NUMERO: 1964A - ANNO: 2016

A P P U N T I

STUDENTE: Iandoli Felice

MATERIA: Hydraulic and thermal machines testing - prof. Dongiovanni

29/9/2015

HYDRAULIC AND THERMAL MACHINES TESTING

- USO DI ARDUINO! (≃ 30 ore di Arduino)
  Da comprare
- Book 1:
  Experimental Method for Engineers (cheap and good for didactical use)
- Book 2: (expensive, good but not good for didactical use)
- No slides shown
- $\boxed{\text{EXAM}}$: 3 hours of time < 1 hour: Arduino + 1/2 hour for check all the work; 1,5 hour: Exercises
  - Written
    - 1st part: Arduino Uno (build a program, an electronic setup and check)
    - 2nd part: Exercises (always the same) ← Statistical Analysis of Data; Electronic; Set a digital acquisition
  - \+
  - Oral: discussion of written exam AND transducers

N.B Program doesn't work ⇒ Exam not passed!

• INTRODUCTION

• Measurement = number + unit of measurement

number ↑ indicated with ~ (tilde) over the letter (in literal writing)

N.B Pronunciation of $\frac{m}{s}$ = meter PER second, meter OVER second

$$\frac{m}{s} \to \frac{Km}{h} \text{ conversion} = \frac{1000}{3600} = 0,2778 \left(= \frac{1}{3,6}\right) = K$$

factor of conversion = K (constant)

K = 1 ⇒ COHERENT UNIT OF MEASUREMENT (with S.I)

- DEFINITION OF U.M

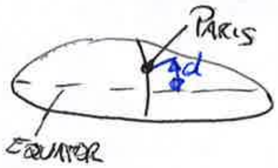
1) time

$$1s = \frac{1}{86400} \quad \text{(at the beginning)}$$

↳ solar year

$1s$ = related to cesium (now)

2) length



$$1\,m = \frac{d}{10\,000\,000}$$
(at the beginning)

where d = distance between Paris and the equator

3) Mass

At the beginning → water

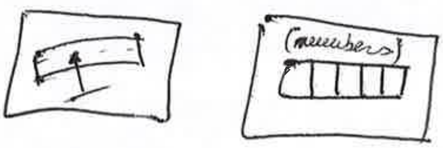
1kg = mass of 1 l of distilled water at 4°C ("Pronunciation: four Celsius degrees")

Now: related to an iridium mass

- CHARACTERISTICS OF INSTRUMENTS

1) READ OUT

Analog   Digital



N.B Analog instrument can do digital read out!

2) READABILITY

It gives an idea of the easyness of the read out

3) LEAST COUNT   N.B Not resolution

It gives us the minimum quantity we can read on the scale (on digital read out it's the last digit)

DA ESAME:

· Is it better to buy a precise instrument or an accurate instrument?

THE PRECISION INSTRUMENT!

It's the most expensive, accuracy can be adjusted by calibrating the instrument (very simple)

To improve the precision you have to repeat the measurement a lot of time (waste of time) and use the statistical analysis

· AN EXAMPLE OF INSTRUMENT CHARACTERISTICS

- Accuracy: 1%

FRS → (200 mA ÷ 1A = 1000 mA ⇒ ACCURACY = 0,01 · 1000 = 10 mA)

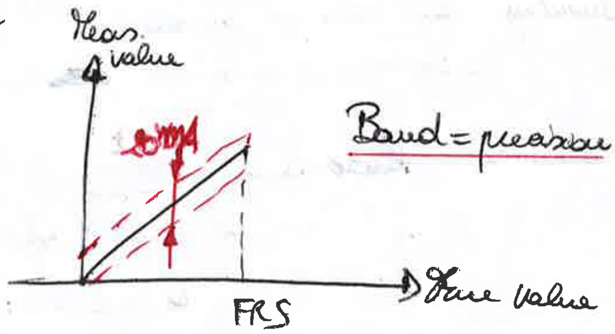
↳ Full Range Scale

- Precision: ± 1% FRS

↓ in the example

20 mA

N.B. Band of precision = 2 · Accuracy (= 20 mA in the example)



Band = precision

5) INDIRECT METHOD

The measurement is performed by measuring an other quantity G

: we can measure more quantity at the same time

(riscaldo ma 20 photo) C. W

Mm eadls

Types of Error

1) Bias Error

It's the most important and dangerous category

The error goes always on the same direction (ex watch)

and for this reason is dangerous. We have to use

another instrument to find it (ex. use another watch)

2) Rough Error

Stupid errors, like writing 10 instead of 20

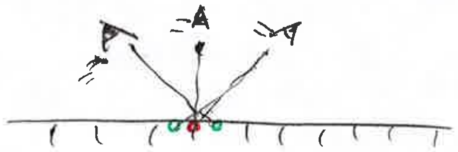
It can be easily found

3) Experimenter's Errors

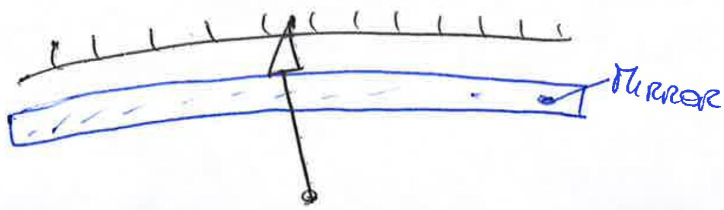
The experimenter "corrects" the error on his way by relationing

input and output (minuto 28)

Common experimenters error : Parallax Error

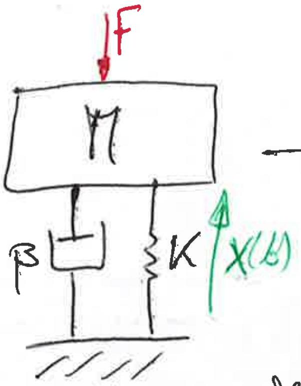


Solution: mirror on the scale plane (you don't see a double image when you are perpendicular)



© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 9 di 496

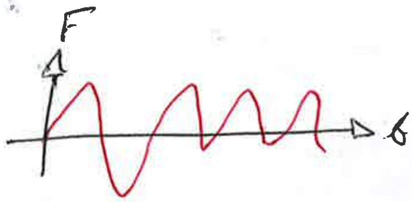
The Dynamic Behavior Of Instruments



$\longrightarrow F = K\Delta x$

measurement

Δ by transducer



- Types Of Mathematical Models

Model => differential equations

- General model

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_1 \frac{dx}{dt} + a_0 x = F(t)$$

- $n = 0$ $\quad a_0 \cdot x = F(t)$ (linear) $\quad$ 0 ORDER
- $n = 1$ $\quad a_1 \frac{dx}{dt} + a_0 x = F(t)$ $\quad$ 1st ORDER (derivata prima)
- $n = 2$ $\quad a_2 \frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = F(t)$ $\quad$ 2nd ORDER (derivata seconda)

N.B. A "0 ORDER" instrument for dynamic measurements doesn't exist! It's ideal

$n = 0 \rightarrow F(t) = a_0 \cdot x(t)$

Can be performed only in STATIC measurements!

1st Order → temperature (no inertia)

2nd Order → stiffness, mass, pressure (mechanical models in general)

30/9/2015

Hydraulic And Thermal Machines Testing

- Model Of Dynamic Behaviour Of The System

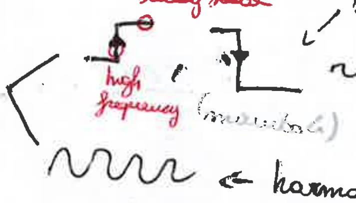
1) $n=0$ (0 order) $\rightarrow$ $a_0\, x(t) = F(t)$

2) $n=1$ (1^{st} order) $\rightarrow$ $\tau \dfrac{dx}{dt} + x = F(t)$ • x = answer of the instrument; F = physical quantity to measure

τ → Time constant

- 1^{st} Order Systems (temperature measurements)

$F(t) =$ step function, steady state, high frequency (mountain) $\rightarrow \tau$; harmonic function

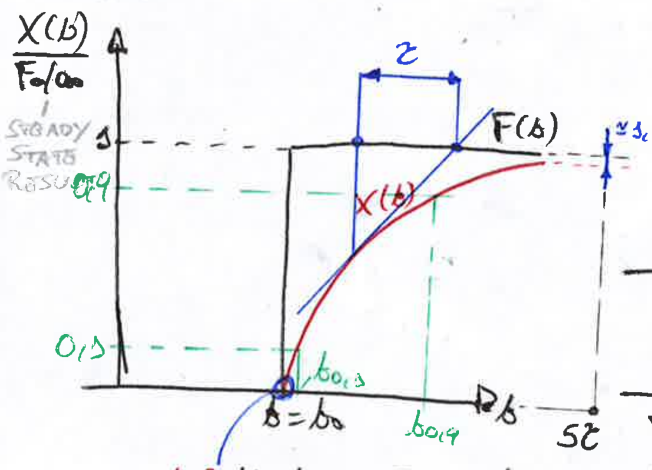


1) Step Function $F(t)$

$$= \begin{cases} 0 & \text{if } t<0 \\ F_0 & \text{if } t\geq 0 \end{cases}$$

in case of step function

Dynamic Result



$$x(t) = \tilde{x}(t) + \hat{x}(t) = \frac{F_0}{a_0} + c_1 e^{-\frac{t}{\tau}}$$

$\tilde{x}(t)$ → Homogeneous Solution; $\hat{x}(t)$ → Particular Solution

$$x(t) = \frac{F_0}{a_0} + \underbrace{\left(x_0 - \frac{F_0}{a_0}\right)}_{c_1} e^{-\frac{t}{\tau}} \quad \text{(rising step)}$$

$$\begin{cases} x(t=0) = 0 \\ x(t) = \frac{F_0}{a_0} + c_1 e^{-\frac{t}{\tau}} \rightarrow c_1 = x_0 - \frac{F_0}{a_0} \end{cases}$$

$$x(t) = \frac{F_0}{a_0} e^{-\frac{t}{\tau}} \quad \text{(falling step)}$$

$$\begin{cases} x(t=0) = \frac{F_0}{a_0} \\ x(t) = \frac{0}{a_0} + c_2 e^{-\frac{t}{\tau}} \rightarrow c_2 = \frac{F_0}{a_0} \end{cases}$$

N.B. Non Null Tangent! ⇒ Discontinuity in $t=0$ (It's different from 2^{nd} order system)

$\hookrightarrow \tan\alpha \neq 0$

N.B. 1) τ Is Constant

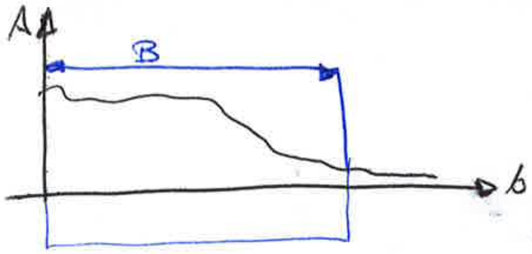
2) $x(t)$ (red) is Always Under $F(t)$ (black)

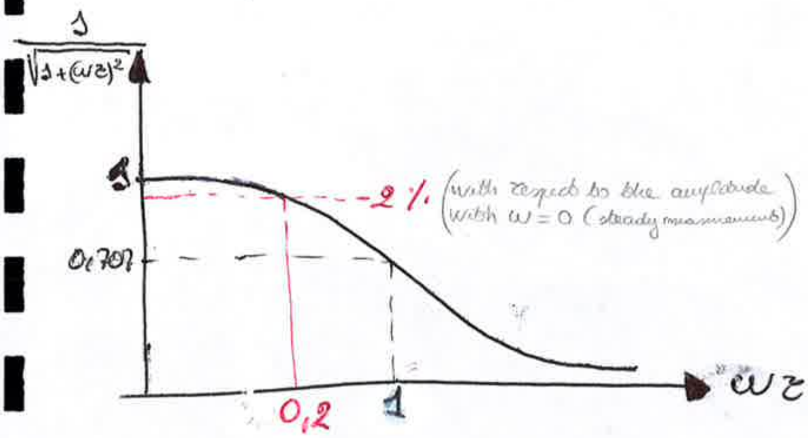
3) at $t = 5\tau$, $F(t) - x(t) \simeq 1,7\%$ ⇒ Variation finished after 5τ

4) We evaluate time at 10% and 90%.

$t_{0,1}$

$t_{0,9}$ → Rising Time: $t_{0,9} - t_{0,1} = \tau \cdot \ln 9 \simeq \tau \cdot 2,18$ → $\ln \frac{0,9}{0,1}$





In order to have a small attenuation (distortion in the amplitude of signal) and small delay (see next page):

$\omega\tau < 0{,}2$ (only for a 1st order system)

From where:

$$\omega < \frac{0{,}2}{\tau}$$

$\omega = 2\pi f$

- Example

$\tau = 100\ ms \longrightarrow B\,?$

$$\omega < \frac{0{,}2}{100 \cdot 10^{-3}} = 2\ rad/s$$

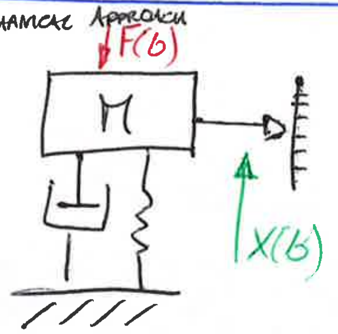
$$\omega = 2\pi f \rightarrow f < \frac{\omega}{2\pi} = \frac{2}{6{,}28} = 0{,}33\ Hz$$

$\Rightarrow B = 0{,}3\ Hz$

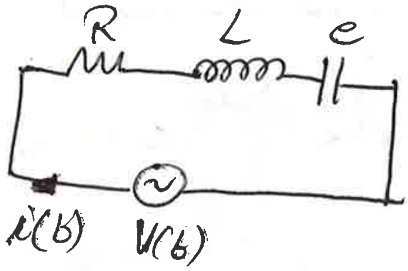
If variation is faster of 0,3 Hz we have to use another instrument

• 2nd ORDER MODEL (mechanical measurements)

• MECHANICAL APPROACH



• ELECTRICAL APPROACH



• DIFFERENTIAL EQUATION

$$M\frac{d^2x}{dt}+\beta\frac{dx}{dt}+Kx=F(t)$$

$$L\frac{d^2i}{dt}+R\frac{di}{dt}+\frac{1}{C}i=\frac{dV}{dt}$$

We introduce:

• NATURAL FREQUENCY

$$\omega_n=\sqrt{\frac{K}{M}} \qquad \omega_n=\sqrt{\frac{1}{LC}}$$

• CRITICAL DAMPING FACTOR

$$\beta_{cr}=2\sqrt{KM} \qquad \beta_{cr}=2\sqrt{\frac{L}{C}}$$

• DIMENSIONLESS DAMPING FACTOR

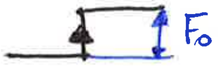
$$\zeta=\frac{\beta}{\beta_{cr}}=\frac{\beta}{2\sqrt{KM}} \qquad \zeta=\frac{R}{2\sqrt{\frac{L}{C}}}$$

And we can write: $\begin{cases}\beta=2\zeta\sqrt{KM}\\ \omega_n=\sqrt{\frac{K}{M}}\end{cases} \rightarrow \frac{\beta}{M}=2\zeta K^{1/2}\cdot M^{1/2}\cdot M^{-1}=2\zeta K^{1/2}M^{-1/2}=$
$=2\zeta\sqrt{\frac{K}{M}}=2\zeta\omega_n$

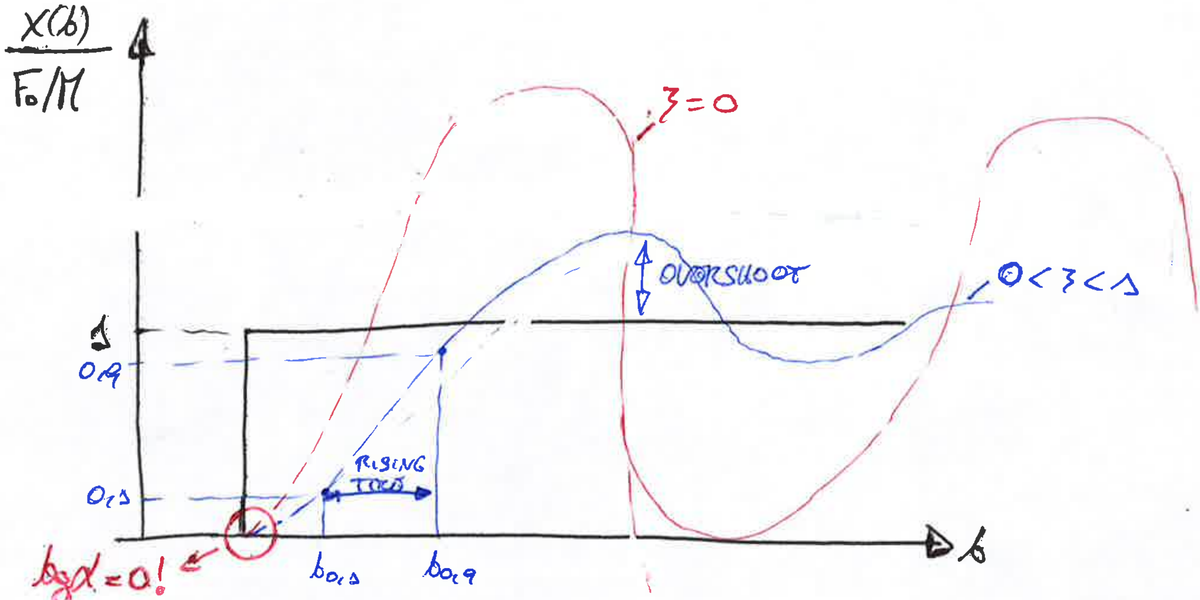
$$\frac{d^2x}{dt^2}+2\zeta\omega_n\frac{dx}{dt}+\omega_n^2x=\frac{F(t)}{M}$$

1) STEP FUNCTION $F(t)$

$F(t)$



$$X(t) = \underbrace{\tilde{X}(t)}_{\text{transient}} + \underbrace{\frac{F_0}{M}}_{\text{steady state}}$$



$tg\,\alpha = 0!$

(in the 1st order system it was $\neq 0$)

- $\zeta = 0 \Rightarrow \tilde{X}(t) = A\cos(\omega_n t)$

  $\Rightarrow$ No measurement possible because the value oscillates

  We must introduce damping!

I have to choose the best damping factor in order to perform the measurement

- **BEST DAMPING FACTOR**

  $\zeta = 0,7$ (In general $\zeta = 0,65 \div 0,70$)

  $\Rightarrow \begin{cases} \text{OVERSHOOT} \simeq 5\% \\ \text{RISING TIME} \simeq 3\,\omega_n t \end{cases}$

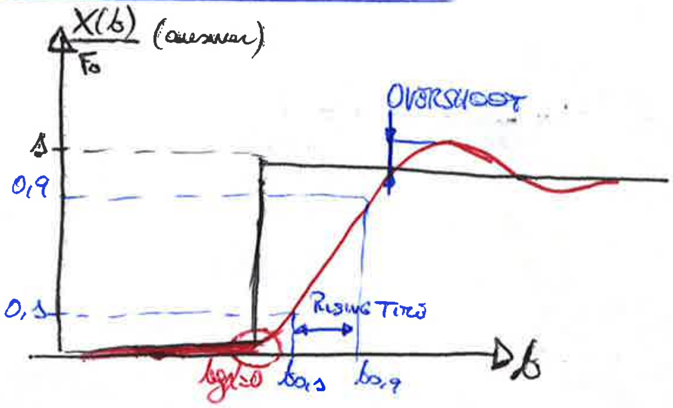
I have to adjust the damping factor of the instrument in order to obtain that overshoot and rising time, so $\zeta \simeq 0,7$

2/10/2015 1

HYDRAULIC AND THERMAL MACHINE TESTING

• 2nd ORDER SYSTEMS (continue)

$m = 2$

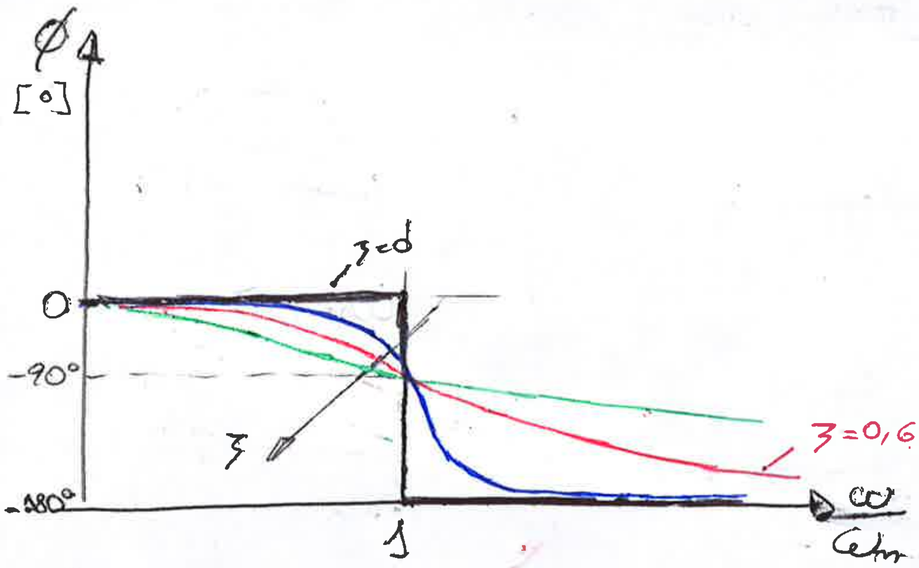


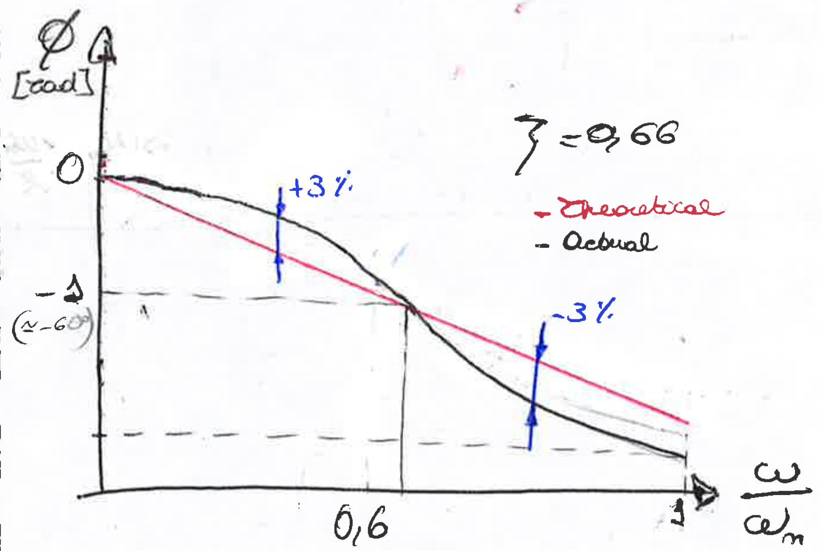
OVERSHOOT $\Rightarrow \zeta = \frac{\beta}{\beta_{cr}} < 1$

- We can find relationships between overshoot and ζ
• RELATIONSHIP BETWEEN OVERSHOOT AND ζ

$$\zeta = 0{,}66 \div 0{,}7 \Rightarrow \text{OVERSHOOT} \simeq 5\%$$

When overshoot is about 5%, I know that $\zeta = 0{,}66 \div 0{,}7$ and it's important to calibrate the instrument by adjusting its damping factor.





- Example

- If I have a signal of 10 kHz, I need an instrument with

$$f_n = 2 \cdot 10 \text{ kHz} = 20 \text{ kHz}$$

(theoretically is $\frac{\omega}{\omega_n} = 0,6 \rightarrow f_n = \frac{\omega}{0,6} = 1,67 \cdot 10 \text{ kHz}$)

• Statistical Analysis

• Type Of Errors

Bias Errors (errori sistematici)
- Zero verification
- Gain / non linearity (sensitivity) ← laboratory is needed to calibrate
- Comparison (with other instruments)
- Critical analysis

Rough Errors (errori di approssimazione)

Experimenter's Errors (errori dello sperimentatore)

Random Errors → they can't be eliminated
they can be reduced by repeating the experiment many times

• Writing The Results

Result = ______ ± ______ ______ ( )

- Most Probable Value
- Uncertainty
- Unit of Measure
- Odds (probability)

- Example

(1 sign. digit)

$p = 147 \pm 2$ mbar (20:1)

- Most Probable Value: 147
- Uncertainty: 2
- U.M: mbar
- Odds (5%): 20:1

N.B The number of significative digits must be in accordance with uncertainty! (It can't be higher than uncertainty)

• How To Write Uncertainty

Uncertainty must be written:
1) for first (the mean value has to be rounded after)
2) With a rounding that gives an error lower than 20 %.

2,456 ⇝ 2
30,56 ⇝ $3 \cdot 10^1$
1,356 ⇝ 1,3 N.B 1 ⇒ 2 digits
13,55 ⇝ $1,4 \cdot 10^1$

Use engineering writing! (same after)

Cifra significativa

La determinazione delle **cifre significative** pone, in maniera implicita, un'espressione numerica all'interno di un intervallo; per esempio per indicare l'errore nella misurazione, l'intervallo di confidenza di una stima o l'errore propagato nel risultato di una successione di calcoli. La loro definizione segue il principio di non indicare più cifre di quelle giustificate dalla precisione della misurazione o di qualsiasi altro processo abbia portato al numero indicato.

Il calcolo della significatività delle cifre di una misura è molto importante, specie quando sono in gioco quantità in correlazione; un caso esemplare è quello delle coppie di Heisenberg (posizione e quantità di moto, per esempio).

Indice

Procedura

Identificazione delle cifre significative

- La *cifra più significativa* è sempre la prima da sinistra che sia diversa da zero;
- La *cifra meno significativa*
  - in un valore intero, è la prima da destra che sia diversa da zero,
  - in un valore con una parte frazionaria, è l'ultima cifra a destra, anche se si tratta di uno zero;
- Le cifre significative sono tutte quelle comprese tra la *più significativa* e la *meno significativa*

Per esempio, `0,00057` ha due cifre significative.

Dal valore alla sua espressione numerica

Dato un valore `K` con un errore `dK` (normalmente indicato con `K ± dK`), si scriverà dK con una o due *cifre significative* e `K` avrà come cifra meno significativa l'omologa in `dK`. Se, per esempio, ci trovassimo a dover scrivere una quantità che abbiamo calcolato o stimato in `14,2856 ± 0,362` potremmo scriverla come `14,3 ± 0,4` o `14,29 ± 0,36`.

Se invece vogliamo indicare il solo valore, senza l'errore, la sua *cifra meno significativa* sarà quella immediatamente superiore alla *cifra più significativa* dell'errore (non indicato). Nel caso in esempio, scriveremmo `14`.

Si noti come la *cifra meno significativa* non viene riportata tale e quale, ma viene arrotondata.

Notazione scientifica

La **notazione scientifica** è un modo conciso di esprimere i numeri reali utilizzando le potenze intere di dieci, ed è usata per numeri molto grandi o molto piccoli. La notazione permette di esprimere quantità fisiche senza includere lunghe file di zeri:

- $10^1 = 10$
- $10^2 = 100$
- $10^3 = 1000$
- $10^6 = 1\ 000\ 000$
- $10^9 = 1\ 000\ 000\ 000$
- $10^{20} = 100\ 000\ 000\ 000\ 000\ 000\ 000$

Oltre alle potenze positive, si possono usare le potenze negative: 10^{-n} è uguale a $1/10^n$ e in decimali si può esprimere con uno 0 seguito dalla virgola, da *n-1 zeri* e da un 1:

- $10^{-1} = 1/10 = 0{,}1$
- $10^{-3} = 1/1000 = 0{,}001$
- $10^{-9} = 1/1\ 000\ 000\ 000 = 0{,}000000001$

Il numero deve essere scritto con una sola cifra diversa da zero prima della virgola e va moltiplicato per una potenza del 10 per far sì che si riproduca il numero originale. In questo modo, un numero molto grande come 156 234 000 000 000 000 000 000 000 000 può essere espresso come $1{,}56234 \times 10^{29}$, e un numero piccolo come 0,0000000000234 può essere scritto come $2{,}34 \times 10^{-11}$ (si mette sempre la virgola dopo la prima cifra, aggiustando di conseguenza l'esponente). Per esempio, la distanza del bordo dell'universo osservabile è circa $4{,}6 \times 10^{26}$ m e la massa di un protone è circa $1{,}67 \times 10^{-27}$ kg. La maggior parte delle calcolatrici e dei programmi per computer presentano i numeri molto grandi e molto piccoli usando la notazione scientifica. Il 10 è normalmente omesso e la lettera E è usata per indicare l'esponente: per esempio, 1,56234 E29. Da notare che questa E non ha relazioni con la costante matematica e.

In termini più generali possiamo dire che un numero *reale* x può essere rappresentato in una base β in questo modo: $x = \pm(.c_1 c_2 c_3 \ldots)\beta^p$

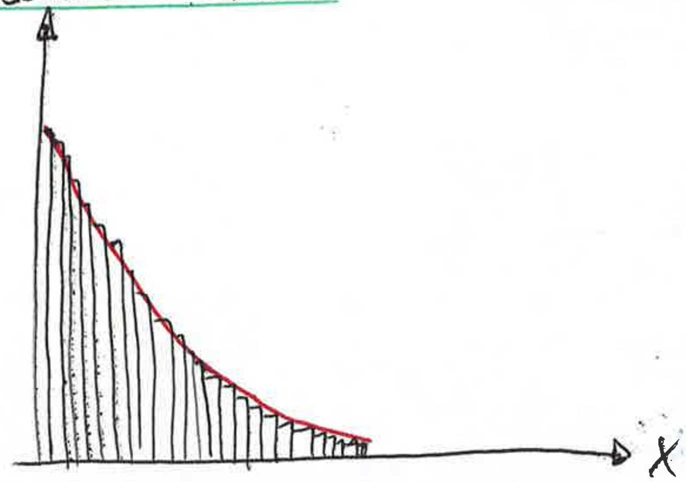
Il "." è detto **punto radice**, mentre le prime cifre della mantissa ($c_1 c_2 c_3$), sono dette **cifre significative** (o ***essenziali***)

La notazione scientifica è molto utile per esprimere le quantità fisiche, perché esse possono essere misurate solo entro certi limiti di errore. Con la notazione scientifica è possibile scrivere solo le cifre significative senza sprecare spazio, e si rendono leggibili testi che trattano quantità molto grandi o molto piccole senza riempirli di zeri.

Di conseguenza, una quantità espressa con la notazione specifica ha un'accuratezza intrinseca a seconda di quante cifre decimali siano state utilizzate: la quantità di $1{,}2340 \times 10^6$ metri significa che la misura reale è compresa tra 1 233 950 e 1 234 050 metri. Se però conoscere la precisione reale è fondamentale ci sono altri metodi.La notazione scientifica inoltre permette di evitare ambiguità relative alla nazione in cui vive lo scrivente: un bilione ha un significato diverso in Europa e in Nord America, mentre 10^9 e 10^{12} sono uguali in tutto il mondo.

[1,23345, 1,23405] $\times 10^6$

- **Normalized Histogram**



Reducing area to 0, we obtain a function

- **Binomial Distribution** (max is 2).

m total number of events

n number of positive events

p probability that the event can happen

- Probability of the n positive events

$$p(n) = \frac{m!}{n!\,(m-n)!}\, p^n (1-p)^{m-n}$$

- Example with 2 tries and 1 coin

$m = 2$ (total events = 2 tries)

$n = 2$ (positive events: H-H, 2 because of the order)

$p = \frac{1}{2}$ (probability of ~~a single positive~~ event)

$$p(2) = \underbrace{\frac{2!}{2!\,0!}}_{1} \underbrace{\left(\frac{1}{2}\right)^2}_{\frac{1}{4}} \underbrace{\left(\frac{1}{2}\right)^0}_{1} = \frac{1}{4}$$

N.B Remember that: $0! = 1$

- Example with 6 numbers, 4 tries ~~and 2~~ positive events

$m = 4$

$n = 2$

$p = 1/6$

$$p(2) = \underbrace{\frac{4!}{2!\,2!}}_{\frac{24}{4} = 6} \underbrace{\left(\frac{1}{6}\right)^2}_{\frac{1}{36}} \cdot \underbrace{\left(\frac{5}{6}\right)^2}_{\frac{25}{36}} = 3$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 31 di 496

• $n+\Delta$ negative errors ($m-(n+\Delta)$ positive errors)

$$\Delta = (m - \underbrace{2n - 2}_{2(n+\Delta)})\delta$$

$$p(n+\Delta) = \frac{m!}{(n+\Delta)!\,(m-n-1)!}\cdot\frac{1}{2^m} = \frac{m!}{(n+1)\,n!\,(m-n-1)!}\,\frac{1}{2^m} =$$

multiplying and dividing for $m-n$

$$= \frac{m!(m-n)\cdot 1/2^m}{(n+1)\,n!\,\underbrace{(m-n)(m-n-1)!}_{(m-n)!}} = \underbrace{\frac{m!}{n!(m-n)!}\,\frac{m-n}{n+1}\,\frac{1}{2^m}}_{p(n)} =$$

$$= \frac{m-n}{n+1}\,p(n)$$

Explanation of Δ

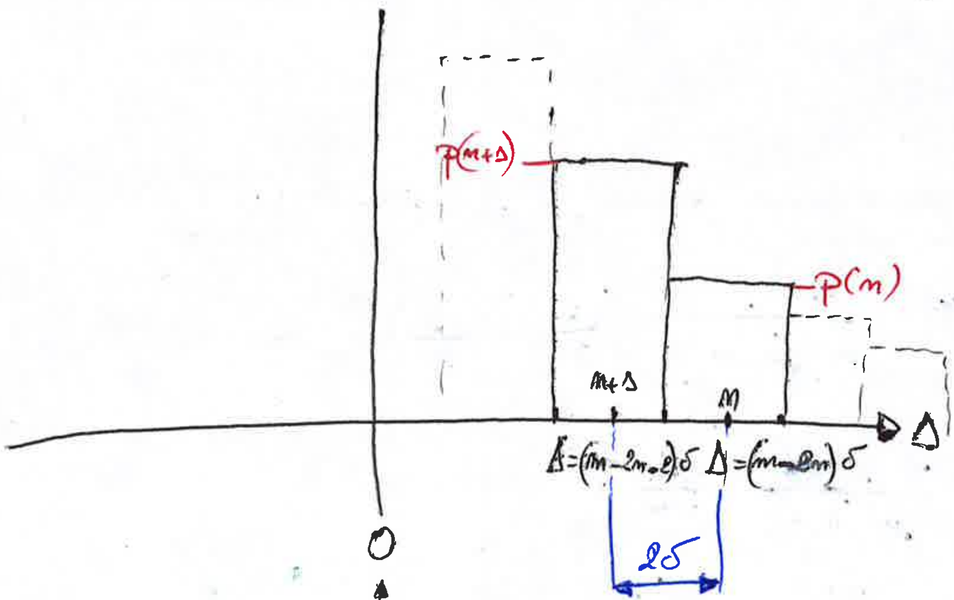
m errors $\longrightarrow$ δ (amplitude of a single error)

Positive $m-n$ | Negative n

$$\Delta = (m-n)\delta + -n\delta = (m-2n)\delta$$

N.B. Δ is the product between:
1) The sum of the positive and negative errors
2) The amplitude of the error

• Histogram of the error



True Value
(I want to represent it but I can't)

(max 7 3)

- Since that δ is a small quantity, we can write the derivative, the approximation of the error distribution (see after)

So, the expression of $\frac{dy}{dx}$ mean 18 3

$$\frac{dy}{dx} = -\lim_{\substack{m \to \infty \\ \delta \to 0}} \frac{m - 2n - \cancel{\Delta}}{2\delta(m + \cancel{\Delta})} y_m \simeq$$

I can neglect it because $m \gg \Delta$

$$\simeq -\lim_{\substack{m \to \infty \\ \delta \to 0}} \frac{m - 2n}{2\delta m} y_n =$$

$$= -\lim_{\substack{m \to \infty \\ \delta \to 0}} \frac{x_m / \delta}{\delta\left(m - \frac{x_m}{\delta}\right)} y_m =$$

substituting the relationship

$$= -\lim_{\substack{m \to \infty \\ \delta \to 0}} \frac{x_m / \delta}{m\delta - \cancel{x_m}} y_m$$

$\to m\delta \gg x_m$

neglect

(OT: I remember that:

$n = 0 \Rightarrow \Delta = m\delta$ (case with no negative error, very low probability))

$$\frac{dy}{dx} \simeq -\lim_{\substack{m \to \infty \\ \delta \to 0}} \frac{x_m y_m}{m \delta^2}$$

• DIFFERENTIAL EQUATION OF THE ERROR DISTRIBUTION IN THE CENTRAL AREA

$$\frac{dy}{dx} = -\frac{xy}{m\delta^2}$$

I define: $2h^2 = \frac{1}{m\delta^2}$

$h \to$ Precision Modulus

$$\Rightarrow \frac{dy}{dx} = -2h^2 xy$$

6/10/2015

Hydraulic And Thermal Machines Testing

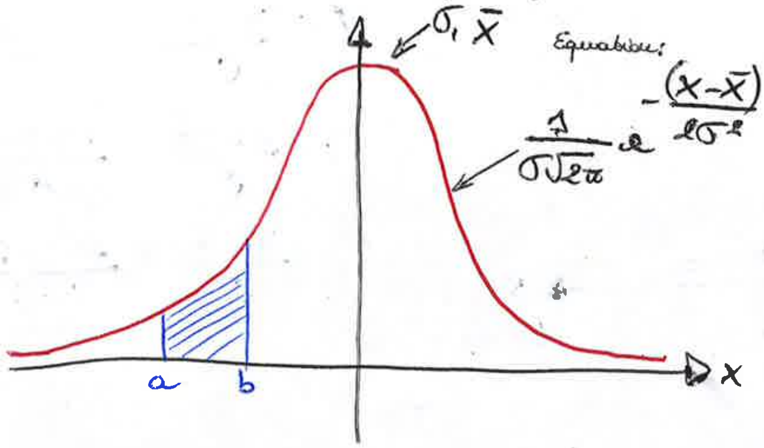
• Error Distribution

Random error distribution can be modeled using the normal (or gaussian) distribution

defined by: $\sigma, \bar{x}$

Equation:

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$



- This distribution can be defined by
  - mean value
  - Standard deviation
- In the central area the gaussian distribution is a very good approximation (not on the queues, where we are not interested in)

- Table Explanation (Standard Normal Distribution) mean S

first term of the interval →

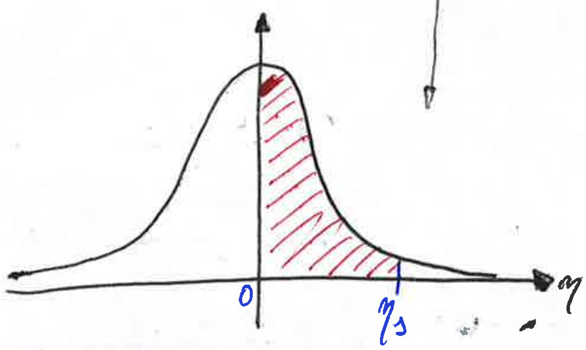
η ↓ last term of the interval

$$f(\eta) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\eta^2}{2}} \qquad \text{where } \eta = \frac{x-\bar{x}}{\sigma}$$

- Next Table → more important because it gives the integral in that interval

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx \quad \text{(in general)}$$

$$g(\eta) = \frac{1}{\sqrt{2\pi}} \int_0^{\eta_1} e^{-\frac{\eta^2}{2}}\, d\eta \quad \text{(in the table)}$$



© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 37 di 496

$$P[12,5 \le x \le 12,6] = P[0 \le \eta \le 1] - P[0 \le \eta \le 0,67] =$$

$$= 0,34134 - 0,24857 \simeq 9\%$$

N.B (see distribution diagram)

• Evaluating True Value and Standard Deviation

mar 22

- Experimental data: $x_1, x_2, \dots, x_n$

$$P(x_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_1 - x_t)^2}{2\sigma^2}}$$

(see the end of 20151002)

N.B x_t, σ DOESN'T CHANGE with the measurements

$$P(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i - x_t)^2}{2\sigma^2}}$$

$$P(x_1, \dots, x_n) = \prod_{i=1}^{N} P(x_i) = \frac{1}{\sigma^N (\sqrt{2\pi})^N} \cdot e^{-\frac{1}{2}\sum_{i=1}^{N}\frac{(x_i - x_t)^2}{\sigma^2}}$$

N.B productory!

N.B not σ_i because σ doesn't change with the measurements

- x_t and σ (unknown) can MAXIMISE PROBABILITY:

• x_t

$$\frac{\partial P(x_1, \dots, x_n)}{\partial x_t} = 0 \quad \Downarrow \quad x_t = f(\,)$$

$$\left(\frac{\partial P(x_1, \dots, x_n)}{\partial \sigma} = 0 \quad \Downarrow \quad \sigma = g(\,) \right)$$

← see next page

$$P \propto \frac{1}{\sigma^N} e^{-\frac{1}{2}\sum_{i=1}^{N}\frac{(x_i - x_t)^2}{\sigma^2}}$$

$$\frac{\partial P}{\partial x_t} \propto e^{-\frac{1}{2}\sum_{i=1}^{N}\frac{(x_i - x_t)^2}{\sigma^2}} \rightarrow \sum_{i=1}^{N} \frac{(x_i - x_t)^2}{\underbrace{\sigma^2}_{cost}} = 0 \rightarrow \sum_{i=1}^{N}(x_i - x_t) = 0$$

NB $\frac{1}{\sigma^N}$ does not depend from x_t!

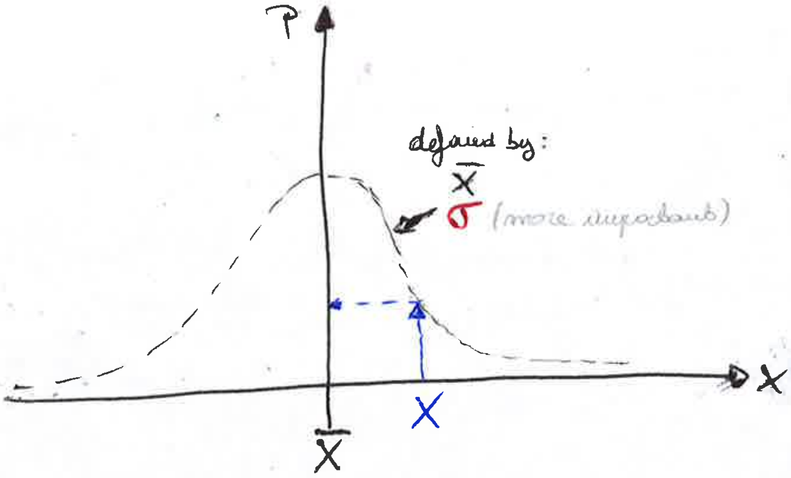
$$\Rightarrow \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} x_t = 0 \rightarrow \sum_{i=1}^{N} x_i = N \cdot x_t$$

$$x_t = \frac{\sum_{i=1}^{N} x_i}{N} \Rightarrow$$

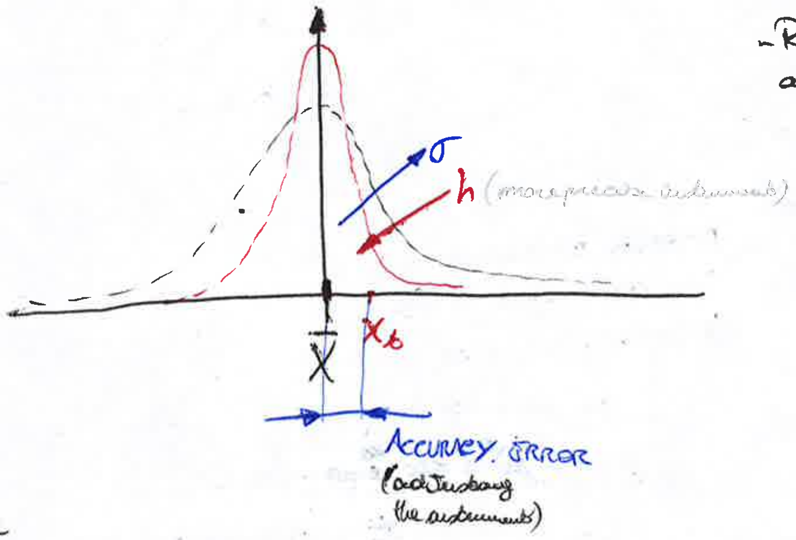
MOST PROBABLE APPROXIMATION OF THE TRUE VALUE = MEAN VALUE

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}}$$

Now we want to evaluate the uncertainty over an interval



- Remember that the area is always equal to 1!



- Example

$\sigma = \pm 1\%$ $(\underbrace{FSR}_{\text{Full Scale Range}})$

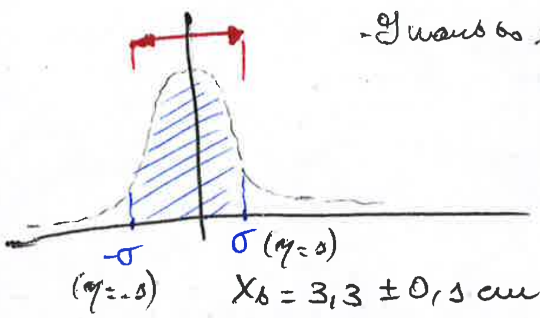
$X_t = 3{,}2$ cm $\Rightarrow$ $\underline{\sigma = 0{,}1 \text{ cm}}$ $(0.01 \cdot 10)$

$FSR = 10$ cm

$N = 1 \longrightarrow X_1 = X = 3{,}3$ cm

- I want to evaluate the probability of being in the red range:

$$\eta = \frac{x - \bar{x}}{\sigma}$$



$\eta_1 = 1$
$\eta_2 = -1$ $\rightarrow p = 68\%$ $(0{,}34134 \cdot 2)$ ← see table of normal distrib.

$X_t = 3{,}3 \pm 0{,}1$ cm (68%) ← NOT CORRECT WAY → see interval of confidence

$\bar{x}$ $\quad \sigma$ = uncertainty

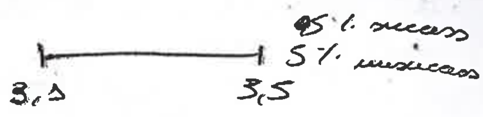
N.B. I haven't got $\bar{x}$ because N = 1

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 41 di 496

$\gamma = 2$

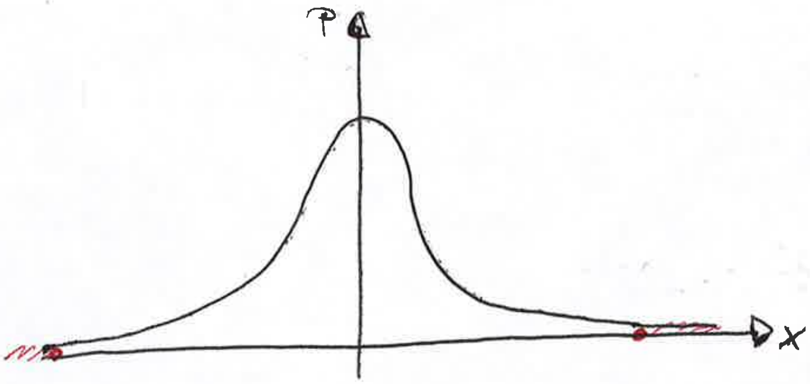
$CL = 95\%$ $\Rightarrow$ $X = 3{,}3 \pm 2 \cdot 0{,}1\ cm$

$SL = 5\%$ $\quad = 3{,}3 \pm 0{,}2\ cm$

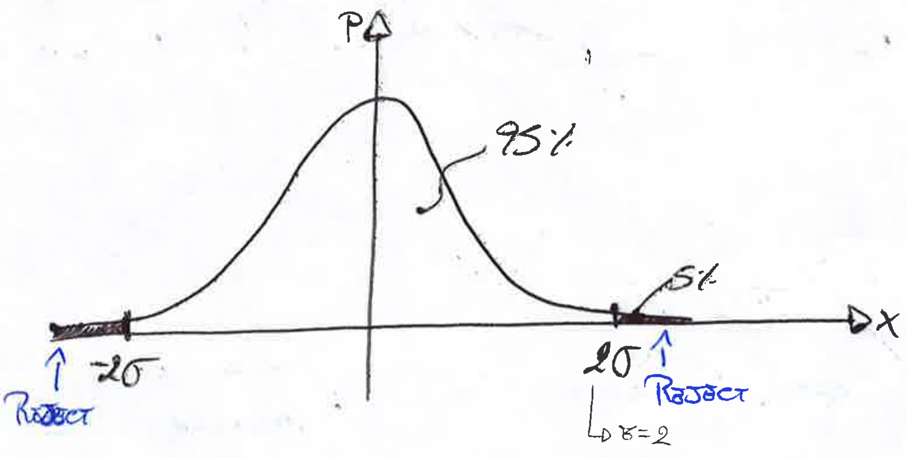


If I want to decrease uncertainty without changing confidence level, I have to change instrument or take more measurements.

• CHAUVENET'S CRITERION



I can obtain measurements with a very low probability (at the queues of the distribution).
So I need a criterion to reject very bad measurements, affected by other errors, for example,

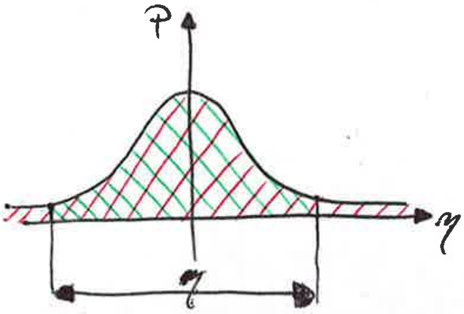


It says:
When the probability to have a measurement in our set of data is lower than $\frac{1}{2N}$, we can reject the measurement.

- Second approach (evaluating the interval of acceptable measurements)

$1-2P(\eta) < \frac{1}{2N} \Rightarrow$ REJECT (for the Chauvenet's criterion)

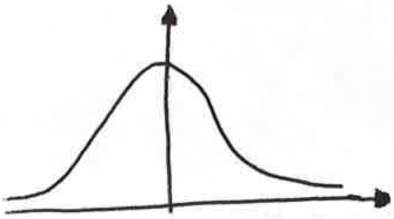
$$2P(\eta) > 1-\frac{1}{2N} = 1-\frac{1}{20} = 0,95$$



$$P(\eta) = \frac{0,95}{2} = 0,475$$

⇓ from the table Normal Distribution

$$\eta = 1,96$$



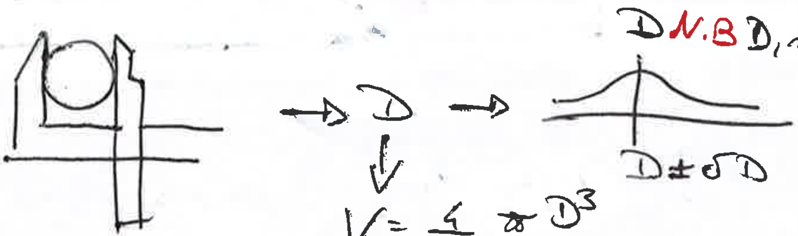
$$\eta = \frac{X-\bar{X}}{\sigma} \rightarrow X = \bar{X} \pm \eta\sigma =$$

$$= 5,621 \pm 1,96 \cdot 0,630$$

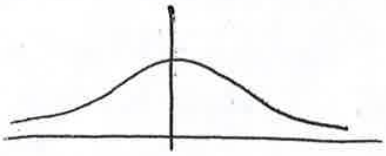
ACCEPTABLE MEASUREMENTS: $[X] = [4,386 \quad 6,856]$

- PROBABILITY GRAPH PAPER

Or



$$V = \frac{4}{3}\pi\frac{D^3}{8}$$



$V = \frac{1}{6}\pi(D \pm \delta D)^3 \Rightarrow$ NOT LINEAR relationship

$$1,1^3 = 1,331 \rightarrow \Delta = +0,331$$

$$0,9^3 = 0,729 \rightarrow \Delta = -0,271$$

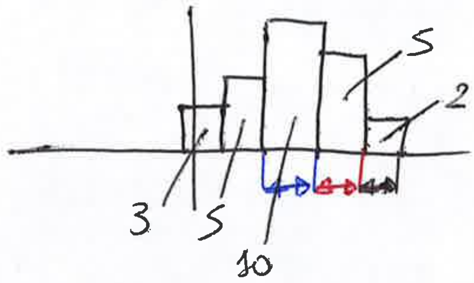
⇒ It's NOT a gaussian distribution

• χ^2 TEST (chi-square test)

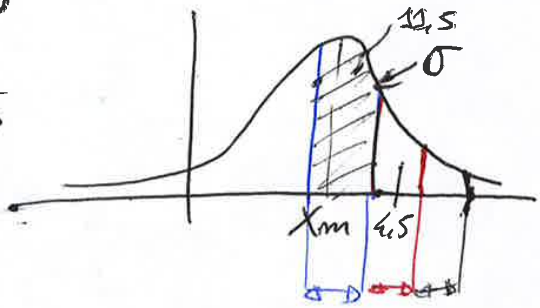
$$\chi^2 = \sum_{i=1}^{N} \frac{\left[\text{NUMBER OF OBSERVED VALUE}_i - \text{NUMBER OF EXPECTED/THEORETICAL VALUE}_i\right]^2}{\text{NUMBER OF THEORETICAL VALUE}_i}$$

- It compares two distributions, an experimental and a theoretical one.

• Experimental data $N = 20$ • Theoretical data



compare



OBSERVED

THEORETICAL

$$\chi^2 \rightarrow 0 \quad \text{if} \quad \#\text{ OBSERVED} \rightarrow \#\text{ THEORETICAL}$$

(χ^2 tends to 0 if the actual distribution is very similar to the theoretical one)

$$\chi^2 = f\left(\text{Goodness of distribution}, N\right)$$

- The χ^2 test can be used to check the validity of various distribution. The probability that our data follow the considered distribution is on the tables or slide:

$$F = \tilde{N} - K$$

F: degree of freedom

K: ↳ number of constraints of the distribution usually $K = 1$

• ANALYSIS OF THE ERROR PROPAGATION

We can evaluate → MAX UNCERTAINTY (deterministic approach) or MAXIMUM ERROR
→ MOST PROBABLE UNCERTAINTY (statistical approach)

1) MAXIMUM ERROR (or uncertainty)

$$X = \bar{X} \pm \delta x = \bar{X} \pm \gamma \sigma$$

$\bar{X}$: most probable = mean value; δx: uncertainty (or error); γ: interval of confidence; σ: standard deviation

• ELEMENTARY CASES

(A) ADDING A CONSTANT QUANTITY ← CONSTANT

• $q = x + A$

$$q = \bar{x} \pm \delta x + A = (\bar{x} + A) \pm \delta x = \bar{q} \pm \delta q$$

$\bar{q} = \bar{x} + A$, $\delta q = \delta x$ ⇒ UNCERTAINTY DOESN'T CHANGE!

(B) PRODUCT WITH A CONSTANT

• $q = A x = A(\bar{x} \pm \delta x) = A\bar{x} \pm A\delta x = \bar{q} \pm \delta q$

$\bar{q} = A\bar{x}$, $\delta q = A\delta x$ ⇒ UNCERTAINTY CHANGES!

(C) SUM OF STOCHASTIC QUANTITIES (randomly determined quantities)

$q = x_1 + x_2$ where • $x_1 = \bar{x}_1 \pm \delta x_1$
• $x_2 = \bar{x}_2 \pm \delta x_2$

$$q = (\bar{x}_1 + \bar{x}_2) \pm (\delta x_1 + \delta x_2) = \bar{q} + \delta q$$

$\bar{q} = \bar{x}_1 + \bar{x}_2$ $\delta q = \delta x_1 + \delta x_2$ ⇒ UNCERTAINTY CHANGES

B) PRODUCT

multiplying and dividing by A

- $q = Ax$ (1)

$$P(x) \propto e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}} = e^{-\frac{(q/A - \bar{x}\frac{A}{A})^2}{2\sigma_x^2}} = e^{-\frac{(q-\bar{x}A)^2}{2A^2\sigma_x^2}}$$

from (1) — doing the least common multiple

$$\bar{q} = A\bar{x}, \quad \sigma_q = A\sigma_x \rightsquigarrow \delta q = A\delta x \Rightarrow \text{UNCERTAINTY CHANGES} \ (2)$$

– / max meas. and most probable give the same results

C) SUM

$$q = x_1 + x_2$$

- $x_1 = \bar{x}_1 \pm \delta x_1$
- $x_2 = \bar{x}_2 \pm \delta x_2$

- Case 1)

$$\bar{x}_1 = \bar{x}_2 = 0 \quad (\text{zero mean})$$

$$P(x_i) \propto e^{-\frac{x_i^2}{2\sigma_{x_i}^2}} \quad \rightarrow \text{because } \bar{x}=0$$

$i = 1, 2$

$$P(x_1, x_2) = \prod_{i=1}^{2} p(x_i) \propto e^{-\frac{x_1^2}{2\sigma_{x_1}^2} - \frac{x_2^2}{2\sigma_{x_2}^2}} =$$

it serves to unify the equation

$$= e^{-\frac{(x_1+x_2)^2}{2(\sigma_{x_1}^2+\sigma_{x_2}^2)} - \underbrace{\frac{(x_1\sigma_{x_2} - x_2\sigma_{x_1})^2}{2\sigma_{x_1}^2\sigma_{x_2}^2(\sigma_{x_1}^2+\sigma_{x_2}^2)}}_{z^2}} =$$

$$= e^{-\frac{(x_1+x_2)^2}{2(\sigma_{x_1}^2+\sigma_{x_2}^2)}} \cdot e^{-\frac{z^2}{2}} = \int_{-\infty}^{\infty} e^{-\frac{(x_1+x_2)^2}{2(\sigma_{x_1}^2+\sigma_{x_2}^2)}} \cdot e^{-\frac{z^2}{2}} dz =$$

in order to eliminate the effect of z^2

$$P(x_1, x_2) \propto e^{-\frac{(x_1+x_2)^2}{2(\sigma_{x_1}^2+\sigma_{x_2}^2)}} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz}_{= 1/\sqrt{2\pi} \ = \text{const}}$$

see the last slide of the error distribution with $h^2 = 1$ and $x^2 = \frac{z^2}{2}$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 51 di 496

$$\boxed{\bar{q} = q(\bar{x}_1, \dots, \bar{x}_n)}$$ Most Probable Mean Value (if $\bar{x}_1, \dots \bar{x}_n \neq 0$)

$$q^2 = \sum_{i=1}^{N} \left(\frac{\partial q}{\partial x_i} \Big|_{\bar{x}_1, \dots \bar{x}_n} \right)^2$$

$$\boxed{\delta q = \sqrt{\sum_{i=1}^{N} \left(\frac{\partial q}{\partial x_i} \Big|_{\bar{x}_1, \dots, \bar{x}_n} \cdot \delta x_i \right)^2} = \gamma \sigma_q}$$ Most Probable Error (or uncertainty) (if $\bar{x}_1, \dots, \bar{x}_n \neq 0$)

$$\boxed{\sigma_q = \sqrt{\sum_{i=1}^{N} \left(\frac{\partial q}{\partial x_i} \Big|_{\bar{x}_1, \dots, \bar{x}_n} \cdot \sigma_{x_i} \right)^2}}$$

sum of the uncertainties of any term of the formula!

Most Probable Value Of Standard Deviation (if $\bar{x}_1, \dots, \bar{x}_n \neq 0$)

1 — Exercise mult 50

$$R_\theta = R_{20} \cdot (1 + \alpha (\theta - 20))$$

x_1, x_2, x_3 — by

$T_{ref} = 20°$

$$R_{20} = 6 (1 \pm 0{,}3\ \%)\ \Omega$$

N.B. of 6Ω!

$$\alpha = 4 \cdot 10^{-3} (1 \pm 0{,}1\ \%)\ C^{-1}$$

$$\theta = \begin{cases} 30 \pm 1\ °C \\ 80 \pm 1\ °C \end{cases}$$

· Requests

1) $R_{30} = \_\_\_\_ \pm \_\_\_\_\ \Omega$

$R_{80} = \_\_\_\_ \pm \_\_\_\_\ \Omega$

2) How to acquire measurements?

· Formulas

$$\bar{q} = q(\bar{x}_1, \dots, \bar{x}_n)$$

$$\delta q = \sqrt{\sum_{i=1}^{N} \left(\frac{\partial q}{\partial x_i} \Big|_{\bar{x}_1, \dots, \bar{x}_n} \cdot \delta x_i \right)^2}$$

$$\delta R_{30} = \sqrt{(1{,}04 \cdot 1{,}8 \cdot 10^{-2})^2 + (60 \cdot 4 \cdot 10^{-5})^2 + (2{,}4 \cdot 10^{-2} \cdot 1)^2} =$$

$$a) = \sqrt{\underbrace{3{,}5 \cdot 10^{-4}}_{R_{20}} + \underbrace{5{,}8 \cdot 10^{-6}}_{\alpha} + \underbrace{5{,}8 \cdot 10^{-4}}_{\theta}} =$$

(two orders of magnitude lower)

$$= 3{,}06 \cdot 10^{-2}\ \Omega = 0{,}0306\ \Omega \simeq 0{,}03\ \Omega$$

↳ using 2 S.D

$$\Rightarrow R_{30} = \overline{R_{30}} \pm \delta R_{30}\ \Omega$$

$$= 6{,}04 \pm 0{,}03\ \Omega$$

PARTIRE DALL' INCERTEZZA!

2) Looking to a)

I have to improve the measurement of θ or R_{20} because they are the ones with a major contribute (they are two orders of magnitude larger)

• $80°C$

$$\left.\frac{\partial R_\theta}{\partial R_{20}}\right|_{\bar{X}} = 1 + 4 \cdot 10^{-3}(80-20) = 1{,}24 \text{ (adimensional)}$$

$$\left.\frac{\partial R_\theta}{\partial \alpha}\right|_{\bar{X}} = 6(80-20) = 360\ \Omega\,°C$$

$$\left.\frac{\partial R_\theta}{\partial \theta}\right|_{\bar{X}} = 6 \cdot 4 \cdot 10^{-3} = 0{,}024\ \frac{\Omega}{°C} = 2{,}4 \cdot 10^{-2}\ \frac{\Omega}{°C}$$

$$\Rightarrow R_{80} = \overline{R_{80}} \pm \delta R_{80} = $$

$$= 6{,}24 \pm 0{,}04\ \Omega$$

$$\delta R_{80} = \sqrt{\left(\left.\frac{\partial R_\theta}{\partial R_{20}}\right|_{\bar{X}} \cdot \delta R_{20}\right)^2 + \left(\left.\frac{\partial R_\theta}{\partial \alpha}\right|_{\bar{X}} \cdot \delta\alpha\right)^2 + \left(\left.\frac{\partial R_\theta}{\partial \theta}\right|_{\bar{X}} \cdot \delta\theta\right)^2} =$$

N.B. don't make approximation, yet!

$$= \sqrt{(1{,}24 \cdot 1{,}8 \cdot 10^{-2})^2 + (360 \cdot 4 \cdot 10^{-5})^2 + (2{,}4 \cdot 10^{-2} \cdot 1)^2} = \sqrt{\underbrace{5{,}0 \cdot 10^{-4}}_{R_{20}} + \underbrace{2{,}1 \cdot 10^{-4}}_{\alpha} + \underbrace{5{,}8 \cdot 10^{-4}}_{\theta}}$$

$$= 3{,}6 \cdot 10^{-2}\ \Omega = 0{,}036\ \Omega \simeq 0{,}04\ \Omega$$

- STANDARD DEVIATION OF THE MEAN

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$ NB In case of

$$\Rightarrow X_b \simeq \bar{x} + \delta \frac{\sigma}{\sqrt{N}}$$ it depends from the instrument

In order to reduce the uncertainty: Use a better instrument with a lower standard deviation / Perform a lot of measurements (more expensive)

- Example (where to apply the weighted arithmetic mean)

3 group of measurements with the same interval of confidence

A) $1,2 \pm 0,3$ $(\delta = 1)$

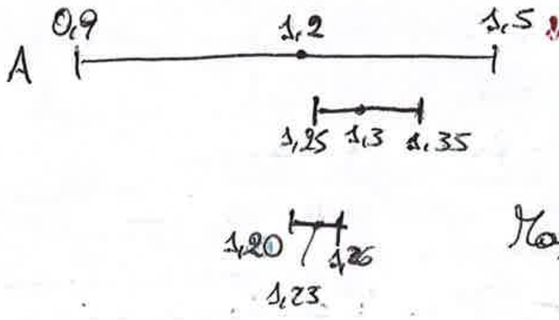
B) $1,30 \pm 0,05$ $(\delta = 1)$

C) $1,23 \pm 0,03$ $(\delta = 1)$

$\xrightarrow{\text{Combining the measurements}}$?

- Firstly, I have to check for bias errors!

I have to look if the results are consistent, if they overlap (2σ si sovrappongono)



Maybe group C is affected by bias error

- Applying the definition of weights to the measurements of the example

$$W_1 = \frac{1}{0{,}3^2} = 11{,}11$$

$$W_2 = \frac{1}{0{,}05^2} = 400$$

$$W_3 = \frac{1}{0{,}03^2} = 1111{,}1$$

- Now, I can calculate the true value:

$$X = \frac{1{,}2 \cdot 11{,}1 + 1{,}3 \cdot 400 + 1{,}23 \cdot 1111{,}1}{1522{,}22} = \leftarrow \text{media pesata}$$

$$= 1{,}2482 \pm \text{uncertainty}$$

- STANDARD DEVIATION OF THE WEIGHTED ARITHMETIC MEAN:

- Now, I have to calculate the uncertainty in the same way of the mean:

$$\bar{X} = \frac{\sum_{i=1}^{N} \bar{x}_i \cdot w_i}{\sum_{i=1}^{N} w_i} \quad , \quad W_i = \frac{1}{\sigma_i^2} \rightarrow \sigma_i = \frac{1}{\sqrt{w_i}}$$

$$\sigma_{\bar{X}} = \sqrt{\sum_{i=1}^{N} \left[\frac{\partial \bar{X}}{\partial \bar{x}_i}\bigg|_{\bar{x}_1,\dots,\bar{x}_n} \cdot \sigma_i \right]^2} = \sqrt{\sum_{i=1}^{N} \left[\frac{\partial}{\partial \bar{x}_i} \left(\frac{\sum_{j=1}^{N} \bar{x}_j \cdot w_j}{\sum_{k=1}^{N} w_k} \right)\bigg|_{\bar{x}_1,\dots,\bar{x}_n} \cdot \sigma_i \right]^2} =$$

$$\begin{cases} \frac{\partial x_j}{\partial x_i} = 1 \\ \sum_{j=1}^{N} w_j = \sum_{i=1}^{N} w_i \end{cases} \rightarrow = \sqrt{\sum_{i=1}^{N} \left[\frac{1}{\sum_{k=1}^{N} w_k} \cdot w_i \cdot \sigma_i \right]^2} =$$

$$\sigma_i = \frac{1}{\sqrt{w_i}} \rightarrow = \sqrt{\frac{1}{\left(\sum_{k=1}^{N} w_k\right)^2} \cdot \sum_{i=1}^{N} \left(w_i \frac{1}{\sqrt{w_i}} \right)^2} =$$

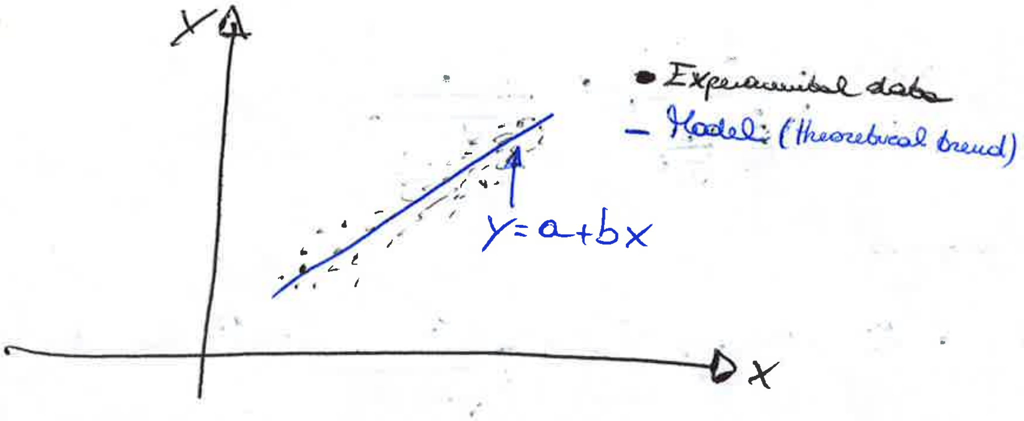
$$\sum_{k=1}^{N} w_k = \sum_{i=1}^{N} w_i = \sqrt{\frac{1}{\left(\sum_{i=1}^{N} w_i\right)^{\cancel{2}}} \cancel{\left(\sum_{i=1}^{N} w_i\right)}} = \sqrt{\frac{1}{\sum_{i=1}^{N} w_i}}$$

STANDARD DEVIATION OF THE WEIGHTED ARITHMETIC MEAN

$$\boxed{\sigma_{\bar{X}_w} = \sqrt{\frac{1}{\sum_{i=1}^{N} w_i}} = \frac{1}{\sqrt{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}}}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 59 di 496

• THE LEAST SQUARE METHOD



- I have to evaluate a and b coefficients in order to maximise the error of the model that approximates my experimental data

$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_m & y_m \end{array}$$

- I suppose that $\delta x << \delta y$ (uncertainty of $x_1 \ldots x_m$ lower than uncertainty of $y_1 \ldots y_m$)

$$y_1, \ldots, y_m \nearrow P(y_i) \propto e^{-\frac{(y_i - y_t)^2}{2\sigma_y^2}}$$

substituting the theoretical trend

$$P(y_1, \ldots, y_m) = \prod_{i=1}^{m} P(y_i) \propto e^{-\sum_{i=1}^{N} \frac{(y_i - y_t)^2}{2\sigma_y^2}} =$$

$$= e^{-\sum_{i=1}^{N} \frac{[y_i - (a + bx_i)]^2}{2\sigma_y^2}}$$

- Now I have to find a and b that maximise $P(y_1, \ldots, y_m)$:

• a

$$\frac{\partial P}{\partial a} \propto \sum_{i=1}^{m} \frac{\cancel{2}[y_i - (a + bx_i)]}{\underbrace{\cancel{2}\sigma_y^2}_{\text{const}}} = 0$$

$$\bullet \sum_{i=1}^{m} y_i \overbrace{- ma}^{-a \cdot \sum_{i=1}^{m} 1} - b \sum_{i=1}^{N} x_i = 0 \quad (1)$$

- Example (for covariance)

$P = C \cdot \omega$ (steady state engine)

- Mean value

1st approach

$$x_1, x_2, \dots, x_n \qquad \bar{x}$$
$$\Rightarrow \qquad q^* = (\bar{x}, \bar{y})$$
$$y_1, y_2, \dots, y_n \qquad \bar{y}$$

2nd approach

$$x_1, x_2, \dots, x_n$$
$$\Rightarrow \quad q_1, q_2, \dots, q_n \rightarrow \bar{q} = \frac{\sum_{i=1}^{n} q_i}{n}$$
$$y_1, y_2, \dots, y_n$$

N.B 1st and 2nd approach are equivalent! Let's analyse why:

- Using the Taylor's Formula:

$$q_i = q(\bar{x}, \bar{y}) + \left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}} (x_i - \bar{x}) + \left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}} (y_i - \bar{y}) \left(+ \dots \text{further orders}\right)$$

$$\bar{q} = \frac{\sum_{i=1}^{n} q_i}{n} = \frac{1}{n} \sum_{i=1}^{n} q(\bar{x}, \bar{y}) + \frac{1}{n} \sum_{i=1}^{n} \left.\frac{\partial q}{\partial x_i}\right|_{\bar{x},\bar{y}} (x_i - \bar{x}) +$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}} (y_i - \bar{y}) + \text{(further orders)} =$$

↳ calculating $\frac{1}{n} \sum_{i=1}^{n} q(\bar{x}, \bar{y})$ and pulling out the derivatives of the sum

$$= \frac{1}{n} \cdot q(\bar{x}, \bar{y}) \underbrace{\sum_{i=1}^{n} 1}_{=n} + \frac{1}{n} \left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}} \sum_{i=1}^{n} (x_i - \bar{x}) + \frac{1}{n} \left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}} \sum_{i=1}^{n} (y_i - \bar{y}) =$$

$$= \frac{1}{\cancel{n}} \cancel{n}\, q(\bar{x}, \bar{y}) + \frac{1}{n} \left.\frac{\partial q}{\partial x}\right|_{\bar{x},\bar{y}} \cancel{\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x}\right)} +$$

$$+ \frac{1}{n} \left.\frac{\partial q}{\partial y}\right|_{\bar{x},\bar{y}} \cancel{\left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \bar{y}\right)} = q(\bar{x}, \bar{y})$$

- Observe that:

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} \overset{\sum_{i=1}^{n} \bar{x} = \bar{x} \sum_{i=1}^{n} 1 = \bar{x} \cdot n}{=} \sum_{i=1}^{n} x_i - \bar{x} \cdot n =$$

↳ $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$

$$= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i}{\cancel{n}} \cancel{n} = 0$$

(the same goes for y)

- Considering a linear dependency between X and Y:

$$X \;,\; Y \;\longrightarrow\; y_i = a + b x_i$$

- I want to demonstrate why two variables are linearly dependent when $r \simeq 1$

So, I start by calculating the terms of r:

$$\sigma_{xy} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) = \left\langle \begin{array}{l} \bar{y} = \sum_{i=1}^{n}\frac{y_i}{n} = \frac{1}{n}\sum_{i=1}^{n} y_i \\ y_i = a + b x_i \end{array}\right.$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})\Big(\underbrace{a+bx_i}_{y_i} - \underbrace{\frac{1}{n}\sum_{i=1}^{n}(a+bx_i)}_{\bar{y}}\Big) =$$

$$\left(\begin{array}{l}\frac{1}{n}\sum(a+bx_i) = \\ = \frac{1}{n}\sum a + \frac{1}{n}\sum b\cdot x_i = \\ = \frac{1}{n}a\sum_{n} 1 + \frac{1}{n}b\sum x_i \\ = a + \frac{1}{n}\sum b x_i\end{array}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})\Big(\cancel{a} + bx_i - \cancel{a} - \frac{1}{n}\sum_{i=1}^{n} b x_i\Big) =$$

remembering that $\dfrac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(bx_i - b\bar{x}) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})b =$$

$$= \frac{1}{n}\, b \sum_{i=1}^{n}(x_i-\bar{x})^2 = b\,\sigma_x^2$$

σ_x^2 (variance)

$$\sigma_y^2 = \frac{1}{n}\sum(y_i-\bar{y})^2 = \frac{1}{n}\sum\left(a + bx_i - \frac{\sum_{i=1}^{n}(a+bx_i)}{n}\right)^2 =$$

same steps as before

$$= \frac{1}{n}\sum_{i=1}^{n}(\cancel{a} + bx_i - \cancel{a} - b\bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}[b(x_i-\bar{x})]^2 =$$

$$= \frac{1}{n}\, b^2 \sum (x_i-\bar{x})^2 = b^2 \sigma_x^2$$

σ_y^2

$$\Rightarrow \sigma_y = |b|\,\sigma_x$$

N.B. r is near 1, usually, ($\simeq 0.95$) but it's not 1 because we are performing experimental measurements.

If it's 1, something is wrong, we can't have no uncertainty (significance level $= 0\%$, see table "Correlation coefficients on slides")

Curve Fitting

- Example

$$y = K e^{cx}$$ ← actual relationship

$$\ln y = \ln K + \ln e^{cx}$$

$$\ln y = \ln K + cx$$

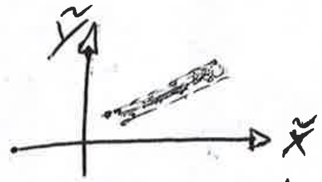
$$\Downarrow$$

$$\tilde{y} = \tilde{a} + \tilde{b}\tilde{x}$$ ← final relationship (linear)

x	y
⋮	⋮

⇝

$\tilde{x}$	$\tilde{y}$
⋮	⋮



⇝ I can apply the least square method and evaluate $\tilde{a}$, $\tilde{b}$

$$\leadsto \begin{cases} \tilde{a} = \ln K \\ \tilde{b} = c \end{cases} \rightarrow \begin{cases} K = e^{\tilde{a}} \\ c = \tilde{b} \end{cases}$$

⇒ I can evaluate the coefficients of the actual relationship!

- Now, I can evaluate the most probable errors

③ $\delta P = \sqrt{(100 \cdot 0{,}1)^2 + (200 \cdot 0{,}1)^2} = \sqrt{100+400} = \sqrt{500} = 22{,}36\ W$

② $\delta P = \sqrt{(20 \cdot 1)^2 + (100 \cdot 0{,}1)^2} = \sqrt{30} = 22{,}36\ W$ → because they combine 3 measurements (I, I, R) or (V, V, R)

① $\delta P = \sqrt{(10 \cdot 1)^2 + (100 \cdot 0{,}1)^2} = \sqrt{200} = 14{,}2\ W$

because it combines only 2 measurements (V, I)

Rule:

I have to set an experiment that combines the least number of measurements

- Results NB remember to start from uncertainty!

③, ② $P = 100 \cdot 10 \pm 2 \cdot 10\ W$ → 1 significant digit

or

$(1000 \pm 2 \cdot 10\ W)$

① $P = 100 \cdot 10 \pm 1{,}4 \cdot 10\ W$ → 2 significant digits

or

$(1000 \pm 1{,}4 \cdot 10\ W)$

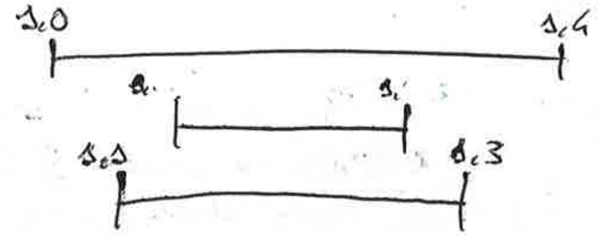
- I have to evaluate the standard deviation of the mean!

$$\delta L = z \cdot \frac{\sigma}{\sqrt{N}} = 1{,}96 \cdot \frac{0{,}599}{\sqrt{10}} = 0{,}371 \text{ mm}$$

$$\Rightarrow \boxed{L = 100{,}1 \pm 0{,}4 \text{ mm}}$$

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 71 di 496

1)



They overlap => they are consistent => No BUS ERRORS!

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 73 di 496

$$\sigma_t = \sqrt{\frac{1}{50} \cdot \sum_{i=1}^{n} \left[(t_i - \bar{t})^2 \cdot n_i \right]} = \sqrt{\frac{42 \cdot 546}{50}} = \cancel{0,922\,s} \; 0,937\,ms$$

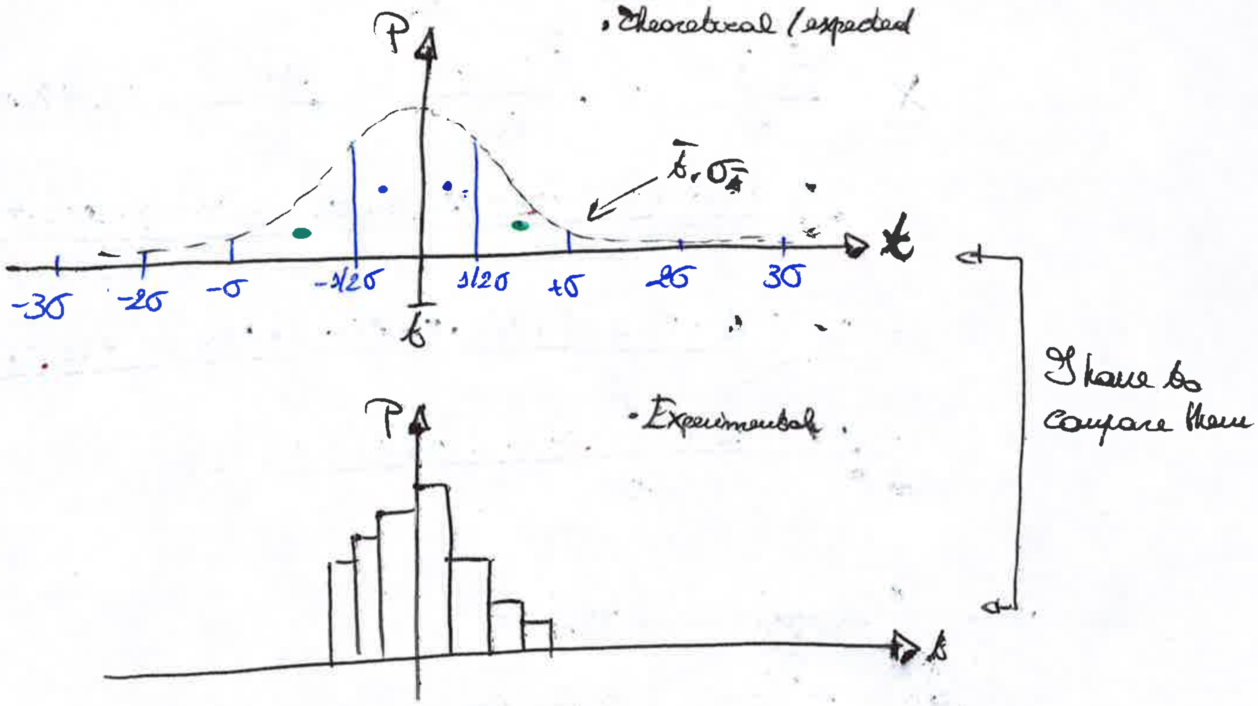
$$t = \bar{t} \pm z \frac{\sigma_t}{\sqrt{N}} =$$

- $z = 1,96$
- $\sigma_t = 0,937\,s$
- $N = 50$

$$\rightarrow z \frac{\sigma_t}{\sqrt{N}} = 0,260\,ms$$

$$= 23,0 \pm 0,3\,ms$$

1)



$$\chi^2 = \sum_{i=1}^{m} \frac{\left[\text{obs. value}_i - \text{th. value}_i \right]^2}{(\text{th. value}_i)}$$

- By using the table Normal Distribution, I can evaluate the expected value (theoretical value$_i$) and compare them with the observed value (from the histogram)

I can estimate how many samples I need in each interval (blue or green part)

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 75 di 496

- Now I have to evaluate how many measurements fall in that intervals (see last page) and, finally I can calculate χ^2:

$$\chi^2 = \frac{(15-9{,}55)^2}{9{,}55} + \frac{(7{,}5-5)^2}{7{,}5} + \frac{(7{,}95-8)^2}{7{,}95} + \frac{(9{,}55-7)^2}{9{,}55} + \frac{(7{,}5-6)^2}{7{,}5} + \frac{(9-7{,}95)^2}{7{,}95}$$

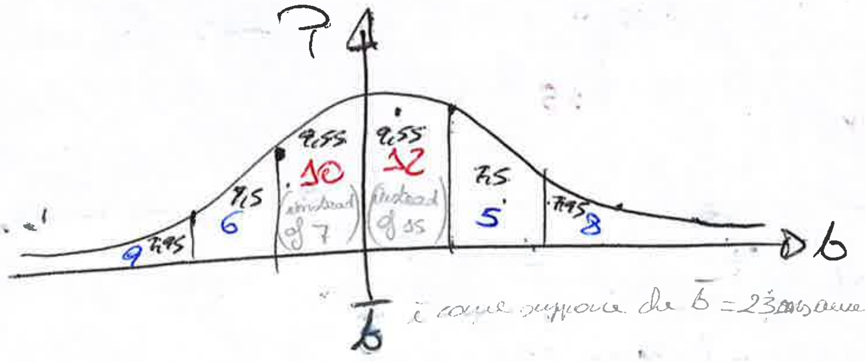
$$= 5{,}06$$

- Now, looking at the table of χ^2 test, knowing that:

$$F = \underbrace{n}_{\text{number of intervals}} - \underbrace{K}_{\text{constraint } (=1 \text{ variable})} = 6 - 1 = 5$$

$\Rightarrow$ Significance level (known χ^2 and F) $\simeq 50\%$ min 30
(Confidence level $\simeq 50\%$ (probability to be a Normal Distribution)

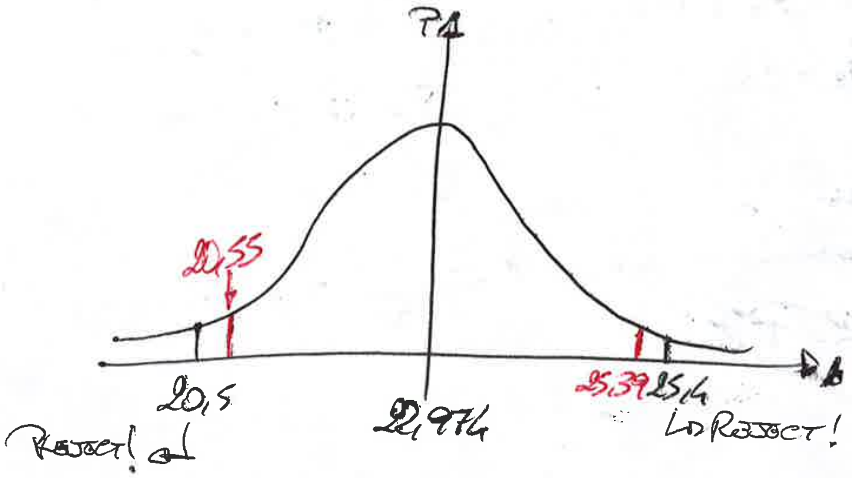
- In order to improve that S.L., I can distribute the values and get a realistic conclusion around $\bar{t}$ in the 2 half equally. I can do it because the variation is lower than uncertainty (it's good sense)



i come suppone che $\bar{b}$ = 23 ms invece di 22,974 ms

$\Rightarrow \chi^{2\prime} \simeq 1{,}55 \Rightarrow$ significance level $\simeq 10\%$ (confidence level $\simeq 90\%$) } of following Normal distribution

– Using the 2nd approach



$$2P(\eta) < 1 - \frac{1}{2N} = 1 - \frac{1}{2 \cdot 50} = 0{,}99$$

$$P(\eta) = \frac{0{,}99}{2} = 0{,}495 \rightarrow \eta = 2{,}58 = \frac{|x - \bar{x}|}{\sigma}$$

$$\Rightarrow |X - \bar{X}| = 2{,}58 \cdot \sigma = 2{,}42$$

$$\Rightarrow \quad 20{,}55 \leq X \leq 25{,}39$$ INTERVAL OF ACCEPTABLE MEASUREMENTS

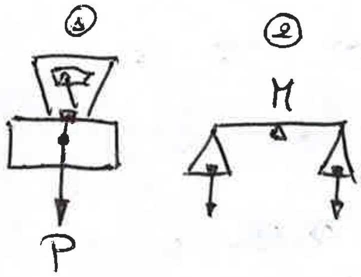
Exercise 6 Da Esame

- DATA

$$g \longrightarrow \frac{\delta g}{g} = \pm 0{,}1\ \%$$

$$\underbrace{P = mg}_{\text{TWO INSTRUMENTS:}} \longrightarrow g = \frac{P}{m} \ ; \ m \simeq 1\,Kg$$

- REQUEST
- Calculate
  - $\frac{\delta P}{P}$
  - $\frac{\delta m}{m}$



$$\delta g = \sqrt{\sum_{i=1}^{m} \left(\frac{\partial g}{\partial x_i}\bigg|_{\bar{x}} \cdot \delta x_i \right)^2}$$

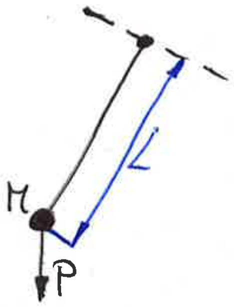
- $\frac{\partial g}{\partial P} = \frac{1}{m}$ $\qquad \delta P = ?$

- $\frac{\partial g}{\partial m} = -\frac{P}{m^2}$ $\qquad \delta m = ?$

$$\delta g = \sqrt{\left(\frac{\partial g}{\partial P}\bigg|_{\bar{P},\bar{m}} \cdot \delta P \right)^2 + \left(\frac{\partial g}{\partial m}\bigg|_{\bar{P},\bar{m}} \cdot \delta m \right)^2}$$

$$\frac{\delta g}{g} = \sqrt{\left[\frac{\partial g}{\partial P}\bigg|_{\bar{P},\bar{m}} \frac{\delta P}{g} \right]^2 + \left[\frac{\partial g}{\partial m}\bigg|_{\bar{P},\bar{m}} \cdot \frac{\delta m}{g} \right]^2}$$

• Esercizio 7



1) $T = f(M, L, P)$? by using the dimensional analysis $\quad P = Mg$

$$T = M^a L^b P^e$$

$[T] = [s]$

$[M] = [Kg]$

$[L] = [m]$

$[P] = [N] = \left[Kg \cdot \frac{m}{s^2}\right]$

$$[s] = [Kg]^a \cdot [m]^b \cdot \left(\left[Kg \cdot \frac{m}{s^2}\right]\right)^e ; \quad [s] = [Kg]^{a+e} \cdot [m]^{b+e} \cdot \underbrace{\left[\frac{1}{s^2}\right]^e}_{s^{-2}} = [Kg]^{a+e} \cdot [m]^{b+e} \cdot [s]^{-2e}$$

$$\begin{cases} a+e=0 \\ b+e=0 \\ -2e=1 \end{cases} \Rightarrow \begin{cases} a=-e=\frac{1}{2} \\ b=-e=\frac{1}{2} \\ e=-\frac{1}{2} \end{cases}$$

$$\Rightarrow T = K \cdot M^{1/2} L^{1/2} P^{-1/2} = K\sqrt{L} \cdot \sqrt{\frac{M}{P}} = K\sqrt{\frac{L}{g}}$$

$$= K\sqrt{\frac{L}{g}}$$

2) $M = 1{,}00 \pm 0{,}01$ Kg

L [mm]	T [s]					
100	0,63	0,64	0,63	0,63	0,64	0,62
150	0,75	0,79	0,78	0,78	0,74	0,76
200	0,91	0,90	0,93	0,92	0,89	0,90
250	1,01	1,03	0,97	0,98	1,02	1,03
300	1,10	1,11	1,09	1,09	1,07	1,09
350	1,20	1,17	1,18	1,19	1,21	1,20
400	1,27	1,25	1,26	1,24	1,27	1,28

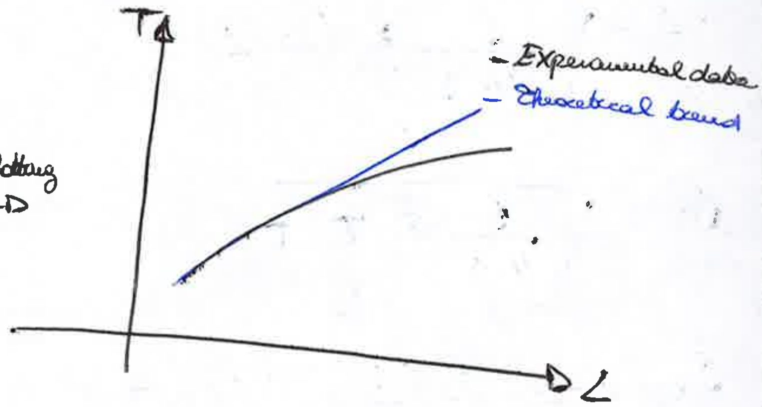
3) I have to apply the least square method with

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$g = 9{,}81 \pm 0{,}02 \ \frac{m}{s^2}$$

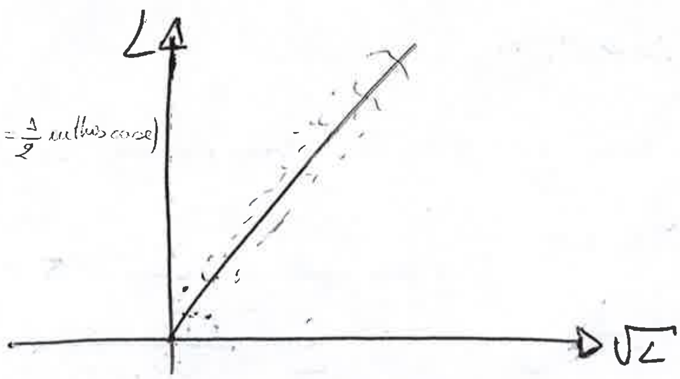
L [mm]	$\bar{T}$ [s]
100	0,635
150	
200	
250	
300	
350	
400	1,262

Plotting →



- Since data trend is not linear, I have to linearize data before using the least square method (better with Excel)

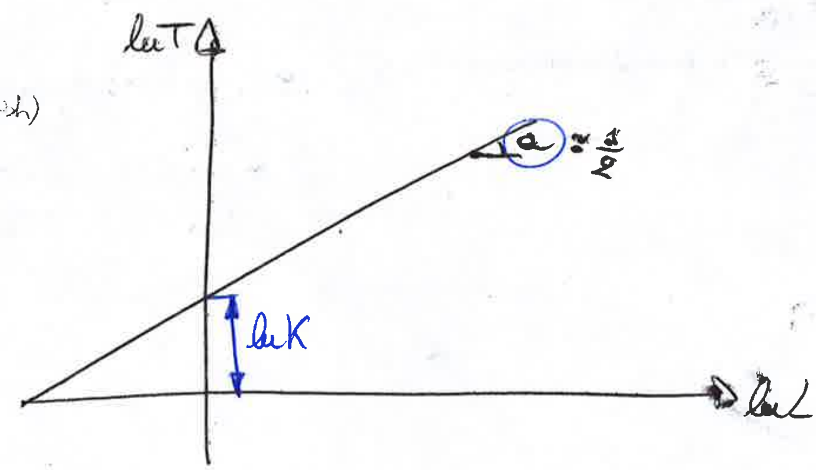
$$T = KL^a \quad (a = \tfrac{1}{2} \text{ in this case})$$



$$\ln T = \underbrace{\ln K}_{\text{Intercetta (in English)}} + a \ln L$$

mar 31 16

$$\ln K = \ln \frac{2\pi}{\sqrt{g}}$$



© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 85 di 496

- I use the Taylor's formula for $\frac{1}{1+x}$ with $x \to 0$

$$f(x) = \frac{1}{1+x} \qquad x \to 0 \quad \Rightarrow f(x) \simeq 1$$

- I linearize that relationship

• GENERAL RULE
Complex relationship, small interval ⇒ LINEARIZE ALWAYS!

$$f(x) = f(x_0) + \frac{\partial f}{\partial x}\Big|_{x_0}(x - x_0) + \dots$$

$$f(0) = 1 \qquad \frac{\partial f}{\partial x}\Big|_{x=0} = -\frac{1}{(1+x)^2}\Big|_0 = -1$$

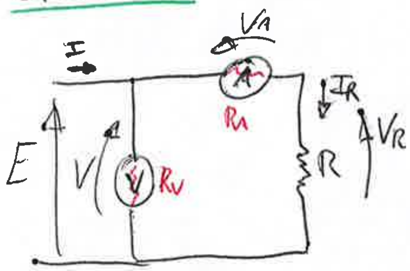
$$\Rightarrow \underbrace{\frac{1}{1+x}}_{\text{"complex"}} \simeq \underbrace{1 - x}_{\text{LINEARIZED (for } x \to 0)} \xrightarrow{\text{in our case}} R = R^* \frac{1}{1+\frac{R}{R_V}} \simeq R^*\left(1 - \frac{R}{R_V}\right)$$

$$\underbrace{R}_{\text{EVALUATED VALUE}} = \underbrace{R^*}_{\text{REAL VALUE}} \underbrace{- R^* \frac{R}{R_V}}_{\text{UNCERTAINTY}}$$

$$\frac{\delta R}{R} = -\frac{R^*}{R_V} \longrightarrow \boxed{\frac{\delta R}{R} = \frac{R}{R_V}}$$

N.B. Sign is not important. Whatever R or R^* is the same!

• UPSTREAM



$$R = \frac{V}{I} = \frac{V_A + V_R}{I_R} = \overbrace{\frac{V_R}{I_R}}^{R^*}\Big(1 + \underbrace{\frac{V_A}{V_R}}_{\delta R \text{ (uncertainty)}}\Big)$$

$$R = R^* + R^* \underbrace{\frac{V_A}{V_R}}_{\frac{\delta R}{R}}$$

- I want $\frac{\delta R}{R}$ in terms of resistance:

3) Evaluate the range of R that can be measured with a systematic error lower than 0,1 %?

UP

$$\frac{\delta R}{R} = \frac{R_A}{R} \leq \frac{0,1}{100} = 1 \cdot 10^{-3}$$

$$R \geq R_A \cdot 1000$$

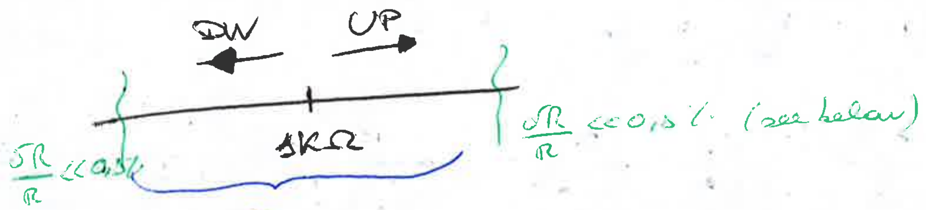
$$R \geq 1 K\Omega$$

DOWN

$$\frac{\delta R}{R} = \frac{R}{R_V} \leq \frac{0,1}{100} = 1 \cdot 10^{-3}$$

$$R \leq 10^{-3} R_V$$

$$R \leq 1 K\Omega$$



$$\frac{\delta R}{R} \leq 0,1 \%$$

- To reduce error < changing layout / statistical analysis (coherent calculations)

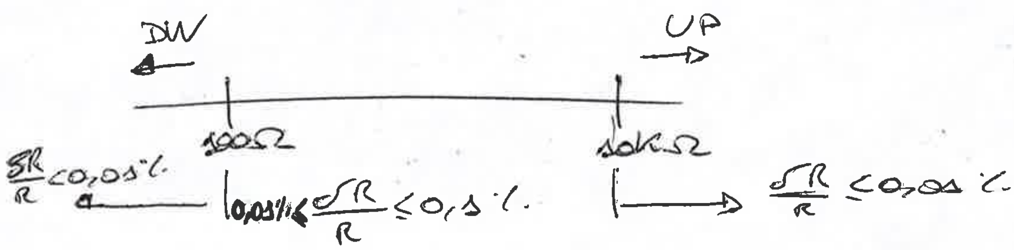
- Limit error $0,01 \% = 10^{-4}$

UP

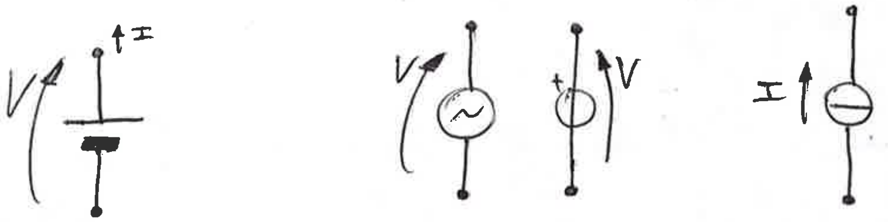
$$R \geq 10000 \cdot R_A = 10 K\Omega$$

DW

$$R \leq 10^{-4} R_V = 100 \Omega$$

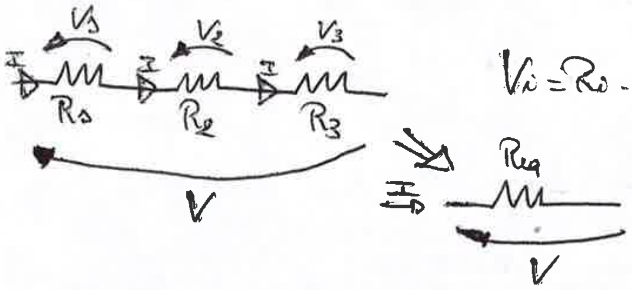


- DC GENERATOR
- VOLTAGE GENERATOR (current doesn't affect voltage)
- CURRENT GENERATOR (Voltage doesn't affect voltage)



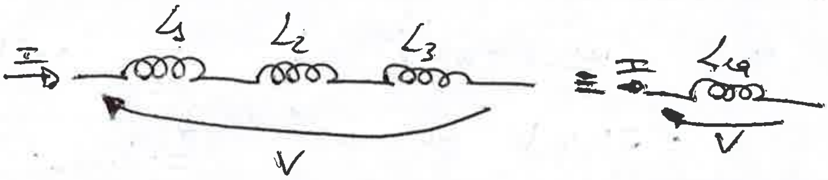
(BATTERY = DC)

- SERIES CONNECTION (serial connection on lists) => the same current passes through each component

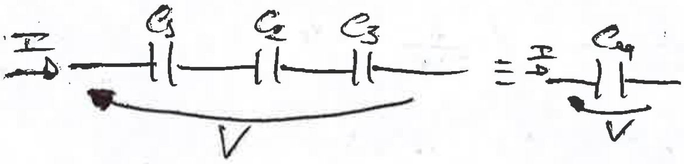


$V_i = R_i \cdot i$

$R_{eq} = \sum_{i=1}^{n} R_i$

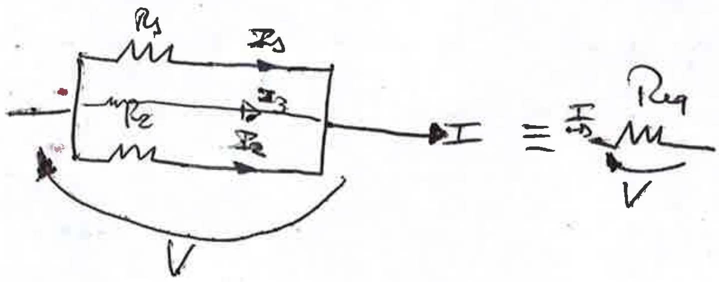


$L_{eq} = \sum_{i=1}^{N} L_i$

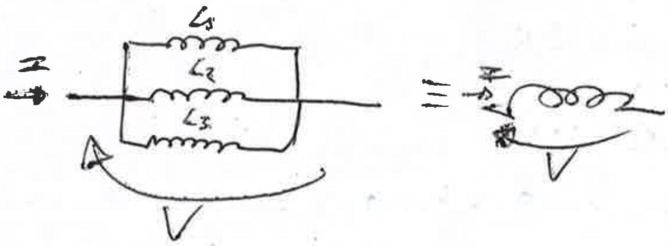


$\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i}$ N.B

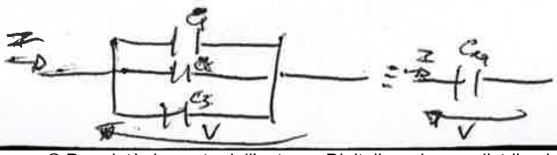
- PARALLEL CONNECTION => the same voltage is applied to each component



$\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i}$



$\frac{1}{L_{eq}} = \sum_{i=1}^{n} \frac{1}{L_i}$



$C_{eq} = \sum_{i=1}^{n} C_i$ N.B

- **IMPEDANCE**: measure of opposition to alternating current

- Ohm's Law

$V = Z \cdot i$

$Z = R + jX$ (resistance + reactance) N.B In addition to the amplitude, phase is introduced

- We will analyse signals in the frequency domain if they are harmonic signals

- **ELECTRICAL ENGINEERING**

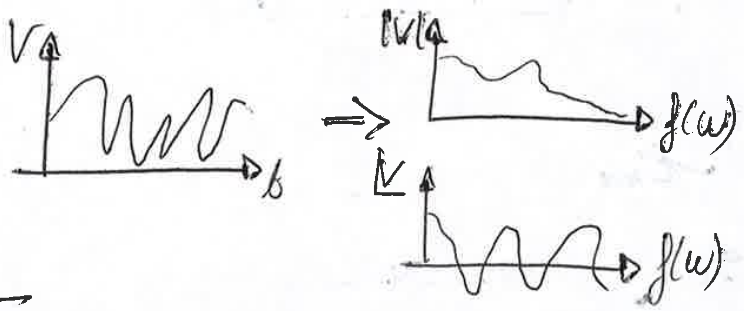
$\omega = CONST$

$X_C = \frac{1}{j\omega C} = -\frac{1}{\omega C}$

$X_L = j\omega L = \omega L$

- **ELECTRONIC ENGINEERING**

$\omega \neq CONST$

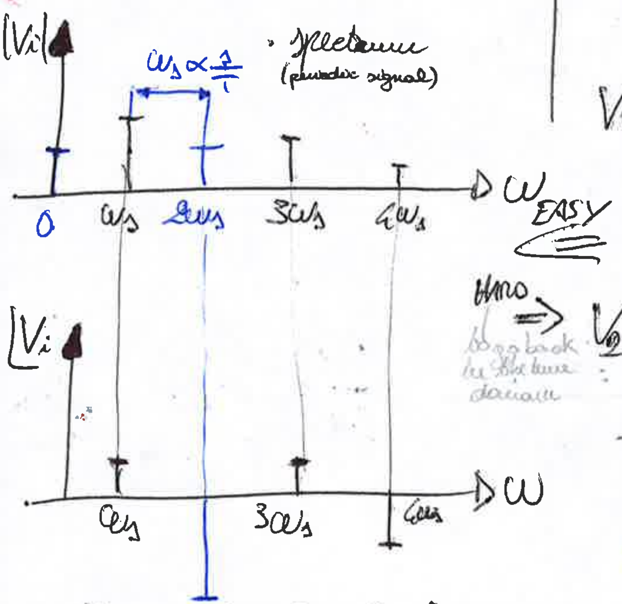


- **FREQUENCY DOMAIN**: shows us how much of the signal lies within each given frequency band over a range of frequencies

banda di frequenze di un determinato dominio di frequenza

amplitude, frequency, phase

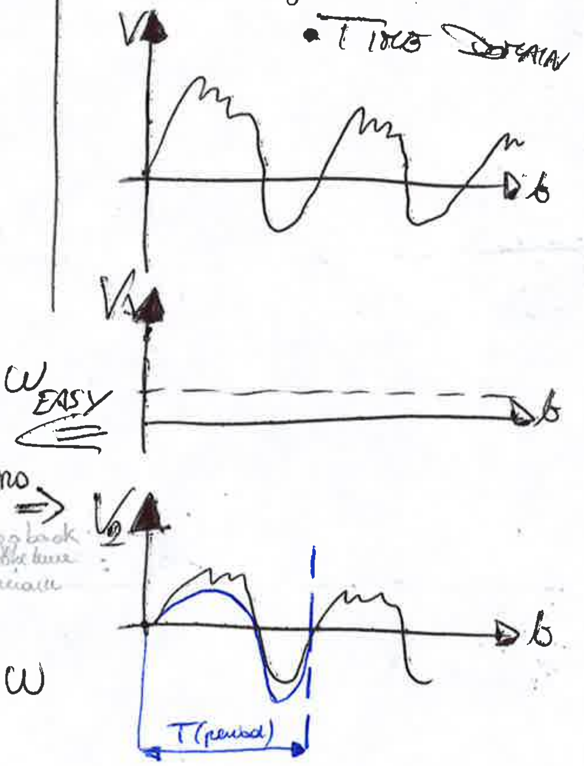
$|V_i| \cos(\omega_i t + \angle V_i)$



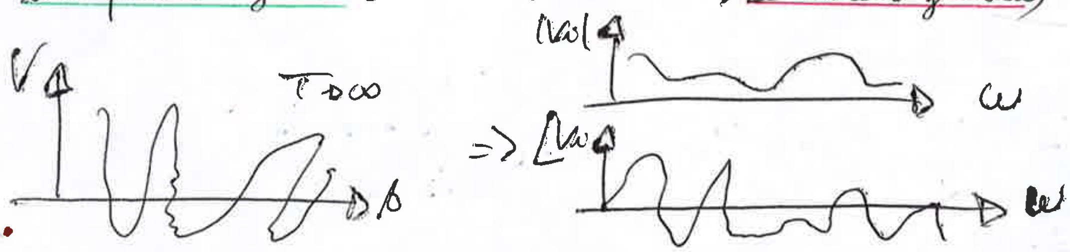
FOURIER

- Periodic signal in the time domain = sum of many harmonics

- TIME DOMAIN



- Non periodic signal ($T \to \infty \Rightarrow \omega \to 0 \Rightarrow$ continuous function)



$\Rightarrow$ Every signal can be considered as a sum of INFINITE HARMONICS and a phase

$$\frac{dV}{dt} - R\frac{di}{dt} - L\frac{di^2}{dt^2} = \frac{i}{C} \quad \text{(deriving)}$$

$$LC\frac{d^2i}{dt^2} + RC\frac{di}{dt} + i = C\frac{dV}{dt} \qquad \left(m\ddot{x} + c\dot{x} + Kx = F(t) \right)$$

- It's hard to use, so I pass to the frequency domain
(because it's a differential equation)

- FREQUENCY DOMAIN

$$V_C = V - V_R - V_L$$

- $V_R = R \cdot i$
- $i = CpV_C \longrightarrow V_C = \frac{i}{Cp}$
- $V_L = Lpi$

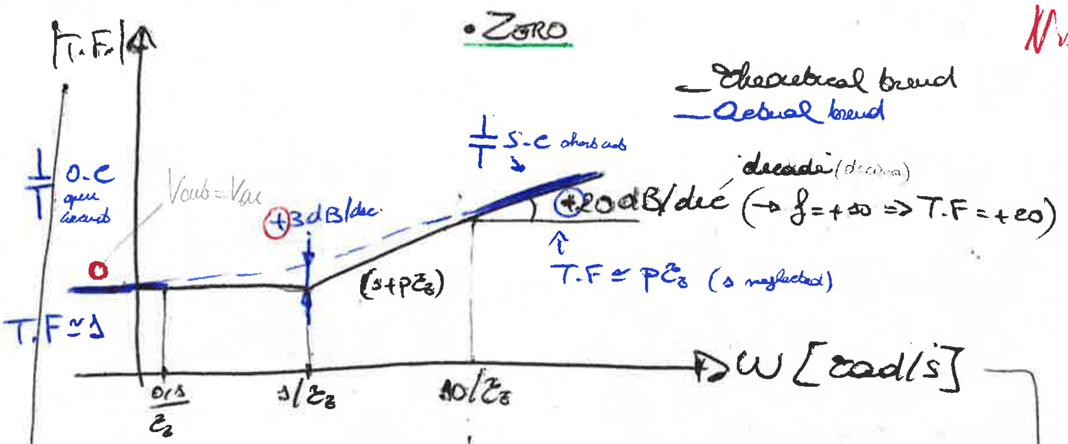
$$\frac{i}{Cp} = V - Ri - Lpi$$

$LCp^2 i + RCpi + i = CpV$ → ALGEBRAIC EQUATION! (Very simple)

Frequency domain shows us the energy of our signal that lies at a particular frequency
In addition, it shows how much of the signal lies within "each given frequency band" over a range of frequencies

• Bode Plot: Poles And Zeros

• Zero

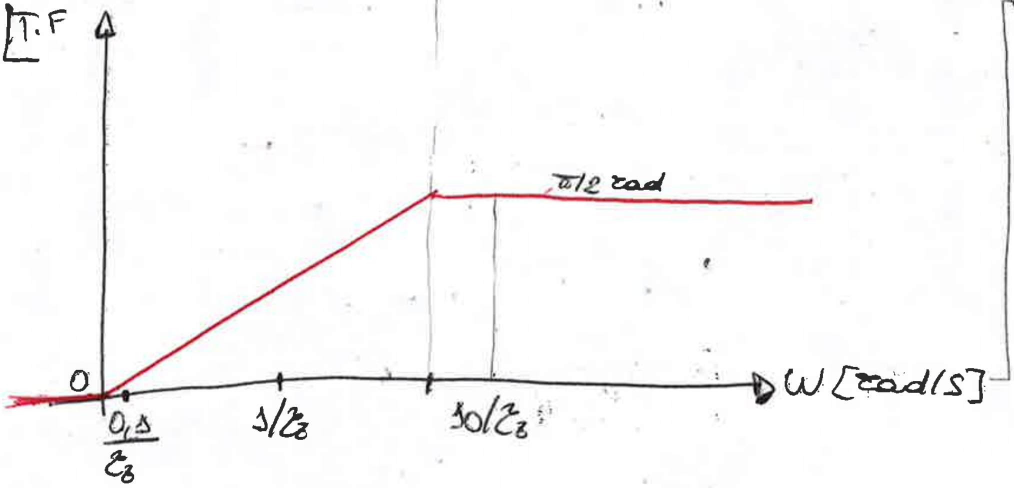


$[dB] = 20 \log_{10} |T.F|$

$[dB_\omega] = 10 \log_{10} \omega$

N.B Ordinate indicates GAIN (so it can be 0)

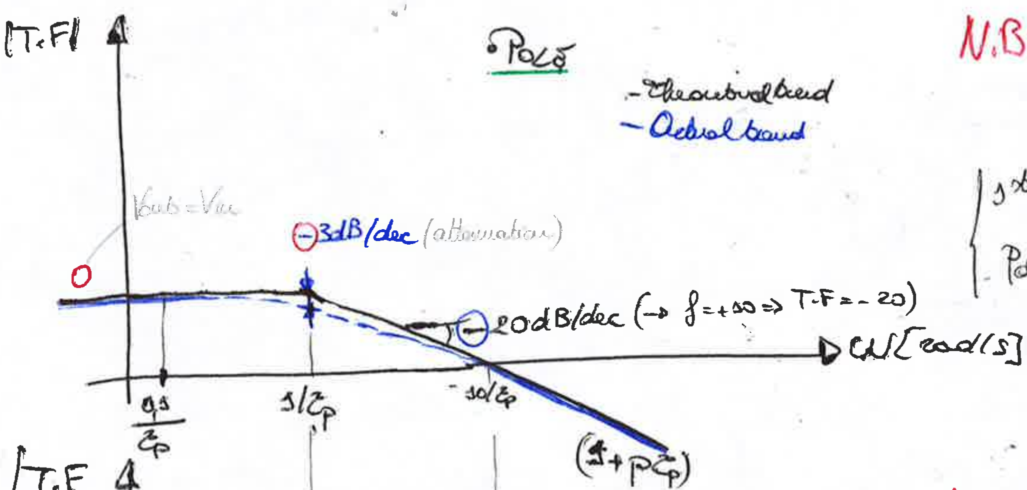
1st Order Zero ⇒ +20 dB/dec Crescent



N.B

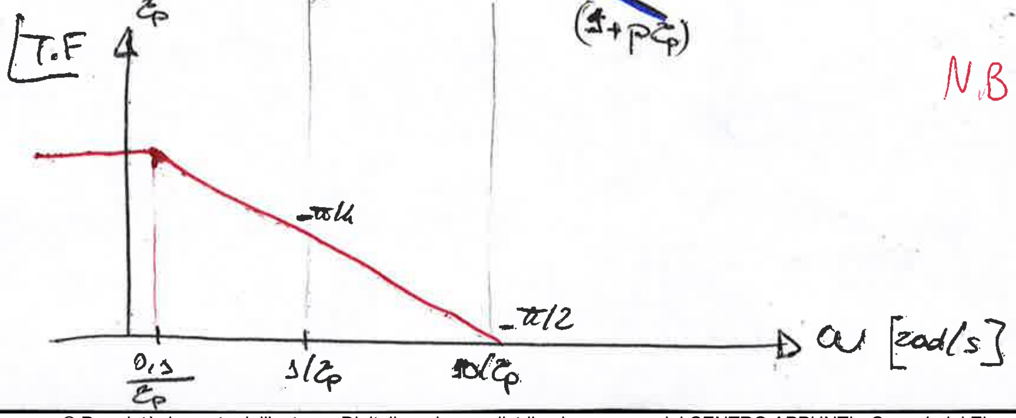
1st Order Zero ⇒ $0 \leq \angle T.F \leq 90°$, $\angle T.F$ crescent

• Pole



N.B Ordinate indicates GAIN (so it can be 0)

1st Order Pole ⇒ −20 dB/dec Decrescent



N.B 1st Order Pole ⇒ $0 \leq \angle T.F \leq 90°$, Decrescent