quad$ e) $f(x,y) = xy+x$ $\quad$ f) $f(x,y) = x^2+x-2$

   g) $f(x,y) = x^2y^3-xy^2$ $\quad$ h) $f(x,y) = \dfrac{x}{y}$ $\quad$ i) $f(x,y) = \dfrac{x+1}{y-1}$

   j) $f(x,y) = \sqrt{xy}$ $\quad$ k) $f(x,y) = e^{x^2-y}$ $\quad$ l) $f(x,y) = \sin(x+2y)$

   m) $f(x,y) = \log(x^2+y^2)$ $\quad$ n) $f(x,y) = e^y \sin x$ $\quad$ o) $f(x,y) = (3x+2y)^4$

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Foglio 2: applicazioni del calcolo differenziale

1. Scrivere l'equazione del piano tangente al grafico delle seguenti funzioni nei punti indicati

   1.1. $f(x,y) = \sin x \cos y$ in $P = (0,0)$. $[z = x]$
   1.2. $f(x,y) = e^{x+y}$ $[z = 1 + x + y]$
   1.3. $f(x,y) = \frac{1}{2}x^2 - y$ in $P = (2,-1,3)$. $[2x - y - z - 2 = 0]$
   1.4. $f(x,y) = x^2 + y^2$ in $P = (1,-2,5)$. $[2x - 4y - z - 5 = 0]$
   1.5. $f(x,y) = e^{x^2-y^2} + \sqrt{1+x^2+y^4}$ in $P = (1,0,f(1,0))$. $[z = e + \sqrt{2} + (e + 1/\sqrt{2})(x-1)]$

2. Scrivere il polinomio di Taylor del secondo ordine centrato nell'origine delle seguenti funzioni

   2.1. $f(x,y) = \cos x + \cos y$ $[2 - (x^2+y^2)/2]$
   2.2. $f(x,y) = \cos x \cos y$ $[1 - (x^2+y^2)/2]$
   2.3. $f(x,y) = \sin(xy)$ $[xy]$
   2.4. $f(x,y) = \sin x \cos y$ $[x]$
   2.5. $f(x,y) = e^{x+y}$ $[1 + x + y + xy + (x^2+y^2)/2]$
   2.6. $f(x,y) = e^{xy}$ $[1 + xy]$
   2.7. $f(x,y) = e^x \sin y$ $[y + xy]$

3. Scrivere il polinomio di Taylor del secondo ordine delle seguenti funzioni centrato nel punto dato

   3.1. $f(x,y) = \frac{1}{2+x-2y}$ in $P_0 = (2,1)$.
   $[\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{2}(y-1) + \frac{1}{8}(x-2)^2 - \frac{1}{2}(x-2)(y-1) + \frac{1}{2}(y-1)^2]$
   3.2. $f(x,y) = \frac{\sin x}{y}$ in $P_0 = (\frac{\pi}{2}, 1)$. $[1 - (y-1) + (y-1)^2 - \frac{1}{2}(x - \pi/2)^2]$

4. Trovare i punti critici delle seguenti funzioni e classificarli

   a) $f(x,y) = x + y - \frac{1}{3}x^3 - \frac{1}{3}y^3$,
   b) $f(x,y) = \frac{1}{2}(x^3 + y^3) - xy$,
   c) $f(x,y) = 2x^2 + 2xy + 3y^2 + y^3$
   d) $f(x,y) = xy - x^2 - y$
   e) $f(x,y) = 2x^2 + y^2 - 4x - 4y$
   f) $f(x,y) = x^2 + 2xy - 4x + 8y$
   g) $f(x,y) = \dfrac{1}{x^2 + 2y^2}$
   h) $f(x,y) = (x^2 + y^2 - 4)(x^2 + 2)$
   i) $f(x,y) = xy^2 e^{-x} + x^2 - 2x$
   j) $f(x,y) = ye^{y-x^2}$

Risposte

$a)$ $(1,1)$ min $(-1,-1)$ max $(-1,1)$, $(1,-1)$ selle; $b)$ $(0,0)$ sella $(\frac{2}{3},\frac{2}{3})$ min
$c)$ $(0,0)$ min $(\frac{5}{6}, -\frac{5}{3})$ sella $d)$ $(1,2)$ sella
$e)$ $(1,2)$ min; $f)$ $(-4,6)$ sella
$g)$ non ha punti critici $h)$ $(0,0)$ sella, $(-1,0)$ $(1,0)$ minimi
$i)$ $(2,0)$ min, $(0,\sqrt{2})$ $(0,-\sqrt{2})$ selle $j)$ $(0,-1)$ min

5. Mostrare che il punto $P = (0,0)$ è critico per la funzione

$$f(x,y) = x^2 + y^2 - cxy$$

per ogni valore del parametro $c \in \mathbb{R}$ e classificarne il tipo al variare di c.

$[(0,0)$ è minimo locale per $c \in [-2,2]$, sella per $c \notin [-2,2]]$

6. (Non standard) Dimostrare che la funzione $f(x,y) = (y - x^2)(y - 2x^2)$ ha un minimo locale se ristretta a *ogni* retta passante per l'origine, ma l'origine *non* è un minimo locale per f.

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$z-1$

2.2) $f(x,y,z)=\dfrac{x^2+y^2+3}{x^2+y^2+z^2}$ $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\neq 0\} \to D=\mathbb{R}^3/\{0,0,0\}$

2.3) $f(x,y,z)=\dfrac{z+2}{xyz}$ $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid xyz\neq 0\} \to D=\mathbb{R}^3/\{x=0 \text{ piano}\}$ e/o $\{y=0 \text{ piano}\}$ e/o $\{z=0 \text{ piano}\}$

2.4) $f(x,y,z)=\sqrt{x+y+z-1}$ $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid x+y+z\geq 1\} \to$ semispazio $x+y+z\geq 1$

2.5) $f(x,y,z)=\log(1-x^2-y^2-z^2)$ $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2<1\}$

↳ interno di una sfera con $r=1$ senza "guscio" $c=(0,0,0)$

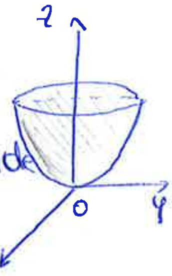
2.6) $f(x,y,z)=\sqrt{x^2+y^2+z^2-4}$ $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\geq 4\}$

{ esterno di sfera con $c=(0,0,0)$, $r=2$ con "guscio" compreso}

2.7) $f(x,y,z)=\log(z-x^2-y^2)$ $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid z>x^2+y^2\}$

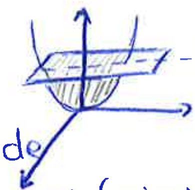
↓

D= interno del paraboloide



2.8) $f(x,y,z)=\dfrac{\sqrt{1-z}}{\sqrt{z-x^2-y^2}}$ $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid 1-z\geq 0 \wedge z>x^2+y^2\}$

$\begin{cases} z\leq 1 \\ z>x^2+y^2 \end{cases}$



interno del paraboloide $z=x^2+y^2$ sotto il piano $z=1$ (piano compreso)

a) $= f(x,y) = e^{x^2-y^2}$

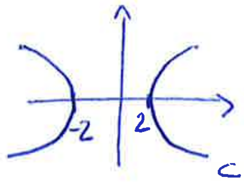
$e^{x^2-y^2} = c$ saranno iperboli, siamo nella forma $f(x,y) = f(ax^2+by^2)$ con $a=b=1$

$\log c = x^2 - y^2$

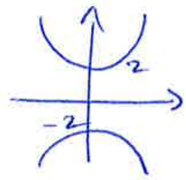
se $c = 0$, $\nexists$

Se $c > 0$, iperboli con v su $x = \sqrt{\log c}$

se $c < 0$, iperboli con v su $y = \sqrt{\log c}$



$c > 0$, $\log c = 4$



$c < 0$, $\sqrt{\log c} = 2$, $\log c = -4$

ESERCIZIO 4

a) $f(x,y) = 2x + 3y - 1 \rightarrow \frac{\partial f}{\partial x} = 2 \quad \frac{\partial f}{\partial y} = 3$

b) $f(x,y) = x^2 + y^2 \rightarrow \frac{\delta f}{\delta x} = 2x \quad \frac{\delta f}{\delta y} = 2y$

c) $f(x,y) = y^2 - x^2 \rightarrow \frac{\delta f}{\delta x} = -2x \quad \frac{\delta f}{\delta y} = 2y$

d) $3x^2 - 4y^3 \rightarrow \frac{\delta f}{\delta x} = 6x \quad \frac{\delta f}{\delta y} = -12y^2$

e) $xy + x \rightarrow \frac{\delta f}{\delta x} = y + 1 \quad \frac{\delta f}{\delta y} = x + 1$

f) $x^2 + x - 2 \rightarrow \frac{\delta f}{\delta x} = 2x + 1 \quad \frac{\delta f}{\delta y} = 0$

g) $x^2y^3 - xy^2 \rightarrow \frac{\delta f}{\delta x} = 2xy^3 - y^2 \quad \frac{\delta f}{\delta y} = 3x^2y^2 - 2xy$

h) $\frac{x}{y} \rightarrow \frac{\delta f}{\delta x} = \frac{1}{y} \quad \frac{\delta f}{\delta y} = -\frac{x}{y^2}$

i) $= \frac{x+1}{y-1} \rightarrow \frac{\delta f}{\delta x} = \frac{1}{y-1} \quad \frac{\delta f}{\delta y} = -\frac{x+1}{(y-1)^2}$

1) $f(x,y)=e^{-x^2+y^4}$, $P_0(1,0)$, $v=(1,1)\to v_N=\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)$

$$\frac{\partial f}{\partial \vec v}(1,0)=\nabla f\circ\vec v$$

$$\nabla f(P_0)=\left(-2x_0e^{-x_0^2+y_0^4},\ 4y_0^3e^{-x_0^2+y_0^4}\right)=$$

$$=(-2e^{-1},0)$$

$$\to \frac{\partial f}{\partial v}(1,0)=\left(-\frac2e,0\right)\cdot\left(\frac1{\sqrt2},\frac1{\sqrt2}\right)=-\frac{2}{\sqrt2}\cdot\frac1e=-\frac{\sqrt2}{e}$$

OPPURE $\to$ $\vec v=\left(\frac1{\sqrt2},\frac1{\sqrt2}\right)$

$$\frac{\partial f}{\partial v}(P_0)=\lim_{h\to0}\frac{f((x_0+\alpha h),(y_0+\beta h))-f(x_0,y_0)}{h}=$$

$$\lim_{h\to0}\frac{\left(1+\frac{h}{\sqrt2},\frac{h}{\sqrt2}\right)-e^{-1}}{h}=\frac{e^{-\left(1+\frac h{\sqrt2}\right)^2+\left(\frac h{\sqrt2}\right)^4}-\frac1e}{h}$$

2) $f(x,y)=e^x\cos y$, $P=(0,0)$, $\vec v=(2,-1)$ $\quad\|\vec v\|=\sqrt{4+1}=\sqrt5$

$$\vec v_N=\left(\frac2{\sqrt5},-\frac1{\sqrt5}\right)$$

$$\frac{\partial f}{\partial v}(P)=\nabla f(P)\cdot\vec v=$$

$$\frac{\partial f}{\partial x}=\cos y\, e^x$$

$$\frac{\partial f}{\partial y}=-e^x\sin y$$

$$\to \frac{\partial f}{\partial v}(P)=(1,0)\cdot\left(\frac2{\sqrt5},\frac2{\sqrt5}\right)=\frac2{\sqrt5}$$

3) $f(x,y)=x^2-xy-2y^2$, $P_0(1,2)$, $\vec v=(\,?\,)$ $\to$ direzione formante $\alpha=\frac\pi3$ con x

$\vec v=\left(\frac12,\frac{\sqrt3}2\right)$ $\to$ versore

$$\frac{\partial f}{\partial v}(P)=\nabla f(P)\cdot\vec v=(2x_0-y_0,-x_0-4y_0)\cdot\left(\frac12,\frac{\sqrt3}2\right)=$$

$$=\cancel{\left(1-\frac{\sqrt3}{2},-\frac12-2\sqrt3\right)\cdot\left(\frac12,\frac{\sqrt3}2\right)=\frac12-\frac{\sqrt3}4-\frac{\sqrt3}4-3=}$$

$$\cancel{=-\frac52-\frac{\sqrt3}2}$$

$$=(\cancel{2}-\cancel{2},-1-8)\cdot\left(\frac12,\frac{\sqrt3}2\right)=-\frac{9\sqrt3}2$$

$$f(x,y,z) = (x-y)(y-z)(z-x)$$

$$\frac{\partial f}{\partial x} \Rightarrow zx - x^2 - yz + xy \longrightarrow \frac{\partial f}{\partial x} = (y-z)(z-2x+y)$$

$$\frac{\partial f}{\partial y} \Rightarrow xy - zx - y^2 + zy \longrightarrow \frac{\partial f}{\partial y} = (z-x)(x-2y+z)$$

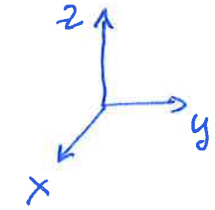
$$\frac{\partial f}{\partial z} \Rightarrow yz - xy - z^2 + zx \longrightarrow \frac{\partial f}{\partial z} = (x-y)(y-2z+x)$$

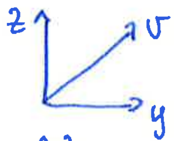
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = \cancel{zy} - 2\cancel{xy} + \cancel{y^2} - \cancel{z^2} + 2\cancel{xz} - \cancel{zy} + \cancel{xz} - 2\cancel{yz} + \cancel{z^2} - \cancel{x^2} + 2\cancel{xy} - \cancel{xz} +$$

$$+ \cancel{xy} - 2\cancel{xz} + \cancel{x^2} - \cancel{y^2} + 2\cancel{yz} - \cancel{xy} = 0$$

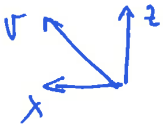
ESERCIZIO 9

1) $f(x,y,z) = x^2 - 3yz + 5 \quad P(1,2,-1)$





$v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\quad\;\; y \quad\; z$



$= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \longrightarrow \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$?

2)

$z = ax + by + c$ dove $a, b =$ direz. di $\vec{v}_N$

$$\nabla f(P) \begin{cases} \frac{\partial f}{\partial x}(P) = 2x = 2 \cdot 1 = 2 \\ \frac{\partial f}{\partial y}(P) = 2y = -4 \end{cases} \rightarrow \vec{v}_N = (2, -4, -1)$$

$z = 2x - 4y + c$ sostituisco P e calcolo c

$5 = 2 + 8 + c \rightarrow c = -5$

$z = 2x - 4y - 5$

5) $f(x,y) = e^{x^2-y^2} + \sqrt{1+x^2+y^4}$ in $P(1, 0, f(1,0)) = (1, 0, e + \sqrt{2})$

$$\nabla f(P) \begin{cases} \frac{\partial f}{\partial x}(P) = 2x\, e^{x^2-y^2} + \frac{1}{2} \frac{1}{\sqrt{1+x^2+y^4}} \cdot 2x = 2e + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot 2 = \\ \qquad = 2e + \frac{1}{\sqrt{2}} \\ \frac{\partial f}{\partial y}(P) = -2y\, e^{x^2-y^2} + \frac{1}{2} \frac{1}{\sqrt{}} \cdot 4y^3 = 0 \end{cases}$$

$z = ax + by + c \rightarrow z = (2e + \frac{1}{\sqrt{2}})x + c$

$e + \sqrt{2} = 2e + \frac{1}{\sqrt{2}} + c$

$-e + \sqrt{2} - \frac{1}{\sqrt{2}} = -e + \frac{1}{\sqrt{2}} = c$

$z = (2e + \frac{1}{\sqrt{2}})x - e + \frac{1}{\sqrt{2}}$ FORSE CALCOLI SBAGLIATI

ESERCIZIO 2 TAYLOR

1) $f(x,y) = \cos x + \cos y$ $P(0,0)$

$\cancel{T(x,y) = f(0,0) + \nabla f(P) \cdot (P - P_0) + \frac{1}{2} Hf(P_0)(P-P_0) +}$

$$T(x,y) = f(0,0) + \frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial f}{\partial y}(P)(y - y_0) + \frac{1}{2}\Big[\frac{\partial^2 f}{\partial x^2}(P)(x-x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(P)(x-x_0)(y-y_0) + \frac{\partial^2 f}{\partial y^2}(P)(y-y_0)^2\Big] + o((x-x_0)^2 + (y-y_0)^2)$$

10

$f(0,0) = 1$

$\frac{\partial f}{\partial x} = e^{x+y} = 1$ $\qquad \frac{\partial^2 f}{\partial x^2} = 1$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = 1$

$\frac{\partial f}{\partial y} = 1$ $\qquad \frac{\partial^2 f}{\partial y^2} = 1$

$$T(x,y) = 1 + x + y + \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + xy$$

6) $f(x,y) = e^{xy}$

$f(0,0) = 1$

$\frac{\partial f}{\partial x} = y e^{xy} = 0$ $\qquad \frac{\partial^2 f}{\partial x^2} = y^2 e^{xy} = 0$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = e^{xy} + yxe^{xy} = 1$

$\frac{\partial f}{\partial y} = x e^{xy} = 0$ $\qquad \frac{\partial^2 f}{\partial y^2} = 0$

$$\rightarrow T(x,y) = 1 + xy$$

7) $f(x,y) = e^x \sin y$

$f(0,0) = 0$

$\frac{\partial f}{\partial x} = \sin y \, e^x = 0$ $\qquad \frac{\partial^2 f}{\partial x^2} = \sin y \, e^x = 0$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = +\cos y \, e^x = 1$

$\frac{\partial f}{\partial y} = \cos y \, e^x = 1$ $\qquad \frac{\partial^2 f}{\partial y^2} = -\sin y \, e^x = 0$

$$\rightarrow T(x,y) = y(1+x) = y + xy$$

ESERCIZIO 3

1) $f(x,y) = \frac{1}{2+x-2y}$ $\quad P(2,1)$

$f(P) = \frac{1}{4-2} = \frac{1}{2}$

$\frac{\partial f}{\partial x}(P) = -\frac{1}{(2+x-2y)^2}\cdot 1 = -\frac{1}{4}$ $\qquad \frac{\partial^2 f}{\partial x^2} = +2\cdot\frac{1}{(2+x-2y)^3} = \frac{1}{4}$

$\frac{\partial f}{\partial y}(P) = +\frac{1}{(2+x-2y)^2}\cdot 2 = \frac{1}{2}$ $\qquad \frac{\partial^2 f}{\partial y^2} = +2\,\frac{1}{(2+x-2y)^3}\cdot 4 = +\frac{8}{8} = 1$

$$\frac{\partial^2 f}{\partial x \partial y} = -2\,\frac{1}{(2+x-2y)^3}\cdot(-2) = \frac{4}{8} = \frac{1}{2}$$

$$T(x,y) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{2}(y-1) + \frac{1}{2}\cdot\frac{1}{4}(x-2)^2 + \frac{1}{2}(y-1)^2 + \frac{1}{2}(x-2)(y-1)$$

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$\frac{\partial \nabla f}{\partial y}$ ... ∂x^2 ... $\partial x \partial y$ ... ∂y^2

$$Hf = \begin{pmatrix} 4 & 2 \\ 2 & 6+6y \end{pmatrix} \rightarrow \det Hf = 24+24y-4 = 20+24y$$

$\det(Hf(P_1)) = 20>0$, $\frac{\partial^2 f}{\partial x^2}(P_1) = 4>0 \rightarrow P_1(0,0)$ MIN

~~$\det(Hf(P_2)) = 20+24(-\frac{5}{3}) = -20<0 \rightarrow P_2(0,-\frac{5}{3})$ SELLA~~

~~$\det(Hf(P_3)) = 20>0 \rightarrow \frac{\partial^2 f}{\partial x^2}>0 \rightarrow P_3(\frac{5}{6},0)$ MIN~~

$\det(Hf(P_4)) = 20-24(\frac{5}{3}) = <0 \rightarrow P_4(\frac{5}{6},-\frac{5}{3})$ SELLA

c) $f(x,y) = \frac{1}{2}(x^3+y^3)-xy$

$\nabla f = 0$

$\frac{\partial f}{\partial x} = \frac{3}{2}x^2-y$

$\frac{\partial f}{\partial y} = \frac{3}{2}y^2-x$

$$\rightarrow \begin{cases} 3x^2 = 2y \rightarrow y = \frac{3}{2}x^2 \;\; \begin{matrix} y=0 \\ y=\frac{2}{3} \end{matrix} \\ 3y^2 = 2x \rightarrow 3(\frac{9}{4}x^4) = 2x \rightarrow x(\frac{27}{4}x^3-2)=0 \;\; \begin{matrix} x=0 \\ x=\frac{2}{3} \end{matrix} \end{cases}$$

$P_1(0,0)$ ~~$P_2(0,\frac{2}{3})$~~ ~~$P_3(\frac{2}{3},0)$~~ $P_4(\frac{2}{3},\frac{2}{3})$

$\frac{\partial^2 f}{\partial x^2} = 3x$ $\quad \frac{\partial^2 f}{\partial x \partial y} = -1 \rightarrow Hf\begin{pmatrix} 3x & -1 \\ -1 & 3y \end{pmatrix} \rightarrow \det(Hf) = 9xy-1$

$\frac{\partial^2 f}{\partial y^2} = 3y$

$\det(Hf(P_1)) = -1 \rightarrow P_1(0,0)$ SELLA

~~$\det(Hf(P_2)) = -1 \rightarrow \det<0 \rightarrow P_2(0,\frac{2}{3})$ SELLA~~

~~$\det(Hf(P_3)) = -1 \rightarrow P_3$ SELLA~~

$\det(Hf(P_4)) = 9\cdot\frac{2}{3}\cdot\frac{2}{3}-1 = 3 \rightarrow \frac{\partial^2 f}{\partial x^2}(P_4) = 2>0 \rightarrow P_4(\frac{2}{3},\frac{2}{3})$ MIN

d) $f(x,y) = xy-x^2-y$

$\nabla f = 0$

$\frac{\partial f}{\partial x} = y-2x$ $\quad \frac{\partial f}{\partial y} = x-1$

$$\begin{cases} x-1=0 \rightarrow x=1 \\ y=2x \rightarrow y=2 \end{cases} \quad P_1(1,2)$$

$\frac{\partial^2 f}{\partial x^2} = -2$ $\quad \frac{\partial^2 f}{\partial x \partial y} = 1 \rightarrow Hf = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \det(Hf) = -1<0$

$\frac{\partial^2 f}{\partial y^2} = 0$

P_1 SELLA

$$\frac{\partial F}{\partial x} = +2x-2+y^2e^{-x}+xe^{-x}$$

$$\frac{\partial F}{\partial y} = xe^{-x}\cdot 2y$$

$$\to \begin{cases} 2x-2+y^2e^{-x}+xe^{-x}=0 \\ xe^{-x}\cancel{2}y=0 \end{cases}$$

$$y^2e^{-x} = 2-2x-xe^{-x}$$

?

MANCA J

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DOPPI

1- $\int_D xy\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ x^2\le y\le 1+x\}$

PER VERT →
$$\int_0^1\left(\int_{x^2}^{1+x} xy\,dy\right)dx = \int_0^1\left(x\frac{y^2}{2}\Big]_{x^2}^{1+x}\right)dx = \int_0^1 x\frac{(1+x)^2}{2} - x\frac{x^4}{2} =$$

$$= \int_0^1 \frac{x}{2}+\frac{x^3}{2}+x^2-\frac{x^5}{2}\,dx = \frac{x^2}{4}+\frac{x^4}{8}+\frac{x^3}{3}-\frac{x^6}{12} = \frac{1}{4}+\frac{1}{8}+\frac{1}{3}-\frac{1}{12} = \cancel{\frac{3}{12}}$$

$$= \frac{3}{8}+\frac{\cancel{3}\,1}{\cancel{12}\,4} = \frac{3+2}{8} = \frac{5}{8}$$

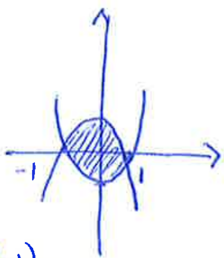
2- $\int$ calcola area di $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 2,\ e^{-x}\le y\le 2-\frac{1}{4}x^2\}$

$$\text{area} = \iint_D dx\,dy = \int_0^2\left(\int_{e^{-x}}^{2-\frac{1}{4}x^2} dy\right)dx = \int_0^2 2-\frac{1}{4}x^2-e^{-x}\,dx =$$

$$= 2x-\frac{x^3}{12}+e^{-x}\Big|_0^2 = 4-\frac{\cancel{8}\,2}{\cancel{12}\,3}+e^{-2}-1 = 3+e^{-2}-\frac{2}{3} = \frac{7}{3}+e^{-2}$$

3- $\int(3y+e^x)\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid x^2-1\le y\le 1-x^2\}$

$$= \int_{-1}^1\left(\int_{x^2-1}^{1-x^2} 3y+e^x\,dy\right)dx = \int_{-1}^1\left(\frac{3}{2}y^2+e^x y\Big]_{x^2-1}^{1-x^2}\right)dx =$$



$$= \int_{-1}^1 \cancel{\frac{3}{2}(1-x^2)^2}+e^x(1-x^2)-\cancel{\frac{3}{2}(x^2-1)^2}-e^x(x^2-1) =$$

$$= \int_{-1}^1 e^x-e^x x^2-e^x x^2+e^x\,dx = \int_{-1}^1 2e^x-2e^x x^2\,dx = 2e^x\Big|_{-1}^1 - \int_{-1}^1 2e^x x^2\,dx$$

$$= 2e-2e^{-1}-\left[2e^x x^2\Big|_{-1}^1-\int_{-1}^1 2e^x\,2x\,dx\right] = \cancel{2e}-\cancel{2e^{-1}}-\cancel{2e}+\cancel{2e^{-1}}+\int_{-1}^1 2e^x\,2x\,dx =$$

$$= 4e^x x\Big|_{-1}^1-\int_{-1}^1 4e^x\,dx = \cancel{4e}+4e^{-1}-\cancel{4e}+4e^{-1} = 8e^{-1}$$

$\iint_D (x-1)e^{x^2+y}\,dxdy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 2,\ -2x+1\le y\le -x^2+1\}$

$$\to \int_0^2 \left(\int_{-2x+1}^{-x^2+1} (x-1)\,e^{x^2+y}dy\right)dx = \int_0^2 \left(xe^{x^2+y} - e^{x^2+y}\right]_{-2x+1}^{-x^2+1} dx =$$

$$= \int_0^2 x\,e^{\cancel{x^2}-\cancel{x^2}+1} - e^{\cancel{x^2}-\cancel{x^2}+1} - xe^{x^2-2x+1} + e^{x^2-2x+1}\,dx =$$

$$= \int_0^2 xe - e \cancel{- x e^{x^2-2x+1}}\ (1-x)\,e^{x^2-2x+1}dx =$$

$$= \frac{x^2}{2}e - ex \Big|_0^2 + \cancel{\int_1^1 -\frac{e^t}{2}dt} \to 0 =$$

chiamo $x^2-2x+1=t$

$dt = 2x-2\,dx$

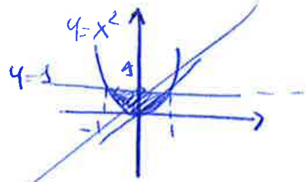
$dx = -\frac{dt}{2}$

se $x\to 2,\ t\to 1$

$x\to 0,\ t\to 1$

$$= \frac{4}{2}e - 2e = 2e - 2e = 0$$

7. $\iint_D |y-x|\,dxdy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid -1\le x\le 1,\ x^2\le y\le 1\}$



PROVO PER VERTICALI

$$\int_{-1}^1 \left(\int_{x^2}^1 |y-x|\,dy\right)dx$$

$|y-x|$

se $y-x\ge 0 \to y\ge x$

se $\le 0 \to x-y\le 0 \to y\ge x$

se $y>0$ e $x>0 \to (y-x) \to \int_{-1}^1 \frac{y^2}{2} - xy \Big|_{x^2}^1 dx =$

$$= \int_{-1}^1 \frac{1}{2} - x - \frac{x^4}{2} + x^3\,dx =$$

$$= \int_{-1}^1 \frac{1}{2}x - \frac{x^2}{2} - \frac{x^5}{10} + \frac{x^4}{4}\Big|_{-1}^1 = \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{10}} + \tfrac{1}{4} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{10}} + \tfrac{1}{4} = \frac{1}{2}$$ NO

oppure

$$\int_{-1}^0 \left(\int_{x^2}^1 x-y\,dy\right)dx + \int_0^1 \left(\int_{x^2}^1 y-x\,dy\right)dx =$$

$$= \int_{-1}^0 \left(xy - \frac{y^2}{2}\Big|_{x^2}^1 = \int_{-1}^0 x - \frac{1}{2} - x^3 + \frac{x^4}{2}\,dx = \frac{x^2}{2} - \frac{1}{2}x - \frac{x^4}{4} + \frac{x^5}{10}\Big|_{-1}^0 =$$

NO

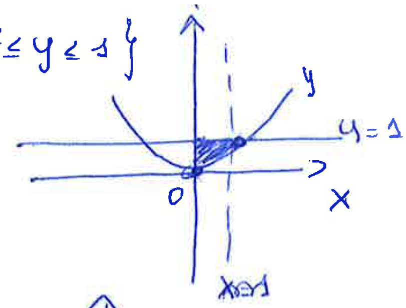
$$= -\frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{10} = -1 + \frac{1}{4} + \frac{1}{10} = \frac{-20+5+2}{20} = -\frac{13}{20} + \int_0^1 \left(\int_{x^2}^1 y-x\,dy\right)dx =$$

$$= -\frac{13}{20} + \int_0^1 \frac{y^2}{2} - xy\Big|_{x^2}^1 = \int_0^1 \frac{1}{2} - x - \frac{x^4}{2} + x^3\,dx$$

$$= -\frac{13}{20} + \left(\frac{1}{2}x - \frac{x^2}{2} - \frac{x^5}{10} + \frac{x^4}{4}\Big|_0^1\right) = -\frac{13}{20} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} - \frac{1}{10} + \frac{1}{4} = \frac{-13-2+5}{20} = -\frac{10}{20} = -\frac{1}{2}$$

3

8) $\int_D |x-y| \, dx\,dy \qquad D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x^2 \le y \le 1\}$

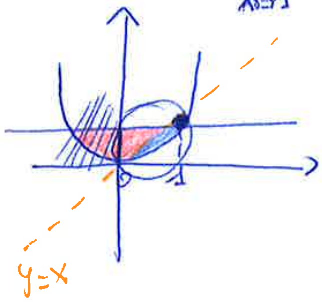


$x - y \ge 0 \to x - y$

$y \le x$

$x - y \le 0 \to -x + y$

$y \ge x$



$D = D_1 + D_2$

$D_1 = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x \le y \le 1\}$

$D_2 = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x^2 \le y \le x\}$

$$\to \int_0^1 \left(\int_x^1 -x + y \, dy \right) dx + \int_0^1 \left(\int_{x^2}^x x - y \, dy \right) dx =$$

$$= \int_0^1 \left(-xy + \frac{y^2}{2} \Big|_x^1 \right) dx + \int_0^1 \left(xy - \frac{y^2}{2} \Big|_{x^2}^x \right) dx =$$

$$= \int_0^1 -x + \frac{1}{2} + x^2 - \frac{x^2}{2} \, dx + \int_0^1 x^2 - \frac{x^2}{2} - x^3 + \frac{x^4}{2} \, dx =$$

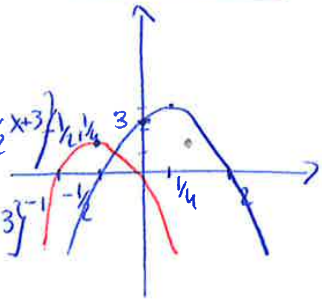
$$= \int_0^1 \frac{x^4}{2} - x^3 + x^2 - x + \frac{1}{2} \, dx = \frac{x^5}{10} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x \Big|_0^1 =$$

$$= \frac{1}{10} - \frac{1}{4} + \frac{1}{3} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} = \frac{2-5}{20} + \frac{1}{3} = -\frac{3}{20} + \frac{1}{3} = \frac{-9+20}{60} = \frac{11}{60}$$

$\int_D x\,dx\,dy$

$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 0,\ -x^2 - x \le y \le -x^2 + \tfrac{1}{2}x + 3\}$$

$$D_2 = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ -x^2 + 2x \le y \le -x^2 + \tfrac{1}{2}x + 3\}$$



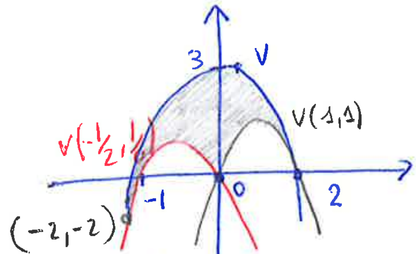
$-\frac{b}{2a} = \frac{1}{2} : (2) = \frac{1}{4}$

$-2x^2 + 1x + 6 = y$

$\Delta = 1 + 2 \cdot 6 \cdot 4 =$
$= 1 + 48 = 49$

$-\frac{\Delta}{4a} = \frac{+49}{+8}$

$$= \int_{D_1} x^2\,dx\,dy + \int_{D_2} x^2\,dx\,dy$$



$$D_1 = \int_{-2}^{0} \left(\int_{-x^2-x}^{-x^2+\frac{1}{2}x+3} x^2\,dy \right) dx = \int_{-2}^{0} \left(x^2(-x^2 + \tfrac{1}{2}x + 3) - x^2(-x^2 - x) \right) dx =$$

$$= \int_{-2}^{0} -x^4 + \tfrac{1}{2}x^3 + 3x^2 + x^4 + x^3\,dx = \left[-\frac{x^5}{5} + \frac{x^4}{8} + x^3 + \frac{x^5}{5} + \frac{x^4}{4} \right]_{-2}^{0} = -\cancel{\frac{32}{5}} - \frac{16}{8} + 8 + \cancel{\frac{32}{5}} - \frac{16}{4}$$

$$= -2 + 8 - 4 = +2$$

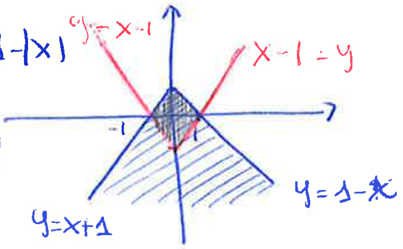
$$D_2 = \int_0^2 \left(\int_{-x^2+x\cdot 2}^{-x^2+\frac{1}{2}x+3} x^2\,dy \right) dx = \int_0^2 \left(x^2(-x^2 + \tfrac{1}{2}x + 3) - x^2(-x^2 + 2x) \right) dx = \int_0^2 -\cancel{x^4} + \tfrac{1}{2}x^3 + 3x^2 + \cancel{x^4} - 2x^3\,dx$$

$$= \left[\frac{x^4}{8} + x^3 - \frac{x^4}{2} \right]_0^2 = \frac{16}{8} + 8 - \frac{16}{2} = 2 + \cancel{8} - 8 = 2$$

$$D_1 + D_2 = 2 + 2 = 4 \quad ✓$$

$$|y| \leq 1 - |x| \}$$

$$-1+|x| \leq y \leq 1-|x|$$



$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid -1 < x \leq 0, -x-1 \leq y \leq x+1\}$$

$$\longrightarrow D_2 = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, x-1 \leq y \leq 1-x\}$$

$$\int_D = \int_{D_1} + \int_{D_2} \rightarrow \int_D x^2y^2\,dx\,dy = \int_{D_1} x^2y^2\,dx\,dy + \int_{D_2} x^2y^2\,dx\,dy = (1)+(2)$$

(1) $$\int_{-1}^{0}\left(\int_{-x-1}^{x-1} x^2y^2\,dy\right)dx = \int_{-1}^{0}\left[x^2\frac{y^3}{3}\right]_{-x-1}^{x+1} dx = \int_{-1}^{0} x^2\frac{(x+1)^3}{3} + x^2\frac{(x+1)^3}{3}\,dx =$$

$$= \frac{2}{3}\int_{-1}^{0} x^2(x^3+1+3x+3x^2)\,dx = \frac{2}{3}\int_{-1}^{0} x^5+x^2+3x^3+3x^4\,dx =$$

$$= \frac{2}{3}\left[\frac{x^6}{6}+\frac{x^3}{3}+\frac{3}{4}x^4+\frac{3}{5}x^5\right]_{-1}^{0} = \frac{2}{3}\left(-\frac{1}{6}+\frac{1}{3}-\frac{3}{4}+\frac{3}{5}\right) = -\frac{1}{9}+\frac{2}{9}-\frac{3}{6}+\frac{2}{5} =$$

$$= \cancel{\frac{-37}{93}\ \frac{3}{6}-\frac{2}{5} = \frac{-10-15-12}{30} = -\frac{37}{30}} \rightarrow \frac{1}{9}-\frac{1}{2}+\frac{2}{5} = \frac{10-45+36}{90} = \frac{1}{90}$$

(2) $$\int_{0}^{1}\left(\int_{x-1}^{1-x} x^2y^2\,dy\right)dx = \int_{0}^{1}\left[x^2\frac{y^3}{3}\right]_{x-1}^{1-x} dx = \int_{0}^{1} x^2\frac{(1-x)^3}{3} + x^2\frac{(-x+1)^3}{3}\,dx =$$

$$= \frac{2}{3}\int_{0}^{1} x^2(1-x)^3\,dx = \frac{2}{3}\int_0^1(-x^3+1-3x+3x^2)x^2\,dx = \frac{2}{3}\int_0^1 -x^5+x^2-3x^3+3x^4\,dx$$

$$= \frac{2}{3}\left[-\frac{x^6}{6}+\frac{x^3}{3}-\frac{3}{4}x^4+\frac{3}{5}x^5\right]_0^1 = \frac{2}{3}\left(-\frac{1}{6}+\frac{1}{3}-\frac{3}{4}+\frac{3}{5}\right) = -\frac{1}{9}+\frac{2}{9}-\frac{1}{2}+\frac{2}{5} =$$

$$= \frac{1}{9}-\frac{1}{2}+\frac{2}{5} = \frac{1}{9}+\frac{-5+4}{10} = \frac{1}{9}-\frac{1}{10} = \frac{10-9}{90} = \frac{1}{90}$$

$\longrightarrow$ quindi $\frac{1}{90}+\frac{1}{90} = \frac{2}{90} = \frac{1}{45}$ ✓

$$\cancel{\int_{-1}^{0}\frac{x^2}{3}\left[x^3+1-3x+3x+x^3+1+3x+3x\right]dx}$$

$$\cancel{= \int_{-1}^{0}\frac{x^2}{3}\left[2x^3+6x\right]dx = \int_{-1}^{0}\frac{x^6}{3}\cdot 2+2x^3\,dx = \frac{2}{3}\left[\frac{x^7}{7}+\frac{x^4}{2}\right]_{-1}^{0} =}$$

$$\cancel{= +\frac{2}{3}\cdot\frac{1}{7}+\frac{1}{2} = \frac{2}{21}+\frac{1}{2} = \frac{4-21}{42} = -\frac{17}{42}}$$

8

4) $\int_D \frac{[illegible]}{x^2+y^2}\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid y\geq x,\ 1\leq x^2+y^2\leq 4\}$

$$\begin{cases} x=\rho\cos\vartheta \\ y=\rho\sin\vartheta \end{cases} \qquad \begin{matrix} 1\leq\rho\leq 2 \\ \frac{\pi}{4}\leq\vartheta\leq\frac{5}{4}\pi \end{matrix}$$

$$= \int_1^2 \rho^2 d\rho \int_{\pi/4}^{5/4\pi} \cos\vartheta\sin^2\vartheta\, d\vartheta = \frac{\rho^3}{3}\Big|_1^2 \left[\int_{\pi/4}^{5/4\pi}\cos\vartheta\, d\vartheta - \int_{\pi/4}^{5/4\pi}\cos^3\vartheta\, d\vartheta\right] =$$

$$= \left(\frac{8}{3}-\frac{1}{3}\right)\left[\sin\vartheta\right]_{\pi/4}^{5/4\pi} - \frac{7}{3}\left[\cancel{\vartheta\cos^3\vartheta - \int\vartheta\cdot 3\cos^2\vartheta(-\sin\vartheta)d\vartheta}\right] =$$

$$= \frac{7}{3}(-\sqrt{2}) - \frac{7}{3}\left[\cancel{\tfrac{3}{8}}\,\sin\vartheta\cos^2\vartheta\Big|_{\pi/4}^{5/4\pi} + \int\sin\vartheta\, 2\sin\vartheta\cos\vartheta\, d\vartheta\right] =$$

$a=\frac{\pi}{4} \quad b=\frac{5}{4}\pi \qquad \frac{d}{d\vartheta}(\cos^2\vartheta = 1-\sin^2\vartheta) = \frac{d}{d\vartheta} \Rightarrow -2\sin\vartheta\cos\vartheta$

$$= -\frac{7\sqrt{2}}{3} - \frac{7}{3}\left[-\frac{\sqrt{2}}{2}\left(\frac{2}{4}\right) - \frac{\sqrt{2}}{2}\cdot\frac{2}{4} + 2\int_a^b \sin^2\vartheta\cos\vartheta\, d\vartheta\right] =$$

$$= -\frac{7\sqrt{2}}{3} + \left(\frac{7\sqrt{2}}{12} + \frac{7\sqrt{2}}{12} - \frac{7}{3}2\left[\sin^2\vartheta(\sin\vartheta) - \int_a^b \sin\vartheta(-2\sin\vartheta\cos\vartheta)d\vartheta\right]\right)$$

$$\sin\vartheta\cos^2\vartheta + \int 2\sin^2\vartheta\cos\vartheta\, d\vartheta = \sin\vartheta\cos^2\vartheta + 2\sin^3\vartheta + 4\int\sin^2\vartheta\cos\vartheta\, d\vartheta$$

quindi $\int\sin^2\vartheta\cos\vartheta\, d\vartheta = \frac{1}{3}(\cancel{\sin\vartheta\cos^2\vartheta} - \cancel{\sin\vartheta\cos^2\vartheta} + 2\sin^3\vartheta)$

$$\rightarrow \boxed{\int\cos^3\vartheta\, d\vartheta = \sin\vartheta\cos^2\vartheta + \frac{2}{3}\sin^3\vartheta}$$

$$\Rightarrow -\frac{7}{3}\sqrt{2} - \frac{7}{3}\left(\sin\vartheta\cos^2\vartheta + \frac{2}{3}\sin^3\vartheta\right]_{\pi/4}^{5/4\pi}\Big) = -\frac{7}{3}\sqrt{2} - \frac{7}{3}\left[-\frac{\sqrt{2}}{2}\left(\frac{2}{4}\right) + \frac{2}{3}\left(-\frac{\sqrt{2}}{2}\right)^3 - \right.$$

$$\left. + \frac{\sqrt{2}}{2}\cdot\frac{2}{4} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] =$$

PER UNA SOLUZIONE NON-IDIOTA CFR. PAG 15

$$= -\frac{7}{3}\sqrt{2} - \frac{7}{3}\left(-\frac{\sqrt{2}}{4} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \frac{\sqrt{2}}{4} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] =$$

$$= -\frac{7}{3}\sqrt{2} + \frac{7}{3}\frac{\sqrt{2}}{2} + \frac{4}{3}\frac{7}{3}\sqrt{2} = \frac{-42\sqrt{2}+21\sqrt{2}+56\sqrt{2}}{18} = \frac{45}{18}\sqrt{2}$$

56 − 42 = 14 + 21 = 45

$$= \frac{15}{6}\sqrt{2} = \frac{5}{2}\sqrt{2}$$

BASTA

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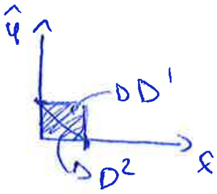
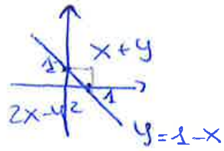
3) $\int_D f(x,y)\,dxdy$ $\qquad D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ 0\le y\le 1\} = [0,1]\times[0,1]$

$\begin{bmatrix}0\\1\end{bmatrix}^{\begin{bmatrix}0 & 1\end{bmatrix}}$ $0\ 1 \to 1$



$\hat{y}\times\hat{y}$

$$f(x)=\begin{cases}2x-y^2 & \text{se } x+y<1\\ x+y & \text{se } x+y\ge 1\end{cases}$$



$$\begin{cases}\int_{D'} x+y\,dxdy \to \int_0^1\left(\int_{1-x}^{1} x+y\,dy\right)dx & (1)\\ \int_{D''} 2x-y^2\,dxdy \to \int_0^1\left(\int_0^{1-x} 2x-y^2\,dy\right)dx & (2)\end{cases}$$

$$(1)\to \int_0^1\left(xy+\frac{y^2}{2}\Big|_{1-x}^{1}\right)dx = \int_0^1 \cancel{x}+\frac{1}{2}-\cancel{x}+x^2-\frac{(1-x)^2}{2}\,dx =$$

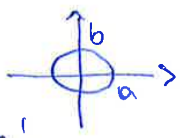
$$=\int_0^1 \cancel{\tfrac{1}{2}}+x^2-\cancel{\tfrac{1}{2}}-\frac{x^2}{2}+x\,dx = \frac{1}{6}x^3+\frac{x^2}{2}\Big|_0^1=\frac{1}{6}+\frac{1}{2}=\frac{4}{6}=\frac{2}{3}$$

$$(2)\ \int_0^1\left(2xy-\frac{y^3}{3}\Big|_0^{1-x}\right)dx=\int_0^1 2x-2x^2-\frac{(1-x)^3}{3}\,dx=\int_0^1 2x-2x^2-\frac{1}{3}+\frac{x^3}{3}+x-\cancel{3}\frac{x^2}{\cancel{3}}\,dx=$$

$$=\int_0^1 3x-\frac{1}{3}-3x^2+\frac{x^3}{3}\,dx=\frac{3}{2}-\frac{1}{3}-1+\frac{1}{12}=\frac{18-4-12+1}{12}=\frac{3}{12}$$

$$\frac{3}{12}+\frac{2}{3}=\frac{3+8}{12}=\frac{11}{12}\ \checkmark$$

4) Area di $D=\{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$



$$\text{Area}=\iint_D dxdy = \int_0^1\int_0^{2\pi} ab\rho\,d\vartheta\,d\rho=\int_0^1 ab\rho\,2\pi\,d\rho = ab\pi$$

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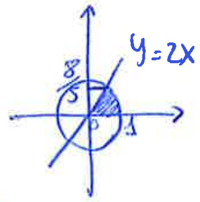
con $C=(0,1)$ e $R=1$

10) $\int_D x\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le 2x,\ x^2+(y-1)^2\le 1\}$



$x^2+y^2-2y\le 0 \rightarrow x^2\le 2y-y^2 \qquad x=\frac{y}{2}$

$$\int_0^b\int_{y/2}^{\sqrt{2y-y^2}} x\,dx\,dy=\int_0^b \frac{x^2}{2}\Big|_{y/2}^{\sqrt{\ }}dy=\int_0^b \frac{2y-y^2}{2}-\left(\frac{y}{2}\right)^2\cdot\frac{1}{2}\,dy$$

Punto b: $\frac{y}{2}=\sqrt{-y^2+2y}$

$\frac{y^2}{4}=-y^2+2y \rightarrow y^2=-4y^2+8y$

$5y^2-8y=0$

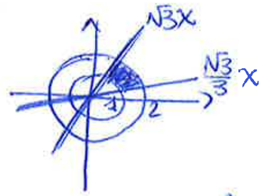
$y_1=0 \qquad y_2=\frac{8}{5}$

Per conti. $\rightarrow \int_0^{8/5}\frac{2y-y^2}{2}-\frac{y^2}{8}\,dy=\frac{y^2}{2}-\frac{y^3}{6}-\frac{y^3}{24}\Big|_0^{8/5}=\frac{64}{25}\cdot\frac{1}{2}-\frac{\cancel{512}\,256}{125}\cdot\frac{1}{\cancel{6}\,3}-\frac{\cancel{512}\,\cancel{128}\,64}{125}\cdot\frac{1}{\cancel{24}\,\cancel{6}\,3}=$

$=\frac{64}{50}-\frac{256}{125\cdot 3}-\frac{64}{125\cdot 3}=\frac{32}{75}$

12) $\int_D |x|\,y\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 4,\ \frac{\sqrt{3}}{3}x\le y\le\sqrt{3}x\}$

uso le coordinate polari



$\begin{cases} x=\rho\cos\vartheta \\ y=\rho\sin\vartheta \end{cases} \qquad |\det J\Phi|=\rho$

$1\le\rho\le 2, \qquad \operatorname{tg}\frac{\sqrt{3}}{3}\le\vartheta\le\operatorname{tg}\sqrt{3}$

$\frac{\pi}{6}\le\vartheta\le\frac{\pi}{3}$

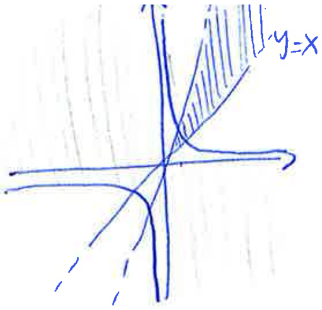
sono nel 1° quadrante $\rightarrow$ tolgo valore assoluto

$$\int_{D'}\rho\cos\vartheta\,\rho\sin\vartheta\,\rho\,d\rho\,d\vartheta=\int_{\pi/6}^{\pi/3}d\vartheta\int_1^2\rho^3\cos\vartheta\sin\vartheta\,d\rho=$$

$$=\int_{\pi/6}^{\pi/3}\cos\vartheta\sin\vartheta\,d\vartheta\ \frac{\rho^4}{4}\Big|_1^2=\left(\frac{16}{4}-\frac{1}{4}\right)=-\frac{\cos^2\vartheta}{2}\Big|_{\pi/6}^{\pi/3}\cdot\frac{15}{4}$$

$$=\left[-\left(\frac{1}{2}\right)^2\cdot\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2\right]\frac{15}{4}=\left[-\frac{1}{8}+\frac{3}{8}\right]\cdot\frac{15}{4}=\frac{\cancel{2}}{8}\cdot\frac{15}{\cancel{4}\,2}=\frac{15}{16}$$

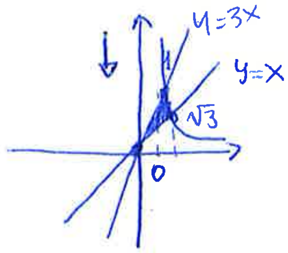
$\iint_D x^2 e^{xy}\,dxdy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid x\le y\le 3x,\ xy\le 3\}$



intersezioni: $y \le \frac{3}{x}$

$\frac{3}{x} = x \to 3 = x^2 \to x = \pm\sqrt{3}$

$\frac{3}{x} = 3x \to 1 = x^2 \to x = \pm 1$



$D_1 = \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ x\le y\le 3x\}$

$D_2 = \{(x,y)\in\mathbb{R}^2 \mid 1\le x\le \sqrt{3},\ x\le y\le \frac{3}{x}\}$

$\to \iint_D x^2 e^{xy}\,dxdy = \iint_{D_1} x^2 e^{xy}\,dxdy + \iint_{D_2} x^2 e^{xy}\,dydy$ (1) (2)

(1) $\int_0^1 \left(\int_x^{3x} x^2 e^{xy}\,dy\right)dx = \int_0^1 x^2 dx \cdot \int_x^{3x} e^{xy}\,dy = \int_0^1 x^2 dx \cdot \left[\frac{e^{xy}}{x}\cdot x^2\right]_x^{3x} =$

$= \int_0^1 x^2 dx \cdot (\cancel{3}x e^{3x^2} - e^{x^2}\cdot x) = \int_0^1 x^3 e^{3x^2} - x^3 e^{x^2}\,dx =$

$= \int_0^1 x^3 e^{3x^2}dx - \int_0^1 x^3 e^{x^2}dx = x^3\frac{e^{3x^2}}{6x}\Big|_0^1 - \int_0^1 \cancel{3}x^2 \cdot \frac{e^{3x^2}}{{}_2\cancel{6x}}\,dx$ → per parti

$= \frac{e^3}{6} - \int_0^1 \frac{xe^{3x^2}}{2}dx = \frac{e^3}{6} - \left[\cancel{x}\frac{e^{3x^2}}{6\cancel{x}}\Big|_0^1 - \int_0^1 \frac{e^{3x^2}}{6x}dx\right] = \frac{e^3}{6} - \frac{e^3}{6} + \frac{1}{6} - \int_0^1 \frac{e^{3x^2}}{6x}dx$

chiamo $t = e^{3x^2}$

$dt = \frac{e^{3x^2}}{6x}dx$

$dt = \frac{t}{6x}dx$

$\frac{dt}{t} = \frac{dx}{6x}$

~~chiamo $t = 3x^2$~~

~~$dt = 6x\,dx$~~

se $x \to 1,\ t \to e^3$

$x \to 0,\ t \to 1$

$= \frac{1}{6} - \int_1^{e^3} \cancel{t}\cdot\frac{dt}{\cancel{t}} = \frac{1}{6} - t\Big|_1^{e^3} =$

$= \frac{1}{6} - e^3 + 1 = \frac{7}{6} - e^3 - \int_0^1 x^3 e^{x^2}dx =$

5

Foglio 3: integrali tripli

1. $\int_\Omega x^2 + y^2 \, dxdydz$, dove Ω è il cubo $[-1,1] \times [-1,1] \times [-1,1]$. [16/3]

2. PROF Sia $\Omega = \left\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2} \leq z \leq 1\right\}$. Calcolare $\int_\Omega z \, dxdydz$ a) per fili, b) per strati, c) passando alle coordinate cilindriche.

3. Calcolare $\int_\Omega z \, dxdydz$, dove Ω è il tetraedro di vertici $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ e $(0,0,1)$. [1/24]

4. PROF $\int_\Omega \frac{dxdydz}{(x+y+z+1)^3}$, dove Ω è il tetraedro delimitato dai piani coordinati e dal piano $x+y+z=1$. [$\log 2/2 - 5/16$]

5. $\int_\Omega \sqrt{x^2+y^2} \, dxdydz$, dove $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2} \leq z \leq 1\}$. [$\pi/6$]

6. $\int_\Omega (6x - 2y) \, dxdydz$, $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 5,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$. [$25\pi/4$]

7. PROF $\int_\Omega \frac{6y^2 z}{x^2+y^2} \, dxdydz$, $\Omega = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 2,\ \sqrt{x^2+y^2} \leq z\right\}$. [$3\pi/2$]

8. $\int_\Omega x^2 \, dxdydz$, $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1,\ z \geq \sqrt{x^2+y^2}\}$. [$\frac{\pi}{5}\left(\frac{8-5\sqrt{2}}{12}\right)$]

9. PROF Calcolare il volume di $\Omega = \left\{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{4} + y^2 \leq 1,\ 1 \leq z \leq 12 - xy\right\}$. [$22\pi$]

10. $\int_\Omega x^2 + y^2 \, dxdydz$. $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1,\ z \geq 0\}$. [$4\pi/15$]

11. Calcolare le coordinate del baricentro di una lamina omogenea (densità = 1) la cui forma è il quarto di ellisse dato dalle condizioni $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, $x \geq 0$ e $y \geq 0$. [$(4a/3\pi, 4b/3\pi)$]

12. Consideriamo il settore circolare descritto in coordinate polari da $\rho \leq 1$, $0 \leq \theta \leq \alpha$. Assumiamo che le densità di massa del settore e del segmento siano costanti e uguali a 1. Per $\alpha \to 0$, il settore "tende" al segmento $\rho \leq 1$, $\theta = 0$. Mostrare che invece il baricentro del settore *non* tende al baricentro del segmento.

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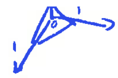
$\int_\Omega z\,dxdydz$ $\quad\Omega =$ tetraedro



per fili $\to \Omega = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D_z,\ \alpha \le z \le \beta\}$

$D_z =$ 

$\alpha = 0$

$\beta = 1-x-y$

$\int \frac{(a+y)^2}{2} = -\frac{(a+y)^3}{6}$

$$\to \int_{D_z} \left(\int_0^{1-x-y} z\,dz \right) dxdy = \int_{D_z} \frac{(1-x-y)^2}{2}\,dxdy$$

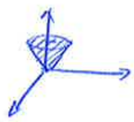
$$D_z = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ y \le 1-x\}$$

$$\to \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\,dydx = \int_0^1 -\frac{(1-x-y)^3}{6}\Big|_0^{1-x} dx =$$

$$= \int_0^1 \cancel{-\frac{(1-x-1+x)^3}{6}} + \frac{(1-x)^3}{6}\,dx = +\int_0^1 \frac{(1-x)^3}{6}\,dx = -\frac{(1-x)^4}{24}\Big|_0^1 =$$

$$= 0 + \frac{1}{24} = \frac{1}{24}$$

5) $\int_\Omega \sqrt{x^2+y^2}\,dxdydz \qquad \Omega = \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2} \le z \le 1\}$



~~per fili~~ per cilindriche $\to \rho, \vartheta, z$

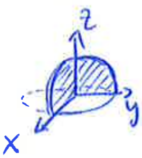
$$0 \le \rho \le 1, \quad 0 \le \vartheta \le 2\pi, \quad \rho \le z \le 1$$

$$\int_0^{2\pi}\int_0^1\int_\rho^1 \rho \cdot \rho\,dz\,d\rho\,d\vartheta = 2\pi \cdot \int_0^1 (\rho^2 z)\Big|_\rho^1 d\rho =$$

$$= 2\pi \int_0^1 \rho^2 - \rho^3\,d\rho = 2\pi\left(\frac{\rho^3}{3} - \frac{\rho^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6\cancel{12}} \cdot \cancel{2}\pi = \frac{\pi}{6}$$

6) $\int_\Omega (6x - 2y)\,dxdydz \qquad \Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 5,\ x \ge 0, y \ge 0, z \ge 0\}$

coordinate sferiche: ρ, ϑ, φ



$$0 \le \rho \le \sqrt{5}$$
$$0 \le \vartheta \le \frac{\pi}{2}$$
$$0 \le \varphi \le \frac{\pi}{2}$$

$$\to \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{5}} \rho^2 \sin\varphi \,(6\rho\sin\varphi\cos\vartheta - 2\rho\sin\varphi\sin\vartheta)\,d\rho\,d\vartheta\,d\varphi$$

$$= \iiint 6\rho^3\sin^2\varphi\cos\vartheta - 2\rho^3\sin^2\varphi\sin\vartheta\,d\rho\,d\vartheta\,d\varphi = \iiint \cancel{2\rho^3(3\cos\vartheta)} = \frac{25}{2}\cdot\frac{\pi}{4}\cdot\left[\sin\vartheta\cdot 3\Big|_0^{\pi/2} + \cos\vartheta\Big|_0^{\pi/2}\right] = \frac{25\pi}{4}$$

$$= \iiint 2\rho^3\sin^2\varphi\,(3\cos\vartheta - \sin\vartheta)\,d\rho\,d\vartheta\,d\varphi = \frac{\rho^4}{2}\Big|_0^{\sqrt{5}} \cdot \frac{\pi}{4} \cdot \int_0^{\pi/2} 3\cos\vartheta - \sin\vartheta\,d\vartheta$$

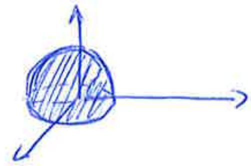
2

w) $\int_\Omega x^2+y^2\, dx\,dy\,dz \qquad \Omega=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 1,\ z\ge 0\}$

$0\le \rho\le 1$

$0\le \vartheta\le 2\pi$

$0\le \varphi\le \pi/2$



$\to \cancel{\int \sqrt{\rho}\cdot\rho^2\sin\varphi\, d\rho\, d\vartheta\, d\varphi = \int_0^{\pi/2}\sin\varphi\, d\varphi \cdot \int_0^{2\pi} d\vartheta \cdot \int_0^1 \rho^{5/2} d\rho = -\cos\varphi\Big|_0^{\pi/2}\cdot 2\pi\cdot\frac{2}{7} =}$

$\cancel{= 1\cdot 2\pi\cdot\frac{2}{7} =}$

ATTENZIONE! NELLE SFERICHE $\rho \to$ DISTANZA PUNTO-ORIGINE $\ne x^2+y^2$! ρ NELLE CILINDRICHE = DIST. PROIEZIONE-O $= x^2+y^2$

$\int (\rho^2\sin^2\varphi\cos^2\vartheta+\rho^2\sin^2\varphi\sin^2\vartheta)\rho^2\sin\varphi\, d\varphi\, d\rho\, d\vartheta =$

$= \int \rho^4\sin^3\varphi\cos^2\vartheta+\rho^4\sin^3\varphi\sin^2\vartheta\, d\ldots = \int \rho^4(\sin^3\varphi(\cos^2\vartheta+\sin^2\vartheta)) =$

$= \frac{\rho^5}{5}\Big|_0^1 \cdot \int_0^{\pi/2}\sin^3\varphi \cdot \vartheta\Big|_0^{2\pi} = \frac{1}{5}\cdot 2\pi\cdot \int_0^{\pi/2}\sin\varphi\, d\varphi =$

$\sin^3\varphi = \sin^2\varphi(\sin\varphi) = (1-\cos^2\varphi)\sin\varphi = \sin\varphi-\sin\varphi\cos^2\varphi$

$= \frac{2}{5}\pi\int \sin\varphi-\sin\varphi\cos^2\varphi = \frac{2}{5}\pi\left(-\cos\varphi\Big|_0^{\pi/2}+\frac{\cos^3\varphi}{3}\Big|_0^{\pi/2}\right) = \frac{2}{5}\pi\left(1-\frac{1}{3}\right) = \frac{2}{5}\pi\cdot\frac{2}{3} = \frac{4}{15}\pi$ ✓

11) BARICENTRO

$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ x\ge 0,\ y\ge 0 \qquad \sigma = \text{densità} = 1$

$x_G = \dfrac{\int x\, dx\,dy}{\int dx\,dy}$

di solito $x_B = \dfrac{\int x\,\sigma(x,y,z)\,dx\,dy\,dz}{\int \sigma(x,y,z)\,dx\,dy\,dz}$

ellittiche

$\begin{cases} x = a\rho\cos\vartheta \\ y = b\rho\sin\vartheta \end{cases}$ Jacob $\to ab\rho$

$0\le\rho\le 1$

$0\le\vartheta\le\pi/2$

$\to \dfrac{\int a\rho\cos\vartheta\cdot ab\rho\, d\rho\, d\vartheta}{\int ab\rho\, d\rho\, d\vartheta} = \dfrac{a^2b\int_0^1\rho^2 d\rho\int_0^{\pi/2}\cos\vartheta\, d\vartheta}{ab\int_0^1\rho\, d\rho\cdot\frac{\pi}{2}} = \dfrac{a^2b\,\frac{1}{3}\cdot 1}{\frac{ab\pi}{4}}$ $\to$ area

$x_B = \frac{a^2b}{3}\cdot\frac{4}{ab\pi} = \frac{4a}{3\pi}$

$y_B = \dfrac{\int y\, dx\,dy}{\text{AREA}} = \frac{ab^2}{3}\cdot\frac{4}{ab\pi} = \frac{4}{3}\frac{b}{\pi}$

Foglio 4: integrali curvilinei

1. Calcolare la lunghezza dell'arco di catenaria dato dal grafico di $f(x) = \cosh x$ per $x \in [-1, 1]$.
[$2\sinh(1)$]

2. Calcolare $\int_\gamma \sqrt{1 + x^2 + 3y}\, ds$, dove γ è l'arco di parabola $f(x) = x^2$ per $0 \le x \le 3$. [39]

3. Calcolare $\int_\gamma x^2 y\, ds$, dove γ è l'arco di circonferenza di equazione $x^2 + y^2 = 1$, situato nel primo e secondo quadrante. [2/3]

4. Calcolare $\int_\gamma z^2\, ds$, dove $\gamma(t) = (\cos t, \sin t, e^t)$, $t \in [0, 2\pi]$. [$\frac{1}{3}\left((1+e^{4\pi})^{3/2} - 2\sqrt{2}\right)$]

5. Calcolare il lavoro compiuto dal campo $F(x, y, z) = (x, xy, xyz)$, lungo la curva $\gamma(t) = (t, t^2, t^3)$, $t \in [0, 1]$, orientata nel senso delle t crescenti. [37/30]

6. Calcolare il lavoro compiuto dal campo $F(x, y) = \left(\frac{x+y}{x^2+y^2}, -\frac{x-y}{x^2+y^2}\right)$ lungo la circonferenza $x^2 + y^2 = 2$, percorsa in senso antiorario. [-2π]

7. Calcolare $\int_\gamma F \cdot dP$, dove $F(x, y) = \left(\frac{1}{|x| + |y|}, \frac{1}{|x| + |y|}\right)$ e γ è il bordo del quadrato di vertici $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ percorso in senso antiorario. [0]

8. Calcolare $\int_\gamma F \cdot dP$, dove $F(x, y) = (-y, x)$ e γ è il bordo, percorso in senso antiorario, dell'insieme $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1,\ y \ge x - 1\}$
[$3\pi/2 + 1$]

4) $\int_\gamma z^2 ds$ $\quad \gamma(t) = (\cos t, \sin t, e^t) \quad t \in [0, 2\pi]$

$d[e^{2x}] = 2e^{2x}$

$f(x,y,z) = (0,0,z^2)$

$f(\gamma(t)) = e^{2t}$

$\gamma'(t) = (-\sin t, +\cos t, e^t) \to \|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t + e^{2t}} = \sqrt{e^{2t}+1}$

$\to \int_0^{2\pi} f(\gamma(t)) \cdot \|\gamma'(t)\| dt = \cancel{\int_0^{2\pi} e^{2t} e^t dt = \int_0^{2\pi} e^{3t} = \frac{e^{4t}}{3}\Big|_0^{2\pi} = \frac{1}{3}(e^{8\pi} - \frac{1}{3})} =$

$= \int_0^{2\pi} e^{2t}(e^{2t}+1)^{1/2} dt = \cancel{\int_0^{2\pi} \frac{1}{2}(e^{2t}+1} \cancel{\int_0^{2\pi} (e^{2t})^{3/2}}$

$\cancel{x = e^{2t}+1}$
$\cancel{dx = 2e^t}$

$\cancel{\frac{1}{2}(x-1)(x)^{1/2} = x^{3/2} - x^{1/2}}$

$(e^{4t}(e^{2t}+1))^{1/2} = (e^{6t} + e^{4t})^{1/2}$ $\quad$ Derivo: $\frac{1}{2} \frac{1}{\sqrt{\ }} \cdot 6e^{5t} \cdot 4e^{3t}$

$x = (e^{2t}+1)^{1/2}$

$dx = \frac{1}{2} \frac{1}{(\ )} \cdot 2e^{2t}$ $\quad \to \int_{\sqrt{2}}^{\sqrt{e^{4\pi}+1}} x \cdot x dx = \frac{x^3}{3}\Big|_{\sqrt{2}}^{\sqrt{\ }} = \frac{1}{3}(e^{4\pi}+1)^{3/2} - \frac{1}{3} 2\sqrt{2} =$

$dx \cdot x = e^{2t} dt$ $\quad \frac{1}{3}[(e^{4\pi}+1)^{3/2} - 2\sqrt{2}]$ ✓

5) W? $\quad F(x,y,z) = (x, xy, xyz)$

$\gamma = (t, t^2, t^3) \quad t \in [0,1]$ $\quad = \int_\gamma F \cdot dP = \int_\gamma F(\gamma(t)) \cdot \gamma'(t) dt$

$F(\gamma(t)) = (t, t^3, t^6)$

$\gamma'(t) = (1, 2t, 3t^2)$ $\quad \to \int_0^1 t + 2t^4 + 3t^8 dt =$

$F(\gamma(t)) \cdot \gamma'(t) = t + 2t^4 + 3t^8$

$= \frac{t^2}{2} + \frac{2}{5}t^5 + \frac{3}{9}t^9 \Big|_0^1 = \frac{1}{2} + \frac{2}{5} + \frac{3}{9} = \frac{45+36+30}{90} =$

$= \frac{111}{90} = \frac{37}{30}$ ✓

6) $F(x,y) = \left(\frac{x+y}{x^2+y^2}, -\frac{x-y}{x^2+y^2} \right)$

$\gamma(t) = (\sqrt{2}\cos t, \sqrt{2}\sin t)$

ANTIORARIO (+)

$F(\gamma(t)) = (\sqrt{2}\cos t + \sqrt{2}\sin t, \sqrt{2}\sin t - \sqrt{2}\cos t) / 2$

$F(\gamma(t)) \cdot \gamma'(t) = \cancel{\beta} - \cancel{4\cos t \sin t} - 2\sin^2 t + \cancel{4\cos t \sin t} - 2\cos^2 t = -2/2 = -1$

$\gamma'(t) = (-\sqrt{2}\sin t, \sqrt{2}\cos t)$

$\int_0^{2\pi} -1 dt = -2\pi$ ✓

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Foglio 5: campi conservativi

1. Stabilire se i campi vettoriali

   a) $F(x, y, z) = (x^2, y, z)$

   b) $G(x, y, z) = (x - xe^z, -y, e^z)$

   sono conservativi in $\mathbb{R}^3$. In caso affermativo, determinarne un potenziale.

   [a) $U(x, y, z) = x^3/3 + y^2/2 + z^2/2$ b) G non è conservativo.]

2. Dato il campo

$$F(x, y) = \left(\frac{(1 - x)(1 + y^2)}{1 + x^2}, \varphi(x)y \right),$$

   determinare tutte le funzioni $\varphi \in C^1(\mathbb{R})$ per cui F è conservativo in $\mathbb{R}^2$.

   [$\varphi(x) = 2 \arctan x - \log(1 + x^2) + C$]

3. Verificare che il campo vettoriale

$$F(x, y, z) = \left(xy - \sin z, \frac{x^2}{2} - \frac{e^y}{z}, \frac{e^y}{z^2} - x \cos z \right)$$

   è conservativo in $D = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ e determinarne i potenziali.

   [$U(x, y, z) = \frac{x^2}{2} y - x \sin z - \frac{e^y}{z} + C$]

4. Verificare che il campo vettoriale $F(x, y) = \left(e^{-y^2}, 1 - 2xye^{-y^2} \right)$ è conservativo in $\mathbb{R}^2$ e determinarne un potenziale.

   [$U(x, y, z) = xe^{-y^2} + y$]

5. Calcolare il lavoro compiuto dal campo $F(x, y) = \left(e^y, xe^y - \frac{1}{y} \right)$, lungo la curva $\gamma : [0, 1] \to \mathbb{R}^2$ definita da $\gamma(t) = (te^t, \log(t^2 + 2))$.

   [$3e + \log\left(\frac{\log 2}{\log 3}\right)$]

6. Sia $\Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$ e sia $F : \Omega \to \mathbb{R}^2$ il campo vettoriale

$$F(x, y) = \left(\frac{x}{\sqrt[3]{x^2 + y^2}}, \frac{y}{\sqrt[3]{x^2 + y^2}} \right).$$

   Sia $\gamma : [0, 2\pi] \to \mathbb{R}^2$ la curva definita da $\gamma(t) = (\cos t, \sin t)$. Calcolare $\int_\gamma F \cdot dP$.

   [0]

7. Dato il campo vettoriale $F(x, y) = \big(\sin(x + 2y) + x \cos(x + 2y) + 3y^2 + 5,\ 2x \cos(x + 2y) + \lambda xy\big)$,

   a) Determinare λ in modo che il campo sia conservativo in $\mathbb{R}^2$.

   b) Con tale valore di λ, trovare un potenziale del campo.

   [$\lambda = 6$, $U(x, y) = x \sin(x + 2y) + 3xy^2 + 5x + C$]

$F(x,y,z) = \left(xy - \sin z, \frac{x^2}{2} - \frac{e^y}{z}, \frac{e^y}{z^2} - x\cos z\right)$, $D = \{(x,y,z) \in \mathbb{R}^3 \mid z > 0\}$

è conservativo in D? Se sì, determina U.

D verticalmente convesso? Sì! $\mathbb{R}^3 - \{z \leq 0\}$

rot F? $\frac{\partial F_2}{\partial x} \overset{?}{=} \frac{\partial F_1}{\partial y} \; ? \to x = x$ ✓ $\frac{\partial F_1}{\partial z} \overset{?}{=} \frac{\partial F_3}{\partial x} \to -\cos z = -\cos z$ ✓

$\frac{\partial F_2}{\partial z} \overset{?}{=} \frac{\partial F_3}{\partial y} \to \frac{e^y}{z^2} = \frac{e^y}{z^2}$ ✓

→ F è conservativo!

$\frac{\partial U}{\partial x} = F_1 \to \quad U = \frac{x^2}{2}y - x\sin z + C(y,z)$

$\frac{\partial U}{\partial y} = F_2 \to \quad \cancel{\frac{x^2}{2}} + \frac{\partial C}{\partial y} = F_2 = \cancel{\frac{x^2}{2}} - \frac{e^y}{z} \to C = -\frac{e^y}{z}$

$U = \frac{x^2}{2}y - x\sin z - \frac{e^y}{z}$

$\frac{\partial U}{\partial z} = F_3 \to \quad \cancel{-\cos z \cdot x} + \cancel{\frac{e^y}{z^2}} = F_3 = \cancel{\frac{e^y}{z^2}} - \cancel{x\cos z} + \frac{\partial C}{\partial z} \to \frac{\partial C}{\partial z} = 0$

$U(x,y,z) = \frac{x^2}{2}y - x\sin z - \frac{e^y}{z} + h$ con $h \in \mathbb{R}$

ESERCIZIO 4

$F(x,y) = \left(e^{-y^2}, 1 - 2xye^{-y^2}\right)$ è conservativo?

$D = \mathbb{R}^2 \to$ semplicemente connesso

$\to \text{rot}(F) = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \to \quad -2ye^{-y^2} \overset{?}{=} -2e^{-y^2}$ ✓

$\text{rot}(F) = (0,0,0)$ è conservativo!

$\to U? \quad \frac{\partial U}{\partial x} = F_1 \to \quad U = xe^{-y^2} + C(y,z)$

$\frac{\partial U}{\partial y} = F_2 \to \quad \cancel{-2yxe^{-y^2}} + \frac{\partial C}{\partial y} = F_2 = 1 - \cancel{2xye^{-y^2}}$

$\frac{\partial C}{\partial y} = 1 \to \quad U = xe^{-y^2} + y + h$

$C = y + h$

$U = xe^{-y^2} + y + h$ con $h \in \mathbb{R}$

$$F(x,y) = \left(\frac{x}{\sqrt[3]{x^2+y^2}} ; \frac{y}{\sqrt[3]{x^2+y^2}}\right)$$

$\gamma : [0, 2\pi] \to \mathbb{R}^2$

↳ $\gamma(t) = (\cos t, \sin t)$. Calcola $\int_\gamma F \cdot dP$.

integr. curvilineo di 2a specie. lavoro di F su γ

Attenzione! $\mathbb{R}$ non è semplicemente connesso! Però $\gamma(t) \to$ 

→ È una curva di Jordan in senso ANTIORARIO → GREEN.

$$\int_\gamma F \cdot dP = \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx dy \qquad \cancel{\frac{\partial F_2}{\partial x} = -2x \frac{1}{(x^2+y^2)^4}}$$

$$\frac{\partial F_2}{\partial x} = D\left[y \cdot (x^2+y^2)^{-1/3}\right] = -\frac{1}{3} y (x^2+y^2)^{-4/3} \cdot 2x$$

$$\frac{\partial F_1}{\partial y} = -\frac{1}{3} x (x^2+y^2)^{-4/3} \cdot 2x \qquad \to \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$$\to \int_D \text{quella} = \int_\gamma F \cdot dP = 0$$

7) $F(x,y) = (\sin(x+2y) + x\cos(x+2y) + 3y^2 + 5, \; 2x\cos(x+2y) + \lambda xy)$

λ affinché F sia conservativo:

$$\text{rot}(F) = 0 \to \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$$\frac{\partial F_2}{\partial x} = \lambda y + 2\cos(x+2y) - 2x\sin(x+2y)$$

$$\frac{\partial F_1}{\partial y} = \cos(x+2y) \cdot 2 - x\sin(x+2y) \cdot 2 + 6y$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \lambda y - 6y = y(\lambda - 6) = 0 \to \lambda = 6 \to F \text{ conservativo } \lambda = 6$$

• Calcolare un potenziale.

x PARTI

$$F_1 = \frac{\partial U}{\partial x} \to \quad U = \int F_1 dx = 5x + 3xy^2 + \int \sin(x+2y) + \int x\cos(x+2y)\, dx =$$

$$= 5x + 3xy^2 - \cancel{\cos(x+2y)} + x\sin(x+2y) + \cancel{\cos(x+2y)} + c(y) =$$

$$= 5x + 3xy^2 + x\sin(x+2y) + c(y)$$

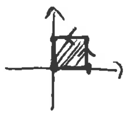
$$F_2 = \frac{\partial U}{\partial y} \to \quad \cancel{6xy} + \cancel{2x\cos(x+2y)} + \frac{\partial c}{\partial y} = F_2 = \cancel{2x\cos(x+2y)} + \cancel{6xy}$$

$$\frac{\partial c}{\partial y} = 0 \to \quad c \text{ COSTANTE}$$

ALLORA $\to U = 5x + 3xy^2 + x\sin(x+2y) + C$ con $C \in \mathbb{R}$

4

1) $\oint_\gamma F \cdot dP$ $\quad F(x,y) = (y^2, x)$ e γ

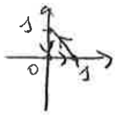


$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\}$

$$\int_\gamma F \cdot dP = \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dxdy = \int_D 1 - 2y \, dxdy$$

$$= \int_0^1 \int_0^1 1 - 2y \, dydx = \int_0^1 y - y^2 \Big|_0^1 dx = \int_0^1 0 \, dx = 0 \quad \checkmark$$

2) $\oint_\gamma F \cdot dP$ $\quad \gamma =$



$F(x,y) = (x^4, xy)$

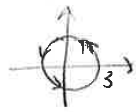
$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1-x\}$

$$\oint_\gamma F \cdot dP = \text{PER GREEN} = \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dxdy =$$

$$= \int_0^1 \int_0^{1-x} y \, dydx = \int_0^1 \frac{(1-x)^2}{2} dx = -\frac{(1-x)^3}{6} \Big|_0^1 = 1/6$$

3) calcola la circuitazione di $F(x,y) = (3y - e^{\sin x}, 7x + \sqrt{y^4+1})$ lungo $x^2+y^2=9$ (ANTIOR)

$\gamma(t) = (3\cos t, 3\sin t)$

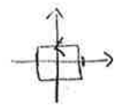


$\rightarrow$ green $\oint_\gamma F \cdot dP = \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dxdy$ $\qquad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 7 - 3 = 4$

$= 4$ area cerchio $= 4 \cdot \pi \cdot 9 = 36\pi$

4) $F(x,y) = (x^3 - xy^3, y^2 - 2xy)$

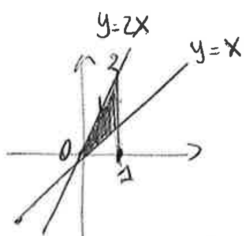
$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -2y + 3xy^2$ $\quad \rightarrow \oint_\gamma F \cdot dP = \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dxdy$ $\qquad \gamma =$



$$= \int_0^2 \int_0^2 3xy^2 - 2y \, dxdy = \int_0^2 \frac{3}{2}x^2y^2 - 2xy \Big|_0^2 dy = \int_0^2 6y^2 - 4y \, dy = 2y^3 - 2y^2 \Big|_0^2 =$$

$= 2 \cdot 8 - 2 \cdot 4 = 16 - 8 = 8 \checkmark$

5) $F(x,y) = (2xy, x)$ $\quad D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, x \le y \le 2x\}$

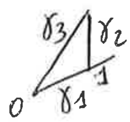


$\int_\gamma F \cdot dP$?

- $\gamma_1 : [0,1] \rightarrow \mathbb{R}^2$ $\quad \gamma_1(t) = (t, t)$

  $\gamma'(t) = (1, 1)$, $F(\gamma(t)) = (2t^2, t)$

  $F(\gamma(t)) \cdot \gamma'(t) = 2t^2 + t$ $\qquad \int_0^1 2t^2 + t \, dt = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$



- $\gamma_2 \Rightarrow$ circuitazione nulla
- $\gamma_3 : [0,1] \rightarrow \mathbb{R}^2$ però antiorario! $\quad \gamma_3(-t+a+b) = \gamma_3(1-t)$

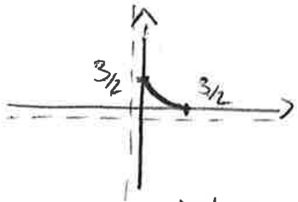
  $\gamma_3(1-t) = (1-t, 2-2t)$, $\gamma_3 \in [0,1]$ $\qquad \gamma_3'(t) = (-1, -2)$ $\qquad F(\gamma_3) = ((2-2t)^2, 1-t)$

  $F(\gamma_3) \cdot \gamma'(t) = -4 - 4t^2 + 8t + 2t - 2 = -4t^2 + 10t - 6$

  $\int_0^1 -4t^2 + 10t - 6 \, dt = -\frac{4}{3} + \frac{10}{2} - 6 = \frac{-8+30-36}{6} = -\frac{14}{6} = -\frac{7}{3}$ $\quad \rightarrow$ circuit $= \frac{7}{6} - \frac{7}{3} = -\frac{7}{6}$

CALCOLA AREA DI K -

$(2y+1)(2x+1)\le 4 \to 4xy+2y+2x+1\le 4$



$4xy+2y \le 4-1-2x$

$y(4x+2)\le 4-1-2x \to y\le \frac{3-2x}{4x+2}$

asintoti $= \begin{cases} -\frac{1}{2} = y \\ x = -\frac{1}{2} \end{cases}$

interseca con $x=0$;

$\to y = \frac{3}{2}$ con $y=0 \to x=\frac{3}{2}$

• AREA DI K $\to \int_0^{3/2} \frac{3-2x}{4x+2}\,dx = \int_0^{3/2} \frac{4}{4x+2} - \frac{1}{2}\,dx = \log(4x+2) - \frac{1}{2}x \Big|_0^{3/2} = \cancel{\log 8 - \frac{3}{4}}$ ~~log~~

$= \log 8 - \frac{3}{4} - \log 2 = \log 4 - \frac{3}{4}$

• $\gamma(t) = \left(t, \frac{3-2t}{4t+2}\right)$ antior. $\gamma(t): [0, \frac{3}{2}] \to \mathbb{R}^2$ curva di Jordan

se $a=3$, $F(x,y) = (x+y, 3x)$

$\int_\gamma F \cdot dP$ = PER GREEN $= \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dx\,dy = \int_D 3-1\,dx\,dy = 2\int_D dx\,dy = 2 \cdot$ AREA K $=$

$= 2\cdot\log 4 - \frac{3}{2}$

• affinché F sia conserv. a?

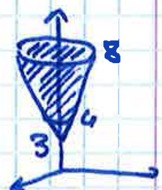
IRROTAZIONALE $\to \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \to a-1=0 \to a=1$

INTEGRALI DI SUPERFICIE

1) $\Sigma = \{(x,y,z)\in\mathbb{R}^3 \mid z=3+3x^2+3y^2,\ 4\le z\le 8\}$. Calcola $\int_\Sigma \frac{4x^2}{(x^2+y^2)\sqrt{1+36(x^2+y^2)}}\,d\sigma$

integrale di superficie di 1ª specie.

$$\int_\Sigma f\,d\sigma = \int_D F(x,y,g(x,y))\cdot\sqrt{1+\|\nabla g\|^2}\,dx\,dy$$



$g(x,y) = 3+3x^2+3y^2 \longrightarrow F(x,y,g(x,y)) = F(x,y)$

$\nabla g = (6x,6y)$, $\sqrt{\|\nabla g\|^2+1} = \sqrt{36x^2+36y^2+1} = \sqrt{*}$

$$\int_\Sigma F\,d\Sigma = \int_D F(x,y,g(x,y))\cdot\sqrt{*}\,dx\,dy = \int_D \frac{4x^2}{x^2+y^2}\,dx\,dy$$

~~coord. cilindriche → ρ,ϑ,z. $0\le\rho\le 2\sqrt{2}$~~

$0\le\vartheta\le 2\pi$

Integrare su D → dominio di Σ in $\mathbb{R}^2$. Nel piano xy con $z=0$:

$3+3x^2+3y^2=0 \longrightarrow 1+x^2+y^2=0$

$\rightarrow 4\le z\le 8 \rightarrow 4\le 3+3x^2+3y^2\le 8 \rightarrow \frac{1}{3}\le x^2+y^2\le\frac{5}{3}$ $\quad(x^2+y^2=\rho^2)$

$\sqrt{\frac{1}{3}}\le\rho\le\sqrt{\frac{5}{3}}$, $\quad 0\le\vartheta\le 2\pi$ $\quad$ COORDIN. POLARI

$$\int_0^{2\pi}\int_{\sqrt{1/3}}^{\sqrt{5/3}} \frac{4\rho^2\cos^2\vartheta}{\rho^2}\,\rho\,d\rho\,d\vartheta = \int_0^{2\pi}\int_{\sqrt{1/3}}^{\sqrt{5/3}} 4\cos^2\vartheta\,\rho\,d\rho\,d\vartheta = \int_0^{2\pi}\left[\frac{4}{2}\rho^2\cos^2\vartheta\right]_{\sqrt{1/3}}^{\sqrt{5/3}}d\vartheta =$$

$$= \int_0^{2\pi}\frac{8}{3}\cos^2\vartheta\,d\vartheta = \frac{8}{3}\pi$$

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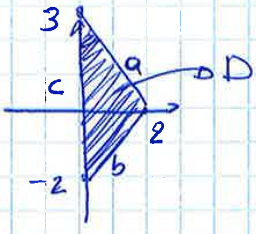
3) $\displaystyle\int_S \frac{z+\cos y}{\sqrt{\cos^2 y+2\sin^2 y+4x^2}}\,dx \qquad S=\{(x,y)\in\mathbb{R}^2 \mid z=x^2-\cos y,\ (x,y)\in T\}$



$$\int_S F\cdot d\sigma = \int_{T=D} F(x,y,g(x,y))\cdot\sqrt{\|\nabla g\|^2+1}\,dx\,dy$$

$$=\int_D x^2\,dx\,dy \quad \text{CFR P.2 X I PASSAGGI}$$

lato a $\to$ $y=3-\frac{3}{2}x$

lato b $\to$ $y=x-2$

$$\to \int_0^2\int_b^a x^2\,dy\,dx=\int_0^2 x^2\left(3-\frac{3}{2}x+2-x\right)dx=\int_0^2 5x^2-\frac{5}{2}x^3\,dx=\frac{5}{3}x^3-\frac{5}{8}x^4\Big|_0^2=\frac{10}{3} \checkmark$$

5) $\displaystyle\int_\Sigma \frac{x^2+y^2}{(1+e^{2z})^{1/2}}\,d\sigma \qquad \Sigma$ sup $g(x,y)=-\frac{1}{2}\log(x^2+y^2)$ tra i cilindri $x^2+y^2-e^{-2}=0$ e $x^2+y^2-1=0$

$$\nabla g=\left(-\frac{x}{x^2+y^2},-\frac{y}{x^2+y^2}\right) \to \|\nabla g\|^2+1=\frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{2}}}+1=1+\frac{1}{x^2+y^2}$$

$$\int_\Sigma f\cdot d\sigma=\int_D F(x,y,g(x,y))\sqrt{\|\nabla g\|^2+1}\,(dx\,dy)=\int_D\frac{x^2+y^2}{(1+e^{2(-\frac{1}{2}\log(x^2+y^2))})^{1/2}}\cdot\sqrt{1+\frac{1}{x^2+y^2}}\,dx\,dy=$$

$$=\int_D\frac{x^2+y^2}{\cancel{\sqrt{1+\frac{1}{x^2+y^2}}}}\cdot\cancel{\sqrt{1+\frac{1}{x^2+y^2}}}\,dx\,dy=\int_D x^2+y^2\,dx\,dy$$

$\sqrt{e^{-2}}\le\rho\le 1$, $\quad 0\le\vartheta\le 2\pi$

$$\int_0^{2\pi}\int_{\sqrt{e^{-2}}}^1\rho^3\,d\rho\,d\vartheta=\int_0^{2\pi}\frac{\rho^4}{4}\Big|_{\sqrt{e^{-2}}}^1=2\pi\cdot\frac{1}{4}\left(1-(e^{-2})^{\frac{1}{2}4}\right)=\frac{\pi}{2}\cdot(1-e^{-4})$$

3) $F(x,y,z) = (4x, -2y^2, z^2)$ uscente da $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1, 0 \leq z \leq 3\}$

$\text{div } f = 4 - 4y + 2z$ $\quad 0 \leq z \leq 3, \; 0 \leq \vartheta \leq 2\pi, \; 0 \leq \rho \leq 1$

$$\int_\Omega 4 - 4y + 2z \, dx\,dy\,dz = \int \rho(4 - 4\rho\sin\vartheta + \cancel{2\rho}\, 2z)\,dz\,d\rho\,d\vartheta =$$

$$\cancel{= \int_0^{2\pi}\int_0^1 \left[4z - 4z\rho\sin\vartheta + z^2\right]_0^3 d\rho\,d\vartheta = \int_0^{2\pi}\int_0^1 12 - 12\rho\sin\vartheta + 9\,d\rho\,d\vartheta =}$$

$$\cancel{= \int_0^{2\pi} 21 - 6\sin\vartheta\,d\vartheta = 42\pi - 6\int_0^{2\pi}\sin\vartheta\,d\vartheta =}$$

$$= \int_0^{2\pi}\int_0^1\int_0^3 4\rho - 4\rho^2\sin\vartheta + 2\rho z\,dz\,d\rho\,d\vartheta = \int_0^{2\pi}\int_0^1 21\rho - 12\rho^2\sin\vartheta\,d\vartheta =$$

$$= \int_0^{2\pi} \frac{21}{2} - 6\sin\vartheta\,d\vartheta = \frac{21}{2}\cdot 2\pi = 21\pi$$

4) $F(x,y,z) = (-xz^2, yz^2, z^2 - zx^2 + zy^2)$

a) Flusso di f uscente da $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1, 0 \leq z \leq 1\}$

$\text{div } F = -\cancel{z^2} + \cancel{z^2} + 2 - x^2 + y^2 = 2 - x^2 + y^2$

$$\rightarrow \int_D \text{div } F\,dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_0^1 2\rho - \rho^3\cos\vartheta + \rho^3\sin\vartheta\,dz\,d\rho\,d\vartheta =$$

$$= \int_0^{2\pi} 1 - \frac{1}{4}\cancel{\cos\vartheta} + \frac{1}{4}\cancel{\sin\vartheta}\,d\vartheta = 2\pi$$

b) $\text{rot } f = 0 \rightarrow \cancel{\frac{\partial f_2}{\partial y} = \frac{\partial f_3}{\partial z}} \rightarrow \begin{vmatrix} x & y & z \\ \partial f_1 & \partial f_2 & \partial f_3 \\ \partial x & \partial y & \partial z \end{vmatrix} \rightarrow \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \rightarrow 2yz = 2yz$ ✓

semplicem. connesso + irrotaz. → CONSERVATIVO

$\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \rightarrow -2xz = -2xz$ ✓

$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \rightarrow 0 = 0$ ✓

Potenziale? $\frac{\partial U}{\partial x} = f_1 \rightarrow U = -\frac{x^2}{2}z^2 + c(y,z)$

$\frac{\partial U}{\partial y} = f_2 \rightarrow \frac{\partial c}{\partial y} = yz^2$

$\rightarrow U = -\frac{x^2}{2}z^2 + \frac{y^2}{2}z^2 + c(z)$

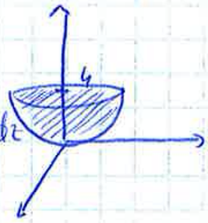
$\frac{\partial U}{\partial z} = f_3 \rightarrow -\cancel{x^2z} + \cancel{y^2z} + \frac{\partial c}{\partial z} = 2z - \cancel{x^2z} + \cancel{y^2z} \rightarrow \frac{\partial c}{\partial z} = 2z \rightarrow c = z^2$

$\rightarrow U = -\frac{x^2}{2}z^2 + \frac{y^2}{2}z^2 + z^2 + A$

2

5) $F(x,y,z) = (2x^3 - 10, 2y^3, e^{z/2}) \to$ flusso di F uscente da $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z \leq 4\}$

$\operatorname{div} F = (6x^2 + 6y^2 + \frac{1}{2}e^{z/2})$



$\to$ Per Gauss $= \int_\Omega \operatorname{div} dx\,dy\,dz = \int 6x^2+6y^2+\frac{1}{2}e^{z/2}\,dx\,dy\,dz$

coord. cilindriche $\to \rho, \vartheta, z$

$\rho^2 \leq z \leq 4, \quad 0 \leq \rho \leq 2, \quad 0 \leq \vartheta \leq 2\pi$

$$\int_0^{2\pi}\int_0^2\int_{\rho^2}^4 (6\rho^2 + \tfrac{1}{2}e^{z/2})\,\rho\,dz\,d\rho\,d\vartheta = \int_0^{2\pi}\int_0^2\int_{\rho^2}^4 6\rho^3 + \tfrac{1}{2}\rho e^{z/2}\,dz\,d\rho\,d\vartheta =$$

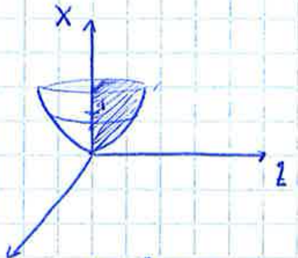
$$= 2\pi \int_0^2 6\rho^3 z + \rho e^{z/2}\Big|_{\rho^2}^4 d\rho = 2\pi \int_0^2 24\rho^3 + \rho e^2 - 6\rho^5 - \rho e^{\rho^2/2}\,d\rho =$$

$$= 2\pi\left(\frac{24}{4}\rho^4 + \frac{e^2}{2}\rho^2 - \frac{6}{6}\rho^6 - e^{\rho^2/2}\right)_0^2 = 2\pi(24\cdot 4 + 2e^2 - 64 - e^2 + 1) =$$

$$= 2\pi(96 - 64 + 1 + e^2) = 2\pi(33 + e^2)$$

6) $F(x,y,z) = (yz, yz^2, zy^2)$ uscente da $D = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq y^2+z^2, z \geq 0, x \leq 1\}$

$\operatorname{div} f = z^2 + y^2$



$\to \int_D \operatorname{div} f\,dx\,dy\,dz = \int_D z^2+y^2\,dx\,dy\,dz =$

Coordinate cilindriche ρ, ϑ, x

$0 \leq \rho \leq 1, \quad 0 \leq \vartheta \leq \pi, \quad y^2+z^2 \leq x \leq 1 \to \rho^2 \leq x \leq 1$

$$\int_0^\pi\int_0^1\int_{\rho^2}^1 \rho^3\,dx\,d\rho\,d\vartheta = \pi\int_0^1 x\rho^3\Big|_{\rho^2}^1 d\rho = \pi\int_0^1 \rho^3 - \rho^5\,d\rho =$$

$$= \pi\left(\frac{\rho^4}{4} - \frac{\rho^6}{6}\right)_0^1 = \pi\left(\frac{1}{4} - \frac{1}{6}\right) = \pi\left(\frac{1}{12}\right) = \frac{\pi}{12}$$

FOGLIO 9: TEOREMA DEL ROTORE (STOKES)

1. Calcolare il flusso del rotore del campo $F(x,y,z) = (2x - y, -yz^2, -y^2z)$ attraverso l'emisfero superiore della sfera $x^2 + y^2 + z^2 = 1$ con normale che punta verso l'alto. [π]

2. Sia Σ la superficie che si ottiene togliendo al bordo del tetraedro individuato dai piani $x = 0$, $y = 0$, $z = 0$ e $x + y + z = 4$ la faccia dove $y = 0$. Calcolare il flusso del rotore del campo $F(x,y,z) = (xz, -y, x^2y)$ attraverso Σ, orientata con normale che punta verso l'interno del tetraedro. [$-32/3$]

3. Sia Σ la superficie definita da $z = \sqrt{1 - x^2 - y^2}$, $y \geq 0$, orientata con normale che punta verso l'alto. Calcolare la circuitazione lungo il bordo di Σ del campo vettoriale $F(x,y,z) = (y + z, z + x, x + y)$. [0]

4. Sia γ la curva intersezione del cilindro $2x^2 + y^2 - 6x = 0$ con piano $x + z = 3$, orientata in modo che la sua proiezione sul piano (x,y) risulti percorsa in verso antiorario. Dato il campo $F(x,y,z) = (z, x, y)$, calcolare $\int_\gamma F \cdot dP$ direttamente ed utilizzando il teorema di Stokes. [$9\pi/\sqrt{2}$]

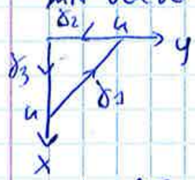
5. Calcolare il flusso del rotore del campo $F(x,y,z) = (ze^{x^2}, 3(x-1), z(x-1))$ attraverso la superficie definita da $x = z^2 + y^2$, $z \geq 0$, $x \leq 1$, con normale che forma un angolo ottuso con l'asse x. [4]

6. Calcolare il flusso del rotore del campo $F(x,y,z) = (ze^{x^2}, 3(x-1), z\sin y)$ attraverso la superficie definita da $x = z^2 + y^2$, $z \geq 0$, $x \leq 1$, con normale orientata in modo da formare un angolo acuto con l'asse x. [$-4 - 2\cos 1 + 2\sin 1$]

2) $F(x,y,z) = (xz, -y, 4x^2)$ $\quad \Sigma \to$ tetr. $- \{y=0\}$

$\cancel{y=0},\ x=0,\ y=0,\ x+y+z=4$

$\hat{n}$ verso l'interno $\ominus$ *



$\gamma_1 = (t, 4-t)$, $\quad \gamma_2 = (0,t)$, $\quad \gamma_3 = (t,0)$

$g(x,y) = z = 4-x-y$

- $\gamma_{1\Sigma} = (t, 4-t, 0)$ $\qquad \gamma'_{1\Sigma} = (1,-1,0)$

  $F(\gamma_{1\Sigma}) = (0, t-4, 4t^2-t^3)$ $\quad \to t \in [0,4]$

  $F(\gamma_{1\Sigma}) \cdot \gamma'_{1\Sigma} = 0 + 4 - t$

  $$① \to \int_0^4 4-t\,dt = 16 - \frac{16}{2} = \boxed{8}$$

- $\gamma_{2\Sigma} = (0, t, 4-t)$ $\qquad \gamma'_{2\Sigma} = (0,1,-1)$

  $f(\gamma_{2\Sigma}) = (0,-t,0)$

  $F(\gamma_{2\Sigma}) \cdot \gamma'_{2\Sigma} = -t$ $\quad \to \int_0^4 -t\,dt = -\boxed{8}$ $\leftarrow ②$

- $\gamma_{3\Sigma} = (t, 0, 4-t)$ $\qquad \gamma'_{3\Sigma} = (1,0,-1)$

  $F(\gamma_{3\Sigma}) = (4t-t^2, 0, 0)$ $\quad$ con $t \in [0,4]$

  $F(\gamma_{3\Sigma}) \cdot \gamma'_{3\Sigma} = 4t - t^2$

  $$③ \to \int_0^4 4t - t^2\,dt = 2t^2 - \frac{t^3}{3}\Big|_0^4 = 32 - \frac{64}{3} = \frac{96-64}{3} = \boxed{\frac{32}{3}}$$

$1+2+3$ con segno $\ominus$ *

$$-\left(\cancel{8} - \cancel{8} + \frac{32}{3}\right) = -\frac{32}{3}$$

2

$\oplus$

6) $F(x,y,z) = \{ze^{x^2}, 3(x-1), z\sin y\}$, $x = y^2+z^2$, $z \geq 0$, $x \leq 1$ $\hat{u}_N$ angolo acuto con x

$\gamma(t) = \cancel{(1,\cos t, \sin t)}\ (1, \cos t, \sin t)$ $\ominus$ ANTIOR.

$F(\gamma(t)) = (\sin t\, e, 0, \sin t \cdot \sin(\cos t))$

$\gamma'(t) = (0, -\sin t, \cos t)$

$F(\gamma(t)) \cdot \gamma'(t) = \cos t \sin t \sin(\cos t)$ $t \in [0, \pi]$

$\int_0^{\pi} \cos t \sin t \sin(\cos t)\, dt$?

$F=(x,y,z)=(z,x,y)$ $\qquad g=-x+3$

$$\text{rot } f=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\hat{x}-\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)\hat{y}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\hat{z}$$

$$\text{rot } f=(1-0)\hat{x}-(0-1)\hat{y}+(1)\hat{z}=\hat{x}+\hat{y}+\hat{z} \rightarrow \text{rot } f=(1,1,1)$$

$$\int_\gamma f\cdot dP=\int_\Sigma \nabla\wedge f\,\hat{u}_N d\Sigma$$

~~div(rot f) = ∇·(∇∧f) = 0~~ $N(x,y)=\left(-\frac{\partial g}{\partial x},-\frac{\partial g}{\partial y},1\right)=(1,0,1)$

$$\begin{cases}2x^2+y^2-6x=0\\ x+z=3\end{cases}\qquad \begin{cases}x^2-3x+\frac{y^2}{2}=0\\ z=3-x\end{cases}$$

$$x^2-3x+\frac{9}{4}+\frac{y^2}{2}=\frac{9}{4}$$
$$\left(x-\frac{3}{2}\right)^2+\frac{y^2}{2}=\frac{9}{4}$$
$$\frac{4}{9}\left(x-\frac{3}{2}\right)^2+\frac{4y^2}{9}\frac{y^2}{2}=1$$

ELLISSE DI $C=\left(\frac{3}{2},0\right)$

$a=\frac{3}{2}\ ,\quad b=\frac{3}{\sqrt{2}}$

coordinate ellittiche

$$\frac{4}{9}\left(x-\frac{3}{2}\right)^2+\frac{2}{9}y^2=1$$

QUASI

$x=\frac{3}{2}+\frac{3}{2}\rho\cos\vartheta$

$y=\frac{3}{\sqrt{2}}\rho\sin\vartheta$

$$|\phi|=ab\rho=\frac{3}{2}\frac{3}{\sqrt{2}}\rho=\frac{9}{2\sqrt{2}}\rho\ \frac{9\sqrt{2}}{4}\rho$$

3.6. $\sum_{n=1}^{\infty}(\arctan|x|+1)^n$ [Divergente]

4. Discutere la convergenza delle seguenti serie telescopiche

4.1. $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})$ [Divergente]

4.2. $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\right)$ [Convergente]

4.3. $\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)$ [Convergente]

4.4. $\sum_{n=1}^{\infty}\left(\log\left(\frac{\pi}{n}\right)-\log\left(\frac{\pi}{n+1}\right)\right)$ [Divergente]

5. Determinare il carattere delle seguenti serie numeriche applicando il criterio del confronto asintotico

5.1. $\sum_{n=0}^{\infty}[\log(n+3)-\log(n+1)]$ [Divergente]

5.2. $\sum_{n=2}^{\infty}\frac{n-\sqrt{n}}{n^3-3n}$ [Convergente]

5.3. $\sum_{n=1}^{\infty}\frac{n+\sqrt[3]{n}}{n^2+n+1}$ [Divergente]

5.4. $\sum_{n=0}^{\infty}\frac{n!}{(n+2)!}$ [Convergente]

5.5. $\sum_{n=0}^{\infty}(\sqrt{4+3^{-n}}-2)$ [Convergente]

5.6. $\sum_{n=1}^{\infty}(e^{n+\frac{1}{3^n}}-e^n)$ [Convergente]

5.7. $\sum_{n=1}^{\infty}(e^{n+\frac{1}{2^n}}-e^n)$ [Divergente]

5.8. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\sin\frac{1}{n}\right)$ [Convergente]

5.9. $\sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right)$ [Convergente]

5.10. $\sum_{n=1}^{\infty}\frac{\log(1+2^n)}{n^2}$ [Divergente]

5.11. $\sum_{n=1}^{\infty}\frac{\log(3^n-1)}{n^3}$ [Convergente]

5.12. $\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-1-\sin\frac{1}{n}\right)$ [Convergente]

5.13. $\sum_{n=1}^{\infty}\frac{1+\sqrt{n}+e^{-\sqrt{n^3}}}{(n+1)^3+1+\cos(n!)}$ [Convergente]

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8.5. $\sum_{n=1}^{\infty} \left(n^{1/n} - 1\right)^n$ [Convergente]

8.6. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ [Convergente]

9. Determinare il carattere delle seguenti serie numeriche applicando il criterio di Leibniz

9.1. $\sum_{n=1}^{\infty} (-1)^n (3^{1/n} - 1)$ [Convergente semplicemente]

9.2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+1}{n^3}$ [Convergente semplicemente]

9.3. $\sum_{n=1}^{\infty} (-1)^n \arctan \frac{1}{2n+1}$ [Converge semplicemente]

9.4. $\sum_{n=1}^{\infty} (-1)^n \left[\frac{\pi}{2} - \arctan(\log n)\right]$ [Converge semplicemente]

10. Stabilire il carattere della serie numerica al variare del parametro $\alpha > 0$

10.1. $\sum_{n=1}^{\infty} \frac{1}{n^\alpha(\sqrt{n}+3)}$ [Converge se e solo se $\alpha > 1/2$]

10.2. $\sum_{n=1}^{\infty} \frac{3n+1}{n^\alpha(n+1)^2}$ [Converge se e solo se $\alpha > 0$]

10.3. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^\alpha - 4}$ [Converge se e solo se $\alpha > 3/2$]

10.4. $\sum_{n=1}^{\infty} \frac{3n-2}{(2n-1)n^\alpha}$ [Converge se e solo se $\alpha > 1$]

10.5. $\sum_{n=1}^{\infty} \frac{4n^2-1}{\sqrt{n^\alpha - 3}}$ [Converge se e solo se $\alpha > 6$]

10.6. $\sum_{n=0}^{\infty} (3^n + 5^n)\alpha^n$ [Converge se e solo se $\alpha < 1/5$]

11. Stabilire il carattere delle seguenti serie numeriche

11.1. $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$ [Diverge]

11.2. $\sum_{n=0}^{\infty} \frac{1+\sqrt{n}}{1+n+n^2}$ [Converge]

11.3. $\sum_{n=0}^{\infty} \frac{1}{2^n - n^3}$ [Converge]

11.4. $\sum_{n=1}^{\infty} \frac{n!+1}{(n+1)!}$ [Diverge]

11.5. $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$ [Diverge]

11.6. $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$ [Converge]

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SERIE NUMERICHE ~ FOGLIO 1

CONDIZIONE NECESS. se serie converge, $\lim_{n \to \infty} a_n = 0$

2.1) $\sum_{n=1}^{\infty} n^2(e^{1/n} - \cos(\frac{1}{n}) - \frac{1}{n})$

$$\lim_{n \to \infty} n^2(e^{1/n} - \cos\frac{1}{n} - \frac{1}{n}) = \lim_{n \to \infty} n^2 \left(1 + \cancel{x} - \cancel{1} + \frac{1}{2x^2} - \cancel{\frac{1}{x}}\right) = \frac{1}{2}$$

DIVERGE → violata la condizione necessaria di convergenza

2.2) $\sum_{n=0}^{\infty} \frac{n^2+2n}{(n+1)(n+2)}$ $\quad \lim_{n \to \infty} \frac{n^2+2n}{(n+1)(n+2)} = 1$ DIVERGE (perché $\not\to 0$ per $n \to \infty$)

2.3) $\sum_{n=1}^{\infty} (\sinh(\frac{1}{n}) - \sin(\frac{1}{n}))$ uso sviluppi di Taylor perché $\frac{1}{n} \to 0$

$$\lim_{n \to \infty} (\sinh(\frac{1}{n}) - \sin(\frac{1}{n})) = \lim_{n \to \infty} x + \frac{x^3}{6} - x + \frac{x^3}{6} = \infty \text{ DIVERGE}$$

2.4) $\sum_{n=1}^{\infty} n^2 \log(1 + \frac{1}{n^2})$

per $x \to 0$, $\log(1+x) \sim x - \frac{x^2}{2} + \frac{x^3}{3}$ $\quad \lim_{n \to \infty} n^2(\frac{1}{n} - \frac{2}{n^2}) = \lim_{n \to \infty} n - 2 = \infty$ DIVERGE

2.5) $\sum_{n=1}^{\infty} n^3 10^{-n}$ $\quad \lim_{n \to \infty} \frac{n^3}{10^n} = 0$

il limite tende a zero quindi la serie non diverge. Non posso però dire nulla sulla convergenza! Il teorema dice che se la serie converge ALLORA il $\lim(n \to \infty)$ è zero! Non il contrario

CARATTERE + calcolo somma → geometriche o telescopiche

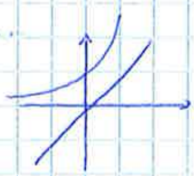
3.1) $\sum_{n=0}^{\infty} \frac{4^n + 5^n}{10^n} = \sum_{n=0}^{\infty} (\frac{4}{10})^n + \sum_{n=0}^{\infty} (\frac{5}{10})^n$

$q_1 = \frac{4}{10} = \frac{2}{5} < 1$ $\quad q_2 = \frac{5}{10} = \frac{1}{2} < 1$ → due serie geom con $|q| < 1$ e quindi convergenti a $\frac{1}{1-q}$

convergenza: ① → $\frac{1}{1-\frac{2}{5}} = \frac{5}{3}$ $\quad$ ② $\frac{1}{1-\frac{1}{2}} = 2$ → quindi $= \frac{11}{3}$

→ la serie converge a $\frac{11}{3}$

3.5) $\sum_{n=0}^{\infty} (xe^{-|x|})^n = \sum_{n=0}^{\infty}\left(\frac{x}{e^{|x|}}\right)^n = 1+\frac{x}{e^{|x|}}+\left(\frac{x}{e^{|x|}}\right)^2 \dots$



ragione $q=\frac{x}{e^{|x|}}<1$

quindi converge a $\frac{1}{1-q}=\frac{1}{1-\frac{x}{e^{|x|}}}=\frac{e^{|x|}}{-x+e^{|x|}}=\frac{e^{|x|}}{e^{|x|}-x}$

3.6) $\sum_{n=1}^{\infty} (\mathrm{arctg}|x|+1)^n$

$\lim_{n\to\infty} (\mathrm{arctg}(x|+1)^n=\infty$ perché $\mathrm{arctg}|x|+1>1$ sempre

serie divergente

SERIE TELESCOPICHE

4.1) $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})$ $\quad b_n=\sqrt{n} \to b_{n+1}-b_n$

sviluppo scrivendo la serie $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})=(\cancel{b_1}-b_0)+(\cancel{b_2}-\cancel{b_1})+(\cancel{b_3}-\cancel{b_2})\dots + b_{n+1}-\cancel{b_n}$

$=b_{n+1}-b_0=\sqrt{n+1}-1$

$\lim_{N\to\infty} S_N=\lim_{N\to\infty}\sqrt{N+1}-1=+\infty$ diverge

4.2) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\right)$ $\quad b_{n+1}-b_n$ con $b_n=\frac{1}{\sqrt{n}}$

scrivo le somme $S_N=(\cancel{b_1}-b_0)+(\cancel{b_2}-\cancel{b_1})+(b_3-\cancel{b_2})+\dots+(b_{n+1}-\cancel{b_n})$

$=b_{n+1}-b_0=\frac{1}{\sqrt{N+1}}-1$

$\lim_{N\to\infty}\frac{1}{\sqrt{N+1}}-1=-1$ convergente

4.3) $\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)$ $\quad b_n-b_{n+1}$

$S_N=(b_0-\cancel{b_1})+(\cancel{b_1}-\cancel{b_2})+(\cancel{b_2}-b_3)+\dots+(\cancel{b_N}-b_{N+1})$

$=b_0-b_{N+1}=1-b_{N+1}$

$\lim_{N\to\infty} 1-\cos\left(\frac{\pi}{n+1}\right)=0$ convergente

5.5) $\sum_{n=0}^{\infty} \left(\sqrt{n+3^{-n}}-2\right)$

$\left(n+\frac{1}{3^n}\right)^{1/2}-2 \sim 2-2=0$ conv

5.6) $\sum_{n=1}^{\infty} \left(e^{n+\frac{1}{3^n}}-e^n\right)$

$e^{n+1/3n}-e^n = e^n\left(e^{\frac{1}{3^n}}-1\right) \sim e^n\left(\cancel{1}+\frac{1}{3^n}-\cancel{1}\right) = \left(\frac{e}{3}\right)^n$

ma $\frac{e}{3}<1$, serie geom. convergente

5.7) $\sum_{n=1}^{\infty} \left(e^{n+\frac{1}{2n}}-e^n\right)$

$e^{n+\frac{1}{2n}}-e^n = e^n\left(e^{\frac{1}{2n}}-1\right) \sim e^n\left(\cancel{1}+\frac{1}{2^n}-\cancel{1}\right) = \left(\frac{e}{2}\right)^n$

dove $\frac{e}{2}>1 \to$ serie geometrica con $q>1$, diverge

5.8) $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\sin\frac{1}{n}\right)$

$\frac{1}{n}-\sin\frac{1}{n} \sim \cancel{\frac{1}{n}}-\cancel{\frac{1}{n}}+\frac{x^3}{6}\ \frac{1}{6n^3}$

la serie $\frac{1}{6}\sum_{n=1}^{\infty} 1/n^3$ converge

5.9) $\sum_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right)$

$1-\cos\frac{1}{n} \sim \cancel{1}-\cancel{1}+\frac{x^2}{2} = \frac{1}{n^2 2}$ converge

5.10) $\sum_{n=1}^{\infty} \frac{\log(1+2^n)}{n^2}$ $\qquad \frac{\log(1+2^n)}{n^2}$

$\log(1+2^n) = \log\left(2^n\left(\frac{1}{2^n}+1\right)\right) = \log 2^n + \log\left(\frac{1}{2^n}+1\right) \sim n$

quindi $\frac{\log}{n^2} \sim \frac{n}{n^2} = \frac{1}{n}$ armonica div.

5.11) $\sum_{n=1}^{\infty} \frac{\log(3^n-1)}{n^3} \sim \frac{n}{n^3} = \frac{1}{n^2}$ converg.

5.12) $\sum_{n=1}^{\infty} \left(e^{1/n}-1-\sin\frac{1}{n}\right) \sim \cancel{1}+\cancel{\frac{1}{n}}-\cancel{1}-\cancel{\frac{1}{n}}+\frac{1}{6n^3}+\frac{1}{n^2 2}$ conv.

CRITERIO DEL RAPPORTO

7.1) $\sum_{n=1}^{\infty} \frac{\log n}{2^n}$ criterio del rapporto

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell <$ se $\ell>1$ diverge; se $\ell<1$ converge

$\lim_{n\to\infty} \frac{\log(n+1)}{2^{n+1}} \cdot \frac{2^n}{\log n} = \frac{\log(n+1)}{\log n} \cdot \frac{1}{2} = \log\left(\frac{n+1}{n}\right) \cdot \frac{1}{2} =$

$= \log\left(1+\frac{1}{n}\right) \frac{1}{2} = 0 < 1$ converge

7.2) $\sum_{n=1}^{\infty} \frac{\log n}{n!}$ criterio del rapporto

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\log(n+1)}{(n+1)!} \cdot \frac{n!}{\log n} = \log\left(\frac{n+1}{n}\right) \cdot \frac{1}{n+1} =$

$= 0 < 1$ conv

7.3) $\sum_{n=1}^{\infty} \frac{n^n}{n!\,2^n}$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!\,2^{n+1}} \cdot \frac{n!\,2^n}{n^n} = \frac{(n+1)^{n+1}}{n^n \cdot 2 \cdot (n+1)} =$

$\lim_{n\to\infty} \frac{(n+1)^n}{n^n \cdot 2} =$ ✓ ASPETTA!

$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{2}\right)^{n+1} \cdot \frac{1}{(\cancel{n+1})!} \cdot \cancel{n!} \left(\frac{2}{n}\right)^n = \frac{(n+1)^n}{2^{n+1}} \cdot \frac{2^n}{n^n} = \frac{(n+1)^n}{n^n \cdot 2}$ ✓

$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \frac{1}{2} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \frac{1}{2} =$ diverge

7.4) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ $\lim_{n\to\infty} \frac{((n+1)!)^3}{(3n+3)!} \cdot \frac{3n!}{(n!)^3} =$

$\lim_{n\to\infty} \frac{[(n+1)(n!)]^3}{(3n+3)(3n+2)(3n+1)} \cdot \frac{1}{(n!)^3} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \lim_{n\to\infty} \frac{n^3}{9 \cdot 3n^3} = \frac{1}{27}$

$\frac{1}{27} < 1$ converge

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CRITERIO DELLA RADICE

8.1) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$ $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n} = e^{-1} < 1$ conv

8.2) $\sum_{n=1}^{\infty} \frac{e^{n^2}}{n^2}$ $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{e^{n}}{n} = \infty$ div

8.3) $\sum_{n=1}^{\infty} \frac{e^{\sqrt{n}}}{n^n}$ $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{(e^{\sqrt{n}})^{1/n}}{(n^n)^{1/n}} = \lim_{n\to\infty} \frac{e^{1/n^2}}{n} = 0$ conv.

8.4) $\sum_{n=1}^{\infty} \left(\frac{n^2-2n}{n^2}\right)^{2n^2}$ $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1-\frac{2n}{n^2}\right)^{2n} = e^{-4} < 1$ conv

8.5) $\sum_{n=1}^{\infty} (n^{1/n}-1)^n$ $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} n^{1/n}-1 = \lim_{n\to\infty} e^{\frac{1}{n}\log n}-1 = \infty$

diverg

$(n^{1/n}-1)^n \sim (n^{1/n-1})^n = n$

8.6) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n} = e^{-1} < 1$ converg.

8.5) $\sum_{n=1}^{\infty} (n^{1/n}-1)^n$ $(n^{1/n}-1)^n = \left(n^{1/n}\left(1-\frac{1}{n^{1/n}}\right)\right)^n \sim (n^{1/n})^n = n$

per criterio radice, $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{n}-1 = 1-1 = 0$ conv ?

CRITERIO DI LEIBNIZ

se $a_n \to 0$ per $n\to\infty$ e a_n decrescente $\to$ la serie A SEGNI ALTERNI converge SEMPLICEMENTE

9.1) $\sum_{n=1}^{\infty} (-1)^n (3^{1/n}-1)$ $a_n = 3^{1/n}-1 \to 0$ per $n\to\infty$? sì

a_n decresce? $f(n) = a_n \to f(x) = 3^{1/x}-1$ decresce

$\to$ per Leibniz la serie conv. SEMPLICEMENTE

9.2) $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{n^2+1}{n^3}\right)$ converge assolutamente?

$\left|\frac{n^2+1}{n^3}\right| \sim \left|\frac{1}{n}\right|$ no. Non posso concludere Leibniz?

10.6) $\sum_{n=0}^{\infty} (3^n + 5^n)\alpha^n$

$$5^n\left(\left(\frac{3}{5}\right)^n + 1\right)\alpha^n \sim 5^n\alpha^n = (5\alpha)^n$$

serie geometrica di ragione $q = 5\alpha$

converge se e solo se $|q| < 1$ (so che $\alpha > 0$) $\to q < 1$

$5\alpha < 1 \to \alpha < \frac{1}{5}$ solo qui ho conv.

ESERCIZI DI RIEPILOGO

11.1) $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$ $\qquad \frac{n^2}{1+n^2} \leq \frac{n^2}{n^2} \leq 1$ posso usare cfr asintotico

$\frac{n^2}{1+n^2} \sim 1$ diverg $\to$ poiché la serie è a termini positivi

la serie quindi ha lo stesso carattere di $\sum_{n=1}^{\infty} 1$ diverge

11.2) $\sum_{n=0}^{\infty} \frac{1+\sqrt{n}}{1+n+n^2}$ $\qquad$ noto che $\frac{1+\sqrt{n}}{1+n+n^2} \sim \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

ma $\frac{3}{2} > 1$ quindi converge

11.3) $\sum_{n=0}^{\infty} \frac{1}{2^n - n^3} \sim \left(\frac{1}{2}\right)^n$ con $q = \frac{1}{2} < 1$ converge

11.4) $\sum_{n=0}^{\infty} \frac{n!+1}{(n+1)!}$ criterio del rapporto

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(n+1)!+1}{(n+2)!} \cdot \frac{(n+1)!}{n!+1} =$$

$$\frac{n!\cdot(n+1)\} + 1}{(n+2)\cancel{(n+1)!}} \cdot \frac{\cancel{(n+1)!}}{n!+1} = \frac{(n+1)!+1}{n(n!+1)+2(n!+1)}$$

aspetta

$$\frac{n!+1}{(n+1)!} = \frac{n!}{(n+1)!} + \frac{1}{(n+1)!} \to$$ applico rapporto a due serie distinte, A e B

A $\to \frac{(n+1)!}{(n+2)!} \cdot \frac{(n+1)!}{n!} = \frac{n+1}{n+1} = 1$

B $\to \frac{1}{(n+2)!} \cdot (n+1)! = \frac{1}{n+2}$

$1 + \frac{1}{n+2} = \frac{n+2+1}{n+2} = \frac{n+3}{n+2}$

$\lim_{n\to\infty} \frac{n+3}{n+2} = 1$ non posso concludere

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