NUMERO: 2387A

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A P P U N T I

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MATERIA: Structural Mechanics II - Teoria + Esercizi + Temi di Esame - Prof. Cornetti

STATICALLY DETERMINATED BEAMS

4/10/17

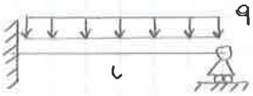
	kinematics	statics
v :	degree of constrain	number of constrain reactions
g :	degree of freedom	number of balance equations

Three alternatives:

$g > v$ mechanism
if $g = v + 1$ → kinematical chain

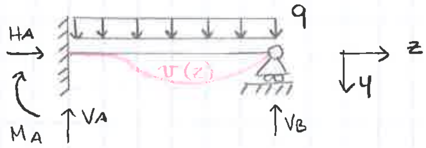
$g = v$ statically determined
number of balace equations leads to solution (enough)

$v > g$ too much unknowns
statically indetermined



$g = 3$
$v = 3 + 1 = 4$
once statically indetermined
there are ∞^1 solutions but it's necessary to choose the one which is also congruent

Methods of solutions



- integration of elastic line

$$\frac{d^4 v}{dz^4} = \frac{q(z)}{EI}$$

$$\frac{d^2 v}{dz^2} = -\frac{M}{EI}$$

$$\frac{d^3 v}{dz^3} = -\frac{T}{EI}$$

Through boundery conditions

$$\begin{cases} v(0) = 0 \\ \varphi(0) = 0 \\ v'(0) = 0 \\ v(\ell) = 0 \\ M(\ell) = 0 \\ \frac{d^2 v}{dz^2} = 0 \end{cases}$$

- method of forces
↳ delete a constrain. (∞ ways)

redondant unknown

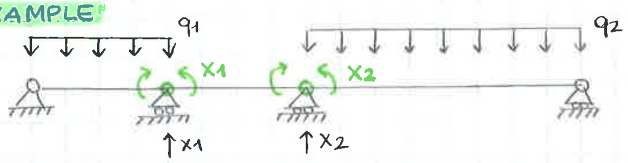
↳ congruent equation
$v_B = 0 = v_B(q) + v_B(X)$ → due to the load, → due to the roller $\} = 0$
↳ vertical displacment

$$v_B(q) = \frac{qL^4}{8EI}$$

$$v_B(X) = -\frac{XL^3}{3EI}$$

$$\frac{qL^4}{8EI} - \frac{XL^3}{3EI} = 0 \Rightarrow X = \frac{3}{8} qL$$

EXAMPLE



$g = 3$
$v = 2+3 = 5$
↳ twice indeterminated
1) two redundant unknowns X_1 and X_2, delating rollers
$v_B = 0$
$v_C = 0$

$$\begin{cases} v_B(q) + v_B(X_1) + v_B(X_2) = 0 \\ v_C(q) + v_C(X_1) + v_C(X_2) = 0 \end{cases}$$

($q \to q_1 + q_2$)

Two equations in two unknowns ⇒ not easy

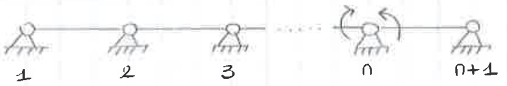
2) innerhings (beam is divided into simply supported beams)
$\varphi_{BA} = \varphi_{BC}$
$\varphi_{CB} = \varphi_{CD}$

$$\begin{cases} \varphi_{BA}(q_1) + \varphi_{BA}(X_1) = \varphi_{BC}(X_1) + \varphi_{BC}(X_2) \\ \varphi_{CB}(X_1) + \varphi_{CB}(X_2) = \varphi_{CD}(q_2) + \varphi_{CD}(X_2) \end{cases}$$

easier than before

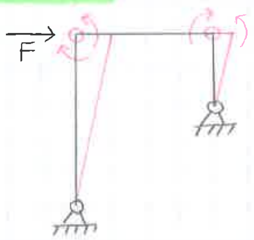
$$\begin{cases} \dfrac{qL^3}{24EI} - \dfrac{X_1L}{3EI} = \dfrac{X_1L}{3EI} + \dfrac{X_2L}{6EI} \\ -\dfrac{X_1L}{6EI} - \dfrac{X_2L}{3EI} = \dfrac{q(2L)^3}{24EI} + \dfrac{X_2(2L)}{3EI} \end{cases}$$

Continous beam over (n+2) supports

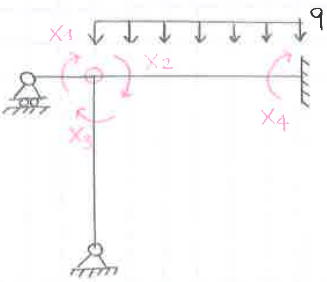


⇒ n unknows
↳ easiest way is to put innerhings and couple of unknowns
↳ n congruent equations in n unknowns
↳ every eq. contains at most 3 unknowns (quite easy) and they are called 3 MOMENT EQUATIONS

EXAMPLE



$g = 3$
$v = 2+2 = 4$
↳ once statically indetermined
↳ associate truss structures is mechanics so the nodes can move
⇒ it's called TRASLATING NODE FRAME

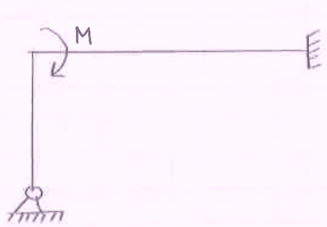


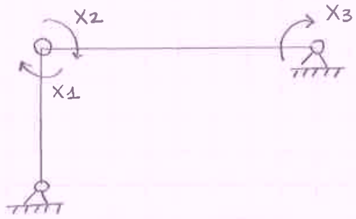
Putting all these ings the system becomes sd (associate truss structures)
$g = 3$ ↳ statically determinated
$v = 3+2+1 = 6$
↳ three time indeterminated

$$\begin{cases} \varphi_{BA} = \varphi_{BD} \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_C = 0 \end{cases}$$

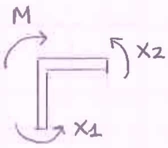
the nodes of frame don't move if the associate truss structures is statically determinated (neglected assial despla

②

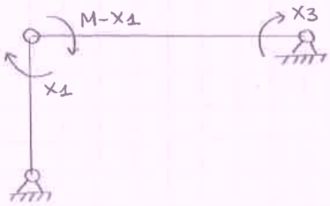


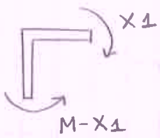


associated truss structure is the same as before, but X_1 and X_2 are different



$M - X_2 - X_1 = 0$
$X_2 = M - X_1$ } only two unknown X_1 and X_3





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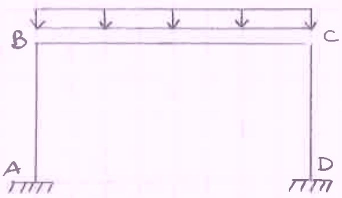
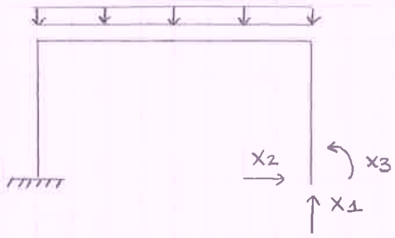
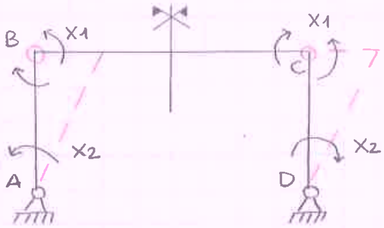




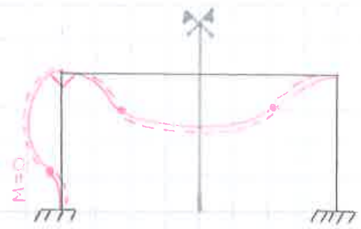
$u_D = 0$
$v_D = 0$
$\varphi_D = 0$

$g = 3$
$v = 3+3 = 6$



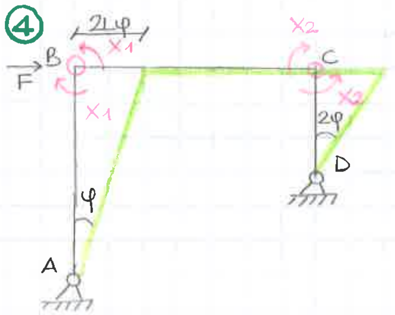
associated truss structure isn't statically determinated
$g = 9$
$v = 4 \cdot 2 = 8$
↳ mechanism, but it's antisimetrical so it cannot happen so it's a rotating-node frame

$\varphi_A = 0$
$\varphi_{BA} = \varphi_{BC}$
$\varphi_{CB} = \varphi_{CD}$ } useless (not needed)
$\varphi_D = 0$

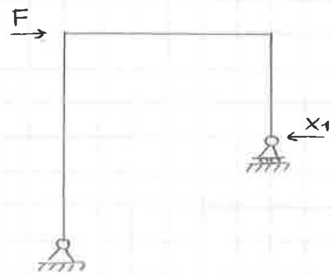


when M=0 there is inflection point
↳ the concavity of the displacement depends on the sign of the moment

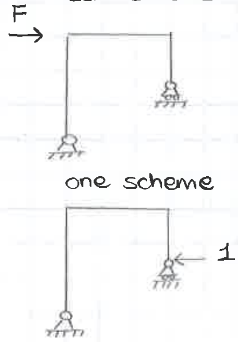
④



MoF



zero scheme



one scheme

$g = 3$
$v = 2+2 = 4$

$u_D = 0$
$\int M_1^2 dz$ } computed
$\int M_0 M_1 dz$ }

↳ once statically indeterminated
↳ a hinge in every node
↳ with the hinge in C the structure becomes statically determined, but the formulary can't be used because ABC isn't a simply supported beam.

$g = 6$
$v = 6$

⟹ so put a hinge in every node

$g = 9$
$v = 8$

mechanism without simmetry so there are two alternative ways → we have to "built" a relationship of balance between X1 and X2

- cardinal mechanical equations
- principle virtual works

imagine a virtual displacement (in a number which is the same of mechanism)
↳ congruent displacement

$x = 2L \, tg\varphi = 2L\varphi$

all the structure moves in a rigid movement

virtual displacement · real forces = virtual forces · real displacement (= 0)

↳ for BC there is not rotation so not work

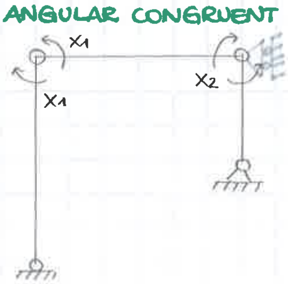
$2\varphi L \cdot F + X_1 \varphi - X_2 \cdot 2\varphi = 0$

work is positive when force and displacement are in the same direction

EQUILIBRIUM EQUATION

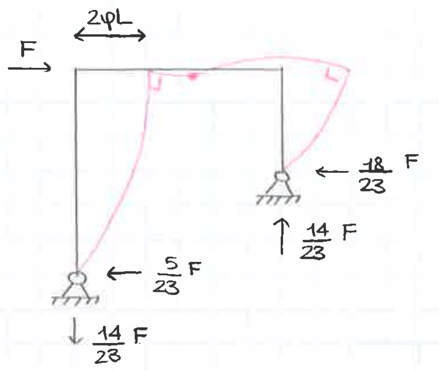
$\forall \varphi: \ \varphi(2FL + X_1 - 2X_2) = 0$ (φ is arbitrary)

ANGULAR CONGRUENT EQUATION



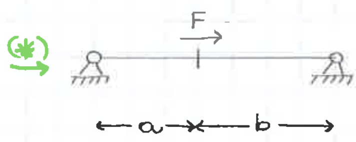
we put a fictitious roller so the structure became not-ruotated node (before, it was a traslating node structure)

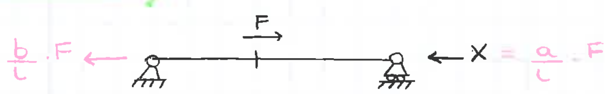
$$\begin{cases} \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = \varphi_{CD} \end{cases}$$



F is equilibrated partly by the right reaction partly by the left one

$2\varphi L = \frac{44}{69}\frac{FL^3}{EI}$





$g = 3$

$v = 2+2 = 4$

statically indeterminated

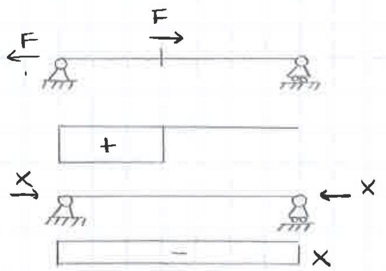
$u_B = F/EA \cdot a$

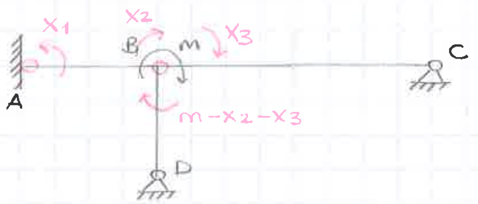
$u_B = -X/EA \cdot L$

$\frac{F}{EA} \cdot a - \frac{X}{EA} L = 0 \Rightarrow X = \frac{a}{L} \cdot F$

X has to be which

$u_B = u_B(F) + u_B(X) = 0$





Degree of internal hinge

$2(n-1)$

$g = 3$

$v = 3+2+2 = 7$

The associate truss structure has:

$g = 9$

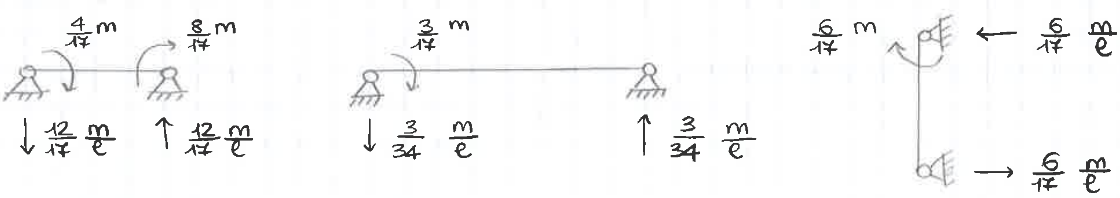
$v = 6+4 = 10$

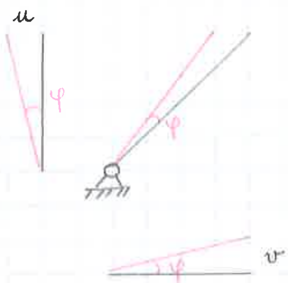
Still statically indeterminated

Three congruent equations:

$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BD} \\ \varphi_{BA} = \varphi_{BC} \end{cases}$$

$$\begin{cases} \frac{X_1 \ell}{EI} + \frac{X_2 \ell}{6EI} = 0 \\ -\frac{X_1 \ell}{6EI} - \frac{X_2 \ell}{3EI} = -\frac{(m - X_2 - X_3)\cdot \ell}{3EI} \\ -\frac{X_1}{6EI} - \frac{X_2 \ell}{3EI} = -\frac{X_3 \cdot 2\ell}{3EI} \end{cases}
\qquad
\begin{cases} X_1 = -\frac{4}{17} m \\ X_2 = \frac{8}{17} m \\ X_3 = \frac{3}{17} m \end{cases}$$





Through these diagrams we can read always the two components of displacement of a generic point of the beam

Movment can be pure rotation or traslation

↳ the centre of rotation lies on the line which is orthogonal to the line of displacement (it can be relative if the constrain is internal)

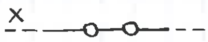
$C \in a$

$C = A$

$C = \infty$ } horizontal translation

↳ of vertical axis

$C = A$



$C \in x$

1° Theorem of kinematical chain

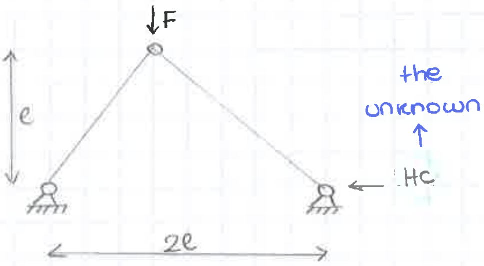
CNES that a beam system is chained is that C_i, C_j and C_{ij} are lined

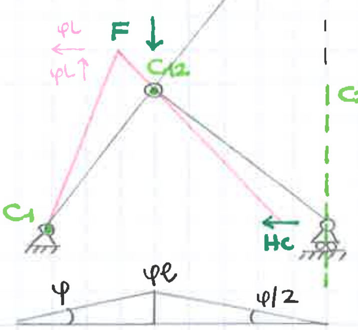
↳ at least 2 beams

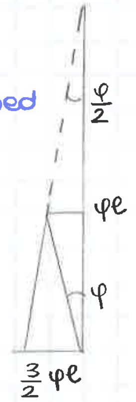
2° Theorem of kinematical chain

CNES that a system is chained is that C_{IJ}, C_{IK}, C_{JK} are lined

↳ at least 3 beams







We can applied the principle virtual work

In C1 e C2 rotation is zero, in C12 the two rotations (right and left) are the same.

$L_{v,ext} = 0$

↳ of virtual forces

$- F\varphi e + H_C \cdot \frac{3}{2} \varphi e = 0$

$\forall \varphi \quad \varphi \left(- Fe + \frac{3}{2} H_C \cdot e\right) = 0 \rightarrow H_C = \frac{2}{3} F$

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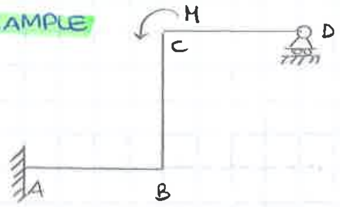
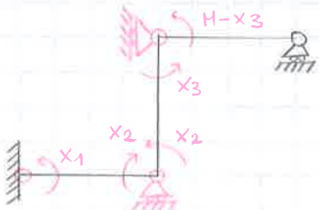
EXAMPLE



$g = 3$
$v = 3 + 1 = 4$
$n = 1$
↳ degree of indeterminated

Associated truss structure

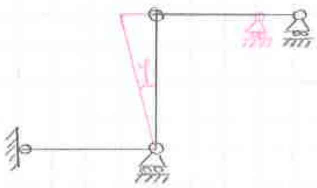


$g = 3 \cdot 3 = 9$
$v = 3 \cdot 2 + 1 = 7$
$\ell = 2$
↳ twice hypostatic

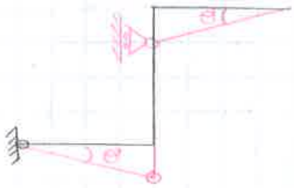
Use a fictious roller to fix the configuration, in this case we need two rollers.
↳ $2\ell + n = 5$ number of unknowns and equations

x_1, x_2, x_3 three static unknowns
φ, θ two kinematical unknowns → arbitrary, not in the equations

removing a roller the structure is a mechanism, a different one for each roller



For the 1° equilibrium equation (x_1, x_2, x_3)



For the 2° equilibrium equation (x_1, x_2, x_3)

They have to be balanced, both of them
→ ℓ equilibrium equations

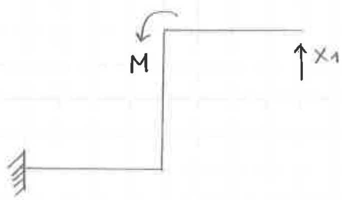
$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = \varphi_{CD} \end{cases}$$

$(n+\ell)$ angular congruent equations

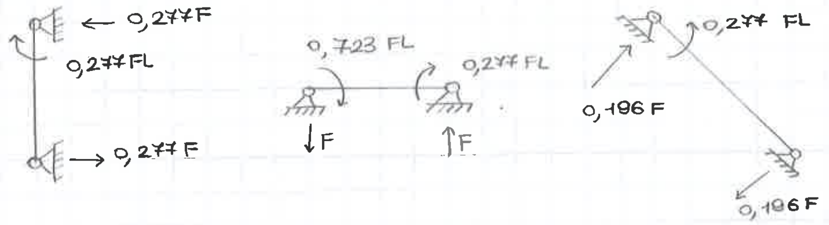
→ θ and φ appears

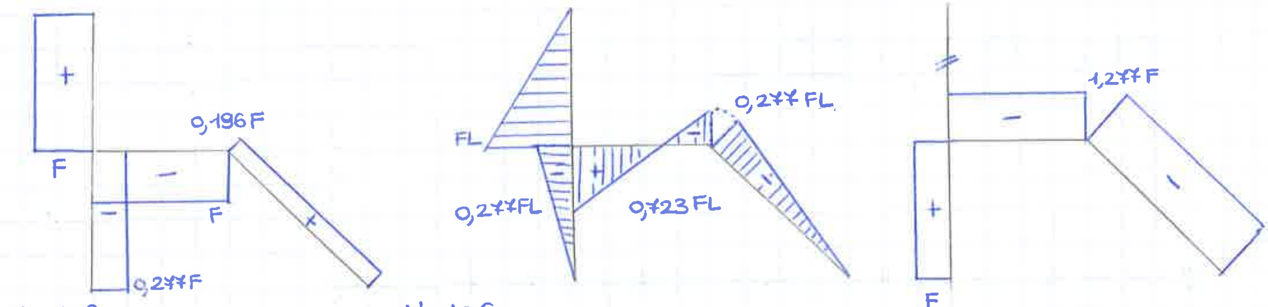
⇒ 5 equations in 5 unknowns $(x_1, x_2, x_3, \varphi, \theta)$

Method of forces



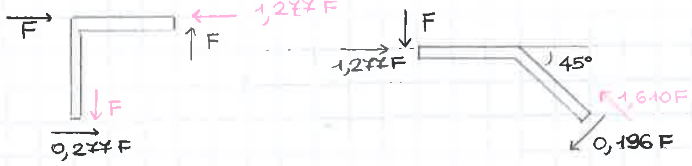
$v_D = 0$ (1 equation in 1 unknown x_1)
$\int M_1^2$
$\int M_0 M_1$





Node B

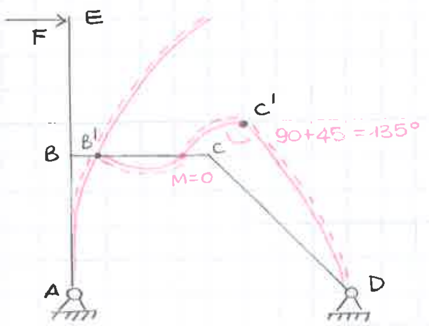
Node C



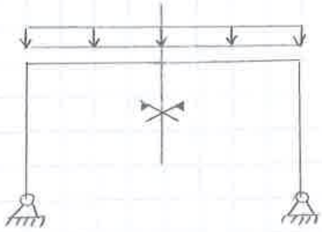
$N_{CD} + \frac{\sqrt{2}}{2} \cdot 1{,}277\,F + \frac{\sqrt{2}}{2}\,F = 0$

$N_{CD} = -1{,}610\,F$

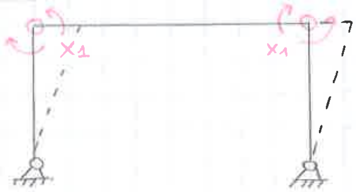
Deflection curve



SYMMETRICAL STRUCTURES UNDER SYMMETRICAL LOADS

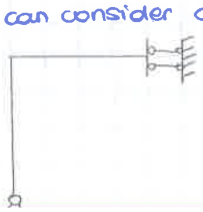


I method



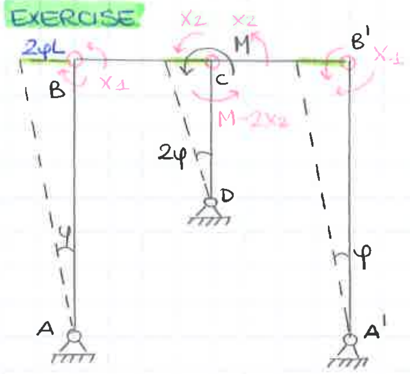
$\varphi_{BA} = \varphi_{BB'}$
Rotating node frame → mechanism has to be symmetrical so it can't happen

=> we can consider only half structure



→ fixed rotation and horizontal movement, but it allows vertical displacement

EXERCISE



Symmetrical structure with antisymmetrical loads

$g = 3 \cdot 5 = 15$
$\vartheta = 14$
$n = 1$

The mechanism is antisymmetrical, according to the loads

Equilibrium equation

$$-X_1\varphi + (M-2x_2)2\varphi - x_1\varphi = 0$$
$$\varphi(-2x_1 - 4x_2 + M) = 0 \quad \forall\varphi$$
$$x_1 + 2x_2 = M$$

Congruent angular equations

$\varphi_{BA} = \varphi_{BC}$ ⟵ $\varphi_{CB} = \varphi_{CD}$
$\varphi_{CB} = \varphi_{CB'}$ ⟶ $\varphi_{B'A'} = \varphi_{B'C}$ (useless because of symmetry)
$0 = 0$

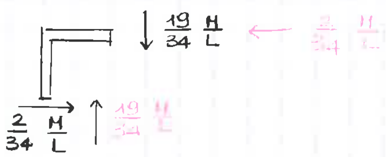
Considering symmetry M unknowns became from four, two.

$$\begin{cases} -\dfrac{x_1 2L}{3EI} + \varphi = \dfrac{x_1 L}{3EI} - \dfrac{x_2 L}{6EI} \\ -\dfrac{x_1 L}{6EI} + \dfrac{x_2 L}{3EI} = \dfrac{(M-2x_2)L}{3EI} + 2\varphi \\ x_1 + 2x_2 = M \end{cases}$$

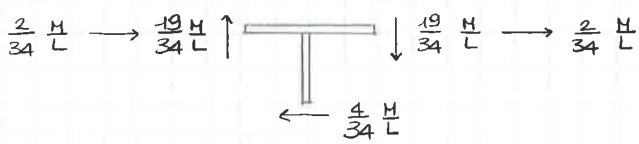
↳ Three equations in three unknowns

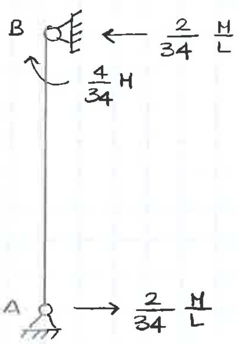
$$\begin{cases} X_1 = \dfrac{4}{34}M \\ X_2 = \dfrac{15}{34}M \\ \varphi = \dfrac{3}{68}\dfrac{mL}{EI} \end{cases}$$

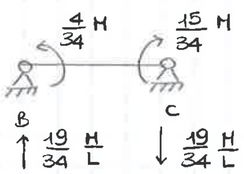
Node B

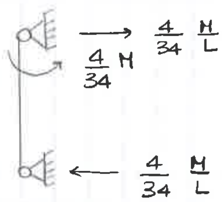


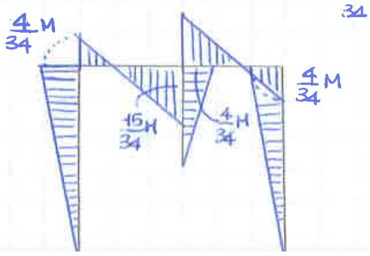
Node C

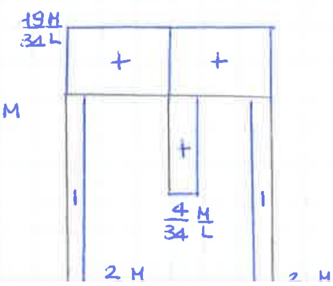


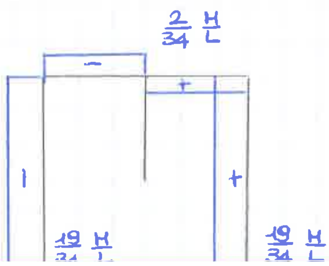












LUSAS

12/10/17

Approximate solutions for difficult problems
Final element method

The structure is semplified introducing a number of nodes (finite number), linked with lines (finite number)
↳ from infinite configuration to finite one
↳ the structure is a **MESH**

The unknowns are the nodal displacements, interpolating them we can compute the displacement everywhere and so the state of stress.

We pass from partial differential equations to ordinal diffential equation (easier to solve)

- vertical axis: orthogonal to the working plane (z)
- depth, width
- cantilever

- Asy is the changing of area, depending on the cross section and shear deformability (Timoshenko beam theory, if we neglect it we use Bernoulli theory)

  σ = Navier's formula
  τ = Jourawsky's formula

  ⇒ .dxf file can be imported in Lusas

- New points
  A (0,0,0)
  B (1,0,0)
- Select all the points and click on "new line"
- Attributes
  ↳ follow the sequence

  1) Mesh: line because it's a 2D problem
     Thin beam because we neglect shear deformation
     Number of divisions increases the precision of the solution
  2) Geometry: cross section properties (line)
  3) Material: isotropic (steel)
  4) Supports
  5) Loading
- Select the beam and assign all the things constructed before
- right click:
  - values of displacement
  - diagrams

Fictisious roller to fixed the structure ([illegible]) and calculate the elastic rotation.

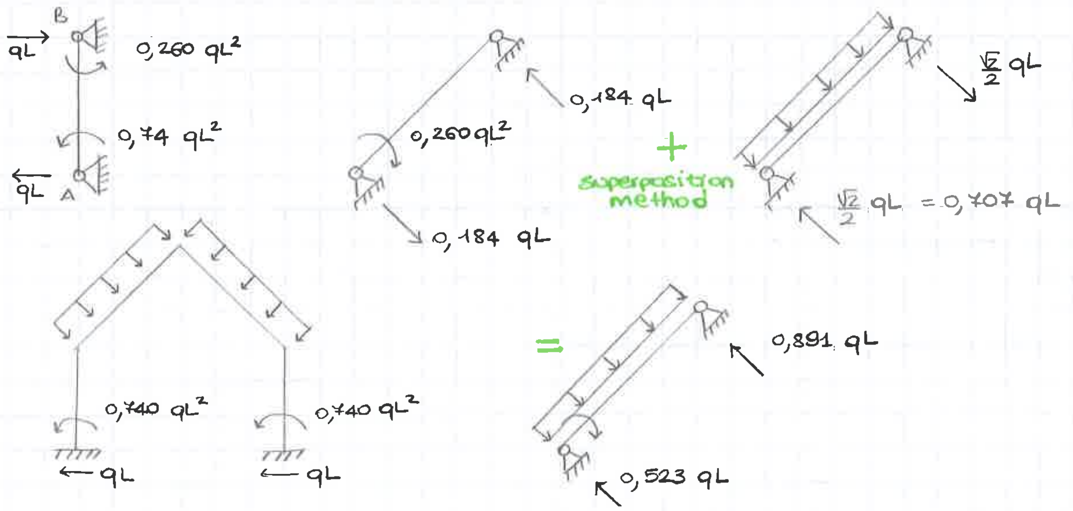
$\varphi_A = 0$

$\varphi_{BA} = \varphi_{BC}$

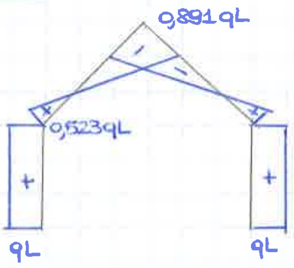
$$\begin{cases} \frac{X_1 L}{3EI} + \frac{X_2 L}{6EI} - \varphi = 0 \\ -\frac{X_2 L}{3EI} - \frac{X_1 L}{6EI} - \varphi = \frac{X_2 (L\sqrt{2})}{3EI} - \frac{q(\sqrt{2}L)^3}{24} \\ X_1 - X_2 = qL^2 \end{cases}$$

↳ resolving system

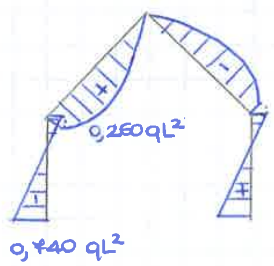
$$\begin{cases} X_1 = \frac{8+9\sqrt{2}}{28} qL^2 \simeq 0{,}740\ qL^2 \\ X_2 \cong -0{,}260\ qL^2 \\ \varphi \simeq 0{,}2035\ \frac{qL^2}{EI} \end{cases}$$

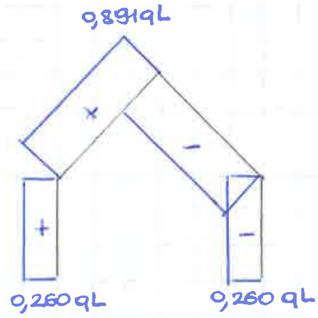


Shear

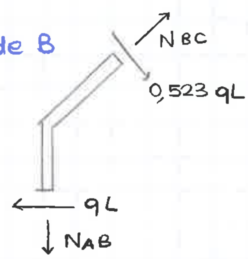


Moment





Node B



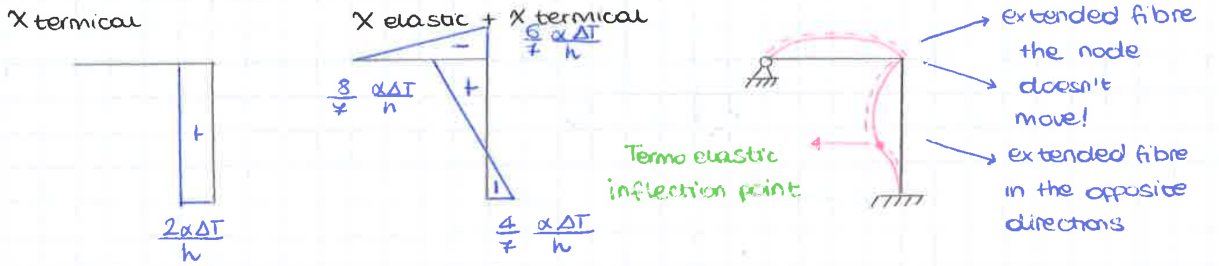
$\xrightarrow{+})$ $-qL + \frac{\sqrt{2}}{2} N_{BC} + \frac{\sqrt{2}}{2} 0{,}523\, qL = 0$

$N_{BC} = (-0{,}523 + \sqrt{2})\, qL \simeq 0{,}891\, qL$

$+\uparrow)$ $0 = 0{,}523\, qL - \frac{\sqrt{2}}{2} qL + \frac{\sqrt{2}}{2} N_{AB}$

$N_{AB} = (1 - 0{,}523\sqrt{2})\, qL \simeq 0{,}260\, qL$

The concavity of deformation doesn't depend on the moment diagram anymore (we have to evaluate the total curvature!)



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$$V_C(F) = \frac{\frac{F}{2}}{EA_2} \cdot L \qquad K_1 = \frac{EA_1}{L}$$

$$V_C(X) = -\frac{XL}{EA_1} - \frac{\frac{X}{2}}{EA_2} \cdot L \qquad K_2 = \frac{EA_2}{L}$$

$$\frac{F}{2EA_2} \cdot L - \frac{XL}{EA_1} - \frac{X}{2K_2} = 0$$

$$\frac{F}{2K_2} - \frac{X}{K_1} - \frac{X}{2K_2} = 0$$

$$(2K_2 + K_1)X = K_1 F$$

$$\boxed{X = \frac{K_1}{K_1 + 2K_2} \cdot F} = N_{CQ}$$

$$\delta_1 = \frac{N_{CQ}}{K_1} = \frac{X}{K_1} = \frac{F}{K_1 + 2K_2} \longrightarrow \text{comparison, we find } \delta_2 \text{ and } K_{TOT}$$

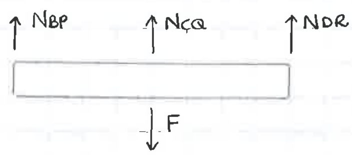
$$\delta_2 = \frac{F}{K_{TOT}}$$

$$K_{TOT} = K_1 + K_2$$

$$N_{BP} = \frac{F - N_{CQ}}{2} = \frac{K_2}{K_1 + 2K_2} \cdot F$$

② Method of displacement

$$v_P = v_Q = v_R = \delta$$



$$N_{BP} + N_{CQ} + N_{DR} - F = 0$$

$$N_{BP} = K_2 \delta = N_{DR}$$

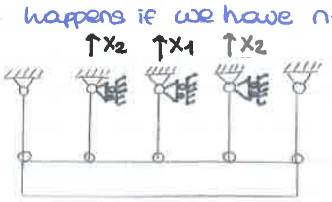
$$N_{QC} = K_1 \delta$$

$$2K_2\delta + K_1\delta = F$$

$$\boxed{\delta = \frac{F}{K_1 + 2K_2}}$$

$$N_{CQ} = K_1 \delta = \frac{K_1}{K_1 + 2K_2} F$$

- What happens if we have n-bars?



If $n = 5$ unknown forces are 2

$$1 - \frac{n-3}{2} = \frac{n-1}{2} = \frac{5-1}{2} = 2$$

→ METHOD OF FORCES ⇒ unknowns
→ METHOD OF DISPLACEMENT ⇒ unknown is always one

$$\sum_{i=1}^{n} N_i = F \qquad N_i = K_i \delta \longrightarrow \text{is the same for every bar}$$

$N(L) = N_J \qquad N(0) = -N_i$

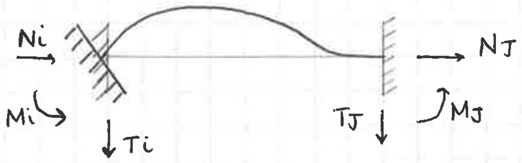
$T(L) = T_J \qquad T(0) = -T_i$

$M(L) = M_J \qquad M(0) = -M_i$

$$\begin{pmatrix} M_i \\ T_i \\ N_i \\ M_J \\ T_J \\ N_J \end{pmatrix} = \begin{pmatrix} \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \frac{6EI}{L^2} & 0 \\ -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 \\ 0 & 0 & \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} \\ \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & \frac{6EI}{L^2} & 0 \\ \frac{6EI}{L} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 \\ 0 & 0 & -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} \end{pmatrix} \begin{pmatrix} \varphi_i \\ \upsilon_i \\ \omega_i \\ \varphi_J \\ \upsilon_J \\ \omega_J \end{pmatrix}$$

vector of reactions [Q] = stiffness matix [K] · vector of nodal displacement [δ]

- The first column is obtained by setting $\varphi_i \neq 0$
  $\upsilon_i = \omega_i = \varphi_J = \upsilon_J = \omega_J = 0$
  $N = k\delta$



$$\frac{d^2\upsilon}{dz^2} = -\frac{M}{EI}$$

$$\frac{d^3\upsilon}{dz^3} = -\frac{T}{EI}$$

$$\frac{d\upsilon}{dz} = -\varphi(z)$$

$$\frac{d^4\upsilon}{dz^4} = \frac{q}{EI} = 0$$

$$\begin{cases} \upsilon(z) = c_1 z^3 + c_2 z^2 + c_3 z + c_4 \\ \upsilon'(z) = 3c_1 z^2 + 2c_2 z + c_3 \\ \upsilon''(z) = 6c_1 z + 2c_2 \\ \upsilon'''(z) = 6c_1 \end{cases} \qquad \begin{cases} \upsilon(0) = 0 \\ \varphi(0) = \varphi_i \\ \upsilon(L) = 0 \\ \varphi(L) = 0 \end{cases}$$

$$\begin{cases} \upsilon(0) = 0 \\ \upsilon'(0) = -\varphi_i \\ \upsilon(L) = 0 \\ \upsilon'(L) = 0 \end{cases} \qquad \begin{cases} c_4 = 0 \\ c_3 = -\varphi_i \\ c_1 L^3 + c_2 L^2 + c_3 L = 0 \\ 3c_1 L^2 + 2c_2 L + c_3 = 0 \end{cases}$$

$$2c_1 L^2 + c_2 L = 0$$

$c_2 = -2c_1 L$

$-c_1 L^2 - \varphi_i = 0$

$\hookrightarrow c_1 = -\frac{\varphi_i}{L^2}$

$c_2 = 2\frac{\varphi_i}{L}$

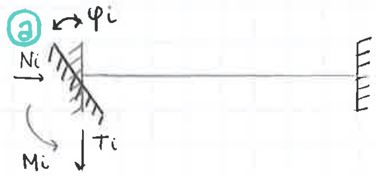
$$M = -EI\,\upsilon''(z) = -EI\left(-\frac{6\varphi_i}{L^2}z + \frac{4\varphi_i}{L}\right) = EI\varphi_i\left(\frac{6}{L^2}z - \frac{4}{L}\right)$$

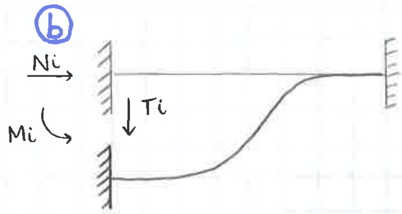
$N_i = \frac{EA}{L} w_i$

$N_J = -\frac{EA}{L} w_i$

$T_i = M_i = T_J = M_J = 0$

- $[K]$ symmetrical





BETTI'S THEOREM:

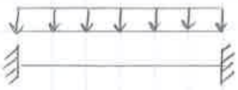
$T_i^{(a)} v_i^{(b)} = M_i^{(b)} \varphi_i^{(a)}$

$v_i^{(b)} = \varphi_i^{(a)} = 1$

$T_i^{(a)} = M_i^{(b)}$

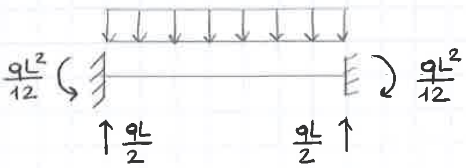
$K_{21} = K_{12}$

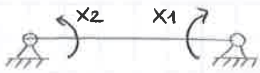
- $\{Q\} = [K]\{\delta\} - \{F\}$ → vector of the equivalent nodal forces



If $\delta = 0$ so $\{Q\} = -\{F\}$

- $\{F\}$ for distributed load

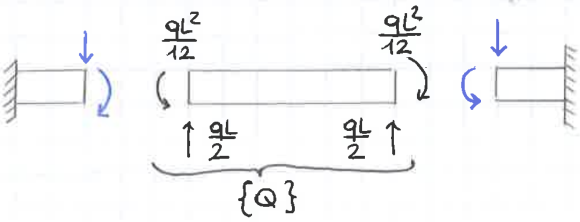




$\varphi_A = 0$

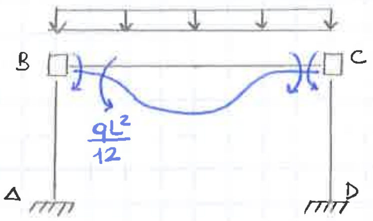
$\frac{X_1 L}{3EI} + \frac{X_1 L}{6EI} - \frac{qL^3}{24EI} = 0$

$X_1 = \frac{qL^2}{12}$



$$\{F\} = \begin{pmatrix} -\frac{qL^2}{12} \\ \frac{qL}{2} \\ 0 \\ -\frac{qL^2}{12} \\ \frac{qL}{2} \end{pmatrix}$$

$\{Q\} = [K]\{\delta\} - \{F\}$



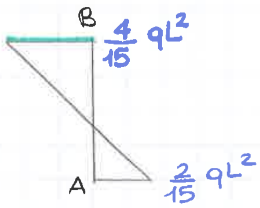
Rotational equilibrium of node B

$$-\frac{2EI}{L}\varphi_B - \frac{4EI}{L}\varphi_B - \frac{EI}{L}\varphi_C - \frac{qL^2}{3} = 0$$

$\varphi_C = -\varphi_B$

$$-\frac{5EI}{L}\varphi_B = \frac{qL^2}{3}$$

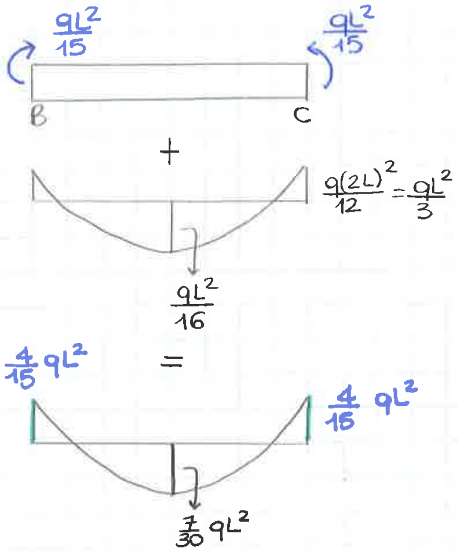
$$\boxed{\varphi_B = -\frac{qL^3}{15EI}}$$



$$M_A = \frac{2EI}{L}\varphi_B = -\frac{2}{15}qL^2$$

(we can use table or matrix to find it)

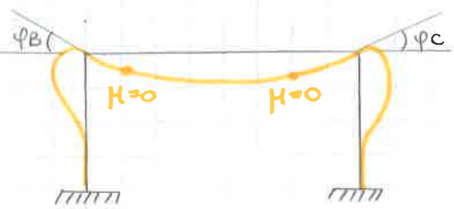
$$M_B = \frac{4EI}{L}\varphi_B = -\frac{4}{15}qL^2$$



$$M_B = \frac{4EI}{2L}\varphi_B + \frac{2EI}{2L}\varphi_C = -\frac{1}{15}qL^2$$

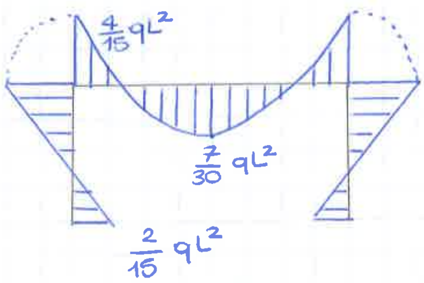
$$M_C = \frac{2EI}{2L}\varphi_C + \frac{4EI}{2L}\varphi_C = \frac{1}{15}qL^2$$

Deformations



In this case displacement is more similar to reality (convincent) because we have had only one equation to solve

Final result



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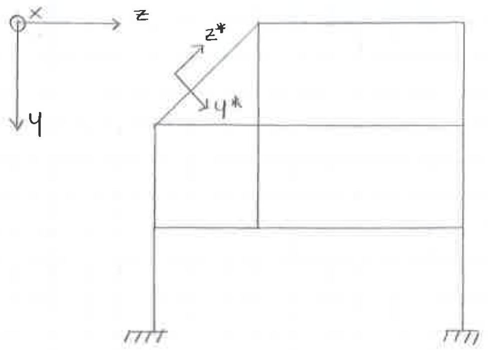


⇒ solve the structure with a code

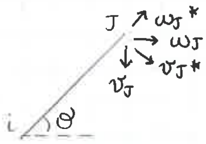
- pass to a local reference system to a global one.

$$[K_e^*]\{\delta_e^*\} - \{F_e^*\} = \{Q_e^*\}$$

(* → local reference system; e → single element)

- $[K_e^*]$: stiffness matrix of the beam
- $\{\delta_e^*\}$: nodal displacement vector
- $\{F_e^*\}$: nodal load equivalent
- $\{Q_e^*\}$: vector of constraint reactions

- we consider this beam



$$v_J^* = v_J \cos\theta + w_J \operatorname{sen}\theta$$
$$w_J^* = -v_J \operatorname{sen}\theta + w_J \cos\theta$$
$$\varphi_J^* = \varphi_J$$

$$\begin{pmatrix} \varphi_J^* \\ v_J^* \\ w_J^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \operatorname{sen}\theta \\ 0 & -\operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varphi_J \\ v_J \\ w_J \end{pmatrix}$$

$$\begin{pmatrix} \varphi_i^* \\ v_i^* \\ w_i^* \\ \varphi_J^* \\ v_J^* \\ w_J^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \operatorname{sen}\theta & 0 & 0 & 0 \\ 0 & -\operatorname{sen}\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \operatorname{sen}\theta \\ 0 & 0 & 0 & 0 & -\operatorname{sen}\theta & 0 \end{pmatrix} \begin{pmatrix} \varphi_i \\ v_i \\ w_i \\ \varphi_J \\ v_J \\ w_J \end{pmatrix}$$

$$[\delta_e^*] = [N]\{\delta_e\}$$

$[N]$: 6x6 (rotation matrix)

- properties of $[N]$

$$[N]^{-1} = [N]^T$$
$$[N]^T[N] = [I]$$

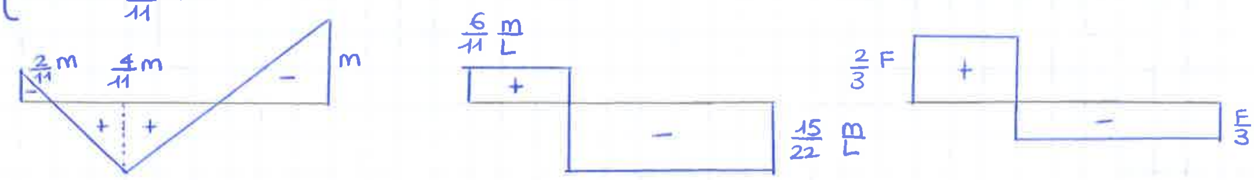
$$\underset{6\times6}{[K_e^*]}\,\underset{6\times6}{[N]}\{\delta_e\} - \underset{6\times6}{[N]}\{F_e\} = [N]\{Q_e\}$$

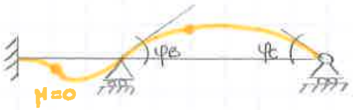
$$[N]^T[K_e^*][N]\{\delta_e\} - \{F_e\} = \{Q_e\}$$

$$[K_e]\{\delta_e\} - \{F_e\} = \{Q_e\}$$

$[K_e]$: STIFFNESS MATRIX in the global reference system.

$$\begin{cases} X_1 = \frac{2}{11} m \\ X_2 = -\frac{4}{11} m \end{cases}$$

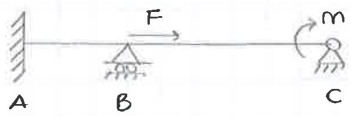




$$\varphi_B = -\frac{2}{11 \cdot 6}\frac{mL}{EI} + \frac{4}{11}\cdot\frac{1}{3}\frac{mL}{EI} = \frac{mL}{11EI}$$

$$\varphi_C = -\frac{m(2L)}{3EI} - \frac{X_2(2L)}{6EI} = -\frac{2}{3}\frac{mL}{EI} + \frac{4}{11\cdot 3}\frac{mL}{EI} = -\frac{18}{33}\frac{mL}{EI} = \frac{6}{11}\frac{mL}{EI}$$

(2) Method of displacement

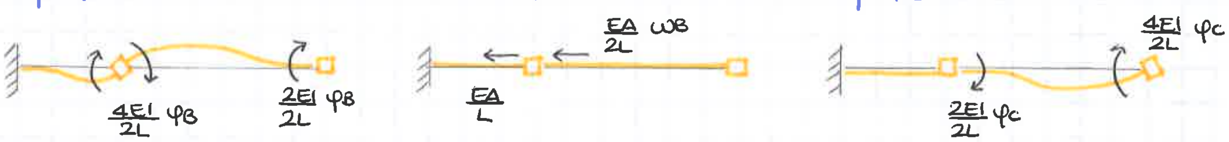


$s = 3$

$\varphi_B; w_B; \varphi_C$ (unknowns)

$$\underset{3\times3}{[k]}\underset{3\times1}{\{\delta\}} = \underset{3\times1}{\{F\}}$$

- $\varphi_B \neq 0$
- $w_B \neq 0$
- $\varphi_C \neq 0$



$$\begin{cases} -\frac{4EI}{L}\varphi_B - \frac{2EI}{L}\varphi_B - \frac{EI}{L}\varphi_C = 0 \\ -\frac{EA}{L}w_B - \frac{EA}{2L}w_B + F = 0 \\ -\frac{EI}{L}\varphi_B - \frac{2EI}{L}\varphi_C - m = 0 \end{cases}$$

This system is actually $[K]\{\delta\} = \{F\}$

$$\begin{cases} 6\varphi_B + \varphi_C = 0 \\ -2w_B - w_B + \frac{2FL}{EA} = 0 \\ -\varphi_B - 2\varphi_C - \frac{mL}{EI} = 0 \end{cases} \qquad \begin{cases} w_B = \frac{2}{3}\frac{FL}{EA} \\ \varphi_C = -\frac{6}{11}\frac{mL}{EI} \\ \varphi_B = \frac{mL}{11EI} \end{cases}$$

Comparison with method of force (same results)

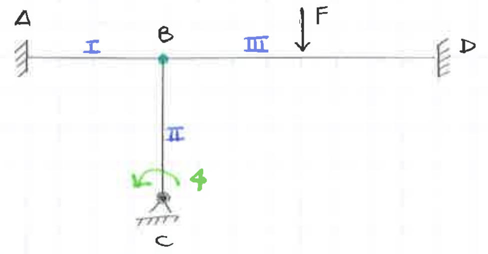
$$M_A = \frac{2EI}{L}\varphi_B = \frac{2}{11} m$$

If $\{Fe\} \neq 0$ $[K]\{\delta\} = \{F\} + \{Fe\}$

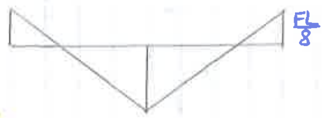
in our case:

$$\{Fe\} = \begin{pmatrix} Feq_1 \\ Feq_2 \\ Feq_3 \end{pmatrix} \qquad \begin{aligned} Feq_1 &= Fe_4^{(I)} + Fe_1^{(II)} \\ Feq_2 &= Fe_6^{(I)} + Fe_3^{(II)} \\ Feq_3 &= Fe_4^{(II)} \end{aligned}$$

EXERCISE



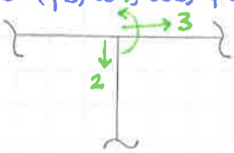
$\{Fe\} = \left(-\frac{FL}{8}; \frac{F}{2}; 0; \frac{FL}{8}; \frac{F}{2}; 0\right)^T$



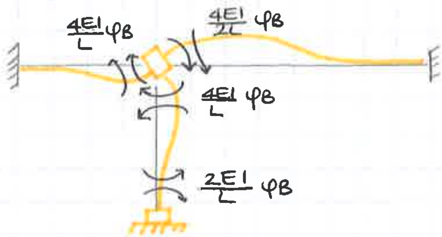
$s = 3 + 1 = 4$

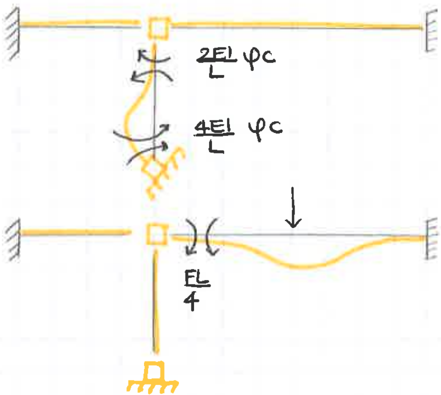
$v - g = 5$

unknowns: $(\varphi_B, v_B = 0, w_B, \varphi_C)$



	φ_i	v_i	w_i	φ_J	v_J	w_J
beam 1	0	0	0	1	2	3
beam 2	4	0	0			
beam 3	1	2	3	0	0	0





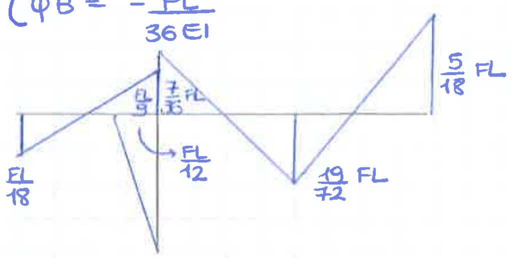
Node B

$$-\frac{4EI}{L}\varphi_B - \frac{2EI}{L}\varphi_B - \frac{4EI}{L}\varphi_B - \frac{2EI}{L}\varphi_C - \frac{FL}{4} = 0$$

Node C

$$-\frac{2EI}{L}\varphi_B - \frac{4EI}{L}\varphi_C = 0$$

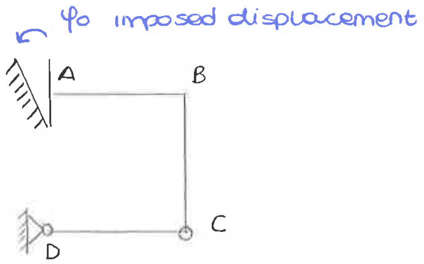
$$\begin{cases} \varphi_C = \frac{1}{72}\frac{FL^2}{EI} \\ \varphi_B = -\frac{FL^2}{36EI} \end{cases}$$



MIXED MODE ~ imposed displacement

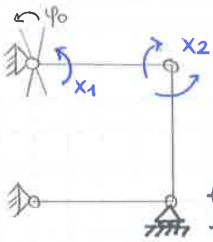
26/10/17

Ex 1

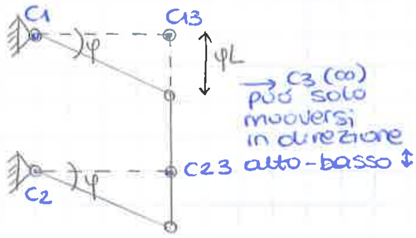


$g = 6$
$v = 7$

- Associate truss structure



fictitious roller to say that the strain is fixed



3 unknowns so 3 equations

PLV

① $\varphi_{AB} = \varphi_0$
② $\varphi_{BA} = \varphi_{BC}$

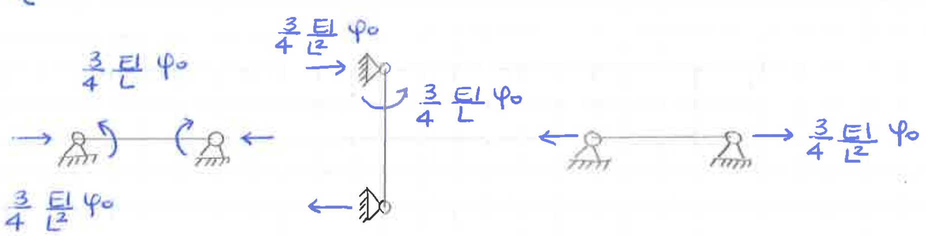
① $\dfrac{X_1 L}{3EI} + \dfrac{X_2 L}{6EI} - \varphi - \varphi_0$

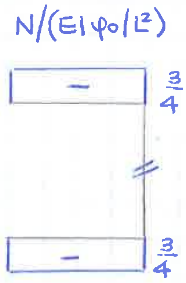
② $-\dfrac{X_2 L}{3EI} - \dfrac{X_1 L}{6EI} - \varphi = \dfrac{X_2 L}{3EI}$

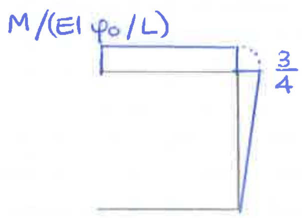
③ equilibrium equations:
$X_2\varphi - X_1\varphi = 0$

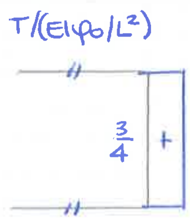
$$\begin{cases} X_1 = X_2 = \dfrac{3}{4}\dfrac{EI}{L}\varphi_0 \\ \varphi = -\dfrac{5}{8}\varphi_0 \end{cases}$$

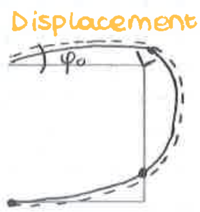
→ it means that the moment is constant in AB
shear force = 0





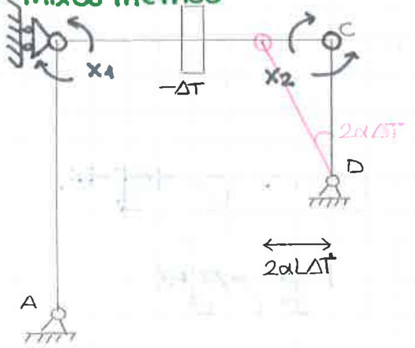




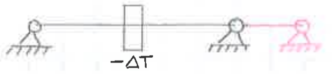


1) EX: mixed method

2/11/17

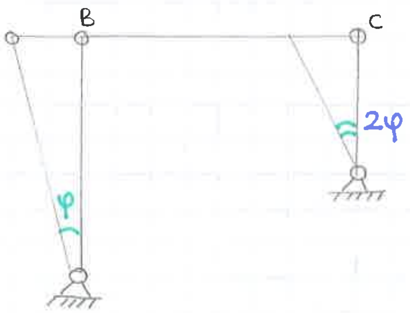


Substitute every internal node with a hinge (= associated truss structure)
$g = 9$
$v = 8$
To make the system fixed we put a fictitious roller
The effect of variation of temperature are:



If we choose to put the roller in C the reduction because of the temperature will be in the opposite part and the angle will be $\alpha\Delta T$ because the beam (AB) is 2L long.

⇒ we choose the angle φ, linked to the displacement in the original truss structure



Three unknowns X_1, X_2 and φ

$$\begin{cases} \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = \varphi_{CD} \\ \text{PVW (principle of virtual works)} \sim \text{couples} \cdot \text{angles} \end{cases}$$

→ the couple X_1, X_2 which acts on BC (it moves only horizontally) do

$$\begin{cases} -\frac{X_1 \cdot (2L)}{3EI} + \varphi = \frac{X_1 \cdot (2L)}{3EI} + \frac{X_2\,(2L)}{6EI} \\ -\frac{X_1\,(2L)}{6EI} - \frac{X_2\,(2L)}{3E} = \frac{X_2 L}{} + 2\alpha\Delta T + 2\varphi \\ X_1\varphi + 2X_2\varphi = 0 \end{cases}$$

$$\begin{cases} X_1 = -\frac{12}{23}\frac{\alpha\Delta T EI}{L} \\ X_2 = -\frac{6}{23}\frac{\alpha\Delta T EI}{L} \\ \varphi = -\frac{18}{23}\frac{\alpha\Delta T EI}{L} \end{cases}$$

dimension $[F \cdot L]$

We have to compute the shear and the axial forces (⇒ all the parts simply supported)

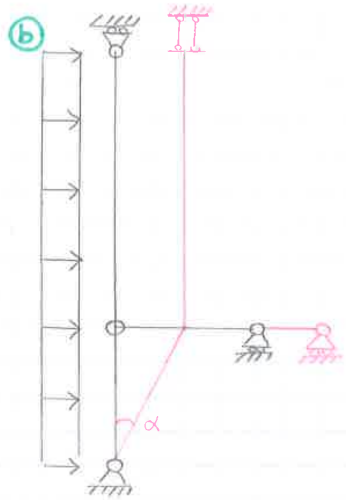
We can try to put a fictitious roller => the motion now is just one (one angle φ)
if we change the position of the roller the motion changes too (one angle θ)

{ two congruent equations
{ two equilibrium equations

PVW (indipendent ~ one related to φ, the other one to θ)

$$\begin{cases} \varphi_{DC} = \varphi_{DB} \\ \varphi_{DB} = \varphi_{DA} \\ PVW_1(\varphi) \\ PVW_2(\theta) \end{cases}$$

We want to prevent the rotation, but to allow the displacement for the second scheme (=> we put a slider) ~ another possible motion



=> the final deflection curve, using (a) or (b) has to be the same.

$M = -EI\, v''(z)$ moment
$T = -EI\, v'''(z)$ shear

⇒ linear elastic model on soil acts a pressure p

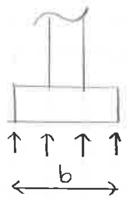
$p = -k v$

$[F/L^2]$ — downwards is positive (vertical displacement [L])

mechanical property of soil (SOIL ELASTIC MODULUS) $[F][L^{-3}]$.

Below the distributed load we have pressure of soil.

$$\frac{d^4v}{dz^4} = \frac{q + q_{SOIL}}{EI}$$



p has a constant distribution → it can be accepted as true (possible)

$q_{SOIL} = p \cdot b = -kb \cdot v = -K v$

equivalent spring stiffness (not a direct property of the soil $= -k \cdot$ geometrical width) $[F][L^{-2}]$

$$\frac{d^4v}{dz^4} = \frac{q - Kv}{EI}$$

$$\frac{d^4v}{dz^4} + \frac{Kv}{EI} = \frac{q(z)}{EI}$$ governing equation of beam on elastic foundation

$$\beta^4 = \frac{K}{4EI}$$

$$\frac{d^4v}{dz^4} + 4\beta^4 v = \frac{q(z)}{EI}$$

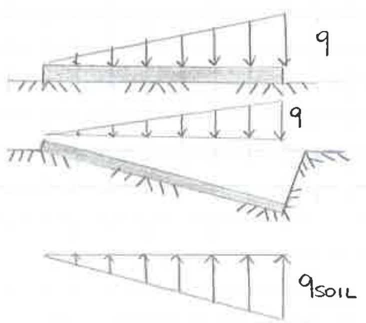
$\beta = [L^{-1}]$

NOTES

1) If q(z) is polynomial of degree less than 3, the solution is:

$v = \frac{q}{K}$

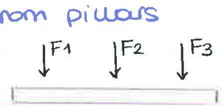
↳ polynomial of degree less than 3 ⇒ not very interesting



v and q are different only for a multiplicative factor
↳ they are actually the same

pressure of soil is equal to q
$q_{SOIL} = -Kv = q$

2) In reality we haven't a distributed load, but concentrated loads which come from pillars



$$-EI\,\vartheta'''(0) = T(0^+) = -\frac{F}{2}$$

$$\vartheta'''(0) = \frac{F}{2EI}$$

$$2\beta^3 \cdot 2c_3 = \frac{F}{2EI}$$

$$c_3 = \frac{F}{8EI\beta^3} = \frac{F}{8EI\beta^4}\cdot\beta = \frac{F}{8EI}\cdot\beta\cdot\frac{4EI}{K}$$

$$c_3 = c_4 = \frac{F}{2K}\beta$$

* $\vartheta(z) = \frac{F\beta}{2K}\underbrace{e^{-\beta z}(\cos\beta z + \operatorname{sen}\beta z)}_{A_{\beta z}}$

$A_{\beta z}$ → argoment of exponential, cos and sen

$$B_{\beta z} = e^{-\beta z}\operatorname{sen}\beta z$$
$$C_{\beta z} = e^{-\beta z}(\cos\beta z - \operatorname{sen}\beta z)$$
$$D_{\beta z} = e^{-\beta z}\cos\beta z$$

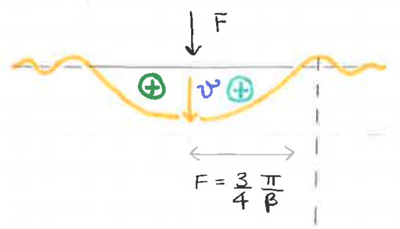
$$\frac{dA_{\beta z}}{dz} = 2\beta B_{\beta z}$$
$$\frac{dB_{\beta z}}{dz} = \beta\cdot C_{\beta z}$$
$$\frac{dC_{\beta z}}{dz} = -2\beta D_{\beta z}$$
$$\frac{dD_{\beta z}}{dz} = -\beta A_{\beta z}$$

* $\varphi(z) = -\vartheta'(z) = \frac{F\beta^2}{K} B_{\beta z}$

* $M(z) = -EI\,\vartheta''(z) = \frac{EIF\beta^3}{K}\cdot C_{\beta z}\cdot\frac{\beta}{\beta} = \frac{F}{4\beta} C_{\beta z}$

* $T(z) = -\frac{F}{2} D_{\beta z}$

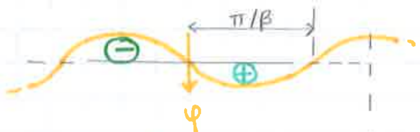
⇒ **solutions**



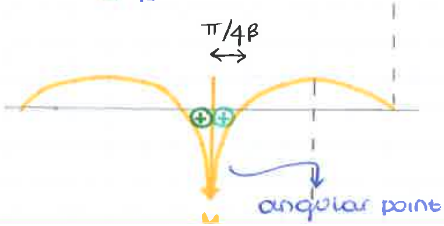
we have the solution only for $z>0$ (the function doesn't exist for $z<0$, we obtain it for simmetry)

$\vartheta = \pm\frac{F\beta}{2K} A_{\beta z}$

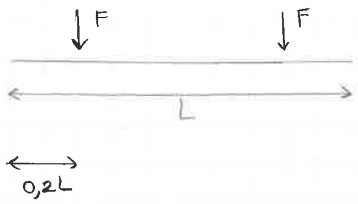
⊕ for the right side
⊕ for the left side (we consider $|z|$, so only the distance)



$\varphi = \pm\frac{F\beta^2}{K} B_{\beta z}$



EX.

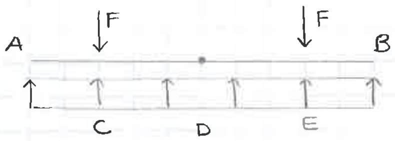


Notes

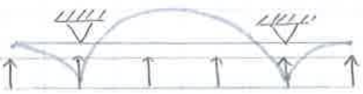
1) If the beam is complaint (→ flexible) and/or the soil is stiff, there is only parameter which rules the problem ⇒ β is large (high k, but small denominator)
   - ↳ The quality of the problem depends only on β
   - ↳ The solicitations dump quickly, moving far away from the loading points (they are low and soil pressure is high)
     - ↳ from C and E
     - ↳ it's a bad behaviour because pressure is concentrated, the beam doesn't spread the load along is length

2) If the beam is stiff and/or the soil is compliant, β is small
   - ↳ all the beam is solicited and so does the soil beneath the beam
   - ↳ soil pressure is more distributed, is low
   - ↳ solicitations (M and T) are very high

3) If the beam is infinitely stiff $\beta \to 0$
   - ↳ it moves as a rigid body
   - ↳ soil pressure is constant
   - ↳ the beam is upside (load is from down to up and pillars are the supports) ~ very solicitated (M and T are very high)



$q_{SOIL} = \frac{2F}{L}$



Moment is maximum

$$M_C = \frac{q(0,2L)^2}{2} = 0,04\ FL = 0,2\ Nm$$

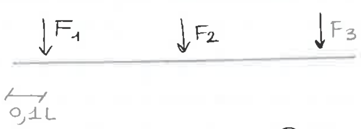
$F = 0,5\,N$

$L = 10\ m$

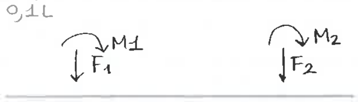
$$M_D = \frac{q(0,5L)^2}{2} - 0,3L = -0,05\ F\ell = -0,5\ Nm$$

HOMEWORK

①



②



we have to restart from the infinite beam with a couple in the middle
- right boundary conditions
- Betti's theorem

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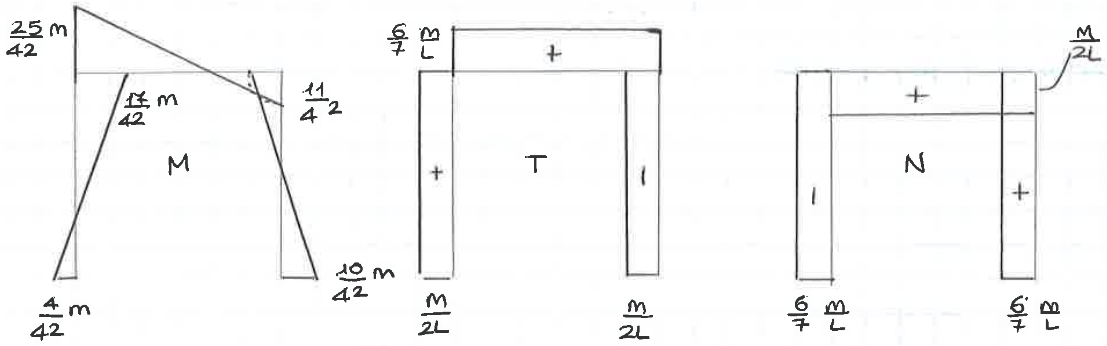
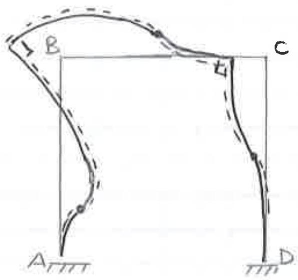
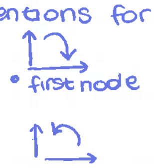




Conventions for signs

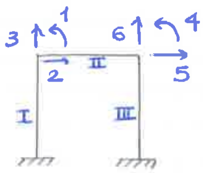


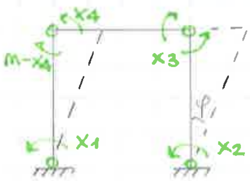
FANTILLI

OUR'S

Incidence matrix

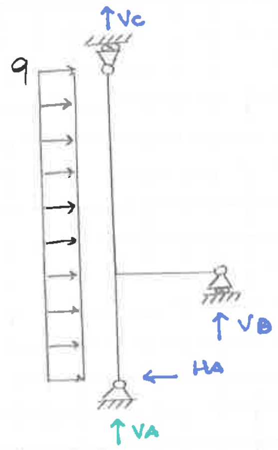
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 0 & 0 & 0 \end{pmatrix} \end{array}$$





1) Method of forces : 3
2) Mixed method : 5
3) Displacement : 3

15/11/17

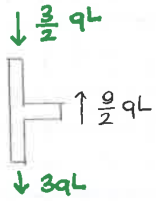


$v - g = 1$
↳ axial statical indeterminate case
Equilibrium around point A

$V_D \cdot L - \frac{9}{2} qL^2 = 0$

$V_D = \frac{9}{2} qL$

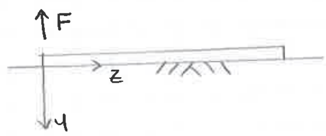
$H_A = 3qL$



$V_A = -\frac{2}{3} V_D = \frac{2}{3} \cdot \frac{9}{2} qL = 3qL$

$V_C = -\frac{1}{3} V_D = -\frac{3}{2} qL$

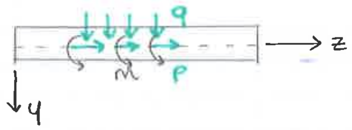
BEAM ON ELASTIC FOUNDATION
↳ force or moment acting on the end of the beam



- If $L < \lambda = 2\pi/\beta$
  ↳ the force affected the all beam
  $v(z) = C_1 e^{\beta z} \cos\beta z + C_2 e^{\beta z} \operatorname{sen}\beta z + C_3 e^{-\beta z} \cos\beta z + C_4 e^{-\beta z} \operatorname{sen}\beta z$
  4 boundery conditions
  $$\begin{cases} M_A = -EI\, v''(0) = 0 \\ T_A = -EI\, v'''(0) = +F \\ M_B = -EI\, v''(L) = 0 \\ T_B = -EI\, v'''(L) = 0 \end{cases}$$
  ⇒ so we can find $C_1 \div C_4$

- If $L > \lambda = 2\pi/\beta$ (**LONG BEAM**)
  ↳ the decay of force is so fast that shear and moment are neglected in B.
  $v(z) = C_3 e^{-\beta z} \cos\beta z + C_4 e^{-\beta z} \operatorname{sen}\beta z$
  ↳ if we put $z = L$ everything (v, φ, M, T) are $= 0$ so the first part is useless
  2 boundery conditions
  $$\begin{cases} M_A = -EI\, v''(0) = 0 \\ T_A = -EI\, v'''(0) = F \end{cases}$$
  $M_B = 0$, $T_B = 0$ } automatically satisfied

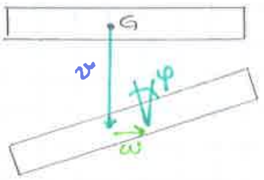
BEAM WITH RECTILINEAR AXIS



EQUILIBRIUM EQUATIONS

$$\begin{pmatrix} \frac{d}{dz} & 0 & 0 \\ 0 & \frac{d}{dz} & 0 \\ -1 & 0 & \frac{d}{dz} \end{pmatrix} \begin{pmatrix} T \\ N \\ M \end{pmatrix} + \begin{pmatrix} q \\ p \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underset{3\times3}{[\delta]^*} \underset{3\times1}{\{Q\}} + \underset{3\times1}{\{F\}} = \underset{3\times1}{\{0\}}$$



KINEMATICAL EQUATIONS

$$\begin{pmatrix} \gamma \\ \varepsilon \\ \chi \end{pmatrix} = \begin{pmatrix} \frac{d}{dz} & 0 & 1 \\ 0 & \frac{d}{dz} & 0 \\ 0 & 0 & \frac{d}{dz} \end{pmatrix} \begin{pmatrix} v \\ w \\ \varphi \end{pmatrix}$$

$$\overset{3\times1}{\{q\}} = \overset{3\times3}{\{\delta\}} \overset{3\times1}{\{\eta\}}$$

a) Force system balanced
b) Displacement sy. congruent
} they mean that these equations are satisfied

PVW

$$\int_0^L \{F_a\}^T \{\eta_b\} dz + \left[\{Q_a\}^T \{\eta_b\}\right]_{z=0}^{z=L} = \int_0^L \{Q_a\}^T \{q_b\} dz$$

DEMONSTRATION

$$\int_0^L (q_a v_b + p_a w_b + m_a \varphi_b) dz + \left[T_a v_b + N_a w_b + M_a \varphi_b\right]_{z=0}^{z=L} =$$

$$F(L) - F(0) = \int_0^L \frac{dF}{dz} dz$$

$$\int_0^L \left[q_a v_b + p_a w_b + m_a \varphi_b + \frac{d}{dz}(T_a v_b + N_a w_b + M_a \varphi_b)\right] dz =$$

= 0 → for equilib. ← = 0 (T_a)

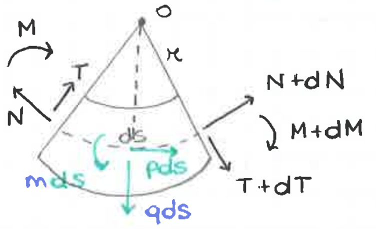
$$= \int_0^L \left[\left(q_a + \frac{dT_a}{dz}\right) v_b + \left(p_a + \frac{dN_a}{dz}\right) w_b + \left(m_a + \frac{dM_a}{dz}\right) \varphi_b + T_a \frac{dv_b}{dz} + N_a \frac{dw_b}{dz} + M_a \frac{d\varphi_b}{dz}\right] dz =$$

$$= \int_0^L \left[T_a \left(\frac{dv_b}{dz} + \varphi_b\right) + N_a \frac{dw_b}{dz} + M_a \frac{d\varphi_b}{dz}\right] dz =$$

$$= \int_0^L (T_a \gamma_b + N_a \varepsilon_b + M_a \chi_b) dz = \int_0^L \{Q_a\}^T \{q_b\} dz$$

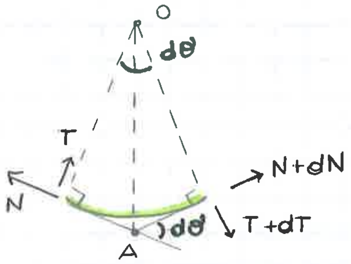
CURVILINEAR BEAM

↳ with curvilinear axis
↳ we consider infinitesimal piece of curvilinear beam (length = ds because it's a curvilinear abscissa)



two sections aren't parallel
$r\,d\theta = ds$

Equilibrium equation



$\xrightarrow{+})$ $(N+dN)\cos\frac{d\theta}{2} - N\cos\frac{d\theta}{2} + (T+dT)\,\text{sen}\frac{d\theta}{2} + T\,\text{sen}\frac{d\theta}{2} + p\,ds = 0$

We consider infinitesimal of 1st order, not the others

$\cos\frac{d\theta}{2} = 1 - \frac{1}{2}\left(\frac{d\theta}{2}\right)^2 \simeq 1$

↳ second order infinitesim

$\text{sen}\frac{d\theta}{2} \simeq \frac{d\theta}{2}$

$dN + T\frac{d\theta}{2} + p\,ds = 0$

$\frac{dN}{ds} + \frac{T}{r} + p = 0$ (1)

$+\downarrow)$ $(T+dT)\cos\frac{d\theta}{2} - T\cos\frac{d\theta}{2} - (N+dN)\,\text{sen}\frac{d\theta}{2} - N\,\text{sen}\frac{d\theta}{2} + q\,ds =$

$dT - N\,d\theta + q\,ds = 0$

$\frac{dT}{ds} - \frac{N}{r} + q = 0$ (2)

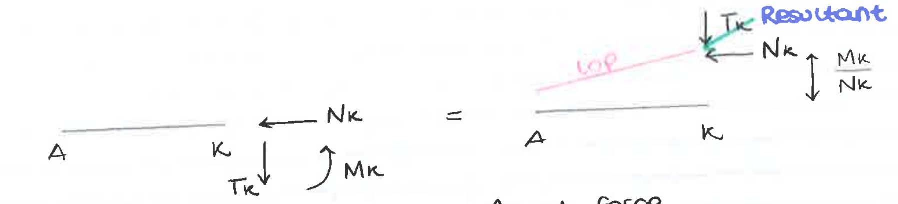
$\overset{\curvearrowleft}{A})$ $(M+dM) - M - T\frac{ds}{2} - T\frac{ds}{2} + m\,ds = 0$

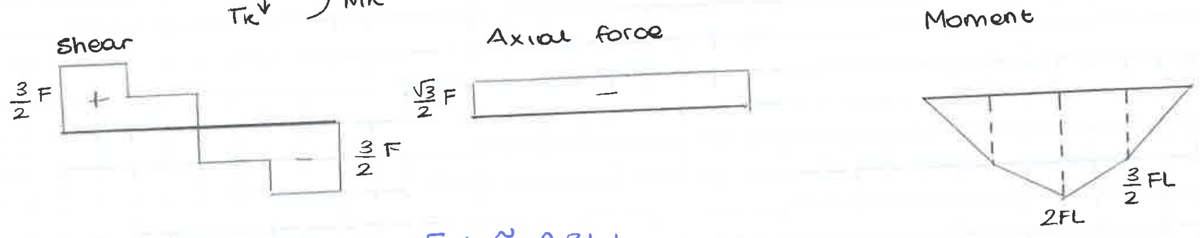
$\frac{dM}{ds} - T + m = 0$ (3)

We can set them in a matrix form

$$\begin{pmatrix} \frac{d}{ds} & -\frac{1}{r} & 0 \\ \frac{1}{r} & \frac{d}{ds} & 0 \\ -1 & 0 & \frac{d}{ds} \end{pmatrix} \begin{pmatrix} T \\ N \\ M \end{pmatrix} + \begin{pmatrix} q \\ p \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$[\delta]^* \{Q\} + \{F\} = \{0\}$



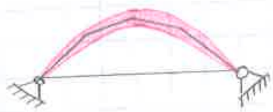


$$e_D = \left|\frac{M_D}{N_D}\right| = \frac{2FL}{\frac{\sqrt{3}}{2}F} = \frac{4\sqrt{3}}{3}L \cong 2{,}31\ L$$

$$e_C = \left|\frac{M_C}{N_C}\right| = \frac{3}{2}FL \cdot \frac{2}{\sqrt{3}F} = \sqrt{3}\ L \simeq 1{,}73\ L$$

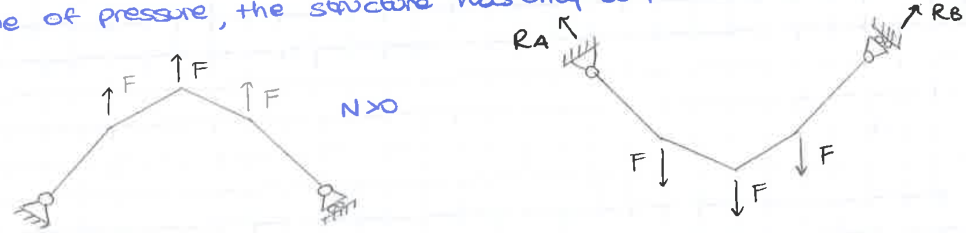
It's better having a beam following the pressure line because in this way $h = 0$ so $M = T = 0$ and the structure is loaded only by axial force

it gives the higher stress so it's a positive condition



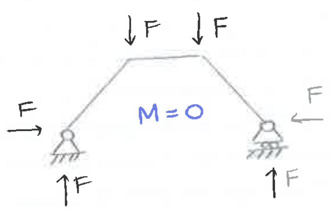
⇒ move from a BEAM to ARCH

If we change the sign of the all forces static characteristics change, but not line of pressure, the structure has only a positive axial force



If we have a rope (not fixed shape) we can reach the condition of being loaded only by axial force so the archs can be designed using loaded ropes, then we mirror them to find the correct arch shape (as in Sagrada Familia)

The presence of the horizontal component of the reaction is of praramount importance



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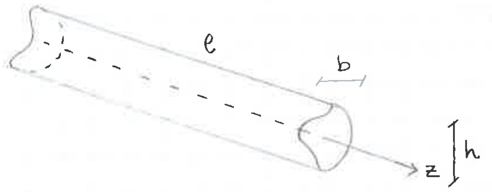
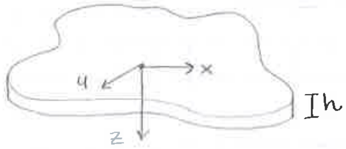
PLATES AND SHELLS

→ a beam is a solid in which one dimension is larger than the other two.
$\ell >> h, b$



1D MODEL: all quantities depend only on z. We integrate on x and y (smallest dim. and obtain f(z))

→ for a plate, thickness is smaller than the other two, we obtain it moving an area along z direction for a small height.



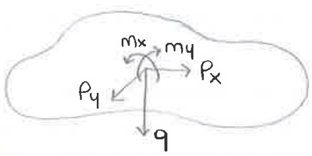
h: plane thickness
$z = 0$ is the middle plane
$z = \pm h/2$ bottom and upper surface of the plate

The model works when $a, b >> h$

2D MODEL: all quantities depend on x,y. We integrate on z (smallest) and obtain f(x,y)

STATIC CHARACTERISTIC

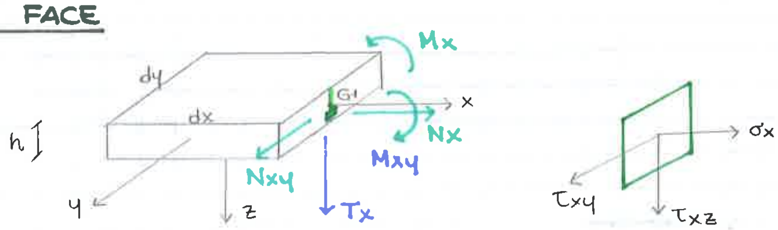
→ in a beam we refer them to the centre of gravity of the section
→ the integral, for the plate, is along a line orthogonal to the areas of the plate
→ beam is represended by a line, plane by middle plane



→ External forces: forces for unit area
P_x and P_y belong to the middle plane, q is orthogonal.
m_x: around y
m_y: around x

Infinitesimal element of a plane

I FACE



Consider forces acting on a segment and reduce them to the center of gravity. We consider another infinitesimal area, the positive directions are along the axis

Resultant of the system
↳ along the whole segment

$$N_x = \int_{-h/2}^{+h/2} \sigma_x \, dz$$

BEAM

$$N_x = \int_A \sigma \, dA$$

Equilibrium equation

Traslation

Along x

$$\left(\frac{\partial N}{\partial x}dx\right)dy + \left(\frac{\partial N_{xy}}{\partial y}dy\right)dx + P_x\,dx\,dy = 0$$

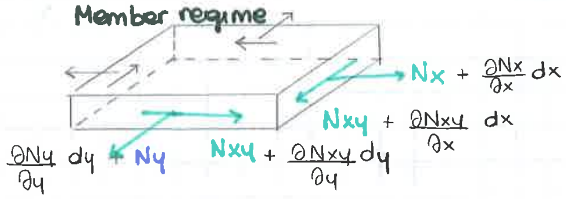
Along y

$$\left(\frac{\partial N}{\partial y}dy\right)dx + \left(\frac{\partial N_{xy}}{\partial x}dx\right)dy + P_y\,dx\,dy = 0$$

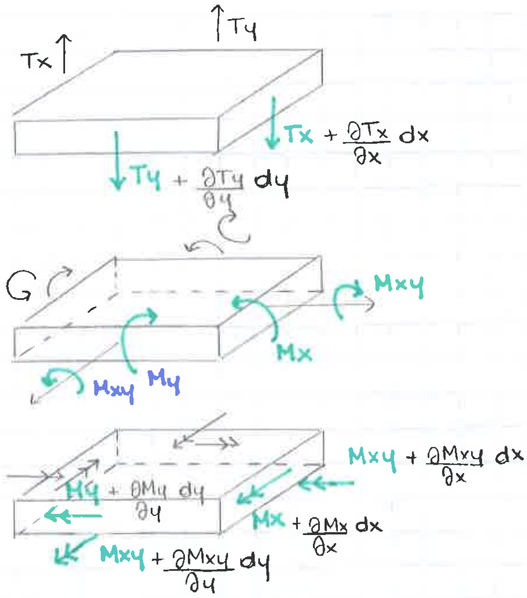
Along z

$$\left(\frac{\partial T_x}{\partial x}dx\right)dy + \left(\frac{\partial T_y}{\partial y}dy\right)dx + q\,dx\,dy = 0$$

Member regime



Bending regime



Rotation

$-T_x\,dx\,dy$

Around y

$$\left(\frac{\partial M_x}{\partial x}dx\right)dy + \left(\frac{\partial M_{xy}}{\partial y}dy\right)dx + m_x\,dx\,dy = 0$$

Around − x

$$\left(\frac{\partial M_y}{\partial y}dy\right)dx + \left(\frac{\partial M_{xy}}{\partial x}dx\right)dy + m_y\,dx\,dy + $$
$$- T_y\,dy\,dx = 0$$

$$\begin{cases} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + P_x = 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + P_y = 0 \\ \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + q = 0 \\ -T_x + \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + m_x = 0 \\ -T_y + \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + m_y = 0 \end{cases}$$

5 equations (rows) and 8 unknowns (columns)

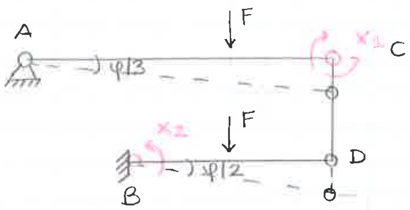
$$\begin{pmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & 0 & -1 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ T_x \\ T_y \\ T_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} + \begin{pmatrix} P_x \\ P_y \\ q \\ m_x \\ m_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underset{5\times 8}{[\delta]^*}\ \underset{8\times 1}{\{Q\}} + \underset{5\times 1}{\{F\}} = \underset{5\times 1}{\{0\}}$$

→ internally because it doesn't depend on boundary conditions but it's a characteristic of the model

↳ it's three times static indeterminated (in beams the problem is determinate)
equilibrium equations aren't enough (we need kinematic and costitutive equations)

1) The membrane problem is uncoupled with bending one (due to kinematic and costitutive equations too)
↳ curvature leads to a coupled problem.



$$\begin{cases} g = 2\cdot 3 = 6 \\ v = 2(A) + 3(B) + 2(D) = 7 \end{cases}$$

$v - g = i = 1$

statically indeterminated

Associated truss structure

$$\begin{cases} g = 3\cdot 3 = 9 \\ v = 2(A) + 2(B) + 2(C) + 2(D) = 8 \end{cases}$$

$g - v = l = 1$

mechanism (φ)

$$\begin{cases} \varphi_{BD} = 0 \\ \varphi_{CA} = \varphi_{CD} \\ PVW \end{cases}$$

BD ↺ X_2
CA ↻ X_1
CD ↺ X_1

$$\begin{cases} \frac{X_2 \cdot 2L}{3EI} - \frac{\varphi}{2} - \frac{F(2L)^2}{16EI} = 0 \\ -\frac{X_1 \cdot 3L}{3EI} - \frac{\varphi}{3} + \frac{2L^2(5L)}{18L\,EI}F = \frac{X_1 L}{3EI} \\ +X_1\frac{\varphi}{3} - X_2\frac{\varphi}{2} + F\frac{\varphi}{3}\cdot 2L + F\frac{\varphi}{2}\cdot L = 0 \end{cases}$$

$$\begin{cases} \frac{2}{3}\frac{X_2 L}{EI} - \frac{\varphi}{2} - \frac{4}{16}\frac{FL^2}{EI} = 0 \quad \cdot\left\{\frac{EI}{L}\right\} \to \frac{2}{3}X_2 - \frac{EI}{2L}\varphi - \frac{FL}{4} = 0 \\ -\frac{X_1 L}{EI} - \frac{\varphi}{3} + \frac{5}{9}\frac{FL^2}{EI} - \frac{X_1 L}{3EI} \quad \cdot\left\{\frac{EI}{L}\right\} \to -X_1 - \frac{1}{3}\frac{EI}{L}\varphi + \frac{5}{9}FL - \frac{X_1}{3} = 0 \\ \frac{X_1}{3} - \frac{X_2}{2} + \frac{2}{3}FL + \frac{1}{2}FL = 0 \end{cases}$$

$$-\frac{4}{3}X_1 - \frac{EI}{3L}\varphi + \frac{5}{9}FL = 0$$

$$\begin{cases} X_2 = \frac{3}{2}\cdot\frac{1}{4}FL + \frac{3}{2}\cdot\frac{1}{2}\frac{EI}{L}\varphi = \frac{3}{8}FL + \frac{3}{4}\frac{EI}{L}\varphi \\ X_1 = -\frac{3}{4}\cdot\frac{1}{3}\frac{EI}{L}\varphi + \frac{3}{4}\cdot\frac{5}{9}FL = -\frac{1}{4}\frac{EI}{L}\varphi + \frac{5}{12}FL \\ -\frac{1}{12}\frac{EI}{L}\varphi + \frac{5}{36}FL - \frac{3}{16}FL - \frac{3}{8}\frac{EI}{L}\varphi + \frac{2}{3}FL + \frac{1}{2}FL = 0 \end{cases}$$

$$\left(-\frac{1}{12} - \frac{3}{8}\right)\frac{EI}{L}\varphi = \left(-\frac{5}{36} + \frac{3}{16} - \frac{2}{3} - \frac{1}{2}\right)FL$$

$$-\frac{11}{24}\frac{EI}{L}\varphi = -\frac{161}{144}FL \to \varphi = \frac{161}{144}\cdot\frac{24}{11}\frac{FL^2}{EI} = \frac{161}{66}\frac{FL^2}{EI}$$

$$\begin{cases} X_1 = -\frac{17}{88}FL \\ X_2 = \frac{97}{44}FL \\ \varphi = \frac{161}{66}\frac{FL^2}{EI} \end{cases}$$

ARCHITETTURA ROMANICA
XI, XII secolo
ARCO A TUTTO SESTO

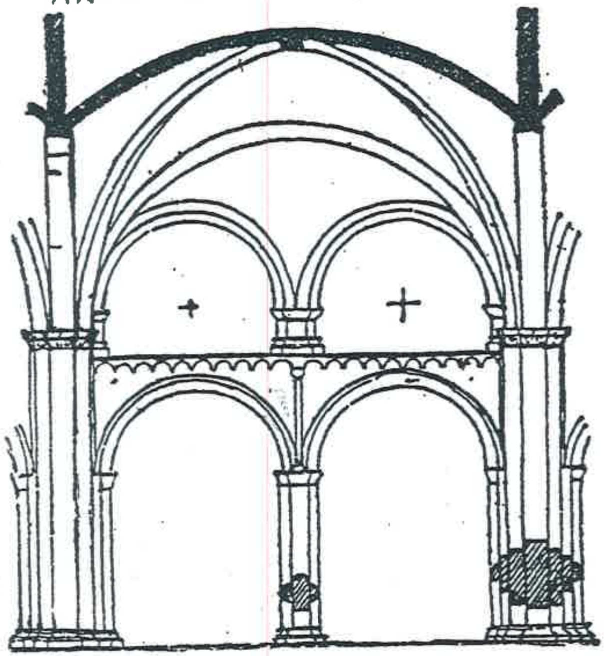
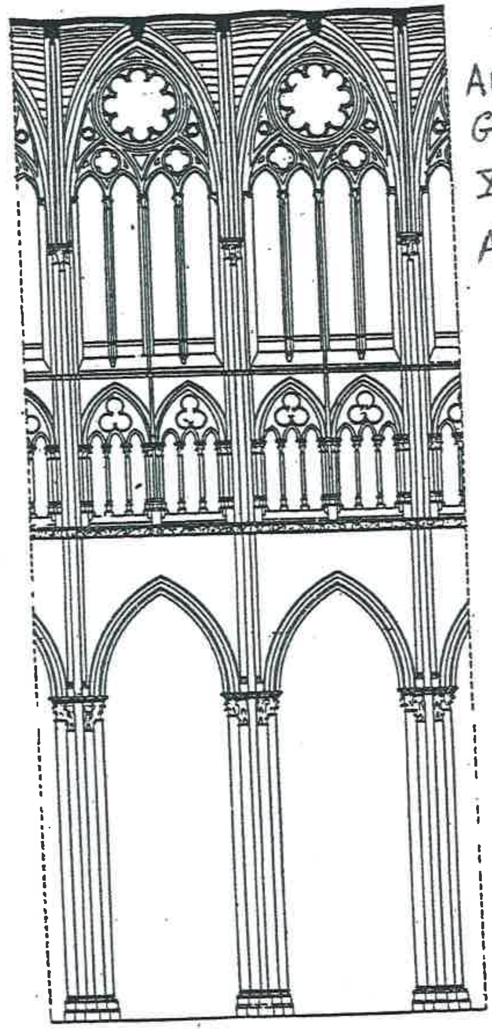


Fig. 212. - *Milano*, S. Ambrogio: particolare di una campata.

ARCHITETTURA GOTICA
XIII, XIV secolo
ARCO A SESTO ACUTI



51 Elevazione della navata della cattedrale di Amiens, com. 1220.

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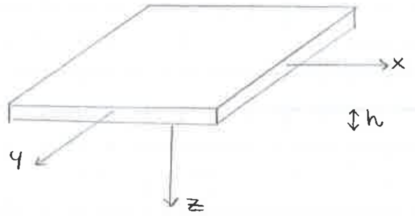
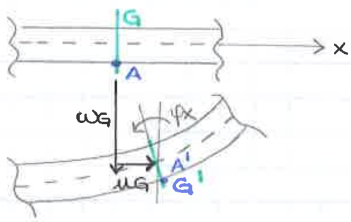
PLATES





Beam theory: cross section remains plane

Plates theory: we don't have a cross section, we only consider a segment orthogonal to the middle plane which remains a segment after deformation.

- piece of plate after deformation
- loading plate has a deformed shape

Position of G becomes G'

The segment rotates, translates, but it remains a segment

- Three translations, even in the orthogonal direction of the xz plane
- Two rotations, even a rotation around x (not only around y = φ_x)

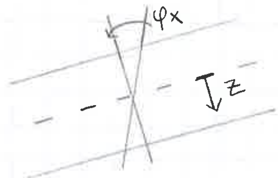
Each point (A) has a displacement and becomes A'

Displacement field

$$\begin{cases} u(x,y,z) = u_G(x,y) + \varphi_x(x,y)\, z \\ v(x,y,z) = v_G(x,y) + \varphi_y(x,y)\, z \\ w(x,y,z) = w_G(x,y) \end{cases}$$

If φ is small vertical displacement is equal for all points

The contribution of φ can be neglected.



$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_G}{\partial x} + \frac{\partial \varphi_x}{\partial x} \cdot z = \varepsilon_{xG} + \chi_x \cdot z$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_G}{\partial y} + \frac{\partial \varphi_y}{\partial y} \cdot z = \varepsilon_{yG} + \chi_y \cdot z$$

→ we use ε because they are related to middle plane and are the characteristics of membrane problem

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \quad \{w_G \text{ is not a function of } z\}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_G}{\partial y} + \frac{\partial v_G}{\partial x} + \left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right) z = \varepsilon_{xyG} + \chi_{xy} \cdot z$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \varphi_x + \frac{\partial w_G}{\partial x} = \gamma_x \quad (= \text{not dependent on } z)$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \varphi_y + \frac{\partial w_G}{\partial y} = \gamma_y$$

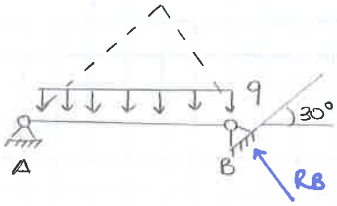
⟹ All the quantities depend on x, y (deformation characteristics) NOT on z

$$\begin{pmatrix} \varepsilon_{xG} \\ \varepsilon_{yG} \\ \varepsilon_{xyG} \\ \gamma_x \\ \gamma_y \\ \chi_x \\ \chi_y \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & 1 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 1 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} u_G \\ v_G \\ w_G \\ \varphi_x \\ \varphi_y \end{pmatrix}$$

membrane quantities (displacements of the middle plane)

Two of three are indipendent (balance, kinematic and PVW) from two systems we can obtain the third one.

PROBLEM OF THE ARCH



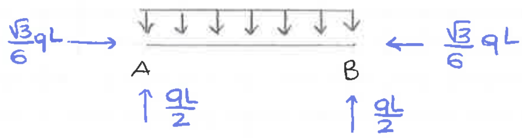
$\circlearrowleft_A)$ $R_B \frac{\sqrt{3}}{2} \cdot L - \frac{qL^2}{2} = 0$

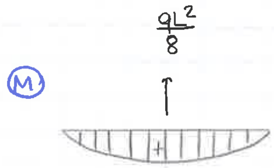
$R_B = \frac{\sqrt{3}}{3} qL$

$H_B = R_B \cos 60° = \frac{\sqrt{3}}{6} qL$

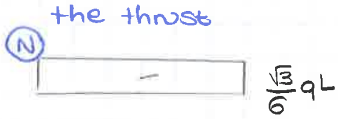
$V_B = R_B \cos 30° = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{3} qL = \frac{qL}{2}$

SOLUTIONS





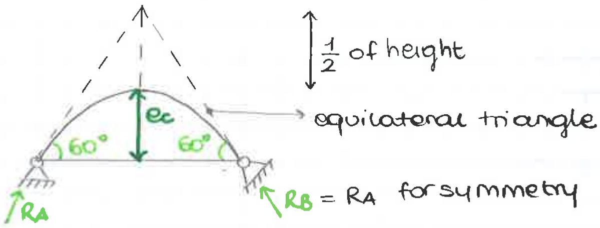
constant compression as the thrust



Thrust : spinta

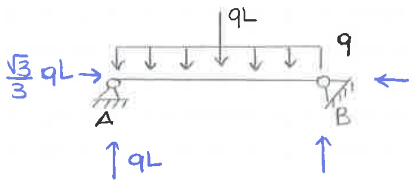
$e = \left| \frac{M}{N} \right|$

$e_C = \left| \frac{M_C}{N_C} \right| = \frac{qL^2}{8} \cdot \frac{6}{\sqrt{3} qL} = \frac{\sqrt{3}}{4} L = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} L$

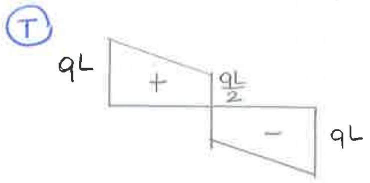


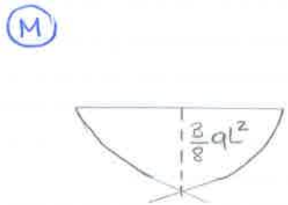
Line of pressure is a parabola and is only compressed (best solution)
↳ tangent to partial resultant

EX 2

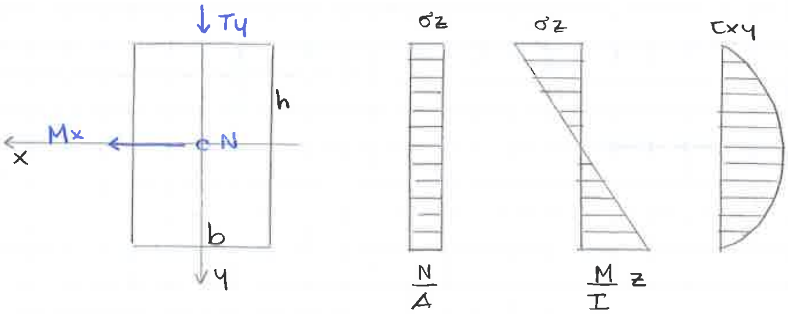


$M_C = qL \cdot \frac{L}{2} - \frac{qL}{2} \cdot \frac{L}{4} = \frac{3}{8} qL^2$





Constitutive equations



$\sigma_z = \frac{N}{A}$

$\sigma_z = \frac{M_x}{I_x} y$

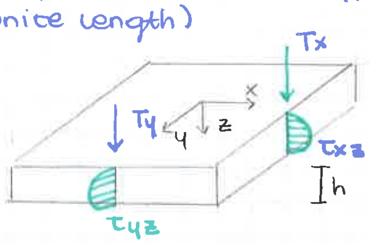
$\tau_{xy} = \frac{6 T_x}{bh^3} \left(\frac{h^2}{4} - y^2 \right)$

$(\tau_{xy})_{max}$ when $y=0$

$\tau_{xy,max} = \frac{3}{2} \frac{T_y}{bh}$

$\gamma_y = \frac{6}{5} \frac{T_y}{Gbh}$

In plate situation T_y/b of the beam section is equal to T_y (= force for unite length)



$$\begin{cases} \tau_{xz} = \frac{6 T_x}{h^3} \left(\frac{h^2}{4} - z^2 \right) \\ \tau_{yz} = \frac{6 T_y}{h^3} \left(\frac{h^2}{4} - z^2 \right) \end{cases}$$

Max values

$\tau_{xz} = \frac{3}{2} \frac{T_x}{h}$

$\tau_{yz} = \frac{3}{2} \frac{T_y}{h}$

$\gamma_y = \frac{6}{5} \frac{T_y}{Gh}$

$\gamma_x = \frac{6}{5} \frac{T_x}{Gh}$

upper and bottom surface are free loaded } $\sigma_z = 0$

h is small related to the other dimensions so σ_z is $=0$ everywhere

$$\begin{cases} \varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \\ \varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E} \\ \gamma_{xy} = \frac{\tau_{xy}}{G} \end{cases}$$

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ T_x \\ T_y \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{pmatrix} \frac{12D}{h^2} & \frac{\nu 12D}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\nu 12D}{h^2} & \frac{12D}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{6D(1-\nu)}{h^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5D(1-\nu)}{h^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5D(1-\nu)}{h^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D & \nu D & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu D & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-\nu}{2} & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{xG} \\ \varepsilon_{yG} \\ \varepsilon_{xyG} \\ \gamma_x \\ \gamma_y \\ \chi_x \\ \chi_y \\ \chi_{xy} \end{pmatrix}$$

Terms out of diagonal are linked to Poisson effect ($\nu \neq 0$)

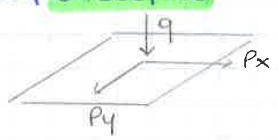
$$\underset{8\times1}{\{Q\}} = \underset{8\times8}{[H]} \underset{8\times1}{\{q\}}$$

$[H]$ ↓ constitutive matrix

Membrane static and kinematic characteristic are uncoupled with shear and bending characteristics (=) even for constitutive equations)

=> Membrane and bending problems are completely uncoupled

- P_x and P_y cause N_x, N_y, N_{xy} and so $\varepsilon_x, \varepsilon_y$ and ε_{xy}
  ↳ u and v (displ. of middle plate)



- q, m_x, m_y cause $T_x, T_y, M_x, M_y, M_{yx}$ so $\gamma_x, \gamma_y, \chi_x, \chi_y, \chi_{xy}$
  ↳ w, φ_x, φ_y

Bending regime

$$\begin{cases} \underset{3\times5}{[\partial]^*} \underset{5\times1}{\{Q\}} + \underset{3\times1}{\{F\}} = \underset{3\times1}{\{0\}} \\ \underset{5\times1}{\{Q\}} = \underset{5\times5}{[H]} \underset{5\times1}{\{q\}} \\ \underset{5\times1}{\{q\}} = \underset{5\times3}{[\partial]} \underset{3\times1}{\{\eta\}} \end{cases}$$

3 eq. 5 unknowns → 2 indeterminate → STATIC

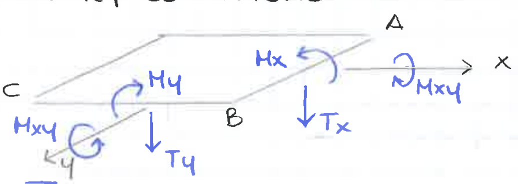
CONSTITUTIVE EQ

KINEMATIC EQ.

$$\underbrace{\underset{3\times5}{[\partial]^*} \underset{5\times5}{[H]} \underset{5\times3}{[\partial]}}_{\underset{3\times3}{[L]}} \underset{3\times1}{\{\eta\}} + \underset{3\times1}{\{F\}} = \underset{3\times1}{\{0\}}$$

System of differential equations, but solved because 3 eq. and 3 unknowns (w, φ_x, φ_y)

BOUNDERY CONDITIONS



On $\overline{AB}$

$T_x = \overline{T_x}$ or $w = 0$

BENDING PROBLEM

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$$\begin{pmatrix} T_x \\ T_y \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} \begin{pmatrix} \partial/\partial x & \partial/\partial y & 0 & 0 & 0 \\ -1 & 0 & \partial/\partial x & 0 & \partial/\partial y \\ 0 & -1 & 0 & \partial/\partial y & \partial/\partial x \end{pmatrix} + \begin{pmatrix} q \\ m_x \\ m_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- equilibrium equations (STATIC)
  ↳ we don't consider membrane problem at all

$$\begin{pmatrix} \gamma_x \\ \gamma_y \\ \chi_x \\ \chi_y \\ \chi_{xy} \end{pmatrix} = \begin{pmatrix} \partial/\partial x & 1 & 0 \\ \partial/\partial y & 0 & 1 \\ 0 & \partial/\partial x & 0 \\ 0 & 0 & \partial/\partial x \\ 0 & \partial/\partial y & \partial/\partial x \end{pmatrix} \begin{pmatrix} w \\ \varphi_x \\ \varphi_y \end{pmatrix}$$

- congruent equation (KINEMATIC)

$$\begin{cases} M_x = D\chi_x + \nu D\chi_y \\ M_y = D\chi_y + \nu D\chi_x \\ M_{xy} = \frac{1-\nu}{2} D\, \chi_{yx} \end{cases}$$

- constitutive equations

Neglect shear deformability is more correct as much as little is thikness of the plate

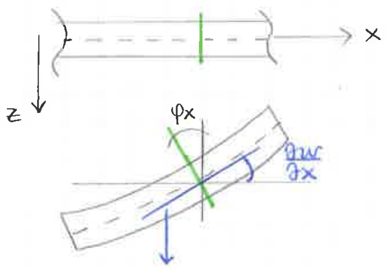
$\gamma_x = 0$

$\varphi_x = -\frac{\partial w}{\partial x}$

$\gamma_y = 0$

$\varphi_y = -\frac{\partial w}{\partial y}$

⇒ only one unknown (because from w we can compute φ_x and φ_y)



tangent to the point (it represents w(x))

Neglecting shear deformability means that after deformation, a segment ⊥ to the middle plane remains a segment ⊥ to the deformed middle plane.
↳ it's a stronger hipotesis than segment which remains a segment

Assume m_x and $m_y = 0$

$$T_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$
$$T_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}$$
} static equations

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + q = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0 \quad (*)$$

Kinematic equations

$$\chi_x = -\frac{\partial^2 w}{\partial x^2}$$

$$\chi_y = -\frac{\partial^2 w}{\partial y^2}$$

$$\chi_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

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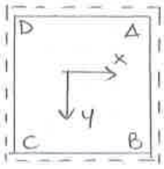
- hinge (simply supported plane) ~ cilindrical hinge



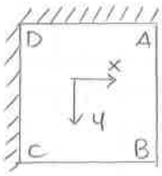
(AB) $\begin{cases} w=0 \\ M_x=0 \\ \varphi_y=0 \end{cases} \rightarrow \chi=0 \rightarrow \begin{cases} w=0 \\ \partial^2 w/\partial x^2 + \nu\, \partial^2 w/\partial y^2 = 0 \\ \partial w/\partial y = 0 \end{cases}$ (obvious ↑) $\begin{cases} w=0 \\ \dfrac{\partial^2 w}{\partial x^2} = 0 \end{cases}$

↳ rotation around x is allowed not y ↳ ortogonal to the size

(BC) $\begin{cases} w=0 \\ M_y=0 \\ \varphi_x=0 \end{cases} \rightarrow \chi=0 \rightarrow \begin{cases} w=0 \\ \partial^2 w/\partial y^2 + \nu\, \partial^2 w/\partial x^2 = 0 \\ \partial w/\partial x = 0 \end{cases}$ $\begin{cases} w=0 \\ \dfrac{\partial^2 w}{\partial y^2} = 0 \end{cases}$

⇒ Two conditions for each size

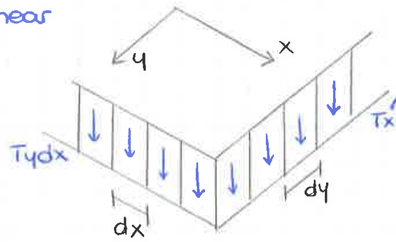
- free borders



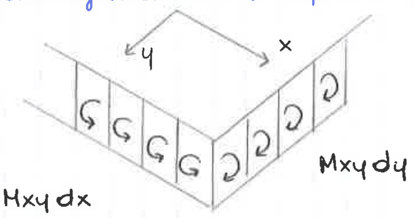
(AB) $\begin{cases} T_x = 0 \\ M_x = 0 \\ M_{xy} = 0 \end{cases}$

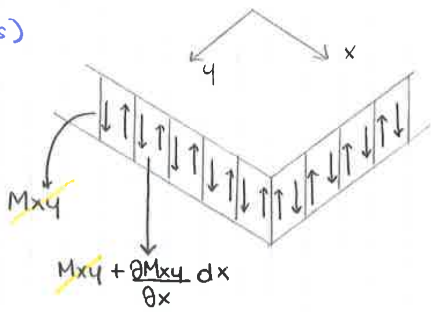
If $\chi = 0$ [...]

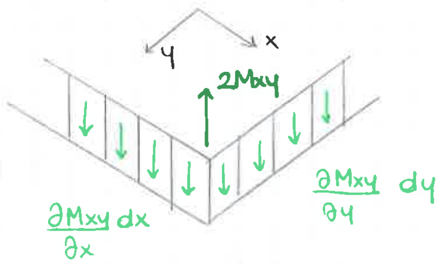
Shear



Twisting moment (3 equivalent systems)







$V_x dy = T_x dy + \dfrac{\partial M_{xy}}{\partial y} dy \Rightarrow T_x + \dfrac{\partial M_{xy}}{\partial y} = V_x$

$V_y dx = T_y dx + \dfrac{\partial M_{xy}}{\partial x} dx \Rightarrow V_y = T_y + \dfrac{\partial M_{xy}}{\partial x}$

Simply supported rectangular plate subjected to a uniform load q

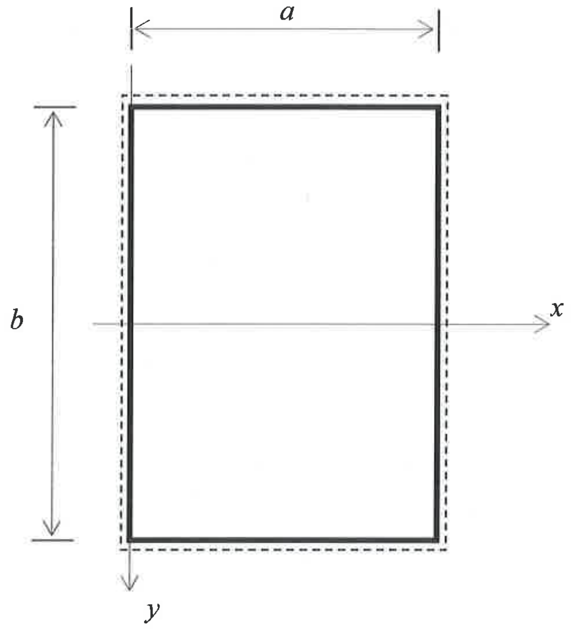


TABLE 6. NUMERICAL FACTORS β' AND β'_1 FOR BENDING MOMENTS OF SIMPLY SUPPORTED RECTANGULAR PLATES UNDER UNIFORM PRESSURE q

$\nu = 0.3,\ b \geq a$

b/a	$M_x = \beta' q a^2,\ y = 0$					$M_y = \beta'_1 q a^2,\ y = 0$				
	$x = 0.1a$	$x = 0.2a$	$x = 0.3a$	$x = 0.4a$	$x = 0.5a$	$x = 0.1a$	$x = 0.2a$	$x = 0.3a$	$x = 0.4a$	$x = 0.5a$
1.0	0.0209	0.0343	0.0424	0.0466	0.0479	0.0168	0.0303	0.0400	0.0459	0.0479
1.1	0.0234	0.0389	0.0486	0.0541	0.0554	0.0172	0.0311	0.0412	0.0475	0.0493
1.2	0.0256	0.0432	0.0545	0.0007	0.0627	0.0174	0.0315	0.0417	0.0480	0.0501
1.3	0.0277	0.0472	0.0599	0.0671	0.0694	0.0175	0.0316	0.0419	0.0482	0.0503
1.4	0.0297	0.0509	0.0649	0.0730	0.0755	0.0175	0.0315	0.0418	0.0481	0.0502
1.5	0.0314	0.0544	0.0695	0.0783	0.0812	0.0173	0.0312	0.0415	0.0478	0.0498
1.6	0.0330	0.0572	0.0736	0.0831	0.0862	0.0171	0.0309	0.0411	0.0472	0.0492
1.7	0.0344	0.0599	0.0773	0.0874	0.0908	0.0169	0.0306	0.0405	0.0466	0.0486
1.8	0.0357	0.0623	0.0806	0.0913	0.0948	0.0167	0.0301	0.0399	0.0459	0.0479
1.9	0.0368	0.0644	0.0835	0.0948	0.0985	0.0165	0.0297	0.0393	0.0451	0.0471
2.0	0.0378	0.0663	0.0861	0.0978	0.1017	0.0162	0.0292	0.0387	0.0444	0.0464
2.5	0.0413	0.0729	0.0952	0.1085	0.1129	0.0152	0.0272	0.0359	0.0412	0.0430
3.0	0.0431	0.0763	0.1000	0.1142	0.1189	0.0145	0.0258	0.0340	0.0390	0.0406
4.0	0.0445	0.0791	0.1038	0.1185	0.1235	0.0138	0.0246	0.0322	0.0369	0.0384
∞	0.0450	0.0800	0.1050	0.1200	0.1250	0.0135	0.0240	0.0315	0.0360	0.0375

→ square plate simply supported

Lastra quadrata (b/a = 1)

↳ symmetrical

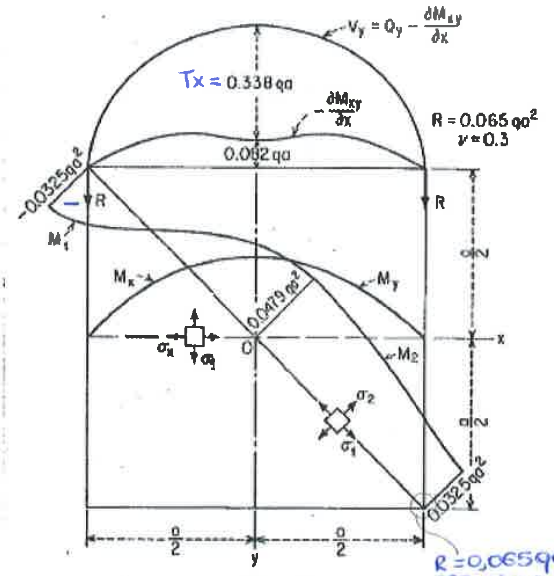


FIG. 63

In the middle M is max. And because of the symmetry is equal in vertical and horizontal directions.

⇒ principal directions rotate of 45° so as we consider the diagonal

R = 0,065qa² (concentrate reaction on the corner)

N.B.: le tabelle ed i grafici precedenti sono presi da *Theory of Plates and Shells* di Timoshenko. Nella sua notazione il segno di M_{xy} è opposto a quello da noi adottato (assume $M_{xy} = -M_{yx}$). I tagli sono indicati con Q_x e Q_y anziché T_x e T_y.

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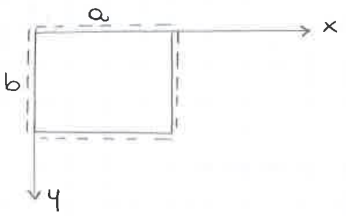
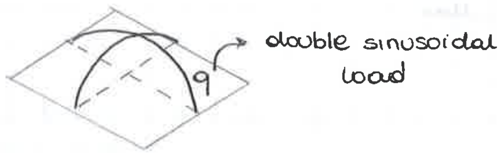
SQUARE PLATE SIMPLY SUPPORTED

$$\nabla^4 w = \frac{q}{D}$$





$$q = q_0 \,\text{sen}\, \frac{\pi x}{a} \,\text{sen}\, \frac{\pi y}{b}$$

↳ sinusoidal load so we can find a solution easily

$$w = c \,\text{sen}\, \frac{\pi x}{a} \,\text{sen}\, \frac{\pi y}{b}$$

$$c = \frac{q_0}{\pi^4 D\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}$$

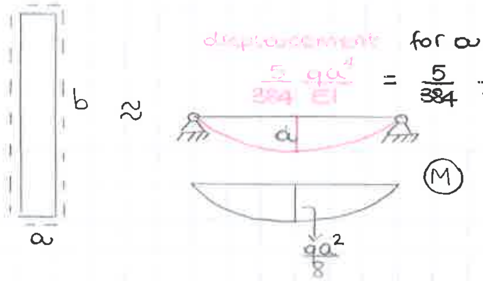
$$w = \frac{q_0}{\pi^4 D\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} \,\text{sen}\, \frac{\pi x}{a} \,\text{sen}\, \frac{\pi y}{b}$$

In general we have different kind of loads (ex. uniform distributed loads)

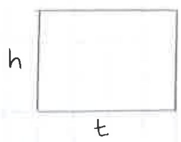
Uniformly distributed load

↳ we use the Fourier's serie for sinusoidal functions so we consider a constant load as the sum of several sinusoidal functions.

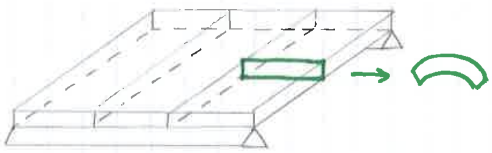
If $b/a \to \infty$ the plate seems to be a beam



displacement $\frac{5}{384}\frac{qa^4}{EI}$ for a rectangular cross section $= \frac{5}{384} \frac{q}{t} \frac{12}{E} \frac{a^4}{h^3}$



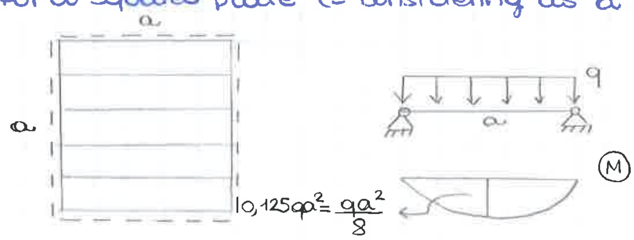
$$w_{max} = \frac{5}{384} \frac{qa^4}{D} = (1-\nu^2) \cdot \frac{5}{385} \frac{qa^4}{E} \frac{12}{h^3}$$



We can consider a plate as a several beam

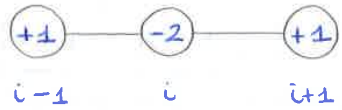
Because of Poisson's effect the plate is stiffer so for stresses it behaves as the beam, for deflection there is a corrective factor $(1-\nu^2)$

For a squared plate (= considering as a union of beams)

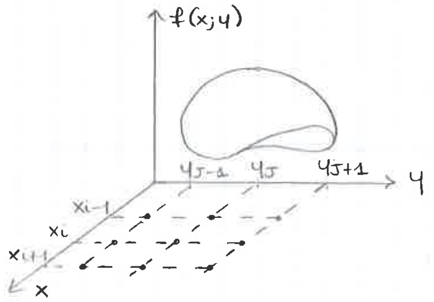


We overestimate the bending moment of 263% so the plate is much stiffer than

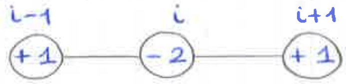
SECOND DERIVATY (molecols)



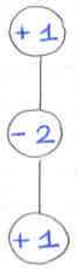
If the function is of two variables, not one, the problem changes.



$$\left(\frac{\partial^2 f}{\partial x_i^2}\right)_{\substack{x=x_i\\ y=y_J}} = \frac{f(x_{i+1}\, y_J) - 2f(x_i\, y_J) + f(x_{i-1}\, y_J)}{\Delta^2}$$

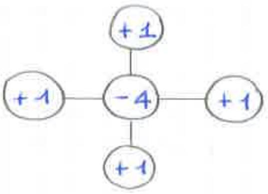


$\left.\frac{\partial^2 f}{\partial x_i^2}\right|_{\substack{x=x_i\\ y=y_J}}$ fixed (x changes)



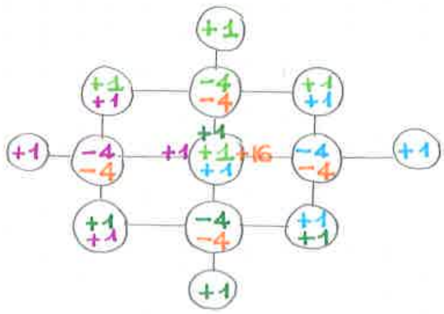
$\left.\frac{\partial^2 f}{\partial y_J^2}\right|_{\substack{x=x_i\\ y=y_J}}$ (fixed) y changes

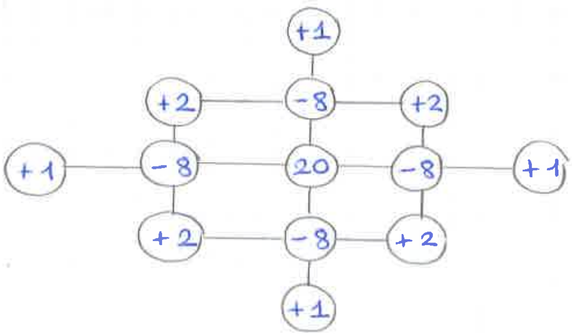
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$



⇒ The sum of all the coefficients is always zero

$$\nabla^4 f = \nabla^2(\nabla^2 f)$$

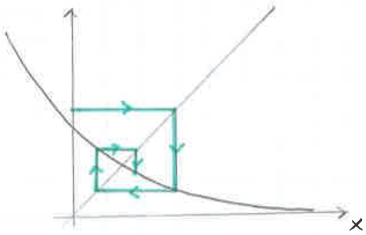




(*) it's the same one (+1)(-4)(+1) multiplied by -4

ITERATIVE METHOD

$x = e^{-x}$



$e^{-x} = y$ and intersection with the line $y = x$ → it converges quickly (ω)

Start with the first value x_0

$x_0 = 0$

$x_1 = e^{-x_0} = 1$

$x_2 = \frac{1}{e} \quad (e^{-x_1})$

$x_3 = e^{-1/e} \quad (e^{-x_2})$

⋮

$x_{i+1} = e^{-x_i}$

Some times it converges to the solution sometimes not, it can be used to solve non linear equations and linear sistem too

↳ we generally use Jacobi method or Gauss-Seidel method to solve linear sistems

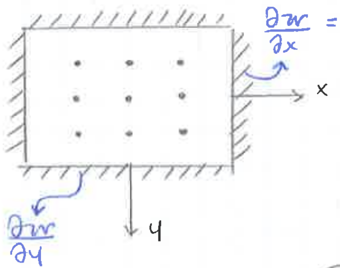
$w_{i+1} = f(w_i)$

still unknowns so we can built an Excel file to do the iterative method and find a solution of the all unknowns (from w_1 to w_{30})

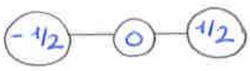
Excel → formula → manual → iterative method (choose precision and n° of iterative steps)

F9 until the solution with the exact precision we want (written in options)

Different supports : clamped beam

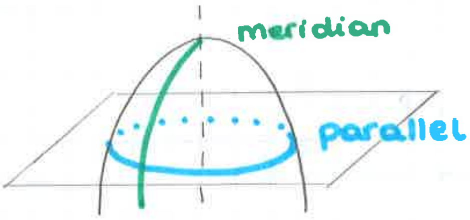


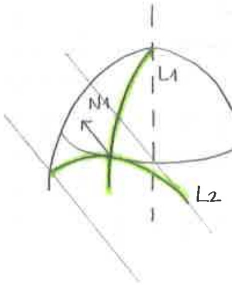
$-\frac{w_{13}}{2} + \frac{w_{15}}{2} = 0$



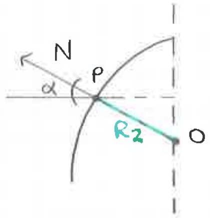
$w_{13} = w_{15}$

as before but with with positive signs.





L1 and L2 are the principal normal sections

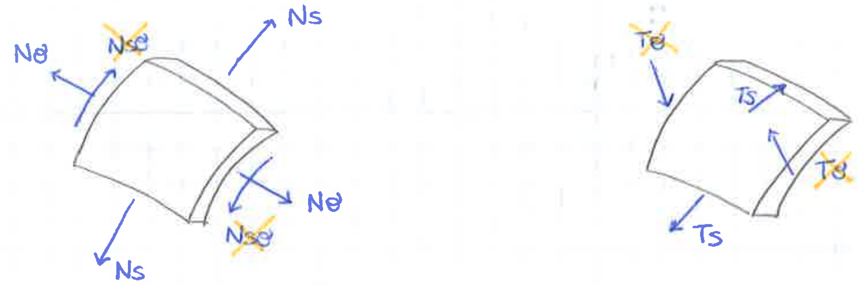


$r = R_2 \cos\alpha$ (MEUSNIER'S THEOREM)

$R_2 = OP$

↳ second radius of curvature

Element of the surface (infinitesimal)



$N_{s\theta}$ is equal to zero because of symmetry, also $T_\theta = 0$ for the same reason (every non symmetrical quantity is $=0$) $\Rightarrow M_{s\theta} = 0$

$$\{Q\} = \begin{pmatrix} N_s \\ N_\theta \\ T_s \\ M_s \\ M_\theta \end{pmatrix}$$

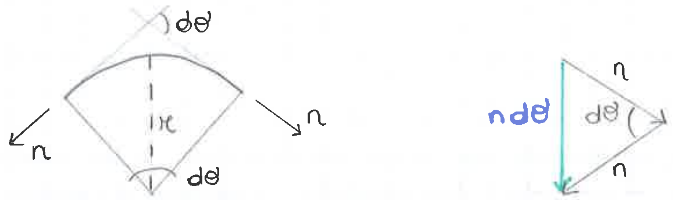
$$\{F\} = \begin{pmatrix} p_s \\ \cancel{p_\theta} \\ q \\ m_s \\ \cancel{m_\theta} \end{pmatrix}$$ → no loads along parallels

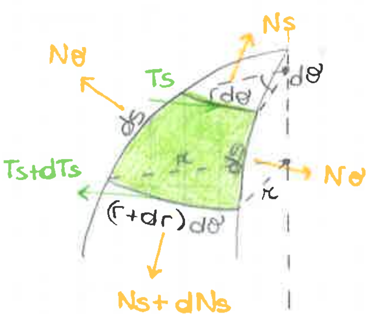
$$\{F\} = \begin{pmatrix} p_s \\ q \\ m_s \end{pmatrix}$$

Equilibrium equations about the meridian

NB

Let's consider an arch of circumference loaded with normal forces





All the quantities depend only on s and they always have to respect the symmetry through the meridian direction

$(M_s + dM_s)(r+dr)\,d\theta - M_s\, r\, d\theta - M_\theta \operatorname{sen}\alpha\, ds\, d\theta - T_s\, r\, d\theta\, ds + m_s\, r\, d\theta\, ds = 0$

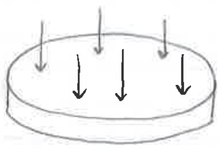
$- M_\theta \operatorname{sen}\alpha\, ds\, d\theta - T_s\, r\, d\theta\, ds + m_s\, r\, d\theta\, ds = 0$

$$\left(\frac{d}{ds} + \frac{\operatorname{sen}\alpha}{r}\right) M_s - \frac{\operatorname{sen}\alpha}{r} M_\theta - T_s + m_s = 0 \qquad (3)$$

$$\begin{pmatrix} \frac{d}{ds} + \frac{\operatorname{sen}\alpha}{r} & -\frac{\operatorname{sen}\alpha}{r} & \frac{1}{R_1} & 0 & 0 \\ -\frac{1}{R_1} & -\frac{1}{R_2} & \frac{d}{ds} + \frac{\operatorname{sen}\alpha}{r} & 0 & 0 \\ 0 & 0 & -1 & \frac{d}{ds} + \frac{\operatorname{sen}\alpha}{r} & -\frac{\operatorname{sen}\alpha}{r} \end{pmatrix} \begin{pmatrix} N_s \\ N_\theta \\ T_s \\ M_s \\ M_\theta \end{pmatrix} + \begin{pmatrix} p_s \\ q \\ m_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underset{3\times5}{[\partial]^*} \{Q\} + \underset{3\times1}{\{F\}} = \underset{3\times1}{\{0\}}$$

Because of (a) the membrain and bending solutions are coupled. If shell becomes flect (R_1 and R_2 are ∞ that terms are zero and the problem is uncoupled ~ it's a plate, a particular shell of revolution)



DISPLACEMENT VECTOR

$$\{\eta\} = \begin{pmatrix} u \\ v \\ w \\ \varphi_s \\ \varphi_\theta \end{pmatrix} \begin{matrix} \text{along } s \\ \text{along } \theta \\ \text{along } N \\ \text{around } +\theta \\ \text{around } -s \end{matrix} \rightarrow \text{displacement of the middle point}$$

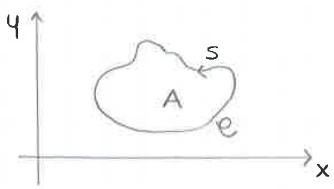
5 component

Because of axial symmetry $v = \varphi_\theta = 0$

$$\{\eta\} = \begin{pmatrix} u \\ w \\ \varphi_s \end{pmatrix}$$

Green's theorem

$$\int_A \frac{\partial f}{\partial x}\, dx\, dy = \oint_e f\, dy$$



If $f = g \cdot h$ (product of two functions)

$$= \int_A -\frac{\partial g}{\partial x} h\, dx\, dy = \int_A g \frac{\partial h}{\partial x}\, dx\, dy - \oint_e g h\, dy$$

PRINCIPAL OF VIRTUAL WORK

- External forces

$$\begin{pmatrix} N_s \\ N_\theta \\ T_s \\ M_s \\ M_\theta \end{pmatrix} = \begin{pmatrix} 12D/h^2 & \nu 12D/h^2 & 0 & 0 & 0 \\ \nu 12D/h^2 & 12D/h^2 & 0 & 0 & 0 \\ 0 & 0 & \frac{5(1-\nu)D}{h^2} & 0 & 0 \\ 0 & 0 & 0 & D & \nu D \\ 0 & 0 & 0 & \nu D & D \end{pmatrix} \begin{pmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_s \\ \chi_s \\ \chi_\theta \end{pmatrix}$$

12D: stiffness of the shell

$$\underset{5\times1}{\{Q\}} = \underset{5\times5}{[H]} \underset{5\times1}{\{q\}}$$

=> Twice static indeterminate but using kinematic and constitutive eq. we can find a solution
- The 2 matrix operator ( ) aren't the trasposal one of the other with finite terms with opposit signs.

$$\underbrace{\underset{3\times5}{[\partial]^*} \underset{5\times5}{[H]} \underset{5\times3}{[\partial]}}_{3\times3} \underset{3\times1}{\{\eta\}} + \underset{3\times1}{\{F\}} = \{0\}$$

→ Three equations in three unknowns

A) Difference between beam and rope :
- beam : bending stiffness $\neq 0$
- rope : bending stiffness $= 0$ or negligible with respect to axial stiffness

↳ we can pass from beam to rope through $h \to 0$ (bending stiffness goes to zero faster than axial stiffness which remains even in the rope)

$I = \frac{bh^3}{12}$ bending stiffness is $\propto I$

Axial stiffness is $\propto$ to A (h^2)

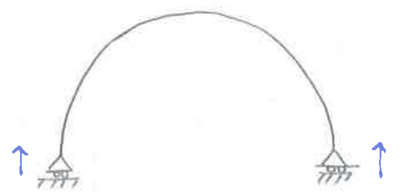
B) Difference between membrane and shell :
- membrane has a bending stiffness $= 0$ $(E = 0)$

$D = \frac{Eh^3}{12(1-\nu^2)}$ axial stiffness is related with D, but we can consider a theoric membrane with only $D = 0$ (bending stiffness)
- shell thicker has a bending stiffness $\neq 0$

=> If $h \to 0$ shell becomes a membrane

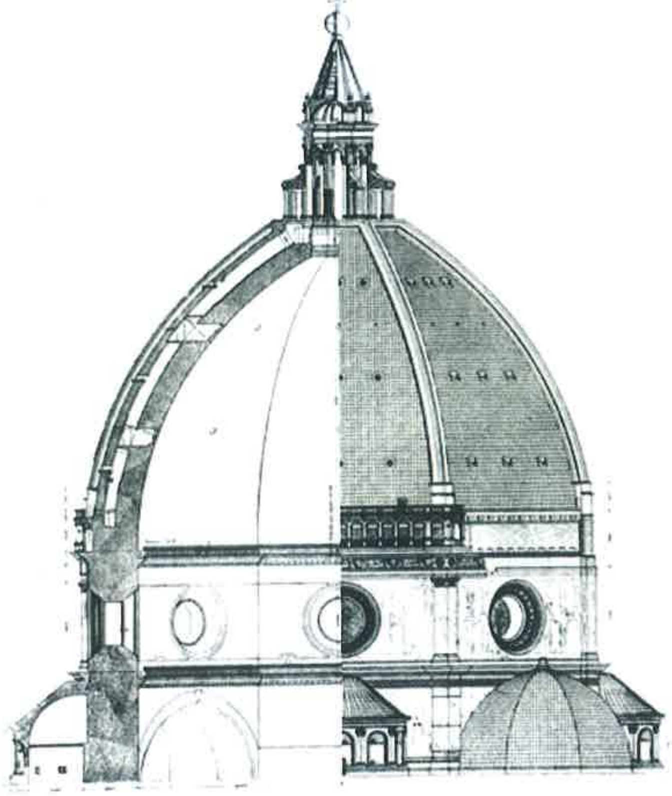
MEMBRANE solution is important because:
1) It's almost equal to shell solutions if constrains are of membrane type
2) It holds also for shells with non membrane constrains sufficiently far from the constrains.

Shell with bending stiffness
Ex Dome

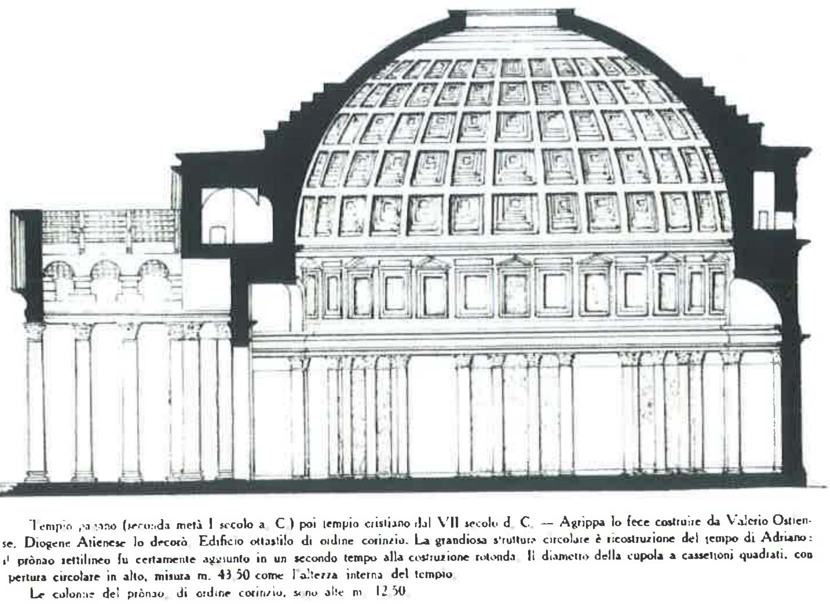


Membrane type constrains: reactions are "membrane" so (N_s) => solution is equal to shells

Constrain reactions are also shear and moment so the solution is different : sinusoidal with exponential decrease so away from constrains the solution is as a membrane (not near them)



PANTHEON
SEZIONE LONGITUDINALE



Tempio pagano (seconda metà I secolo a. C.) poi tempio cristiano dal VII secolo d. C. — Agrippa lo fece costruire da Valerio Ostiense. Diogene Atienese lo decorò. Edificio ottastilo di ordine corinzio. La grandiosa struttura circolare è ricostruzione del tempo di Adriano: il prònao rettilineo fu certamente aggiunto in un secondo tempo alla costruzione rotonda. Il diametro della cupola a cassettoni quadrati, con pertura circolare in alto, misura m. 43.50 come l'altezza interna del tempio.

Le colonne del prònao, di ordine corinzio, sono alte m. 12.50.

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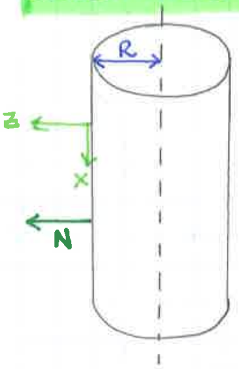
$$\begin{pmatrix} \frac{d}{ds} + \frac{\sin\alpha}{r} & -\frac{\sin\alpha}{r} \\ -\frac{1}{R_1} & -\frac{1}{R_2} \end{pmatrix} \begin{pmatrix} N_s \\ N_\theta \end{pmatrix} + \begin{pmatrix} p_s \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_s \\ \varepsilon_\theta \end{pmatrix} = \begin{pmatrix} \frac{d}{ds} & \frac{1}{R_1} \\ \frac{\sin\alpha}{r} & \frac{1}{R_2} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_s \\ \varepsilon_\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{Eh} & -\frac{\nu}{Eh} \\ -\frac{\nu}{Eh} & \frac{1}{Eh} \end{pmatrix} \begin{pmatrix} N_s \\ N_\theta \end{pmatrix}$$

Cilindrical membrane: meridian is a straight line (s becomes x)
↳ under constant internal pressure



$R_1 = \infty$
$\alpha = 0$
$r = R_2 \cos\alpha = R$
↳ the parallel is a principal normal section (N belongs to its plane, only for cilindrical membrane)

STATIC EQUATIONS

$$\begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} \begin{pmatrix} N_x \\ N_\theta \end{pmatrix} + \begin{pmatrix} p_x \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_\theta \end{pmatrix} = \begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & \frac{1}{R} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

1) $\frac{dN_x}{dx} + p_x = 0$

We assume $p_x = 0$ ~ practicle assumption

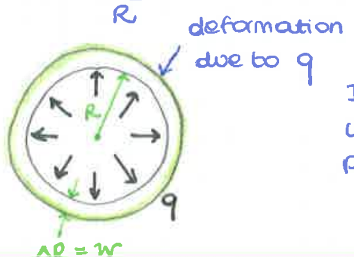
$N_x = cost$

2) $-\frac{N_\theta}{R} + q$

$N_\theta = qR$

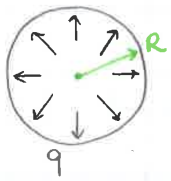
⟹ The problem is static determinate (2 eq. in 2 unknowns)

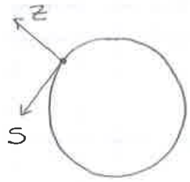
3) $\varepsilon_\theta = \frac{w}{R}$



It's like a pipe with an internal pressure q

Spherical membrane under internal pressure





Displacement along s is u, along z is w

Because of symmetry $u=0$ and for the same reason $N_s = N_\theta = N$

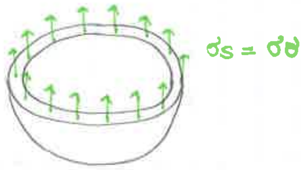
$R_1 = R_2 = R$

↳ radius of meridians and parallels is the same as the sphere

2nd equation:

$$-\frac{N}{R} - \frac{N}{R} + q = 0$$

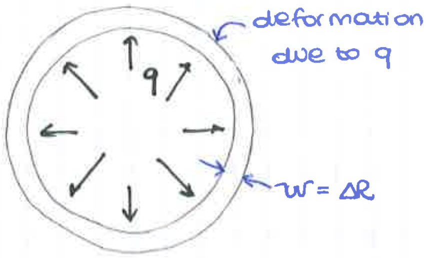
$$N = \frac{qR}{2}$$



$$\sigma_s = \sigma_\theta = \frac{N}{h} = \frac{qR}{2h}$$

↳ constant everywhere

↳ smaller is h higher is σ



$$\varepsilon_\theta = \frac{w}{R} = \frac{1-\nu}{Eh} \cdot N = \frac{1-\nu}{Eh} \cdot \frac{qR}{2h}$$

$$q = \underbrace{\frac{2Eh^2}{(1-\nu)R^2}}_{\text{STIFFNESS}} w$$

Spherical shell is stiffer than cilindrical one

- According to Von Mises for a plane stress case

$$\sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$$

$\sigma_1 = \sigma_2 = \frac{qR}{2h}$ in our case (sphere)

$$\sigma_{eq} = \frac{qR}{2h} < \sigma_{adm}$$

$$\boxed{q_{cr} = \frac{2h}{R}\sigma_{adm}}$$

↳ critical pressure which leads to crisis

- For a cilindrical case:

$$\sigma_\theta = \frac{qR}{h}$$

$$\sigma_x = \frac{qR}{2h}$$

$$\sigma_{eq} = \frac{qR}{h}\sqrt{1 + \frac{1}{4} - \frac{1}{2}} = \frac{\sqrt{3}}{2}\frac{qR}{h}$$

$$\boxed{q_{cr} = \frac{2\sqrt{3}}{3}\frac{h}{R}\sigma_{adm}}$$

this value is higher, the cilindrical shell is stronger

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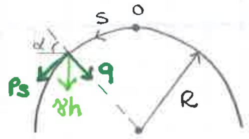
DOME SUBJECTED TO SELF WEIGHT

$$\begin{pmatrix} \frac{d}{ds} + \frac{\operatorname{sen}\alpha}{r} & -\frac{\operatorname{sen}\alpha}{r} \\ -\frac{1}{R} & -\frac{1}{R} \end{pmatrix} \begin{pmatrix} N_s \\ N_\theta \end{pmatrix} = -\begin{pmatrix} p_s \\ q \end{pmatrix} \quad * \rightarrow$$

$R_1 = R_2 = R$

$r = R\cos\alpha$



γ: weight per unit volume (N/m^3)

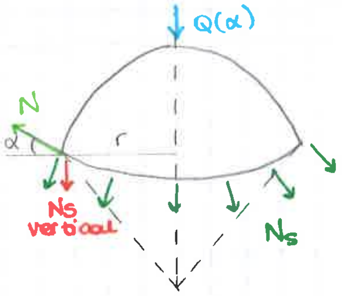
γh: weight per unit surface (N/m^2)

$$\begin{cases} p_s = \gamma h \cos\alpha \\ q = -\gamma h \operatorname{sen}\alpha \end{cases}$$

$*$ ← 2 eq in 2 unknown

First equation is complicated because it's differential

So partial equilibrium in vertical direction for a part of the dome



Ns is a force for unit lenght so we need to multiply for the lenght of parallel

$r = R\cos\alpha$

$Q(\alpha) = \gamma h \, A(\alpha)$

$N_s \cos\alpha \; 2\pi r + Q(\alpha) = 0$

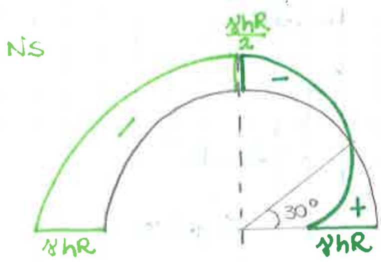
$N_s = -\dfrac{Q(\alpha)}{2\pi R\cos^2\alpha} = -\dfrac{\gamma h (1-\operatorname{sen}\alpha) 2\pi R^2}{2\pi R \cos^2\alpha}$ $\quad \cos^2\alpha = (1-\operatorname{sen}\alpha)(1+\operatorname{sen}\alpha)$

$N_s = -\dfrac{\gamma h R}{1+\operatorname{sen}\alpha}$

To compute N_θ we use the second equation

$\dfrac{N_s}{R} + \dfrac{N_\theta}{R} = q$

$N_\theta = qR - N_s = -\gamma h R \operatorname{sen}\alpha + \dfrac{\gamma h R}{1+\operatorname{sen}\alpha} = \gamma h R \, \dfrac{1-\operatorname{sen}\alpha-\operatorname{sen}^2\alpha}{1+\operatorname{sen}\alpha}$



Quantities are functions of α → $N_\theta = f(\alpha)$ and $N_s = f(\alpha)$

$\alpha = 90°$ we are at the pole, $\alpha = 0°$ we are at the equator in both sides

Meridian is always compressed $N_s < 0$. Along the parallel in a certain point the force Nq becomes positive ~ problems for structures which don't resist to tensile stresses

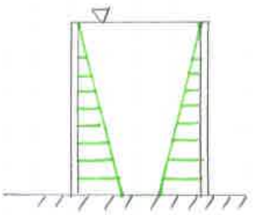
$$\varepsilon_x = \frac{N_x}{Eh} - \nu \frac{N_\theta}{Eh}$$

$$\varepsilon_\theta = \frac{N_\theta}{Eh} - \nu \frac{N_x}{Eh}$$

$$M_x = D(\chi_x + \nu \chi_\theta)$$

$$M_\theta = D(\chi_\theta + \nu \chi_x)$$

↳ CONSTITUTIVE EQUATIONS



- consider a tank
- isostatic pressure, directed along the normal
- P_x is the self weight but negligible with respect to the internal pressure
  ↳ it's truly zero on the top of the tank
- If $N_x = 0 = $ cost is zero everywhere

$$\varepsilon_\theta = \frac{w}{R}$$

$$\gamma_x = \frac{dw}{dx} + \varphi_x \Rightarrow \text{(HP)}$$

↳ shear deformability is negligible as in the beam

$$\gamma_x = 0$$

$$\varphi_x = -\frac{dw}{dx}$$

$$\chi_x = \frac{d\varphi}{dx} = -\frac{d^2w}{dx^2}$$

$$\chi_\theta = 0$$

From equilibrium equations we have:

$$-\frac{N_\theta}{R} + \frac{d^2M_x}{dx^2} + q = 0 \quad (3)$$

$$m_x = 0$$

$$T_x = \frac{dM_x}{dx}$$

From constitutive eq.

$$M_x = -D\frac{d^2w}{dx^2} \quad (2)$$

$$\varepsilon_\theta = \frac{N_\theta}{Eh} = \frac{w}{R}$$

$$N_\theta = Eh\frac{w}{R} \quad (1)$$

(1) and (2) in (3)

$$-Eh\frac{w}{R^2} - D\frac{d^4w}{dx^4} + q = 0$$

$$D\frac{d^4w}{dx^4} + \frac{Eh}{R^2}w = q(x)$$

⇒ the same of the beam in elastic foundation

$$EI\frac{d^4v}{dx^4} + kv = q(x)$$

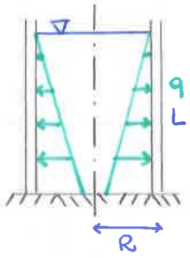
TRANSFORMATION

$EI \rightarrow D$

k (stiffness of the spring equivalent to the ground) $\rightarrow \frac{Eh}{R^2}$ stiffness of cilindrical membrane so of a ring)

So we can study the meridian as a beam on elastic foundation where the stiffness is given by the parallel around it.

Cilindrical tank under hydrostatic pressure 13/12/17



γ: unit weight of the liquid
h: thikness of the plate
—·— axis of revolution

we want to obtain static characteristic
Loads are hydrostatic pressures
↳ Neglect self weight of cilindre

$q(x) = \gamma x$

$w^{IV} + 4\beta^4 w = \frac{q(x)}{D}$

w : displacement in radius direction
D : bending stiffness of the plate

$D = \frac{Eh^3}{12(1-\nu^2)}$

$w^{IV} + \frac{Eh}{R^2 D} w = \frac{\gamma x}{D}$

↳ Differential non homogeneous equation not similar to elastic foundation where we are interested in concentrated loads and so the equation is homogeneous

1° solution (particular solution)

$w^{IV} = 0$

$\tilde{w} = \frac{\gamma R^2}{Eh} x$

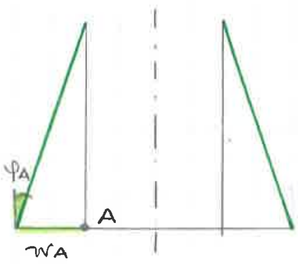
2° solution
↳ particular solution + homogeneous solution (= beam on elastic foundation)

$\tilde{\varphi}_x = -\frac{d\tilde{w}}{dx} = -\frac{\gamma R^2}{Eh}$

$\tilde{M}_x = \tilde{T}_x = 0$
↳ they are ∝ to 2nd and 3rd derivaties (=0) because $\varphi_x = cost.$

$\tilde{N}_\theta = Eh \frac{\tilde{w}}{R} \neq 0$

so φ_x is only the **MEMBRANE SOLUTION** (M=T=0), but the real solution is different: constrains aren't membrane ones as it is produced by the solution, they aren't respected.



$\begin{cases} w_A \neq 0 \\ \varphi_A \neq 0 \end{cases}$

But a clamped edge building supports lead to displacements = 0

$\begin{cases} w_A = 0 \\ \varphi_A = 0 \end{cases}$

2° solution (valid for any L)

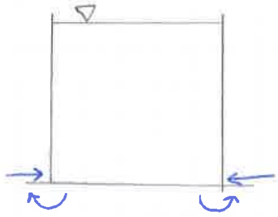
$w(x) = \frac{\gamma R^2}{Eh} x + C_1 e^{\beta z} \cos\beta z + C_2 e^{\beta z} \operatorname{sen}\beta z + C_3 e^{-\beta z} \cos\beta z + C_4 e^{-\beta z} \operatorname{sen}\beta z$

↳ we need 4 boundery conditions to compute C_1, C_2, C_3, C_4

$M(0) = 0$ $w''(0) = 0$

$T(0) = 0$ $w'''(0) = 0$

T_A provides a force which avoid opening of the tank due to internal pressure.



Beam on elastic foundation
↳ exponential decay along L so x becomes L−x

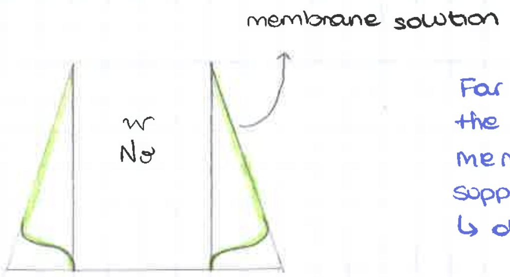
$$w(x) = \frac{\gamma R^2}{Eh} x + T_A \cdot \frac{1}{2D\beta^3} e^{-\beta(L-x)} \cos\beta(L-x) - M_A \frac{1}{2D\beta^2} e^{-\beta(L-x)} [\cos\beta(L-x) - \text{sen}\,\beta(L-x)]$$

We can compute φ, M_x, T_x by derivation

$M_\theta = \nu M_x$

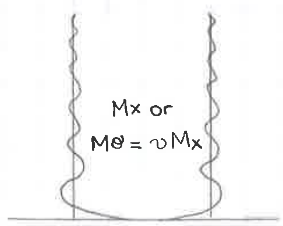
$N_\theta = \frac{Eh}{R} w(x)$

DISPLACEMENT



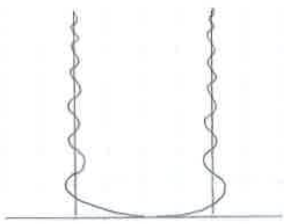
Far away from supports the behaviour is the membrane one, near supports is different
↳ different solution

MOMENT



exponential decay of sinusoidal function

SHEAR



⟹ shear and moment which appear near supports desappear quickly away from supports themselves. And it's equivalent to say that the cilindre is long

Equilibrium of rotation:

$(M_r + dM_r)(r+dr)d\theta - M_r \cdot r d\theta - M_\theta d\theta dr - T_r r d\theta dr + m_r r d\theta dr = 0$

- 1° order derivaties delete each others
- 2° order derivaties values are the only which remain, divided by $r d\theta dr$

$$\frac{dM_r}{dr} + \frac{M_r}{r} - \frac{M_\theta}{r} - T_r + m_r = 0 \quad (2)$$

(1) and (2) are the **STATIC EQUATIONS**

we need to find kinematic equation

$$\begin{pmatrix} \frac{d}{dr} + \frac{1}{r} & 0 & 0 \\ -1 & \frac{d}{dr} + \frac{1}{r} & -\frac{1}{r} \end{pmatrix} \begin{pmatrix} T_r \\ M_r \\ M_\theta \end{pmatrix} + \begin{pmatrix} q \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(m) always $= 0$

transpose of the kinematic matrix with finite terms with opposite signs

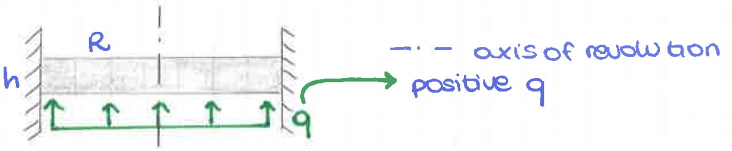
$$\begin{pmatrix} \gamma_r \\ \chi_r \\ \chi_\theta \end{pmatrix} = \begin{pmatrix} \frac{d}{dr} & +1 \\ 0 & \frac{d}{dr} \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} w \\ \varphi_r \end{pmatrix}$$

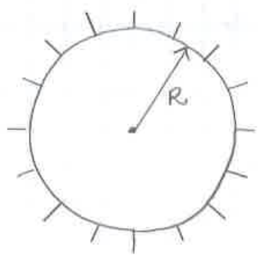
$$\begin{cases} M_r = D(\chi_r + \nu \chi_\theta) \\ M_\theta = D(\chi_\theta + \nu \chi_r) \end{cases}$$

- $\gamma_r = 0$, $\varphi_r = -\frac{dw}{dr}$
- $\chi_r = \frac{d\varphi_r}{dr}$
- $\chi_\theta = \frac{\varphi_r}{r}$

(*)

Clamped circular plate under unifor distributed load q





(1) $\frac{dT_r}{dr} + \frac{T_r}{r} + q = 0$

$$\frac{1}{r} \cdot \frac{d}{dr}(r T_r) = -q$$

$$\frac{d}{dr}(r T_r) = -qr$$

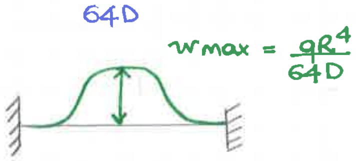
$\frac{dw}{dr} = -\varphi_r$ we know φ_r so integrating its expression we can compute w

$w(r=R) = 0$

$$w = -\frac{q}{16D}\left(\frac{R^2r^2}{2} - \frac{r^4}{4} + C_4\right)$$

For boundary condition $w(R)=0$ we compute C_4

$$w = \frac{q}{64D}(R^4 - 2r^2R^2 + r^4) = \frac{q}{64D}(R^2 - r^2)^2$$



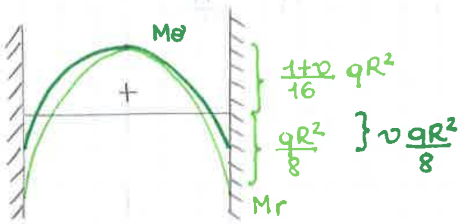
$$\begin{cases} M_r = \frac{q}{16}\left[(R^2 - 3r^2) + \nu(R^2 - r^2)\right] \\ M_\theta = \frac{q}{16}\left[(R^2 - r^2) + \nu(R^2 - 3r^2)\right] \end{cases}$$

$$\begin{cases} M_r(0) = \frac{1+\nu}{16} qR^2 \\ M_\theta(0) = M_r(0) \end{cases}$$

↳ in the middle they have the same value

$$\begin{cases} M_r(R) = -\frac{qR^2}{8} \\ M_\theta(R) = -\nu\frac{qR^2}{8} \end{cases}$$

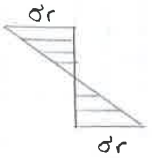
The max positive value is in the middle, the max absolute is negative one and it's reached on the borders where there are supports



Navier equation to find stresses

$$\sigma_r = \frac{M_r z}{h^3/12}$$

There isn't b because M is for unit length



σ_r min or max are achieved when $z = \pm h/2$ and their values are

$$\sigma_{r\,min/max} = \frac{6M_r}{h^2}$$

↳ it's achieved also when $r = R$

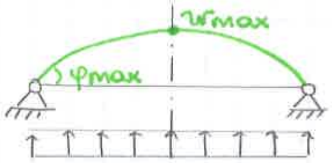
$$\sigma_{r\,min/max} = \frac{3}{4}\frac{qR^2}{h^2}$$

$$w = -\frac{Mr^2}{2(1+\nu)D} + C_3$$

$$w(R) = 0$$

$$w = \frac{M}{2(1+\nu)D}(R^2 - r^2)$$

Superposition of effects with $M = qR^2/8$
↳ simply supported circular plate under uniform load q



$$w_{max} = \frac{MR^2}{2(1+\nu)D} \quad \text{when } M = \frac{qR^2}{8} \quad + \text{ the } w_{max} \text{ for the clamped case}$$

$$= \frac{qR^4}{16(1+\nu)D} + \frac{qR^4}{64D} = \frac{qR^4}{64D(1+\nu)}[5+\nu]$$

It's about 4 times the value for the clamped case (depending on the value of $\nu \simeq 0,3$) which is the stiffer one.

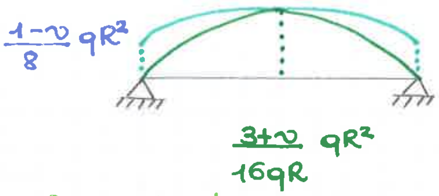
$$\varphi_{max} = \frac{qR^3}{2(1+\nu)D}$$

$$M_r(0) = M_\theta(0) = \frac{1+\nu}{16}qR^2 + \frac{qR^2}{8} = \frac{3+\nu}{16}qR^2$$

$$M_r(R) = 0$$

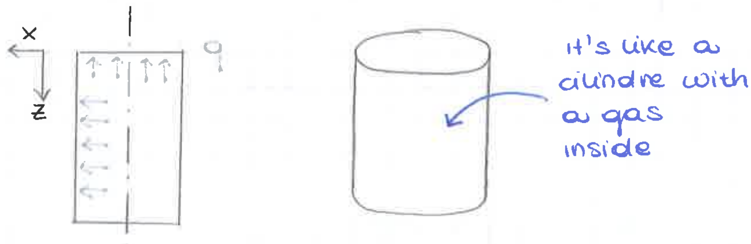
$$M_\theta(R) = -\nu\frac{qR^2}{8} + \frac{qR^2}{8} = \frac{(1-\nu)}{8}qR^2$$

Bending moment diagram

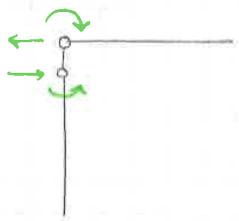


Pressurized cilindrical vessel with caps:

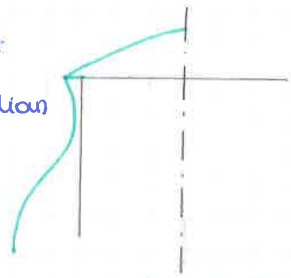
1) Flat faced caps
2) Emisphericau caps



- No hydrostatic, consant everywhere
- With bending stiffness
- We remove a constrain to achieve a known membrane case then we introduce shear and moment to balance and make congruent with the initial system



Disp. of meridian



displacement of the cap

For away we obtain the membrain solution

it's an intermediate solution between the clamped shell and the simply supported one

Moment

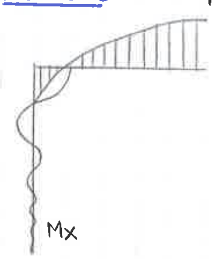
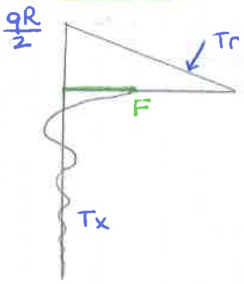


plate is flected always under bending

on the cilindre solicitations goes rapidly to zero (membrane solution)

the border value is M for both plate and cilindre

Shear Tx



far away we have only Hθ (?)

From eq. (2)
$M = 0$
From eq. (1)

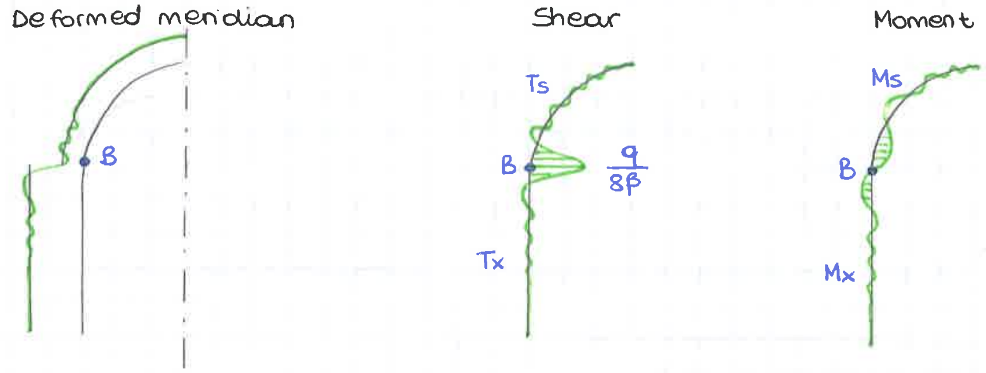
$$\frac{F}{D\beta^3} = \frac{R^2 q}{Eh} \quad \frac{2-\nu-1+\nu}{2}$$

$$F = \frac{R^2 q D \beta^3}{2Eh} = \frac{R^2 q D \beta^4}{2Eh\beta}$$

$$\beta^4 = \frac{Eh}{4DR^2}$$

$$F = \frac{R^2 q D}{2Eh\beta} \cdot \frac{Eh}{4DR^2}$$

$$\begin{cases} F = \dfrac{q}{8\beta} \\ M = 0 \end{cases}$$



Far away we have membrane solution

$N_\theta = qR$ for the cilindre → so in point B there is
$N_\theta = \frac{qR}{2}$ for sphere — a transition for these two values

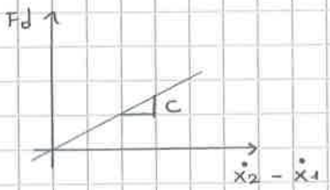
Similar to the plot of w (displacement), in fact $N_\theta \propto w$
$N_\theta = Eh \frac{w}{R}$

⇒ The force F appears only because of the different elements which compose the structure.

Usually we have membran solutions except for:
- constrains which aren't membrane ones
- different linked geometrical elements in junction points

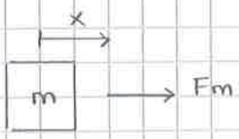
↳ generation of shear and moment which quickly disappear far away from these points and become membrane again.

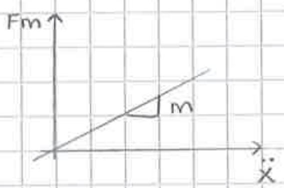
c: is coefficient of dumping (Ns/m)



3) Discret mass

$F_m = m\ddot{x}$ (second Newton's law)





m is the **DISCRETE MASS** which behaves as a rigid body not elastic

=> the analysis related to traslations (x) or rotations (φ) is the same

Second order linear system

①

- spring
- dash pot
- mass

Free body diagram (isolating the mass we compute forces on it)

$F(t) - F_s(t) - F_d(t) = m\ddot{x}$

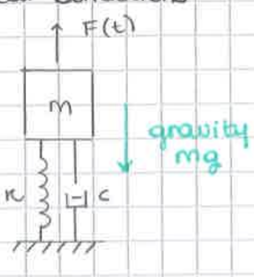
$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$

↳ second order system in which m, c and k are constant and are parameters of the system (it has only one degree of freedom x)

↳ neglecting the mass it becomes first order system (unusual)

the position of the mass depends only on the equilibrium equations so the unstretched position concides with equilibrium position

② Vertical directions



↥ y(t) for undeformed configuration

↥ x(t) for equilibrium configuration ↕ X_{ST}

$y(t) - x(t) = X_{ST}$

where $X = \sqrt{A^2 + B^2}$

$X = \sqrt{\left(\frac{\dot{x}_0}{\omega}\right)^2 + x_0^2}$ amplitude

$\varphi = \text{arctg} \frac{A}{B} = \text{arctg} \frac{\dot{x}_0}{\omega x_0}$ phase angle

=> solution can be even put in the form

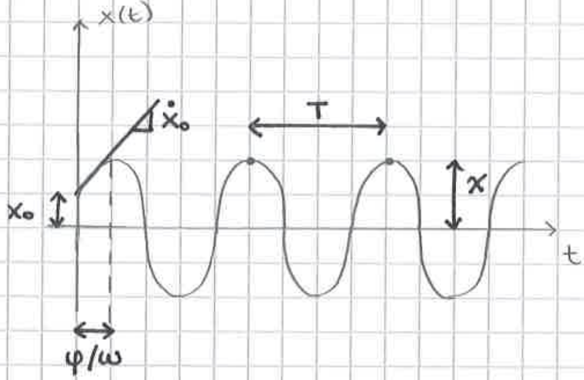
$x(t) = X \text{ sen}(\omega t - \varphi)$

in this case

$\varphi = \text{arctg} \frac{B}{A}$

$f = \frac{\omega}{2\pi}$ ordinary frequency

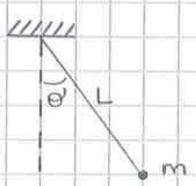
$T = \frac{2\pi}{\omega}$ period



Initial conditions should be ≠ 0 (at least one) because F=0 so only things that cause vibrations are initial conditions themselves.

EX

↳ without damping we have a HARMONIC OSCILLATOR (PENDOLUM)



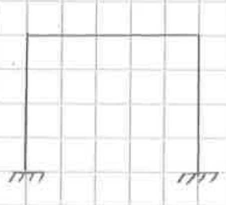
Degree of freedom is the rotation θ not the motion

$\ddot{\theta} + \omega \theta = 0$

$\omega^2 = \frac{g}{L}$

EX

↳ Shear - type frame



$I_h >> I_v$

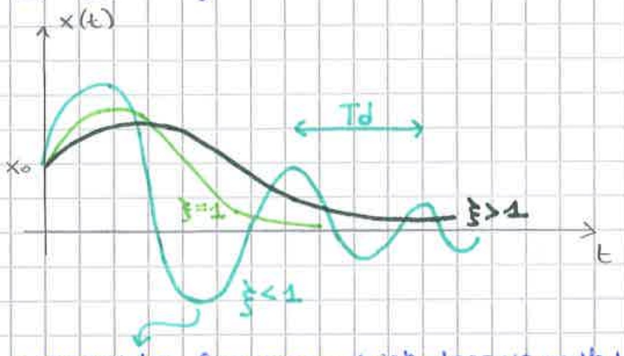
So the total mass is concentrated in the cross member and the apparats have a stiffness equal to k.

$$x(t) = \left(c_1 e^{i\omega_D t} + c_2 e^{-i\omega_D t}\right) e^{-\xi\omega t}$$
$$= (A \,\mathrm{sen}\, \omega_D t + B \cos \omega_D t)\, e^{-\xi\omega t} = X \cos(\omega_D t - \varphi)\, e^{-\xi\omega t}$$

The costants A, B or X can be expressed through initial conditions

$$A = \frac{\dot{x}_0 + x_0 \xi\omega}{\omega_D}$$
$$B = x_0$$
$$X = \sqrt{A^2 + B^2}$$
$$\mathrm{tg}\,\varphi = \frac{B}{A}$$

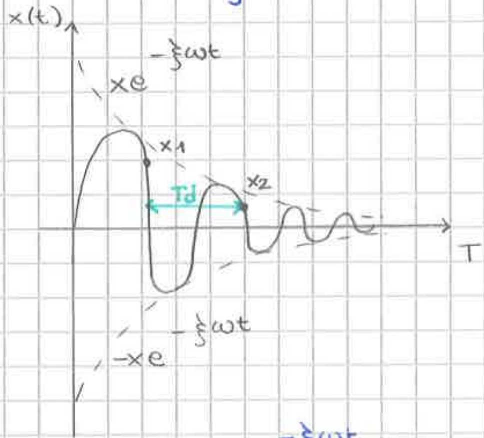
If $\omega_D < \omega$, $T_d > T$



sinusoidal function which decays with time

LOGARITHMIC DECREMENT

- true only for $\xi < 1$
- evaluation ξ in a esperimental way, considering exponential decay



The curves $\pm x e^{-\xi\omega t}$ provide the envelope bounding the oscillatory responce. As $T \to \infty$, $x(t) \to 0$

A convinient measure of the amount of damping in a single degree of freedom system is provided by the extent to which the amplitude has fallen during one complete cycle of vibration.

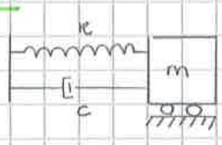
$$T_d = \frac{2\pi}{\omega_D}$$

$$\frac{x_1}{x_2} = \frac{X e^{-\xi\omega t_1} \cos(\omega_D T_1 - \varphi)}{X e^{-\xi\omega t_2} \cos(\omega_D t_2 - \varphi)}$$

$$t_2 = t_1 + T_d$$

21/12/17

EX



$m = 100$ kg
$T_d = 2$ s
$x_2 = \frac{x_1}{16}$

$\delta = \ln \frac{x_1}{x_2} = \ln 16$

$\xi = \frac{1}{2\pi} \delta = 0,44$ it's true if ξ is small enough

$\delta = \frac{2\pi \xi}{\sqrt{1-\xi^2}}$

$\xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$ → this relationship is always true → $\xi = 0,404$ (10% of difference, it's not negligible)

If $\xi \to 0$ $\xi = \frac{\delta}{2\pi}$, but in this case it's not sufficiently small

$\omega = \frac{\delta}{\xi T_d} = 3,434$ rad/s

$\begin{cases} \omega = 2\pi f \\ f = \frac{\omega}{2\pi} \end{cases}$

$\frac{k}{m} = \omega^2$ definition of natural frequency

$k = 1179,2$ N/m

$\frac{c}{m} = 2 \xi \omega$

$c = 277,47$ Ns/m

Free response ($F=0$)
Each degree of freedom correspond to an order differential equation
$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$
↳ $F=0$ so excitations depend on initial conditions

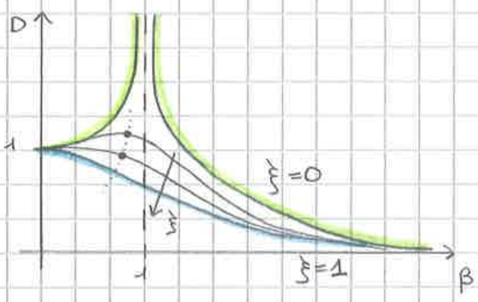
Forced systems ($F \neq 0$)
↳ FORCED VIBRATIONS

$m\ddot{x}(t) + c\dot{x}(t) + k\,x(t) = F(t)$
↳ the solution depends on the type of $F(t)$ ~ harmonic excitations
↳ $F(t) = F \text{ sen}(\omega_f t)$
↳ ω_f frequency of external force ≠ natural frequency of the system so how the solution changes with the ratio between these two frequencies

Equation of motion (dividing by m the initial equation)
$\ddot{x}(t) + 2\xi\omega\,\dot{x}(t) + \omega^2 x(t) = \frac{F}{m} \text{ sen}(\omega_f t)$

The general solution is the sum of
- x_p particular solution which depends on loads
- x_c solution given by free vibration (complementary solution when $F=0$)

$x_p(t) = C_1 \text{ sen } \omega_f t + C_2 \cos \omega_f t$
The damped system in general isn't in phase with the loading (so the particular solution if F is sinusoidal (x_p) isn't just sinusoidal so we add cos)



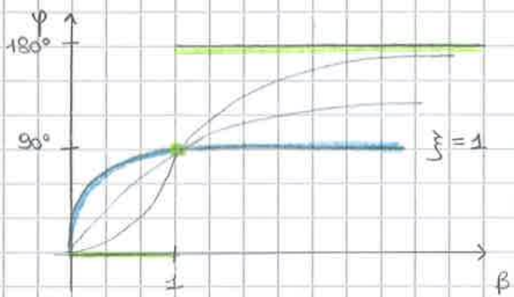
Different curves related to different values of ξ

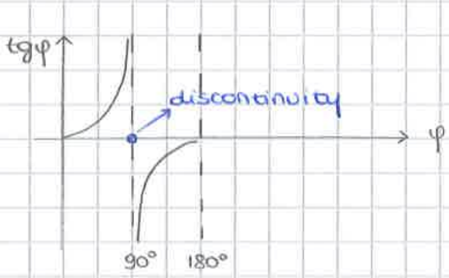
- the general solution when $\xi=0$ (so $c=0$)

  $$x(t) = \frac{F}{k}\frac{1}{1-\beta^2}\,\text{sen}\,\omega_f t + (A\,\text{sen}\,\omega t + B\cos\omega t)$$

  If the force has a frequency = frequency of the system, the amplitude tents to infinite so the structure tents to collapse (RESONANCE)
- $\xi \neq 0$

  curves have picks ≠1 and their positions define the LOCUS OF THE MAXIMA until $\xi=1$ in which the curve decreseas monotonically





LOCUS OF MAXIMUM

$$\frac{d}{dx}\frac{1}{\sqrt{f(x)}} \propto f'(x)$$

$$\frac{d}{d\beta}\left[(1-\beta^2)^2 + (2\xi\beta)^2\right] = -2\beta\cdot 2(1-\beta^2) + 8\xi^2\beta = 0$$

$$\beta_{peak} = \sqrt{1-2\xi^2}$$

$\xi < \frac{1}{\sqrt{2}}$ for larger values we haven't a max anymore ($\xi=1$ ~ monotonic function)

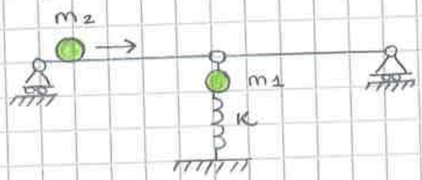
$$D_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\omega_{f,ris} = \omega\sqrt{1-2\xi^2}$$

↳ max amplitude or risonance frequency

Equation of motion is the same, but it can be solved not analitically, but numerically.

EXAMPLE
$m = m(t)$
$c = c(t)$



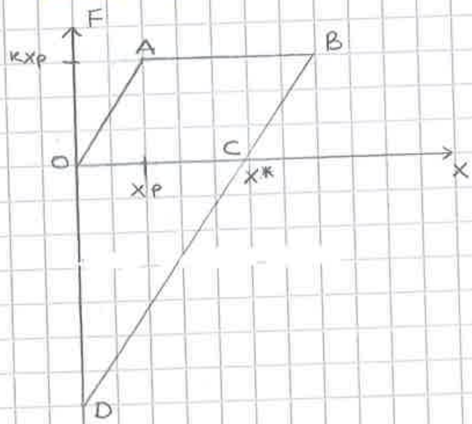
aproximation of a bridge with a moving load

The mass of the oscillator (m1) depends on the position of m2, even c is influenced by the position and the velocity of m2

EXAMPLE
$k = k(t)$
Non linearity is linked to the consitutive load. The spring is a structure.
There are two non linearity related to the spring

- **Geometric non linearity**
  ↳ high displacements, hypotesis of small displacements is removed, it's linked with instability
- **Physical non linearity**
  ↳ spring hasn't an elastic behaviour, but it feels plasticity too.

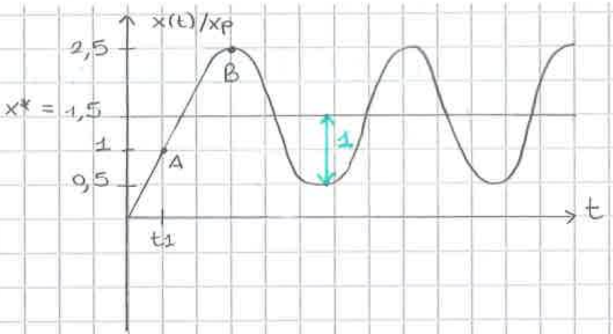
ELASTO-PLASTIC OSCILLATOR
- suppose $c = 0$ (no damping)
- $x_0 = 0$ (initial displacement is equal to zero)
- $\dot{x}_0 > 0$ (initial velocity is grater than zero)
- Constitutive load is perfectly plastic



OA: elastic part
AB: plastic behaviour
x*: permanent displacement
BD: second elastic phase

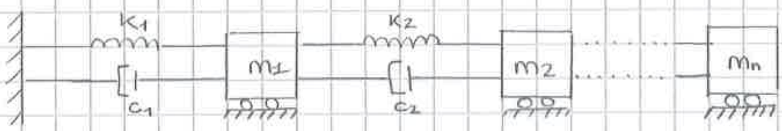
The elastic force $kx(t)$ can be replaced by $k[x(t) - x^*]$ during the elastic phase. Here x^* denotes the permanent displacement. During the plastic phases we have to replace $kx(t)$ with $\pm x_p$ depending on the solution.

- 1ST PHASE (OA)
  $x^* = 0$
  Only a mass and a spring: equation of motion is
  $m\ddot{x}(t) + kx(t) = 0$
  Solution is expressed in terms of Asen + Bcos which A and B depending on initial conditions



$\dot{x}_0 = 2\omega x_p$

MDOF SYSTEMS
↳ multi degree of freedom



m, c, k aren't constant, not scalars, but matrices

MODAL ANALYSIS ([C] = 0 and [F] = 0)

$[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{0\}$

[K] and [M] are N×N matrices
Positive defined, symmetric and generally not diagonal

Equation of motion
↳ in terms of components

$\sum_{J=1}^{N} M_{iJ}\,\ddot{\delta}_J(t) + \sum_{J=1}^{N} K_{iJ}\,\delta_J(t) = 0$

$i = 1 \dots N$

Solution is in terms of:
$\{\delta(t)\} = \{\delta\} f(t)$

Separating space problem $\{\delta\}$ (it's not a function of time anymore) and time problem $f(t)$. So all the coordinates in the space vary in the same way as load in time.

We substitute $\{\delta(t)\}$ in equation of motion

$[M]\{\delta\}\ddot{f}(t) + [K]\{\delta\} f(t) = \{0\}$ (*)

$\ddot{f}(t) + \dfrac{\{\delta\}^T [K] \{\delta\}}{\{\delta\}^T [M] \{\delta\}} f(t) = 0$

If K and M are positive defined the ratio is positive for sure and it can be called ω^2 (= generalization of $\omega^2 = K/m$ in single degree of freedom systems)

$\ddot{f}(t) + \omega^2 f(t) = 0$
↳ equation of harmonic motion which has a well known solutions

$f(t) = A\cos\omega t + B\,\mathrm{sen}\,\omega t = C\cos(\omega t - \varphi)$
↳ it has to be substituted in the first equation (*)

$([K] - \omega^2[M])\{\delta\} = \{0\}$

TIME PROBLEM

$\ddot{f}_i + \omega_i^2 f_i = 0 \qquad i = 1 \dots N$

$f_i(t) = A_i \cos\omega_i t + B_i \operatorname{sen}\omega_i t$

General solution of the problem takes the following form:

$\{\delta(t)\} = \sum_{i=1}^{N} \{\delta_i\} (A_i \cos\omega_i t + B_i \operatorname{sen}\omega_i t)$

A_i and B_i can be evaluated through initial conditions

$\{\delta(0)\} = \{\delta_0\} = \sum_{i=1}^{N} A_i \{\bar{\delta}_i\} \qquad J = 1, \dots, N$

$\{\dot{\delta}(0)\} = \{\dot{\delta}_0\} = \sum_{i=1}^{N} B_i \omega_i \{\bar{\delta}_i\}$

$A_J = \{\delta_0\}^T [M] \{\bar{\delta}_J\}$

$B_J = \frac{1}{\omega_i} \{\dot{\delta}_0\}^T [k] \{\bar{\delta}_J\}$

$\{\delta_0\} = a \{\bar{\delta}_i\}$

$\{\dot{\delta}_0\} = \{0\}$

$A_J = \begin{cases} a & \text{if } i = J \\ 0 & \text{if } i \neq J \end{cases}$

$B_J = 0$

$\{\delta(t)\} = \{\bar{\delta}_i\} \cos\omega t = \{\delta_0\} \cos\omega t$

Modal analysis

$[M]\{\ddot{\delta}(t)\} + [k]\{\delta(t)\} = \{0\}$

Equation of motion can be rewritten through modal matrix

$[\Delta] = [\{\bar{\delta}_1\}][\{\bar{\delta}_2\}] \dots [\{\bar{\delta}_n\}]$

$\{\delta(t)\} = [\Delta]\{f\}$

$\underbrace{[\Delta]^T [M] [\Delta]}_{[I]} \{\ddot{f}\} + \underbrace{[\Delta]^T [k] [\Delta]}_{\Lambda} \{f\} = \{0\}$

$[I]$ Identity matrix — Λ Diagonal matrix with eigen values $\omega_1^2, \omega_2^2, \dots$

$\{\ddot{f}\} + [\Lambda]\{f\} = 0$

$\ddot{f}_i + \omega_i^2 f_i$

in case of neglecting damping the solution (eq. of motion) is uncoupled (there are N indipendent equation)

- Throug orthogonality
- Every vibration mode is expressed with a linear combination of eigen vectors

$[K]\{x_i\} = \omega_i^2 [M]\{x_i\}$ $\quad i = 1,2$

In components:

1) $(k_{11} - \omega_i^2 m_{11}) x_{i1} + (k_{12} - \omega_i^2 m_{12}) x_{i2} = 0$
2) $(k_{21} - \omega_i^2 m_{21}) x_{i1} + (k_{22} - \omega_i^2 m_{22}) x_{i2} = 0$

Previous equations become different, dividing both equations by k

$$\left(2 - \frac{\omega_i^2 m}{k}\right) x_{i,1} - x_{i,2} = 0 \qquad i = 1,2$$

$$-x_{i1} + 2\left(1 - \frac{\omega_i^2 m}{k}\right) x_{i2} = 0$$

Two components for x_1 and two for x_2. It's homogeneous system and solve only two equations not every four of them

$i = 1$

$$x_{12} = \left(2 - \frac{\omega_1^2 m}{k}\right) x_{11} = 1,366\, x_{1,1}$$

$$\{x_1\} = \begin{Bmatrix} 1,0000 \\ 1,366 \end{Bmatrix}$$

$i = 2$

$$x_{22} = \left(2 - \frac{\omega_2^2 m}{k}\right) x_{2,1} = -0,366\, x_{21}$$

$$\{x_2\} = \begin{Bmatrix} 1,0000 \\ -0,366 \end{Bmatrix}$$

Eigen vector are defined for a multiplicative factor so we need a normalization to find a unique value (**MODAL ANALYSIS**)

$$\{x_1\}^T [M] \{x_1\} = \begin{Bmatrix} 1,000 \\ 1,366 \end{Bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1,000 \\ 1,366 \end{Bmatrix}$$

$= 4,7320\ m = m_1$ (the first modal mass)
$= 1,2679\ m = m_2$

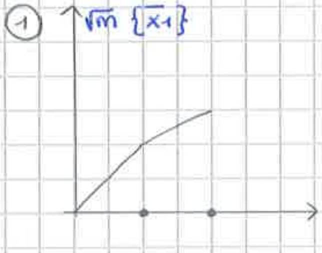
$$\{\bar{x}_1\} = \frac{\{x_1\}}{\sqrt{m_1}} = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0,4597 \\ 0,6280 \end{Bmatrix}$$

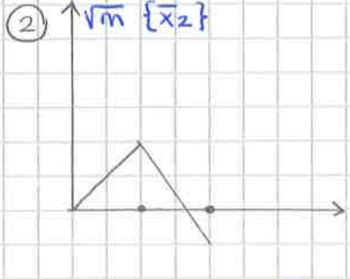
Both components are positive so both of them move rightwards

$$\{\bar{x}_2\} = \frac{\{x_2\}}{\sqrt{m_2}} = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0,8881 \\ -0,3251 \end{Bmatrix}$$

First spring stretch, third enshorts so second spring enshorts

These are the two eigenvectors





The first spring and the third one are going to stretch, but the second value is > than the first one so even the second sping stretch

These vectors form a base so each oscillating mode is expressed with a linear combination of them

MDOF SYSTEMS:

Shear type frames

17/01/18

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{0\}$$

1) Solve problem in the space

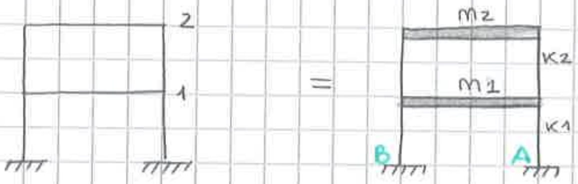
$$([K] - \omega^2 [M])(\delta) = 0$$

det = 0 to find eigenvalues (ω^2) then substituting it in the equation lead to find eigenvectors (δ) ⇒ operation normalization

2) $\ddot{f} + \omega^2 f = 0$ (armonic equation of motion)

$$\{\delta(t)\} = \Sigma \{\bar{\delta}_i\} \cdot (A_i \cos \omega_i T + B_i \operatorname{sen} \omega_i T)$$

SHEAR TYPE FRAMES



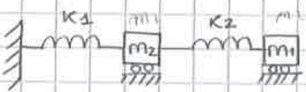
$I_{x1} > I_{x2}$

↳ of horizontal elements (1) > than vertical elements (2)

m_1 and m_2 are concentrated in the horizontal elements, instead of k which are considered in vertical elements ↳ infinity stiff

$$k_i = 2 \cdot \frac{12 EI_i}{h_i^2}$$ SHEAR STIFFNESS

because it's the sum of (A) and (B)



EQUATIONS OF MOTION

$$m_1 \ddot{\delta}_1 = k_1 \delta_1 + k_2 (\delta_2 - \delta_1)$$
$$m_2 \ddot{\delta}_2 = - k_2 (\delta_2 - \delta_1)$$

In a matrix form we have:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det([K] - \omega^2 [M]) = 0$$

$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

↳ second order equation in ω^2

If $k_1 = k_2 = k$ and $m_1 = m_2 = m$ we have:

$$\omega^4 - 3 \frac{k}{m} \omega^2 + \frac{k^2}{m^2} = 0$$

$$\omega^2 = \frac{3 \pm \sqrt{3}^{\,5}}{2} \frac{k}{m} =$$

- FLEXURAL PROBLEM

$$EI \frac{d^4 v}{dz^2} = 0$$

$0 < z < L$

v it's a third order function

$$v(z) = \frac{1}{6} c_1 z^3 + \frac{1}{2} c_2 z^2 + c_3 z + c_4$$

c_i = function of generalized nodal displacement
↳ translation and rotation

$$-\left.\frac{dv}{dz}\right|_{z=0} = \varphi_1$$

$$v(0) = v_1$$

$$-\left.\frac{dv}{dz}\right|_{z=L} = \varphi_2$$

$$v(L) = v_2$$

$$c_1 = -\frac{6}{L^3}(-2v_1 + L\varphi_1 + 2v_2 + L\varphi_2)$$

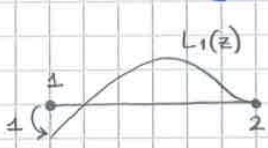
$$c_2 = \frac{2}{L^2}(-3v_1 + 2L\varphi_1 + 3v_2 + L\varphi_2)$$

$$c_3 = -\varphi_1$$

$$c_4 = v_1$$

$$v(z) = L_1(z)\varphi_1 + L_2(z)v_1 + L_3(z)\varphi_2 + L_4(z)v_2$$

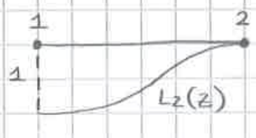
$$L_1(z) = -L\left[\frac{z}{L} - 2\left(\frac{z}{L}\right)^2 + \left(\frac{z}{L}\right)^3\right]$$



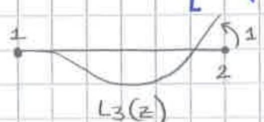
rotation in 1 = 1
rotation in 2 = 0

} all these functions have 1 displacement = 1 and one = 0

$$L_2(z) = \left[1 - 3\left(\frac{z}{L}\right)^2 + 2\left(\frac{z}{L}\right)^3\right]$$



$$L_3(z) = -L\left[-\left(\frac{z}{L}\right)^2 + \left(\frac{z}{L}\right)^3\right]$$



$$L_4(z) = \left[3\left(\frac{z}{L}\right)^2 - 2\left(\frac{z}{L}\right)^3\right]$$

$$[M] = \frac{\mu L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\frac{\text{mass}}{\text{length}} \cdot \text{length} = \text{mass}$$

- FLEXURAL PROBLEM

$$v(z,t) = \{L(z)\}^T \{v(z)\}$$

Potential energy

$$\mathcal{U}(t) = \frac{1}{2}\int_0^L M\chi \, dz = \frac{1}{2}\int_0^L EI \left[\frac{\partial^2 w(z,t)}{\partial z^2}\right]^2 dz$$

$$M = EI\chi$$

$$\mathcal{U}(t) = \frac{1}{2}\int_0^L \{v(t)\}^T [k] \{v(t)\}$$

$$[k] = \int_0^L EI \{L''(z)\}\{L''(z)\}^T dz$$

$$\{L''(z)\} = \frac{z}{L}\begin{pmatrix} 2 - 3z/L \\ -3/L + 6z/L^2 \\ 1 - 3z/L \\ 3/L - 6z/L \end{pmatrix}$$

$$[k] = \frac{4EI}{L^2}\int_0^L \begin{pmatrix} 2 - 3z/L \\ -3/L + 6z/L^2 \\ 1 - 3z/L \\ 3/L - 6z/L \end{pmatrix}\begin{pmatrix} 2 - 3z/L \\ -3/L + 6z/L^2 \\ 1 - 3z/L \\ 3/L - 6z/L \end{pmatrix}^T dz$$

$$k_{11} = \frac{4EI}{L^2}\int_0^L \left(2 - \frac{3z}{L}\right)^2 dz = \frac{4EI}{L}$$

$$k_{13} = k_{31} = \frac{4EI}{L^2}\int_0^L \left(2 - \frac{3z}{L}\right)\left(1 - \frac{3z}{L}\right) dz = \frac{2EI}{L}$$

$$k_{24} = k_{42} = \frac{4EI}{L^2}\int_0^L \left(-\frac{3}{L} + \frac{6z}{L}\right)\left(\frac{3}{L} - \frac{6z}{L}\right) dz = -\frac{12EI}{L}$$

$$[k] = EI \begin{bmatrix} \frac{4}{L} & -\frac{6}{L^2} & \frac{8}{L} & \frac{6}{L^2} \\ & \frac{12}{L^3} & -\frac{6}{L^2} & -\frac{12}{L^3} \\ & & \frac{4}{L} & \frac{6}{L^2} \\ \text{sym.} & & & \frac{12}{L^3} \end{bmatrix}$$

Kinetical energy

$$T = \frac{1}{2}\int_0^L \mu \left[\frac{\partial w(z,t)}{\partial t}\right]^2 = \frac{1}{2}\{\dot{v}(t)\}[M]\{\dot{v}(t)\}^T$$

$$[M] = \int_0^L \mu \{L(z)\}\{L(z)\}^T dz$$

$$[M] = \mu \int_0^L \begin{Bmatrix} L_1(z) \\ L_2(z) \\ L_3(z) \\ L_4(z) \end{Bmatrix}\begin{Bmatrix} L_1(z) \\ L_2(z) \\ L_3(z) \\ L_4(z) \end{Bmatrix}^T dz$$

$$\frac{EI}{\mu}\frac{d^4\eta}{dz^4}\cdot\frac{1}{\eta} = -\frac{d^2f}{dt^2}\cdot\frac{1}{f} = \omega^2$$

↳ They must be constant ($=\omega^2$) to respect this equation

(1) $\frac{d^2f}{dt^2} + \omega^2 f = 0$ → familiar free vibration expression for undumped single degree of freedom system
$f(t) = A\cos\omega t + B\,\text{sen}\,\omega t$
A and B can be expressed through initial conditions
$f(t) = f(0)\cos\omega t + \frac{\dot f(0)}{\omega}\,\text{sen}\,\omega t$

(2) $\frac{d^4\eta}{dz^4} - \alpha^4\eta = 0$

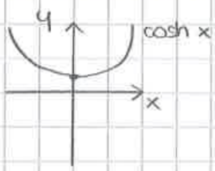
$$\alpha = \sqrt[4]{\frac{\mu\omega^2}{EI}}$$

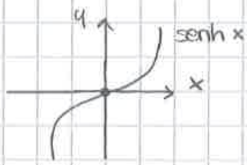
↓ Can be solved a solution of the form $\eta(z) = G\exp(st)$
$(s^4 - \alpha^4)G\exp(st) = 0$
$s_{12} = \pm i\alpha$
$s_{34} = \pm\alpha$

$$\eta(z) = G_1\exp(i\alpha z) + G_2\exp(-i\alpha z) + G_3\exp(\alpha z) + G_4\exp(-\alpha z)$$

$\cos hx = \frac{e^x + e^{x}}{2}$

$\text{sen}\,hx = \frac{e^x - e^{-x}}{2}$





$$\eta(z) = C\cos\alpha z + D\sin\alpha z + E\cosh\alpha z + F\,\text{senh}\,\alpha z$$

$C, D, E, F = f(G_1, G_2, G_3, G_4)$
↳ depend on boundery conditions

- $\eta_2 = \text{sen} \frac{2\pi}{L} z$

$\omega_2 = 4 \frac{\pi^2}{L^2} \sqrt{\frac{EI}{\mu}}$

- $\omega_3 = 9 \frac{\pi^2}{L^2} \sqrt{\frac{EI}{\mu}}$

$\eta_3 = \text{sen} \frac{9\pi}{L} z$

(NB) As for discrete systems, mode shape are defined up to a moltiplicative constants in continous systems.
↳ They are orthogonal and satisfy the following condition:

$$\int_0^L \mu(z)\, \eta_K(z)\, \eta_J(z)\, dz = 0 \qquad K \neq J$$

if $K = J$ η needs to be normalized

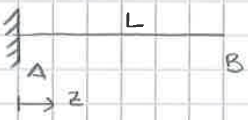
The solution is:

$$v(z,t) = \sum_{n=1}^{\infty} \eta_n(z)\, f_n(t)$$

where the number n is chosen depending on the problem

EX 2

Cantilever beam



$\eta(0) = 0$ $\qquad C + E = 0$

$\eta'(0) = 0$ $\qquad D + F = 0$

$\eta'''(L) = 0$

↳ $\alpha^3 (C \text{sen}\,\alpha L - D \cos \alpha L + E \,\text{senh}\,\alpha L + F \cosh \alpha L) = 0$

$\eta''(L) = 0$

↳ $\alpha^2 (-C \cos\alpha L - D\, \text{sen}\alpha L + E \cosh\alpha L + F\, \text{senh}\alpha L) = 0$

$$\begin{cases} C = -E \\ D = -F \end{cases}$$

The latter two equation can be written in:

$$\begin{bmatrix} \cos\alpha L + \cosh\alpha L & \text{sen}\alpha L + \text{sen}\,\alpha L \\ -\text{sen}\alpha L + \text{senh}\alpha L & \cos\alpha L + \cosh\alpha L \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

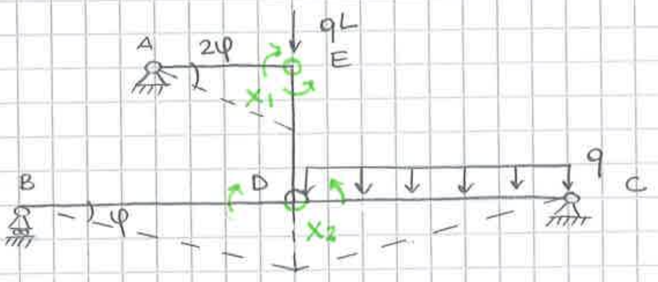
The determinant must be $= 0$

$$(\cos\alpha L + \cosh\alpha L)^2 - (\text{sen}^2 h\alpha L - \text{sen}^2 \alpha L) = 0$$

$$\cos\alpha L \cosh\alpha L = -1$$

$\alpha_1 L = 1{,}875$

$\alpha_2 L = 4{,}694$



$$\begin{cases} g = 2 \cdot 3 = 6 \\ v = 2(A) + 1(B) + 2(D) + 2(C) = 7 \end{cases}$$

$v - g = i = 1$

Associated truss structure

$$\begin{cases} g = 4 \cdot 3 = 12 \\ v = 2(A) + 2(E) + 1(B) + 4(D) + 2(C) = 11 \end{cases}$$

$g - v = \ell = 1$

It's a mechanism

$$\begin{cases} \varphi EA = \varphi ED \\ \varphi DB = \varphi DC \\ PLV \end{cases}$$

EA ↷ ED ↶(+) DB ↷ DC ↶(+)

$$\begin{cases} -\frac{X_1 L}{3EI} - 2\varphi = \frac{X_1 L}{3EI} \\ -\frac{X_2 \cdot 2L}{3EI} - \varphi = \frac{X_2 \cdot 2L}{3EI} + \varphi - \frac{q(2L)^3}{24EI} \\ X_1 \cdot 2\varphi + X_2 \varphi + X_2 \varphi + 2qL \cdot \varphi L + qL \cdot 2\varphi L = 0 \end{cases}$$

$\frac{2}{3} \frac{X_1 L}{EI} = -2\varphi \Rightarrow \varphi = -\frac{X_1 L}{3EI}$

$-\frac{4}{3} \frac{X_2 L}{EI} + \frac{2}{3} \frac{X_1 L}{EI} + \frac{qL^3}{3EI} = 0$

$\hookrightarrow -\frac{4}{3} X_2 + \frac{2}{3} X_1 + \frac{q}{3} L^2 = 0$

$X_1 = -\frac{q}{2} L^2 + 2X_2$

$2X_1 + 2X_2 + 4qL^2 = 0$

$X_1 + X_2 + 2qL^2 = 0$

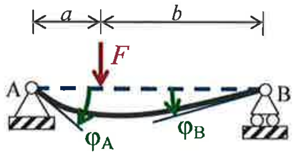
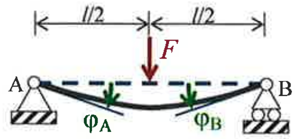
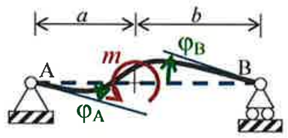
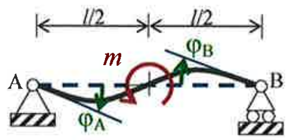
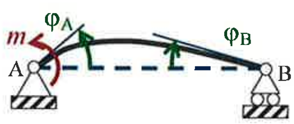
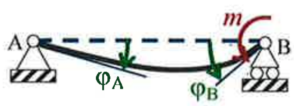
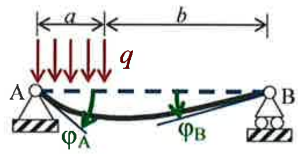
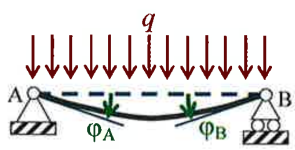
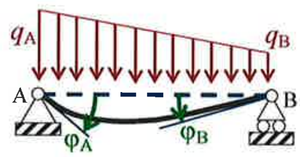
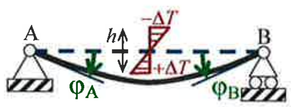
$-\frac{q}{2} L^2 + 2X_2 + X_2 + 2qL^2 = 0$

$3X_2 = \left(-2 + \frac{1}{2}\right) qL^2$

$$\begin{cases} X_2 = -\frac{1}{2} qL^2 \\ X_1 = -\frac{qL^2}{2} - qL^2 = -\frac{3}{2} qL^2 \end{cases}$$

Rotations for simply supported beams

Notes: Beams are l-long, the cross section height is h and the inertia moment is I. Material has Young modulus E and coefficient of thermal expansion α. Positive values in expressions (right) corresponds to the arrow directions in drawings (left) and vice-versa.

	$\varphi_A = \frac{ab(l+b)}{6lEI}F$ $\varphi_B = \frac{ab(l+a)}{6lEI}F$		$\varphi_A = \frac{Fl^2}{16EI}$ $\varphi_B = \frac{Fl^2}{16EI}$
	$\varphi_A = \frac{l^2 - 3b^2}{6lEI}m$ $\varphi_B = \frac{l^2 - 3a^2}{6lEI}m$		$\varphi_A = \frac{ml}{24EI}$ $\varphi_B = \frac{ml}{24EI}$
			$\varphi_A = \frac{ml}{3EI}$ $\varphi_B = \frac{ml}{6EI}$
			$\varphi_A = \frac{ml}{6EI}$ $\varphi_B = \frac{ml}{3EI}$
	$\varphi_A = \frac{qa^2(a+2b)^2}{24lEI}$ $\varphi_B = \frac{qa^2(2l^2 - a^2)}{24lEI}$		$\varphi_A = \frac{ql^3}{24EI}$ $\varphi_B = \frac{ql^3}{24EI}$
	$\varphi_A = \frac{(8q_A + 7q_B)l^3}{360EI}$ $\varphi_B = \frac{(7q_A + 8q_B)l^3}{360EI}$		
	$\varphi_A = \alpha\Delta T\frac{l}{h}$ $\varphi_B = \alpha\Delta T\frac{l}{h}$		

Notes to formulary
"Reactions for double-clamped beams"

The formulary provides the reactions for a double-clamped beam under common load conditions.

The formulary is useful when facing a structure with the method of displacement. With the method of displacement, one need to know the forces acting on the frame nodes: these forces are equal in modulus and opposite in directions to reactions given in the formulary.

The first six cases are related to support settlements (i.e. imposed displacements): the corresponding reactions coincide with the columns of the beam stiffness matrix. Last row is about thermal effects. The remaining cases are common loads: at the left there are general cases; at the right specific cases frequently met in engineering practice.

For what concerns the signs, positive values in the reaction expressions (right) correspond to the direction of the arrows (left) and vice-versa.

For what concerns last case ("butterfly" thermal deformation), note that only if the center of mass lies at half the cross section height h (as, e.g., in sections with two axes of symmetry), the temperature variation does not yield axial reactions.

Note to formulary
"Rotations for simply supported beams"

The formulary provides the rotations for simply supported beams under common load conditions.

The formulary is useful when facing a structure with the so-called mixed method.

To provide static determinacy, we set a hinge at left edge. However, since there are no axial loads, hinge and roller are equivalent (and thus they can be exchanged).

At the left there are general cases; at the right, specific cases frequently met in engineering practice.

For what concerns the signs, positive values in the rotation expressions (right) correspond to the direction of the arrows (left) and vice-versa.

For what concerns last case ("butterfly" thermal deformation), note that only if the center of mass lies at half the cross section height h (as, e.g., in sections with two axes of symmetry), the temperature variation does not yield axial dilation/contraction.

Scienza delle costruzioni
Telai a nodi fissi

SCIENZA DELLE COSTRUZIONI

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Scienza delle costruzioni
Telai a nodi fissi

TELAI A NODI FISSI

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Scienza delle costruzioni
Telai a nodi fissi

Argomenti della lezione

- → **Cedimenti vincolari elastici**
- → **Metodo misto**
  - → **Introduzione**
  - → **Telai a nodi fissi**

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Cedimenti vincolari elastici

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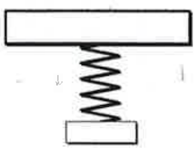
Telai a nodi fissi

La reazione è proporzionale all'entità del cedimento

Il vincolo può essere schematizzato con una molla di rigidezza k



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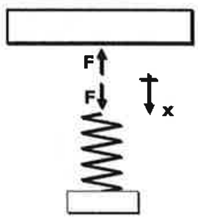
La reazione è proporzionale all'entità del cedimento

Il vincolo può essere schematizzato con una molla di rigidezza k



$F = -k\,x$

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$$\int_S M_1^2 dz = \frac{l^3}{6} \qquad \int_S M_1 M_0 dz = -\frac{5}{24} q l^4$$

Equazione di congruenza

$$-\frac{X_1}{k} = -\frac{5}{24}\frac{ql^4}{EI} + \frac{X_1 l^3}{EI}$$

$$X_1 = 2ql \frac{\frac{5}{48}}{\frac{EI/l^3}{k} + \frac{1}{6}}$$

Il termine $(EI/l^3)/k$ rappresenta il rapporto tra le rigidezze della trave e del vincolo

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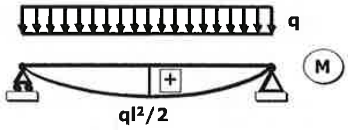
Casi limite

La trave è molto più rigida del vincolo: $(EI/l^3)/k \to \infty$ → $X_1 = 0$



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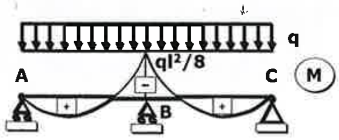
Casi limite

Il vincolo è molto più rigido della trave: $(EI/l^3)/k \to 0$ → $X_1 = \frac{5}{8}(2ql)$



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Introduzione al metodo misto

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Telai a nodi fissi

Telaio: sistema di travi a molti gradi di iperstaticità

Nei telai, l'uso del PLV per la scrittura delle equazioni di congruenza risulta essere oneroso

Se il telaio è n volte iperstatico, occorre calcolare $(n^2+3n)/2$ integrali

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Metodo misto

Si inseriscono cerniere in tutti i nodi-incastro del telaio

La struttura così ottenuta è detta reticolare associata al telaio

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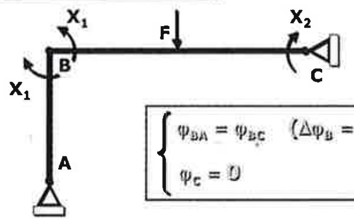
Equazioni di congruenza angolare



$$\begin{cases} \varphi_{BA} = \varphi_{BC} & (\Delta\varphi_B = 0) \\ \varphi_C = 0 \end{cases}$$

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Telai a nodi fissi

La reticolare associata è costituita da tante travi doppiamente appoggiate le cui rotazioni sono note dagli schemi elementari

La scrittura delle equazioni di congruenza angolare è immediata!

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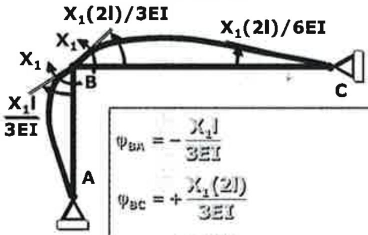
Telai a nodi fissi



$$\varphi_{BA} = -\frac{X_1 l}{3EI}$$
$$\varphi_{BC} = +\frac{X_1(2l)}{3EI}$$
$$\varphi_C = -\frac{X_1(2l)}{6EI}$$

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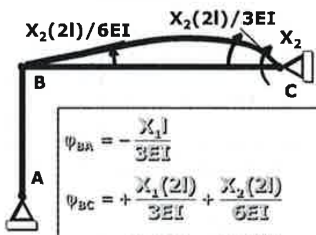
Telai a nodi fissi



$$\varphi_{BA} = -\frac{X_1 l}{3EI}$$
$$\varphi_{BC} = +\frac{X_1(2l)}{3EI} + \frac{X_2(2l)}{6EI}$$
$$\varphi_C = -\frac{X_1(2l)}{6EI} - \frac{X_2(2l)}{3EI}$$

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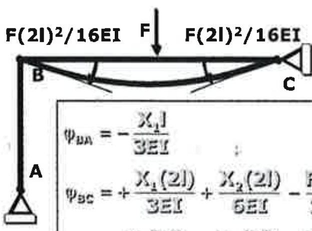
Telai a nodi fissi



$$\varphi_{BA} = -\frac{X_1 l}{3EI}$$
$$\varphi_{BC} = +\frac{X_1(2l)}{3EI} + \frac{X_2(2l)}{6EI} - \frac{F(2l)^2}{16EI}$$
$$\varphi_C = -\frac{X_1(2l)}{6EI} - \frac{X_2(2l)}{3EI} + \frac{F(2l)^2}{16EI}$$

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Telai a nodi fissi

Equazioni di congruenza angolare

$$\begin{cases} \varphi_{BA} = \varphi_{BC} \\ \varphi_C = 0 \end{cases}$$

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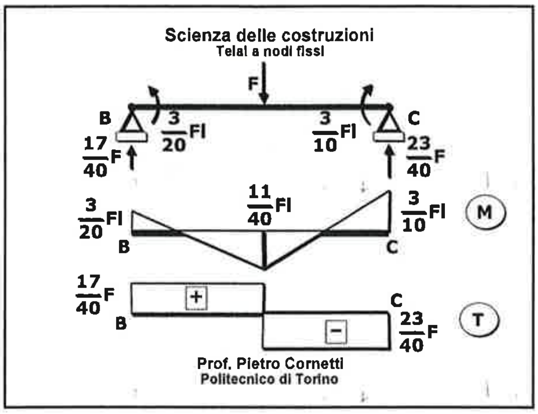
Scienza delle costruzioni
Telai a nodi fissi
F
B
C
3/20 Fl
3/10 Fl
17/40 F
23/40 F
11/40 Fl
M
T
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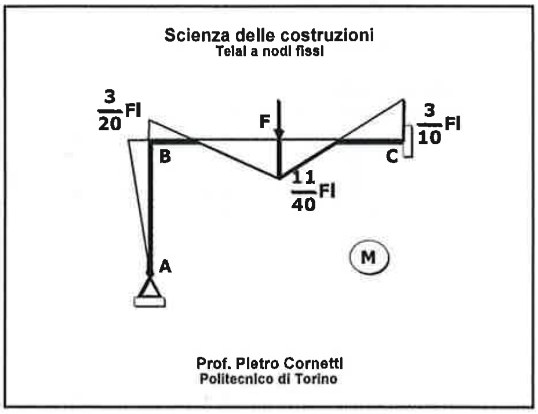
Scienza delle costruzioni
Telai a nodi fissi
3/20 Fl
F
3/10 Fl
B
C
11/40 Fl
A
M
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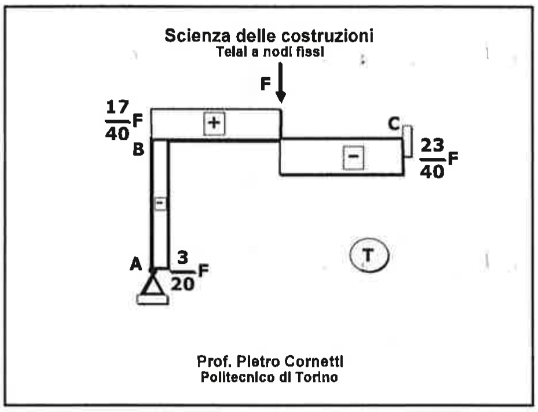
Scienza delle costruzioni
Telai a nodi fissi
F
17/40 F
B
C
23/40 F
A
3/20 F
T
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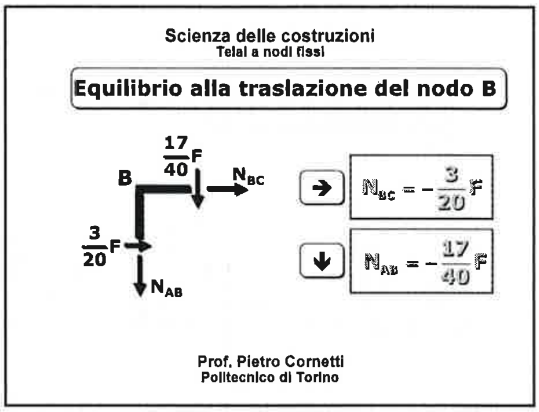
Scienza delle costruzioni
Telai a nodi fissi
Equilibrio alla traslazione del nodo B
17/40 F
B
N_BC
3/20 F
N_AB
N_BC = -3/20 F
N_AB = -17/40 F
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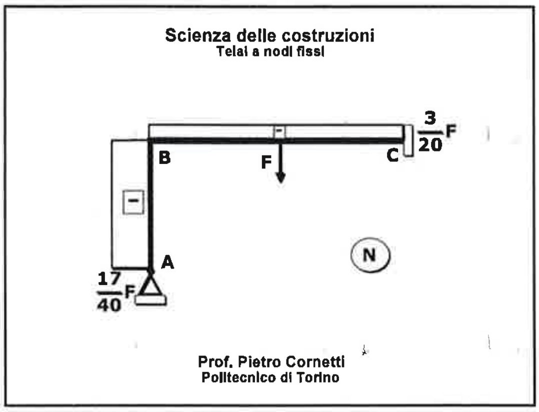
Scienza delle costruzioni
Telai a nodi fissi
3/20 F
B
F
C
A
17/40 F
N
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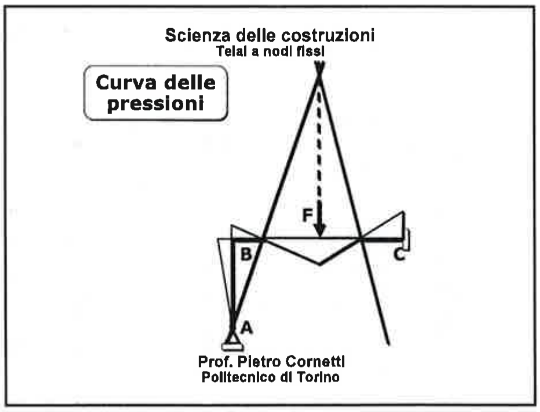
Scienza delle costruzioni
Telai a nodi fissi
Curva delle pressioni
F
B
C
A
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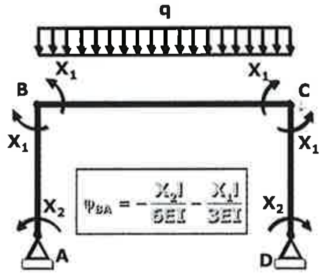
Scienza delle costruzioni
Telai a nodi fissi



$$\varphi_{BA} = -\frac{X_2 l}{6EI} - \frac{X_1 l}{3EI}$$

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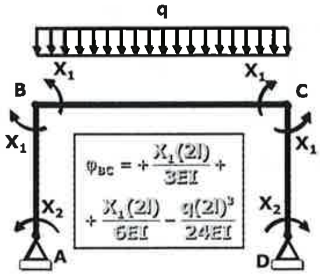
Scienza delle costruzioni
Telai a nodi fissi



$$\varphi_{BC} = +\frac{X_1(2l)}{3EI} + \frac{X_1(2l)}{6EI} - \frac{q(2l)^3}{24EI}$$

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Scienza delle costruzioni
Telai a nodi fissi

Equazioni di congruenza angolare

$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BC} \end{cases}$$

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Scienza delle costruzioni
Telai a nodi fissi

Equazioni di congruenza angolare

$$\begin{cases} +\frac{X_2 l}{3EI} + \frac{X_1 l}{6EI} = 0 \\ -\frac{X_2 l}{6EI} - \frac{X_1 l}{3EI} = +\frac{X_1(2l)}{3EI} + \frac{X_1(2l)}{6EI} - \frac{q(2l)^3}{24EI} \end{cases}$$

Sistema lineare nelle due incognite iperstatiche X_1 e X_2

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Telai a nodi fissi

Equazioni di congruenza angolare

$$\begin{cases} X_1 = +\frac{4}{15} ql^2 \\ X_2 = -\frac{2}{15} ql^2 \end{cases}$$

Sistema lineare nelle due incognite iperstatiche X_1 e X_2

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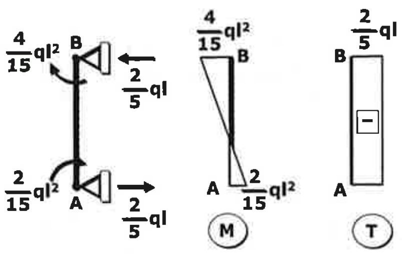
Telai a nodi fissi



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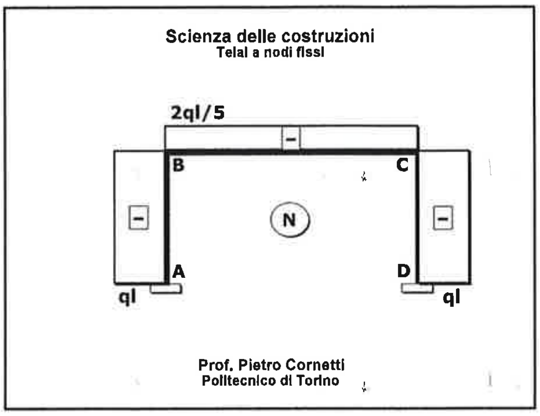
Scienza delle costruzioni
Telai a nodi fissi
2ql/5
B
C
N
A
D
ql
ql
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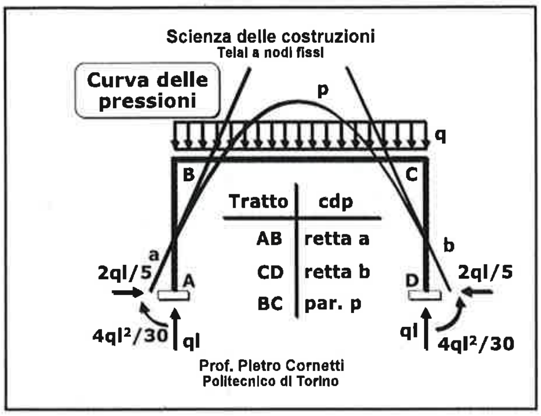
Scienza delle costruzioni
Telai a nodi fissi
Curva delle pressioni
p
q
B
C
Tratto | cdp
AB | retta a
CD | retta b
BC | par. p
a
b
2ql/5
2ql/5
A
D
4ql²/30
ql
ql
4ql²/30
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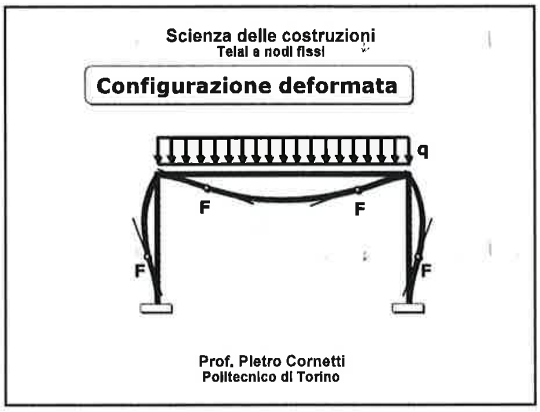
Scienza delle costruzioni
Telai a nodi fissi
Configurazione deformata
q
F
F
F
F
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SCIENZA DELLE COSTRUZIONI

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IL METODO MISTO

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Argomenti della lezione

- **Calcolo di un telaio a nodi fissi**
- **Il metodo misto per i telai a nodi mobili**
  - **Introduzione**
  - **Equilibrio e congruenza**
  - **Esempio**

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Calcolo di un telaio a nodi fissi

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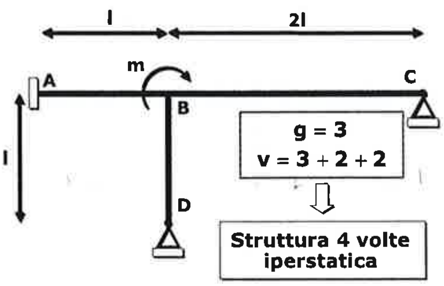

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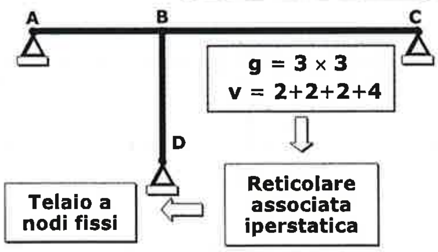
Reticolare associata



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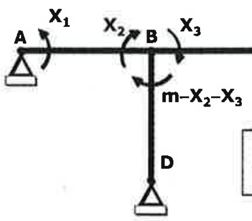
Scienza delle costruzioni
Il metodo misto

Equazioni di congruenza angolare



$$\varphi_{BC} = -\frac{X_3(2l)}{3EI}$$

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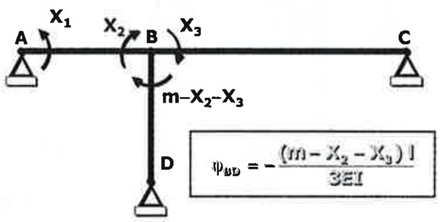
Scienza delle costruzioni
Il metodo misto

Equazioni di congruenza angolare



$$\varphi_{BD} = -\frac{(m - X_2 - X_3)\,l}{3EI}$$

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Scienza delle costruzioni
Il metodo misto

Equazioni di congruenza angolare

$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_{BA} = \varphi_{BD} \end{cases}$$

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Il metodo misto

Equazioni di congruenza angolare ⇨ **Sistema lineare nelle tre incognite iperstatiche X_1, X_2, X_3**

$$\begin{cases} +\frac{X_1 l}{3EI} + \frac{X_2 l}{6EI} = 0 \\ -\frac{X_1 l}{6EI} - \frac{X_2 l}{3EI} = -\frac{X_3(2l)}{3EI} \\ -\frac{X_1 l}{6EI} - \frac{X_2 l}{3EI} = -\frac{(m - X_2 - X_3)\,l}{3EI} \end{cases}$$

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Scienza delle costruzioni
Il metodo misto

Equazioni di congruenza angolare ⇨ **Sistema lineare nelle tre incognite iperstatiche X_1, X_2, X_3**

$$\begin{cases} X_1 = -\frac{4}{17}m \\ X_2 = +\frac{8}{17}m \\ X_3 = +\frac{3}{17}m \end{cases}$$

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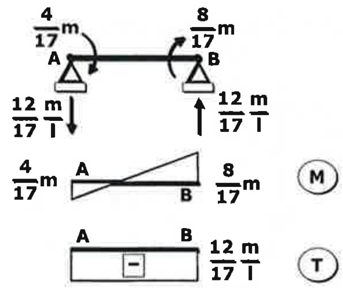
Il metodo misto



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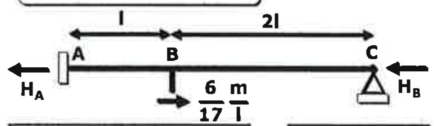
Il metodo misto

Iperstaticità assiale



$$H_A = \frac{2}{3}\left(\frac{6}{17}\frac{m}{l}\right) = \frac{4}{17}\frac{m}{l}$$
$$H_B = \frac{1}{3}\left(\frac{6}{17}\frac{m}{l}\right) = \frac{2}{17}\frac{m}{l}$$
⇨
$$N_{AB} = +\frac{4}{17}\frac{m}{l}$$
$$N_{BC} = -\frac{2}{17}\frac{m}{l}$$

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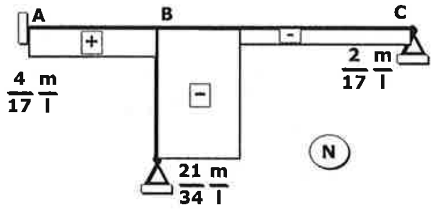
Scienza delle costruzioni
Il metodo misto



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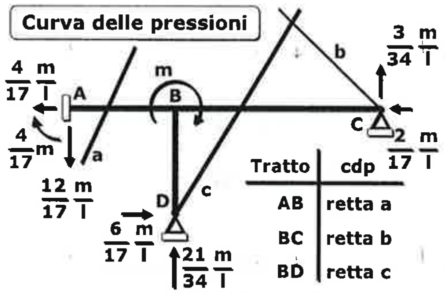
Scienza delle costruzioni
Il metodo misto

Curva delle pressioni



Tratto	cdp
AB	retta a
BC	retta b
BD	retta c

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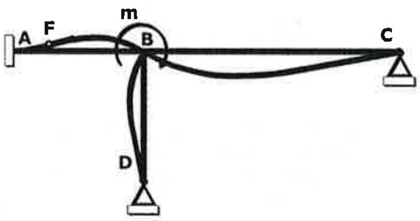
Scienza delle costruzioni
Il metodo misto

Configurazione deformata



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Scienza delle costruzioni
Il metodo misto

Osservazioni

- ➔ Per i telai a nodi fissi il metodo misto è un caso particolare del metodo delle forze
- ➔ Le scrittura delle equazioni di congruenza è molto rapida se sono note le rotazioni elementari (tabelle)

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Scienza delle costruzioni
Il metodo misto

Telai a nodi mobili

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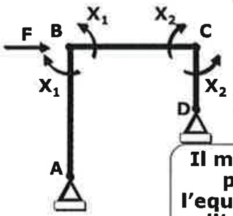
Reticolare associata



La reticolare associata, labile, è però in equilibrio!

Il modo più rapido per imporre l'equilibrio consiste nell'applicare il PLV

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Applicazione del PLV

Sistema di forze (a) reale (r)

Sistema di spostamenti (b) fittizio (f) costituito dal cinematismo della reticolare associata

Il lavoro virtuale interno è nullo ⇨ Nell'equazione di equilibrio compaiono solo le X_i ed i carichi

$L_{vi} = 0$

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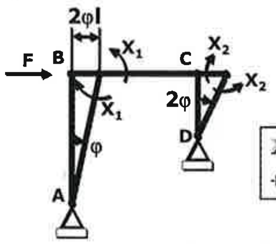
Cinematismo



PLV

$$X_1\varphi - X_2 2\varphi + F\,2\varphi l = 0$$

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$$(X_1 - 2X_2 + 2Fl)\varphi = 0$$

L'equazione è valida per ∀φ (φ è virtuale)

$$X_1 - 2X_2 + 2Fl = 0$$

Equazione di equilibrio

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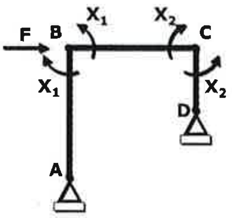

Equazioni di congruenza angolare

$$\begin{cases} \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = \varphi_{CD} \end{cases}$$

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Rotazioni

Contributo della deformabilità delle travi (letto sulla reticolare a nodi bloccati)

Contributo dello spostamento rigido della reticolare associata (letto sul cinematismo)

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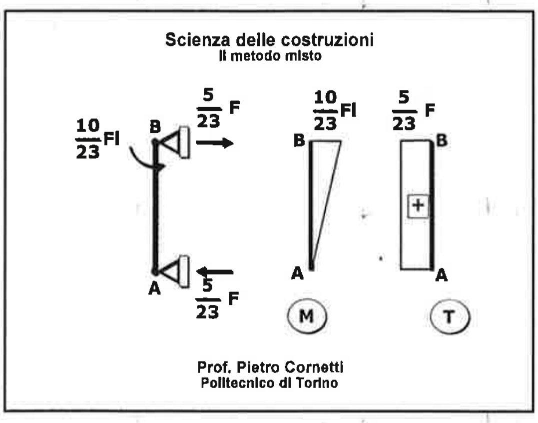
Scienza delle costruzioni
Il metodo misto
10/23 Fl
5/23 F
10/23 Fl
5/23 F
B
A
5/23 F
M
T
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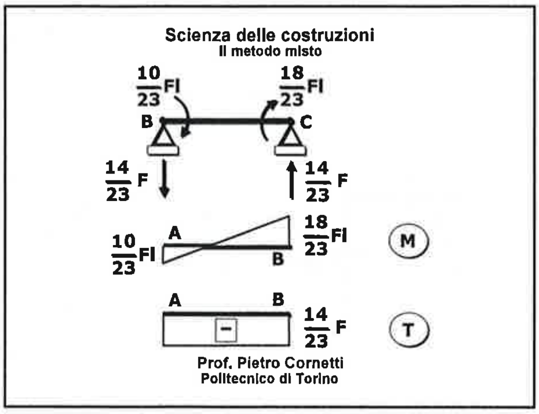
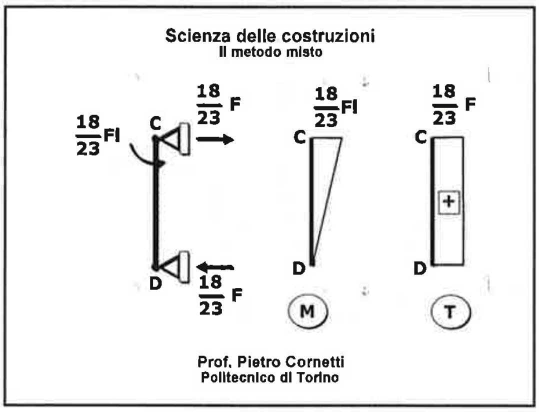

Scienza delle costruzioni
Il metodo misto
10/23 Fl
18/23 Fl
B
C
14/23 F
14/23 F
A
B
18/23 Fl
10/23 Fl
M
14/23 F
T
Prof. Pietro Cornetti
Politecnico di Torino


Scienza delle costruzioni
Il metodo misto
18/23 Fl
18/23 F
18/23 Fl
18/23 F
C
D
18/23 F
M
T
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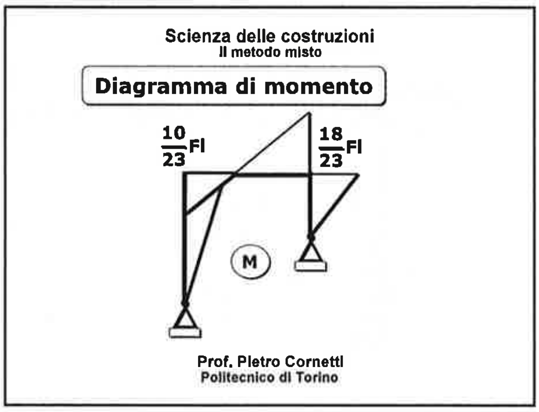
Scienza delle costruzioni
Il metodo misto
Diagramma di momento
10/23 Fl
18/23 Fl
M
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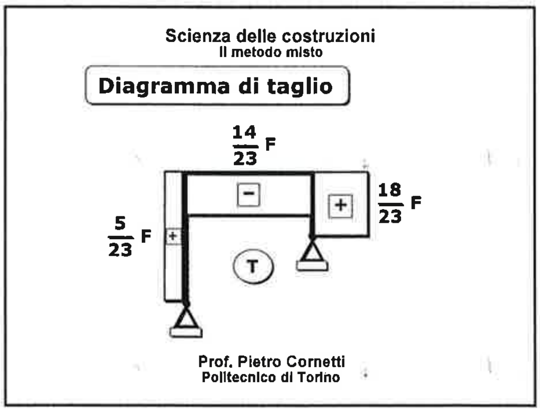
Scienza delle costruzioni
Il metodo misto
Diagramma di taglio
14/23 F
18/23 F
5/23 F
T
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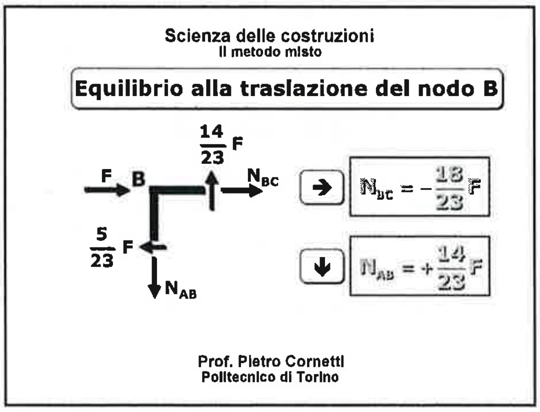
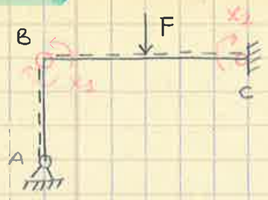

Scienza delle costruzioni
Il metodo misto
Equilibrio alla traslazione del nodo B
14/23 F
F
B
N_BC
5/23 F
N_AB
N_BC = −18/23 F
N_AB = +14/23 F
Prof. Pietro Cornetti
Politecnico di Torino

- **Ex 1**



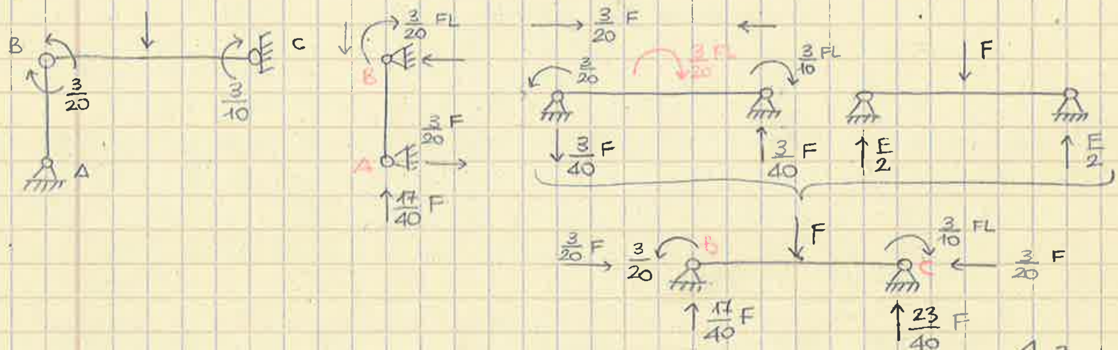
$g = 3$ because it's only one body
$v = 5$ (3 for the fixed support, 2 for the hinge)
we built the associate truss structure, putting a hinge in every node
↳ we need two equations
$g = 6$
$v = 2+2+2 = 6$
statically determinated

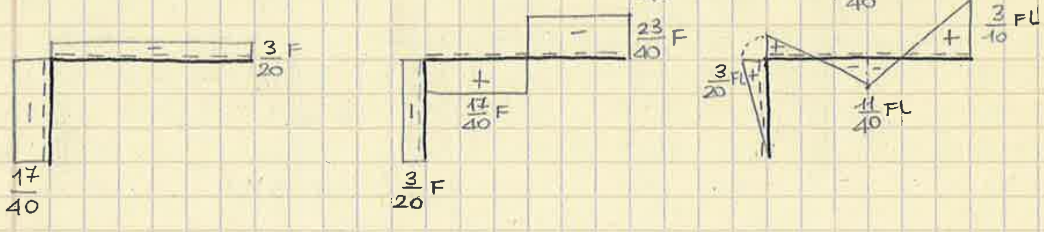
$$\begin{cases} \varphi_{BA} = \varphi_{BC} \\ \varphi_C = 0 \end{cases}$$

$$\begin{cases} -\dfrac{X_1 L}{3EI} = -\dfrac{F(2L)^2}{16EI} + \dfrac{X_2(2L)}{6EI} + \dfrac{X_1(2L)}{3EI} \\ -\dfrac{X_2(2L)}{3EI} + \dfrac{F(2L)^2}{16EI} - \dfrac{X_1(2L)}{6EI} = 0 \end{cases}$$

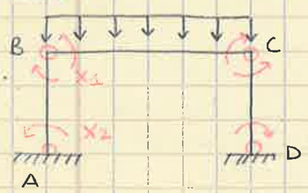
$$\begin{cases} -\dfrac{X_1}{3} = -\dfrac{FL}{4} + \dfrac{X_2}{3} + \dfrac{2}{3}X_1 \\ -\dfrac{2}{3}X_2 + \dfrac{FL}{4} - \dfrac{X_1}{3} = 0 \end{cases} \qquad \begin{cases} \dfrac{X_2}{3} = \dfrac{FL}{8} - \dfrac{X_1}{6} \\ -\dfrac{FL}{4} - \dfrac{X_1}{6} + \dfrac{FL}{8} + \dfrac{2}{3}X_1 + \dfrac{X_1}{3} = 0 \end{cases}$$

$$\begin{cases} \dfrac{5}{6}X_1 = +\dfrac{FL}{8}\cdot\dfrac{6}{5} = +\dfrac{3}{20}FL \\ \dfrac{X_2}{3} = \dfrac{FL}{8} - \dfrac{1}{40}FL = \dfrac{4}{40}FL = \dfrac{1}{10}FL \rightarrow X_2 = \dfrac{3}{10}FL \end{cases}$$





- **Ex 2**

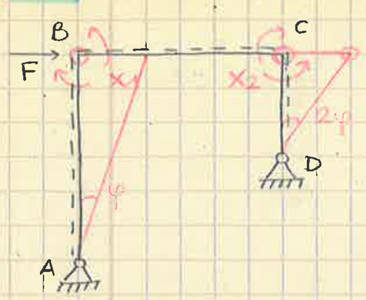


$g = 3$ $\quad$ $g = 9$ $\quad$ $\ell = 1$
$v = 6$ $\quad$ $v = 8$

The associated truss structure is a mechanism but because it's symmetrical the mechanism doesn't happen ~ The inner reactions are symmetrical too so they can be computed considering only half structure

$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BC} \end{cases}$$

• EX 3



$g = 3$
$\vartheta = 4$
$\ell = 1$ (it is a mechanism)
If we consider the associated truss structure we have:
$g = 9$
$\vartheta = 8$
$\ell = 1$
↳ The mechanism has to be considered

PVW (virtual displacements · real forces = real displacement · virtual forces = 0)

$$\begin{cases} \varphi \cdot 2L \cdot F + X_1 \varphi - X_2 \cdot 2\varphi = 0 \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = \varphi_{CD} \end{cases}$$

(B) AB ↻ BC ↺(+)

(C) BC ↻ CD ↺(+)

$$\begin{cases} 2LF + X_1 - 2X_2 = 0 \\ -\frac{X_1 (2L)}{3EI} - \varphi = \frac{X_1 (2L)}{3EI} + \frac{X_2 (2L)}{3 \cdot 2 EI} & \left\{ \cdot \frac{3EI}{2L} \right\} \\ -\frac{X_1 (2L)}{3 \cdot 2 EI} - \frac{X_2 (2L)}{3EI} = \frac{X_2 L}{3EI} - 2\varphi & \left\{ \cdot \frac{3EI}{L} \right\} \end{cases}$$

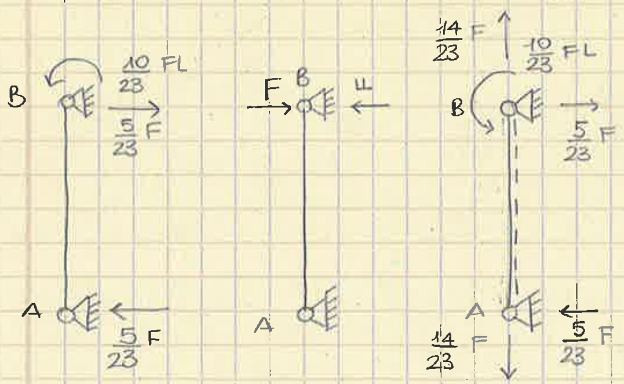
$$\begin{cases} X_1 = 2X_2 - 2LF \\ -X_1 - \frac{3\varphi EI}{2L} - X_1 - \frac{X_2}{2} = 0 \\ -X_1 - 2X_2 - X_2 + \frac{6\varphi EI}{L} = 0 \end{cases} \qquad \begin{cases} X_1 = 2X_2 - 2LF \\ -2X_2 + 2LF - 3X_2 + \frac{6}{L}\varphi EI = 0 \\ -2X_1 - \frac{3}{2}\frac{EI}{L}\varphi - \frac{X_2}{2} = 0 \end{cases}$$

↳ $-4X_2 + 4LF$

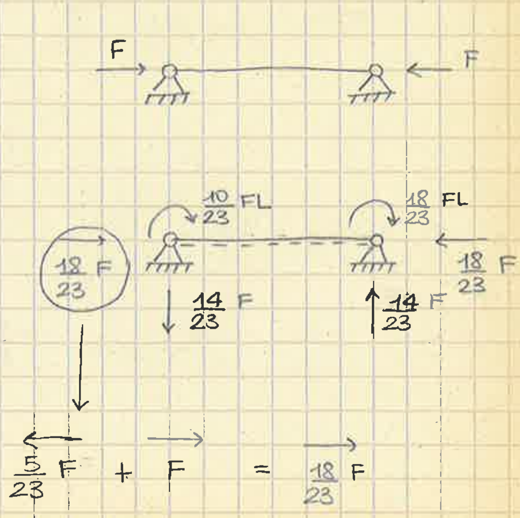
$$\begin{cases} X_1 = 2X_2 - 2LF \\ 5X_2 = \frac{2LF}{5} + \frac{6}{5}\frac{EI}{L}\varphi \\ -\frac{8}{5}LF - \frac{24}{5}\frac{EI}{L}\varphi + 4LF - \frac{3}{2}\frac{EI}{L}\varphi - \frac{LF}{5} - \frac{3}{5}\frac{EI}{L}\varphi = 0 \end{cases}$$

$$\left(-\frac{24}{5} - \frac{3}{2} - \frac{3}{5}\right)\frac{EI}{L}\varphi = \left(\frac{8}{5} - 4 + \frac{1}{5}\right)LF$$

$$\begin{cases} +\frac{69}{10}\varphi = +\frac{11}{5}\frac{FL^2}{EI} \rightarrow \varphi = \frac{11}{5} \cdot \frac{10}{69}\frac{FL^2}{EI} = \frac{22}{69}\frac{FL^2}{EI} \\ X_2 = \frac{2}{5}LF + \frac{6}{5} \cdot \frac{22}{69} \cdot FL = \frac{18}{23}FL \\ X_1 = \frac{36}{23}FL - 2FL = -\frac{10}{23}FL \end{cases}$$



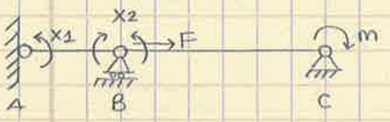
$$\begin{cases} V_A + V_B = F \\ V_A \cdot L = 0 \end{cases} \qquad \begin{cases} V_A = 0 \\ V_B = F \end{cases}$$



• Ex 4

$$\begin{cases} g = 3 \\ \nu = 3 + 2 + 1 = 6 \end{cases}$$
$i = 3$

1) Method of forces: we built the associate truss structure



φA) +
φB) AB −
BC +

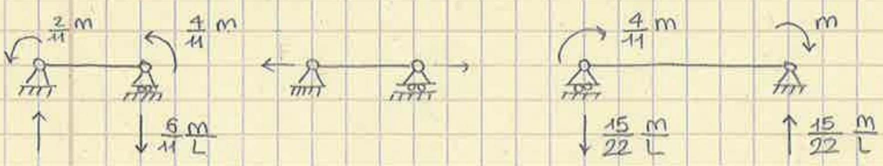
$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BC} \end{cases}$$

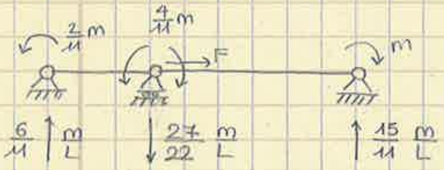
$$\begin{cases} \frac{X_1 L}{3EI} + \frac{X_2 L}{6EI} = 0 & \left\{ \cdot \frac{3EI}{L} \right\} \\ -\frac{X_1 L}{6EI} - \frac{X_2 L}{3EI} = \frac{X_2 \cdot 2L}{3EI} + \frac{m \cdot 2L}{3EI} & \left\{ \frac{3EI}{L} \right\} \end{cases}$$

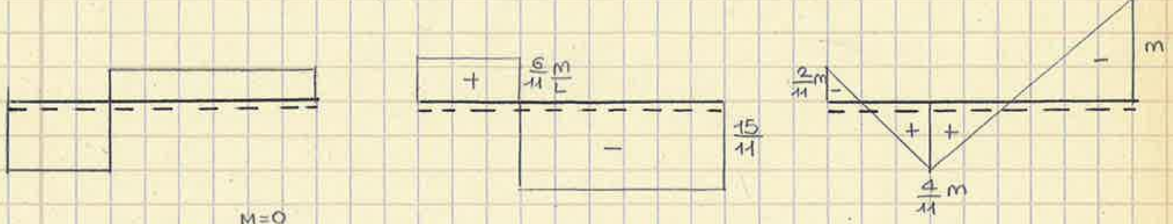
$$\begin{cases} X_1 + \frac{X_2}{2} = 0 \\ -\frac{X_1}{2} - X_2 = 2X_2 + m \end{cases}$$

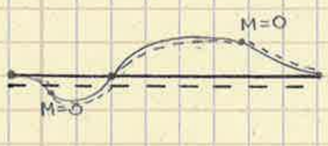
$$\begin{cases} X_2 = -2X_1 \\ -\frac{X_1}{2} + 2X_1 + 4X_1 = m \end{cases}$$

$$\begin{cases} X_1 = \frac{2}{11} m \\ X_2 = -\frac{4}{11} m \end{cases}$$

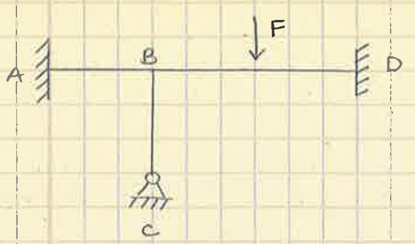




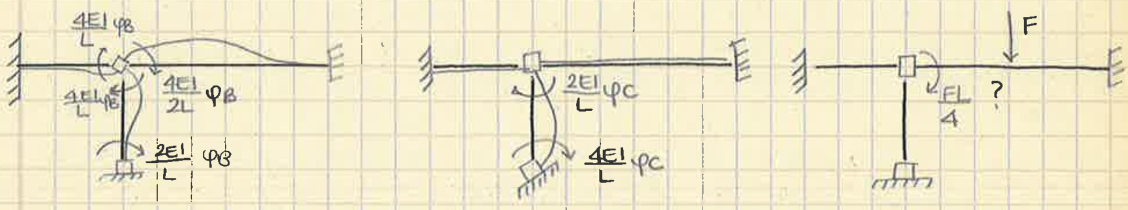




• EX 5 ?



$\varphi_C, \varphi_B, \omega_B, \vartheta_B$



$$\begin{cases} -\frac{4EI}{L}\varphi_B - \frac{2EI}{L}\varphi_B - \frac{4EI}{L}\varphi_B - \frac{FL}{4} - \frac{2EI}{L}\varphi_C = 0 \\ -\frac{2EI}{L}\varphi_B - \frac{4EI}{L}\varphi_C = 0 \end{cases}$$

$$\begin{cases} -\varphi_B - 2\varphi_C = 0 \\ -2\varphi_B - \varphi_B - 2\varphi_B - \frac{FL^2}{8EI} - \varphi_C = 0 \end{cases}$$

$$\begin{cases} \varphi_B = -2\varphi_C \\ 10\varphi_C - \varphi_C = \frac{FL^2}{8EI} \end{cases}$$

$$\begin{cases} \varphi_C = \frac{FL^2}{72EI} \\ \varphi_B = -\frac{FL^2}{36EI} \end{cases}$$

$M_A = \frac{2EI}{L} \cdot \varphi_B = -\frac{1}{18} FL$

$M_{BA} = \frac{4EI}{L}\varphi_B = -\frac{1}{9} FL$

$M_{BC} = \frac{4EI}{L}\varphi_B + \frac{2EI}{L}\varphi_C = -\frac{1}{12} FL$

$M_{BD} = \frac{4EI}{2L}\varphi_B + \frac{FL}{4} = \frac{7}{36} FL$

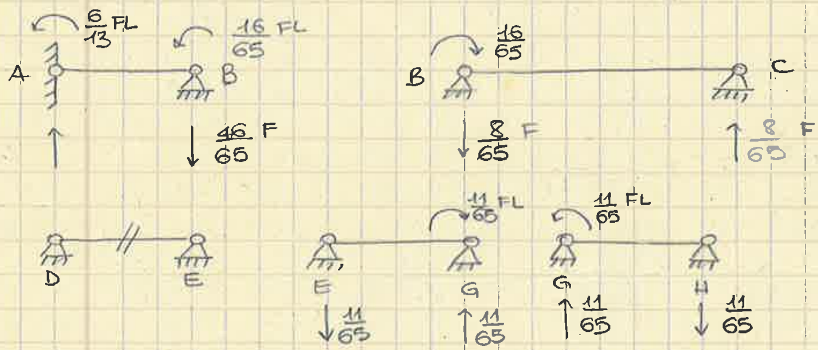
$M_D = \frac{2EI}{L}\varphi_B = -\frac{1}{18} FL$

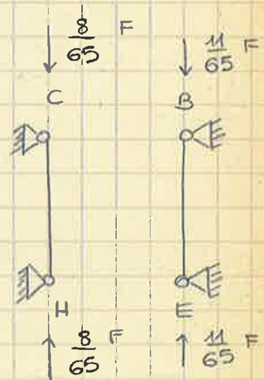
$M_C = \frac{2EI}{L}\varphi_B + \frac{4EI}{L}\varphi_C = 0$

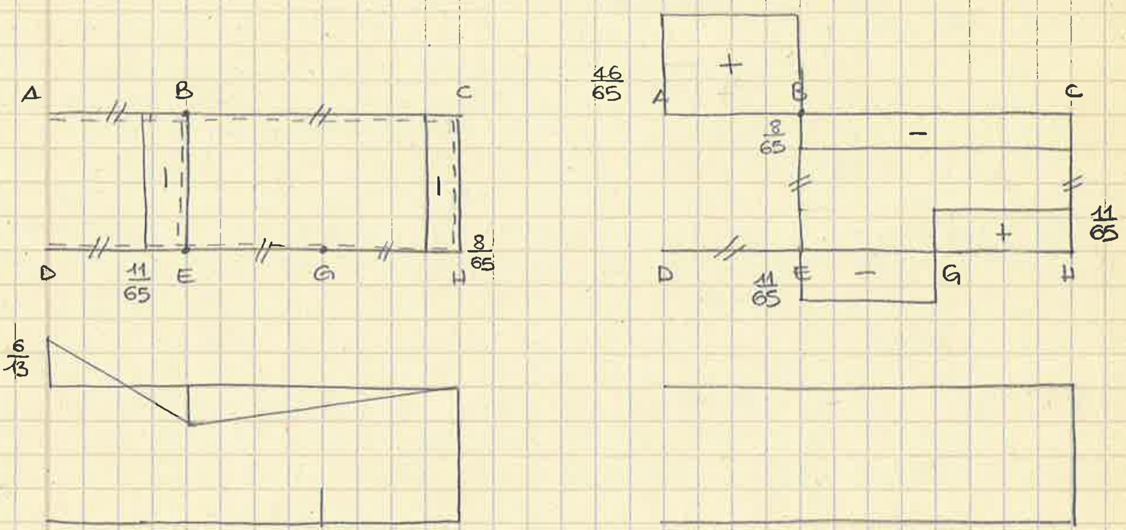
$$\begin{cases} X_3 = \frac{3}{2}\frac{EI}{L}\varphi \\ X_1 = 3\frac{EI}{L}\varphi - \frac{X_2}{2} \\ -\frac{3}{2}\frac{EI}{L}\varphi + \frac{X_2}{4} - X_2 - \frac{3EI}{L}\varphi - 2X_2 - \frac{3}{2}\frac{EI}{L}\varphi = 0 \\ -X_1 + FL + \frac{3}{2}X_2 - \frac{3}{2}\frac{EI}{L}\varphi = 0 \end{cases}$$

$$\begin{cases} \left(\frac{1}{4} - 1 - 2\right)X_2 = \left(\frac{3}{2} + 3 + \frac{3}{2}\right)\frac{EI}{L}\varphi \Rightarrow X_2 = \left(6 \cdot -\frac{4}{11}\right)\frac{EI}{L}\varphi = -\frac{24}{11}\frac{EI}{L}\varphi \\ X_1 = \left(3 + \frac{12}{11}\right)\frac{EI}{L}\varphi = \frac{45}{11}\frac{EI}{L}\varphi \\ X_3 = \frac{3}{2}\frac{EI}{L}\varphi \\ \left(-\frac{45}{11} - \frac{3}{2}\cdot\frac{24}{11} - \frac{3}{2}\right)\frac{EI}{L}\varphi = -FL \end{cases}$$

$$\begin{cases} \varphi = \frac{22}{195}\frac{FL^2}{EI} \\ X_1 = \frac{45}{11}\cdot\frac{22}{195}FL = \frac{6}{13}FL \\ X_2 = -\frac{24}{11}\cdot\frac{22}{195}FL = -\frac{16}{65}FL \\ X_3 = \frac{3}{2}\cdot\frac{22}{195}FL = \frac{11}{65}FL \end{cases}$$







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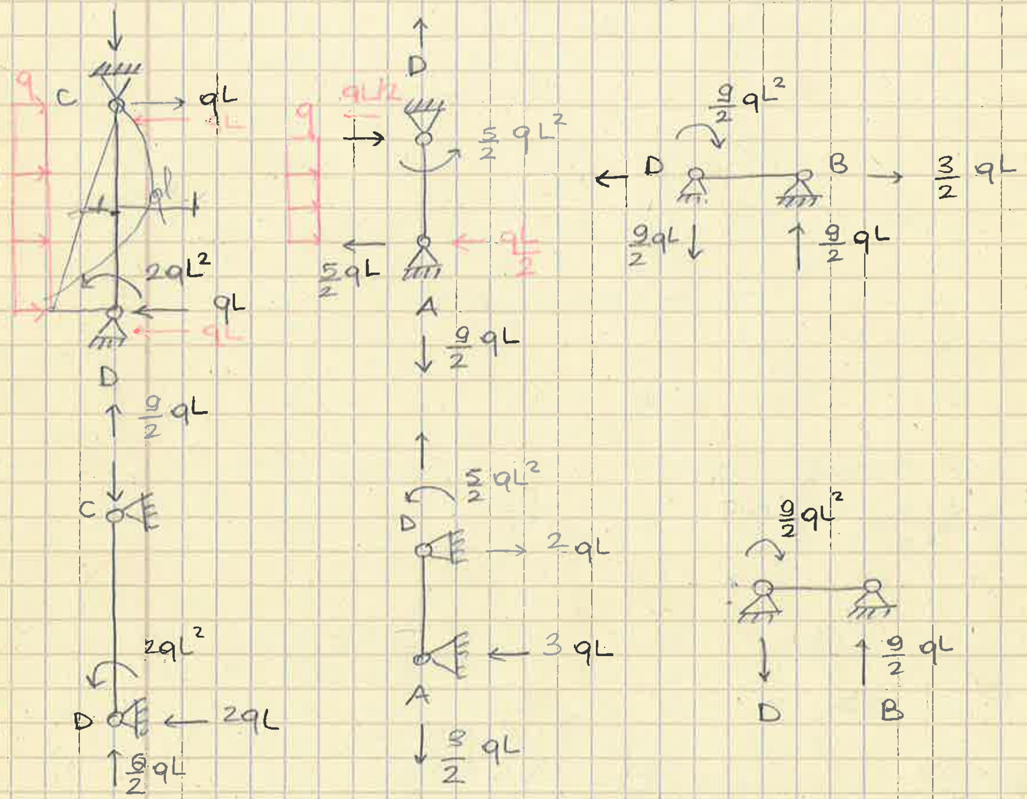
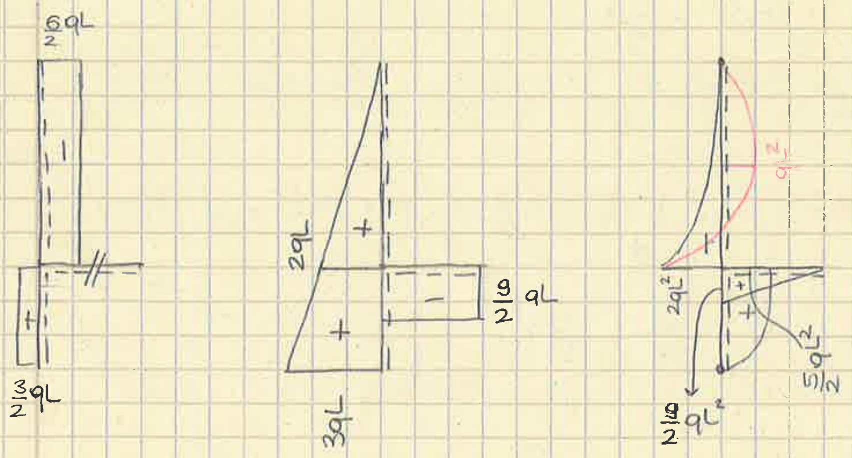
$$\begin{cases} 4qL^2 - qL^2 + \frac{9}{2}qL^2 = -\frac{3\varphi EI}{L} \\ 4qL^2 - qL^2 + \frac{9}{2}qL^2 = \frac{3\theta' EI}{L} \\ X_1 = -2qL^2 \\ X_2 = \frac{9}{2}qL^2 \end{cases}$$

$$\begin{cases} \frac{15}{2}qL^2 = +\frac{3\varphi EI}{L} \Rightarrow \varphi = -\frac{5}{2}\frac{qL^3}{EI} \\ \frac{15}{2}qL^2 = \frac{3\theta EI}{L} \Rightarrow \theta = \frac{5}{2}\frac{qL^3}{EI} \\ X_1 = -2qL^2 \\ X_2 = \frac{9}{2}qL^2 \end{cases}$$

$$X_1 + X_2 = \left(\frac{9}{2} - 2\right)qL^2 = \frac{5}{2}qL^2$$

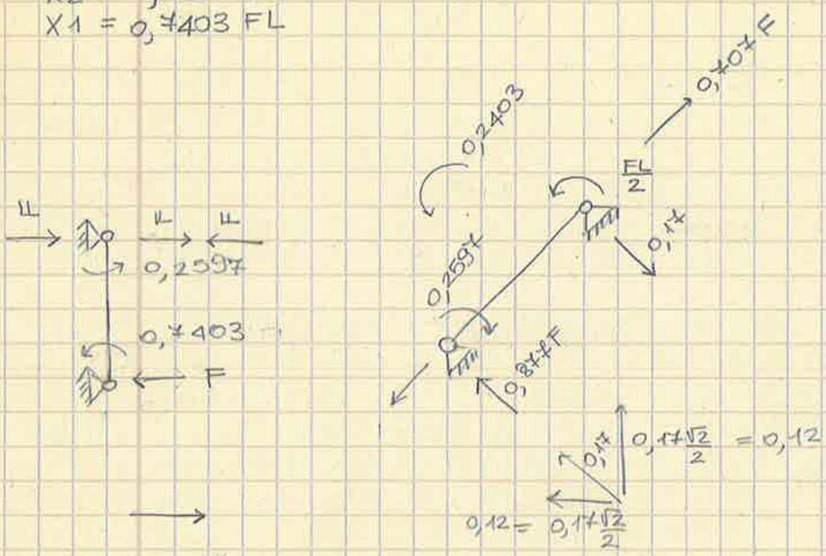


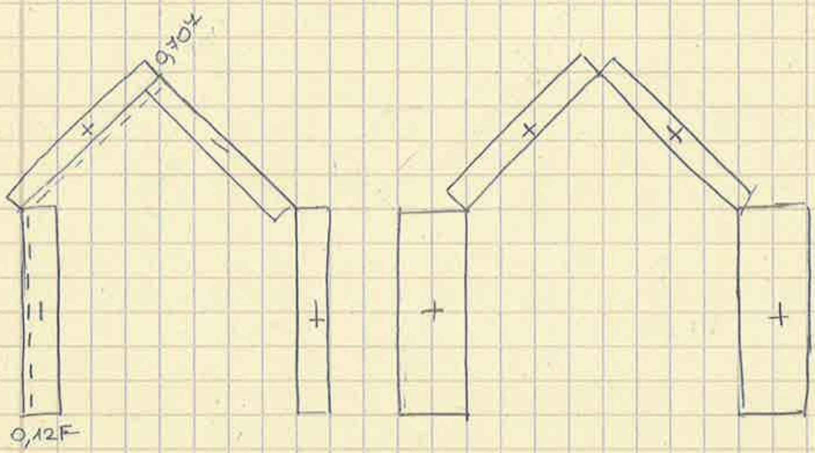


$\varphi = 0,2035 \frac{FL^2}{EI}$

$X_2 = -0,2597\ FL$

$X_1 = 0,7403\ FL$





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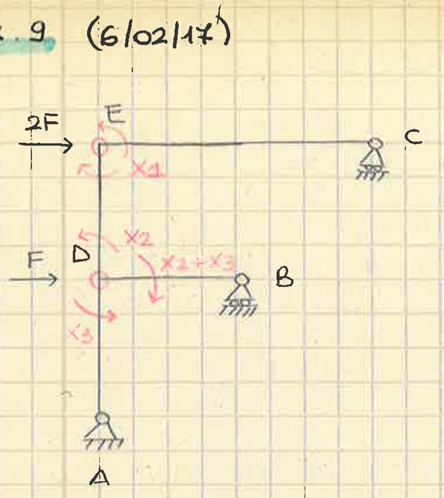
• Ex. 9 (6/02/17)



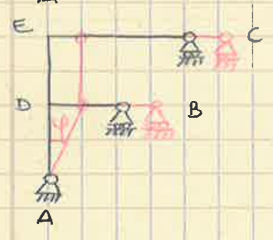
$$\begin{cases} g = 3 \\ v = 2(A) + 1(B) + 1(C) = 4 \end{cases}$$

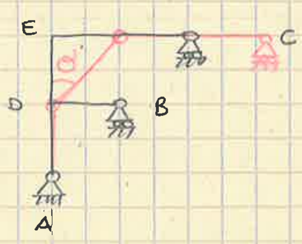
$v - g = 1$

Associated truss structure

$$\begin{cases} g = 4 \cdot 3 = 12 \\ v = 2(A) + 1(B) + 1(C) + 4(D) + 2(E) = 10 \end{cases}$$

$g - v = 2$





$$\begin{cases} \varphi_{DA} = \varphi_{DB} \\ \varphi_{ED} = \varphi_{EC} \\ \varphi_{DE} = \varphi_{DB} \\ PVW\ (\varphi) \\ PVW\ (\vartheta) \end{cases}$$

DA ↺(+) DB ↻ ED ↻ EC ↺(+) DE ↺(+)

$$\begin{cases} \frac{X_3 L}{3EI} - \varphi = -\frac{(X_2 + X_3)L}{3EI} & \left\{\frac{3EI}{L}\right\} \\ -\frac{X_1 L}{3EI} - \vartheta - \frac{X_2 L}{6EI} = \frac{X_1 2L}{3EI} & \left\{\frac{3EI}{L}\right\} \\ \frac{X_2 L}{3EI} - \vartheta + \frac{X_1 L}{6EI} = -\frac{(X_2 + X_3)L}{3EI} & \left\{\frac{3EI}{L}\right\} \\ -X_3 \varphi + F\varphi L + 2F\varphi L = 0 \\ -X_2 \vartheta + X_1 \vartheta + 2F\vartheta L = 0 \end{cases}$$

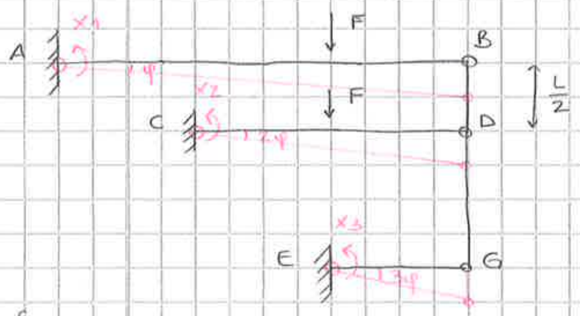
$$\begin{cases} X_3 + X_2 + X_3 = \frac{3EI}{L}\varphi & \longrightarrow & X_3 = \frac{3EI}{2L}\varphi - \frac{X_2}{2} \\ -X_1 - \frac{X_2}{2} - 2X_1 = \frac{3EI}{L}\vartheta & \longrightarrow & X_1 = -\frac{X_2}{6} - \frac{EI}{L}\vartheta \\ X_2 + \frac{X_1}{2} + X_2 + X_3 = \frac{3EI}{L}\vartheta \end{cases}$$

$$2X_2 - \frac{X_2}{12} - \frac{EI}{2L}\vartheta + \frac{3}{2}\frac{EI}{L}\varphi - \frac{X_2}{2} - \frac{3EI}{L}\vartheta = 0$$

$$\left(2 - \frac{1}{12} - \frac{1}{2}\right)X_2 = \left(+\frac{1}{2} + 3\right)\frac{EI}{L}\vartheta - \frac{3}{2}\frac{EI}{L}\varphi$$

$$\begin{cases} \frac{17}{12}X_2 = \frac{7}{2}\frac{EI}{L}\vartheta - \frac{3}{2}\frac{EI}{L}\varphi \longrightarrow X_2 = \frac{42}{29}\frac{EI}{L}\vartheta - \frac{18}{29}\frac{EI}{L}\varphi \\ X_1 = -\frac{7}{29}\frac{EI}{L}\vartheta + \frac{3}{29}\frac{EI}{L}\varphi - \frac{EI}{L}\vartheta = -\frac{36}{29}\frac{EI}{L}\vartheta + \frac{3}{29}\frac{EI}{L}\varphi \\ X_3 = \frac{3}{2}\frac{EI}{L}\varphi - \frac{21}{29}\frac{EI}{L}\vartheta + \frac{9}{29}\frac{EI}{L}\varphi = \frac{105}{58}\frac{EI}{L}\varphi - \frac{21}{29}\frac{EI}{L}\vartheta \end{cases}$$

- **EX 10** (2,8)



$$\begin{cases} g = 5 \times 3 = 15 \\ v = 3(A) + 2(B) + 3(C) + 4(D) + 3(E) + 2(G) = 17 \end{cases}$$

$v - g = 2$

Associated truss structure

$$\begin{cases} g = 5 \times 3 = 15 \\ v = 2(A) + 2(B) + 2(C) + 4(D) + 2(E) + 2(G) = 14 \end{cases}$$

$g - v = 1$ (once mechanics)

$$\begin{cases} \varphi_A = 0 \\ \varphi_C = 0 \\ \varphi_E = 0 \\ PLV \end{cases}$$

AB (+ ↺)

$\frac{4}{9}\frac{FL^2}{EI}$

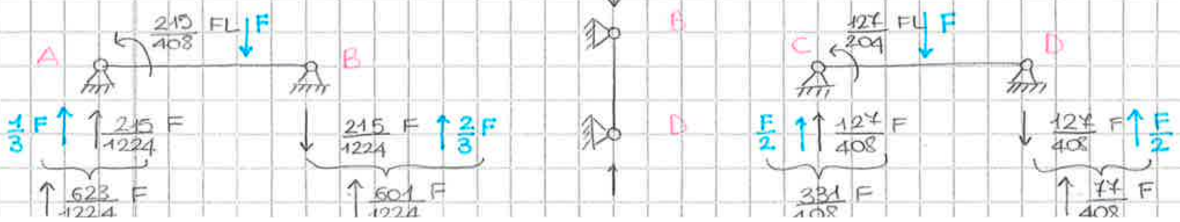
$$\begin{cases} \frac{X_1 \cdot 3L}{3EI} - \varphi - \frac{(2L)^2 \cdot 4L}{9 \cdot 18 L EI} \cdot F = 0 \\ \frac{X_2 \cdot 2L}{3EI} - 2\varphi - \frac{L^2 \cdot 3L}{4 \cdot 12 L EI} F = 0 \\ \frac{X_3 \cdot L}{3EI} - 3\varphi = 0 \\ -X_1\varphi - X_2 \cdot 2\varphi - X_3 \cdot 3\varphi + F 2L\varphi + F 2\varphi L = 0 \end{cases}$$

$$\begin{cases} X_3 = \frac{9EI}{L}\varphi \\ X_1 - \frac{\varphi EI}{L} - \frac{4}{9}FL = 0 \longrightarrow X_1 = \frac{4}{9}FL + \frac{\varphi EI}{L} \\ \frac{2}{3}X_2 - \frac{2EI}{L}\varphi - \frac{1}{4}FL = 0 \longrightarrow X_2 = \frac{3}{8}FL + \frac{3EI}{L}\varphi \\ -\frac{4}{9}FL - \frac{EI}{L}\varphi - \frac{3}{4}FL - \frac{6EI}{L}\varphi - \frac{27EI}{L}\varphi + 4FL = 0 \end{cases}$$

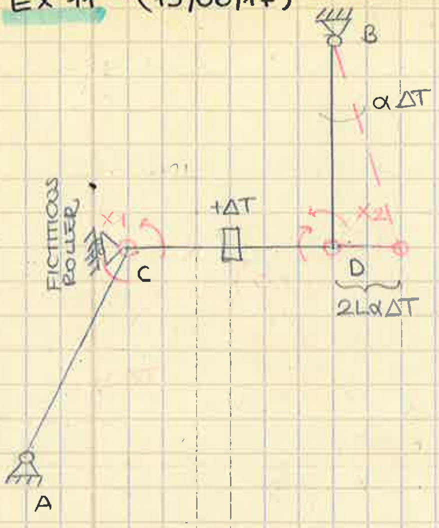
$$\left(-1 - 6 - 27\right)\frac{EI}{L}\varphi = \left(\frac{4}{9} + \frac{3}{4} - 4\right)FL$$

$$\begin{cases} \varphi = +\frac{101}{36} \cdot \frac{1}{34} \frac{FL^2}{EI} = \frac{101}{1224}\frac{FL^2}{EI} \\ X_1 = \frac{4}{9}FL + \frac{101}{1224}FL = \frac{215}{408}FL \\ X_2 = \frac{3}{8}FL + \frac{303}{1224}FL = \frac{127}{204}FL \\ X_3 = \frac{101}{136}FL \end{cases}$$

$\frac{601}{1224}F - \frac{77}{408}F = \frac{185}{612}F$



• Ex 11 (19/06/17)



$$\begin{cases} g = 3 \\ v = 2(A) + 2(B) = 4 \end{cases}$$

$v - g = i = 1$

Associated truss structure

$$\begin{cases} g = 3 \cdot 3 = 9 \\ v = 2(A) + 2(B) + 2(C) + 2(D) = 8 \end{cases}$$

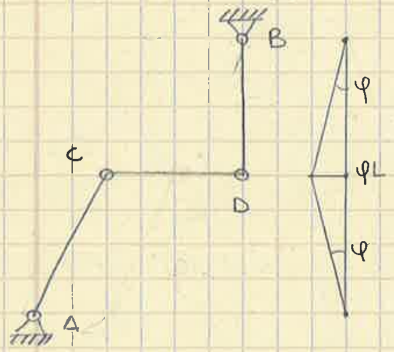
$g - v = 1$

$AC = \sqrt{(2L)^2 + L^2} = \sqrt{5}\,L$

AC (−)
CD (+)
DC (−)
DB (+)

$$\begin{cases} \varphi_{CA} = \varphi_{CD} \\ \varphi_{DC} = \varphi_{DB} \\ \text{PLV} \end{cases}$$

$$\begin{cases} -\frac{X_1 \cdot \sqrt{5}L}{3EI} + \varphi = \frac{X_1 2L}{3EI} + \frac{X_2 L}{3EI} - \frac{\varphi}{2} & \left\{\cdot \frac{3EI}{L}\right\} \\ -\frac{X_1 \cdot L}{3EI} - \frac{X_2 \cdot 2L}{3EI} - \frac{\varphi}{2} = \frac{X_2 2L}{3EI} - \varphi + \alpha\Delta T & \left\{\frac{3EI}{L}\right\} \\ -X_1 \varphi^{3} - X_1 \frac{\varphi}{2} + X_2 \frac{\varphi}{2} - X_2\varphi = 0 \end{cases}$$



$$\begin{cases} -X_1\sqrt{5} + 3EI\frac{\varphi}{L} = 2X_1 + \overbrace{X_2}^{-3X_1} - \frac{3}{2}EI\frac{\varphi}{L} \\ -X_1 - 2X_2 - \frac{3}{2}EI\frac{\varphi}{L} = 2X_2 - 3EI\frac{\varphi}{L} + \frac{3EI\alpha\Delta T}{L} \\ -\frac{3}{2}X_1 - \frac{X_2}{2} = 0 \qquad X_2 = -3X_1 \end{cases}$$

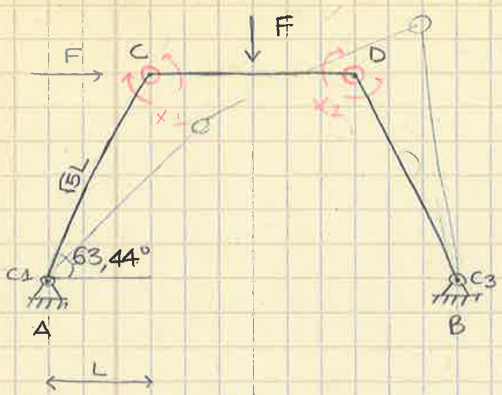
$$(-\sqrt{5} - 2 + 3)X_1 = \left(-\frac{3}{2} - 3\right) EI\frac{\varphi}{L}$$

$$+1{,}24\,X_1 = +4{,}5\,EI\frac{\varphi}{L}$$

$$X_1 = +\frac{4{,}5}{1{,}24}EI\frac{\varphi}{L} = 3{,}63\,EI\frac{\varphi}{L}$$

$$X_2 = 10{,}89\,EI\frac{\varphi}{L}$$

- Ex 12 (2/03/17)



$$\begin{cases} g = 3 \\ v = 2(A) + 2(B) = 4 \end{cases}$$

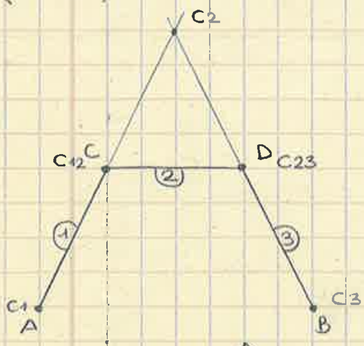
$v - g = i = 1$

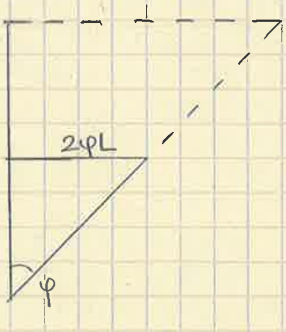
Associated truss structure

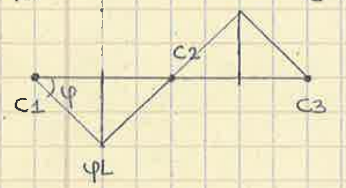
$$\begin{cases} g = 3 \cdot 3 = 9 \\ v = 2(A) + 2(C) + 2(D) + 2(B) = 8 \end{cases}$$

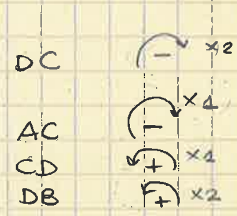
$g - v = l = 1$

One mechanism









$$\begin{cases} \varphi_{AC} = \varphi_{CD} \\ \varphi_{CD} = \varphi_{DB} \\ PLV \end{cases}$$

$$\begin{cases} -\dfrac{X_1 \sqrt{5} L}{3EI} - \varphi = \dfrac{X_1\, 2L}{3EI} - \dfrac{F(2L)^2}{16EI} + \dfrac{X_2\, 2L}{6EI} + \varphi & \cdot \left\{\dfrac{EI}{L}\right\} \\ -\dfrac{X_1\, 2L}{6EI} + \dfrac{4FL^2}{16EI} - \dfrac{X_2\, 2L}{3EI} + \varphi = \dfrac{X_2 \sqrt{5} L}{3EI} + \varphi & \cdot \left\{\dfrac{EI}{L}\right\} \\ 2X_1\varphi - 2X_2\varphi + F2\varphi L = 0 \end{cases}$$

$$\begin{cases} X_1 - X_2 + FL = 0 \qquad X_1 = X_2 - FL \\ -\dfrac{(X_2 - FL)\sqrt{5}}{3} - \dfrac{\varphi EI}{L} - \dfrac{(X_2 - FL)\cdot 2}{3} + \dfrac{FL}{4} - \dfrac{X_2 \cdot 2}{6} - \dfrac{\varphi EI}{L} = 0 \\ -\dfrac{(X_2 - FL)\cdot 2}{6} + \dfrac{FL}{4} - \dfrac{X_2 \cdot 2}{3} + \dfrac{\varphi EI}{L} - \dfrac{X_2\sqrt{5}}{3} + \dfrac{\varphi EI}{L} = 0 \end{cases}$$

$X_1 = X_2 - FL$

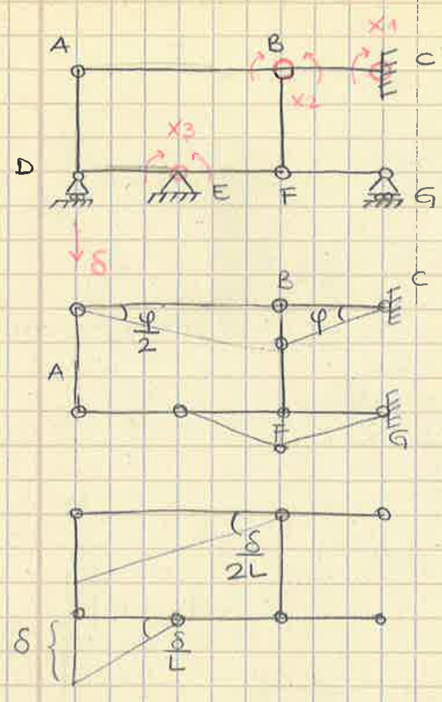
$$-X_2 \frac{\sqrt{5}}{3} + \frac{FL\sqrt{5}}{3} - \frac{2\varphi EI}{L} - X_2 \cdot \frac{2}{3} + FL \cdot \frac{2}{3} + \frac{FL}{4} - \frac{X_2}{3} = 0$$

$$\frac{2\varphi EI}{L} = X_2\left(-\frac{\sqrt{5}}{3} - \frac{2}{3} - \frac{1}{3}\right) + FL\left(\frac{\sqrt{5}}{3} + \frac{2}{3} + \frac{1}{4}\right)$$

$$\varphi = -X_2 \frac{(\sqrt{5}+3)}{3} \cdot \frac{L}{2EI} + FL \frac{4\sqrt{5}+11}{12} \cdot \frac{L}{2EI}$$

$$\varphi = -X_2 L \frac{(\sqrt{5}+3)}{6EI} + \frac{FL^2}{EI} \frac{4\sqrt{5}+11}{24}$$

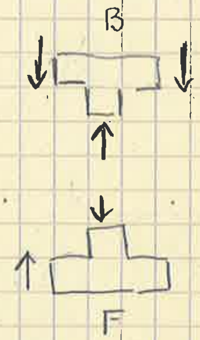
- EX 13 (12/09/16)



$$\begin{cases} g = 3 \cdot 5 = 15 \\ v = 2(A) + 3(C) + 2(B) \\ \quad 3(D) + 2(E) + 4(F) + 1(G) = 17 \end{cases}$$

$$\begin{cases} g = 3 \cdot 7 = 21 \\ v = 2(A) + 4(B) + 2(C) + 3(D) \\ \quad + 2(E) + 4(F) + 1(G) = 18 \end{cases}$$

CB x_1
BA x_2
BC x_2



$$\begin{cases} \varphi_C = 0 \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_{ED} = \varphi_{EF} \\ PLV \end{cases}$$

$$\begin{cases} -\frac{x_1 L}{3EI} - \frac{x_2 L}{6EI} + \varphi = 0 & \left\{\frac{3EI}{L}\right\} \\ -\frac{x_2 \cdot 2L}{3EI} - \frac{\varphi}{2} + \frac{\delta}{2L} = \frac{x_2 L}{3EI} + \frac{x_1 L}{6EI} + \varphi & \left\{\frac{3EI}{L}\right\} \\ -\frac{x_3 L}{3EI} + \frac{\delta}{L} = \frac{x_3 L}{3EI} - \varphi \\ -x_1 \varphi + x_2 \varphi + x_2 \frac{\varphi}{2} - x_3 \varphi = 0 \end{cases}$$

$$\frac{2}{3}\frac{x_3 L}{EI} = \frac{\delta}{L} + \varphi \Rightarrow \varphi = \frac{2}{3}\frac{x_3 L}{EI} - \frac{\delta}{L}$$

(1) $x_1 = -\frac{x_2}{2} + 2x_3 - \frac{3\delta EI}{L^2}$

$$-x_2 \cdot 2 - \frac{3}{2}\frac{\varphi EI}{L} + \frac{3\delta EI}{2L^2} = x_2 + \frac{x_1}{2} + \frac{\varphi 3EI}{L}$$

$$-3x_2 - \frac{x_1}{2} + \frac{3}{2}\frac{\delta EI}{L^2} = \frac{9}{2}\frac{\varphi EI}{L}$$

$$-3x_2 + \frac{x_2}{4} - x_3 + \frac{3}{2}\frac{\delta EI}{L^2} + \frac{3}{2}\frac{\delta EI}{L^2} = \frac{9}{2} \cdot \frac{EI}{L}\left(\frac{2}{3}\frac{x_3 L}{EI} - \frac{\delta}{L}\right)$$

$$-\frac{11}{4}x_2 - x_3 + \frac{3\delta EI}{L^2} = 3x_3 - \frac{9}{2}\frac{\delta EI}{L^2}$$

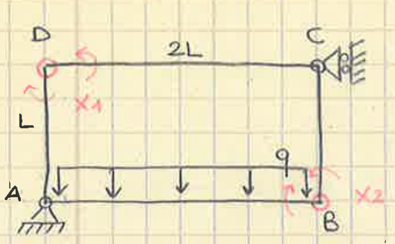
$$-\frac{11}{4}x_2 - 4x_3 = -\frac{15}{2}\frac{\delta EI}{L^2}$$

(2) $x_3 = -\frac{11}{16}x_2 + \frac{15}{8}\frac{\delta EI}{L^2}$

$$\frac{x_2}{2} + \frac{11}{8}x_2 - \frac{15}{4}\frac{\delta EI}{L^2} + \frac{3\delta EI}{L^2} + \frac{3}{2}x_2 + \frac{11}{16}x_2 - \frac{15}{8}\frac{\delta EI}{L^2} = 0$$

$$\left(\frac{1}{2} + \frac{11}{8} + \frac{3}{2} + \frac{11}{16}\right)x_2 = \left(\frac{15}{4} - 3 + \frac{15}{8}\right)\frac{\delta EI}{L^2}$$

- **EX 14** (15/06/15)



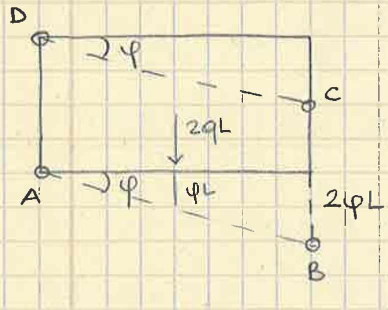
INTERNO

$$\begin{cases} g = 3 \cdot 2 = 6 \\ i - e = 3c - 2d - s = 3 - 0 - 2 = 1 \end{cases}$$

ESTERNO

$$\begin{cases} g = 3 \\ v = 2(A) + 1(C) = 3 \end{cases}$$

$v - g = 0$



DA ↻(−) X_1
DC ↺(+) X_1
BA ↻(−) X_2
BC ↺(+) X_2

$$\begin{cases} \varphi_{DA} = \varphi_{DC} \\ \varphi_{BA} = \varphi_{BC} \\ PLV \end{cases}$$

$$\begin{cases} -\dfrac{X_1 L}{3EI} = \dfrac{X_1 2L}{3EI} - \varphi \\ -\dfrac{X_2 \cdot 2L}{3EI} + \dfrac{q(2L)^3}{24EI} - \varphi = \dfrac{X_2 \cdot L}{3EI} \\ -X_1\varphi + X_2\varphi + 2qL \cdot \varphi L = 0 \end{cases}$$

$$\varphi = \frac{L}{3EI}(2X_1 + X_1) = \frac{X_1 L}{EI}$$

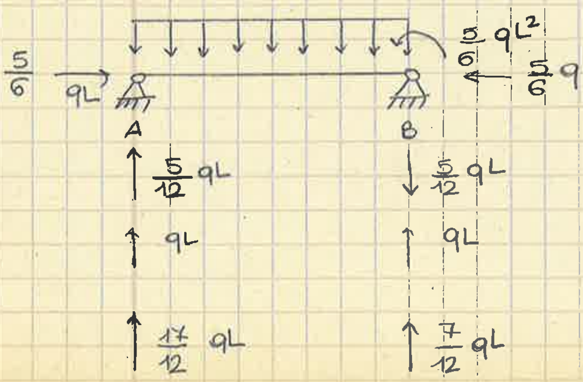
$$\varphi = -\frac{3X_2 \cdot L}{3EI} + \frac{qL^{3\,2}}{3EI} - \frac{X_1 L}{EI} = 0$$

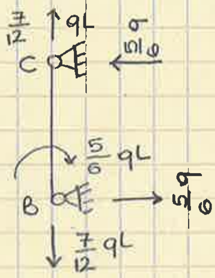
$$X_1 = -X_2 + \frac{qL^2}{3}$$

$$+X_2 - \frac{qL^2}{3} + X_2 + 2qL^2 = 0$$

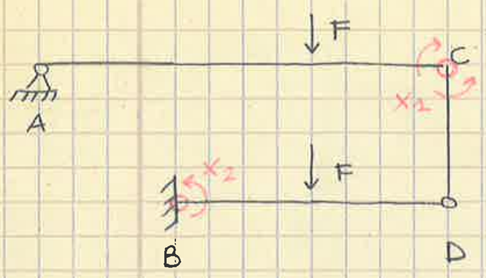
$$2X_2 = -\frac{5}{3}qL^2 \cdot \frac{1}{2}$$

$$\begin{cases} X_2 = -\dfrac{5}{6}qL^2 \\ X_1 = +\dfrac{5}{6}qL^2 + \dfrac{qL^2}{3} = \dfrac{5+2}{6}qL^2 = \dfrac{7}{6}qL^2 \end{cases}$$





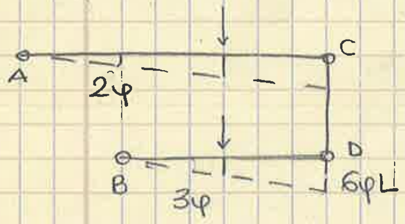
• EX 15 (16/09/13)



$$\begin{cases} g = 2\cdot 3 = 6 \\ v = 2(A) + 2(D) + 3(B) = 7 \\ i = 1 \end{cases}$$

Associated truss structure

$$\begin{cases} g = 3\cdot 3 = 9 \\ v = 2(A) + 2(B) + 2(C) + 2(D) = 8 \\ e = 1 \end{cases}$$



$\ell = 3L$

$a = 2L$

$b = L$

CA (−)

CD (+)

BD (+)

$$\begin{cases} \varphi_{CA} = \varphi_{CD} \\ \varphi_B = 0 \\ PLV \end{cases}$$

$$\begin{cases} -\dfrac{X_1\cdot 3L}{3EI} + \dfrac{2L^2\cdot 5L}{18L\cdot EI}F - 2\varphi = \dfrac{X_1 L}{3EI} \\ \dfrac{X_2\cdot 2L}{3EI} - \dfrac{F\cdot 4L^2}{16EI} - 3\varphi = 0 \\ F\cdot 2\varphi\cdot 2L + F\,3\varphi L + X_1\,2\varphi - X_2\,3\varphi = 0 \end{cases}$$

$$\varphi = \frac{2}{9}\frac{X_2 L}{EI} - \frac{FL^2}{12EI}$$

$$-\frac{X_1 L}{EI} - \frac{X_1 L}{3EI} + \frac{5}{9}\frac{FL^2}{EI} - \frac{4}{9}\frac{X_2 L}{EI} + \frac{FL^2}{6EI} = 0$$

$$-X_1 - \frac{X_1}{3} + \frac{5}{9}FL - \frac{4}{9}X_2 + \frac{FL}{6} = 0$$

$$-\frac{4}{3}X_1 - \frac{4}{9}X_2 + \frac{13}{18}FL = 0$$

$$X_1 = -\frac{4}{9}\cdot\frac{3}{4}X_2 + \frac{13}{18}\cdot\frac{3}{4}FL = -\frac{1}{3}X_2 + \frac{13}{24}FL$$

$$4FL + 3FL + \frac{13}{12}FL - \frac{2}{3}X_2 - 3X_2 = 0$$

$$\frac{11}{3}X_2 = \frac{97}{12}FL$$

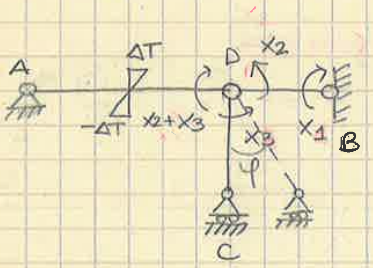
$$\begin{cases} X_2 = \dfrac{97}{44}FL \\ X_1 = -\dfrac{1}{3}\cdot\dfrac{97}{44}FL + \dfrac{13}{24}FL = -\dfrac{17}{88}FL \end{cases}$$

$N_2 = 0$

$N_3 = \frac{F}{4}$

$N_1 = \frac{F}{4}(\sqrt{2} - 1)$

- Ex (8/7/13)



$$\begin{cases} \varphi_{BD} = 0 \\ \varphi_{DB} = \varphi_{DC} \\ \varphi_{DC} = \varphi_{DA} \\ PLV \end{cases}$$

BD ↷
DB ↶(+)
DC ↶(+)
DA ↷

$$\begin{cases} -\frac{X_1 L}{3EI} - \frac{X_2 L}{6EI} = 0 \\ \frac{X_2 L}{3EI} + \frac{X_1 L}{6EI} = \frac{X_3 L}{3EI} + \varphi \\ \frac{X_3 L}{3EI} + \varphi = -\frac{(X_2+X_3)\,2L}{3EI} - \alpha\Delta T \frac{2L}{h} \\ X_3\varphi = 0 \end{cases}$$

$$\begin{cases} X_3 = 0 \\ X_2 = -2X_1 \\ X_2 + \frac{X_1}{2} = \varphi\frac{3EI}{L} \rightarrow -2X_1 + \frac{X_1}{2} = \varphi\frac{3EI}{L} \rightarrow \frac{\varphi EI}{L} = -\frac{1}{2}X_1 \\ -\frac{X_1}{2} = -\frac{2}{3}X_2 - 2\alpha\Delta T\frac{L}{h} \end{cases}$$

$X_3 = 0$

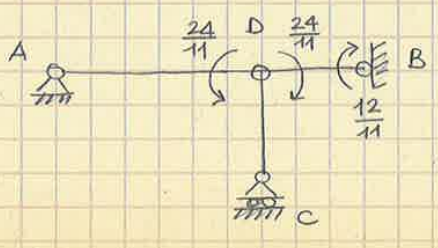
$X_2 = -2X_1$

$\frac{\varphi EI}{L} = -\frac{X_1}{2}$

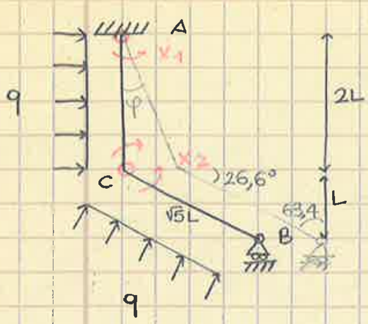
$+\frac{X_1}{2} + \frac{4}{3}X_1 = +2\alpha\Delta T\frac{L}{h}$

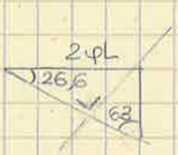
$\frac{3+8}{6}X_1 = 2\alpha\Delta T\frac{L}{h}$

$$\begin{cases} X_1 = \frac{12}{11}\alpha\Delta T\frac{L}{h} \\ X_2 = -\frac{24}{11}\alpha\Delta T\frac{L}{h} \\ X_3 = 0 \end{cases}$$



• EX 16





$$\begin{cases} g = 3 \\ v = 3(A) + 1(B) \\ i = 1 \end{cases}$$

Associated truss structure

$$\begin{cases} g = 3 \cdot 2 = 6 \\ v = 2(A) + 1(B) + 2(C) \\ \ell = 1 \end{cases}$$

$$\begin{cases} \varphi_A = 0 \\ \varphi_{CA} = \varphi_{CB} \\ PLV \end{cases}$$

AC (+)
CA (−)
CB (+)

$$\begin{cases} \dfrac{X_1 2L}{3EI} + \dfrac{X_2 \cdot 2L}{6EI} + \dfrac{q(2L)^3}{24EI} + \varphi = 0 \quad \cdot \left\{\dfrac{3EI}{L}\right\} \\ -\dfrac{X_2 \cdot 2L}{3EI} - \dfrac{X_1 \cdot 2L}{6EI} + \varphi - \dfrac{q(2L)^3}{24EI} = \dfrac{X_2 \sqrt{5}L}{3EI} + \dfrac{q(\sqrt{5}L)^3}{24EI} \\ X_1\varphi - X_2\varphi + 2qL \cdot \varphi L + q\sqrt{5}L \cdot 0{,}9\varphi L = 0 \end{cases}$$

($q\sqrt{5}L \cdot 0{,}9\varphi L = 2qL\varphi L$)

↳ spostamento ⊥ CB $= 2\varphi L \operatorname{sen} 26{,}6° = 2 \cdot \varphi \cdot L \cdot 0{,}45$

$$\begin{cases} 2X_1 + X_2 + qL^2 + \dfrac{3\varphi EI}{L} = 0 \\ -\dfrac{2}{3}X_2 - \dfrac{1}{3}X_1 + \dfrac{\varphi EI}{L} - \dfrac{qL^2}{3} - \dfrac{\sqrt{5}X_2}{3} - \dfrac{5\sqrt{5}qL^2}{24} = 0 \\ X_1 - X_2 + 2qL^2 + 2qL^2 = 0 \end{cases}$$

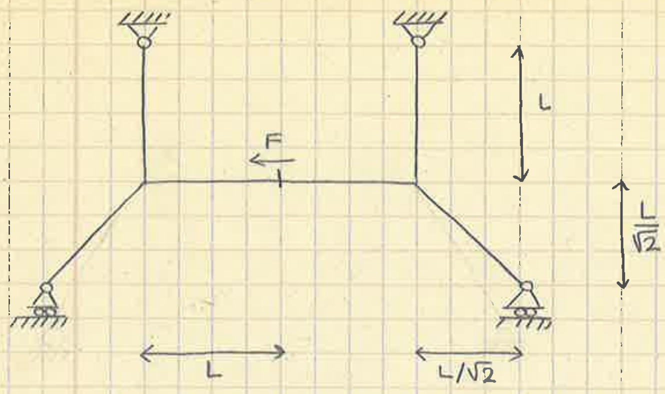
$$\begin{cases} \dfrac{\varphi EI}{L} = -\dfrac{2}{3}X_1 - \dfrac{X_2}{3} - \dfrac{qL^2}{3} \\ -\dfrac{2}{3}X_2 - \dfrac{X_1}{3} - \dfrac{2}{3}X_1 - \dfrac{X_2}{3} - \dfrac{qL^2}{3} - \dfrac{qL^2}{3} - \dfrac{\sqrt{5}}{3}X_2 - \dfrac{5\sqrt{5}}{24}qL^2 = 0 \\ \left(-\dfrac{2}{3} - \dfrac{1}{3} - \dfrac{\sqrt{5}}{3}\right)X_2 + \left(-\dfrac{1}{3} - \dfrac{2}{3}\right)X_1 + \left(-\dfrac{1}{3} - \dfrac{1}{3} - \dfrac{5\sqrt{5}}{24}\right)qL^2 = 0 \end{cases}$$

$$\begin{cases} X_1 = -\dfrac{3+\sqrt{5}}{3}X_2 - \dfrac{16+5\sqrt{5}}{24}qL^2 \\ -\dfrac{3+\sqrt{5}}{3}X_2 - \dfrac{16+5\sqrt{5}}{24}qL^2 - X_2 + 4qL^2 = 0 \\ \left(-\dfrac{3+\sqrt{5}}{3} - 1\right)X_2 + \left(4 - \dfrac{16+5\sqrt{5}}{24}\right)qL^2 = 0 \end{cases}$$

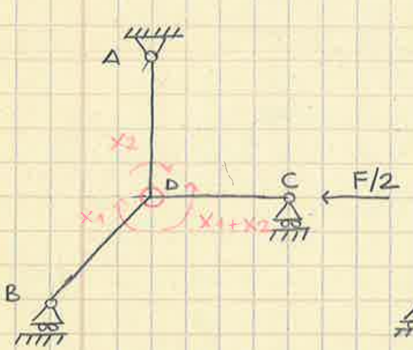
$$\frac{6+\sqrt{5}}{3}X_2 = \frac{80-5\sqrt{5}}{24}qL^2$$

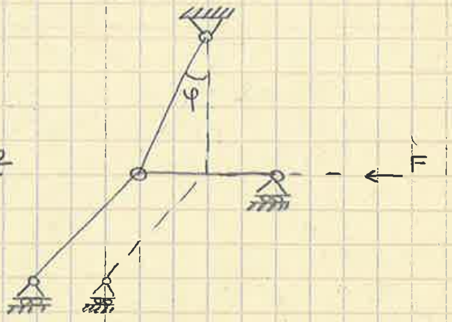
$$\begin{cases} X_2 = 1{,}044\, qL^2 \\ X_1 = -1{,}822\,qL^2 - 1{,}132\,qL^2 = -2{,}954\,qL^2 \end{cases}$$

• EX 17 5/09/12



The structure is symmetrical, antisymmetrical loaded so we can consider only a half of it





$$\begin{cases} \varphi_{DB} = \varphi_{DA} \\ \varphi_{DC} = \varphi_{DA} \\ PLV \end{cases}$$

DB (−)
DA (−)
DC (+)

$$\begin{cases} -\frac{X_1 \cdot L}{3EI} = -\frac{X_2 L}{3EI} - \varphi \quad ✓ \\ \frac{(X_1+X_2)\cdot L}{3EI} = -\frac{X_2 L}{3EI} - \varphi \quad ✓ \\ X_2 \varphi + \frac{F}{2}\varphi L = 0 \quad ✓ \end{cases}$$

$$X_2 = -FL/2$$

$$+\frac{X_1 L}{3EI} = -\frac{FL^2}{6EI} + \varphi$$

$$\varphi = \frac{X_1 L}{3EI} + \frac{FL^2}{6EI}$$

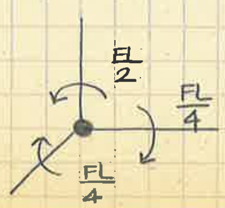
$$\frac{X_1 L}{3EI} + \frac{X_2 L}{3EI} + \frac{X_2 L}{3EI} + \frac{X_1 L}{3EI} + \frac{FL^2}{6EI} = 0$$

$$2X_1 + 2X_2 + \frac{FL}{2} = 0$$

$$4X_1 + 4X_2 + FL = 0$$

$$4X_1 - 2FL + FL = 0$$

$$\begin{cases} X_1 = \frac{FL}{4} \\ X_2 = -\frac{FL}{2} \\ X_1 + X_2 = -\frac{FL}{4} \end{cases}$$



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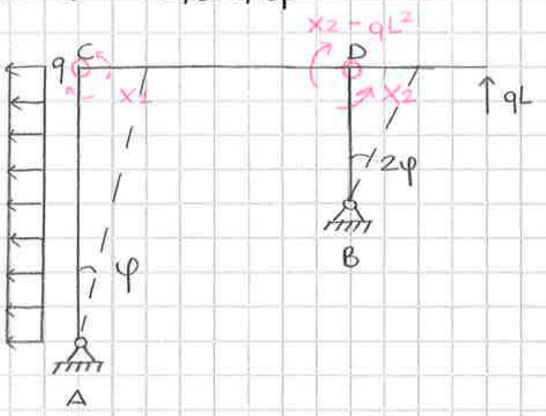
• Ex 18 20/02/14



$$\begin{cases} g = 3 \\ v = 2(A) + 2(B) = 4 \\ i = 1 \end{cases}$$

Associated truss structure

$$\begin{cases} g = 3 \cdot 4 = 12 \\ v = 2(A) + 2(B) + 2(C) + 4(D) = 10 \\ \ell = 2 \end{cases}$$

$$\begin{cases} \varphi_{CA} = \varphi_{CD} \\ \varphi_{DB} = \varphi_{DC} \\ \text{PLV} \end{cases}$$

CA (−) CD (+) DB (+) DC (−)

$$\begin{cases} -\dfrac{X_1 2L}{3EI} - \dfrac{q(2L)^3}{24EI} - \varphi = \dfrac{X_1 2L}{3EI} + \dfrac{(X_2 - qL^2)\cdot 2L}{6EI} \\ \dfrac{X_2 \cdot L}{3EI} - 2\varphi = -\dfrac{(X_2 - qL^2)2L}{3EI} - \dfrac{X_1 \cdot 2L}{6EI} \\ X_1\varphi - X_2 2\varphi - 2qL \cdot \varphi L = 0 \end{cases}$$

$$\begin{cases} -\dfrac{2}{3}X_1 - \dfrac{qL^2}{3} - \dfrac{\varphi EI}{L} - \dfrac{2}{3}X_1 - \dfrac{X_2}{3} + \dfrac{qL^2}{3} = 0 \\ \dfrac{X_2}{3} - \dfrac{2\varphi EI}{L} + \dfrac{2}{3}X_2 - \dfrac{2}{3}qL^2 + \dfrac{X_1}{3} = 0 \\ X_1 - 2X_2 - 2qL^2 = 0 \end{cases}$$

$$\begin{cases} -\dfrac{4}{3}X_1 - \dfrac{X_2}{3} = \dfrac{\varphi EI}{L} \\ \dfrac{X_2}{3} + \dfrac{8}{3}X_1 + \dfrac{2}{3}X_2 + \dfrac{2}{3}X_2 - \dfrac{2}{3}qL^2 + \dfrac{X_1}{3} = 0 \\ X_1 - 2X_2 - 2qL^2 = 0 \end{cases}$$

$$\begin{cases} \dfrac{5}{3}X_2 + 3X_1 - \dfrac{2}{3}qL^2 = 0 \\ \dfrac{2}{9}qL^2 - \dfrac{5}{9}X_2 - 2X_2 - 2qL^2 = 0 \end{cases} \qquad X_1 = \frac{2}{9}qL^2 - \frac{5}{9}X_2$$

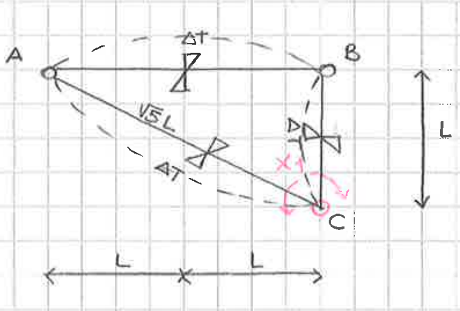
$$\left(-\frac{5}{9} - 2\right)X_2 = \left(2 - \frac{2}{9}\right)qL^2$$

$$-\frac{23}{9}X_2 = \frac{16}{9}qL^2$$

$$X_2 = -\frac{16}{23}qL^2$$

$$X_1 = \frac{2}{9}qL^2 + \frac{5}{9}\cdot\frac{16}{23}qL^2 = \frac{14}{23}qL^2$$

• Ex 19 13/06/16

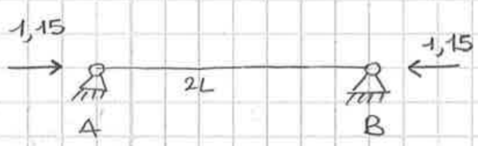


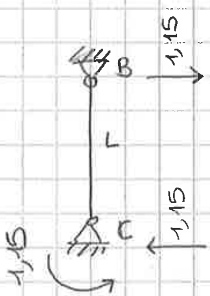
$\varphi_{CA} = \varphi_{CB}$

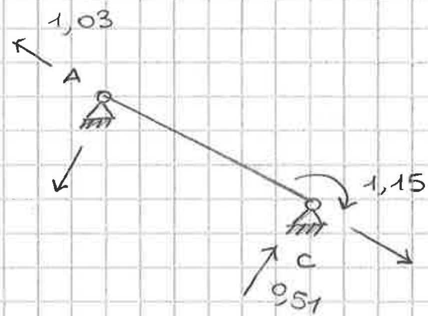
$$\frac{X_1 \sqrt{5} \cdot L}{3EI} + \alpha \Delta T \frac{\sqrt{5} L}{h} = - \frac{X_1 \cdot L}{3EI} + \alpha \Delta T \frac{L}{h}$$

$$X_1 (\sqrt{5}+1) \frac{L}{3EI} = \frac{\alpha \Delta T L}{h} (1-\sqrt{5})$$

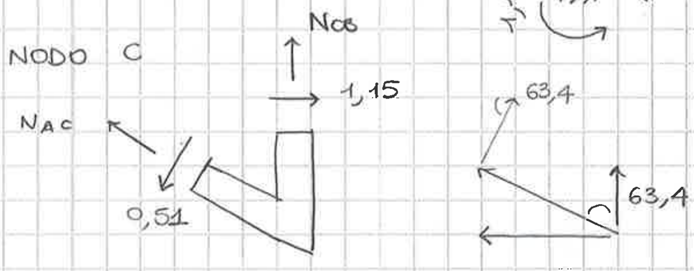
$$X_1 = \frac{\alpha \Delta T}{h} \cdot 3EI \frac{(1-\sqrt{5})}{(1+\sqrt{5})} = -1{,}15 \frac{\alpha \Delta T EI}{h}$$







NODO C

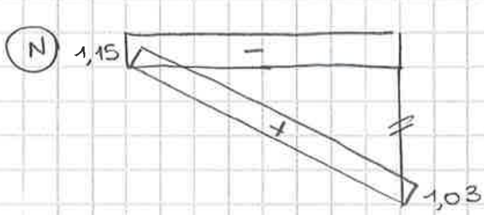


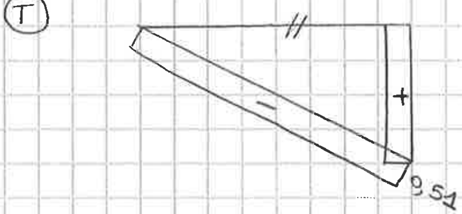
$N_{CB} + N_{AC} \cos 63{,}4 - 0{,}51 \operatorname{sen} 63{,}4 = 0$

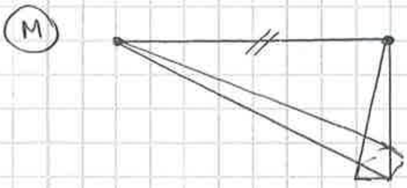
$N_{AC} \operatorname{sen} 63{,}4 - 1{,}15 + 0{,}51 \cos 63{,}4 = 0$

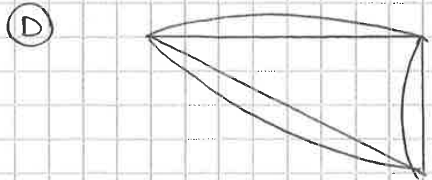
$N_{AC} = 1{,}03$

$N_{CB} = 0$

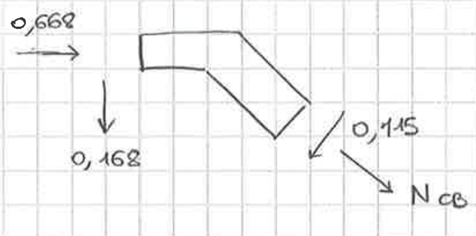






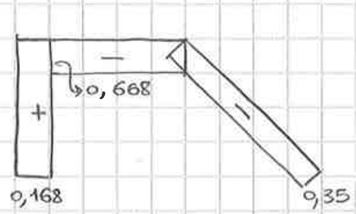


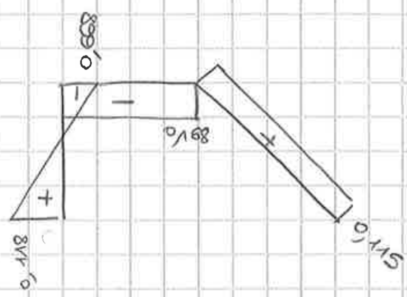
NODO C

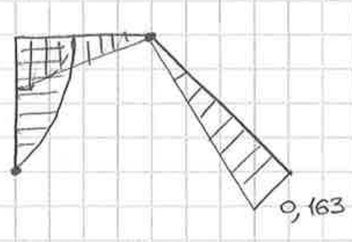


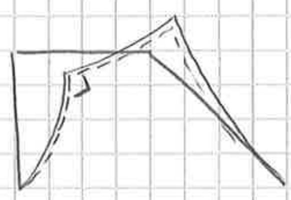
N_{CB} sen 45° + 0,168 + 0,115 sen45 = 0

N_{CB} = − 0,35

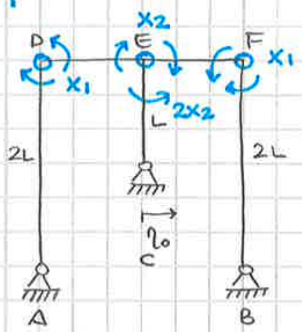








3/03/14



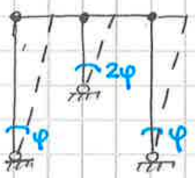
$$\begin{cases} g = 3 \\ v = 2(A) + 2(B) + 2(C) = 6 \end{cases}$$

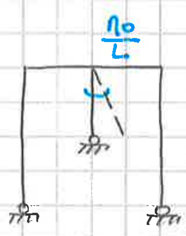
It's statically indeterminated

Associated truss structure

$$\begin{cases} g = 3 \cdot 5 = 15 \\ v = 2(A) + 2(B) + 2(C) + 2(D) + 2(F) + 4(E) = 14 \end{cases}$$

$g - v = l = 1$





It's symmetrical loaded in antisimmetrical way so unknowns are equal and opposite due to simmetry

$$\begin{cases} \varphi_{DA} = \varphi_{DE} \\ \varphi_{ED} = \varphi_{EC} \\ PLV \end{cases}$$

DA ↻(−) DE ↺(+) ED ↻(−) EC ↺(+)

$$\begin{cases} -\frac{X_1 \cdot 2L}{3EI} - \varphi = \frac{X_1 \cdot L}{3EI} + \frac{X_2 L}{6EI} \quad ✓ \\ -\frac{X_2 L}{3EI} - \frac{X_1 \cdot L}{6EI} = \frac{2X_2 L}{3EI} - 2\varphi + \frac{\eta_0}{L} \\ 2X_1\varphi - 2X_2 \cdot 2\varphi = 0 \quad ✓ \end{cases}$$

$$X_1 = 2X_2$$

$$\varphi = \Big(\underbrace{-\frac{2}{3}X_1 - \frac{1}{3}X_1}_{-X_1} - \frac{X_2}{6}\Big)\frac{L}{EI} =$$

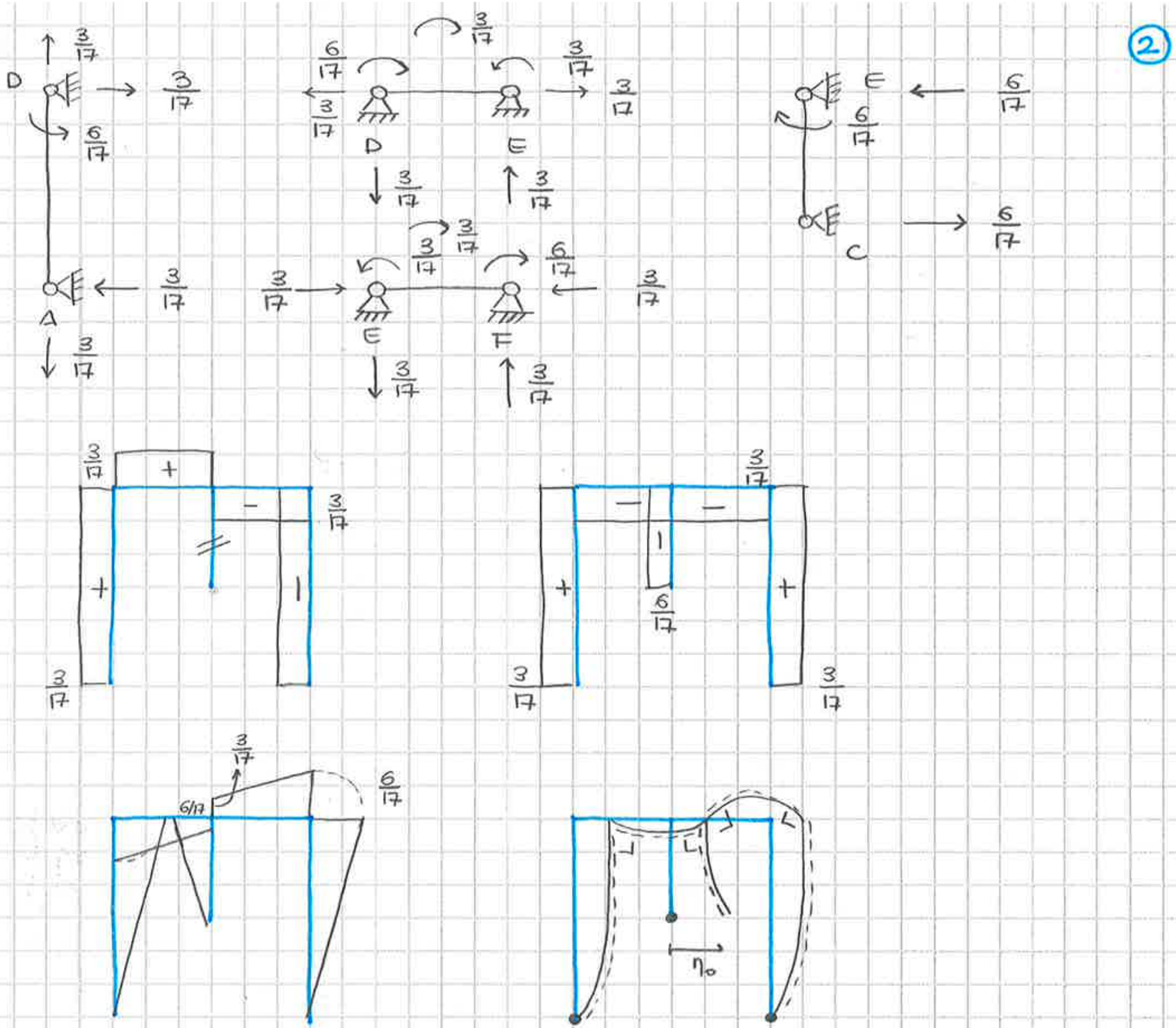
$$= \left(-2X_2 - \frac{X_2}{6}\right)\frac{L}{EI} = -\frac{13}{6}\frac{X_2 L}{EI}$$

$$\left(-\frac{X_2}{3} - \frac{X_1}{6} - \frac{2}{3}X_2 - \frac{13}{6}X_2\right)\frac{L}{EI} = \frac{\eta_0}{L}\frac{EI}{L}$$

$$\left(-\frac{1}{3} - \frac{1}{3} - \frac{2}{3} - \frac{13}{6}\right)X_2 = \frac{\eta_0 EI}{L^2}$$

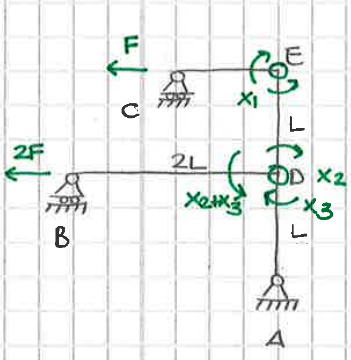
$$-\frac{17}{3}X_2 = \frac{\eta_0 EI}{L^2}$$

$$\begin{cases} X_2 = -\frac{3}{17}\frac{\eta_0 EI}{L^2} \\ X_1 = -\frac{6}{17}\frac{\eta_0 EI}{L^2} \end{cases}$$



2
D
A
E
F
C
3/17
6/17
+
−
η_0

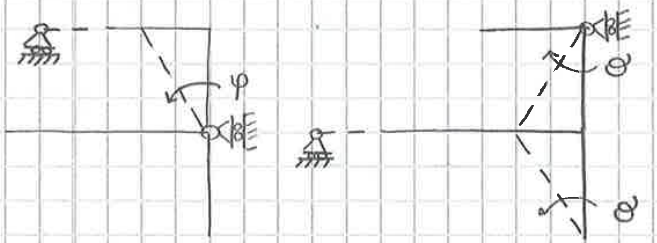
6/02/17



$$\begin{cases} g = 3 \\ v = 2(A) + 1(B) + 1(C) = 4 \end{cases}$$
$v - g = 1$

Associated truss structure

$$\begin{cases} g = 4 \cdot 3 = 12 \\ v = 2(A) + 1(B) + 1(C) + 2(E) + 4(D) = 10 \end{cases}$$
$g - v = 2$



$$\begin{cases} \varphi_{EC} = \varphi_{ED} \\ \varphi_{DE} = \varphi_{DB} \\ \varphi_{DE} = \varphi_{DA} \\ PLV(\varphi) \\ PLV(\theta) \end{cases}$$

EC (−) ED (+)
DE (−) DB (+)
DA (−)

$$\begin{cases} -\frac{X_1 L}{3EI} = \frac{X_1 L}{3EI} + \frac{X_2 L}{6EI} + \varphi - \theta \quad \checkmark \\ -\frac{X_2 L}{3EI} - \frac{X_1 L}{6EI} + \varphi - \theta = \frac{(X_2+X_3)\cdot 2L}{3EI} \quad \checkmark \\ -\frac{X_2 L}{3EI} - \frac{X_1 L}{6EI} + \varphi - \theta = -\frac{X_3 \cdot L}{3EI} + \theta \quad \text{changed} \checkmark \\ X_1\varphi - X_2\varphi + F\varphi L = 0 \quad \checkmark \\ -X_1\theta + X_2\theta - X_3\theta + 2F\theta L = 0 \end{cases}$$

$$\frac{(X_2+X_3)2L}{3EI} = -\frac{X_3 L}{3EI} + \theta$$

$$\hookrightarrow \theta = \frac{2X_2 L}{3EI} + \frac{2X_3 L}{3EI} + \frac{X_3 L}{3EI} = \left(\frac{2}{3}X_2 + X_3\right)\frac{L}{EI}$$

$$-\frac{1}{3}X_2 - \frac{X_1}{6} + \varphi - \frac{2}{3}X_2 - X_3 - \frac{2}{3}X_2 - \frac{2}{3}X_3 = 0$$

$$\hookrightarrow \left(-\frac{1}{3} - \frac{2}{3} - \frac{2}{3}\right)X_2 - \frac{X_1}{6} - \frac{5}{3}X_3 = -\varphi$$

$$-\frac{5}{3}X_2 - \frac{X_1}{6} - \frac{5}{3}X_3 = -\varphi$$

$$\varphi = \frac{5}{3}X_2 + \frac{X_1}{6} + \frac{5}{3}X_3$$

(2)

$$-\frac{x_1}{3} - \frac{x_1}{3} - \frac{x_2}{6} - \frac{5}{3}x_2 - \frac{x_1}{6} - \frac{5}{3}x_3 + \frac{2}{3}x_2 + x_3 = 0$$

$$\left(-\frac{1}{3} - \frac{1}{3} - \frac{1}{6}\right)x_1 + \left(-\frac{1}{6} - \frac{5}{3} + \frac{2}{3}\right)x_2 + \left(-\frac{5}{3} + 1\right)x_3 = 0$$

$$-\frac{5}{6}x_1 - \frac{7}{6}x_2 - \frac{2}{3}x_3 = 0$$

$$x_3 = -\frac{7}{4}x_2 - \frac{5}{4}x_1 = -\frac{7}{4}x_2 - \frac{5}{4}x_2 + \frac{5}{4}FL = -3x_2 + \frac{5}{4}FL$$

$$x_1 - x_2 + FL = 0$$

$$x_1 = x_2 - FL$$

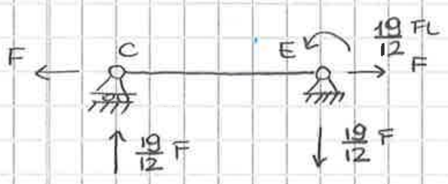
$$-x_1 + x_2 - x_3 + 2FL = 0$$

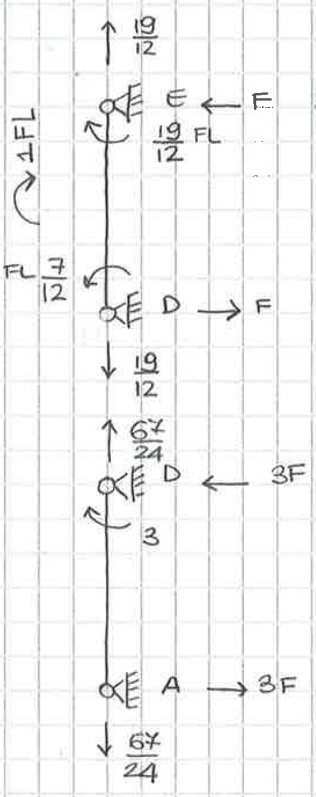
$$-x_2 + FL + x_2 + 3x_2 - \frac{5}{4}FL + 2FL = 0$$

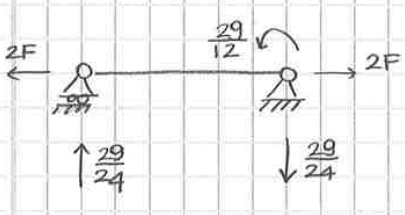
$$3x_2 = -3FL + \frac{5}{4}FL = -\frac{7}{12}FL$$

$$x_2 = -\frac{7}{12}FL \qquad x_3 = 3 \qquad \Big\}\ x_2 + x_3 = \frac{29}{12}$$

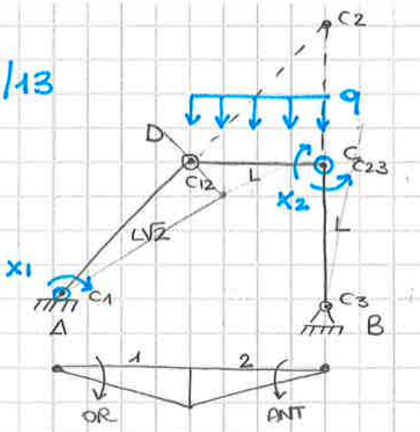
$$x_1 = -\frac{19}{12}FL$$

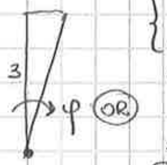






7/02/13 (1)





$$\begin{cases} g = 3 \cdot 2 = 6 \\ v = 3(A) + 2(B) + 2(D) = 7 \end{cases}$$

$v - g = i = 1$

Associated truss structure

$$\begin{cases} g = 3 \cdot 3 = 9 \\ v = 2(A) + 2(B) + 2(C) + 2(D) = 8 \end{cases}$$

$g - v = l = 1$

$$\begin{cases} \varphi_{CB} = \varphi_{CD} \\ \varphi_{AD} = 0 \\ PLV \end{cases}$$

CB (+)
CD
AD (−)

$$\begin{cases} \dfrac{X_2 \cdot L}{3EI} - \varphi = -\dfrac{X_2 L}{3EI} + \dfrac{qL^3}{24EI} + \varphi \\ -\dfrac{X_1 L\sqrt{2}}{3EI} - \varphi = 0 \\ X_1\varphi - X_2\varphi - X_2\varphi + qL \cdot \varphi \cdot \dfrac{L}{2} = 0 \end{cases}$$

$$\varphi = -\frac{X_1 L\sqrt{2}}{3EI}$$

$$\frac{2}{3}\frac{X_2 L}{EI} - \frac{qL^{3\,2}}{24EI_{\,8EI}} + \frac{2X_1 L\sqrt{2}}{3EI} = 0$$

$$\hookrightarrow 2X_2 - \frac{qL^2}{8} + 2\sqrt{2}X_1 = 0$$

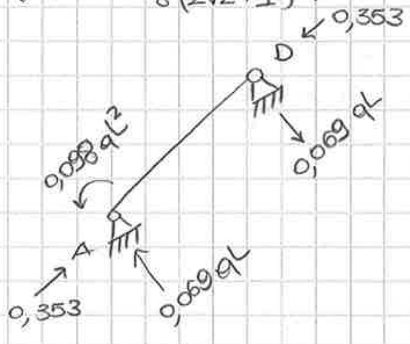
$$X_2 = \frac{qL^2}{16} - \sqrt{2}X_1$$

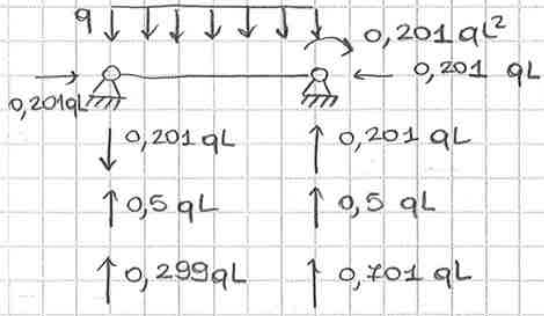
$$X_1 - 2X_2 + \frac{qL^2}{2} = 0$$

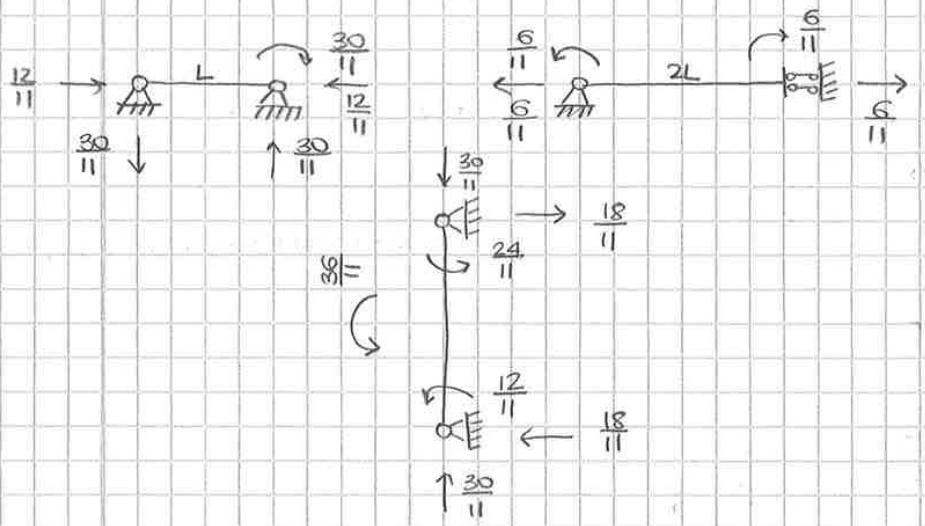
$$X_1 - \frac{qL^2}{8} + 2\sqrt{2}X_1 + \frac{qL^2}{2} = 0$$

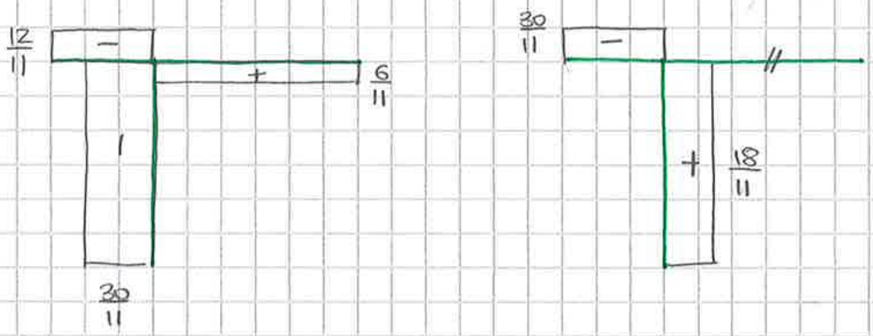
$$(1 + 2\sqrt{2})X_1 = \frac{qL^2}{8} - \frac{qL^2}{2} = \frac{1-4}{8}qL^2 = -\frac{3}{8}qL^2$$

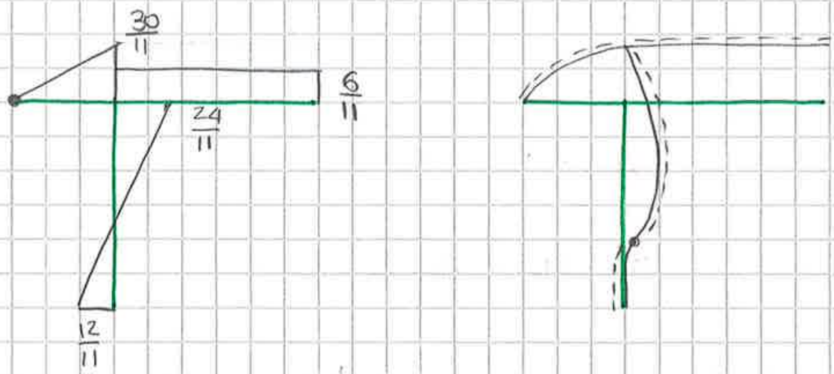
$$\begin{cases} X_1 = \dfrac{-3}{8(2\sqrt{2}+1)}qL^2 = -0{,}098\,qL^2 \\ X_2 = \left(\dfrac{1}{16} + \dfrac{3\sqrt{2}}{8(2\sqrt{2}+1)}\right)qL^2 = 0{,}201\,qL^2 \end{cases}$$

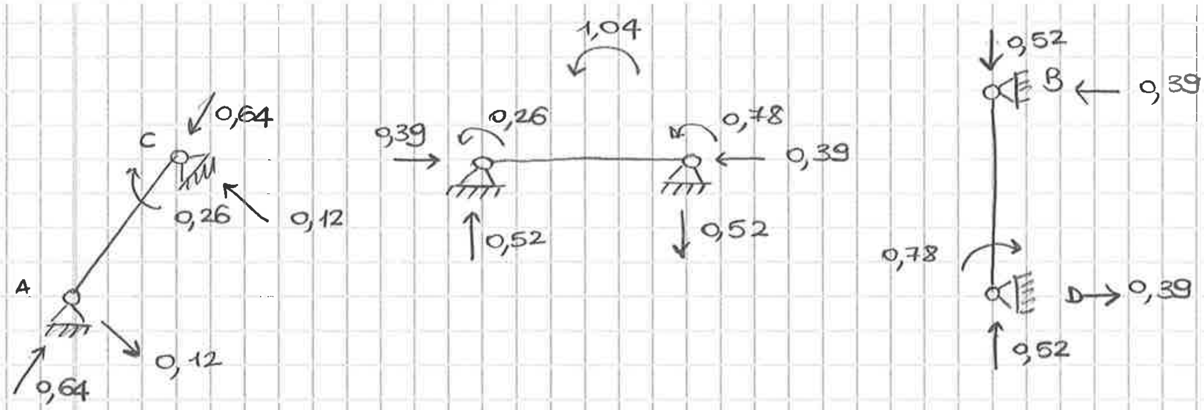


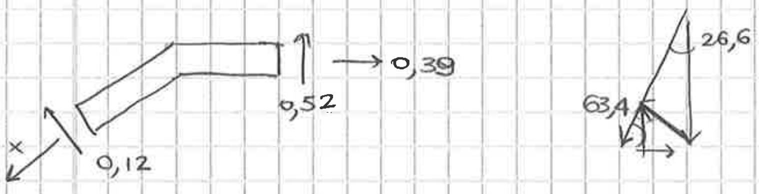






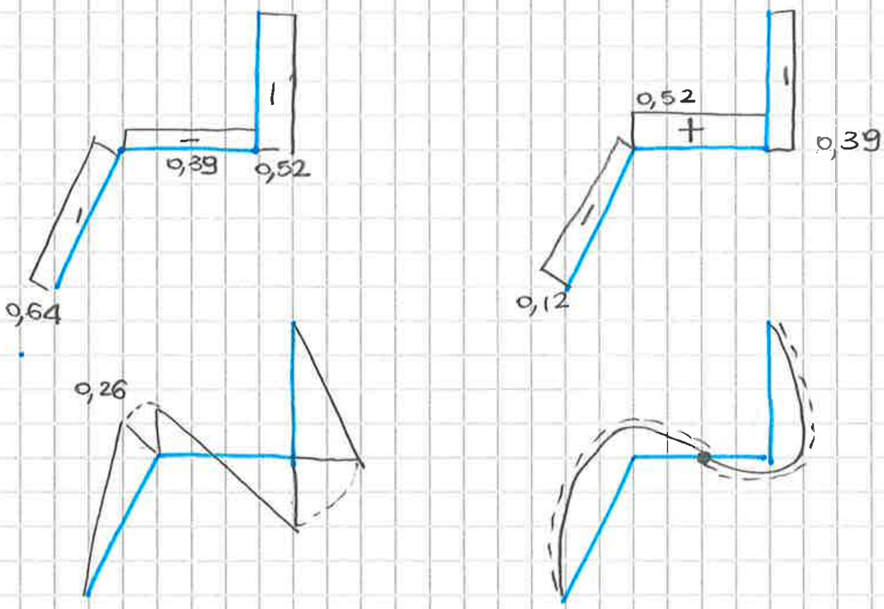






+↑) $0,52 + 0,12 \cos 63,4 - x \cos 26,6 = 0$

$$x = \frac{0,52 + 0,12 \cos 63,4}{\cos 26,6} = 0,64$$



21.09.11

venerdì 5 gennaio 2018 21:58

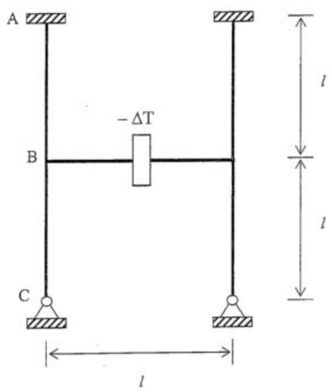
Ok

iPad 21:59 100%

2011 - Settembre.pdf

21 Settembre 2011

COGNOME :	CORSO DI LAUREA :
NOME:	MATRICOLA:



1. Tracciare in scala i diagrammi di M, N e T.
2. Tracciare la curva delle pressioni.
3. Tracciare la deformata elastica.

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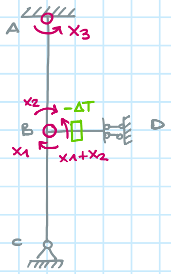
It's a symmetrical structure loaded in a symmetrical way, we can consider only half of it.



$$\begin{cases} g = 3 \\ v = 3(A) + 2(C) + 2(D) = 7 \end{cases}$$

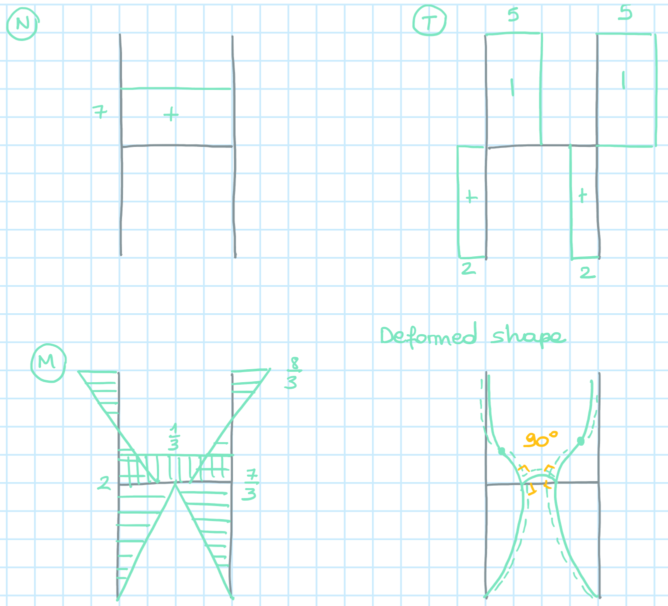
$v - g = i = 4$

Associated truss structure

$$\begin{cases} g = 3 \cdot 3 = 9 \\ v = 2(A) + 4(B) + 2(C) + 2(D) = 10 \end{cases}$$

$v - g = i = 1$

It isn't a mechanism it moves only for the ΔT



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$$\begin{cases} g = 3 \cdot 3 = 9 \\ v = 2(A) + 2(B) + 2(C) + 4(D) = 10 \end{cases}$$

$$\begin{cases} \varphi_{DA} = \varphi_{DB} \\ \varphi_{DA} = \varphi_{DC} \\ \varphi_{CD} = 0 \end{cases}$$

DA DB DC CD

$$\begin{cases} -\frac{X_2 L}{3EI} - \frac{\eta_0}{L} = -\frac{X_3 L}{3 \cdot 3EI} + \frac{3\eta_0}{L} & \cdot \left\{\frac{3EI}{L}\right\} \\ -\frac{X_2 L}{3EI} - \frac{\eta_0}{L} = \frac{(X_2+X_3)L}{3EI} + \frac{X_1 L}{6EI} & \cdot \left\{\frac{3EI}{L}\right\} \\ -\frac{X_1 L}{3EI} - \frac{(X_2+X_3)L}{6EI} = 0 & \cdot \left\{\frac{3EI}{L}\right\} \end{cases}$$

$$\begin{cases} -X_1 - \frac{X_2}{2} - \frac{X_3}{2} = 0 \\ -X_2 - \frac{3EI}{L^2} \cdot \eta_0 - X_2 - X_3 - \frac{X_1}{2} = 0 \\ -X_2 - \frac{3EI}{L^2} \cdot \eta_0 + \frac{X_3}{3} - \frac{9EI}{L^2}\eta_0 = 0 \end{cases}$$

$$X_2 = \frac{X_3}{3} - \frac{12EI}{L^2}\eta_0$$

$$-\frac{X_3}{3} + \frac{12EI}{L^2}\eta_0 - \frac{3EI}{L^2}\eta_0 - \frac{X_3}{3} + \frac{12EI}{L^2}\eta_0 - X_3 - \frac{X_1}{2} = 0$$

$$-\frac{2}{3}X_3 - X_3 + \frac{21}{L^2}EI\,\eta_0 - \frac{X_1}{2} = 0$$

$$-\frac{5}{3}X_3 + \frac{21EI}{L^2}\eta_0 = \frac{X_1}{2}$$

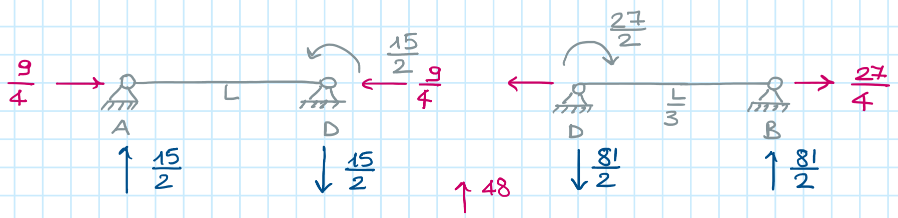
$$X_1 = -\frac{10}{3}X_3 + \frac{42EI}{L^2}\eta_0$$

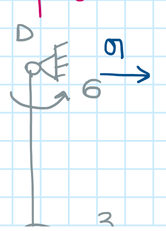
$$\frac{10}{3}X_3 - \frac{42EI}{L^2}\eta_0 - \frac{X_3}{6} + \frac{6EI}{L^2}\eta_0 - \frac{X_3}{2} = 0$$

$$\left(\frac{10}{3} - \frac{1}{6} - \frac{1}{2}\right)X_3 = \frac{36EI}{L^2}\eta_0$$

$$\frac{8}{3}X_3 = \frac{36EI}{L^2}\eta_0$$

$$\begin{cases} X_3 = \frac{27}{2}\frac{EI}{L^2}\eta_0 \\ X_2 = \left(\frac{9}{2} - 12\right)\frac{EI}{L^2}\eta_0 = -\frac{15}{2}\frac{EI}{L^2}\eta_0 \\ X_1 = \left(-\frac{10}{3} \cdot \frac{27}{2} + 42\right)\frac{EI}{L^2}\eta_0 = -\frac{3EI}{L^2}\eta_0 \end{cases}$$





Lo sforzo normale si distribuisce in maniera inversa rispetto alla lunghezza. Più una

$$L + \frac{L}{3} = \frac{4}{3}L$$

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23.02.11 (B)

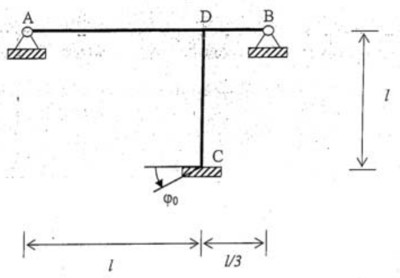
lunedì 29 gennaio 2018 16:11

Ok

23 Febbraio 2011

COGNOME :	CORSO DI LAUREA :
NOME:	MATRICOLA:

FILA B



1. Tracciare in scala i diagrammi di M, N e T.
2. Tracciare la curva delle pressioni.
3. Tracciare la deformata elastica.

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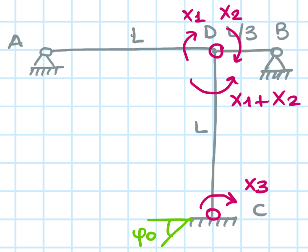
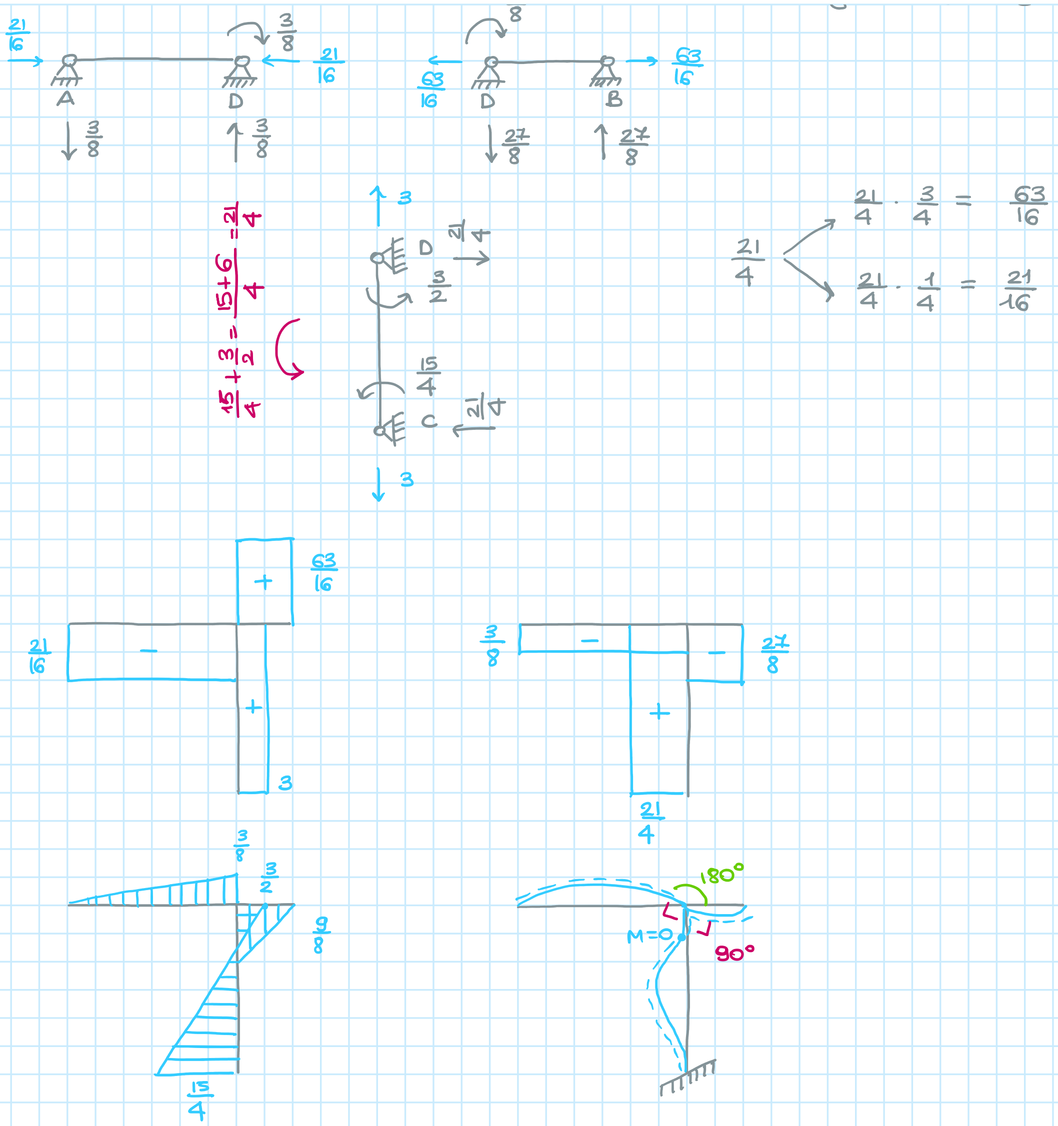
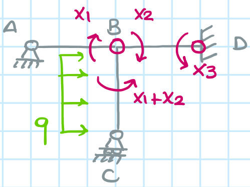
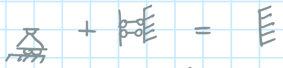


$$\begin{cases} g = 3 \cdot 1 = 3 \\ v = 2(A) + 2(B) + 3(C) = 7 \end{cases}$$

$v - g = 4$


A
D
B
C
15/4 + 3/2 = (15+6)/4 = 21/4
21/4 · 3/4 = 63/16
21/4 · 1/4 = 21/16
M=0
180°
90°



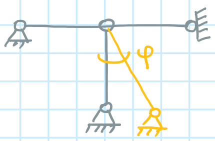


rotazione no (bipendolo)
trasl. ↔ no (bipendolo)
trasl ↕ no (carrello)
↳ no movements are allowed

$$\begin{cases} g = 3\cdot 3 = 9 \\ v = 2(A) + 4(B) + 1(C) + 2(D) = 9 \end{cases}$$

$g - v = 0 \quad (i = \ell)$

↳ it's a mechanism, but also a statically indeterminate structure



$$\begin{cases} \varphi_{BA} = \varphi_{BD} \\ \varphi_{DB} = 0 \\ \varphi_{BA} = \varphi_{BC} \\ PLV \end{cases}$$

BA (−) BD (−) DB (+) BC (+)

$$\begin{cases} -\frac{X_1 L}{3EI} = -\frac{X_2 L}{3EI} - \frac{X_3 L}{6EI} \\ \frac{X_3 L}{3EI} + \frac{X_2 L}{6EI} = 0 \\ -\frac{X_1 L}{3EI} = \frac{(X_1+X_2)L}{3EI} + \varphi + \frac{qL^3}{24EI} \\ (X_1+X_2)\varphi + qL \cdot \frac{\varphi L}{2} = 0 \end{cases}$$

$X_3 + \frac{X_2}{2} = 0$

↳ $X_2 = -2X_3$

$-X_1 + X_2 + \frac{X_3}{2} = 0$

↳ $-X_1 - 2X_3 + \frac{X_3}{2} = 0$

$X_1 = \frac{-4+1}{2} X_3 = -\frac{3}{2} X_3$

$\varphi = -\frac{1}{3} X_1 \frac{L}{EI} - \frac{1}{3} X_1 \frac{L}{EI} - \frac{X_2}{3}\frac{L}{EI} - \frac{qL^3}{24EI}$

$= -\frac{2}{3}\frac{L}{EI} X_1 - \frac{1}{3}\frac{L}{EI} X_2 - \frac{qL^3}{24EI}$

$= -\frac{2}{3}\left(-\frac{3}{2}X_3\right)\frac{L}{EI} - \frac{1}{3}\left(-2X_3\right)\frac{L}{EI} - \frac{qL^3}{24EI}$

$= \frac{X_3 L}{EI} + \frac{2}{3} X_3 \frac{L}{EI} - \frac{qL^3}{24EI}$

$= \frac{3+2}{3} X_3 \frac{L}{EI} - \frac{qL^3}{24EI}$

$= \frac{5}{3}\frac{L}{EI} X_3 - \frac{qL^3}{24EI}$

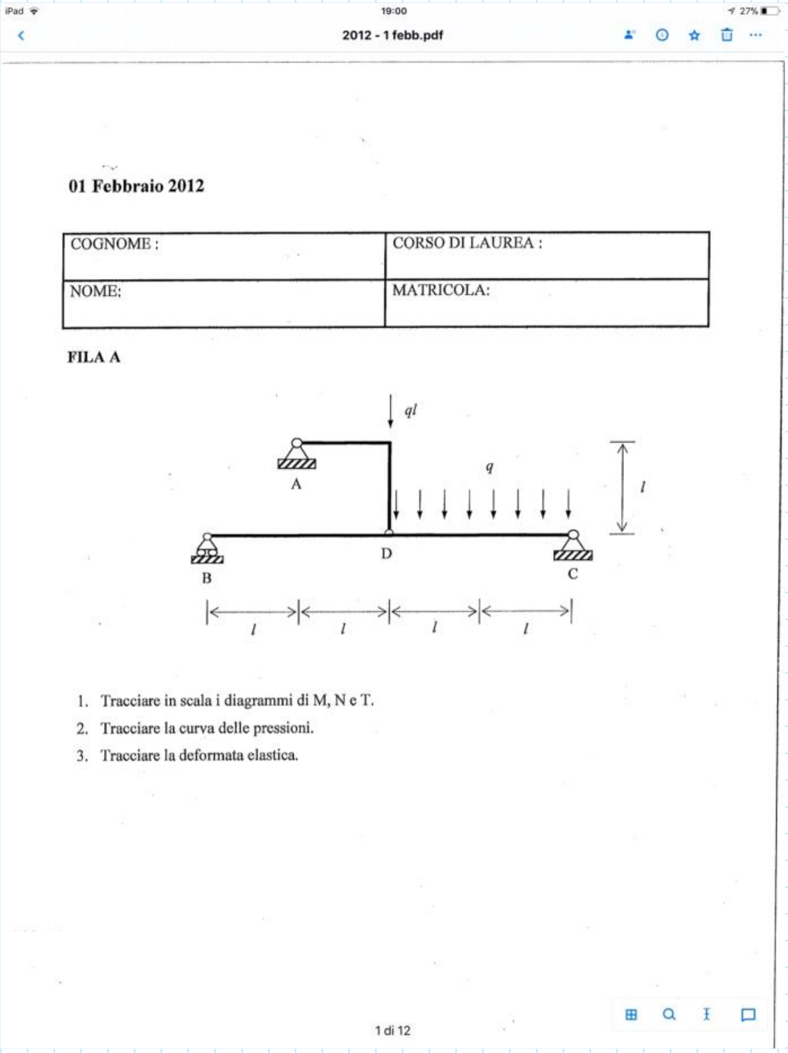
$-\frac{3}{2}X_3 - 2X_3 + \frac{qL^2}{2} = 0$

$\frac{-3-4}{2} X_3 = -\frac{qL^2}{2}$

$X_3 = \frac{qL^2}{7}$

01.02.12 (A)

sabato 30 dicembre 2017 19:01



iPad 19:00 27%

2012 - 1 febb.pdf

01 Febbraio 2012

COGNOME :	CORSO DI LAUREA :
NOME:	MATRICOLA:

FILA A

1. Tracciare in scala i diagrammi di M, N e T.
2. Tracciare la curva delle pressioni.
3. Tracciare la deformata elastica.

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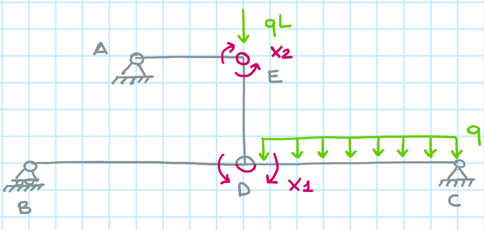


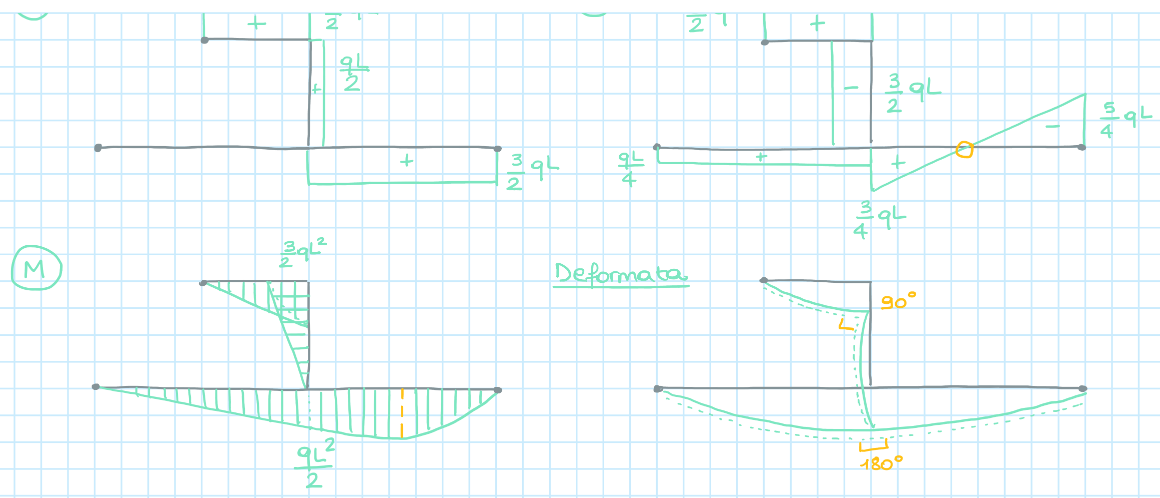
$$\begin{cases} g = 2 \cdot 3 = 6 \\ v = 2(A) + 1(B) + 2(C) + 2(D) = 7 \end{cases}$$

$v - g = i = 1$

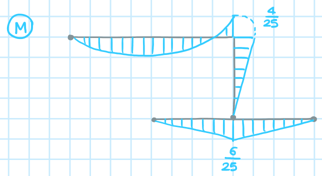
Associated truss structure

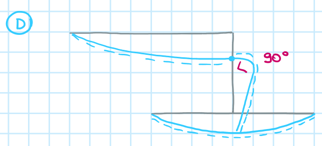
$$\begin{cases} g = 3 \cdot 4 = 12 \\ v = 2(A) + 1(B) + 2(C) + 4(D) + 2(E) = 11 \end{cases}$$

$g - v = \ell = 1$



qL/2
3/2 qL
3/2 qL
qL/4
5/4 qL
3/4 qL
M
3/2 qL²
qL²/2
Deformata
90°
180°





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$$\begin{cases} -\dfrac{X_3 L}{3EI} - \dfrac{(X_1+X_2)L}{6EI} = 0 & \left\{\cdot \dfrac{3EI}{L}\right\} \\ -\dfrac{X_1 L}{3EI} - \alpha \Delta T \dfrac{L}{H} = -\dfrac{X_2 L}{2EI} & \left\{\cdot \dfrac{EI}{L}\right\} \\ -\dfrac{X_2 L}{2EI} = \dfrac{(X_1+X_2)L}{3EI} + \dfrac{X_3 L}{6EI} & \left\{\cdot \dfrac{EI}{L}\right\} \end{cases}$$

$$\begin{cases} -X_3 - \dfrac{X_1}{2} - \dfrac{X_2}{2} = 0 \\ -\dfrac{X_1}{3} - \dfrac{EI\alpha\Delta TL}{H} + \dfrac{X_2}{2} = 0 \\ -\dfrac{X_2}{2} - \dfrac{X_1}{3} - \dfrac{X_2}{3} - \dfrac{X_3}{6} = 0 \end{cases}$$

$$\begin{cases} X_3 = -\dfrac{X_1}{2} - \dfrac{X_2}{2} = -\dfrac{3}{4}X_2 + \dfrac{3}{2}\dfrac{EI\Delta TL}{H}\alpha - \dfrac{X_2}{2} = -\dfrac{5}{4}X_2 + \dfrac{3}{2}\dfrac{EI\Delta TL}{H}\alpha \\ X_1 = \dfrac{3}{2}X_2 - \dfrac{3EI\Delta TL}{H}\alpha \\ -\dfrac{X_2}{2} - \dfrac{X_2}{2} + \dfrac{EI\Delta TL}{H}\alpha - \dfrac{X_2}{3} + \dfrac{5}{24}X_2 - \dfrac{1}{4}\dfrac{EI\Delta TL}{H}\alpha \end{cases}$$

$$\left(-\frac{1}{2} - \frac{1}{2} - \frac{1}{3} + \frac{5}{24}\right)X_2 = \left(\frac{1}{4} - 1\right)\frac{EI\Delta TL}{H}\alpha$$

$$\frac{-12-12-8+5}{24}X_2 = \frac{1-4}{4}\frac{EI\Delta TL}{H}\alpha$$

$$\frac{27}{24}X_2 = \frac{3}{4}\frac{EI\Delta TL\alpha}{H}$$

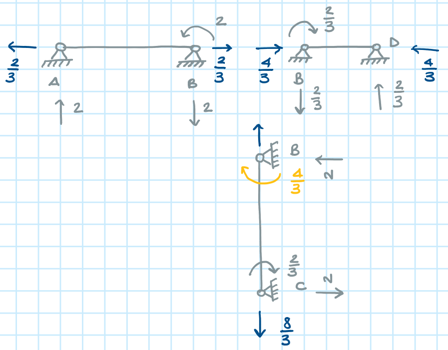
$$X_2 = \underbrace{\frac{3}{4}\,\frac{8}{9}}_{}\frac{EI\Delta TL\alpha}{H}$$

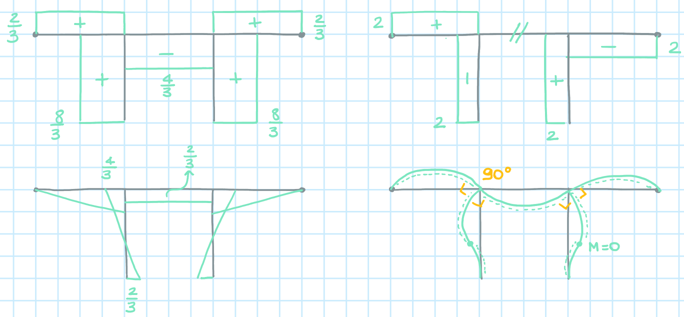
$$\frac{2}{3}\frac{EI\Delta TL\alpha}{H}$$

$$X_1 = -\frac{2EI\Delta TL}{H}\alpha$$

$$X_1 + X_2 = -\frac{4}{3}\frac{EI\Delta TL}{H}\alpha$$

$$X_3 = \left(-\frac{5}{4}\cdot\frac{2}{3} + \frac{3}{2}\right)\frac{EI\Delta TL}{H}\alpha = \frac{2}{3}\frac{EI\Delta TL}{H}\alpha$$





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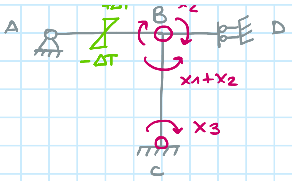


$$\begin{cases} g = 3 \cdot 3 = 9 \\ \nu = 2(A) + 4(B) + 2(C) + 2(D) = 10 \end{cases}$$

$\nu - g = 1$

It's still a statically indeterminated structure

$$\begin{cases} \varphi_{BA} = \varphi_{BD} \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = 0 \end{cases}$$

BA ↷ BD ↷ BC ↶(+) CB ↷

$$\begin{cases} -\dfrac{X_1 L}{3EI} - \dfrac{\alpha \Delta T L}{H} = -\dfrac{X_2 \cdot L}{2EI} \\ -\dfrac{X_1 L}{3EI} - \dfrac{\alpha \Delta T L}{H} = \dfrac{(X_1+X_2)L}{3EI} + \dfrac{X_3 L}{6EI} \\ -\dfrac{X_3 L}{3EI} - \dfrac{(X_1+X_2)L}{6EI} = 0 \end{cases}$$

$$-X_3 - \frac{X_1}{2} - \frac{X_2}{2} = 0$$

$$\hookrightarrow X_3 = -\frac{X_1}{2} - \frac{X_2}{2} = \frac{3}{2}\frac{\alpha \Delta T EI}{h} - \frac{3}{4}X_2 - \frac{X_2}{2} = \frac{3}{2}\frac{\alpha \Delta T EI}{h} - \frac{5}{4}X_2$$

$$-\frac{X_1}{3} - \frac{\alpha \Delta T EI}{h} + \frac{X_2}{2} = 0$$

$$X_1 = -\frac{3\alpha \Delta T EI}{h} + \frac{3}{2}X_2$$

$$-X_1 - \frac{3\alpha \Delta T EI}{h} - X_1 - X_2 - \frac{X_3}{2} = 0$$

$$\frac{3\alpha \Delta T EI}{h} - \frac{3}{2}X_2 - \frac{3\alpha \Delta T EI}{h} + \frac{3\alpha \Delta T EI}{h} - \frac{3}{2}X_2 - X_2 - \frac{3}{4}\frac{\alpha \Delta T EI}{h} + \frac{5}{8}X_2 = 0$$

$$\left(-\frac{3}{2} - \frac{3}{2} - 1 + \frac{5}{8}\right)X_2 = \left(-3 + \frac{3}{4}\right)\frac{\alpha \Delta T EI}{h}$$

$$\frac{-12-12-8+5}{8}X_2 = \frac{-12+3}{4}\frac{\alpha \Delta T EI}{h}$$

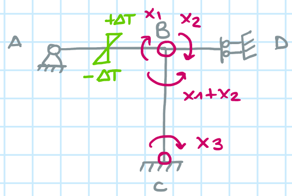
$$-\frac{27}{8}X_2 = -\frac{9}{4} \cdot \frac{8}{27}\frac{\alpha \Delta T EI}{h}$$

$$X_2 = \frac{2}{3}\frac{\alpha \Delta T EI}{h}$$

$$X_1 = -2\frac{\alpha \Delta T EI}{h}$$

$$-2 + \frac{2}{3} = \frac{-6+2}{3} = -\frac{4}{3}$$

$$X_3 = \left(\frac{3}{2} - \frac{5}{4} \cdot \frac{2}{3}\right)\frac{\alpha \Delta T EI}{h} = \frac{+9-5}{6}\frac{\alpha \Delta T EI}{h} = \frac{2}{3}\frac{\alpha \Delta T EI}{h}$$



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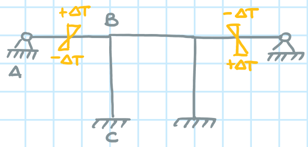
24.02.12 (B)

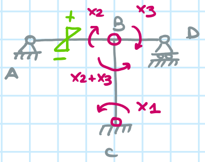
sabato 3 febbraio 2018 12:59



$$\begin{cases} g = 3 \\ v = 2(A) + 3(C) + 3(D) + 2(E) = 10 \end{cases}$$

$v - g = 7$

Associated truss structure



$$\begin{cases} g = 3 \cdot 3 = 9 \\ v = 2(A) + 4(B) + 2(C) + 1(D) = 9 \end{cases}$$

Statically determinated

$$\begin{cases} \varphi_{BA} = \varphi_{BD} \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = 0 \end{cases}$$

BA BD BC CB

$$\begin{cases} -\frac{x_2 L}{3EI} - \frac{\alpha \Delta T L}{h} = -\frac{x_3 L}{6EI} \\ -\frac{x_2 L}{3EI} - \frac{\alpha \Delta T L}{h} = \frac{(x_2+x_3)L}{3EI} - \frac{x_1 L}{6EI} \\ \frac{x_1 L}{3EI} - \frac{(x_3+x_2)L}{6EI} = 0 \end{cases}$$

$x_1 - \frac{x_3}{2} - \frac{x_2}{2} = 0$

$\hookrightarrow x_1 = \frac{x_2}{2} + \frac{x_3}{2} = \frac{x_3}{4} - \frac{3}{2}\frac{\alpha \Delta T EI}{h} + \frac{x_3}{2} = \frac{3}{4}x_3 - \frac{3}{2}\frac{\alpha \Delta T EI}{h}$

$x_2 = \frac{x_3}{2} - \frac{\alpha \Delta T \cdot 3EI}{h}$

$-x_2 - 3\frac{\alpha \Delta T EI}{h} - x_2 - x_3 + \frac{x_1}{2} = 0$

$-2x_2 + \frac{x_1}{2} - x_3 = \frac{3\alpha \Delta T EI}{h}$

$-x_3 + 6\frac{\alpha \Delta T EI}{h} + \frac{3}{8}x_3 - \frac{3}{4}\frac{\alpha \Delta T EI}{h} - x_3 = \frac{3\alpha \Delta T EI}{h}$

$\left(-1 + \frac{3}{8} - 1\right)x_3 = \left(3 + \frac{3}{4} - 6\right)\frac{\alpha \Delta T EI}{h}$

$\frac{-8+3-8}{8}x_3 = \frac{12+3-24}{4}\frac{\alpha \Delta T EI}{h}$

$-\frac{13}{8}x_3 = -\frac{9}{4}\frac{\alpha \Delta T EI}{h}$

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