NUMERO: 2307A ANNO: 2018

A P P U N T I

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MATERIA: Structural Mechanics II, Teoria + Esercizi + Temi di Esame risolti - Prof. Cornetti - Sapora

4/10/17 A

STRUCTURAL

MECHANICS II

By L.M.C.

Pietro Cornetti DISEG DOOR 1 3rd floor

BEFORE:

sforzo deformazione

STRESS, STRAIN

LINEAR ELASTICITY

STATICALLY DETERMINATE BEAM SYSTEM

STATICALLY INDETERMINATE BEAM SYSTEM (METHOD OF FORCES)

NOW:

(1°) ☒ 1D STRUCTURES

⊛ ☒ STATICALLY INDETERMINATE STRUCTURES:

- ☒ • METHOD OF FORCES
- ☒ • METHOD OF DISPLACEMENT (degli spostamenti)
- ☒ • MIXED METHOD

↗ from RC course

☒ HOW TO IMPLEMENT THE METHOD (THE SOLUTION) IN A PC CODE.

☒ CURVED BEAMS (ARCHES)

☒ BEAM ON ELASTIC FOUNDATION

(2°) ☒ 2D STRUCTURES

☒ PLATES

☒ MEMBRANES

☒ SHELLS (TANKS under pressure)

EXAM

1 EXERCISE WRITTEN TO BE SOLVED WITH ONE OF THE METHODS ⊛

+ ORAL THEORY

+ EXERCISE IN THE LAIB

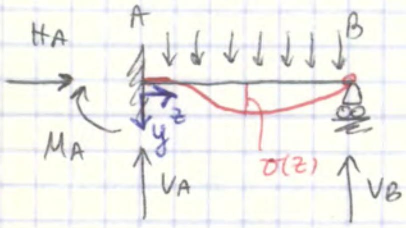
BOOK

"Calcolo dei telai piani" Carpinteri

"Structural Mechanics" Carpinteri

"Theory of plates and shells" Timoshenko

METHOD OF FORCES



① METHOD: integration of the deflection curve of the elastic line:

derivazione

$$\frac{d^4 v}{dz^4} = \frac{q(z)}{EI}$$

I = moment of inertia

E = elastic coeffic.

$$\frac{d^3 v}{dz^3} = -\frac{T(z)}{EI} \qquad \frac{d^2 v}{dz} = -\varphi(z)$$

T = shear force

M = moment

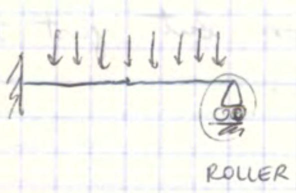
$$\frac{d^2 v}{dz^2} = -\frac{M(z)}{EI}$$

v(z) = vertical displacement

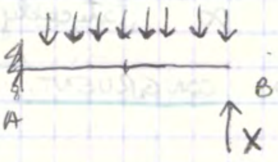
+ BOUNDARY CONDITION which must be suitable:

$$\begin{cases} v_A(0) = 0 \\ \varphi_A(0) = 0 \rightarrow v'(0) = 0 \end{cases} \qquad \begin{cases} v_B(\ell) = 0 \\ M_B(\ell) = 0 \rightarrow v''(\ell) = 0 \end{cases}$$

② METHOD OF FORCES



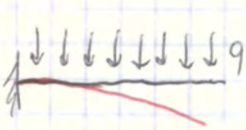
delete 1 constrain and put a force x + conditions



$v_B = 0$

due to the load

⇒ I have the equation: $v_B(q) + v_B(x) = 0$ ⇒ divido in 2 sistemi

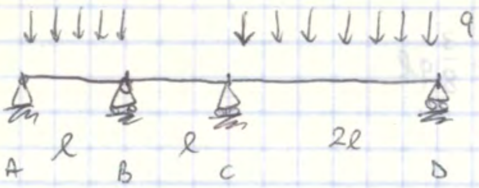


$$v_B(q) = \frac{q\ell^4}{8EI}$$

ℓ raised to 3

$$v_B(F) = \frac{F\ell^3}{3EI} \Rightarrow v_B(x) = -\frac{x\ell^3}{3EI}$$

Ex

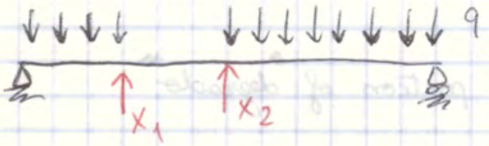


$g = 3$

$v = 5$

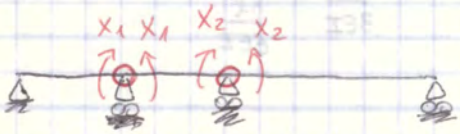
⇒ Twice stat indet.

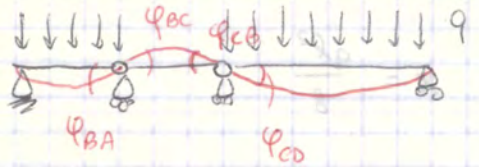
Remove B and C constraint



$$\begin{cases} v_B = 0 & v_B(q) + v_B(X_1) + v_B(X_2) = 0 \\ v_C = 0 & v_C(q) + v_C(X_1) + v_C(X_2) = 0 \end{cases}$$

I can choose to put 2 hinges (it's easier to calculate)





$$\begin{cases} \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = \varphi_{CD} \end{cases} \Rightarrow \begin{cases} ① \; \varphi_{BA}(q) + \varphi_{BA}(X_1) = \varphi_{BC}(X_1) + \varphi_{BC}(X_2) \\ ② \; \varphi_{CB}(X_1) + \varphi_{CB}(X_2) = \varphi_{CD}(q) + \varphi_{CD}(X_2) \end{cases}$$

Now the load q influence only part of unknowns

"Prima la q influenzava entrambe le forze X_1 e X_2, ma ora solo le φ più vicine a loro".

① $$\frac{ql^3}{24EI} \ominus \frac{X_1 l}{3EI} = \frac{X_1 l}{3EI} + \frac{X_2 l}{6EI}$$

CLOCKWISE
⇓
NEGATIVE } I can choose (↷ + or + ↶)

(also if it is "CONCORDE" CON LA FORZA) ? THE ROTATION OF THE ANGLE φ

Ⓐ degrees before $g = 3$

$v = 3 + 2 + 1 = 6$

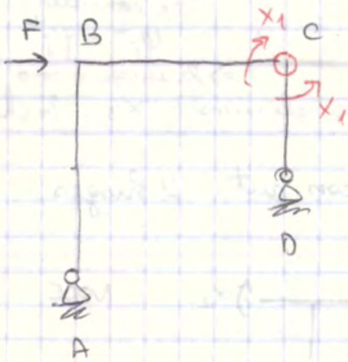
Three times stat indet

But degrees with 2 hinges ⇒ static. determ.

$$\begin{cases} \varphi_{BA} = \varphi_{BD} \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_{C} = 0 \end{cases}$$

If the truss struct is statically determinated ⇒ I can neglect the axial deformation. And also NODES DON'T MOVE, so it's called ROTATING - NODE FRAME (they can only rotate). We can solve only using congruent equations.

Ⓑ

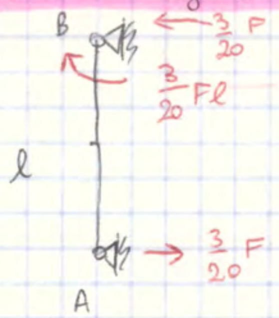


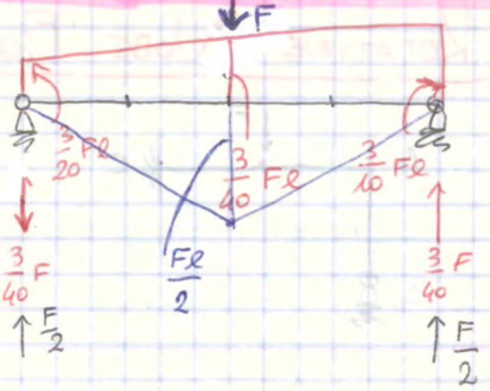
$\varphi_{CB} = \varphi_{CD}$ → I can find it's formula from tables (formulari delle strutture tipo).

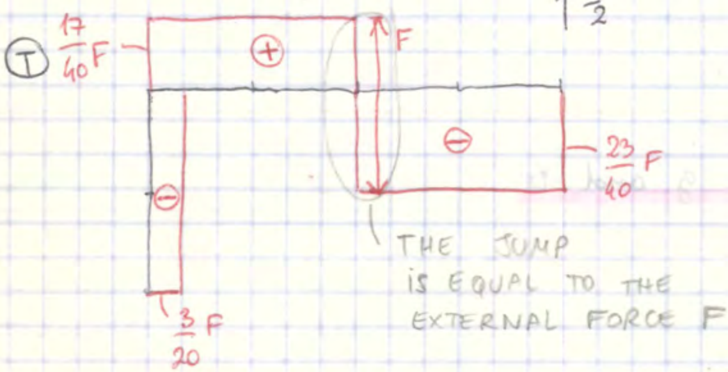
↓

I don't know it's formula/value

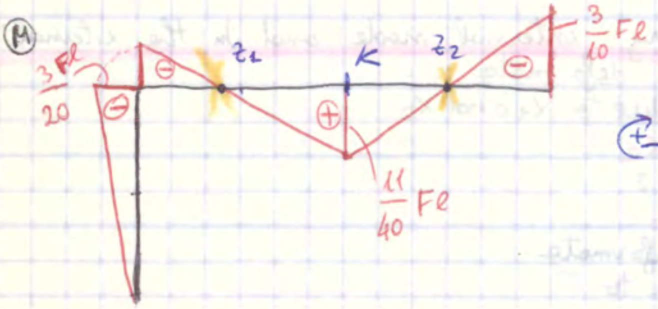
5) Diagrams → find the reactions : SUPERPOSITION METHOD





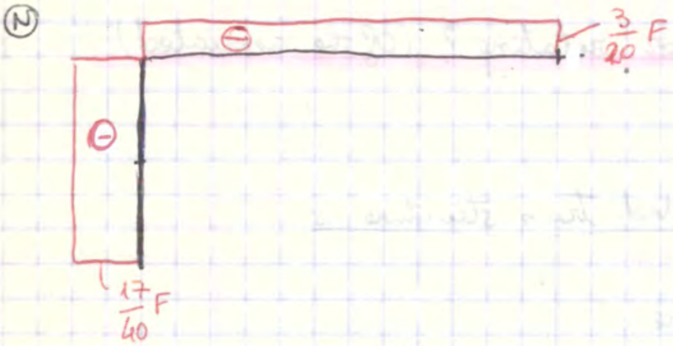


THE JUMP IS EQUAL TO THE EXTERNAL FORCE F



PAY ATTENTION TO CONCENTRATED FORCES IN THE MIDDLE OF THE BEAM:

$$M_K = -\frac{3}{20}Fl + \frac{17}{40}Fl = \frac{-6+17}{40}Fl = \frac{11}{40}Fl$$

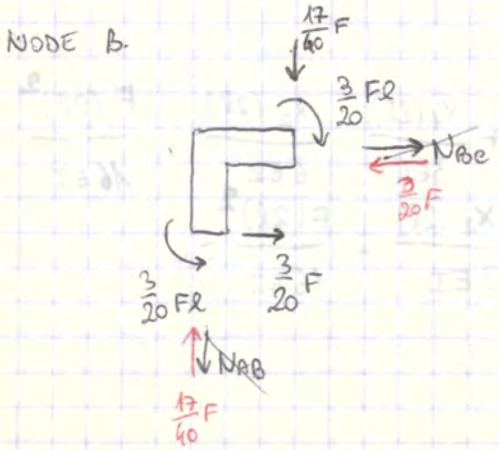


! Perché quello che ho trovato sono le reazioni e i diagrammi dei momenti X_1 e X_2, X_3... a cui devo sovrapporre le reazioni / diagrammi delle F, m, q esterne.

$z_1 = \frac{3}{34}l$

$z_2 = \frac{34}{23}l$ B → C

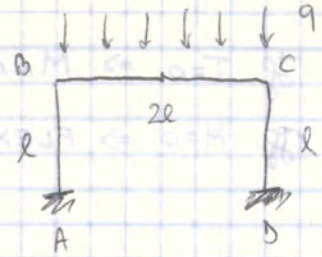
oppure $z_2 = \frac{12}{23}l$ B ← C



$N_{BC} = -\frac{3}{20}F$

$N_{AB} = -\frac{17}{40}F$

Ex

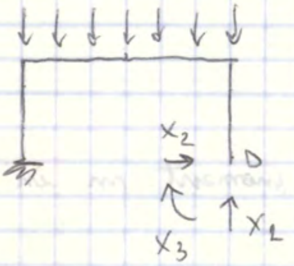


$g=3$

$v=3+3=6$

3 times stat. indet

⇒ Associated truss struct:



$u_D=0$

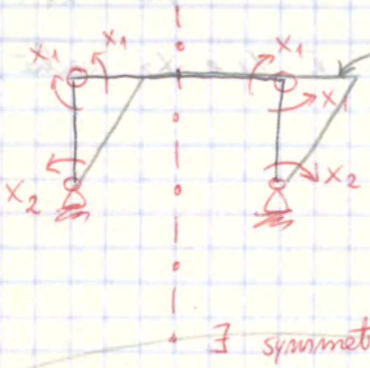
$v_D=0$

$\varphi_D=0$

⇒ Evaluate M_0, M_1, M_2, M_3

Compute integrals → ≈ 9 integrals

⇒ What happens with "hinge" method:



MECHANISM BUT ANTISYMMETRICAL

↓

BUT IT MUST BE SYMM.

$g = 3\times3 = 9$

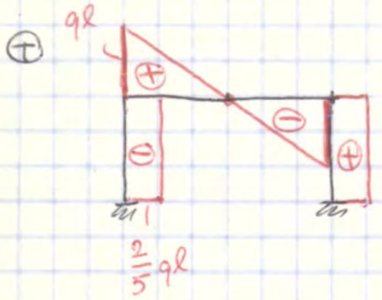
$v = 4\times2 = 8$

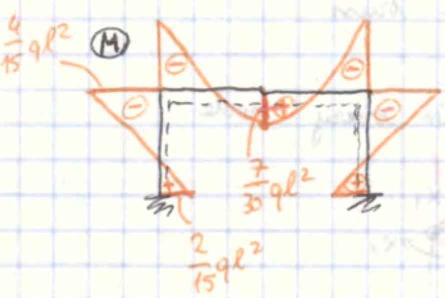
⇒ It's a mechanism.

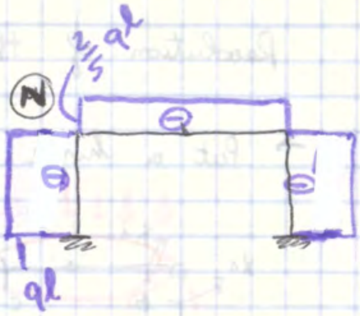
∃ symmetry ⇒ just 2 unknowns x_1 and x_2 !

It can't exist ⇒ So because of symmetry it's a rotating-node frame!

$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BC} \\ \cancel{\varphi_{CB} = \varphi_{CD}} \rightarrow \text{are useless because of symmetry} \\ \cancel{\varphi_D = 0} \end{cases}$$



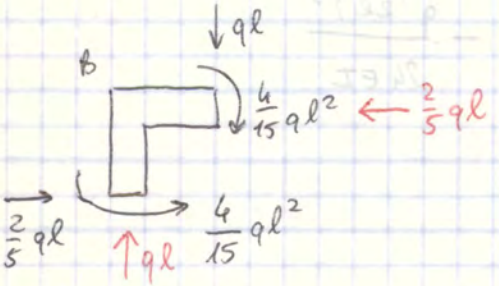




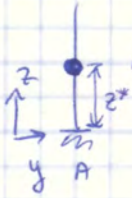
ANTYSIMMETRICAL

If a function $Y(x)$ is symmetrical even $\Rightarrow$ $Y(x)'$ is odd and $Y''(x)$ is even again and $Y'''(x)$ is odd.

$\Rightarrow$ $M(z)$ is even $\Rightarrow$ $T(z)$ is odd.



! AT which z^*, $M(z^*)=0$?

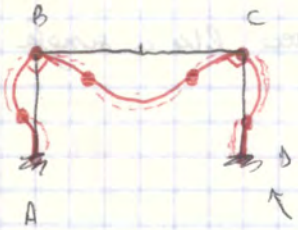


$M(z^*) = 0 = \frac{2}{15}ql^2 - \frac{2}{5}qlz^*$

$\Rightarrow \frac{1}{3}l = z^*$

$M(z') = 0 = -\frac{4}{15}ql^2 + qlz'$

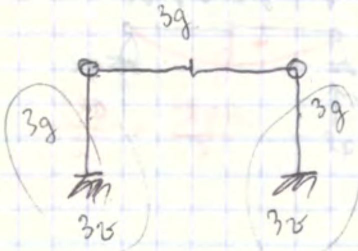
$\Rightarrow \frac{4}{15}l = z'$



Tangent vertical.

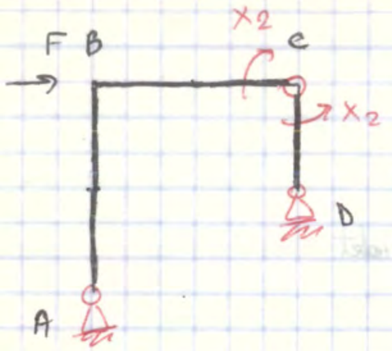
A, B, C, D don't move (also because it's rotating-node frame)

!



$g = 9$

$v = \cancel{2}+2+3+3 = 10$



$g=6$

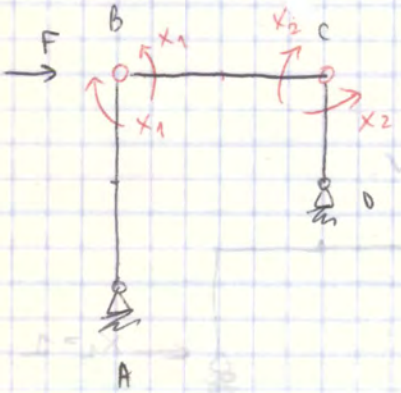
$v=6$

$\varphi_{CB} = \varphi_{CD}$

→ Put a hinge in C ⇒ but it's still difficult

→ Another method :

Put a hinge in each node :



$g = 9$

$v = 8$

But it's 1 hypostatic

↳ it's kinematical mechanism.

⇛ I can do 2 corrections:

① Say that X_1 and X_2 are in equilibrium through 2 ways

- → cardinal equation
- or → PLV

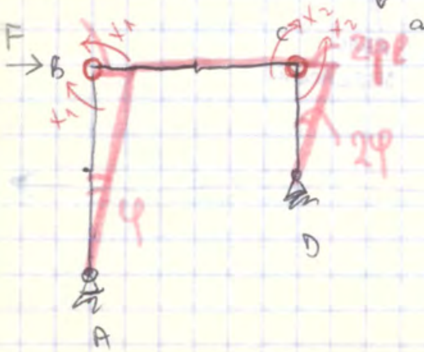
PLV : it's sufficient 1 variable to fix the mechanism :

↓

arbitrary angle φ (small variation)

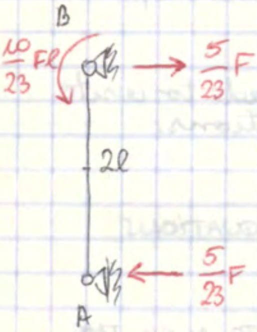
⇒ $tg\varphi \cong \varphi$

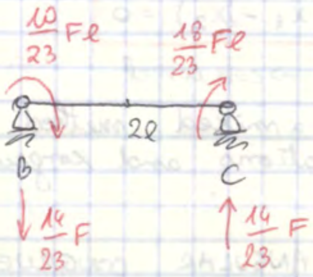
2φl

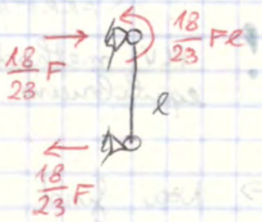


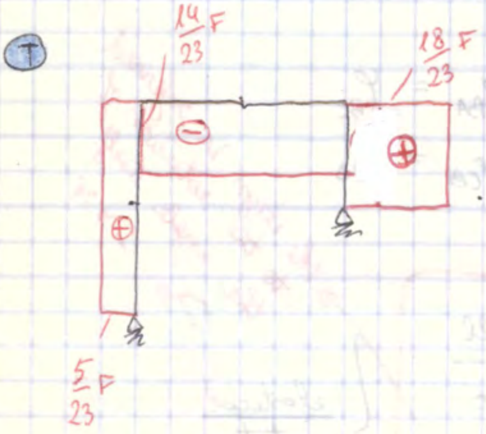
There is a rotation in A with an angle φ. B moves of a distance of 2φl so also C moves by 2φl because it's a rigid traslation. So D rotates by 2φ.

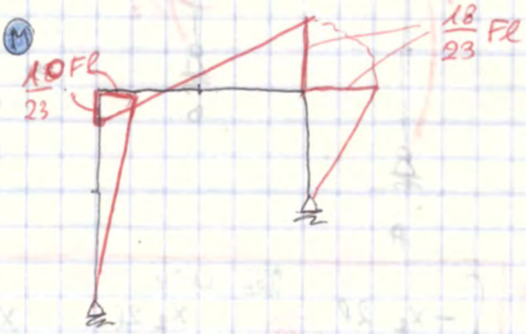
$$\Rightarrow \begin{cases} X_1 = -\dfrac{10}{23}F\ell \\ X_2 = +\dfrac{18}{23}F\ell \\ \varphi = +\dfrac{22}{69}\cdot\dfrac{F\ell^2}{EI} \end{cases}$$

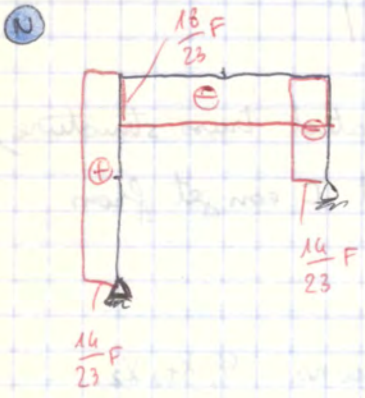


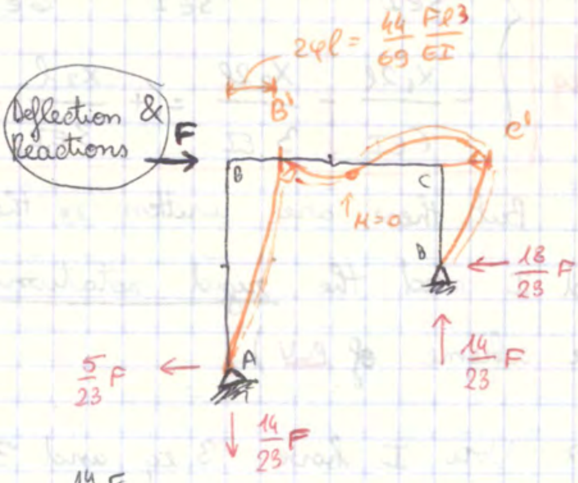


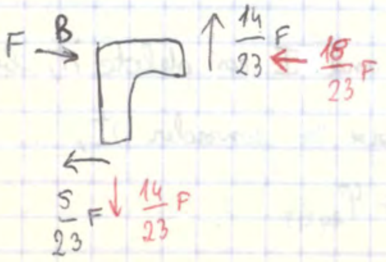


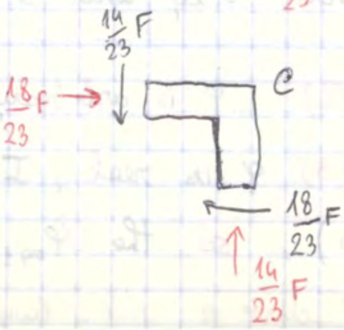




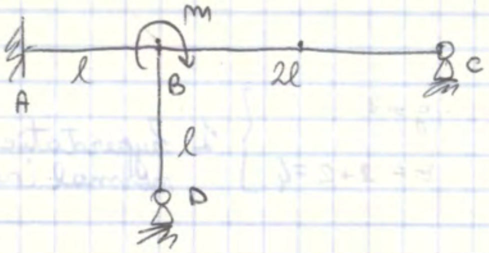








Ex

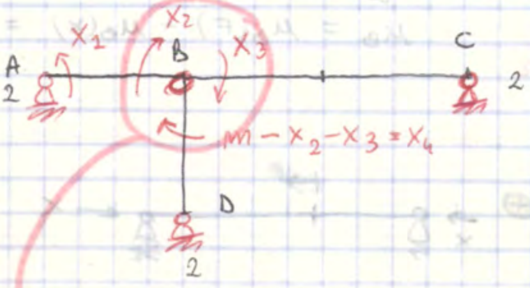


$g = 3$

$v = 7$

⇒ redundant reactions are 4.

⇒



$g = 9$

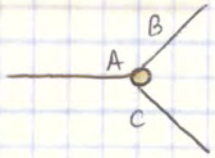
$v = 6 + 4 = 10$

NR DI VINCOLI della cerniera nel nodo

$$2 \cdot (n-1) = 2 \cdot (3-1) = 4$$

n = numero di aste concorrenti nel nodo.

ex:



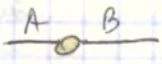
$n = 3$

In order to evaluate the number of constrains you have to count how many kinematical equations you can write:

$$\begin{cases} u_A = u_B \\ v_A = v_B \\ u_A = u_C \\ v_A = v_C \end{cases}$$

! Guarda strutture reticolari

Infact:



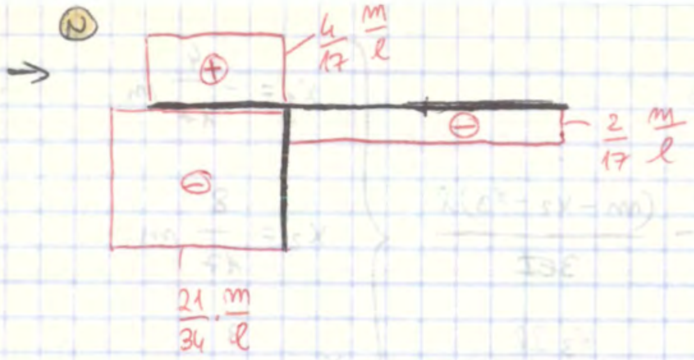
$n = 2$

⇒ $\begin{cases} u_A = u_B \\ v_A = v_B \end{cases}$ → only 2

⇒ $v = 2 \cdot (n-1) = 2(2-1) = 2$

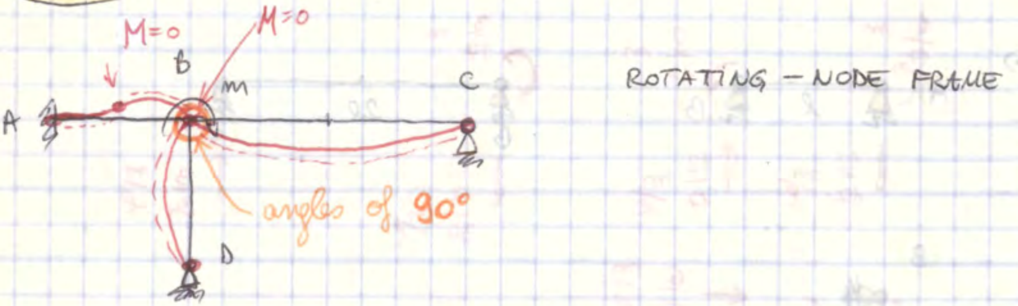
⇝ Return to our problem: Write congruence equations:

$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BD} \\ \varphi_{BD} = \varphi_{BC} \end{cases}$$



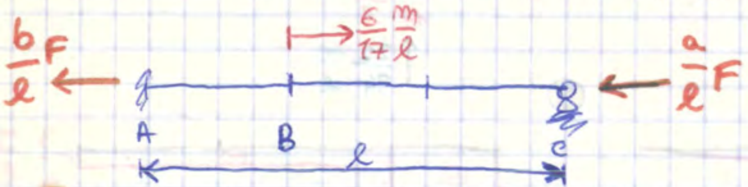
"La struttura risultava ipostatica per la normale (forza (assiale)), e avendo trovato prima una soluzione generale per la strutt. 1 volta iperstatica per la normale, l'abbiamo applicata anche in questo caso."

Deflection



How to find N_{BC} and N_{AB}? (STEPS):

① Consider the beam ABC with $\frac{6}{17}\frac{m}{l} = F$ applied in B.



② Using the scheme, we know that in A we have $\leftarrow \frac{b}{l}F$ and in C we have $\leftarrow \frac{a}{l}F$

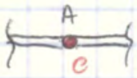
③ The beam ABC must be considered as long as l → so $a = \frac{1}{3}l$ and $b = \frac{2}{3}l$

④ So the reactions are:

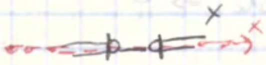
$$\leftarrow N_{BC} = \frac{a}{l}F = \frac{1}{3}\cdot\frac{6}{17}\cdot\frac{m}{l} = \frac{2}{17}\frac{m}{l}$$

$$\leftarrow N_{AB} = \frac{b}{l}F = \frac{2}{3}\cdot\frac{6}{17}\cdot\frac{m}{l} = \frac{4}{17}\cdot\frac{m}{l}$$

CENTER OF RELATIVE ROTATIONS



c = A

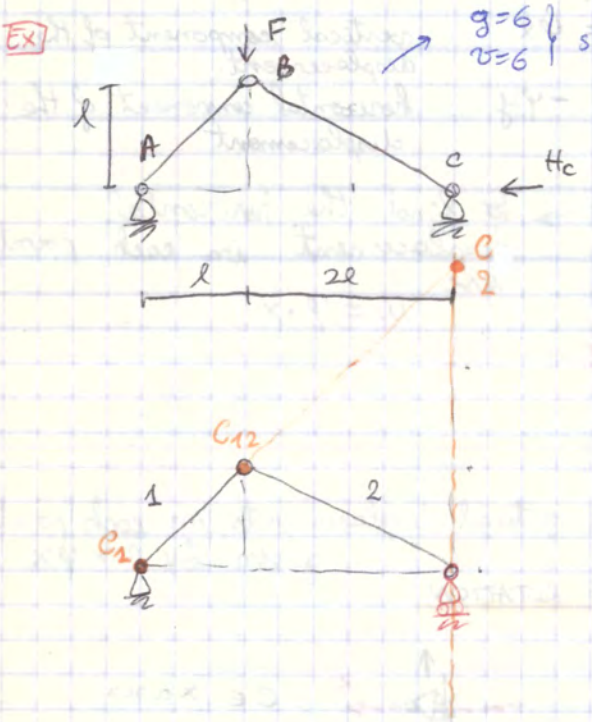


c ∈ x

1^{ST} TH. OF KINEMATICAL CHAIN → C_i, C_j, C_{ij} must be aligned
2 corps i, j

2^{ND} TH OF " .... → C_{ij}, C_{ik}, C_{jk} must be aligned.
3 corps i, j, k

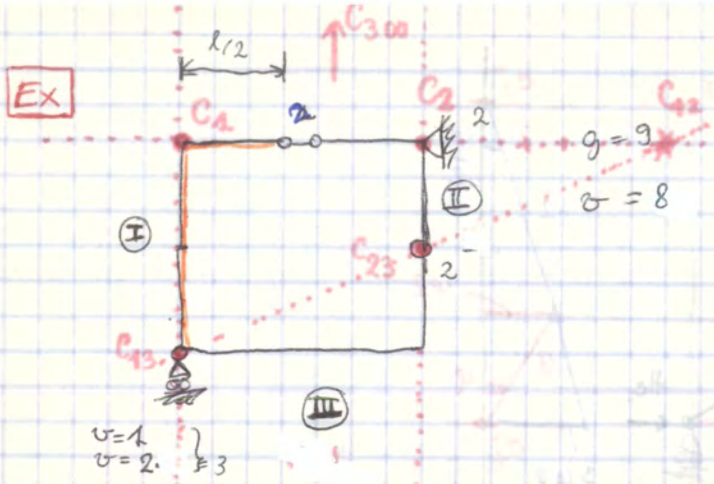
Ex



g=6 } static determ.
v=6

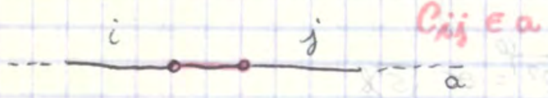
degradation of constrain c. in order to find the reaction Hc, by applying the PLV =0

11/10/2017



Is it a kinematical chain? Yes → 1 degree of freedom.

⇒ Determine the POLE OF MOTION.



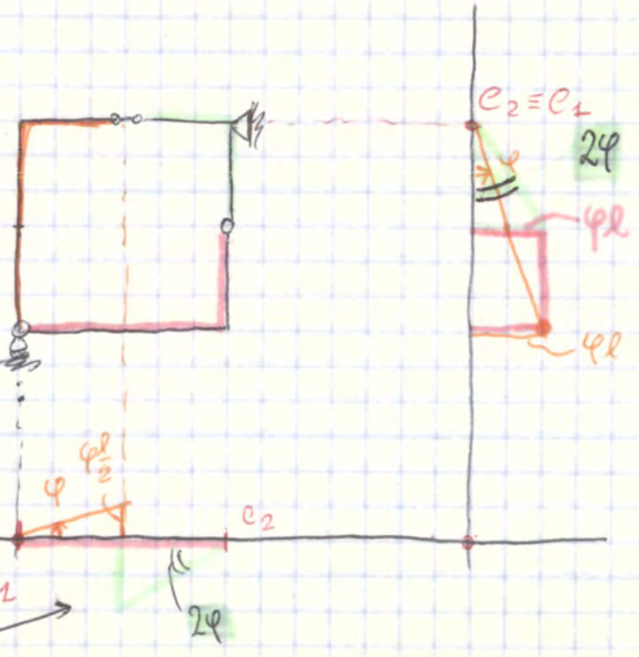
On the same line:

C_{12}, C_{23}, C_{13}

C_{12}, C_1, C_2

C_{13}, C_1, C_3

! $C_3 \to \infty$ ⇒ Body 3 traslates in the ⊥ direction WRT the direction of C_3. ⇒ Horizontally.



∃ discontinuity because -o-o- allows rotation between beams

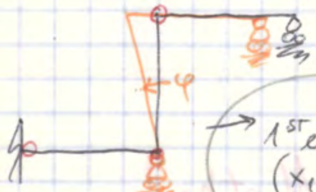
⊕ II can go down ⇒ put 

⊕ structure can move horizontally ⇒ put 

$m+2l = 1+2\cdot 2=5$

⇒ 5 unknowns and 5 equations : $X_1, X_2, X_3, \varphi_1, \varphi_2$

⊕ ! Don't forget the moments (from the degradation)
|||
putting hinges.



1st equil equat. ⊕
(x_1, x_2, x_3)

! Also the kinematical unknowns : φ and θ derived from the 2 mechanisms.

(l) equil equations

2nd equil equat.
(x_1, x_2, x_3)

⊕ angular congruence equation.

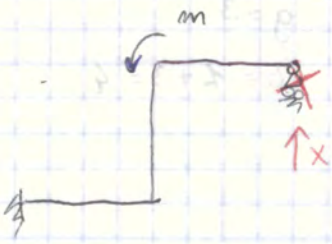
$$\begin{cases} \varphi_A = 0 \\ \varphi_{BA} = \varphi_{BC} \\ \varphi_{CB} = \varphi_{CD} \end{cases}$$

(m + l) congruence equations

⇒ 5 unknowns ⇔ 5 equations $(X_1, X_2, X_3, \varphi, \theta)$

→ BUT THE MIXED METHOD IS TOO LONG.

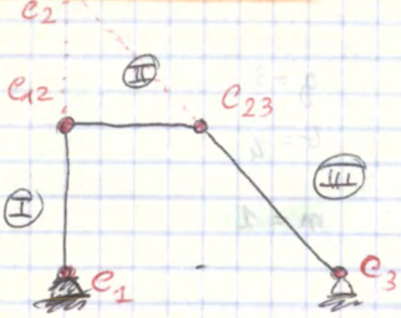
⇒ It's better to use the METHOD OF FORCES.



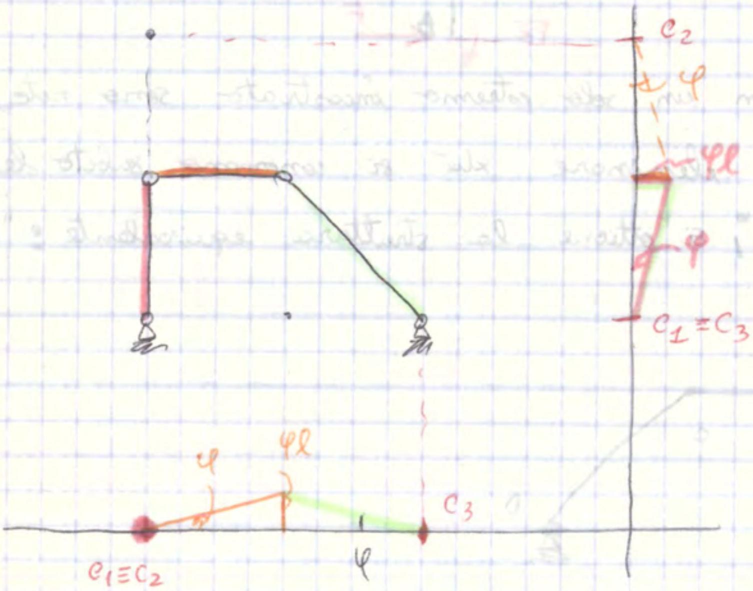
$\uparrow X + v_D = 0$

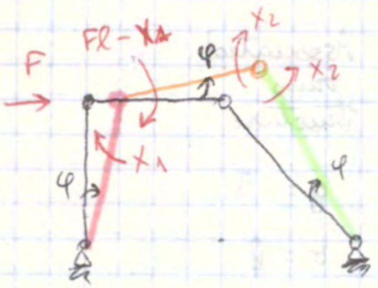
1 unknown ⇒ 1 equat

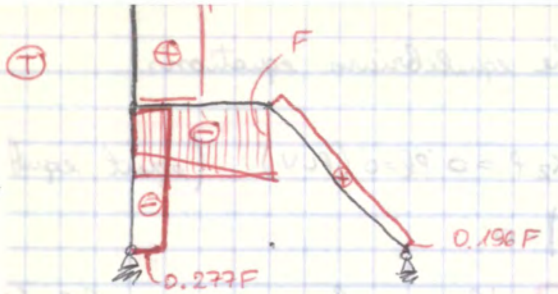
② Define the mechanism.

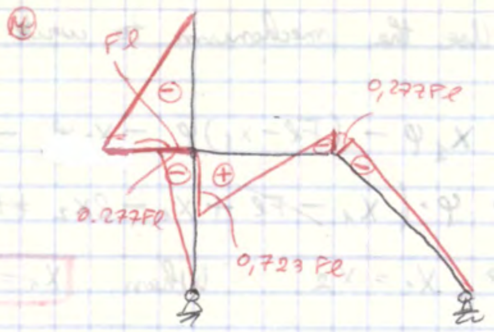


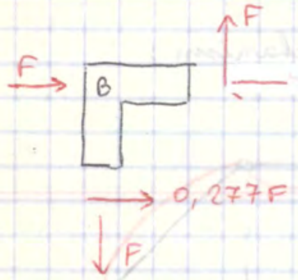
" In questo caso C_{13} non c'è perché non è una struttura chiusa".

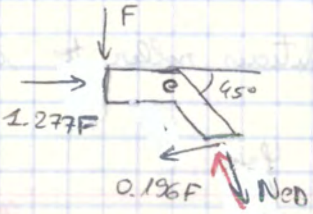








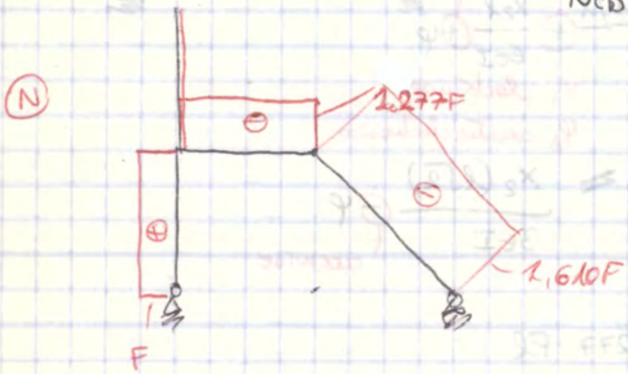


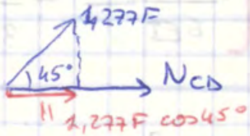


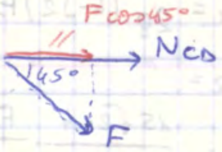
EQUILIBR.: $N_{CD} + \frac{\sqrt{2}}{2} \cdot 1,277F + \frac{\sqrt{2}}{2}F = 0$

$N_{CD} = -1,610F$

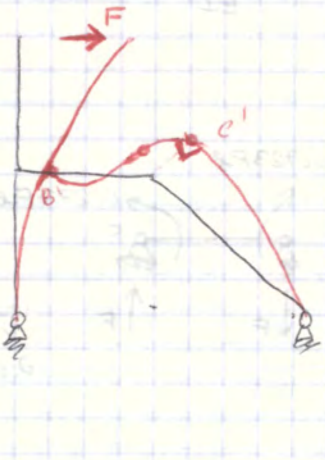
↓ This is the projection of 1,277F and F on the line N_{CD}:



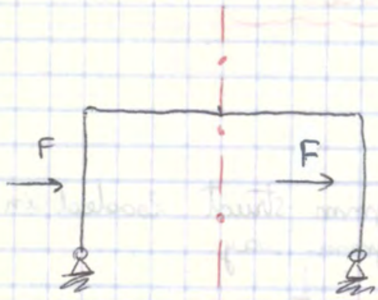




Defl. curve



Symmetrical structure loaded in ANTISYMMETRICAL way



v antisymm
φ symm
M antisymm
T symm

⇓

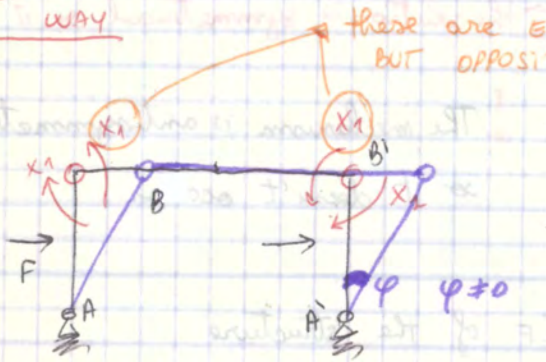
on the axis of symm:

$v = 0$

$M = 0$

2 WAYS ⇐

1st WAY



– Put a hinge in all the nodes

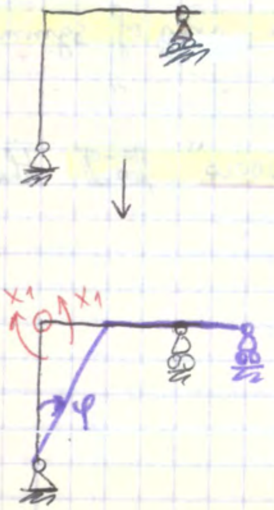
→ All structure but exploiting the symmetry

$$\left\{\begin{array}{l}\varphi_{BA} = \varphi_{BB'} \\ PLV\end{array}\right\} \rightarrow (X_1, \varphi)$$

2nd WAY Study HALF structure

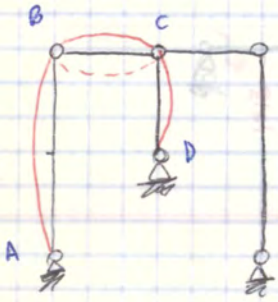
→ Put a roller on the axis

↳ $v = 0$
$M = 0$



$$\frac{-X_1 \cdot 2\ell}{3EI} + \varphi = \frac{X_1 \ell}{3EI} - \frac{X_2 \ell}{6EI}$$

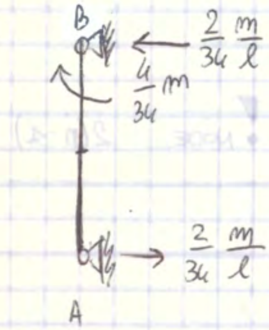
$$\frac{-X_1 \ell}{6EI} + \frac{X_2 \ell}{3EI} = \frac{(m - 2X_2)\ell}{3EI} + 2\varphi$$

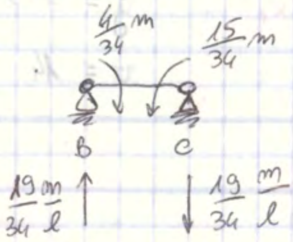


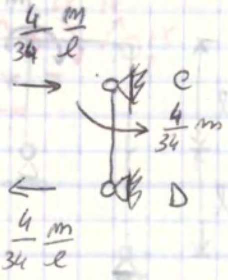
$$X_1 = \frac{4}{34} m$$

$$X_2 = \frac{15}{34} m$$

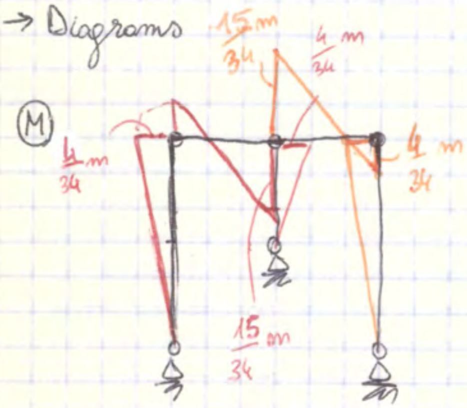
$$\varphi = \frac{3}{68} \cdot \frac{m\ell}{EI}$$



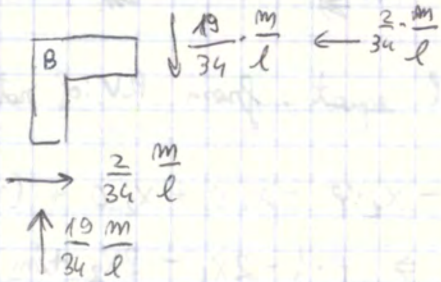


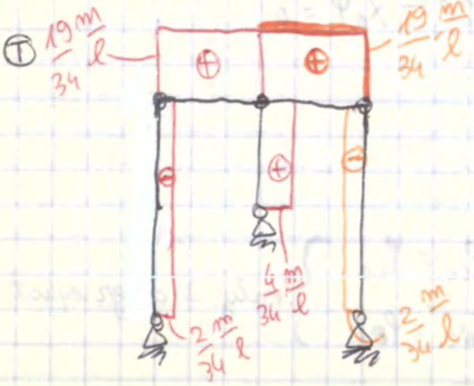


→ Diagrams



To draw the yellow part, remember that the diagram is antisymmetric.





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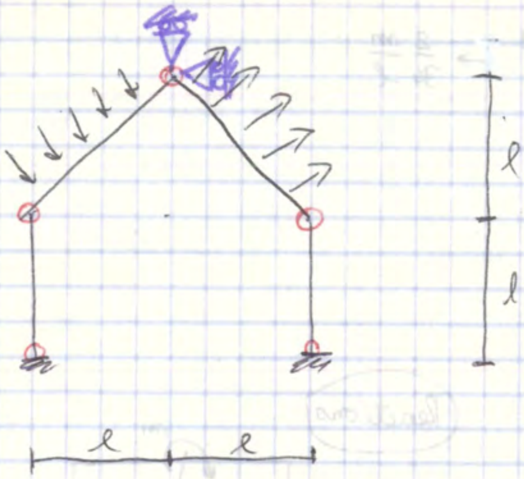


$g = 3$

$v = 3+3 = 6$

↓ PUT HINGES

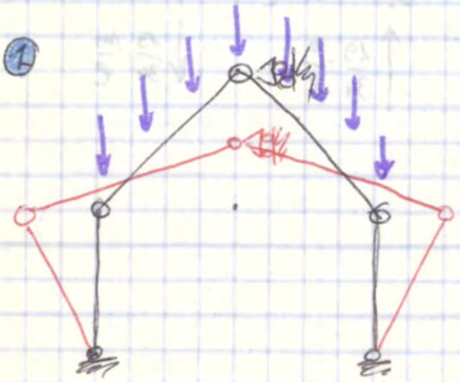
$g = 12$

$v = 10$

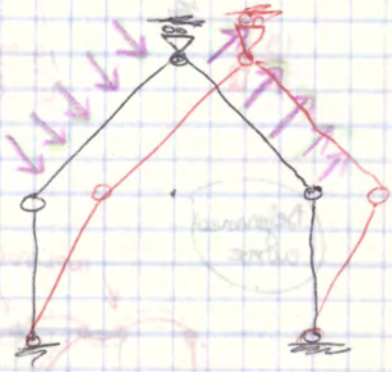
→ 2 hypostatic

To avoid mechanisms I put 2 rollers ⇒ struct statically determined :

①



②



SYMMETRICAL

↓

The load will be

ANTISYMMETRICAL

↓

The load will be the initial one

↓ pass to the half of the structure

①

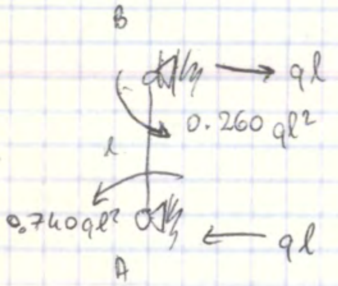
②

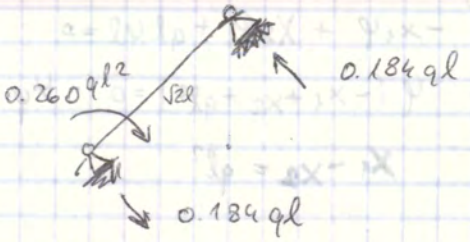
$$\Rightarrow \quad X_1 = \frac{8+9\sqrt{2}}{28}\, ql^2 \simeq 0.740\, ql^2$$

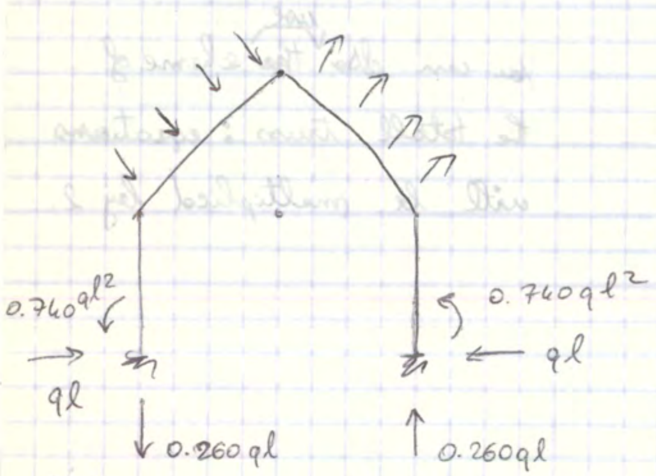
$$X_2 \cong -0.260\, ql^2$$

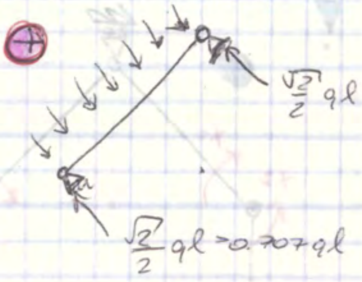
$$\varphi \cong 0.2035\, \frac{ql^3}{EI}$$

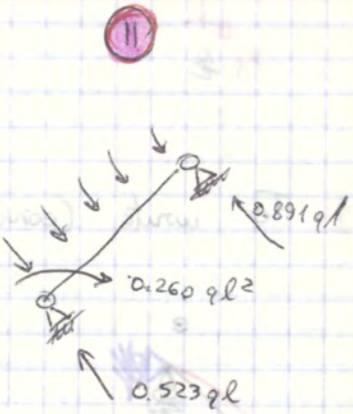
⇒ Diagrams + reactions



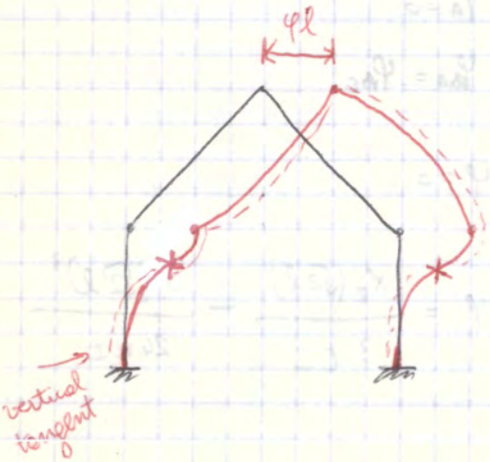




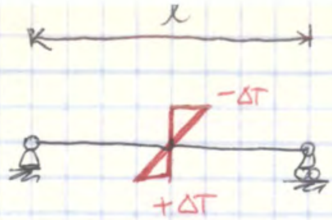




Deformation:

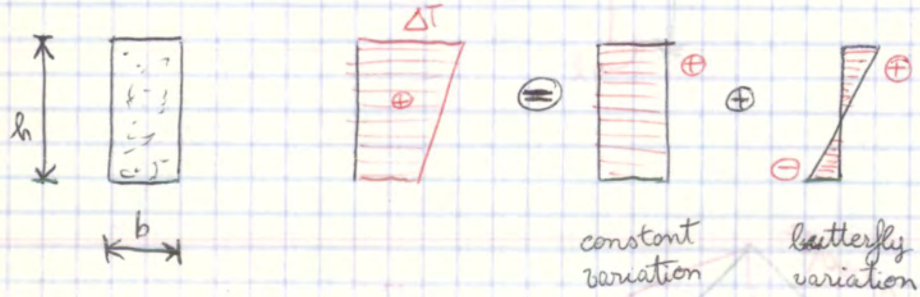


Ex

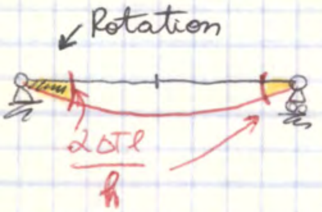


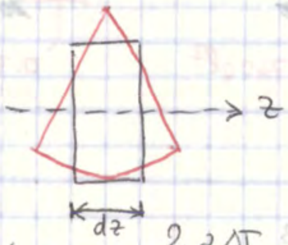
α = coefficient of thermal expansion.

Rectangular CROSS SECTION



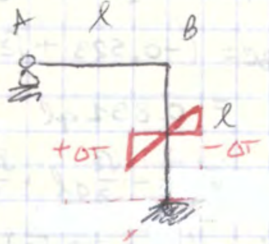
If the beam is statically determined, it can accomplish the ΔT without any creation of other stress.





Effect of temperature : $\chi_{term} = \frac{2\alpha\Delta T}{h}$

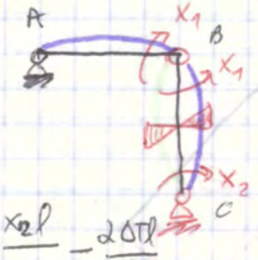
Ex



$g = 3$

$v = 2+3 = 5$

Associated truss structure



⇒ Congruence equations :

$$\begin{cases} \varphi_{BA} = \varphi_{BC} \\ \varphi_C = 0 \end{cases} \qquad \begin{cases} \frac{-X_1 l}{3EI} = \frac{X_1 l}{3EI} + \frac{X_2 l}{6EI} - \frac{\alpha\Delta T l}{h} \\ \bigcirc\!\!\!\!-\,\frac{X_1 l}{6EI} \;\bigcirc\!\!\!\!-\,\frac{X_2 l}{3EI} \;\boxed{+}\;\frac{\alpha\Delta T}{h} = 0 \end{cases}$$

clockwise

counterclockwise

18/10/2017

METHODS

FORCES

DEGREE

$(v-g)$ degree of static indeterminacy

UNKNOWNS

$(v-g)$ constraint reactions statically indeterminate

STRATEGY

Choose among the $\infty^{(v-g)}$ static admissible solution the only one which is also congruent.

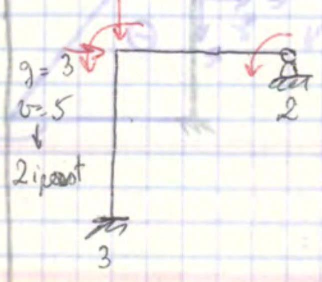
EQUATIONS

$(v-g)$ congruence equations dual to the unknown reactions.

DISPLACEMENTS

DEGREE

s : degree of kinematical indeterminacy.



$s = 1$ ROTATION

$s + 3$ in the NODE

$s = 4$

In the nodes of the frame you don't know the displacement

⇒ Here there are 4 unknowns

UNKNOWNS

s nodal displacement (generalized)

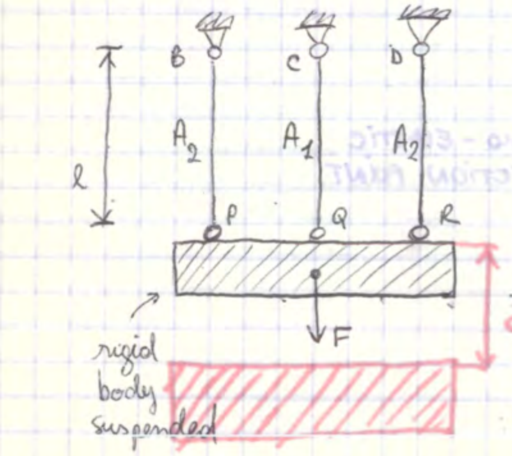
STRATEGY

Choose among the $\infty^{(s)}$ kinematically admissible solutions, the only one which is also balanced.

EQUATIONS

s equil. equations which are dual to the unknown nodal displacements

Ex Parallel arranged bar system.



Symmetric.

How much force is carried on A_1 and on A_2?

δ = DISPLACEMENT = ?

1) Find reactions in B, C, D

2) Find $\delta = \frac{R_{B,C,D}}{K}$

$$\frac{F}{2EA_2}\ell - \frac{X\ell}{EA_1} - \frac{X\ell}{2EA_2} = 0$$

$k_1 = \frac{EA_1}{\ell}$

$k_2 = \frac{EA_2}{\ell}$

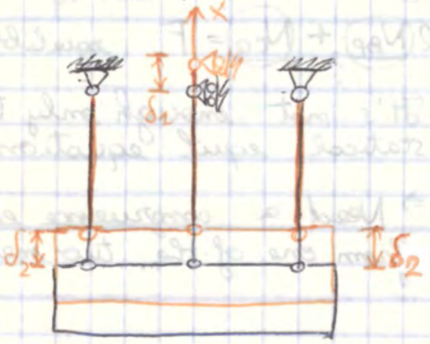
$$\frac{F}{2k_2} - \frac{X}{k_1} - \frac{X}{2k_2} = 0$$

$$(2k_2 + k_1)X = k_1 F$$

$$X = \boxed{\frac{k_1}{k_1 + 2k_2}} F \quad \simeq \text{analoga alla formula del cls}$$

< 1

Ⓐ "Gli spostamenti non si sommano" → Displacements of the same type don't sum up.



$$-\frac{\cancel{X/2}}{EA_2}\ell$$

È SBAGLIATO

$$\Rightarrow \delta_{\text{FORCES}} = \frac{N_{CQ}}{k_1} = \frac{X}{k_1} = \frac{F}{k_1 + 2k_2} = \frac{F}{k_{TOT}}$$

$$X = N_{CQ} = \frac{k_1}{k_1 + 2k_2} F$$

$$N_{BQ} = \frac{F - N_{CQ}}{2} = \frac{k_2}{k_1 + 2k_2} F$$

⇒ 1) Eq of equilibrium (vertical) + congruence $v_C^{X,F} = 0$ ⇒ 3×3
2) Find $X = N_{CQ}$, then $N_{BP} = N_{DR}$
3) Find δ.

$$\boxed{\begin{array}{l} 1 = \text{UNKNOWN} \\ \quad (\text{M. DISPL}) \end{array}}$$

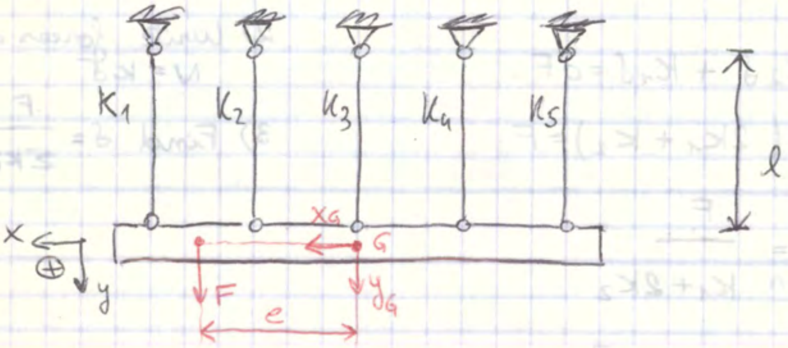
⇓

$$\sum_{i=1}^{n} N_i = F \qquad \sum_{i=1}^{n} k_i \delta = F$$

$$\Rightarrow \begin{cases} \delta = \dfrac{F}{\sum_{i=1}^{n} k_i} \quad , \quad N_i = \left(\dfrac{k_i}{\sum_{i=1}^{n} k_i} \right) F \\ k_i = \text{axial stiffness of the bar} \Rightarrow k_i = \dfrac{EA_i}{l_i} \end{cases}$$

COEFFICIENT OF DISTRIBUTION

$\boxed{Ex}$

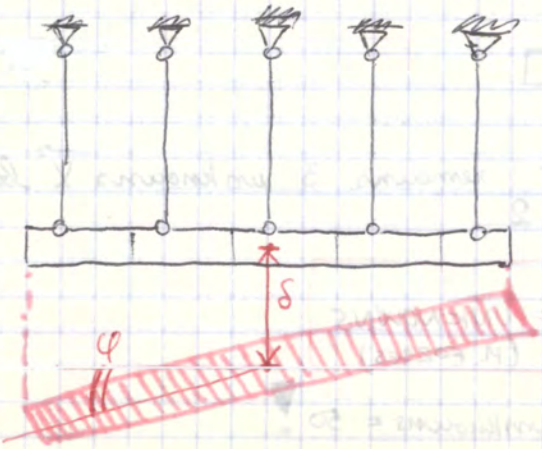


$$k_i = \frac{E_i A_i}{l} \qquad s = 2$$

e = eccentricity.

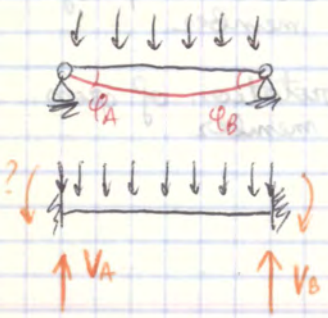
NO SYMMETRY

⇒ Rigid Body can move and rotate

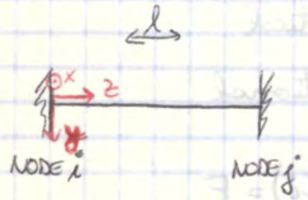


Method of forces → rotations for simply supported beams

Method of displacement → reactions for double clamped beams due to imposed displ.



E
A
I



E, A, I.

reactions

In nodes i and j I take the same positive directions.

$N(0) = -N_i$ $\qquad N(l) = N_j$

$T(0) = -T_i$ $\qquad T(l) = T_j$

$M(0) = -M_i$ $\qquad M(l) = M_j$

Matrix stiffness of the beam

$$\underbrace{\begin{pmatrix} M_i \\ T_i \\ N_i \\ M_j \\ T_j \\ N_j \end{pmatrix}}_{\text{VECTOR OF REACTIONS}} = \underbrace{\begin{pmatrix} & & & & & \\ & & & & & \\ & & ? & & & \\ & & & & & \\ & & & & & \\ \underbrace{\;\;}_{C1} & \underbrace{\;\;}_{C2} & & & & \end{pmatrix}}_{\text{STIFFNESS MATRIX}} \cdot \begin{pmatrix} \varphi_i \\ v_i \\ w_i \\ \varphi_j \\ v_j \\ w_j \end{pmatrix}$$

$$\Rightarrow \begin{cases} M = -EI\,v''(z) = -EI\left(-\dfrac{6\varphi_i}{\ell^2}\cdot z + \dfrac{4\varphi_i}{\ell}\right) = \\ \quad = EI\,\varphi_i\left(\dfrac{6}{\ell^2}\cdot z - \dfrac{4}{\ell}\right) \\ \\ T = \dfrac{6EI}{\ell^2}\cdot\varphi_i \quad \text{constant} \end{cases}$$

$$\Rightarrow M_i = -M(0) = -\frac{4EI}{\ell}\varphi_i$$

$$T_i = -T(0) = -\frac{6EI}{\ell^2}\varphi_i$$

$$N_i = -N(0) = 0$$

$$M_j = M(\ell) = \frac{2EI}{\ell}\varphi_i$$

$$T_j = T(\ell) = \frac{6EI}{\ell^2}\varphi_i$$

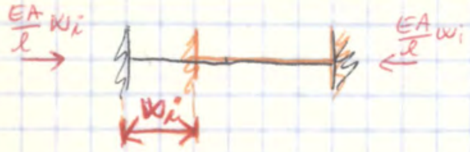
$$N_j = N(\ell) = 0.$$

→ In this way we found the reactions in the double clamped beam if only $\varphi_i \neq 0$

⇒ the column 1 will be:

$$\begin{matrix} \dfrac{4EI}{\ell} \\ \\ \dfrac{-6EI}{\ell^2} \\ \\ 0 \\ \\ \dfrac{2EI}{\ell} \\ \\ \dfrac{6EI}{\ell^2} \\ \\ 0 \end{matrix}$$

→ OBTAIN COLUMN 3 [C3] : $w_i \neq 0$



$N = k \cdot \delta = k \cdot w = \frac{EA}{\ell} w$

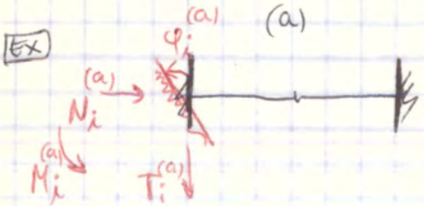
$N_i = + \frac{EA}{\ell} w_i \qquad N_j = - \frac{EA}{\ell} w_i$

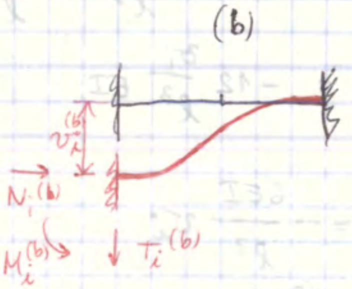
$T_i = M_i = T_j = M_j = 0$

$\Rightarrow \boxed{C3} : \begin{matrix} 0 \\ 0 \\ \frac{EA}{\ell} \\ 0 \\ 0 \\ -\frac{EA}{\ell} \end{matrix}$

⇒ [K] is symmetrical as a consequence of Betty's th.

Ex





$F^{(a)} \delta^{(b)} = F^{(b)} \delta^{(a)}$

$\Rightarrow T_i^{(a)} \cdot v_i^{(b)} = M_i^{(b)} \cdot \varphi_i^{(a)}$ → only forces working here

If $v_i^{(b)} = \varphi_i^{(a)} = 1$

$\Rightarrow T_i^{(a)} = M_i^{(b)}$

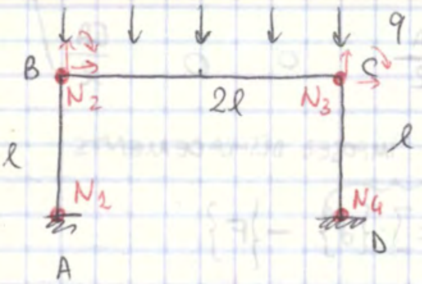
$\boxed{K_{21} = K_{12}}$ → proved that [K] is symmetrical

$$\Rightarrow \{F\} = \begin{pmatrix} -\frac{ql^2}{12} \\ \frac{ql}{2} \\ 0 \\ \frac{ql^2}{12} \\ \frac{ql}{2} \\ 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} \text{NODE A} \\ \text{nodal forces.} \\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} \text{NODE B} \end{matrix}$$

It's negative ↙

19/10/17

Ex



$g = 3$
$v = 6$

$s = 6$ (N_2, N_3)
↓
4 nodes. < 2 completely clamped
2 we don't know
↓
3 movements

- One can neglect axial deformability
- $v_B = 0$; $v_C = 0$

$\mu_B = \mu_C$
$\varphi_B = \varphi_C$
$\Rightarrow s = 3$

BUT:

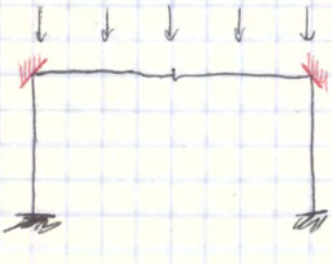
- symmetry : $v_B = v_C$ symm $\rightarrow 0 = 0$

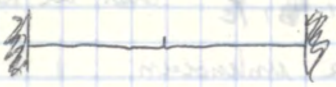
$\mu_B = -\mu_C$ antisymm. $\rightarrow \mu_B = \mu_C = 0$

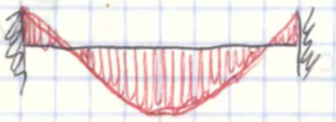
$\varphi_B = -\varphi_C$ antisymm.

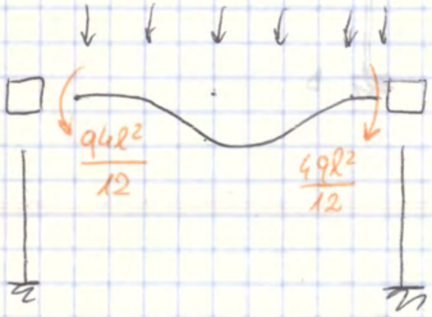
$\Rightarrow s = 1$ because $|\varphi_B| = |\varphi_C|$

→ Solve: Put a fictitious support in the node, to fix the nodal displacements.









→ Rotational equil of node B (unloaded) ⇒

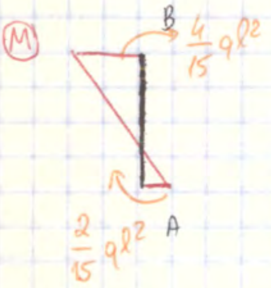
$$\sum M_i = 0 \qquad -\frac{2EI}{\ell}\varphi_B - \frac{4EI}{\ell}\varphi_B - \frac{EI}{\ell}\varphi_C - \frac{q\ell^{23}}{3} = 0$$

$$\varphi_B = -\varphi_C \qquad -2EI\varphi_B - 4EI\varphi_B + EI\varphi_B - \frac{q\ell^3}{3} = 0$$

$$-5EI\varphi_B = \frac{q\ell^3}{3} \Rightarrow \boxed{\varphi_B = -\frac{q\ell^3}{15EI}}$$

→ Diagrams

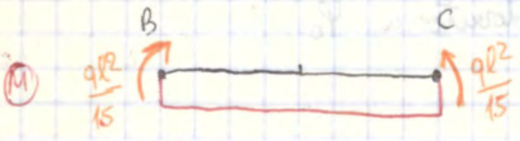
$$M_A = \frac{2EI}{\ell}\varphi_B = -\frac{2}{15}q\ell^2$$ → negative → clockwise (new convention)

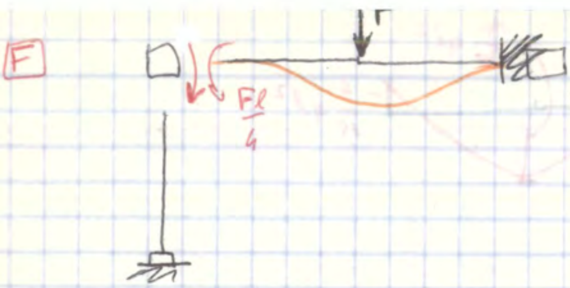


$$M_B = \frac{4EI}{\ell}\varphi_B = -\frac{4}{15}q\ell^2$$

$$M_B = -\frac{1}{15}q\ell^2$$

$$M_C = \frac{1}{15}q\ell^2$$





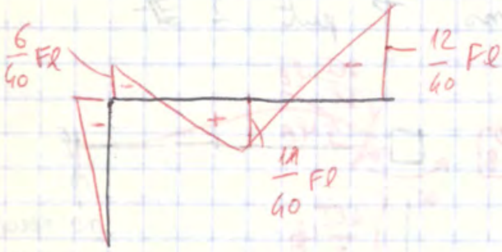
⇒ equil equations on nodes A and B

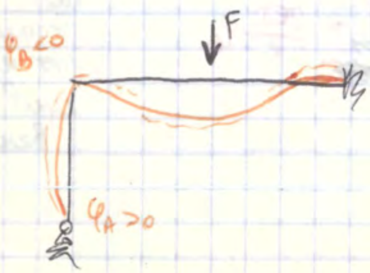
Ⓐ $-\frac{4EI}{\ell}\varphi_A - \frac{2EI}{\ell}\varphi_B = 0 \rightarrow 2\varphi_A + \varphi_B = 0$

Ⓑ $-\frac{2EI}{\ell}\varphi_A - \frac{4EI}{\ell}\varphi_B - \frac{2EI}{\ell}\varphi_B - \frac{F\ell}{4} = 0$

$$\Rightarrow \begin{cases} \varphi_A = \frac{F\ell^2}{40EI} \\ \varphi_B = -\frac{F\ell^2}{20EI} \end{cases}$$

⇒ Diagrams





! The higher is the number of constraints, the easier is to use method of displacement.

Ex

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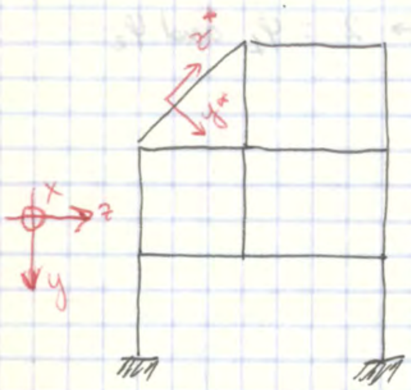


A structure like this is difficult to solve using the method of forces: it has parts that are statically determ. and others which are indeterminated.

All the CODES use the method of displacement.
~~But we want to~~
Use the fondamental equation in a global RS.

$$[K]^*_{(e)}\ \{\delta\}^*_e - \{F\}^*_e = \{Q\}^{*}_e \longrightarrow \text{in the local RS.}$$

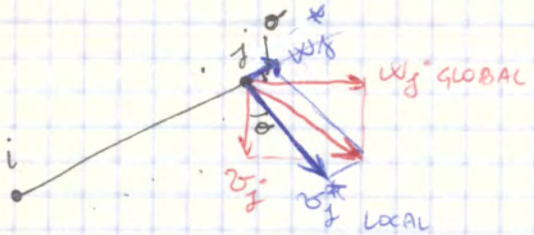
(e) ↳ of the element

STIFFNESS MATRIX OF THE BEAM

NODAL DISPLACEM VECTOR (UNKNOWN)

NODAL LOAD EQUIVALENT

CONSTRAINT REACTION



Pass from local to global components: demonstration

$$v_j^* = v_j \cos\theta + w_j \cdot \sin\theta$$

$$w_j^* = -v_j \sin\theta + w_j \cos\theta$$

$$\varphi_j^* = \varphi_j$$

$$\Rightarrow \begin{pmatrix} \varphi_j^* \\ v_j^* \\ w_j^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \varphi_j \\ v_j \\ w_j \end{pmatrix}$$

MATRICE DI ROTAZIONE

$$\Rightarrow \quad \sum_e \{Q_e\} = \{F\}$$

Write the equation for each beam:

$$+ \quad [K_1]\cdot\{\delta_1\} - \{F_1\} = \{Q_1\}$$

$$+ \quad [K_2]\cdot\{\delta_2\} - \{F_2\} = \{Q_2\}$$

$$\vdots$$

$$+ \quad [K_m]\cdot\{\delta_m\} - \{F_m\} = \{Q_m\}$$

from $e = 1, \dots, m$ = nr of beams.

$$\underbrace{[K]}_{S\times S}\cdot\underbrace{\{\delta\}}_{S\times 1} - \underbrace{\{F_{eq}\}}_{S\times 1} = \underbrace{\{F\}}_{S\times 1}$$

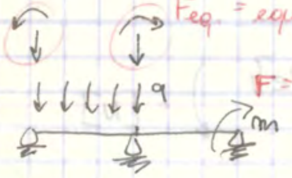
$\{F\}$ → VECTOR OF NODAL FORCES (REACTIONS)

$\{F_{eq}\}$ → VECTOR OF NODAL FORCES EQUIVALENT TO DISTRIBUTED LOADS

where S = degree of kinematical indeterminacy

$[K]$ → STIFFNESS MATRIX OF THE STRUCTURE

$\{\delta\}$ → NODAL DISPLACEM. UNKNOWN

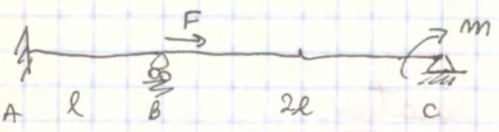


F_{eq} = equivalent nodal force due to load q

F = nodal force, applied directly on the node

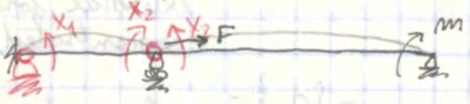
Ex Beam solved with all methods.

- METHOD OF FORCES



g = 3
v = 6 } 3 times iperstatic

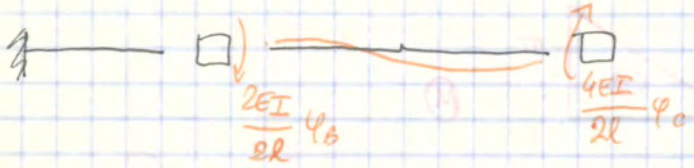
→ Put a hinge in each node



$\vec{F}$ doesn't appear in the angular congruence equations.

$$+ \quad \begin{cases} \varphi_A = 0 \\ \varphi_{AB} = \varphi_{BC} = \varphi_B \end{cases} \begin{cases} \dfrac{X_1 l}{3EI} + \dfrac{X_2 l}{6EI} = 0 \\[2ex] \dfrac{-X_1 l}{6EI} - \dfrac{X_2 l}{3EI} = \dfrac{X_2 l}{3EI} + \dfrac{m\,2l}{6EI} \end{cases}$$

③



$\varphi_C \neq 0$

$$-\frac{4EI}{l}\varphi_B - \frac{2EI}{l}\cdot\varphi_B - \frac{EI}{l}\cdot\varphi_C = 0$$

$$-\frac{EA}{l}w_B - \frac{EA}{2l}w_B + F = 0$$

$$-\frac{EI}{l}\varphi_B - \frac{2EI}{l}\varphi_C - m = 0$$

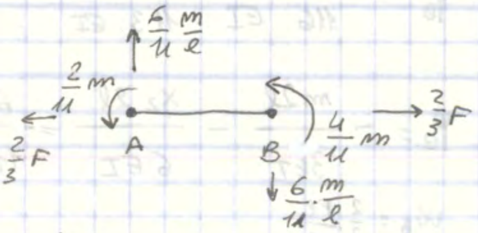
$$\Rightarrow \begin{cases} 6\cdot\varphi_B + \cdot\varphi_C = 0 \\ -2w_B - w_B + \frac{2Fl}{EA} = 0 \\ -\varphi_B - 2\varphi_C - \frac{ml}{EI} = 0 \end{cases} \qquad \begin{cases} w_B = \frac{2}{3}\frac{Fl}{EA} \\ \varphi_C = -6\varphi_B = -\frac{6}{11}\frac{ml}{EI} \\ \varphi_B = \frac{ml}{11EI} \end{cases}$$

$$\Rightarrow M_A = \frac{2EI}{l}\cdot\varphi_B = \frac{2}{11}m$$

$$M_B = \frac{4EI}{l}\varphi_B = \frac{4}{11}m$$

$$T_A = -\frac{6EI}{l^2}\varphi_B = -\frac{6}{11}\cdot\frac{m}{l}$$

$$N_A = -\frac{EA}{l}\cdot w_B = -\frac{2}{3}F$$



} equations got with the stiffness equation.

⇒ All the values are the same of the previous method, but signs change according to the RS.

⇒ Write the $[K]$ matrix, global $\boxed{1}$ is l & $\boxed{2}$ is $2l$

$$\begin{pmatrix} \frac{4EI}{l}+\frac{4EI}{2l} & 0 & \frac{2EI}{2l} \\ 0 & \frac{EA}{l}+\frac{EA}{2l} & 0 \\ \frac{EI}{l} & 0 & \frac{4EI}{2l} \end{pmatrix} \begin{pmatrix} \varphi_B \\ w_B \\ \varphi_C \end{pmatrix} = \begin{pmatrix} M_B = 0 \\ N = F \\ M_C = -m \end{pmatrix} = \begin{pmatrix} 0 \\ F \\ -m \end{pmatrix}$$

→ Terms took from the $[K]\{\delta\} = \{\underset{\overset{\shortparallel}{Q}}{F}\}$ equation

$[K]$ is SYMMETRIC because $[K_e]$ for $e = 1, 2$ (beam) are symmetric

$$\Rightarrow \begin{cases} \left(\frac{4EI}{l}+\frac{4EI}{2l}\right)\varphi_B + \frac{2EI}{2l}\varphi_C = 0 \\ \left(\frac{EA}{l}+\frac{EA}{2l}\right) w_B = F \rightarrow 1 \times 1 \ (w_B) \\ \frac{EI}{l}\varphi_B + \frac{4EI}{2l}\varphi_C = -m \end{cases}$$

2×2 → unknowns φ_B & φ_C

If $\{F_e\} \neq 0$, it means that if there were distributed loads

$$\Rightarrow [K]\{\delta\} = \{F\} + \{F_{eq}\}$$

where $\{F_{eq}\} = \begin{pmatrix} F_{eq1} \\ F_{eq2} \\ F_{eq3} \end{pmatrix}$

$F_{eq\textcircled{1}} = F_{e\textcircled{4}}^{(1)} + F_{e\textcircled{1}}^{(2)}$ → obtained from the forces F applied on each (e) beam.

$F_{eq\textcircled{2}} = F_{e\textcircled{6}}^{(1)} + F_{e\textcircled{3}}^{(2)}$

$F_{eq\textcircled{3}} = F_{e\textcircled{4}}^{(2)}$ → The F_{eq3} acting on node C and giving the unknown φ_C, is due to the force F applied on the 2nd beam

↓ pedix comes from the incidence matrix.

⇒ NODE B : equilibrium

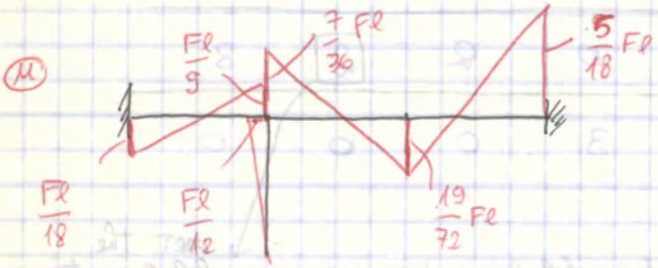
$$-\frac{4EI}{\ell}\varphi_B - \frac{2EI}{\ell}\varphi_B - \frac{4EI}{\ell}\varphi_B - \frac{2EI}{\ell}\varphi_C - \frac{F\ell}{4} = 0$$

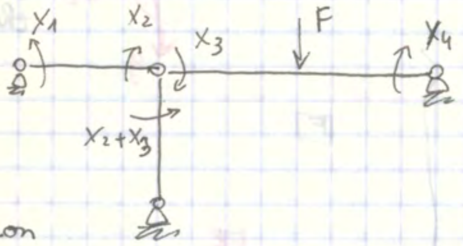
NODE C : equilibrium

$$-\frac{2EI}{\ell}\varphi_B - \frac{4EI}{\ell}\varphi_C = 0$$

$$\Rightarrow \begin{cases} \varphi_C = \frac{1}{72}\frac{F\ell^2}{EI} \\ \varphi_B - \frac{F\ell^2}{36\,EI} \end{cases}$$

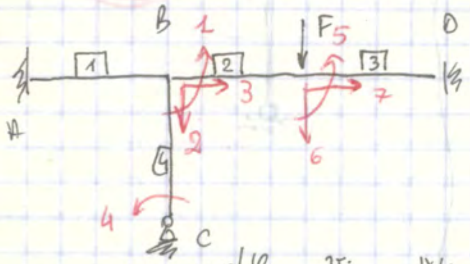
⇒ Diagrams





$v - g = 5$

in the other case the situation would be like this



!!! In the Fantilli's code you can't implement concentrated forces $\vec{F}$ so you have to divide in more pieces the beam

unknowns / Beam	φ_i 1	v_i 2	w_i 3	φ_j 4	v_j 5	w_j 6
1	0	0	0	1	2	3
2	4	0	0	1	2	3
3	1	2	3	5	6	7

$$\begin{cases} \dfrac{X_1 l}{3EI} + \dfrac{X_2 l}{6EI} - \varphi = \varphi_0 \\[2ex] -\dfrac{X_2 l}{3EI} - \dfrac{X_1 l}{6EI} - \varphi = \dfrac{X_2 l}{3EI} \end{cases} \xrightarrow{PLV} X_2\varphi - X_1\varphi = 0$$

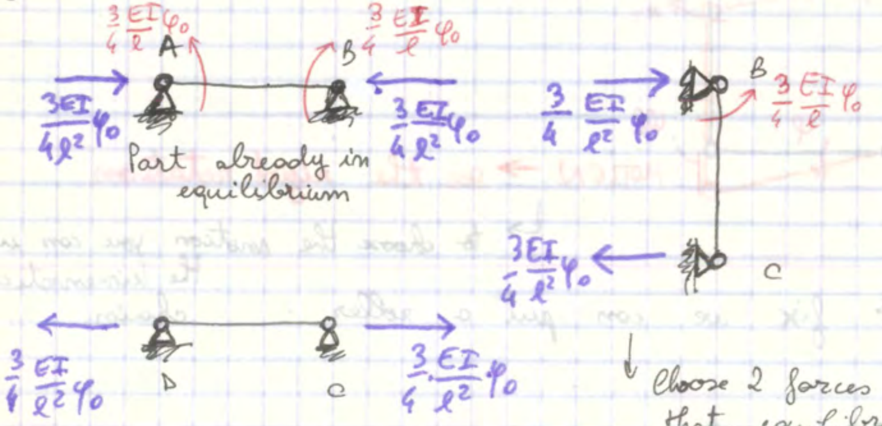
X_2 acting on BC is equal to 0.

$$X_1 = X_2 = \frac{3}{4}\cdot\frac{EI}{l}\varphi_0 \;,\quad \varphi = -\frac{5}{8}\varphi_0$$

$T_{AB} = 0 \rightarrow N_{BC} = 0 \rightarrow T_{CD} = 0 \Rightarrow$ No shear $\Rightarrow$ no moment on AB & CD

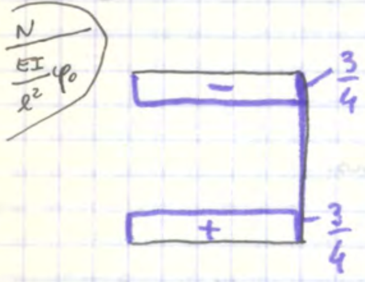
↳ Results consistent with the structure

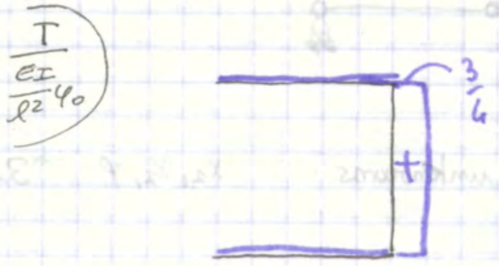
→ Diagrams of forces

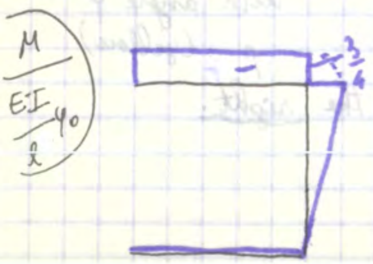


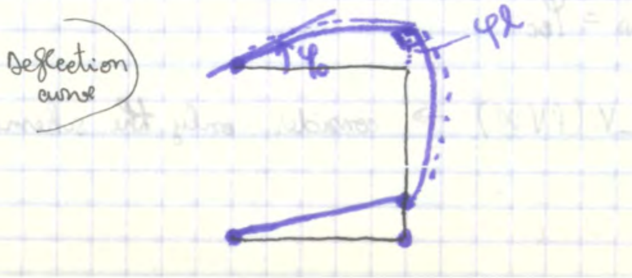
↓ Choose 2 forces that equilibrate the moment (●)

→ Diagrams of sollecitazions









$$\begin{cases} -\dfrac{X_1 l}{3EI} - \dfrac{\delta_0}{l} + \varphi = \dfrac{X_1 l}{3EI} + \dfrac{X_2 l}{6EI} \\ -\dfrac{X_2 l}{3EI} - \dfrac{X_1 l}{3EI} = \dfrac{X_2 l}{3EI} + \varphi \\ PLV:\ -X_1 \varphi + X_2 \varphi = 0 \end{cases}$$

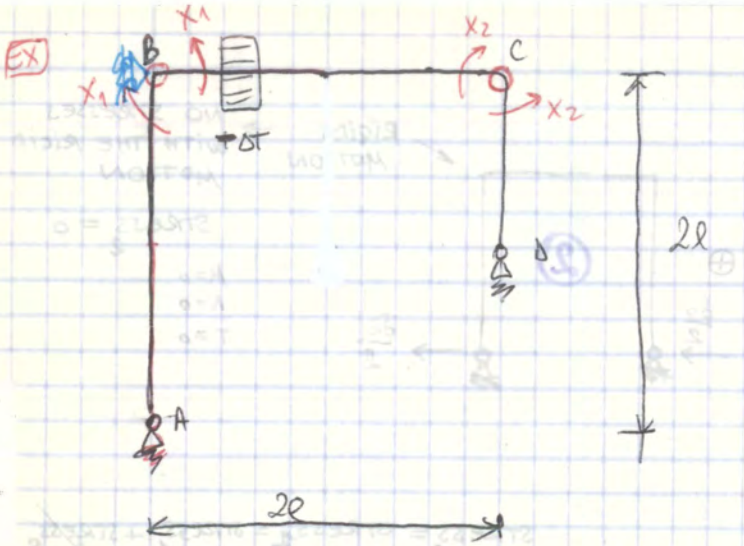
PLV = real <u>effects</u> • virtual <u>causes</u>

⇒ solve the exercise

∇ δ_0 = real force

2/11/17

EX



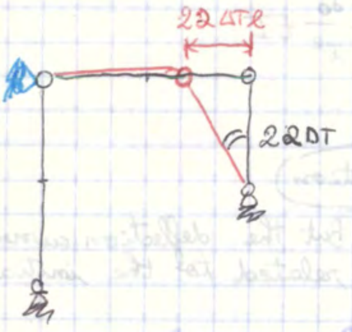
→ Put 2 hinges in the 2 nodes : { g = 9 ; v = 8

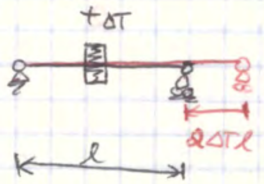
↳ It's 1 hypostatic : study the fixed scheme and the motion related.

To underline that the scheme is fixed I can put a fictitious support in B. (or in C).

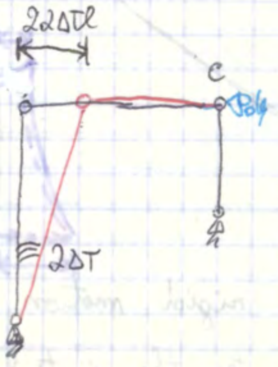
Since the ΔT is negative, the thermal load reduces the length.

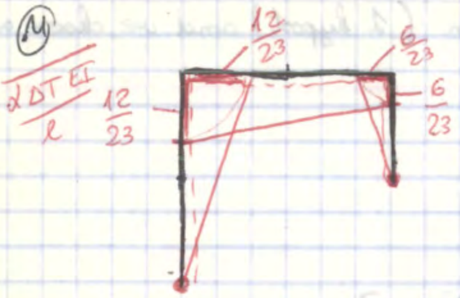
?

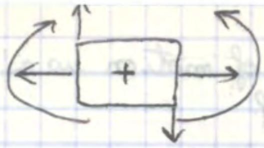




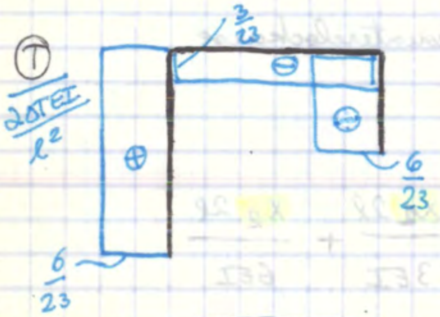
? If I put in C

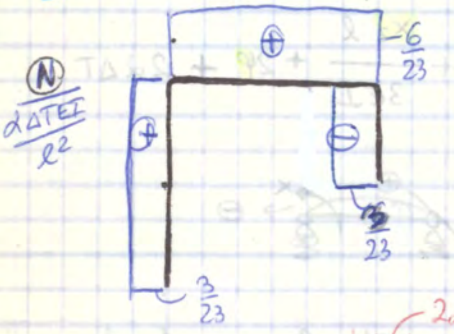


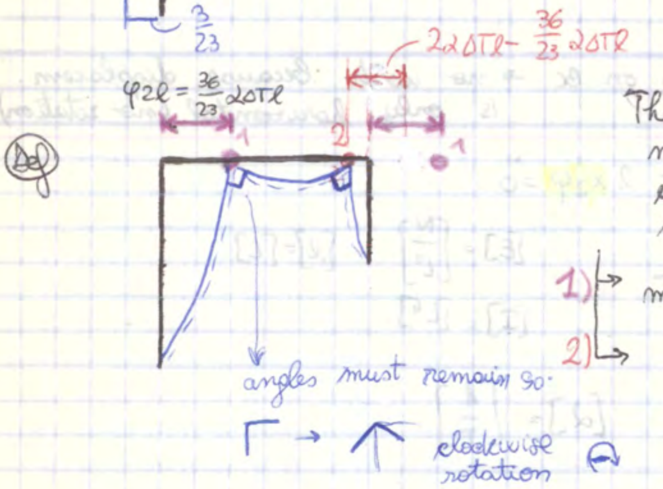




Bending moment : linear
0 in A and D







The nodes move : traslating node frame : because the associated truss structure is not statically determ.

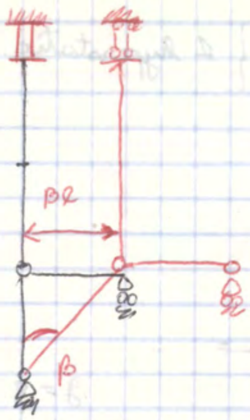
1) move $\overset{B,C}{}$ by $2l\varphi = \frac{36}{23}\alpha\Delta T l$ →

2) " C by $2l\alpha\Delta T$ ←

What happens if I put the fictitious roller support in C?

→ Solve (solution must be the same).

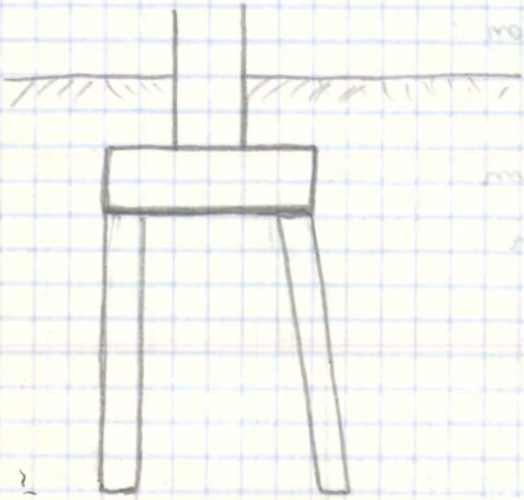
(2.2)



Deflection curve ① + → (2.1)
→ (2.2)

Or even more :

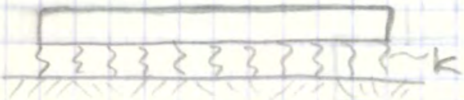
PILES FOUNDATION



WINKLER MODEL

Soil is modelled as a weight on springs of resistance k : soil can be compressed or tractioned; springs are indipendent ⇒ these aren't the real behaviour but the result is quite good.

The model is LINEAR - ELASTIC.



1) Start from eq. of deflection curve:

$$\frac{d^2 v}{dz^4} = \frac{q(z)}{EI}$$

$v = v(z)$ vertical displacement

$\varphi(z) = -v'(z)$ rotation

$M(z) = -EI \cdot v''(z)$ moment

$T(z) = -EI\, v'''(z)$ shear.

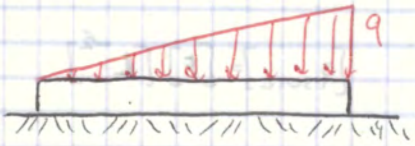
Solution : $v_{particular} + v_{homogeneous}$.

!① If $q(z)$ is a polinomial of degree less than 3 ⇒ the solution is :

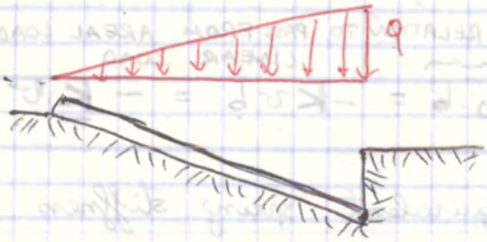
$$v = \frac{q}{k}$$ because also $v(z)$

would be a polynomial of degree ≤ 3

so $\frac{d^4 v}{dz^4} = 0$ and $\frac{q}{EI} = \frac{q}{EI} \Rightarrow 0 = 0$.



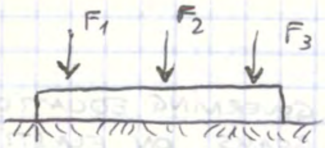
BEAM UNLOADED.



$q_{soil} = -kv = -q$

If The solution of q = polynomial is not interesting.

② We want to know concentrated forces :



2.1) Determine $q_{soil} < q_{soil\ THRESHOLD}$ (pressure on soil)

2.2) M, T → determine moment and shear in the beam.

$$\Rightarrow v(z) = c_1 \cdot e^{\overset{+1}{\beta z}} \cdot \cos(\overset{+i}{\beta z}) + c_2\, e^{\overset{+1}{\beta z}} \sin(\overset{-i}{\beta z}) +$$

$$+ c_3\, e^{\overset{-1}{-\beta z}} \cos(\overset{+i}{\beta z}) + c_4\, e^{\overset{-1}{-\beta z}} \sin(\overset{-i}{\beta z})$$

↳ the 4 λ are transformed into the sum of this

$$\left(\begin{array}{l} e^{+i\beta z} = \cos(\beta z) \\ e^{-i\beta z} = \sin(\beta z) \end{array} \right)$$

⇓

- Find c_1, c_2, c_3, c_4:

$$v'(z) = \beta\Big[(c_1 + c_2)e^{\beta z}\cos(\beta z) + (-c_1 + c_2)e^{\beta z}\sin(\beta z) +$$

$$+ (-c_3 + c_4)e^{-\beta z}\cos(\beta z) + (-c_3 - c_4)e^{-\beta z}\sin(\beta z)\Big]$$

$$v''(z) = 2\beta^2\Big[c_2 e^{\beta z}\cos(\beta z) - c_1 e^{\beta z}\sin(\beta z) -$$

$$- c_4 e^{-\beta z}\cos(\beta z) + c_3 e^{-\beta z}\sin(\beta z)\Big]$$

$$v'''(z) = 2\beta^3\Big[(c_2 - c_1)e^{\beta z}\cos(\beta z) + (-c_2 - c_1)e^{\beta z}\sin(\beta z) +$$

$$+ (c_4 + c_3)e^{-\beta z}\cos(\beta z) + (c_4 - c_3)e^{-\beta z}\sin(\beta z)\Big]$$

$$v^{IV}(z) = -4\beta^4\, v(z) \quad \checkmark$$

BOUNDARY CONDITION 1:

Since beam is ∞ : $\lim_{z\to\infty} v(z) = 0$ for a concentrated load $\vec{F}$

$$\Rightarrow e^{-\beta z} \xrightarrow{z\to\infty} 0$$

$$e^{\beta z} \xrightarrow{z\to\infty} \infty$$

⇒ So I need the result such that $v(z) = 0$ ⇒ $c_1 = c_2 = 0$

" Se c_1 e c_2 = 0 anche l'esponenziale che va all'∞ sarà = 0 per $z \to \infty$.

$$A_{\beta z} = e^{-\beta z}\left(\cos(\beta z) + \sin(\beta z)\right)$$

$$B_{\beta z} = e^{-\beta z} \sin(\beta z)$$

$$C_{\beta z} = e^{-\beta z}\left(\cos(\beta z) - \sin(\beta z)\right)$$

$$D_{\beta z} = e^{-\beta z} \cos(\beta z)$$

↳ Function related by derivation:

$$\frac{dA}{dz} = -2\beta B$$

$$\frac{dB}{dz} = \beta C$$

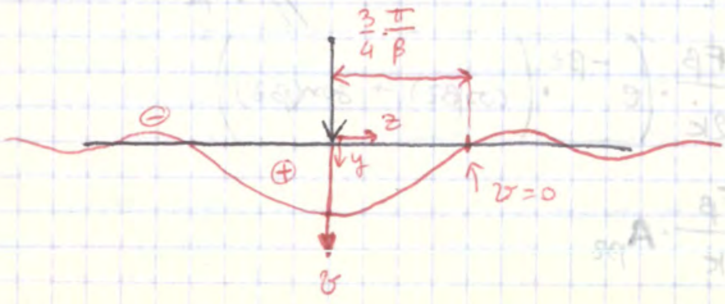
$$\frac{dC}{dz} = -2\beta D$$

$$\frac{dD}{dz} = -\beta A$$

!!! These functions are NOT VALID FOR $z<0$. We plot only because we know ∃ symmetry.

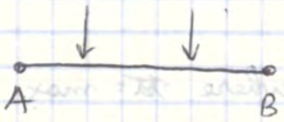
SOLUTION

$$\Rightarrow \begin{cases} \varphi(z) = -\dfrac{dv}{dz} = \dfrac{F\beta^2}{k} B_{\beta z} \\[2ex] M(z) = -EI\, v''(z) = \dfrac{EIF\beta^3}{k} \cdot C_{\beta z} \cdot \dfrac{\beta}{\beta} = \dfrac{F}{4\beta} C_{\beta z} \\[2ex] T(z) = -EI\, v'''(z) = \dfrac{dM(z)}{dz} = -\dfrac{F}{2} D_{\beta z} \end{cases}$$

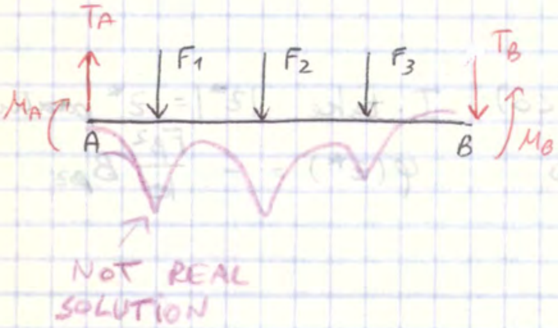


BUT THIS IS NOT THE FINAL SOLUTION :

- Beams are finite
- Forces are more than 1 :



BEAM OF FINITE LENGTH

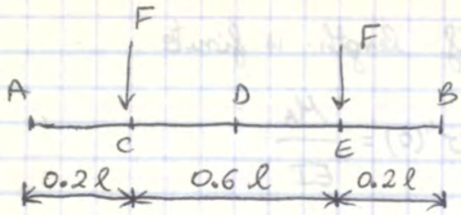


If the beam was ∞ ⇒ $T_A, T_B, M_A, M_B = 0$ but it's not real.

1) Effect of superposition on ∞ beam : calculate M, T for F_1, F_2, F_2 separately and then sum : in the extremes I get values ≠ 0.

2) Find solution finite length

3) Sum of ① + ②. → now in the extremes I get 0 values because from the beginning I wanted to have 0.



$$\beta = \sqrt[4]{\frac{k}{4EI}}$$

big numerator ↑

Comments

1) If the beam is compliant $\overset{\text{or/}}{\text{and}}$ the soil is stiff.

flexible → small denominator

⇒ β is large ⇒ the sollecitations damps quicly while moving far from C and E

⇒ beam sollecitations are low → soil pressure is high

Bad behaviour.

soil pressure high to contrast this.

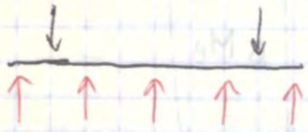
ex: filo on cattedra.

2) If the beam is stiff and/or the soil is compliant

⇒ β small ⇒ all the soil beneath the beam and the beam itself is all sollecitated. → soil pressure is low. (load more distributed) → moment and shear are high in the beam

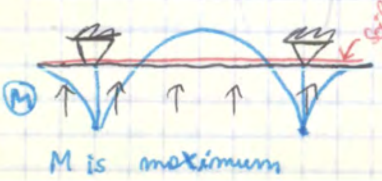
Ex beam on bed

3) Beam is ∞ stiff ⇒ β→0 ⇒ q_{soil} = constant



$q_{soil} = \frac{2F}{l}$ is minimum

soil pressure is uniform (-0.2)

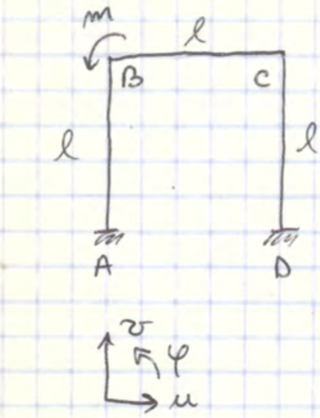


The beam acts as a support

Ex

METHOD OF DISPLACEMENT

9/11/17



$\Delta = 6$

u_B $\cancel{v_B}$ φ_B { unknowns 3 for B

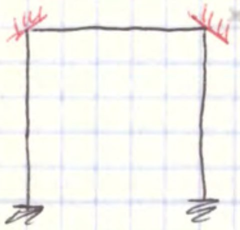
u_C $\cancel{v_C}$ φ_C 3 for C

$$\begin{cases} v_B = 0 \\ v_C = 0 \\ u_B = u_C \end{cases}$$ Axial indeformation

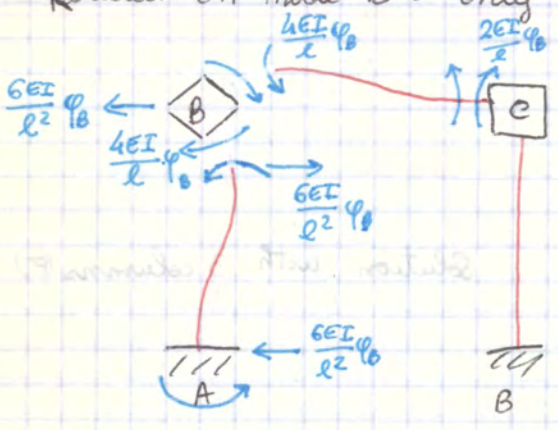
↓

if node B move horizontally

→ Apply superposition of effects: put fictitious fixed supports . The solution is given by 3 equil equations



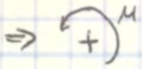
- Rotation on node B: only node B rotates (no traslat)

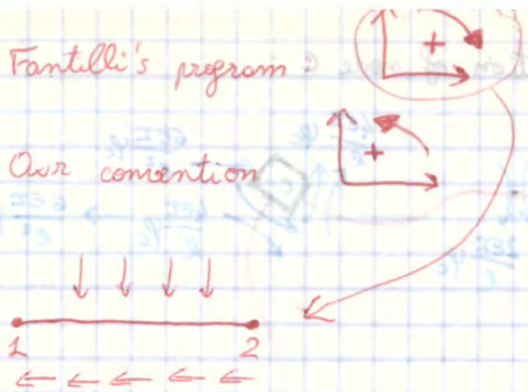


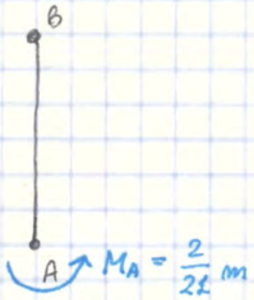
AB, BC deflect

Now write equilibrium of moments and forces: to find them look on the formulary.

I need to write only horizonta forces ⇄ because $u \neq 0$ and $v = 0$.



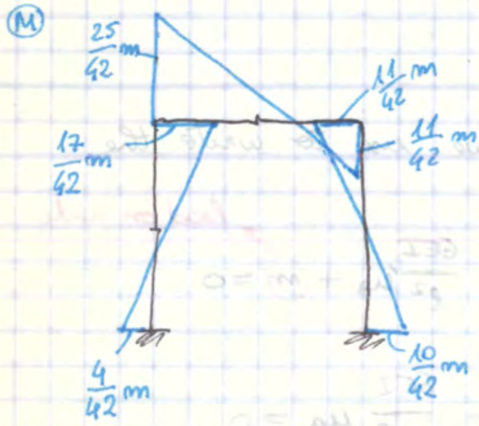


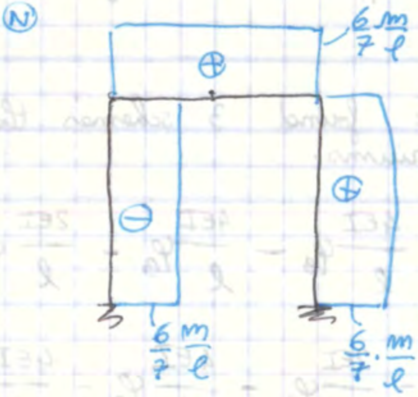


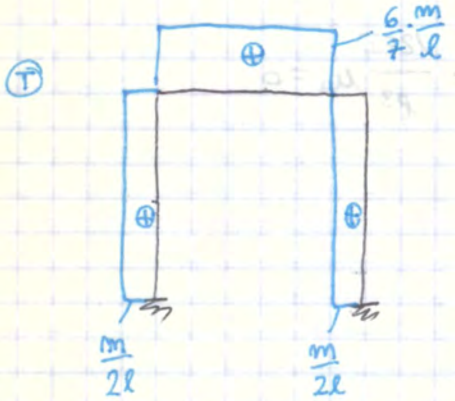
$$M_A = \underbrace{\frac{2EI}{l}\varphi_B + \frac{6EI}{l^2}u_B}_{\text{from relation of the stiffness matrix on the beam}} =$$

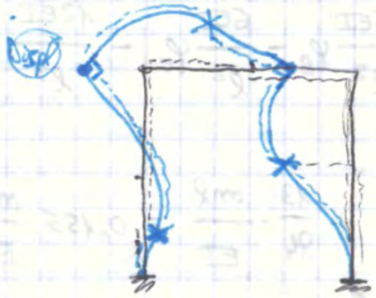
$$= \frac{\cancel{2}EI}{\cancel{l}} \cdot \frac{13}{\underset{42}{\cancel{84}}}\frac{ml}{\cancel{EI}} + \frac{\overset{3}{\cancel{6}}\cancel{EI}}{\cancel{l^2}}(-)\frac{1}{\underset{14}{\cancel{88}}}\frac{ml^2}{\cancel{EI}} = \frac{13}{42}m - \frac{3}{14}m = \frac{2}{21}m$$

The same calculation must be done for B, C, D.



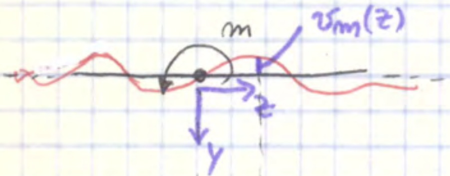


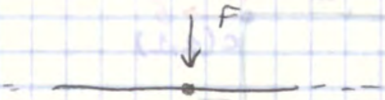




C rotates by small quantity clockwise 90 → 90

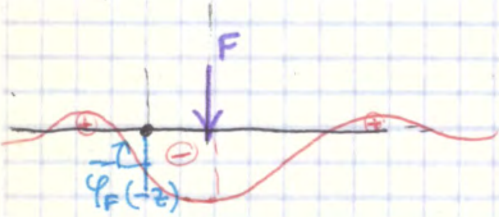
BEAM ON ELASTIC FOUNDATION





$$\varphi(z) = \overset{+}{\boxed{-}} \frac{F\beta^2}{k} B_{\beta z}$$

To evaluate $v_m(z)$ I can apply Betti's theorem: apply a force in the point where I want to compute $v_m(z)$. In that point I have a rotation due to force F.



⇒ TH: $F \cdot v_m(z) = m\varphi_F(-z)$

$$\Rightarrow v_m(z) = \frac{m}{F}\varphi(\ominus z) = \boxed{-}\,\frac{m}{\cancel{F}} \cdot \frac{\cancel{F}\beta^2}{k} B_{\beta z}$$

because for $z<0$ for F, I take ⊖

$$\Rightarrow v_m(z) = \substack{-\\+}\,\frac{m\beta^2}{k} B_{\beta z}$$

derive ↓

$$\varphi = \substack{+\\+}\,\frac{m\beta^3}{k} C_{\beta z}$$

↓

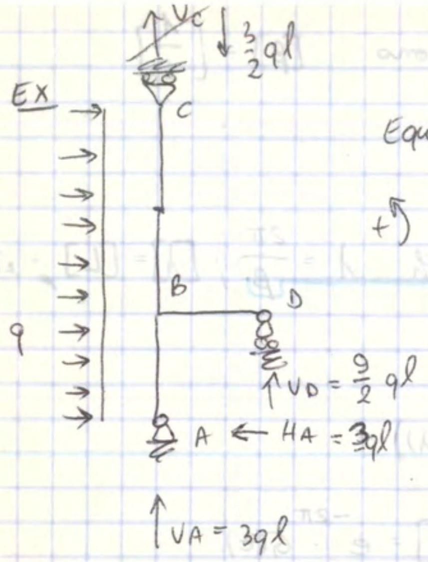
$$M = \substack{-\\+}\,\frac{m}{2} D_{\beta z} \left(= \substack{-\\+}\,\frac{2m\beta^4}{k} EI\, D_{\beta z}\right)$$

↓

$$T = \substack{+\\+}\,\frac{m\beta}{2} A_{\beta z}$$

15/11/17

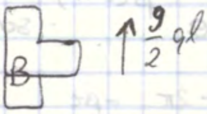
EX

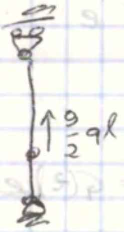


Equil around A:

$+\circlearrowleft$ $V_D l - \frac{9}{2} q l^2 = 0$ $V_D = \frac{9}{2} q l^2$

$\Rightarrow H_A = 3ql$



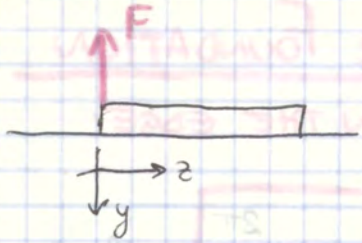


$V_A = \frac{2}{3} V_D = \frac{2}{3} \cdot \frac{9}{2} ql = 3ql$

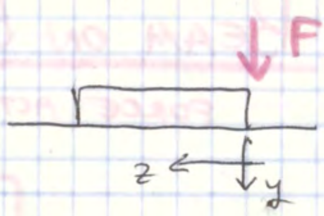
$V_C = -\frac{1}{3} V_D = \frac{-1}{\cancel{3}} \cdot \frac{\overset{3}{\cancel{9}}}{2} ql = -\frac{3}{2} ql$

This exercises was easier to solve in this way

UPPER SIGN



LOWER SIGN



SOLUTION FOR A MOMENT APPLYED

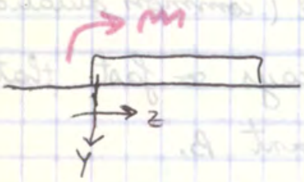
$$v(z) = \begin{smallmatrix}-\\-\end{smallmatrix} \frac{m}{2EI\beta^2} \cdot C_{\beta z}$$

$$\varphi(z) = \begin{smallmatrix}-\\+\end{smallmatrix} \frac{m}{EI\beta} \cdot D_{\beta z}$$

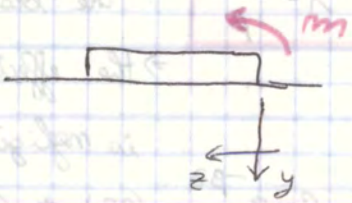
$$M(z) = \begin{smallmatrix}+\\+\end{smallmatrix} m \cdot A_{\beta z}$$

$$T(z) = \begin{smallmatrix}-\\+\end{smallmatrix} 2\beta m \cdot B_{\beta z}$$

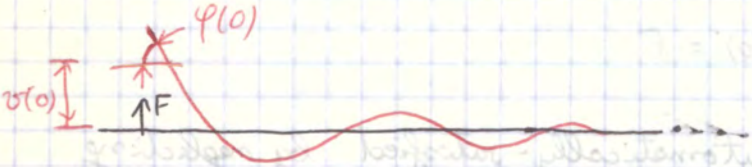
UPPER SIGN



LOWER SIGN



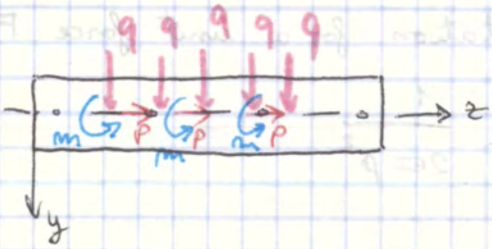
COMPLIANCE



Def: Compliance = displacement or rotation for a unit force or moment

$$\lambda_{FF} = \frac{v(0)}{F} = \frac{\text{displacement}}{\text{force}}$$

BEAM WITH RECTILINEAR AXIS

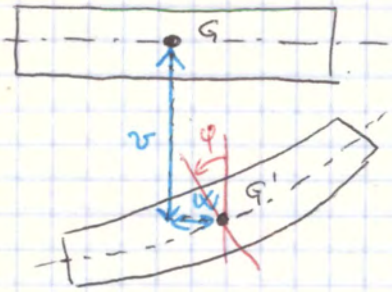


EQUILIBRIUM EQUATIONS

$$\begin{pmatrix} \frac{d}{dz} & 0 & 0 \\ 0 & \frac{d}{dz} & 0 \\ -1 & 0 & \frac{d}{dz} \end{pmatrix} \begin{pmatrix} T \\ N \\ M \end{pmatrix} + \begin{pmatrix} q \\ p \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underset{3\times3}{[\partial]^*} \cdot \underset{3\times1}{\{Q\}} + \underset{3\times1}{\{\mathcal{F}\}} = \underset{3\times1}{\{0\}}$$

KINEMATIC EQUATIONS



$$[\partial]^* = [\partial]^T$$

$-1 \longrightarrow +1$

$$\begin{pmatrix} \gamma \\ \varepsilon \\ \chi \end{pmatrix} = \begin{pmatrix} d/dz & 0 & +1 \\ 0 & d/dz & 0 \\ 0 & 0 & d/dz \end{pmatrix} \cdot \begin{pmatrix} v \\ w \\ \varphi \end{pmatrix}$$

$$\{q\} = [\partial] \cdot \{\eta\}$$

$$= \int_0^{\ell} \left(T_A \gamma_B + N_A \varepsilon_B + M_A \chi_B \right) dz =$$

$$= \int_0^{\ell} \{Q_A\}^T \{q_B\} \, dz \quad \checkmark$$

This principle will be used for plates and shells.

CONSTITUTIVE EQUATIONS

$$\begin{pmatrix} \gamma \\ \varepsilon \\ \chi \end{pmatrix} = \begin{pmatrix} \frac{t_y}{GA} & 0 & 0 \\ 0 & \frac{1}{EA} & 0 \\ 0 & 0 & \frac{1}{EI} \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ M \end{pmatrix}$$

$$\underset{3\times1}{\{q\}} = \underset{3\times3}{[H]^{-1}} \cdot \underset{3\times1}{\{Q\}}$$

G = tangential elastic modulus

A = area

For rectilinear beams : $\overset{N}{\text{axial}}$ and $\overset{T, M}{\text{bending}}$ problems are UNCOUPLED:

$$p \rightarrow N \rightarrow \varepsilon \rightarrow w \qquad \text{AXIAL PROBLEM}$$

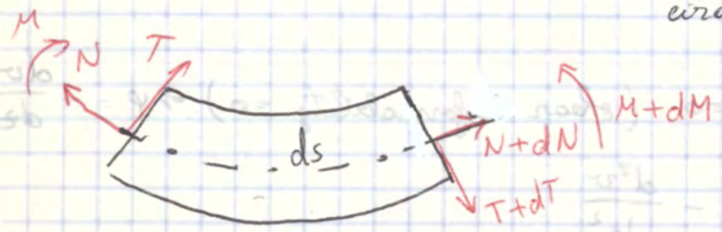
$$\begin{pmatrix} q \\ m \end{pmatrix} \rightarrow \begin{pmatrix} T \\ M \end{pmatrix} \rightarrow \begin{pmatrix} \gamma \\ \chi \end{pmatrix} \rightarrow \begin{pmatrix} v \\ \varphi \end{pmatrix} \qquad \text{BENDING PROBLEM.}$$

BEAMS WITH CURVILINEAR AXIS

Obtain equil equation : infinitesimal piece :

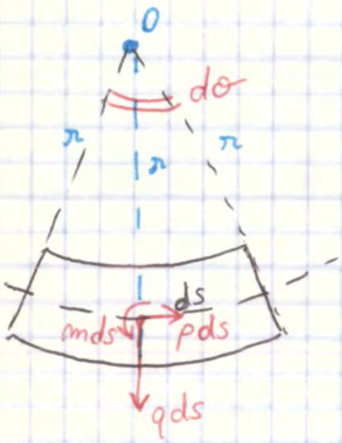
$\downarrow$

arch of circumference

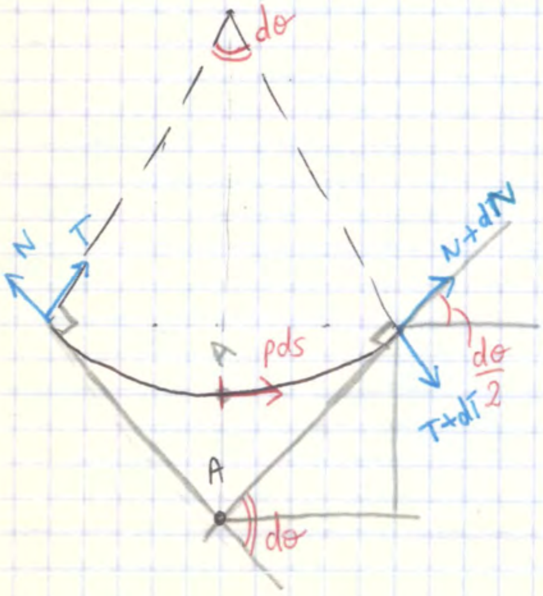


r = radius of circumf ≡ radius of curvature

ds = curvilinear abscissa



$r\,d\theta = ds$



$$\xrightarrow{+} (N+dN)\cos\left(\frac{d\theta}{2}\right) +$$

$$+ (-N)\cos\left(\frac{d\theta}{2}\right) +$$

$$+ (T+dT)\sin\left(\frac{d\theta}{2}\right) +$$

$$+ T\sin\left(\frac{d\theta}{2}\right) + pds = 0$$

$$! \cos\left(\frac{d\theta}{2}\right) \simeq 1 - \underbrace{\frac{1}{2}\left(\frac{d\theta}{2}\right)^2}_{\text{II order}} \simeq 1$$

$$! \sin\left(\frac{d\theta}{2}\right) \simeq \frac{d\theta}{2}$$

KINEMATIC EQUATIONS.

The $[D]^*$ is transposed with sign of the finite terms changed. $-1 \Leftrightarrow +1$

$$\begin{pmatrix} \gamma \\ \varepsilon \\ \chi \end{pmatrix} = \begin{bmatrix} d/ds & -\frac{1}{r} & +1 \\ +\frac{1}{r} & d/ds & 0 \\ 0 & 0 & d/ds \end{bmatrix} \begin{pmatrix} v \\ w \\ \varphi \end{pmatrix}$$

$$\{q\} = [D] \cdot \{\eta\}$$

CONSTITUTIVE EQUATIONS

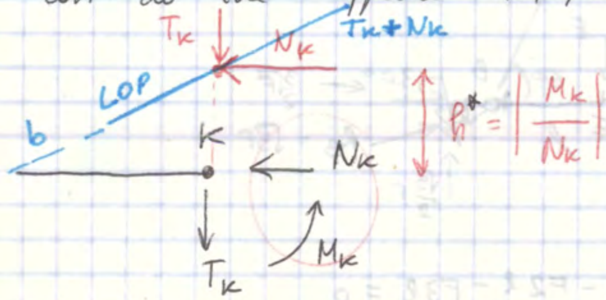
if $r \gg h$ $\Rightarrow$ use the same constitutive equations of the rectilinear beam.

$$\{Q\} = [H]\{q\}$$

The term $\left(\frac{1}{r}\right)$ couples the axial and the bending problems in the curvilinear beams.

$$\boxed{\begin{aligned} |N_k| &= R_1 \cos\alpha \\ |T_k| &= R_1 \sin\alpha \\ |M_k| &= R_1 \cdot h \end{aligned}}$$

I can do the opposite $N, T, M \rightarrow LP$?



LOP

↓

I know N,T,M and I want to find the point of LoP corresponding to the conditions in K :

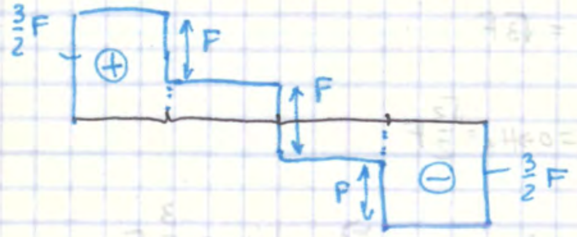
for equilibrium

$$N_K \cdot h^* = M_K$$

$$\Rightarrow h^* = \frac{M_K}{N_K}$$

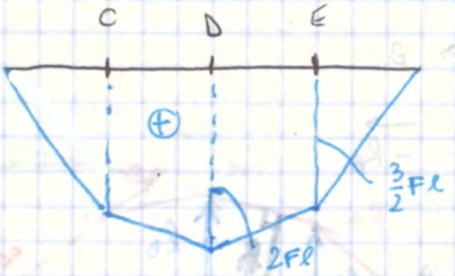
⇒

Ⓣ



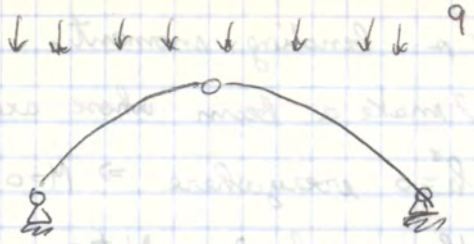
Ⓝ

Ⓜ



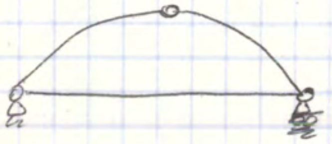
$$\ell_D = \left|\frac{M_D}{N_D}\right| = \frac{2F\ell}{\frac{\sqrt{3}}{2}F} = \frac{4\sqrt{3}}{3}\ell = 2.31\ell$$

$$\ell_C = \left|\frac{M_C}{N_C}\right| = \frac{3/2\,F\ell}{\frac{\sqrt{3}}{2}F} = \sqrt{3}\ell = 1.73\ell$$



3 HINGES ARCH

For q distributed, the optimal shape is PARABOLIC



STATICALLY INDET ARCHES

$$\Rightarrow \begin{cases} \frac{dT}{ds} - \frac{N}{r} + q = 0 \\ \frac{T}{r} + \frac{dN}{ds} + p = 0 \\ -T + \frac{dM}{ds} + m = 0 \end{cases}$$

$EI \nearrow^{0}$ $\quad I \propto h^3$

$EA \quad A \propto h^2$

if bending stiffnen $\to 0$ $[EI \to 0]$

$\Rightarrow \quad M = EI \cdot \chi = 0 \quad \Rightarrow \quad T = 0$

$$\Rightarrow \begin{cases} \frac{-N}{r} + q = 0 \\ \frac{dN}{ds} + p = 0 \end{cases}$$

1 unknown

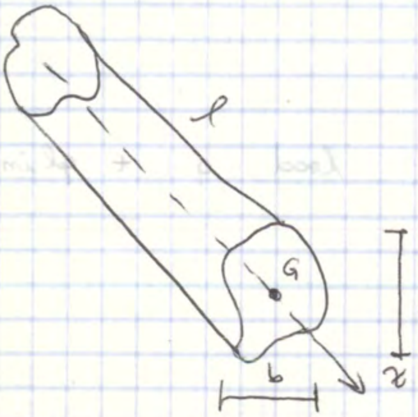
2 equations

$\Rightarrow$ SOLUTION IMPOSSIBLE

BUT $\exists$ SOLUTION ONLY BEAM AXIS = LOP.

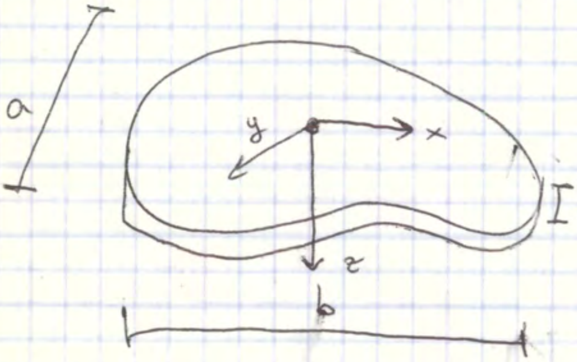
PLATES & SHELLS

16/11/17



$l \gg h, b$

$N = \int_A \sigma_z \, dA$ integrate on the directions which are smaller in order to have dependency only on z



h = plate thickness

$z = 0$ is in the middle plane

$z = \pm \frac{h}{2}$ bottom and upper surface of the plate

$a, b \gg h$

2D MODEL because all quantities depend only on (x, y)

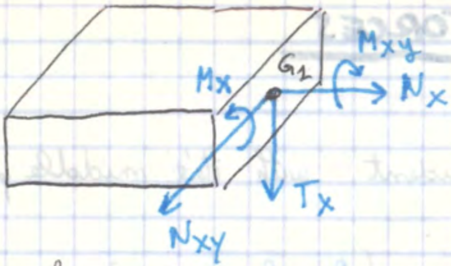
! σ_{xx}

x (first): stress on the face having as normal the x - axis

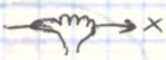
x (second): direction along which this stress is acting

τ_{xy}

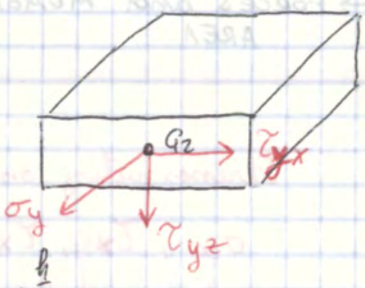
τ_{yz}



$$M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\cdot z \cdot dz$$ positive around −x



②



$$N_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \, dz$$ $[N,T] = \left[\frac{N}{m}\right]$ = force per unit length

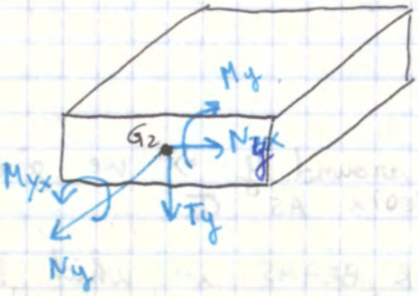
$$N = \int_{-\frac{h}{2}}^{h/2} \tau \quad dz$$

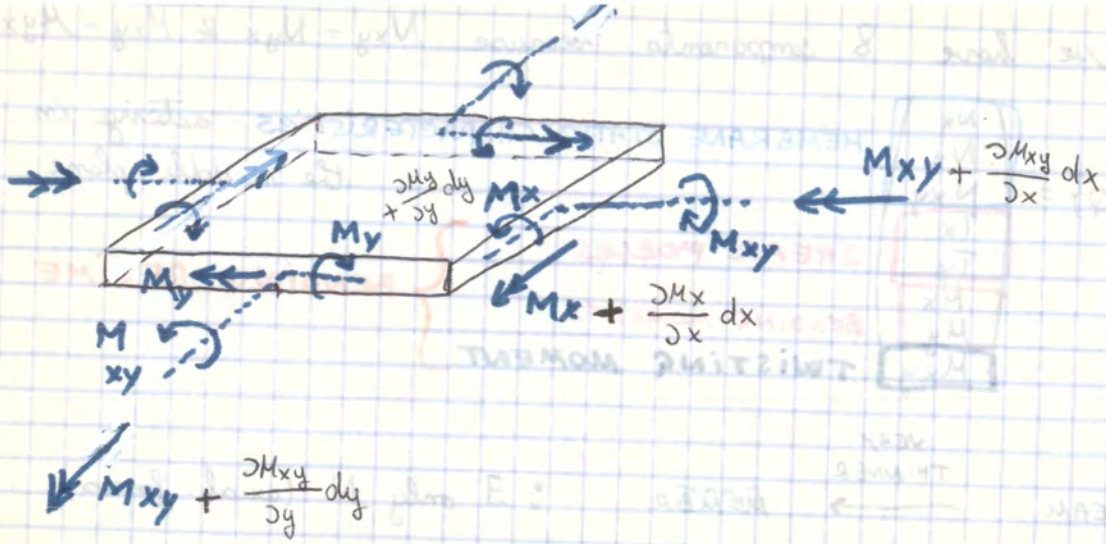
$$T_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} \, dz$$

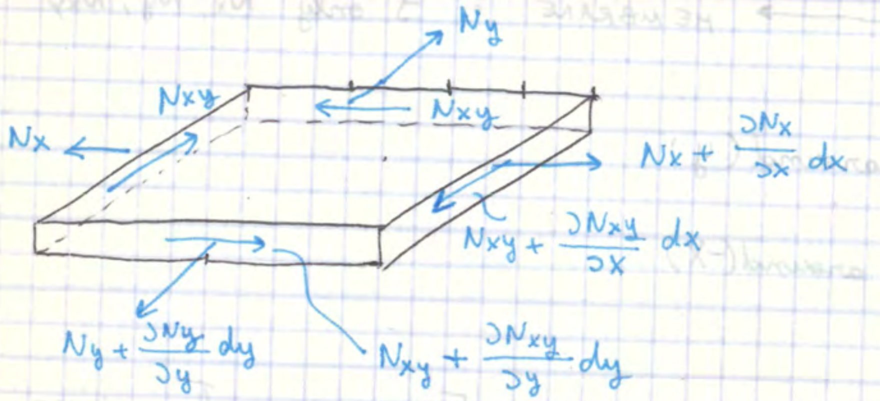
$$\left[\frac{N\cdot m}{m}\right] = [M]$$

$$M_y = \int_{-\frac{h}{2}}^{h/2} \sigma_y z \, dz$$ around (−x)

$$M_{yx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yx}\cdot z \cdot dz$$ around (+y)







MULTIPLIED BY LENGTH TO HAVE [N]

X) $\left(\frac{\partial N_x}{\partial x} dx\right)\boxed{dy} + \left(\frac{\partial N_{xy}}{\partial y} dy\right)\boxed{dx} + p_x \boxed{dx\,dy} = 0$

y) $\left(\frac{\partial N_y}{\partial y} dy\right)dx + \left(\frac{\partial N_{xy}}{\partial x} dx\right)dy + p_y\, dx\, dy = 0$

z) $\left(\frac{\partial T_x}{\partial x} dx\right)dy + \left(\frac{\partial T_y}{\partial y} dy\right)dx + q\, dx\, dy = 0$

Around (+y) :

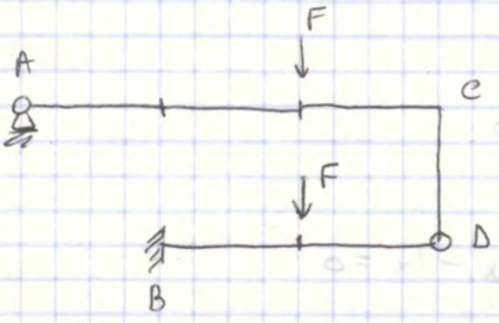
$$\left(\frac{\partial M_x}{\partial x} dx\right)dy + \left(\frac{\partial M_{xy}}{\partial y} dy\right)dx + m_x\, dx\, dy - T_x\, dy\, dx = 0$$

Around (−x)

$$\left(\frac{\partial M_y}{\partial y} dy\right)dx + \left(\frac{\partial M_{xy}}{\partial x} dx\right)dy + m_y\, dx\, dy - T_y\, dx\, dy = 0$$

(= BEAM) THE 2 REGIMES ARE UNCOUPLED, but if plates have curvature ⇒ the problems will be coupled.

EX



$g = 2 \cdot 3 = 6$

$v = 3+2+2 = 7$

} 1 hyperstatic

WRONG WAY

~~1) Put a hinge in C~~

$g = 3 \times 3 = 9$

$v = 3 + 2 \cdot 3 = 9$

~~Associated structure is statically determinated~~
~~→ rotating nodes frame~~
~~⇒ Use only congruence equations~~

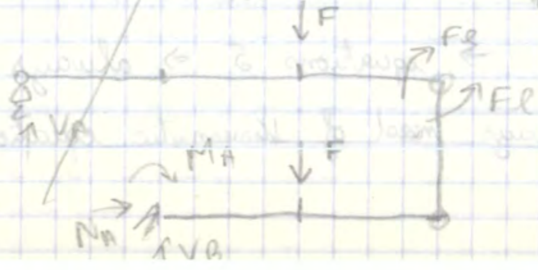
$\varphi_{CA} = \varphi_{CD}$

$\varphi_{CA}(F) + \varphi_{CA}(x) = \varphi_{CD}(X)$

$$+\frac{F\,ab(\ell+a)}{6\ell^2 EI} + \frac{2x\ell}{3EI} = -\frac{2x\ell}{3EI}$$

$$x = -\frac{F\,ab(\ell+a)}{4\ell^2} = \frac{-F\,\ell\,2\ell\,(2\ell)}{4\ell^2} = -2F\ell$$

⇒



$$5F\ell + 18X_1 - \frac{9EI}{\ell^2}\varphi = 0$$

$$-3F\ell - 24X_2 - \frac{18EI}{\ell^2}\varphi = 0$$

$$7F\ell - 2X_1 + 3X_2 = 0$$

$$X_1 = \frac{7}{2}F\ell + \frac{3}{2}X_2$$

$$5F\ell + \overset{9}{\cancel{18}}\left(\frac{7}{2}F\ell + \frac{3}{2}X_2\right) - \frac{9EI}{\ell^2}\varphi = 0$$

$$F\ell(5+9\cdot7) + 27X_2 - \frac{9}{\ell^2}EI\varphi = 0$$

$$X_2 = \frac{-F\ell(5+9\cdot7)}{27} + \frac{9}{27}\frac{EI}{\ell^2}\varphi$$

$$-3F\ell - 24\left[-\frac{(5+9\cdot7)}{27}F\ell + \frac{1}{3}\frac{EI}{\ell^2}\varphi\right] - \frac{18}{\ell^2}EI\varphi = 0$$

$$-3F\ell + \frac{24(5+9\cdot7)}{27}F\ell - 8\frac{EI}{\ell^2}\varphi - 18\frac{EI}{\ell^2}\varphi = 0$$

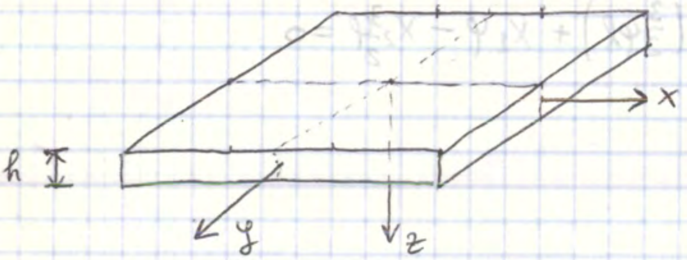
$$F\ell\left(\frac{24(5+9\cdot7)}{27} - 3\right)\frac{\ell^2}{EI}\,\frac{1}{26}\overset{-26}{=}\varphi$$

$$\frac{24\cdot68 - 81}{27\cdot26}$$

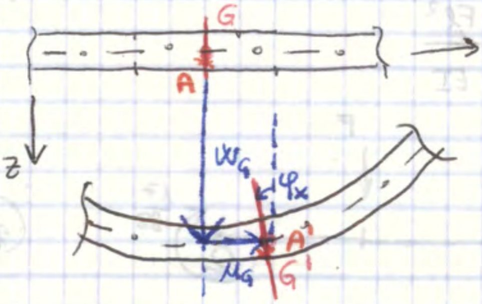
22/11/17

KINEMATIC EQUATIONS FOR PLATES

A segment orthogonal to the middle plane remains a segment after deformation.



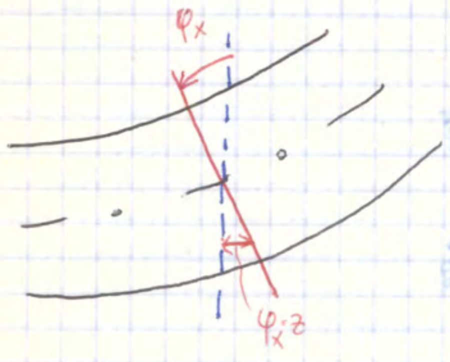
Piece of plate on which we put a load.



- The rotation φ_x is around +y-axis.
- ∃ also displacement along y-axis: v_G.
- ∃ " rotation around x, z.

Displacement of POINTS:

$$u(x,y,z) = u_G(x,y) + \varphi_x \cdot (x,y) \cdot z$$



$$\begin{pmatrix}\varepsilon_{xG}\\ \varepsilon_{yG}\\ \varepsilon_{xyG}\\ \gamma_x\\ \gamma_y\\ \chi_x\\ \chi_y\\ \chi_{xy}\end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{\partial}{\partial x} & +1 & 0\\ 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & +1\\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y}\\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{pmatrix} \cdot \begin{pmatrix}u_G\\ v_G\\ w_G\\ \varphi_x\\ \varphi_y\end{pmatrix}$$

$$\underset{8\times1}{\{g\}} = \underset{8\times5}{[\partial]} \cdot \underset{5\times1}{\{m\}}$$

PRINCIPLE OF VIRTUAL WORKS

System A and B are not related. They must satisfy the 2 types of equations.

$$\left.\begin{aligned} &[\partial]^T\{Q_A\} + \{F_A\} = \{0\}\\ &\{q_B\} = [\partial]\{m_B\}\end{aligned}\right\} \quad \mathcal{L}_{v,EXT} = \mathcal{L}_{v,INT}$$

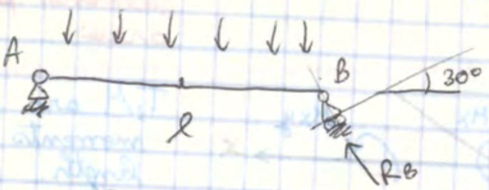
GREEN THEOREM (2D)

⇓

(GAUSS THEOREM (3D))

$$\oint_C g\,dy - f\,dx = \int_A \left(\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\right) dA$$

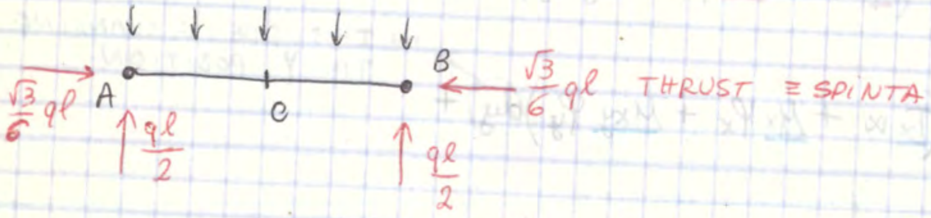
EX) ABOUT ARCHES

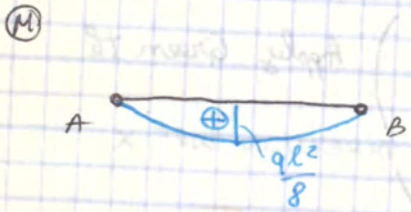


$$A \circlearrowleft \quad R_B \cdot \frac{\sqrt{3}}{2} l - \frac{ql^2}{2} = 0 \quad \Rightarrow \quad R_B = \frac{\sqrt{3}}{3} ql$$

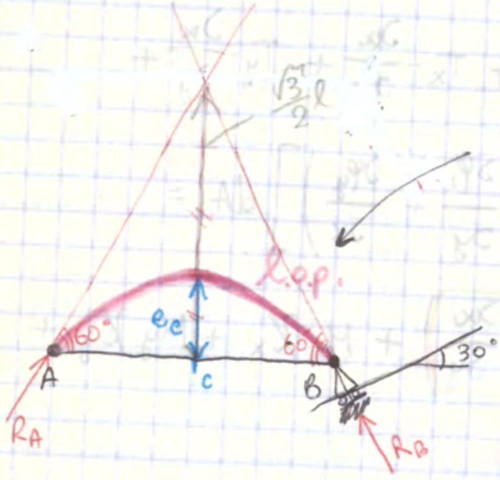
$$H_B = R_{B_{\parallel}} = R_B \cdot \cos(60°) = \frac{\sqrt{3}}{6} ql$$

$$V_B = R_{B_{\perp}} = R_B \cdot \cos(30°) = \frac{ql}{3}$$





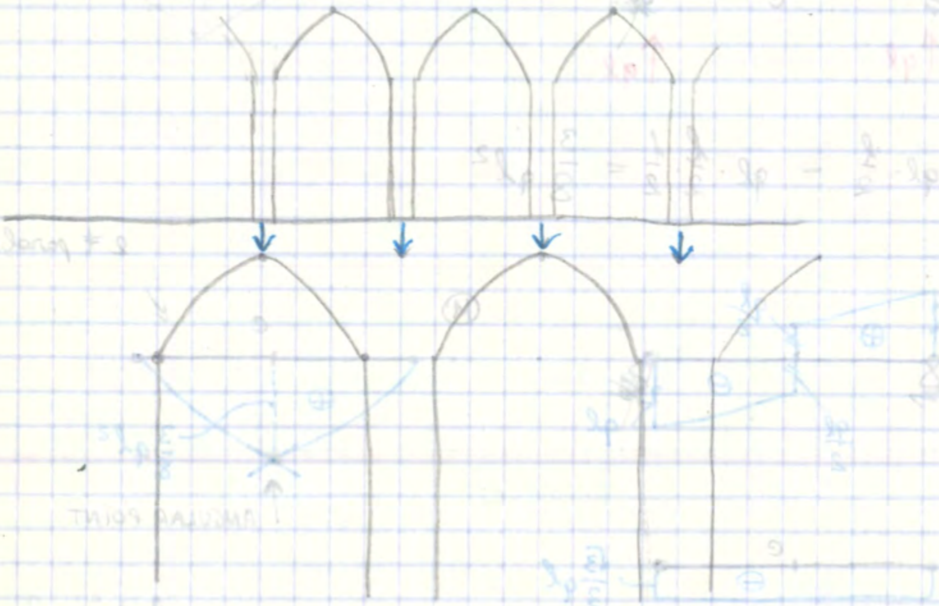
Distances $e = \left|\frac{M}{N}\right| \Rightarrow$ in point C : $e_C = \left|\frac{M_C}{N_C}\right| = \frac{ql^2}{\cancel{8}_4} \cdot \frac{\overset{3}{\cancel{6}}}{\sqrt{3}\,ql} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}l\right)$

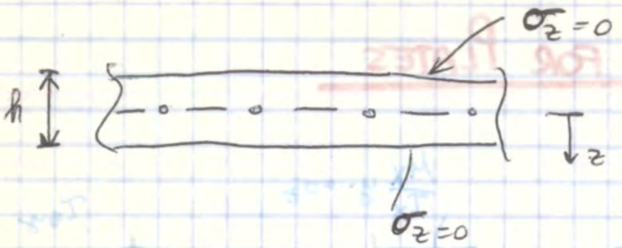


This is a parabola.
If the beam's axis follows the l.o.p ⇒ its better because it will be subjected only to compression.

ex bridges.

GOTHIC ART $\overline{\text{XIII}}$, $\overline{\text{XIV}}$ cent





Assumption:

$\sigma_z = 0$ everywhere

$$\Rightarrow \begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \nu \cdot \dfrac{\sigma_y}{E} - \nu \dfrac{\sigma_z}{E} \\ \varepsilon_y = \dfrac{\sigma_y}{E} - \nu \dfrac{\sigma_x}{E} - \nu \cdot \dfrac{\sigma_z}{E} \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G} \end{cases}$$

$$\begin{cases} \sigma_x = \dfrac{E}{1-\nu^2} \left(\varepsilon_x + \nu \varepsilon_y \right) \\ \sigma_y = \dfrac{E}{1-\nu^2} \left(\varepsilon_y + \nu \varepsilon_x \right) \\ \tau_{xy} = G \cdot \gamma_{xy} \end{cases}$$

$$\begin{cases} \varepsilon_x = \varepsilon_{xG} + \chi_x \cdot z \\ \varepsilon_y = \varepsilon_{yG} + \chi_y \cdot z \\ \gamma_{xy} = \varepsilon_{xyG} + \chi_{xy} \cdot z \end{cases}$$

$$\begin{cases} \sigma_x = \dfrac{E}{1-\nu^2} \left(\varepsilon_{xG} + \nu \varepsilon_{yG} \right) + \dfrac{E}{1-\nu^2} \left(\chi_x + \nu \chi_y \right) z \\ \sigma_y = \dfrac{E}{1-\nu^2} \left(\varepsilon_{yG} + \nu \varepsilon_{xG} \right) + \dfrac{E}{1-\nu^2} \left(\chi_y + \nu \chi_x \right) z \\ \tau_{xy} = G \cdot \varepsilon_{xyG} + G \chi_{xy} z \end{cases}$$

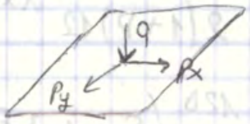
$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz = \frac{Eh}{1-\nu^2} \left(\varepsilon_{xG} + \nu \varepsilon_{yG} \right)$$

$$N_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} G_y \, dy = \frac{Eh}{1-\nu^2} \left(\varepsilon_{yG} + \nu \varepsilon_{xG} \right)$$

$$N_{xy} = \int_{-h}^{\frac{h}{2}} \tau_{xy} \, dz = Gh \cdot \varepsilon_{xyG}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ T_x \\ T_y \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{pmatrix} \frac{12D}{h^2} & \nu\cdot\frac{12D}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu\frac{12D}{h^2} & \frac{12D}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{6D(1-\nu)}{h^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5D(1-\nu)}{h^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5D(1-\nu)}{h^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D & \nu D & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu D & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{D(1-\nu)}{2} \end{pmatrix} \begin{Bmatrix} \varepsilon_{xG} \\ \varepsilon_{yG} \\ \varepsilon_{xyG} \\ \gamma_x \\ \gamma_y \\ \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix}$$

$$\underset{8\times1}{\{Q\}} = \underset{8\times8}{[H]}\ \underset{8\times1}{\{q\}}$$



$\Rightarrow$ Bending and membrane regime are uncoupled

$P_x, P_y \Rightarrow N_x, N_y, N_{xy} \Rightarrow \varepsilon_x, \varepsilon_y, \varepsilon_z \Rightarrow u, v.$

$q, \underbrace{(m_x, m_y)}_{\text{don't usually met.}} \Rightarrow T_x, T_y, M_x, M_y, M_{xy} \Rightarrow \gamma_x, \gamma_y, \chi_x, \chi_y, \chi_{xy}$

$\Rightarrow w, \varphi_x, \varphi_y$

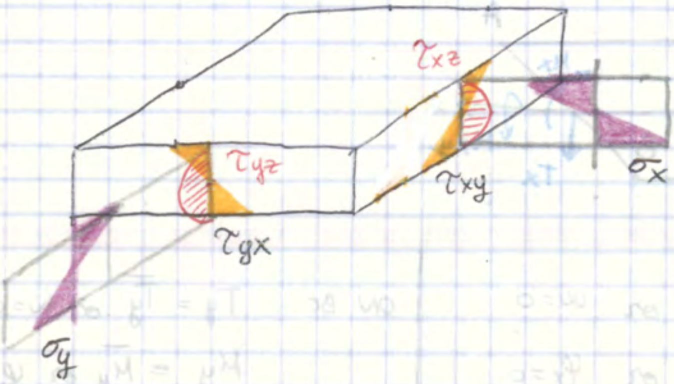
STATIC $\underset{3\times5}{[\partial]^*} \cdot \underset{5\times1}{\{Q\}} + \underset{3\times1}{\{F\}} = \underset{3\times1}{\{0\}}$ twice STATICALLY INDET

~~KINEM~~ $\underset{5\times1}{\{Q\}} = \underset{5\times5}{[H]} \cdot \underset{5\times1}{\{q\}}$

KINET $\underset{5\times1}{\{q\}} = \underset{5\times3}{[\partial]}\ \underset{3\times1}{\{\eta\}}$

How to evaluate stresses?

$$\sigma_x = \frac{M_x}{\frac{h^3}{12}} \cdot z \quad ; \quad \sigma_y = \frac{12\, M_y}{h^3} \cdot z \quad ; \quad \tau_{xy} = \frac{12\, M_{xy}}{h^3} \cdot z$$



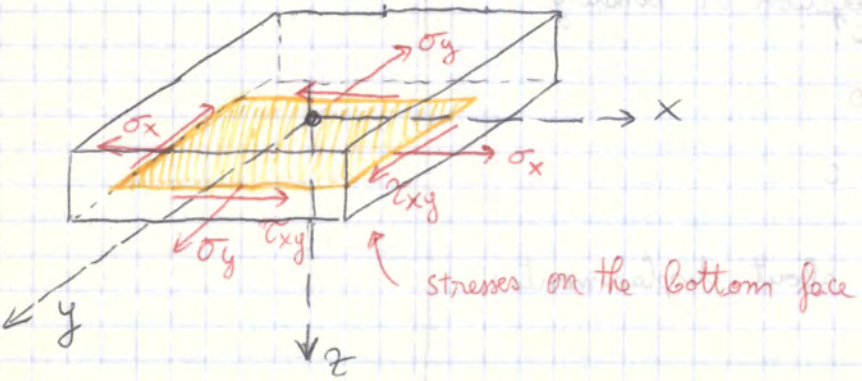
Maximum / Minimum values are achieved for $z = \pm \frac{h}{2}$

$$(\sigma_x)_{max} = \frac{6\, M_x}{h^2}$$

$$(\sigma_y)_{max} = \frac{6\, M_y}{h^2}$$

$$(\tau_{xy})_{max} = \frac{6\, M_{xy}}{h^2}$$

τ_{yz} is negligible



PLATES UNDER BENDING

23/11/17

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ -1 & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & -1 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} T_x \\ T_y \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} + \begin{pmatrix} q \\ m_x \\ m_y \end{pmatrix} = 0$$

Semplification: NEGLECT THE SHEAR DEFORMABILITY FOR h (small)

$$\rightarrow \begin{cases} \gamma_x = 0 \\ \gamma_y = 0 \end{cases}$$

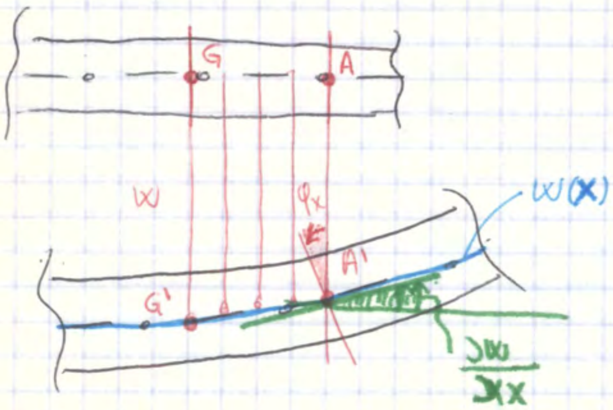
$$\begin{pmatrix} \gamma_x \\ \gamma_y \\ \chi_x \\ \chi_y \\ \chi_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 1 & 0 \\ \frac{\partial}{\partial y} & 0 & 1 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} w \\ \varphi_x \\ \varphi_y \end{pmatrix}$$

! $\left.\begin{matrix} m_x = 0 \\ m_y = 0 \end{matrix}\right\}$ as usual

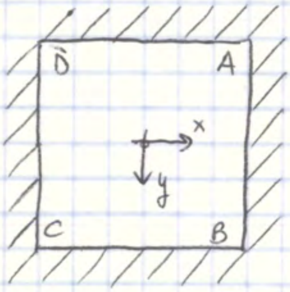
(semplification

$$\Rightarrow \begin{cases} \varphi_x = -\dfrac{\partial w}{\partial x} \\ \varphi_y = -\dfrac{\partial w}{\partial y} \end{cases}$$

Neglecting shear deformability means that after deformation a segment ⊥ to the middle plane remains a segment and ⊥ to the middle plane.



BOUNDARY CONDITIONS



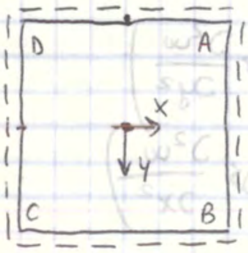
$\overline{AB}$ FOR $\gamma_{x,y} \neq 0$ $= \begin{cases} w = 0 \\ \varphi_x = 0 \\ \varphi_y = 0 \end{cases}$ $\not\Rightarrow$ $\overline{AB}$ FOR $\gamma_{x,y} = 0$ $\begin{cases} \boxed{w = 0 \\ \frac{\partial w}{\partial x} = 0} \\ \frac{\partial w}{\partial y} = 0 \end{cases}$

INDIPENDENT

$\frac{\partial w}{\partial x} \neq 0$, $\frac{\partial w}{\partial y} = 0$

meaningless because since $w = 0$, there is no φ_y

$\overline{CB} = \begin{cases} w = 0 \\ \varphi_x = 0 \\ \varphi_y = 0 \end{cases} \Rightarrow \begin{cases} \boxed{w = 0} \\ \frac{\partial w}{\partial x} = 0 \\ \boxed{\frac{\partial w}{\partial y} = 0} \end{cases}$



$\overline{AB} = \begin{cases} w = 0 \\ \varphi_x \neq 0 \Rightarrow M_x = 0 \\ \varphi_y = 0 \end{cases}$ $\begin{cases} w = 0 \\ \frac{\partial w}{\partial y} = 0 \\ \frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} = 0 \end{cases}$

$\gamma = 0$

$\Rightarrow \boxed{\begin{matrix} w = 0 \\ \frac{\partial^2 w}{\partial x^2} = 0 \end{matrix}}$

$\overline{BC}$:

$\gamma \neq 0$ $\begin{cases} w = 0 \\ M_y = 0 \\ \varphi_x = 0 \end{cases}$

$\gamma = 0$ $\begin{cases} w = 0 \\ \frac{\partial^2 w}{\partial y^2} + \cancel{\nu \frac{\partial^2 w}{\partial x^2}} = 0 \\ \cancel{\frac{\partial w}{\partial x}} = 0 \end{cases}$ $\Rightarrow \begin{cases} w = 0 \\ \frac{\partial^2 w}{\partial y^2} = 0 \end{cases}$

$$\overline{CB} = \begin{cases} M_y = 0 \\ T_y + \dfrac{\partial M_{xy}}{\partial x} = 0 \end{cases}$$

$$V_x = T_x + \frac{\partial M_{xy}}{\partial y} = \frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} =$$

$$= -D\left\{\left(\frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial y^2 \partial x}\right) + 2(1-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right\}$$

$$V_x = 0 \Rightarrow \frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2} = 0$$

$$M_x = 0 \Rightarrow \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0$$ $\overline{AB}$ // y - axis

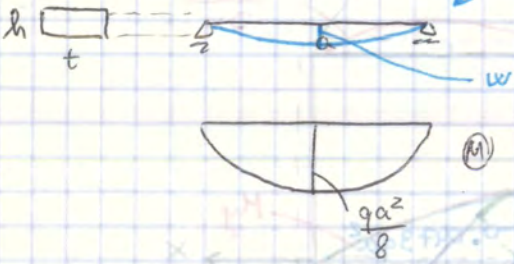
Solution: with uniformely distributed load = Fourier series
↳ Timoshenko's book $\qquad$ ||| trigonometric series

→ TABLES GIVEN BY PROF

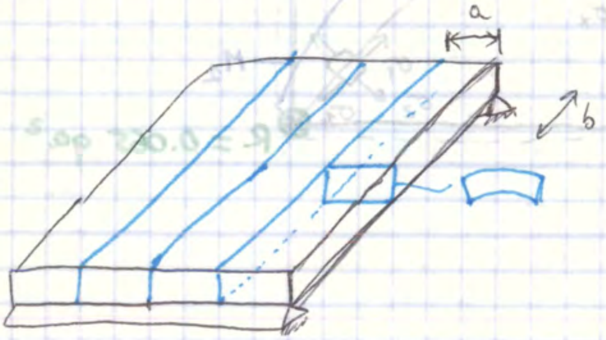
⇒ TABLE 8 $\quad$ if $\frac{b}{a} \rightarrow \infty$ ⇛ shape

⇒ it is like a beam

$$\frac{5}{384}\cdot\frac{qa^4}{D} = \frac{(1-\nu^2)5}{384}\cdot\frac{q\,a^4\,12}{E h^3}$$

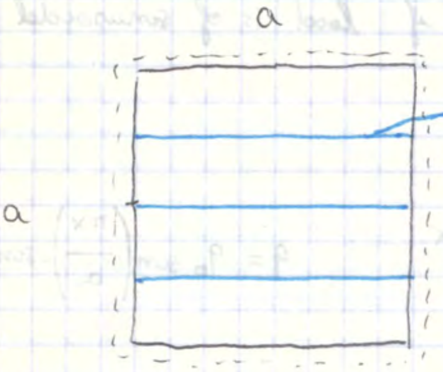


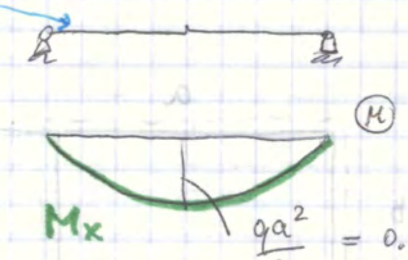
$$w = \frac{5}{384}\cdot\frac{qa^4}{EI} = \frac{5}{384}\underbrace{\left(\frac{q}{t}\right)}_{\frac{kN}{m^2}}\cdot\frac{12.h^3}{E}$$



if $\frac{b}{a} \rightarrow \infty \geq 5, \dots$

You can neglect the effect of the plate because it's acting like a beam.





$\frac{qa^2}{8} = 0,125\cdot qa^2$ NOT REAL

BUT Ⓜ REAL IN THE MIDDLE is $0,0473\,qa^2$

⇒ the difference is an error of 263%. So the plate is stiffer than several beams put one near each other.

Approximation of 2nd derivative : 1st derivative

in P

in K

$$f''(x_i) = \frac{f'\left(x_i + \frac{\Delta}{2}\right) - f'\left(x_i - \frac{\Delta}{2}\right)}{\Delta} =$$

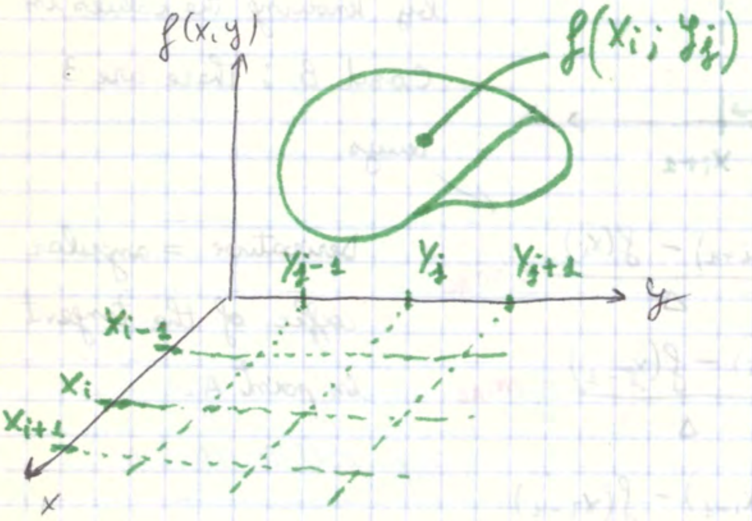
$$= \frac{\frac{f(x_{i+1}) - f(x_i)}{\Delta} - \frac{f(x_i) - f(x_{i-1})}{\Delta}}{\Delta} =$$

$$= \frac{(+1)\, f(x_{i+1}) \; (-)\, 2f(x_i) \overset{(+1)}{+} f(x_{i-1})}{\Delta^2}$$

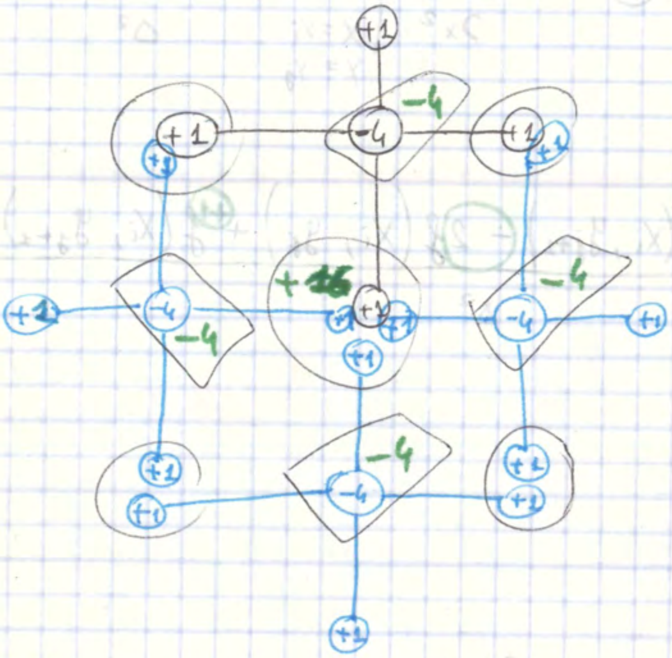
Rapresentation:

$$f''(x_i) \cong \frac{\text{"molecule"}}{\Delta^2}$$

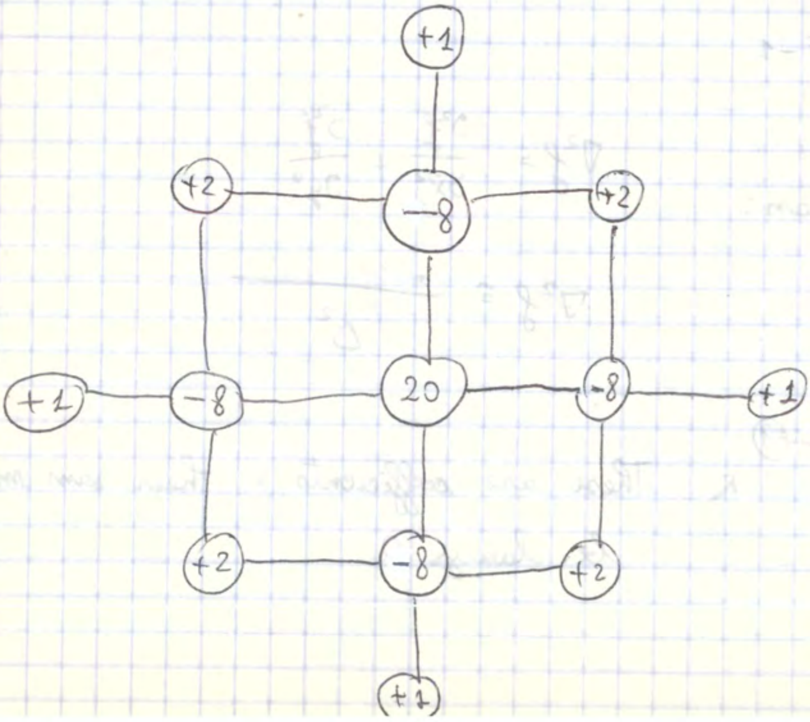
But our function has 2 variables.



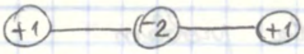
$$\Rightarrow \nabla^4 f = \nabla^2\left(\nabla^2 f\right) \cong \frac{\quad}{\Delta^4}$$



⇓



$\Rightarrow \quad \chi_x = \frac{\partial^2 w}{\partial x^2} = 0$



in point 14 $\quad w_{13} \underbrace{- 2w_{14}}_{=0} + w_{15} = 0 \Rightarrow w_{15} = -w_{13}$

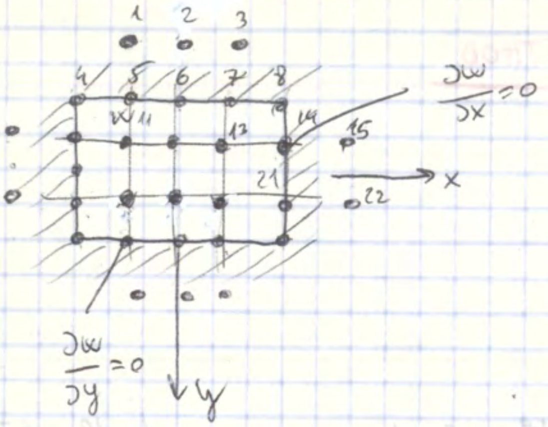
$$\Rightarrow \left.\begin{array}{ll} w_1 = -w_{11} & w_{15} = -w_{13} \\ w_2 = -w_{12} & w_{22} = -w_{20} \\ w_3 = -w_{13} & \cdots \end{array}\right\} \text{ in total 10 equations.}$$

$\Rightarrow$ Total unknowns 30

$\Rightarrow$ equations $6 + 14 + 10 = 30$!

$\Rightarrow$ The system can be solved 30×30

CLAMPED SIDE



$$w_{13} = w_{15}$$

$$\frac{-w_{13}}{2} + \frac{w_{15}}{2} = 0$$

~ HOMEWORK ~

30 nodes → unknowns

‖

6 internal equations

\+

14 border equat from condition : $w_{4,5,6,\dots} = 0$

\+

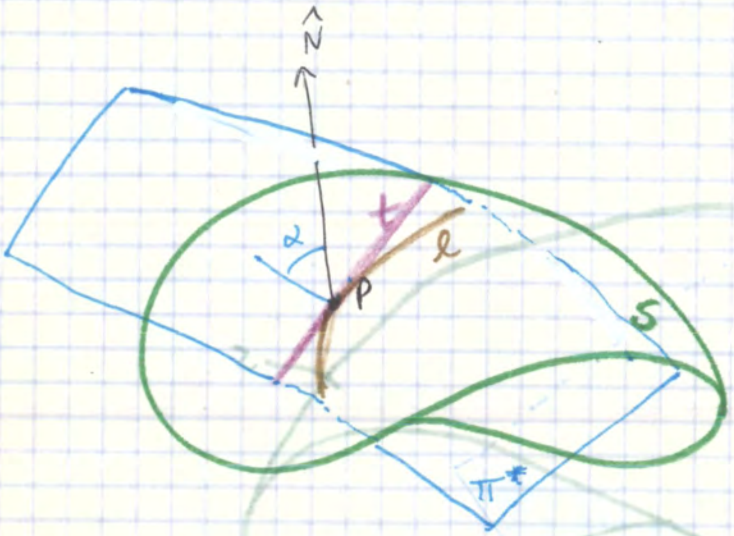
10 border equat from condition :

$\varphi_x = \frac{\partial w}{\partial x} = 0$ ⇒ superpose the molecule of first derivative

$\left[\varphi_y = \frac{\partial w}{\partial y} = 0\right]$ on points 14, 21....

ex point (14) : $0 \cdot w_{14} - \frac{1}{2} w_{13} + \frac{1}{2} w_{15} = 0$

$\Rightarrow w_{13} = w_{15}$



L and ℓ have the same tangent t.

$t \in \pi$, $\hat{N} \notin \pi$, α is angle between $\hat{N}$ and π.

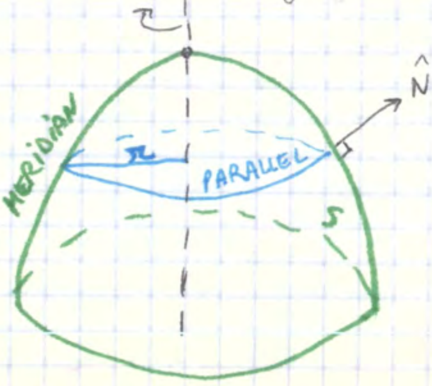
THEOREM MEUSNIER : $r = R\cos\alpha$

r = raggio di ℓ in P. of curvature

R = raggio of curv. of L in P.

SHELLS OF REVOLUTION

Are obtained by taking a line and rotate it in space



Meridians ⊥ Parallels.

Meridians ↓ Principal Normal Sections ⇓ R_1

Parallels ↓ Are not usually normal sections

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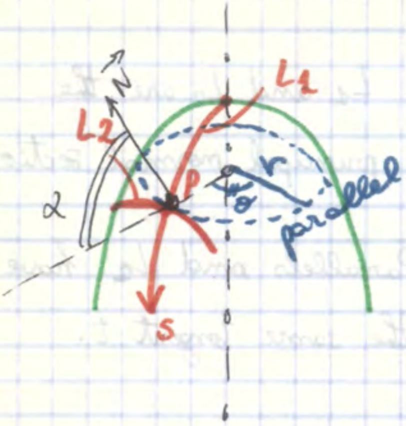


L_1 : meridian principal normal section.
↳ R_1 =

L_2 = principal normal section
↳ R_2

The parallel shares the same tangent as L_2 in P.

$L_2 \perp L_1$ in P

Parallel $\perp L_1$ in P.

α = angle between $\vec{N}$ and horizontal plane passing through P.

ALTITUDE $\alpha = 90°$ → we're in the pole

$\alpha = 0°$ ⇒ we're at the equatore

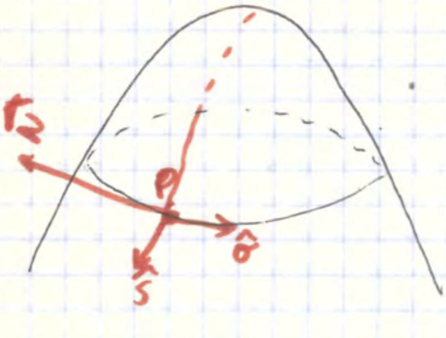
θ = angle between P and r : LONGITUDE.

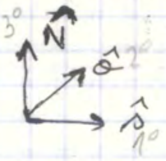
The coordinates of P are $P = P(\alpha, \theta)$ or $P = P(s, \theta)$

s = curvilinear abscissa on the meridian

∃ LOCAL REFERENCE SYSTEM $(\hat{s}, \hat{\theta}, \hat{N})$ analogous to

$(\hat{i}, \hat{j}, \hat{k}) \equiv (x, y, z)$





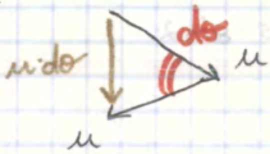
Right hand rule.

∇ Meusnier theorem :

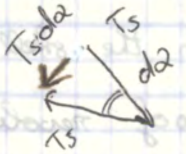
$r = R_2 \cdot \cos\alpha$

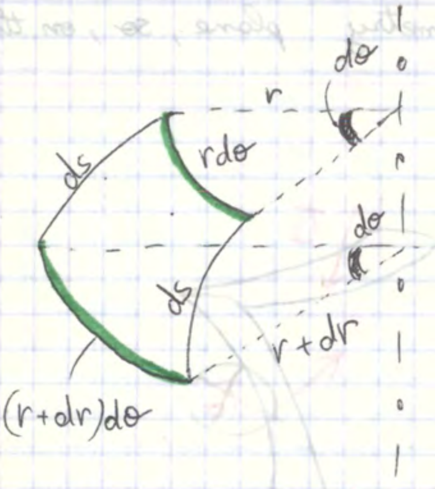
$$\Rightarrow \quad \{\mathcal{F}\} = \begin{Bmatrix} p_s \\ p_\theta \\ q \\ m_s \\ m_\theta \end{Bmatrix} \quad \Rightarrow \quad \{\mathcal{F}\} = \begin{Bmatrix} p_s \\ q \\ m \end{Bmatrix}$$

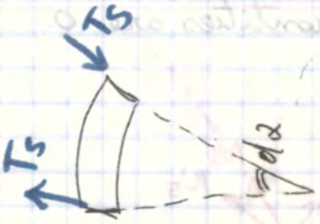
! ARCH



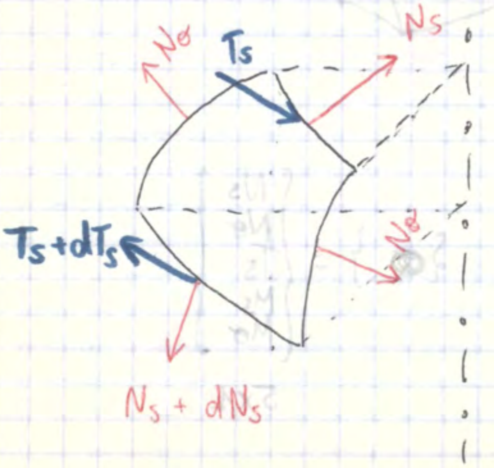
EQUILIBRIUM EQUATIONS

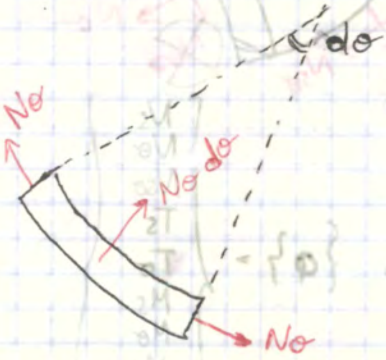




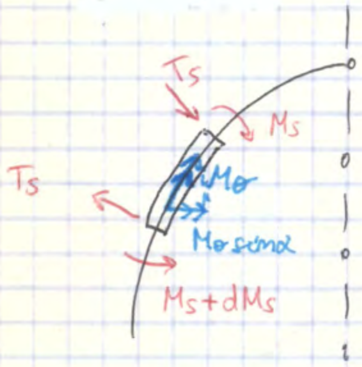


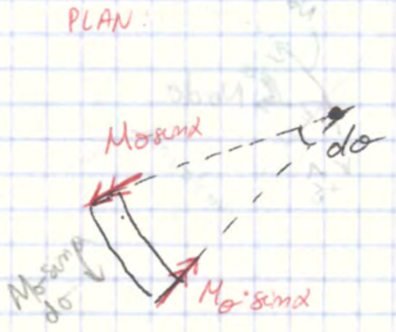
PLAN: TOP VIEW





Equilibrium along parallel : 0=0





$$(M_s + dM_s)(r+dr)\,d\theta - M_s \cdot r \cdot d\theta - M_\theta \sin\alpha \, ds \, d\theta -$$

$$- T_s \cdot r \cdot d\theta \cdot ds + m_s \, r \, d\theta \, ds = 0$$

$$\Rightarrow \left(\frac{d}{ds} + \frac{\sin\alpha}{r}\right) M_s - \frac{\sin\alpha}{r} M_\theta - T_s + m_s = 0$$

⇒ MATRIX

MEMBRANE

$$\begin{pmatrix} \frac{d}{ds} + \frac{\sin\alpha}{r} & -\frac{\sin\alpha}{r} & +\frac{1}{R_1} & 0 & 0 \\ -\frac{1}{R_1} & -\frac{1}{R_2} & \frac{d}{ds} + \frac{\sin\alpha}{r} & 0 & 0 \\ 0 & 0 & -1 & \frac{d}{ds} + \frac{\sin\alpha}{r} & -\frac{\sin\alpha}{r} \end{pmatrix} \begin{pmatrix} N_s \\ N_\theta \\ T_s \\ M_s \\ M_\theta \end{pmatrix} + \begin{pmatrix} p_s \\ q \\ m_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

BENDING REGIME

$$\underset{3\times5}{[\partial]^*} \cdot \underset{5\times1}{\{Q\}} + \underset{3\times1}{\{\mathcal{F}\}} = \underset{3\times1}{\{0\}}$$

Membrane regime and bending regime are COUPLED.

$$\mathcal{L}_{V,EXT} = \int_S (p_s u + q w + m_s \varphi_s)\, r\, d\theta\, ds +$$

$$+ \oint_{\mathcal{C}} (N_s \cdot u + T_s w + M_s \varphi_s)\, r\, d\theta =$$

ON BORDER = CLOSED PATH

$$= \int_S (N_s \varepsilon_s + N_\theta \varepsilon_\theta + T_s \gamma_s + M_s \chi_s + M_\theta \chi_\theta)\, r\, d\theta\, ds =$$

$$= \mathcal{L}_{V,INT}$$

→ Substitute the static equations to $p_s, \dots$; and apply Green's theorem :

$$-\int \frac{dN_s}{ds} \cdot u \; r\, ds\, d\theta = \int N_s \frac{d}{ds}(u \cdot r)\, ds\, d\theta +$$

$$-\oint_{\mathcal{C}} N_s u r\, d\theta =$$

Green's terms

$$\begin{cases} g = N_s \\ h = u \cdot r \\ A \to S \\ x \to s \\ y \to \theta \end{cases}$$

$$\frac{dr}{ds}$$

$$= \int N_s \left(\frac{du}{ds} + \cancel{\frac{\sin\alpha}{r}}\right) r\, ds\, d\theta - \cancel{\oint_{\mathcal{C}} N_s u r\, d\theta}$$

cancel each other

→ DO THE SAME FOR q and φ_s

$$\Rightarrow \mathcal{L}_{V,EXT} = \int_S \Bigg[N_s \underbrace{\left(\frac{du}{ds} + \frac{w}{R_1}\right)}_{\varepsilon_s} + N_\theta \underbrace{\left(\frac{\sin\alpha}{r} u + \frac{w}{R_2}\right)}_{\varepsilon_\theta} +$$

$$+ T_s \underbrace{\left(\frac{dw}{ds} + \frac{u}{R_1} + \varphi_s\right)}_{\gamma_s} + M_s \underbrace{\left(\frac{d\varphi_s}{ds}\right)}_{\chi_s} +$$

$$+ M_\theta \underbrace{\left(\frac{\sin\alpha}{r} \varphi_s\right)}_{\chi_\theta} \Bigg] r\, d\theta\, ds = \mathcal{L}_{V,INT}$$

In order to have the equality between integrals, the quantities in the parenthesis are equal to the deformations ■

$h \to 0$

BEAMS | ROPES

Bending stiffness $\neq 0$

Bending stiffness $= 0$
(or negligible WRT axial stiffness).

As $h \to 0$: bending stiff $\to 0$ faster than axial stiffness.
because:

$$I = \frac{bh^3}{12} \quad , \quad A = b \cdot h$$

MEMBRANES : is a shell whose bending stiff. $= 0$

$$D = \frac{E \cdot h^3}{12(1-\nu^2)}$$

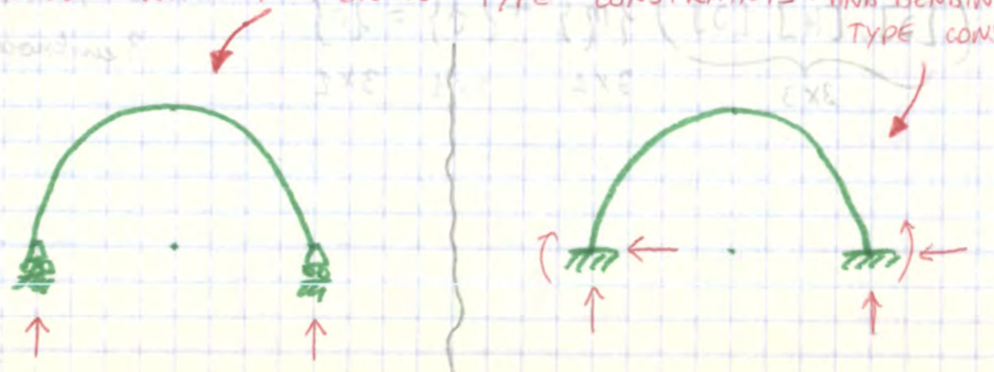
(the same of plate theory if radius of curvature is very large)

! It's an abstraction (mathematical) to say that a shell is a membrane : it's used for very thin shells where h is negligible.

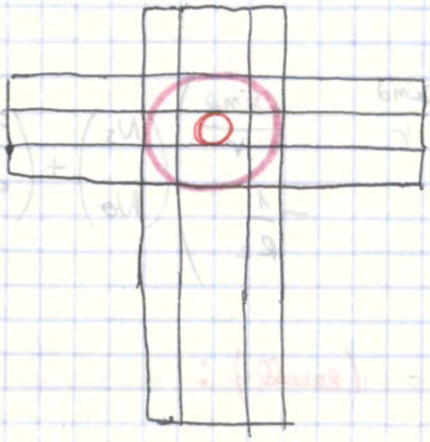
Membrane solution is important also because:

1) It is almost equal to shell solution if constraints are of membrane type.
2) It holds also for shells with non - membrane constraints sufficiently far from the constraints.

Ex| SHELL WITH MEMBRANE TYPE CONSTRAINTS | AND BENDING TYPE CONSTRAINTS



Ex cathedral of Parma : build 1294



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Recall:

$$\begin{pmatrix} \frac{d}{ds}+\frac{\sin\alpha}{r} & -\frac{\sin\alpha}{r} \\ -\frac{1}{R_1} & -\frac{1}{R_2}\end{pmatrix}\begin{pmatrix}N_s\\ N_\theta\end{pmatrix}+\begin{pmatrix}p_s\\ q\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\varepsilon_s\\ \varepsilon_\theta\end{pmatrix}=\begin{pmatrix} d/ds & 1/R_1 \\ \frac{\sin\alpha}{r} & \frac{1}{R_2}\end{pmatrix}\begin{pmatrix}u\\ w\end{pmatrix}$$

$$\begin{pmatrix}\varepsilon_s\\ \varepsilon_\theta\end{pmatrix}=\begin{pmatrix} \frac{1}{Eh} & -\frac{\nu}{Eh} \\ -\frac{\nu}{Eh} & \frac{1}{Eh}\end{pmatrix}\begin{pmatrix}N_s\\ N_\theta\end{pmatrix}$$

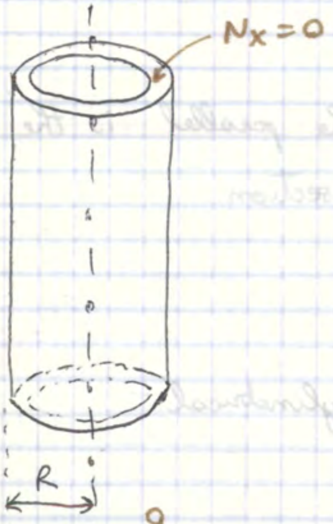
$$\Rightarrow \varepsilon_\sigma = \frac{\sigma_\sigma}{E} - \nu\frac{\sigma_x}{E} = \frac{N_\sigma}{Eh} - \nu\frac{N_x}{Eh} = \frac{w}{R}$$

$$\boxed{w = \frac{R}{E\cdot h}\left(qR - \nu N_x\right)}$$

Displacement in z direction.

∞ PIPE WITHOUT BOTTOMS (EMPTY)

① CASE



$N_x = 0 = \text{constant}$

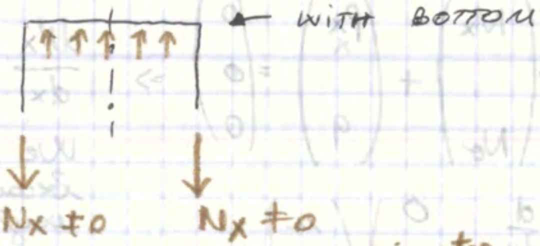
$$\Rightarrow w = \frac{R}{Eh}(qR) = \Delta R$$

$$\Rightarrow q = \underbrace{\frac{Eh}{R^2}}\cdot w$$

STIFFNESS OF THE CILINDER

The smaller is the radius, the stiffer is the cylindrical membrane.

② CASE



$$N_x \cdot \overset{\text{line = perimeter}}{\overbrace{2\pi R}}^{\text{surface where } N_x \text{ is acting}} = q\,\underbrace{\pi R^2}_{\text{area of bottom.}}$$

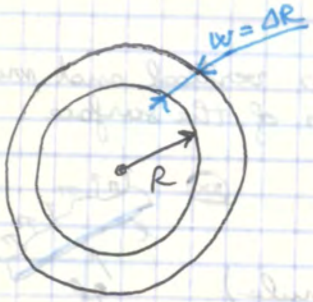
$$\left[\frac{N}{m}\right]\cdot[m] \qquad \left[\frac{N}{m^2}\right]\cdot[m^2]$$

$$\Rightarrow w = \Delta R = \frac{qR^2}{Eh}\left(1-\frac{\nu}{2}\right)$$

$$\Rightarrow q = \underbrace{\frac{2Eh}{(2-\nu)R^2}}_{\text{STIFFNESS WITH BOTTOMS.}}\cdot w \qquad \nu > 0$$

This cylinder is stiffer than the one without bottoms.

Find the stiffness : it's stiffer than cylinder



$$\varepsilon_\theta = \frac{w}{R} = \frac{1-\nu}{Eh} N = \frac{1-\nu}{Eh}\cdot\frac{qR}{2h}$$

$$\Rightarrow q = \boxed{\frac{2Eh^2}{(1-\nu)R^2}}\, w$$

stiffness of spherical shape membrane.

Which is the internal pressure leading to the failure a spherical shell ?

$$(\sigma_{eq})_{\text{VON MISES}} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} = \frac{qR}{2h} < \sigma_{\text{admissible}}$$

$$\sigma_1 = \sigma_2 = \frac{qR}{2h}$$

$$\Rightarrow \quad q_{CRIT} = \frac{2h}{R}\sigma_{\text{admissible}}$$

And for cylindrical ?

$$\sigma_\theta = \frac{qR}{h} = \sigma_1$$

$$\sigma_x = \frac{qR}{2h} = \sigma_2$$

(vertical)

because ∄ shear in these directions

$$\Rightarrow (\sigma_{eq})_{V.M} = \frac{qR}{h}\sqrt{1+\frac{1}{4}-\frac{1}{2}} = \frac{\sqrt{3}}{2}\cdot\frac{qR}{h}$$

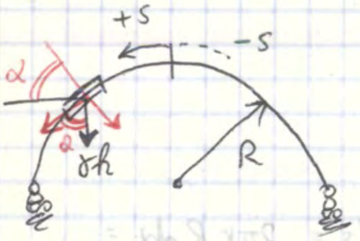
$$\Rightarrow \quad q_{CRIT} = \frac{2}{\sqrt{3}}\cdot\frac{h}{R}\sigma_{adm} = \frac{2\sqrt{3}}{3}\cdot\frac{h}{R}\sigma_{adm}$$

→ Dome subjected to self weight

$$\begin{pmatrix} \frac{d}{ds} + \frac{\sin\alpha}{r} & -\frac{\sin\alpha}{r} \\ -\frac{1}{R} & -\frac{1}{R} \end{pmatrix} \begin{pmatrix} N_s \\ N_\theta \end{pmatrix} = -\begin{pmatrix} p_s \\ q \end{pmatrix}$$

$R_1 = R_2 = R$

$r = R\cos\alpha$



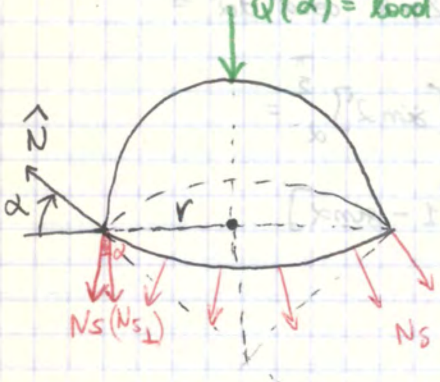
I want components of γh along $\hat{s}$ and along $\hat{\theta}$

$$\begin{cases} p_s = +\gamma h\cos\alpha \\ q = -\gamma h\sin\alpha \end{cases}$$

2 x 2 system

If we write an equilibrium of a part of the dome, we find:

Q(α) = load of the portion of the dome.



$Q(\alpha) = \gamma h \cdot A(\alpha)$

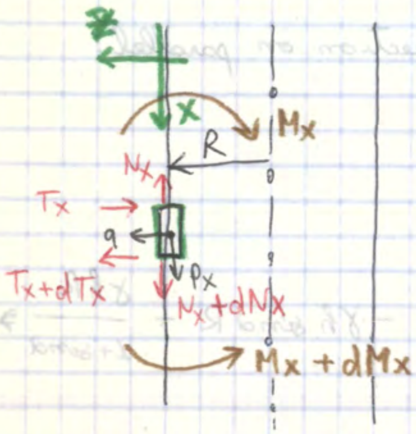
$N_{s,\perp} = \cdot\cos\alpha\, N_s$

$\Rightarrow \quad N_s \cdot \cos\alpha \cdot 2\pi \cdot r + Q(\alpha) = 0$

$r = R\cos\alpha$

$$N_s = -\frac{Q(\alpha)}{2\pi R\cos^2\alpha} = \frac{-\gamma h(1-\sin\alpha)\cancel{2}\cancel{\pi}R^2}{\cos^2\alpha\ \cancel{2R\pi}} \Rightarrow \cdot\frac{(1+\sin\alpha)}{(1+\sin\alpha)}$$

Cylindrical Shells



$\alpha = 0$

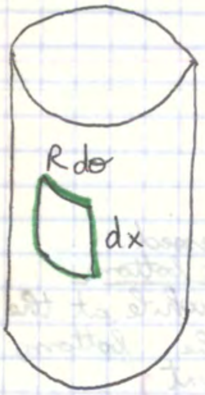
$R_1 = \infty$

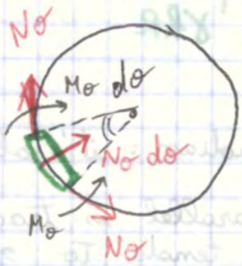
$R_2 = r = R$

$s = x$

MATRIX OF EQUILIBRIUM:

$$\begin{pmatrix} \frac{d}{dx} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R} & \frac{d}{dx} & 0 & 0 \\ 0 & 0 & -1 & \frac{d}{dx} & 0 \end{pmatrix} \begin{pmatrix} N_x \\ N_\theta \\ T_x \\ M_x \\ M_\theta \end{pmatrix} + \begin{pmatrix} p_x \\ q \\ m_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$





? $N_\theta\, d\theta$ = resultant of the two N_θ

Equations along meridian

$$(N_x + dN_x)\, R d\theta - N_x\, R d\theta + p_x \cdot R \cdot d\theta\, dx = 0$$

$$\Rightarrow \quad \frac{dN_x}{dx} + p_x = 0$$

CONSTITUTIVE EQUATIONS

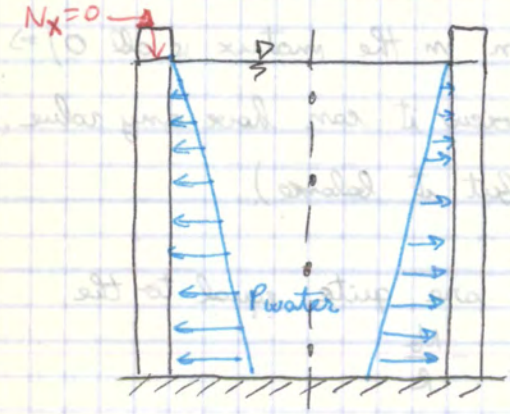
$$\varepsilon_x = \frac{N_x}{Eh} - \nu \frac{N_\theta}{Eh}$$

$$\varepsilon_\theta = \frac{N_\theta}{Eh} - \nu \frac{N_x}{Eh}$$

$$M_x = D\left(\chi_x + \nu\chi_\theta\right)$$

$$M_\theta = D\left(\chi_\theta + \nu\chi_x\right)$$

⇒ Now PROBLEM = TANK FILLED WITH WATER



$p_x = 0 \Rightarrow N_x = 0 = \text{constant}$

$$\Rightarrow \varepsilon_\theta = \frac{w}{R}$$

$$\gamma_x = \frac{dw}{dx} + \varphi_x \boxed{=0}$$

$$\chi_x = \frac{d\varphi_x}{dx}$$

$$\chi_\theta = 0$$

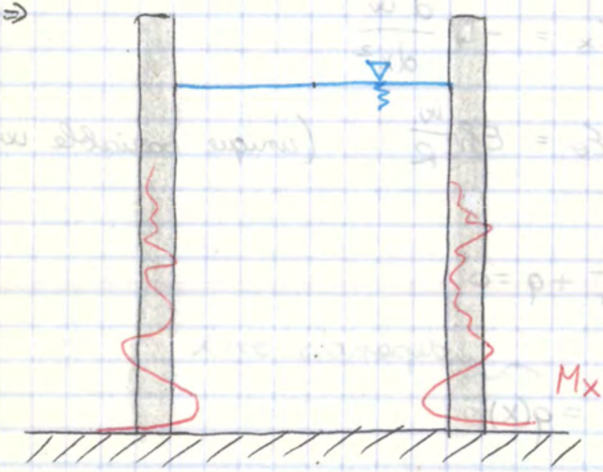
Assume to neglect shear deformability (to simplify as for beams)

$$\Rightarrow \varphi_x = -\frac{dw}{dx}$$

$$\Rightarrow \chi_x = -\frac{d^2w}{dx^2}$$

⇒ $$\boxed{\frac{d^4 w}{dx^4} + 4\beta^4 \cdot w = \frac{q(x)}{D}}$$

⇒

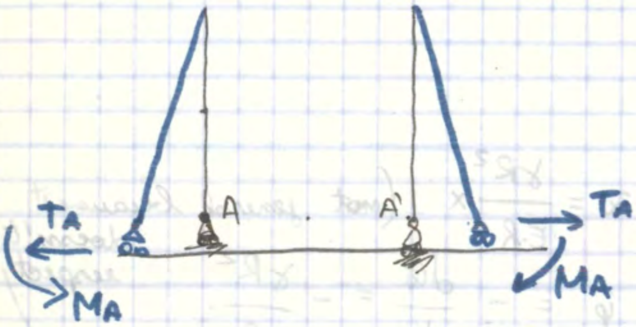


Bending moment Mx quickly vanishes, close to the border their values must be checked.

⇒ CONDITIONS :

$$\begin{cases} M(0)=0 \Rightarrow w''(0)=0 \\ T(0)=0 \Rightarrow w'''(0)=0 \\ w'(\ell)=0 \\ w(\ell)=0 \end{cases}$$

→ Simplification : Solution valid for $\ell > \lambda = \frac{2\pi}{\beta}$



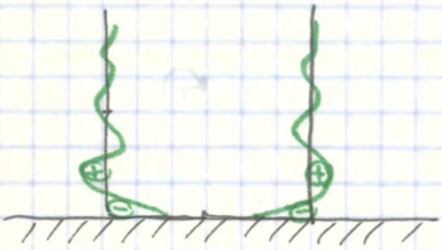
I put in A and A' a roller in order to have M=0 and T=0, but ∃ forces T_A and $M_A \neq 0$, whose values are such that M and T will be =0.

⇒ Use compliances :

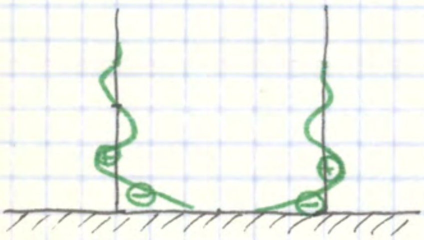
$$\begin{cases} w_A = 0 \\ \varphi_A = 0 \end{cases} \quad \begin{cases} \lambda_{FF} \cdot T_A + \lambda_{FM} M_A + \tilde{w}_A = 0 \\ \lambda_{MF} T_A + \lambda_{MM} M_A + \tilde{\varphi}_A = 0 \end{cases}$$

	Beam on elastic foundation	Cylindrycal Shells
λ_{FF}	$\mp \frac{1}{2EI\beta^3}$	$\mp \frac{1}{2D\beta^3}$
$\lambda_{FM}=\lambda_{MF}$	$= \frac{1}{2EI\beta^2}$	$= \frac{1}{2D\beta^2}$
λ	$= \frac{1}{}$	$\mp \frac{1}{D\beta}$

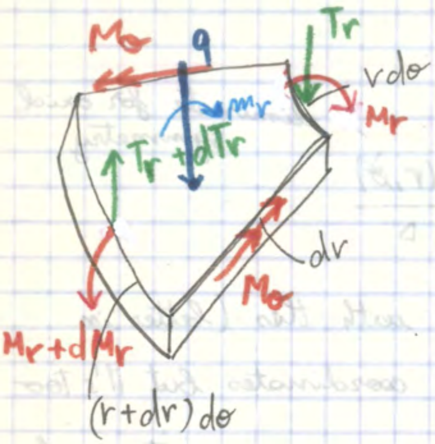
$\left.\begin{matrix}M_x\\ M_\theta\end{matrix}\right\rangle$ differ only by a constant

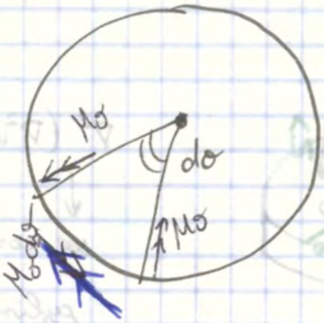


T_x



Close to the constraint, ∃ M_x and T_x but they vanish ⇒ so, far away from the support, the solution is a membrane.





Static equations:

$$(T_r + dT_r)(r+dr)\,d\theta - T_r \cdot r \cdot d\theta + q r \cdot d\theta \cdot dr = 0$$

$$\Rightarrow \frac{dT_r}{dr} + \frac{T_r}{r} + q = 0$$

$$(M_r + dM_r)(r+dr)\,d\theta - M_r \cdot r \cdot d\theta - M_\theta\, d\theta\, dr - T_r \cdot r \cdot d\theta + m_r \cdot r \cdot d\theta \cdot dr = 0$$

$$\Rightarrow \frac{dM_r}{dr} + \frac{M_r}{r} - \frac{M_\theta}{r} - T_r + m_r = 0$$

Equilibrium equations:

$$\begin{array}{c} s1 \\ s2 \end{array}\begin{pmatrix} \frac{d}{dr} + \frac{1}{r} & 0 & 0 \\ -1 & \frac{d}{dr} + \frac{1}{r} & -\frac{1}{r} \end{pmatrix}\begin{pmatrix} T_r \\ M_r \\ M_\theta \end{pmatrix} + \begin{pmatrix} q \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Kinematic equations:

$$\begin{pmatrix} \gamma_r \\ \chi_r \\ \chi_\theta \end{pmatrix} = \begin{pmatrix} \frac{d}{dr} & +1 \\ 0 & \frac{d}{dr} \\ 0 & \frac{1}{r} \end{pmatrix} \cdot \begin{pmatrix} w \\ \varphi_r \end{pmatrix}$$

$$\frac{d^2\varphi_r}{dr^2} + \frac{d}{dr}\left(\frac{1}{r}\varphi_r\right) = -\frac{qr}{2D}$$

$$\frac{d}{dr}\left[\frac{d\varphi_r}{dr} + \frac{\varphi_r}{r}\right] = -\frac{qr}{2D}$$

$$\frac{d}{dr}\left[\frac{1}{r}\cdot\frac{d}{dr}(r\varphi_r)\right] = -\frac{qr}{2D} \quad \Big\downarrow \int$$

$$\frac{1}{r}\cdot\frac{d}{dr}(r\varphi_r) = -\frac{qr^2}{4D} + c_2$$

$$\frac{d}{dr}(r\varphi_r) = -\frac{qr^3}{4D} + c_2\cdot r \quad \Big\downarrow \int$$

$$r\varphi_r = -\frac{qr^4}{16D} + c_2\cdot\frac{r^2}{2} + c_3$$

$$\varphi_r = -\frac{qr^3}{16D} + \frac{c_2}{2}r + \frac{c_3}{r}$$

$$\hookrightarrow \left.\begin{array}{l}\varphi_r(R) = 0 \\ \varphi_r(0) = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}c_3 = 0 \\ c_2 = \dfrac{qR^2}{8D}\end{array}\right.$$

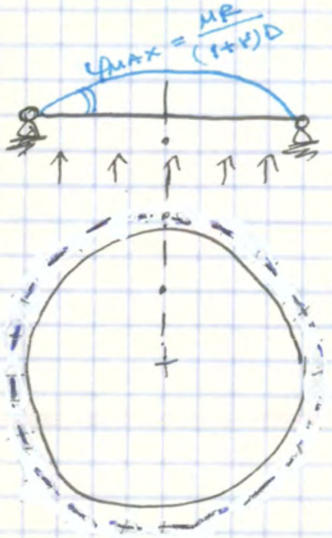
$$\Rightarrow \quad \varphi_r = \frac{q}{16\cdot D}\left(R^2\cdot r - r^3\right) = \frac{qr}{16D}\left(R^2 - r^2\right)$$

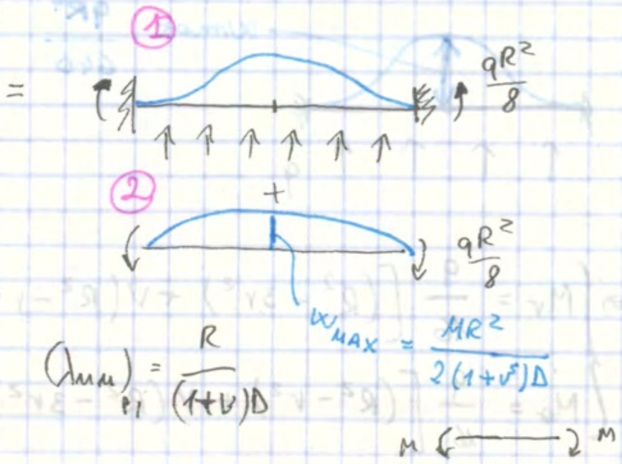
$$\Rightarrow \frac{d\varphi_r}{dr} = \frac{q}{16D}\left(R^2 - 3r^2\right)$$

$$\Rightarrow \quad \frac{dw}{dr} = -\varphi_r \quad \Rightarrow w(R) = 0 \quad \Rightarrow c_4 = -\frac{1}{4}R^4$$

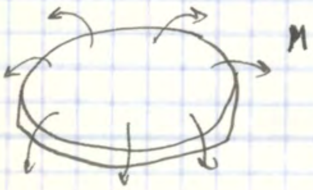
$$\Rightarrow \quad w = -\frac{q}{16D}\left(\frac{R^2r^2}{2} - \frac{r^4}{4} + c_4\right)$$

ANOTHER CASE





$$(\varphi_{MAX})_{P_1} = \frac{R}{(1+\nu)D}$$



$$q = 0$$

$$\frac{d}{dr}\left[\frac{1}{r}\cdot\frac{d}{dr}(r\varphi_r)\right] = 0$$

$$\frac{d}{dr}(r\varphi_r) = C_1 r \quad \rightarrow \oplus \ \varphi_r(0) = 0 \Rightarrow C_2 = 0$$

$$\varphi_r = C_1\cdot\frac{r}{2} + \cancel{\frac{C_2}{r}} = C_1\cdot\frac{r}{2}$$

$$\Rightarrow M_r = D\cdot\left(\frac{d\varphi_r}{dr} + \nu\frac{\varphi_r}{r}\right) = D\cdot\left(\frac{C_1}{2} + \nu\frac{C_1}{2}\right) = \frac{(1+\nu)C_1}{2}\cdot D$$

$$M_r(r=R) = M \Rightarrow C_1 = \frac{2M}{(1+\nu)D}$$

$$\Rightarrow M_r = M = M_\theta$$

$$\varphi_r = \frac{M_r}{(1+\nu)D} = -\frac{dw}{dr}$$

$$w = -\frac{Mr^2}{2(1+\nu)D} + C_3$$

$$w(R) = 0$$

$$\Rightarrow w = \frac{M}{2(1+\nu)D}\cdot(R^2 - r^2)$$

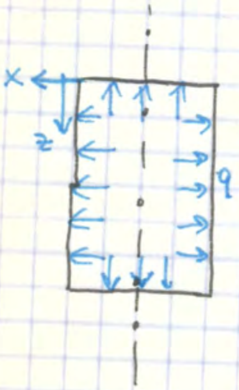
PRESSURED CYLINDRICAL VESSELS

WITH CAPS

↙ ① FLAT - FACED CAPS

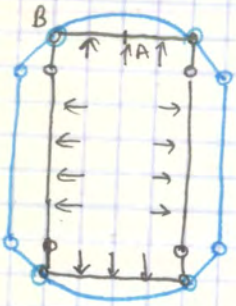
↘ ② HEMISPHERICAL CAPS

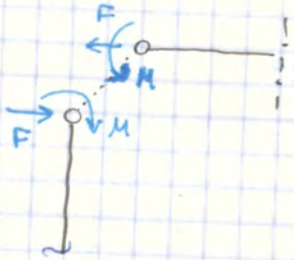
① FLAT -FACED CAPS



We have solved in the membrane case

Imagine to put connecting roads between walls and caps. that transmit only axial forces.





The two forces M and F are such that the movements w and φ must be the same for cylinder and plate.

$$\begin{cases} (w_B)_{plate} = (w_B)_{cylinder} \\ (\varphi_B)_{plate} = (\varphi_B)_{cyl.} \end{cases}$$

② HEMISPHERICAL CAPS

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Chapter 1-2-3

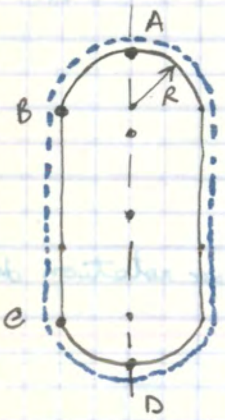
Chapter 4

Chapter 5-6

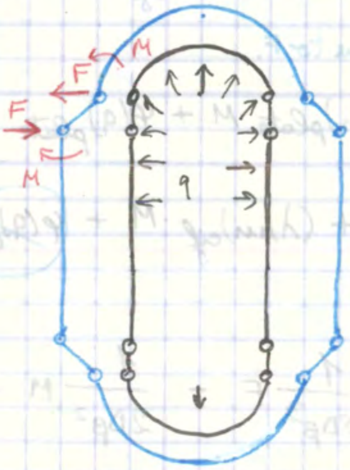
$$\lambda_{FF} = \mp \frac{1}{2D\beta^3}$$

$$\lambda_{FM} = \lambda_{MF} = = \frac{1}{2D\beta^2}$$

$$\lambda_{MM} = \mp \frac{1}{D\beta}$$



Put a connecting rope :



The membrane solution is not the true one.

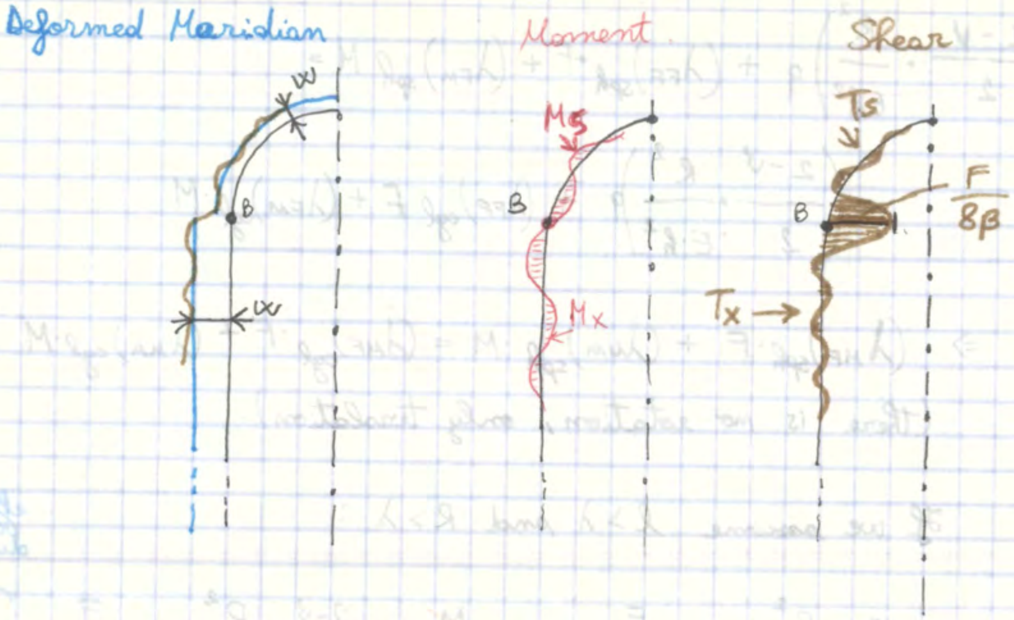
$$q = \overbrace{\left[\frac{2}{1-\nu} \cdot \frac{Eh}{R^2}\right]}^{STIFFNESS} \cdot w$$ for sphere

stiffer than:

$$q = \left[\frac{2}{2-\nu} \cdot \frac{Eh}{R^2}\right] \cdot w$$ for cylinder

2 CONGRUENCE EQUATIONS

$$\Rightarrow \begin{cases} (w_B)_{sph} = (w_B)_{cyl} \\ (\varphi_B)_{sph} = (\varphi_B)_{cyl} \end{cases}$$



⇒ $N_\theta = qR$ for the cylinder
$N_\theta = \frac{qR}{2}$ for the sphere } in point B there is a transition between them.

$N_x = \frac{qR}{2}$ for both.

? There is a peak value of the shear in point B due to the combination of sphere and cylinder:

⇒ a shell of revolution made by different elements, has peak values in the points of junction (without application of external forces).

In general for shells it is used the membrane theory.

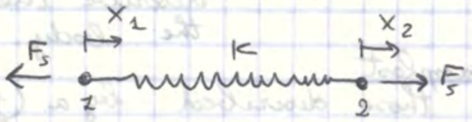
After peak values, the actions usually dissipate far away from these peaks.

DISCRETE SYSTEMS

CHARACTERISTIC OF THEIR COMPONENTS

The elements constituting a discrete system are of 3 types:
- those relating forces to displacements (i)
- those relating forces to velocities (ii)
- those relating forces to accelerations (iii)

(i) These are described by massless SPRINGS:



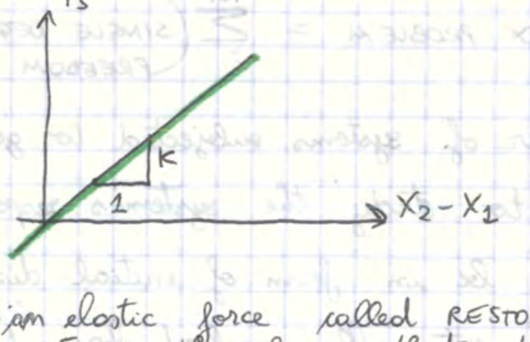
It connects 2 points, and is subjected to two equal and opposite forces.

k = spring stiffness $\left[\frac{N}{m}\right]$.

The spring undergoes an elongation equal to the difference between the end points' displacements. $(x_2 - x_1)$

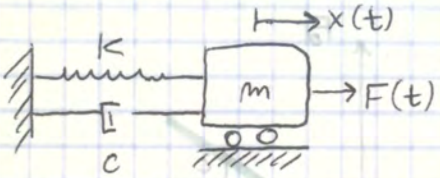
If $(x_2 - x_1)$ is sufficiently small ⇒ we are in the frame of linearity → we can approximate:

$$F_s = k \cdot (x_2 - x_1)$$



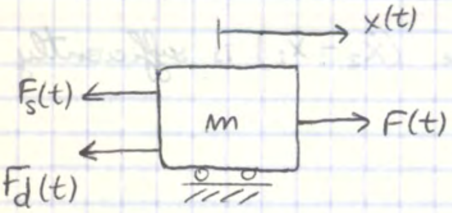
F_s is an elastic force called RESTORING FORCE: for a stretched spring, F_s is the force that tends the spring to return to the unstretched configuration (which in many cases coincides with the equilibrium configuration).

SECOND - ORDER LINEAR SYSTEMS



SPRING - DAMPER - MASS SYSTEM

FREE BODY DIAGRAM: isolate the mass and find all forces applied on it.



(gravity here can be neglected WRT horizontal direction)

2^{nd} Newton Law : $F(t) - F_s(t) - F_d(t) = m \cdot \ddot{x}(t)$

$\Rightarrow \quad m\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = F(t)$

m, c, k = constant in time

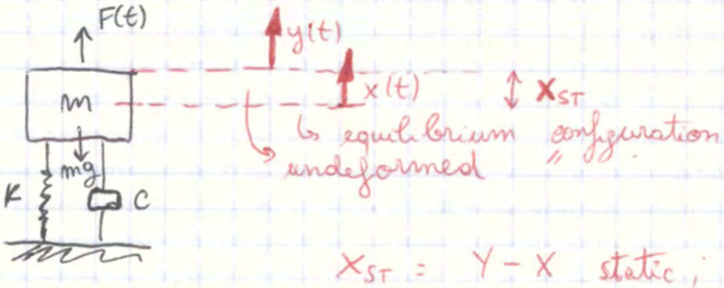
1 equat necessary to describe motion

⇑

single degree of freedom ⇔ 1 coord that describes the motion : x(t) = displacement of the mass.

In this case the position of the unstretched spring coincides with the equilibrium position.

What about the following system? → Consider gravity $F_g = mg$



$X_{ST} = Y - X$ static, constant

FREE VIBRATIONS : F=0

$$m\ddot{x} + c\dot{x} + kx = 0$$

- Homogeneous
- Solution : exponential form : $x(t) = C' e^{st}$

$$\Rightarrow (ms^2 + cs + k) \cdot C' \cdot e^{st} = 0 \quad \Big\downarrow : mC'e^{st}$$

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0 \qquad \omega^2 = \frac{k}{m} \geq 0$$

$$s^2 + \frac{c}{m} \cdot s + \omega^2 = 0$$ Characteristic equation.

∇ ω = natural frequency $\left[\frac{1}{s}\right]$

Now we study this equation.

- UNDAMPED CASE : $c = 0$

$$s^2 + \omega^2 = 0$$

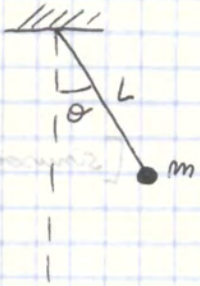
$$s = \pm\sqrt{-\omega^2} = \pm i \cdot \omega$$ complex solutions $i = \sqrt{-1}$

$$\Rightarrow x(t) = c_1 e^{i\omega t} + c_2 \cdot e^{-i\omega t}$$

∇ $$e^{\pm i\omega t} = \cos(\omega t) \pm i \cdot \sin(\omega t)$$

$\Rightarrow$ System's response
$$\begin{cases} x(t) = A \cdot \sin(\omega t) + B \cdot \cos(\omega t) \\ A = c_1 + c_2 \\ B = c_2 - c_1 \\ c_1, c_2 \text{ are initial conditions.} \end{cases}$$

Ex HARMONIC OSCILLATOR - PENDULUM

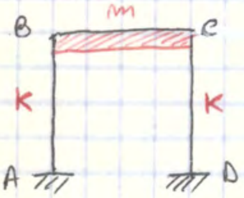


$$\ddot{\theta} + \omega^2 \theta = 0$$

$$\omega^2 = \frac{g}{L}$$

The degree of freedom is the angle

SHEAR - TYPE FRAME



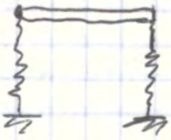
The moment of inertia of BC, cross-member is much greater than the others. (uprights)

$$I_{HORIZ} \gg I_{VERT.}$$

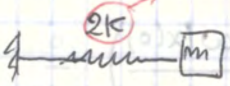
$$K = \frac{12 EI}{\ell^3}$$ shear stiffness.

BC behaves as a rigid body that traslates, and the deformation (so the stiffness) is concentrated into AB and CD. (⚠ this is an approximation)

⇒



⇒



because ∃ 2 springs.

CASE 2: CRITICAL DAMPING

$$s_1 = s_2 = -\frac{c}{2m} = -\omega$$

$\Rightarrow$ $x(t) = (C_1 + C_2 t)\, e^{-\omega t}$ no vibrations / oscillations

$\Rightarrow$ There is an exponential decay toward stability

By means of initial conditions, we have:

$$x(t) = \left[X(0) \cdot (1 + \omega t) + \dot{x}(0) t \right] \cdot e^{-\omega t}$$

CASE 3: UNDER-DUMPED SYSTEMS

$$s_{1/2} = -\xi\omega \pm i \cdot \omega_D$$

$\omega_D = \omega \sqrt{\underbrace{1-\xi^2}_{>0}}$ damped frequency

$$\Rightarrow x(t) = \left(C_1 e^{i\omega_D t} + C_2 e^{-i\omega_D t} \right) e^{-\xi\omega t} =$$

$$= \left[A \cdot \sin(\omega_D t) + B \cdot \cos(\omega_D t) \right] e^{-\xi\omega t} =$$

$$= X \cdot \cos(\omega_D t - \varphi) \cdot e^{-\xi\omega t}$$ Oscillates and decays as exponential

The constants (A, B) or (X, φ) can be expressed through the initial conditions.

$$A = \frac{\dot{x}(0) + x(0) \cdot \xi \cdot \omega}{\omega_D}$$

$$B = X(0)$$

$X = \sqrt{A^2 + B^2}$ $\tan\varphi = \frac{B}{A}$

CYCLE OF VIBRATION

$$\frac{X_1}{X_2} = \frac{X e^{-\xi\omega t_1} \cos(\omega_D t_1 - \varphi)}{X e^{-\xi\omega t_2} \cos(\omega_D t_2 - \varphi)}$$

$$t_2 = t_1 + T_D$$

$$T_D = \frac{2\pi}{\omega_D}$$

$$\Rightarrow \cos(\omega_D t_2 - \varphi) = \cos\left(\omega_D t_1 + \frac{2\pi}{\omega_D}\omega_D - \varphi\right) =$$

$$= \cos(\omega_D t_1 - \varphi)$$

$$\Rightarrow \frac{X_1}{X_2} = \frac{e^{-\xi\omega t_1}}{e^{-\xi\omega(t_1+T_d)}} = e^{+\xi\omega T_d} \qquad \xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

Introduce $\delta = \ln\left(\frac{X_1}{X_2}\right) = \ln\left(e^{-\xi\omega T_d}\right) = \xi\omega T_D = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$

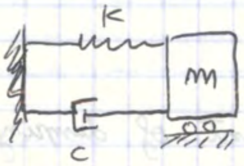
$\boxed{\text{small enough} \Rightarrow \text{tends to } 0}$

→ For low values of ξ, we finally have

$$\delta = 2\pi\xi \Rightarrow \xi = \frac{1}{2\pi} \cdot \ln \cdot \left(\frac{X_1}{X_2}\right)$$

By observing the decreasing of the amplitude we an measure value of ξ.

Ex



$m = 100\ Kg$

$T_d = 2.5\ s$ dumped period

$X_2 = \frac{X_1}{16}$

ξ, ω, K, c ?

dumping factor, natural frequency, stiffnes, dumping constant.

21/12/17

FORCED SYSTEMS $F(t) \neq 0$

$$m\ddot{x}(t) + c\dot{x}(t) + k\cdot x(t) = F(t)$$

- Solution depends on type of $F(t)$.
- HARMONIC EXCITATIONS:

$$F(t) = F\cdot \sin(\omega_F t) \quad \left[F\cdot\cos(\omega_F t) \text{ is the same} \right.$$

The system has its own frequency. The force also has its own. We study how these frequencies interact together and how is the system's response.

Rewrite eq. of motion:

$$\ddot{x}(t) + 2\omega\xi\cdot\dot{x}(t) + \omega^2\cdot x(t) = \frac{F}{m}\cdot\sin(\omega_F t)$$

⇒ General solution is the sum of X_p = particular solution that depends on the loading $F(t)$, and a solution X_c = depends on the free vibration.

$$X_p(t) = C_1 \sin(\omega_F t) + C_2 (\cos\omega_F t)$$

! Damped systems are not in phase with the loading so the X_p has not only form of $\sin(\omega_F t)$ but it has also the $\cos(\omega_F t)$ component [NOT in phase means they have not the same maxima and minima].

$$X_c(t) = \left[A\sin(\omega_D t) + B\cos(\omega_D t) \right]\cdot e^{-\xi\omega t}$$

! As $t\to\infty \Rightarrow X_c(t)\to 0$ because of exponential decay. and only X_p survives. ⇒ It's called TRANSIENT RESPONSE

! If $c = 0$, we have ⇒ $\xi = 0$ ⇒ $\omega_D = \omega$

$$\Rightarrow x(t) = \left\{ A\sin(\omega t) + B\cos(\omega t) \right\} + \frac{F}{k}\cdot\frac{1}{1-\beta^2}\cdot\sin(\omega_F t)$$

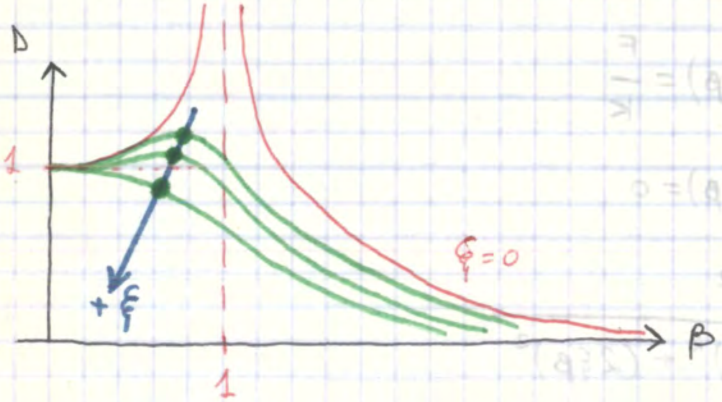
! The steady - state response can be written as (in general for $c \neq 0$)

$$x_p(t) = X \sin(\omega_F t - \varphi)$$

$$X = \frac{F}{k}\cdot\overset{D}{\boxed{\frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}}}$$ amplitude

$$\varphi = \arctan\left(\frac{2\xi\beta}{1-\beta^2}\right)$$ phase angle

Introducing : $D = \dfrac{X}{\frac{F}{k}}$ DYNAMIC MAGNIFICATION FACTOR



RED GRAPH

! Considering $c = 0$, just $x_p(t) = \frac{F}{k}\cdot\frac{1}{1-\beta^2}\cdot\sin(\omega_F t)$ +

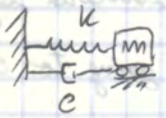
$D = D(\beta)$ $+ [A\sin(\omega t) + B\cos(\omega t)]$.

For $\beta \to 0$

If I apply a force on a system with a force equal to the one of the system $\omega_F = \omega$ ⇒ the system will collapse : RESONANCE SITUATION : amplitude → ∞.

10/01/18

Recall



→F(t) single degree of freedom system

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

- case 1 : $c=0$, $F=0$ harmonic oscillator
- case 2 : $c\neq0$, $F=0$ damping factor
- case 3 : $c\neq0$, $F=\sin\omega_F t$ external loading, with frequency ω_t

⇒ $\beta = \frac{\omega_F}{\omega}$, resonance

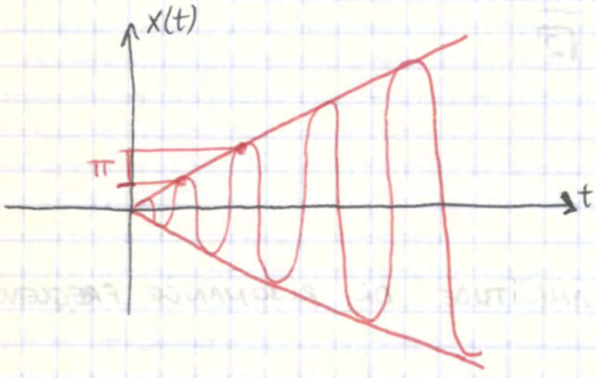
Let us consider the case of a harmonic oscillator at resonance
For $\beta = 1$ we have :

$$x(t) = e^{-\xi\omega t}(A\sin\omega_D t + B\cos\omega_D t) - \frac{F}{k}\cdot\frac{\cos\omega t}{k2\xi}$$

For zero damping ($\xi=0$), after imposing the initial conditions
$x(0) = \dot{x}(0) = 0$, we have:

$$x(t) = \frac{F}{2k}(-\omega t\cdot\cos\omega t + \sin\omega t)$$ (Response of the system)

as t→ increases ⇒ only the first term remains.



The amplitude of the response tends to increase. (divergent)

Types of LOADINGS : pulse, periodic, generic.

≈ bomb

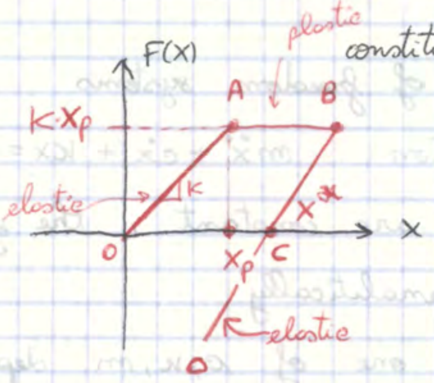
≈ Earthquake

ELASTO - PLASTO OSCILLATOR

HP $c = 0$

$x_0 = 0$

$\dot{x}_0 > 0$



constitutive law of the spring

$x^* =$ permanent displacement.

⇒ same eq of motion: but the elastic force changes (of the spring)

$F_{el} = k \cdot x(t)$ is replaced by $k(x(t) - x^*)$ during the elastic phases (OA) & (BD)

During the plastic phases we replace $kx(t)$ with $\pm k_p$. (AB)

1^{st} phase (OA) : $x^* = 0$

The eq of motion writes $m\ddot{x}(t) + k \cdot x(t) = 0$

After imposing the initial conditions $(x(0) = 0 \,,\ \dot{x}(0) > 0)$

we have that $x(t) = \frac{\dot{x}_0}{\omega} \sin(\omega t)$

2^{nd} phase (AB) : if $\frac{\dot{x}_0}{\omega} > x_p$ so we enter the plastic phase, at the instant t_1, such that :

$$x_p = \frac{\dot{x}_0}{\omega} \sin(\omega t_1)$$

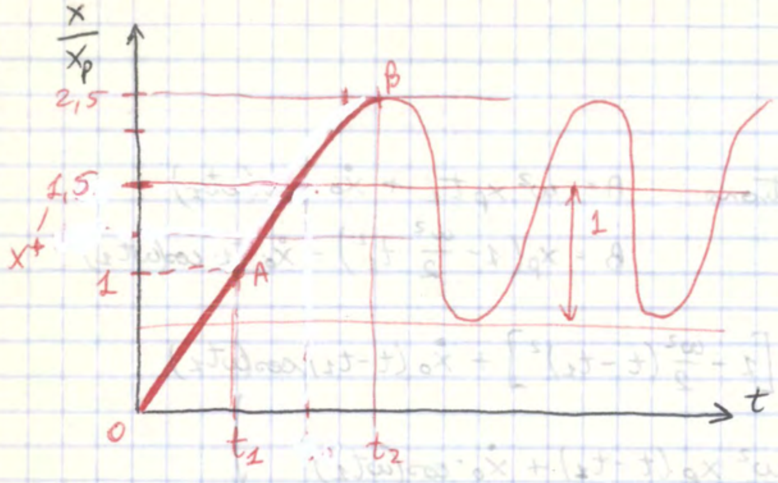
$$\Rightarrow \quad t_1 = \frac{1}{\omega} \cdot \arcsin\left(\frac{x_p \omega}{\dot{x}_0}\right)$$

During the period $t_1 < t < t_2$, we have

$$m\ddot{x}(t) + k x_p = 0 \qquad \Rightarrow m\ddot{x}(t) = \overset{\text{constant}}{-k x_p} \qquad \text{Eq of motion.}$$

$$\Rightarrow \quad x(t) = -\omega^2 x_p \frac{t^2}{2} At + B$$

The constants A and B can be evaluated from the initial conditions of the plastic phase at $t = t_1$.



$\dot{x}_0 = 2\omega x_p$

By substituting $f(t)$ solution into the first equation ⊛, we have

MATRIX

$\boxed{\left([K]-\omega^2[M]\right)}\{\delta\}=\{0\}$ This is only a problem in space

→ This is also a problem in EIGENVALUES and EIGENVECTORS. with unknowns ω^2 which represents the eigenvalues and $\{\delta\}$ which represents the eigenvectors.

ω^2 is evaluated by imposing the condition that the matrix cannot be inverted otherwise if it's inverted we get the trivial solution $\{\delta\}=\{0\}$ = no motion

! If $\det B=0 \Rightarrow \exists\, B^{-1}$ $\qquad \underbrace{(A-\lambda I)}_{B}\vec{x}=\vec{0}$

So the condition is :

$$\det\left([K]-\omega^2[M]\right)=0$$

⇒ Then we can estimate the eigenvectors which are defined up to a multiplicative factor.

[If $\{\delta\}$ is a solution, also $c\{\delta\}$ is a solution]

The eigenvector $\{\delta\}$ posses 2 fundamental properties:

1) they are linearly INDIPENDENT and form a BASIS :

EVERY VIBRATING MODE CAN BE EXPRESSED AS A COMBINATION, LINEAR, OF THE EIGENVECTORS

2) they are ORTHOGONAL WRT $[K]$ and $[M]$:

ex. let us consider 2 different eigenvalues $\omega_j^2 \neq \omega_k^2$:

$$\{\delta_k\}^T[K]\{\delta_j\}=\omega_j^2\cdot\{\delta_k\}^T[M]\{\delta_j\}$$

$$\{\delta_j\}^T[K]\{\delta_k\}=\omega_k^2\cdot\{\delta_j\}^T[M]\{\delta_k\}$$

$$\Rightarrow \quad \ddot{f}_i + \omega^2 f_i = 0$$

$$\Rightarrow \quad f_i(t) = A_i \cos(\omega_i t) + B_i \sin(\omega_i t)$$

This means that the general solution of the problem takes the following form:

$$\{\delta(t)\} = \sum_{i=1}^{N} \{\bar{\delta}_i\} \cdot \left(A_i \cos(\omega_i t) + B_i \sin(\omega_i t)\right)$$

The coefficients Ai and Bi can be evaluated through the initial conditions:

$$\{\delta(0)\} = \{\delta_0\} = \sum_{i=1}^{N} A_i \{\bar{\delta}_i\}$$

$$\{\dot{\delta}(0)\} = \{\dot{\delta}_0\} = \sum_{i=1}^{N} B_i \omega_i \{\bar{\delta}_i\}$$

$$\Rightarrow \quad A_j = \{\delta_0\}^T [M] \{\bar{\delta}_j\}$$

$$B_j = \frac{1}{\omega_j} \{\dot{\delta}_0\}^T [K] \{\bar{\delta}_j\} \qquad j = 1, .., N$$

Finally:

$$\left.\begin{aligned} \{\delta_0\} &= \overset{\text{proportional}}{\boxed{a}} \cdot \overset{\text{eigenvector}}{\boxed{\{\bar{\delta}_i\}}} \\ \{\dot{\delta}_0\} &= \{0\} \end{aligned}\right\} \text{initial conditions.}$$

$$A_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

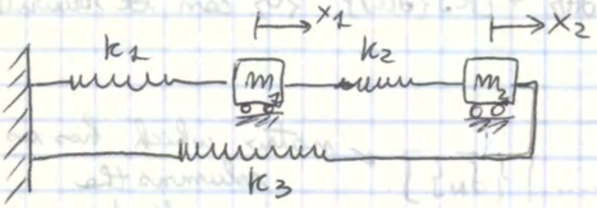
$$B_j = 0$$

$$\Rightarrow \quad \{\delta(t)\} = a \{\bar{\delta}_i\} \cos(\omega_i t) = \{\delta_0\} \cos(\omega_i t)$$

If the system is made vibrating proportional to the one of the eigenvector, it keeps constant in time.

11/01/18

Ex MULTI-DEGREE OF FREEDOM SYSTEM

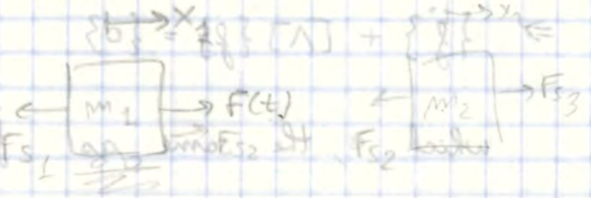


① Write the equations of motion : 2×2 because there are 2 masses [2 equations in 2 unknowns]

K_1 related x_1

K_2 " x_1 and x_2

K_3 " x_2



1) Eq is about m_1:

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) = 0$$

if $(x_2 - x_1) > 0$ (allungamento) ⇒ PUT ⊖ $\boxed{1} \rightarrow F_{s_2}$

$$\Rightarrow m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

2) About m_2:

$$m_2 \ddot{x}_2 + k_3 x_2 + k_2 (x_2 - x_1) = 0$$

if $(x_2 - x_1) > 0$ (WRT m_2 é un accorciamento) ⇒ PUT ⊕

$$\Rightarrow m_2 \ddot{x}_2 - k_2 x_2 + (k_2 + k_3) x_2 = 0$$

② By supposing $k_1 = k_2 = k_3 = k$: $m_1 = m$ and $m_2 = 2m$, evaluate the frequencies of the system and the corresponding vibrating modes

③° Evaluate the eigenvectors:

$$[K]\{X_i\} = \omega_i^2 [M]\{X_i\} \qquad i = 1, 2$$

In components we have:

$$\begin{cases} (K_{11} - \omega_i^2 m_{11})\, X_{i,1} + (K_{12} - \omega_i^2 m_{12})\, X_{i,2} = 0 \\ (K_{21} - \omega_i^2 m_{21})\, X_{i,2} + (K_{22} - \omega_i^2 m_{22})\, X_{i,2} = 0 \end{cases}$$

These are the general equations:

→ in our problem we can simplify them and get: (divide also by K)

$$① \quad \left(2 - \frac{\omega_i^2 m}{K}\right) X_{i,1} - X_{i,2} = 0$$

$$i = 1, 2$$

$$② \quad -X_{i,1} + 2 \cdot \left(1 - \frac{\omega_i^2 m}{K}\right) X_{i,2} = 0$$

Since they are homogeneous we can solve easily only the first.

$\boxed{i=1}$ $$X_{1,2} = \left(2 - \frac{\omega_1^2 m}{K}\right) \cdot X_{1,1} = 1,366 \cdot X_{1,1}$$

↓ Fix the first component = 1 and find the 2nd component

$$\Rightarrow \quad 1^{st} \text{ eigenvector} \quad \{X_1\} = \begin{Bmatrix} 1,000 \\ 1,366 \end{Bmatrix}$$

$\boxed{i=2}$ $$X_{2,2} = \left(2 - \frac{\omega_2^2 m}{K}\right) X_{2,1} = -0,366 \cdot X_{2,1}$$

Fix the first component = 1

$$\Rightarrow \quad 2^{nd} \text{ eigenvector} \quad \{X_2\} = \begin{Bmatrix} 1,000 \\ -0,366 \end{Bmatrix}$$

⑤° Introduce time and initial conditions: evaluate

$\dot{x}_1(0) = V_0$

$x_1(0) = x_2(0) = \dot{x}_2(0) = 0$

Remind the general solution: it take the following form:

$$x(t) = \sum_{i=1}^{N} \{\bar{X}_i\}\Big[\{\bar{X}_0\}^T[M]\{\bar{X}_i\}\cos(\omega_i t) +$$

$$+ \frac{1}{\omega_i}\{\dot{X}_0\}^T[\mathbf{K}]\{\bar{X}_i\}\sin(\omega_i t)\Big]$$

↳ which is the linear combination of the normalized eigenvectors.

⇒ In our exercise we have: $\{x_0\} = \{0\}$:

$$\{x(t)\} = \sum_{i=1}^{2}\{\bar{X}_i\}\left[\frac{1}{\omega_i}\underbrace{\boxed{\{\dot{X}_0\}^T[M]\{\bar{X}_i\}}}_{\text{must be evaluated}}\sin(\omega_i t)\right]$$

!

$$\frac{1}{\omega_1}\{\dot{X}_0\}^T[M]\{\bar{X}_1\} = \frac{1}{0{,}7962\cdot\sqrt{K/m}}\,\frac{1}{\sqrt{m}}\begin{Bmatrix}V_0\\0\end{Bmatrix}^T\begin{bmatrix}m&0\\0&2m\end{bmatrix}\begin{Bmatrix}0{,}4597\\0{,}6280\end{Bmatrix} =$$

$$= 0{,}5774\,\frac{mV_0}{\sqrt{K}}$$

$$\frac{1}{\omega_2}\{\dot{X}(0)\}^T\cdot[M]\cdot\{\bar{X}_2\} = 0{,}5774\,\frac{mV_0}{\sqrt{K}}$$

And after a few manipulations we have:

$$\{x(t)\} = V_0\sqrt{\frac{m}{k}}\left(\begin{Bmatrix}0{,}2654\\0{,}3626\end{Bmatrix}\sin\left(0{,}7962\sqrt{\frac{K}{m}}\,t\right)+\begin{Bmatrix}0{,}5127\\-0{,}1877\end{Bmatrix}\sin\left(1{,}5382\sqrt{\frac{K}{m}}\,t\right)\right)$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det\left([K] - \omega^2 [M]\right) = 0$$

$$\Rightarrow \omega^4 - \left(\frac{k_1+k_2}{m} + \frac{k_2}{m}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

If $\begin{cases} k_1 = k_2 = k \\ m_1 = m_2 = m \end{cases}$

$$\Rightarrow \omega^4 - \frac{3k}{m}\omega^2 + \frac{k^2}{m^2} = 0$$

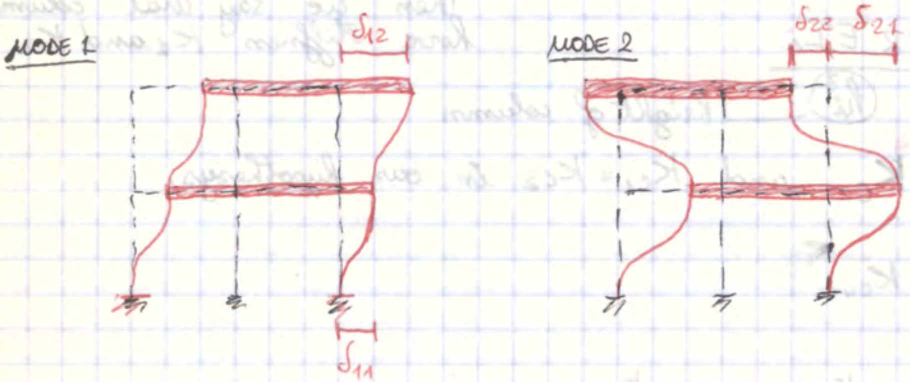
$$\Rightarrow \quad \omega^2 = \frac{3 \pm \sqrt{5}}{2} \cdot \frac{k}{m} = \begin{cases} \omega_1 = 0{,}618 \sqrt{\frac{k}{m}} \\ \omega_2 = 1{,}618 \sqrt{\frac{k}{m}} \end{cases}$$

Finally we evaluate eigenvectors:

$$\delta_{11} = 0{,}618 \cdot \delta_{12}$$

$$\delta_{22} = -0{,}618 \cdot \delta_{21}$$

And the modal shapes are:



?: The concentration of the mass is possible for simple cases. But in general it's not possible, which means that [M] and [K] are full matrices (not only diagonal).

$\Rightarrow$ How to evaluate them?

Each point of the bar has a displacement which is an interpolation of w_1 and w_2, which are our unknowns.

Now do the same for the flexural problem.

- Flexural problem.

Diff equation:

$$EI \cdot \frac{d^4 v}{d v^4} = 0$$

Form of $v(z)$: $v(z) = \frac{1}{6} C_1 z^3 + \frac{1}{2} c_2 z^2 + c_3 z + c_4$
↓
order 3

Again C_1, C_2, C_3, C_4 are the integration constants that we find by imposing the following conditions:

$$-\frac{dv}{dz}\Big|_{z=0} = \varphi_1 \; ; \; v(0) = v_1 \; ; \; -\frac{dv}{dz}\Big|_{z=l} = \varphi_2 \; ; \; v(l) = v_2$$

$$\Rightarrow \begin{cases} c_1 = -\frac{6}{l^3}\left(-2v_1 + l\varphi_1 + 2v_2 + l\varphi_2\right) \\ c_2 = \frac{2}{l^2}\left(-3v_1 + 2l\varphi_1 + 3v_2 + l\varphi_2\right) \\ c_3 = -\varphi_1 \\ c_4 = v_1 \end{cases}$$

After some analytical manipulations, we get:

$$v(z) = L_1(z)\varphi_1 + L_2(z) v_1 + L_3(z)\varphi_2 + L_4(z) v_2$$

where we have: THE SHAPE FUNCTIONS, CALLED CUBIC SPLINES.

$$\Rightarrow \quad U = \frac{1}{2}\int_0^\ell EA\,\{w(t)\}^T\{L'(z)\}\cdot\{L'(z)\}^T\{w(t)\}\,dz$$

! $\frac{\partial w(z,t)}{\partial z}dz$ is the derivative WRT space of the eq ①, and it's applied in particular to the shape function $L(z)$.
Since it's a vector · $\bar{v}^2 = \bar{v}\cdot\bar{v}$

$$\Rightarrow \quad U = \frac{1}{2}\{w(t)\}^T[K]\{w(t)\}$$

where $[K] = \int_0^\ell EA\,\{L'(z)\}\cdot\{L'(z)\}^T dz$

And knowing that

$$\{L'(z)\} = \frac{d}{dz}\begin{Bmatrix}1-\frac{z}{\ell}\\ \frac{z}{\ell}\end{Bmatrix} = \frac{1}{\ell}\cdot\begin{Bmatrix}-1\\1\end{Bmatrix}$$

⇒ so we find $[K]$:

$$[K] = \int_0^\ell EA\,\frac{1}{\ell^2}\begin{Bmatrix}-1\\1\end{Bmatrix}\begin{Bmatrix}-1\\1\end{Bmatrix}^T\cdot dz = \frac{EA}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Now to evaluate the $[M]$ we use Kinetic energy: T

$$T = \frac{1}{2}\int_0^\ell \underbrace{\mu}_{\substack{\text{mass}\\ \text{density}}:\left[\frac{kg}{m}\right]}\left[\frac{\partial w(z,t)}{\partial t}\right]^2 dz =$$

$$= \frac{1}{2}\int_0^\ell \mu\cdot\{\dot{w}(t)\}^T\{L(z)\}\cdot\{L(z)\}^T\{\dot{w}(t)\}\,dz =$$

$$= \frac{1}{2}\{\dot{w}(t)\}^T[M]\cdot\{w(t)\}$$

where:

$$[M] = \int_0^\ell \mu\,\{L(z)\}^T\{L(z)\}\,dz = \mu\int_0^\ell\begin{Bmatrix}1-\frac{z}{\ell}\\ \frac{z}{\ell}\end{Bmatrix}\cdot\begin{Bmatrix}1-\frac{z}{\ell}\\ \frac{z}{\ell}\end{Bmatrix}dz =$$