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NUMERO: 1935A - ANNO: 2016

A P P U N T I

STUDENTE: Partiti Sofia

MATERIA: Analysis - Prof. Boieri

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BASIC NOTIONS 1 - Sets and some Logic

Operating with sets (8-9/16)

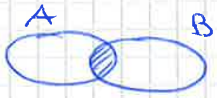
- **Union** $A \cup B$

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

- **Intersection** $A \cap B$

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

If $A \cap B = \emptyset \Rightarrow$ disjoint



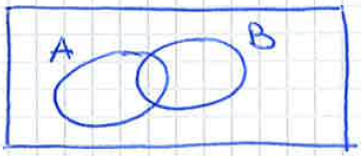
- **Difference** $A \setminus B$

$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

- **Symmetric difference** $A \triangle B$

$A \triangle B = (A \setminus B) \cup (B \setminus A)$

$A \triangle B = (A \cup B) \setminus (A \cap B)$



$$\left.\begin{aligned} \complement(A \cup B) &= (\complement A) \cap (\complement B) \\ \complement(A \cap B) &= (\complement A) \cup (\complement B) \end{aligned}\right\} \text{DE MORGAN'S LAWS}$$

Propositions (10-11/16)

$P; Q; R \ldots$ P / True / False

$x > 1$ IS NOT a proposition

- **Negation** $\neg P$

P	$\neg P$
T	F
F	T

(linked to the complement)

- **Conjunction** P and Q $(P \wedge Q)$

P	Q	P And Q
T	T	T
T	F	F
F	T	F
F	F	F

(linked to the intersection)

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EXAMPLES OF PROOFS

→ direct proof

If n is EVEN $\Rightarrow n^2$ is EVEN

$n = 2k \quad n^2 = 4k^2 = 2(2k^2)$

→ contrapositive proof $P \Rightarrow Q \equiv (\neg Q) \Rightarrow (\neg P)$

If n^2 is EVEN $\Rightarrow n$ is EVEN

n is ODD $\Rightarrow n^2$ is ODD

$n = 2k+1 \quad n^2 = 4k^2+4k+1 = 4(k^2+k)+1 \rightsquigarrow$ ODD

→ contradiction proof $P \wedge (\neg Q) \Rightarrow$ contraddiction

If n^2 is EVEN $\Rightarrow n^2$ is EVEN

n is ODD $\Rightarrow n = 2k+1 \Rightarrow n^2 = 4k^2+4k+1 \Rightarrow n^2$ is ODD

Predicates with two or more variables (16/16)

one variable ⇝ property

2 or more " , " ⇝ relation

$x > 1$ → I CAN NOT SAY T or F

1) subst. x with a value $3 > 1$ T $\quad -3 > 1$ F

2) putting a quantifier

$\forall x : p(x)$ universal quantifier $p(x)$ is T for all x belonging to a set

es: $\forall x \in \mathbb{R} : x^2 + 1 > 0$ T $\quad \forall x \in \mathbb{R} : x^2 - 1 > 0$ F

$\exists x : p(x)$ existence quant.

there exists at least one x such that $p(x)$ holds

negation of a quantifier

$\neg(\forall x : p(x)) = \exists x : \neg p(x)$ counter example

$\neg(\exists x : p(x)) = \forall x : \neg p(x)$

$p(x;y)$ = the person x can do the job y

$\exists x \forall y : p(x;y)$ it exists an x that can do all the job

$\forall y \exists x : p(x;y)$ for all jobs can do it exists at least one person that

$\neg(\exists x \forall y : p(x;y)) = \forall x \exists y : \neg p(x;y)$

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4 Product

$z = a + ib$
$w = c + id$ $\rightarrow zw = (ac - bd) + (ad + bc)i$

$z = i \rightarrow a = 0 \wedge b = 1$
$w = i \rightarrow c = 0 \wedge d = 1$
$\rightarrow i^2 = (0 - 1) + i(0 + 0) \rightarrow i^2 = -1$

$$\begin{array}{r} a + ib \\ c + id \\ \hline ac + ibc \\ \quad iad + (-bd) \\ \hline ac - bd + i(ad + bc) \end{array}$$

c.v.d.

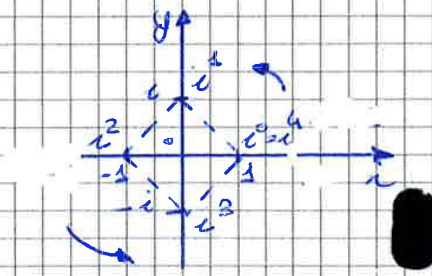
5
$z + 0 = z$
$z \cdot 1 = z$

6
$z \neq 0 \quad \exists!\ w \in \mathbb{C}$ such that $zw = 1 \rightsquigarrow w = \frac{1}{z}$ inverse of z

$z = ai + b \quad w = x + iy$
$zw = 1 \rightarrow$ FIND $w = \frac{1}{z} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$

7 Powers of i

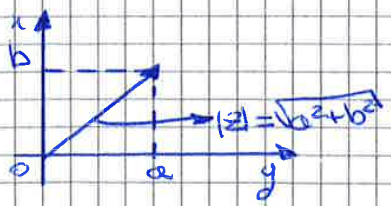
$i^0 = 1 \qquad i^2 = -1 \qquad i^3 = i^2 i = -i$
$i^1 = i \qquad i^4 = i^2 i^2 = 1 \qquad i^5 = i^4 i = i$



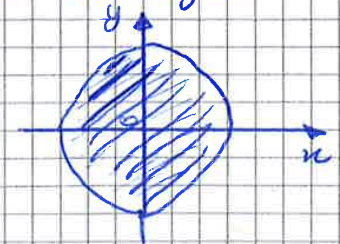
ex
$i^{37} = i^{36}\, i = i$

MODULUS

$z = a + ib \qquad |z| = \sqrt{a^2 + b^2}$
$|z|$ distance of z from the origin

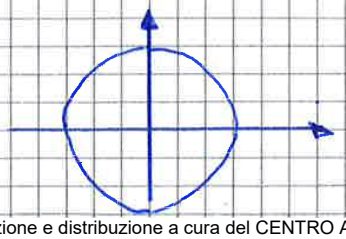


ex
$\{z \in \mathbb{C} : |z| \leq 1\}$



disc center 0 radius 1
$\rightsquigarrow$ unit disc

$\{z \in \mathbb{C} : |z| = 1\}$



circle center 0 radius 1
↓
circonferenza $\rightsquigarrow$ unit circle

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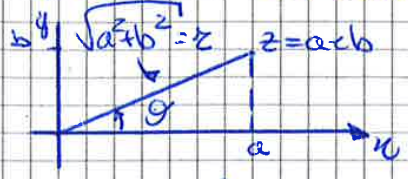
ex

$$\frac{1-i}{2-i} = \frac{(1-i)(1+i)}{5} = \frac{3-i}{5}$$

Trigonometric form

$z = a+ib$ cartesian form



$|z| = r$

ϑ from positive real axis counter-clockwise

$\text{Arg}\, z = \arg z$ (in this course)

$a = r\cos\vartheta$

$b = r\sin\vartheta$

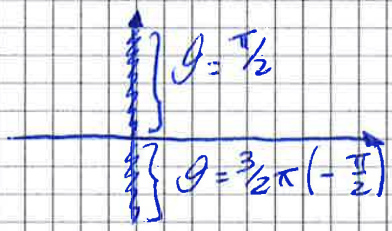
$\Rightarrow z = \underbrace{a+ib}_{\text{cartesian form}} = r\cos\vartheta + ir\sin\vartheta = \underbrace{r(\cos\vartheta + i\sin\vartheta)}_{\text{trigonometric form}}$

$$\begin{cases} r = \sqrt{a^2+b^2} \\ \cos\vartheta = \frac{a}{r} = \frac{a}{\sqrt{a^2+b^2}} \\ \sin\vartheta = \frac{b}{r} = \frac{b}{\sqrt{a^2+b^2}} \end{cases}$$

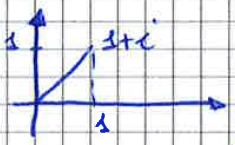
OR

$$\tan\vartheta = \frac{b}{a}$$

$$\begin{cases} \vartheta = \arctan\frac{b}{a} & \text{IF } a > 0 \\ \vartheta = \arctan\frac{b}{a} + \pi & \text{IF } a < 0 \end{cases}$$



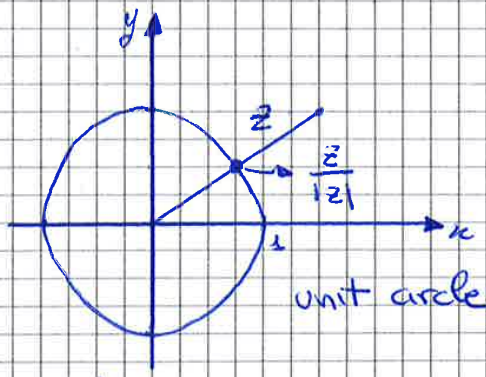
ex



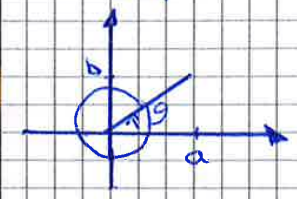
$|z| = r = \sqrt{2}$ $\quad \sin\vartheta = \frac{1}{\sqrt{2}}$ $\quad \cos\vartheta = \frac{1}{\sqrt{2}}$

$\Rightarrow \vartheta = \pi/4$ $\quad \vartheta = \arctan 1 = \pi/4$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$



unit circle



$a+ib$ is unique

r is unique

ϑ up to a multiple of 2π

$z = r(\cos\vartheta + i\sin\vartheta)$

$w = s(\cos\varphi + i\sin\varphi)$

$zw = rs\left[\cos\vartheta\cos\varphi - \sin\vartheta\sin\varphi + i(\cos\vartheta\sin\varphi + \sin\vartheta\cos\varphi)\right]$

$\Rightarrow zw = rs\left[\cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi)\right]$

argument of zw is the sum of the arguments

$|zw|$ is the product of the moduli

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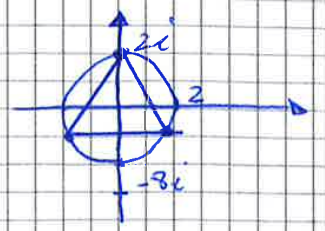
$z^n = 1$ n even $-1, 1$ n odd 1 → in $\mathbb{R}$ → in $\mathbb{C}$

n solutions
vertices of a regular polygon (n sides) inscribed in the unit circle

ex

$w = \sqrt[3]{-8i} \rightarrow z^3 = -8i$



$|w| = s = 8 \quad \cos\varphi = 0 \quad \sin\varphi = -1 \Rightarrow \varphi = \frac{3}{2}\pi$

$z^3 = 8 \rightsquigarrow r = 2$

$z = 2\left[\cos\left(\frac{\frac{3}{2}\pi}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{3}{2}\pi}{3} + \frac{2k\pi}{3}\right)\right]$

$= 2\left[\cos\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right)\right] \quad k = 0, 1, 2$

$z_0 = 2i$

$z_1 = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

$z_2 = \sqrt{3} - i$

EULER'S formula

$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta \qquad \vartheta \in \mathbb{R}$

$e^{i\pi} = -1$

$e^{i\pi} + 1 = 0$

$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$

$e^{-i\vartheta} = \cos\vartheta - i\sin\vartheta$

$\cos\vartheta = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2} \qquad \sin\vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$ EULER

$z = r(\cos\vartheta + i\sin\vartheta) = re^{i\vartheta}$ exponential form for a complex

$w = s(\cos\varphi + i\sin\varphi) = se^{i\varphi}$

$zw = rs\left[\cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi)\right] = rse^{i(\vartheta+\varphi)} = (re^{i\vartheta})(se^{i\varphi})$

$\frac{z}{w} = \frac{r}{s}e^{i(\vartheta-\varphi)}$

COMPLEX EXPONENTIAL

$z = x + iy$

$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin\varphi)$

Properties

1) $z = x \rightarrow e^z = e^x$ when z is real I get the real expo

2) $|e^z| = e^x$

3) $e^{z_1+z_2} = e^{z_1} e^{z_2}$

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Sets of numbers

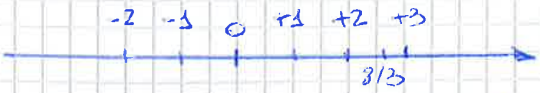
Operations

$\mathbb{N} = \{0,1,2,3,\dots\}$ + ·

$\mathbb{Z} = \{\dots -3,0,3,\dots\}$ + − ·

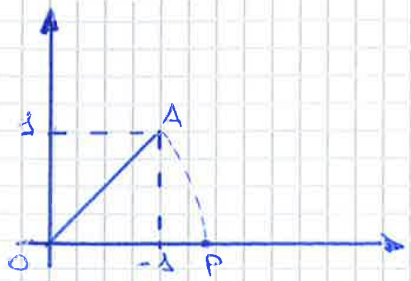
$\mathbb{Q} = \{\frac{p}{q}, p,q \in \mathbb{Z}, q \neq 0\}$ + − · /

$x^2 = 3$ NO sol. in $\mathbb{Q}$



$\forall x \in \mathbb{Q}$ $\exists P$ on line (unique) corresponding to x

The opposite is not true: there are points for which there is not a corresponding $\mathbb{Q}$



There is NO p/q such that $\frac{p^2}{q^2} = 2$

$\Rightarrow p^2/q^2 = \sqrt{2}$

$\overline{OP}^2 = \overline{OA}^2 = 1^2 + 1^2 = 2$

Property of Pitagora

PROOF contradiction

suppose $\exists\, p/q$ $p^2/q^2 = 2$ p, q no common factors

$p^2 = 2q^2 \Rightarrow p^2$ is EVEN $\Rightarrow p$ is EVEN $(p = 2k) \Rightarrow$

$\Rightarrow 4k^2 = 2q^2 \Rightarrow 2k^2 = q^2 \Rightarrow q^2$ is EVEN $\Rightarrow q$ is EVEN $(q = 2k)$

is not possible because we said that p, q have no common factors (but they are both even so divisible for 2)

We need the set $\mathbb{R}$ in order to have:

$\forall x \in \mathbb{R} \rightarrow \exists$ one P on the line

$\forall$ point the line $\rightarrow \exists$ one $x \in \mathbb{R}$] $\mathbb{R}$ is complete

$\mathbb{R}$ — $\mathbb{Q}$ rational

$\mathbb{R} \setminus \mathbb{Q}$ non rational numbers

In a line, in each segment, there are both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ $\Rightarrow$ so NO intervals on $\mathbb{R}$ containing only $\mathbb{Q}$ or only $\mathbb{R} \setminus \mathbb{Q}$)

Order

→ Total order

$\forall x, y \in \mathbb{R}$ — $x < y$ / $x = y$ \ $x > y$

connection between order and operation

$\forall x,y \in \mathbb{R}$ $\forall z \in \mathbb{R}$ $x \leq y \Rightarrow x + z \leq y + z$

$\forall x,y \in \mathbb{R}$ $\forall z \in \mathbb{R}, z \geq 0$ $x \leq y \Rightarrow xz \leq yz$

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X, Y two non empty sets

ORDERED PAIR $(x;y)$

↳ set of 2 elements where we specify that x is the 1st and y is 2nd

$\{x;y\} = \{y;x\}$ $\quad (x,y) \neq (y;x)$ in general

$(x;y) = (x';y') \iff \begin{cases} x = x' \\ y = y' \end{cases}$

Cartesian product

$X \times Y = \{(x,y) : x \in X, y \in Y\}$

$X = \{1;2\}$ $\quad Y = \{a;b\}$

$X \times Y = \{(1;a), (1,b), (2,a), (2,b)\}$

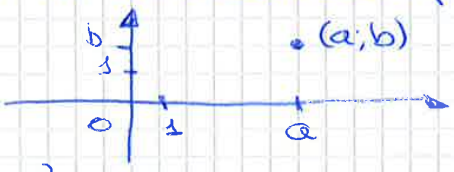
$Y \times X = \{(a,1); (a,2); (b,1), (b,2)\}$

X n elem. $\\$ Y m elem. $\Rightarrow X \times Y$ have $n \cdot m$ elem.

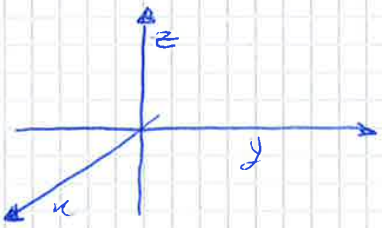
$X \times X = X^2$

es. $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \iff$ cartesian plane

$(a;b) \in \mathbb{R}^2$



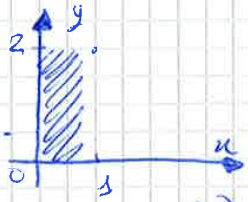
$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 \iff$ euclidean space

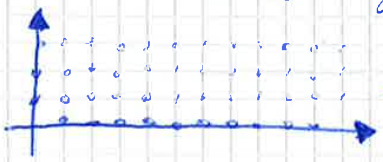


APPLICATION

$[0;1] \times [0,2] \subseteq \mathbb{R}^2$

$\{(x;y) \in \mathbb{R}^2 : x \in [0;1] \wedge y [0;2]\}$



$\mathbb{N}^2 =$  $= \{(x;y) \in \mathbb{R}^2 : x \in \mathbb{N} \wedge y \in \mathbb{N}\}$

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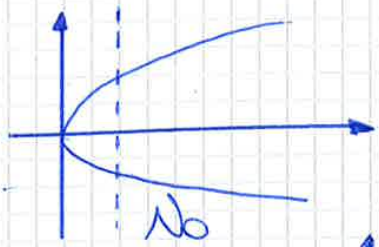
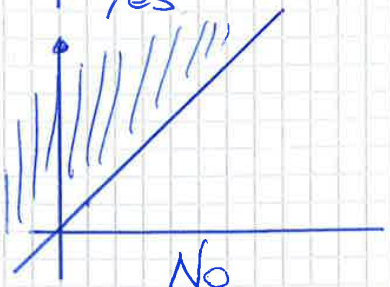
$\Gamma(f) \subseteq \mathbb{R}^2$ given $A \subseteq \mathbb{R}^2$, A is $\Gamma(f)$ for function f

Yes



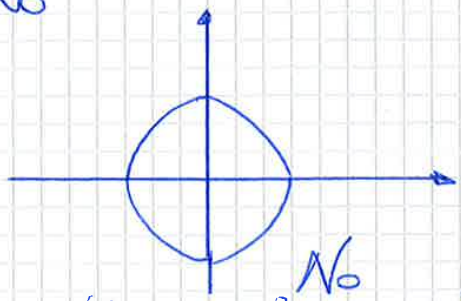
No

The intersection of A with a vertical line must be empty or one point.



No

$\{(x;y) \in \mathbb{R}^2 \; y > x\}$



No

$\{(x;y) \in \mathbb{R}^2 \; x^2+y^2 = 1\}$

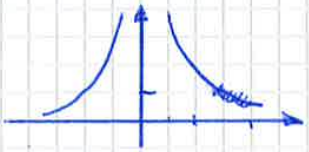
IF X $\text{dom } f \subseteq X$ and Y $\text{im } f \subseteq Y$

$A \subseteq X \quad f(A) = \{y = f(x) \in Y : x \in A\}$ IMAGE OF A UNDER f

$B \subseteq X \quad f^{-1}(B) = \{x \in \text{dom } f : f(x) \in B\}$ PRE-IMAGE OF B UNDER f

es

$f(x) = \frac{1}{x^2}$



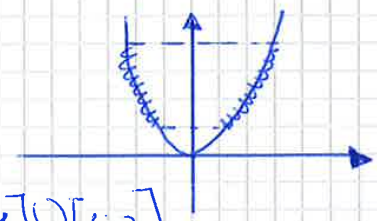
$A = [1;5]$

$f(A) = f([1;5]) = \left[\frac{1}{25}; 1\right]$

es

$f(x) = x^4$

$B = [1;16]$



$f^{-1}(B) = [-2;-1] \cup [1;2]$

DEF INJECTIVITY

$A \subseteq \text{im } f$ f injective in A

$\forall x_1, x_2 \in A \quad x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$

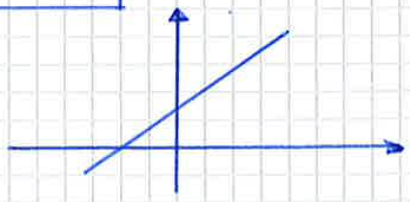
$\forall x_1, x_2 \in A \quad f(x_1) = f(x_2) \rightarrow x_1 = x_2$

es

$f(x) = 3x+5 \quad \mathbb{R}$

$3x_1+5 = 3x_2+5 \Rightarrow x_1 = x_2$

$\Rightarrow$ so it is injective



AT MOST ONE

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Properties of $\mathbb{R}$

DEF MAXIMUM of a funct.

$X \subseteq A$, $X \neq \emptyset$

X has a maximum x_m $\exists x_m \in X$ $\forall x \in X$ $x \leq x_m$

DEF MINIMUM of a funct.

$X \subseteq A$, $X \neq \emptyset$

X has a minimum x_m $\exists x_m \in X$ $\forall x \in X$ $x \geq x_m$

If X is finite (a finite number of elem.) then X has a max and a min

⇒ NOT true if x has an infinite numb. of elements

es

$X = [a; b]$ $\min X = a$ $\max X = b$

$X = [a; b)$ $\min X = a$ $\max X \nexists$

$X = \mathbb{N}$ $\min X = 0$ $\max X \nexists$

DEF X IS BOUDED FROM ABOVE

$\exists b \in \mathbb{R}$ $\forall x \in X$ $x \leq b$

b is called the upper bound for x

DEF X IS BOUDED FROM BELOW

$\exists c \in \mathbb{R}$ $\forall x \in X$ $x \geq c$

c is called the lower bound for x

es

$X = [0; 3]$

$[3; +\infty)$ ⇝ set upper bounds ↞ $\emptyset$

$(-\infty; 0]$ ⇝ set lower bounds ↞ $(-\infty; 0]$

es

$X = \mathbb{N}$

↞ $\emptyset$

↞ $\{0\}$

es

$X = \mathbb{Z}$

↞ $\emptyset$

↞ $\emptyset$

es

$\left\{ x \in \mathbb{R} \quad x = \frac{n}{n+1} \quad n \in \mathbb{N} \right\}$

set upper bounds $[1; +\infty)$

set lower bounds $(-\infty; 0]$

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es

$A = \{x \in \mathbb{Q} \quad x \geq 0, x^2 < 2\} \subseteq \mathbb{Q}$

A bounded from above (2 is an upper bound)

$\sup A \notin$

es

$A = \{x \in \mathbb{R} : x = (-1)^n \frac{n}{n+1}, n \in \mathbb{N}\}$

upper bounds $[1; +\infty)$ $\sup A = 1$

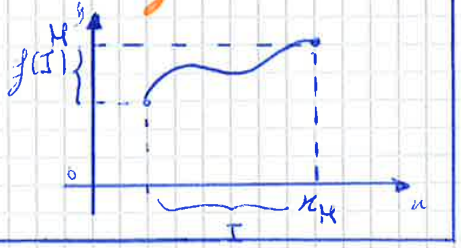
lower bounds $(-\infty; -1]$ $\min A = -1$

DEF absolute (global) maximum for f on I

$f : I \subseteq \text{dom} \to \mathbb{R}$

conditions:

1) $\exists x_M \in I \quad f(x_M) = M$
2) $\forall x \in I \quad f(x) \leq M$

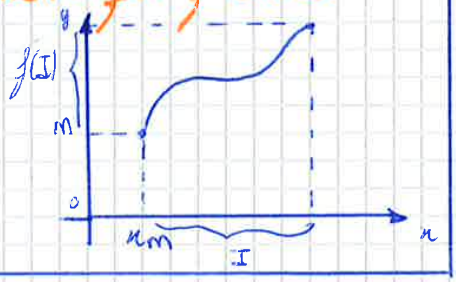


DEF absolute (global) minimum for f on I

$f : I \subseteq \text{dom} \to \mathbb{R}$

conditions:

1) $\exists x_m \in I \quad f(x_m) = m$
2) $\forall x \in I \quad f(x) \geq m$



Remarks

M and m are max and min of $f(I)$

The absolute max (min) do NOT always exist

es

$f(x) = x \quad I = (1; 2)$

no max abs.
no min abs.

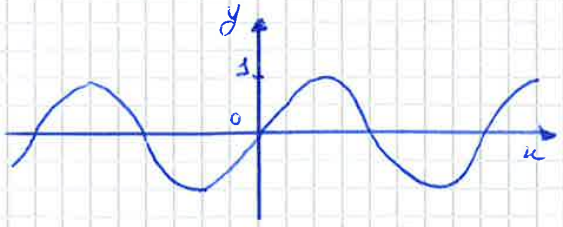
Remark

The absolute max (min), if it exist, is UNIQUE, but in general max point is not necessary unique

es

$y = \sin x$

The max is unique (= 1) but there are ∞ max points



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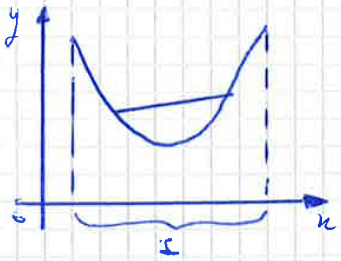
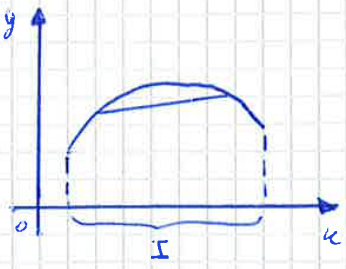
Convex on I



Concave on I

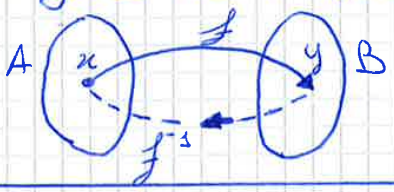


Inverse f

DEF Inverse f

$f: A \subseteq \text{dom} f \quad B = f(A)$

The inverse function f^{-1} of f is the function that relate to $y \in B$ the unique value $x \in A$ such that $y = f(x)$



The conditions are { injective, surjective } bijective

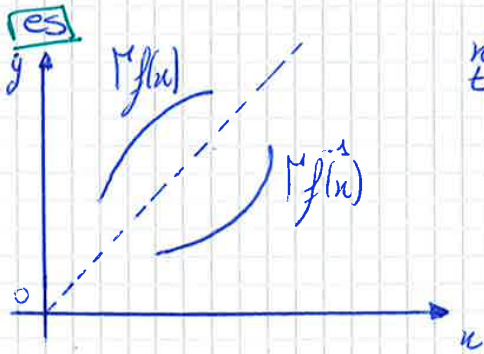
Remark

The inverse function f^{-1} has nothing to do with $\frac{1}{f(x)}$ (reciprocal)

FOUR PROPERTIES

- $\text{dom} f^{-1} = B = \text{im} f$
- $\text{im} f^{-1} = A = \text{dom} f$
- $\forall x \in A \quad f^{-1}(f(x)) = x$
- $\forall y \in B \quad f(f^{-1}(y)) = y$

es



$x \leftrightarrow y$ symmetry with respect to the line $y = x$

Given A, B, f 3 problems

1) Does f^{-1} exist?
2) $\Gamma f^{-1}(x)$
3) "formula"

It's impossible to write the inverse of $y = x + e^x$ as $y = \dots$

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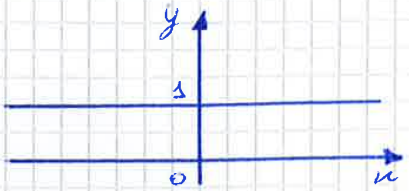
$y = x^{\frac{n}{m}} \quad y = \sqrt[m]{x^n} \quad n \in \mathbb{N}, m \in \mathbb{N} \setminus \{0\}$

m odd
$\text{dom} f : \mathbb{R}$

m even
$\text{dom} f : [0; +\infty)$

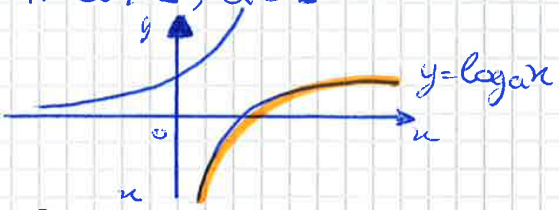
Exponential $\neq$ $y = a^x \quad a > 0$

IF $a = 1$



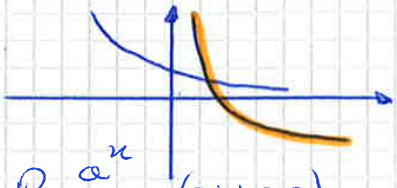
$\mathbb{R} \longrightarrow \{1\}$ NOT INJ.

IF $a \neq 1, a > 1$



$\mathbb{R} \xrightarrow{a^x} (0; +\infty)$ $\quad \log_a x$

IF $a > 1$



$\mathbb{R} \xrightarrow{a^x} (0; +\infty)$ $\quad \log_a x$

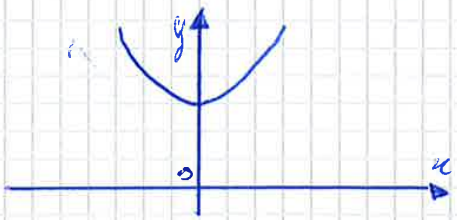
IF $a = e$ (napier's number)

$\mathbb{R} \xrightarrow{e^x} (0; +\infty)$ $\quad \ln x = \log_e x$ (natural logarithm)

Hyperbolic cos of x

$$y = \frac{e^x + e^{-x}}{2} = \cos \ln x$$

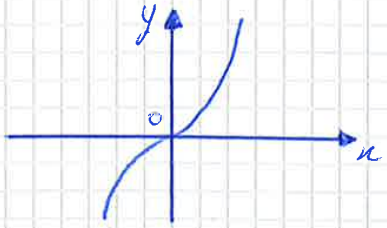
$\mathbb{R} \longrightarrow [1; +\infty)$ EVEN



Hyperbolic sin of x

$$y = \frac{e^x - e^{-x}}{2} = \sin \ln x$$

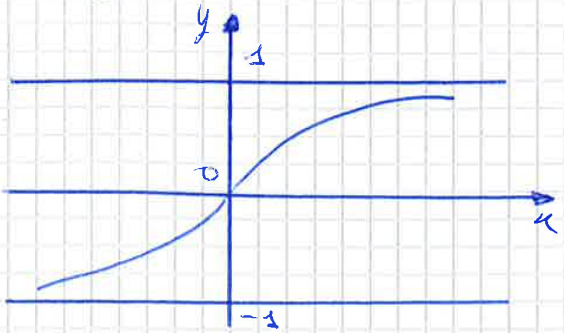
$\mathbb{R} \longrightarrow \mathbb{R}$ ODD



Hyperbolic tan of x

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tan \ln x$$

$\mathbb{R} \longrightarrow (-1; 1)$



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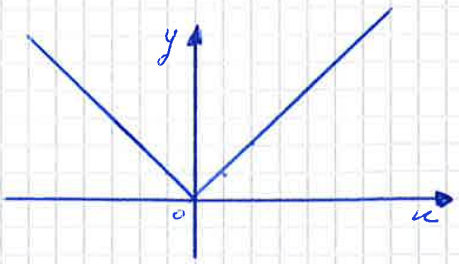
Absolute value

$$f(x) = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

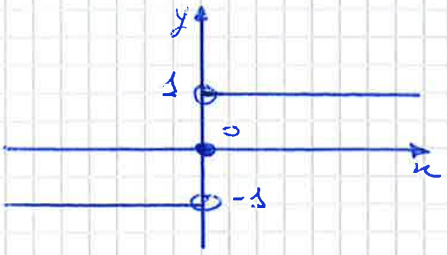
$\mathbb{R} \longrightarrow [0; +\infty)$



Signum function

$$f(x) = \operatorname{sign} x = \operatorname{sgn} x = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$\mathbb{R} \longrightarrow \{-1, 0, 1\}$



N.B

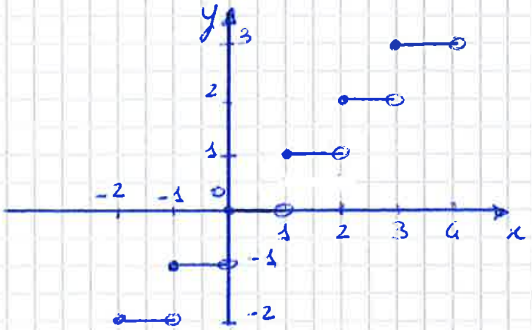
$\operatorname{sign} x \neq \frac{x}{|x|}$ this function in 0 is not defined

Integer part of x

$f(x) = [x]$ Greatest integer $\leq x$

$[0] = 0 \quad [0,35] = 0$

$\mathbb{R} \longrightarrow \mathbb{Z}$

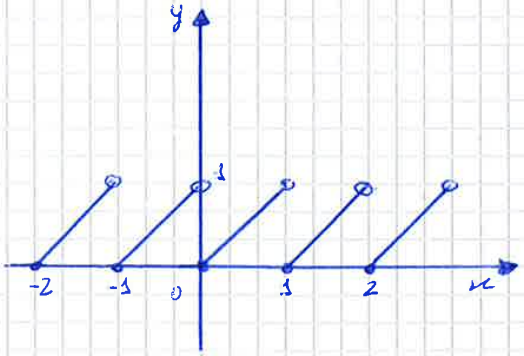


Mantissa function

$f(x) = x - [x]$

	$[x]$	mant
$0 \leq x < 1$	0	x
$1 \leq x < 2$	1	$x-1$
$2 \leq x < 3$	2	$x-2$

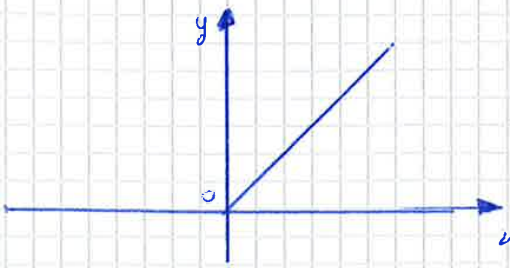
$\mathbb{R} \longrightarrow [0; 1)$



Positive part

$$f(x) = x^+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$[0; +\infty) \longrightarrow [0; +\infty)$



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ex

$f(x) = x^2$
$g(x) = \sin x$ $\Rightarrow g(f(x)) = \sin x^2$ $\quad f(g(x)) = \sin^2 x$

$\text{dom}\, g(f(x)) = \mathbb{R}$ $\quad \text{im}\, g(f(x)) = [-1;1]$
$\text{dom}\, f(g(x)) = \mathbb{R}$ $\quad \text{im}\, f(g(x)) = [-1;1]$

ex

$f(x) = \sqrt[3]{x}$
$g(x) = 2^x$ $\Rightarrow g(f(x)) = 2^{\sqrt[3]{x}}$ $\quad \mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g(f(x))} (0;+\infty)$

Monotone fct. incr. because $f(x)$ and $g(x)$ are monotone fct. incr.

INCREASING / DECREASING

f	g	$g \circ f$
I	I	I
I	D	D
D	I	D
D	D	I

If g is ↗ (I) then f and $g \circ f$ have the same monotonicity.

If g is ↘ (D) then f and $g \circ f$ have opposite monotonicity.

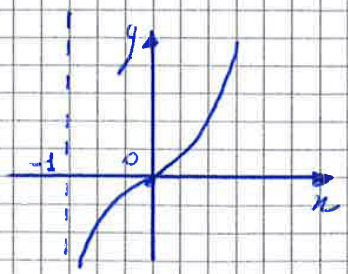
EVEN / ODD

f	g	$g \circ f$
E	E	E
E	O	
O	E	
O	O	O

ex

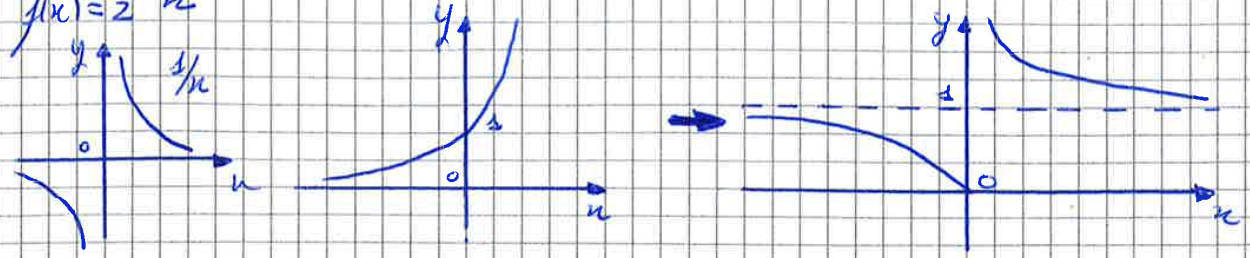
$h(x) = \log(x^3+1)$ $\Rightarrow g(x) = x^3+1$ e $f(x) = \log x$

dom: $(-1;+\infty)$



rex

$f(x) = 2^{\frac{1}{x}}$



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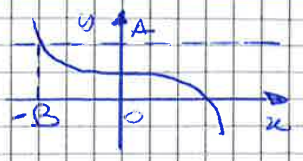
DEF $\lim_{x\to-\infty} f(x) = +\infty$

$\forall A>0 \quad \exists B>0$

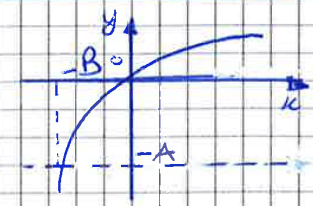
$\forall x: x\in \text{dom} f,\ x<-B \Rightarrow f(x)>A$



DEF $\lim_{x\to-\infty} f(x) = -\infty$

$\forall A>0 \quad \exists B>0$

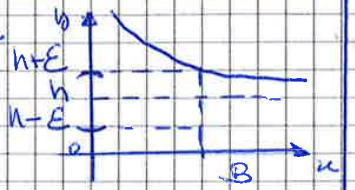
$\forall x: x\in \text{dom} f,\ x<-B \Rightarrow f(x)<-A$



DEF $\lim_{x\to+\infty} f(x) = \ell$ Finite limit $\ell\in\mathbb{R}$

$\forall \varepsilon>0 \quad \exists B>0$

$\forall x: x\in \text{dom} f,\ x>B \Rightarrow |f(x)-\ell|<\varepsilon$



$\lim_{x\to-\infty} f(x) = \ell\in\mathbb{R} \Rightarrow y=\ell$ is left horizontal asymptote

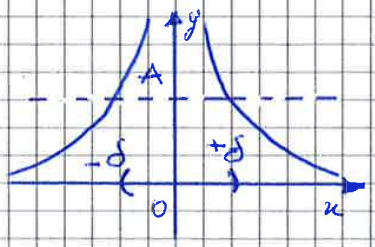
$\lim_{x\to+\infty} f(x) = \ell\in\mathbb{R} \Rightarrow y=\ell$ is right horizontal asymptote

DEF $\lim_{x\to x_0} f(x) = +\infty$

$\forall A>0 \quad \exists \delta>0$

$\forall x: x\in \text{dom} f,\ 0<|x-x_0|<\delta \Rightarrow f(x)>A$

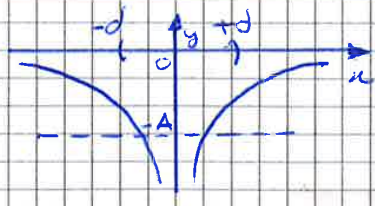
so we not consider $x=x_0$



DEF $\lim_{x\to x_0} f(x) = -\infty$

$\forall A>0 \quad \exists \delta>0$

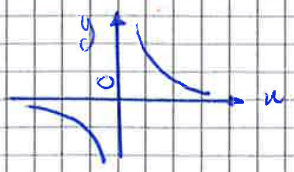
$\forall x: x\in \text{dom} f,\ 0<|x-x_0|<\delta \Rightarrow f(x)<-A$



ex $y = 1/x$

The limit $x\to 0$ is not $+\infty$ or $-\infty$

→ one-sided limits



DEF $\lim_{x\to x_0^+} f(x) = +\infty$

$\forall A>0 \quad \exists \delta>0$

$\forall x: x\in \text{dom} f,\ 0<x-x_0<\delta \Rightarrow f(x)>A$

Right limit

check

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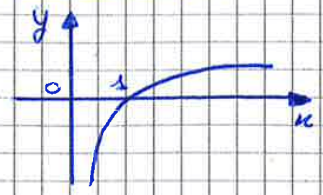
$y = \log_a x$

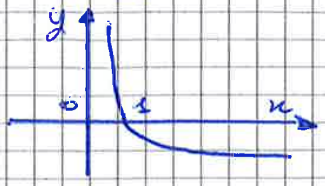
$0 < a < 1$



$x \to +\infty \rightsquigarrow +\infty$

$x \to 0^+ \rightsquigarrow -\infty$

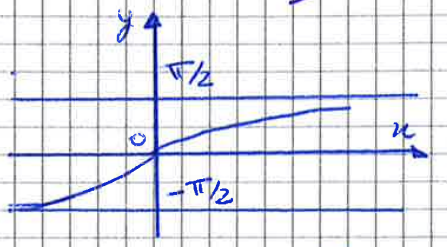
$a > 1$



$x \to +\infty \rightsquigarrow -\infty$

$x \to 0^+ \rightsquigarrow +\infty$

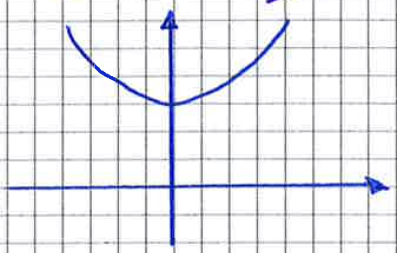
$y = \operatorname{arcotan} x$



$x \to +\infty \rightsquigarrow \pi/2$

$x \to -\infty \rightsquigarrow -\pi/2$

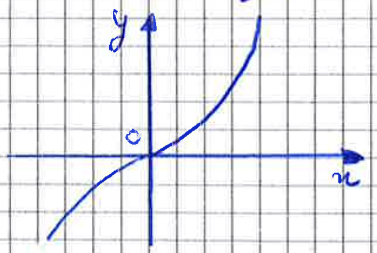
$y = \cos \ln x$



$x \to +\infty \rightsquigarrow +\infty$

$x \to -\infty \rightsquigarrow +\infty$

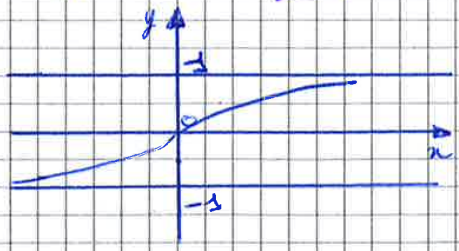
$y = \sin \ln x$



$x \to +\infty \rightsquigarrow +\infty$

$x \to -\infty \rightsquigarrow -\infty$

$y = \tan \ln x$



$x \to +\infty \rightsquigarrow 1$

$x \to -\infty \rightsquigarrow -1$

If $f(x) = \sin x$ or $\tan x$ or $\cos x \Rightarrow \lim_{x \to \pm\infty} f(x) \nexists$

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DEF $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$

$\forall I_\varepsilon(\ell) \quad \exists I_\delta(x_0)$

$\forall x : x \in \text{dom} f, x \in I_\delta(x_0) - \{x_0\} \Rightarrow f(x) \in I_\varepsilon(\ell)$

point $\begin{cases} x \to -\infty \\ x \to +\infty \\ x \to x_0 \end{cases}$ $\Rightarrow$ I called the "point" γ

$\lim \begin{cases} -\infty \\ +\infty \\ \ell \end{cases}$ $\Rightarrow$ I called the "point" λ

DEF $\lim_{x \to \gamma} f(x) = \lambda$

$\forall I(\lambda) \quad \exists I(\gamma)$

$\forall x : x \in \text{dom} f, x \in I(\gamma) - \{\gamma\} \Rightarrow f(x) \in I(\lambda)$

not apply if $\gamma = \pm\infty$

Three examples with the same behaviour

ex 1

$f(x) = x^3 + 1$

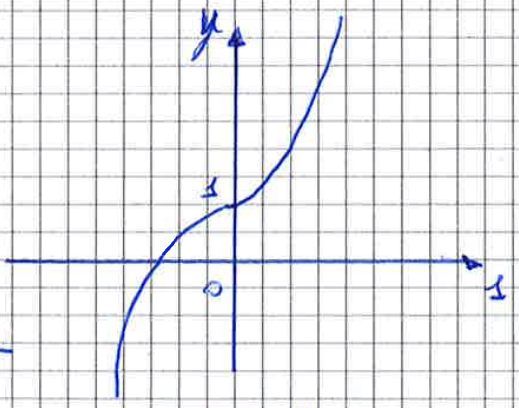
$\lim_{x \to 0} x^3 + 1 = 1$

check with the definition

$\forall \varepsilon > 0 \quad \exists \delta > 0$

$\forall x : 0 < |x| < \delta \rightarrow |x^3 + 1 - 1| < \varepsilon$

$|x^3| = |x|^3 < \varepsilon \xrightarrow{\text{ex with}} \delta \leq \sqrt[3]{\varepsilon}$



The value of the limit is exactly the same of the f

DEF f continuos at x_0

f defined on $I(x_0)$ [so also in x_0]

IF $\lim_{x \to x_0} f(x) = f(x_0)$

$\Rightarrow$ so f is continuos at x_0

DEF f continuos at x_0

IF $\forall \varepsilon > 0 \quad \exists \delta > 0$

$\forall x : x \in \text{dom} f, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

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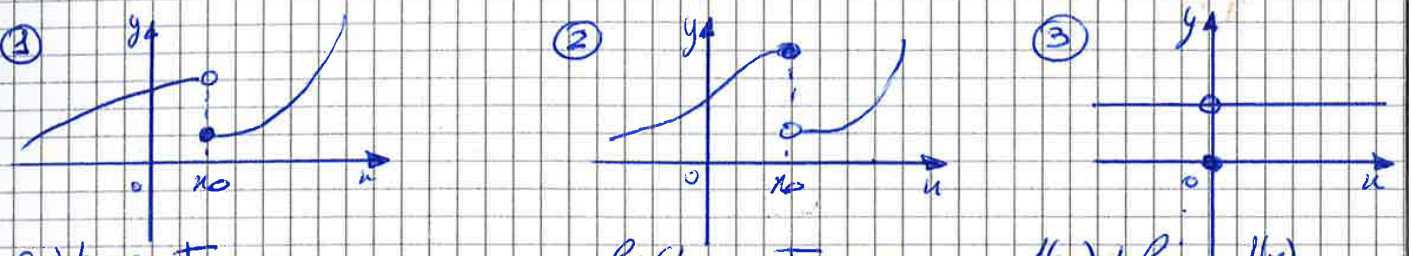
$\lim_{x\to x_0^+} f(x) = l_1 \in \mathbb{R}$
$\lim_{x\to x_0^-} f(x) = l_2 \in \mathbb{R}$
$\Rightarrow x_0$ is jump discontinuity for f

The quantity

$$\left(\lim_{x\to x_0^+} f(x)\right) - \left(\lim_{x\to x_0^-} f(x)\right)$$

is called **the jump of f at x_0**



right continuous

left continuous

$f(x_0) \neq \lim_{x\to x_0^+} f(x)$ ∧ $\neq \lim_{x\to x_0^+} f(x)$

jump discontinuity = **discontinuity of the 1st kind**

all the other possible cases will be called **discontinuity of the 2nd kind**

Algebra of finite limits

$\lim_{x\to\gamma} f(x) = l \in \mathbb{R}$ $\quad \lim_{x\to\gamma} g(x) = m \in \mathbb{R}$

1) $\lim_{x\to\gamma} f(x) \pm g(x) = l \pm m$

2) $\lim_{x\to\gamma} f(x)g(x) = lm$

3) $\lim_{x\to\gamma} \frac{f(x)}{g(x)} = \frac{l}{m}$ $\quad (\text{if } m \neq 0)$

$f \cdot g$ defined on $I(x_0)$

$f \cdot g$ continuous at x_0

1) $f \pm g$ is continuous at x_0

2) fg is continuous at x_0

3) $\frac{f}{g}$ is continuous at x_0 $\quad (\text{if } g(x_0) \neq 0)$

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D] $y = e^x \quad f \in C^0(\mathbb{R})$

$\sin(\ln x)$, $\cos(\ln x)$, $\tan(\ln x)$ $\in C^0(\mathbb{R})$

$\cos(\ln x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(e^x + \frac{1}{e^x}\right) \Rightarrow \in C^0(\mathbb{R})$ ($e^x \in C^0(\mathbb{R})$, $\frac{1}{e^x} \in C^0(\mathbb{R})$)

$y = \tan x \quad f \in C^0\left(\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi\right\}\right) \quad k \in \mathbb{Z}$

TH

f continuous and invertible on an interval I; $J = f(I)$ → f^{-1} continuous on J

E] $y = \log_a x \quad f \in C^0(\mathbb{R})$

ex of NON continuous function

$\operatorname{sign} x$; $[x]$; $H(x)$

TH

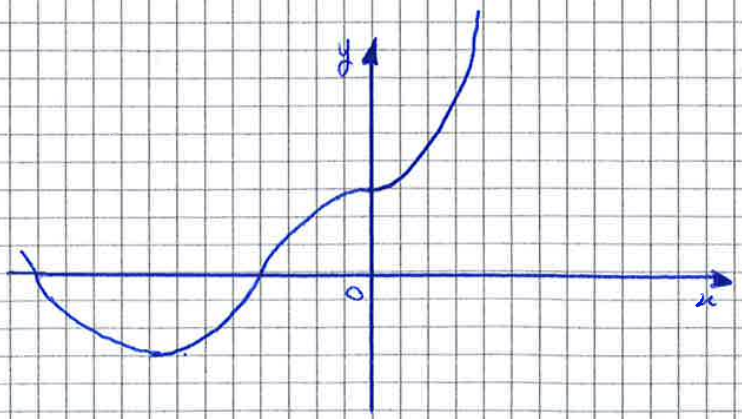
$f(x), g(x)$ defined on $I(x_0) \setminus \{x_0\}$
$f(x), g(x)$ continuous on $I(x_0) \setminus \{x_0\}$
$\lim_{x \to x_0} f(x) = \lambda$
→ $\lim_{x \to x_0} g(x) = \lambda$ check

TH

$f(x), g(x)$ defined on $I(x_0)$
$f(x)$ continuous at x_0
→ $g(x)$ continuous at x_0

ex

$$f(x) = \begin{cases} \cos x & x \le 0 \\ 1 + x^2 & x > 0 \end{cases}$$



(a) continuous at $x_0 = 1$?
$f(x)$ continuous $\forall x_0 > 0$
(b) continuous at $x_0 = -1$?
$f(x)$ continuous $\forall x_0 < 0$
(c) continuous at $x_0 = 0$

$\lim_{x \to 0^+} (1 + x^2) = 1 = \lim_{x \to 0^-} \cos x = 1 = f(0)$ → $f(x)$ continuous at x_0
$\Rightarrow f(x) \in C^0(\mathbb{R})$

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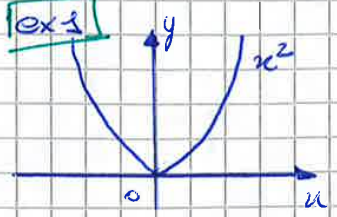
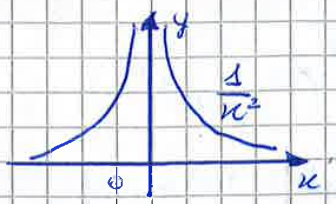
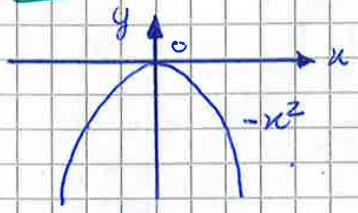
ex 1

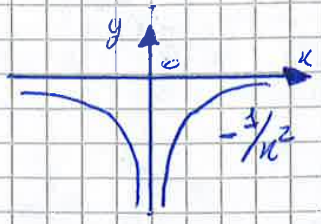




$g(x) \to 0$
$g(x) > 0 \quad I(\gamma) \setminus \{\gamma\}$
$1/g(x) \to +\infty$

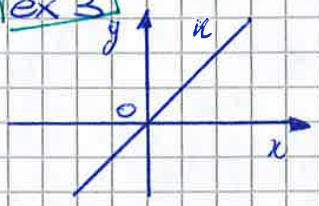
ex 2

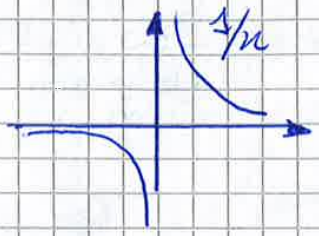




$g(x) \to 0$
$g(x) < 0 \quad I(\gamma) \setminus \{\gamma\}$
$1/g(x) \to -\infty$

ex 3





$g(x) \to 0$
$1/g(x)$ No definite sign (check for limit at right and at the left)

ex

$$\lim_{x \to 1} \frac{5}{\log_3 x} \quad \nexists$$

because

$$\lim_{x \to 1^-} \frac{5}{\log_3 x} = -\infty \qquad \lim_{x \to 1^+} \frac{5}{\log_3 x} = +\infty$$

QUOTIENT

$f(x)$	$g(x)$	$\frac{f(x)}{g(x)}$
0	0	Indeterminate form
ℓ	m	ℓ/m
0	$\pm\infty$	0
$\pm\infty$	0	∞
$\pm\infty$	$\pm\infty$	Indeterminate form

ex

$$\lim_{x \to +\infty} x^3 - 3x^2 + 5 = \lim_{x \to +\infty} x^3\left(-\frac{3}{x} + \frac{5}{x^3} + 1\right) = +\infty$$

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TH2 Substitution theorem

f defined on $I(x_0)$
f continuous at $f(x_0)$
$\lim_{x\to\gamma} f(x) = f(x_0)$

$\rightarrow \lim_{x\to x_0} g(f(x)) = g\left(\lim_{x\to x_0} f(x)\right) = g(f(x_0))$ (continuous)

The composition of 2 continuous functions is always a continuous function.

ex

$\lim_{x\to -3} \log_2(1-x^3)$ $\quad$ dom$h = (-\infty; 1)$ $\Rightarrow$ -3 is included in dom$h$ so no problem

$h \in C^0((-\infty;1))$

$\lim_{x\to -3} \log_2(1-x^3) = \log_2 28$

TH3 Substitution Theorem

$\lim_{x\to\gamma} f(x) = +\infty\ [-\infty]$
$\lim_{y\to +\infty[-\infty]} g(y)\ \exists$

$\rightarrow \lim_{x\to\gamma} g(f(x)) = \lim_{y\to +\infty[-\infty]} g(y)$

ex

$\lim_{x\to \pi/2^-} e^{-\tan x} = \lim_{y\to -\infty} e^y = 0$

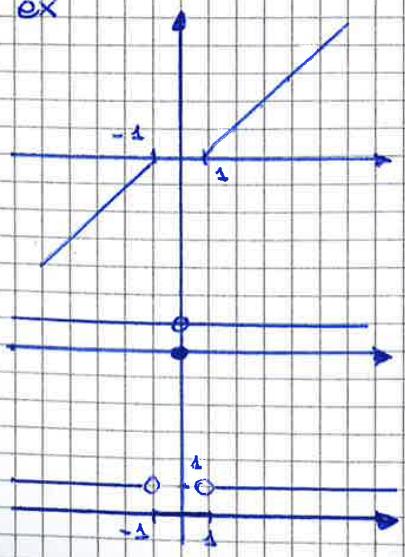
$y = \tan x \quad \lim_{x\to \pi/2^-} -\tan x = -\infty$

Remark

$\lim_{x\to x_0} f(x) = \ell$
$\lim_{y\to \ell} g(x) = m$

$\not\rightarrow$ (No) $\lim_{x\to x_0} g(f(x)) = m$

ex



$$f(x) = \begin{cases} x+1 & x < -1 \\ 0 & -1 \le x < 1 \\ x-1 & x \ge 1 \end{cases}$$

$g(x) = \text{sign}\, x^2$

$g \circ f \quad \lim_{x\to 0} g(f(x)) = 0$

applying substitution theorem

$\lim_{x\to 0} g(f(x)) = \lim_{y\to 0} g(y) = 1$

$\neq$ → so the theorem doesn't work in this case

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$-\ell < f(x) - \ell < \ell \rightarrow 0 < f(x) < 2\ell \rightarrow f(x) > 0$ c.v.d.

caso $\ell = +\infty$ INFINITE case

$\forall A > 0 \quad \exists I(\gamma)$

$\forall x : x \in \text{dom} f, x \in I(\gamma) \setminus \{\gamma\} \rightarrow f(x) > A$

but $A > 0 \rightarrow f(x) > A > 0 \rightarrow f(x) > 0$ c.v.d.

Remark

$f(x) > 0 \quad \forall x \in I(\gamma) \setminus \{\gamma\}$

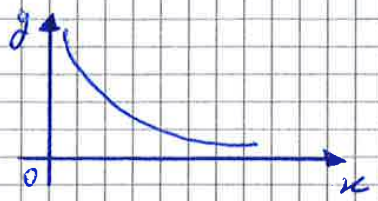
$\lim_{x \to \gamma} f(x) \ \exists$

NO $\rightarrow \lim_{x \to \gamma} f(x) > 0$

ex

$y = \frac{1}{x} \quad (0; +\infty)$

$\lim_{x \to +\infty} f(x) = 0$



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DEF

If f differentiable at x_0 the line

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is the tangent line (to the graph of f at x_0)

properties $\begin{cases} \rightsquigarrow \text{passes at } (x_0; f(x_0)) \\ \rightsquigarrow \text{slope } f'(x_0) \end{cases}$

DEF

$I \subseteq \text{dom } f$, f differentiable at x_0, $\forall x_0 \in I$ $\Rightarrow$ f differentiable on I

$x \longrightarrow f'(x)$

first derivative (function)

Relation between continuity and differentiability at x_0

TH

f defined on $I(x_0)$

f differentiable at x_0 $\Rightarrow$ f continuos at x_0

Remark

f NOT continuos at x_0 $\Rightarrow$ f NOT differentiable at x_0

Remark

It is NOT true that

f continuos at x_0 $\not\Rightarrow$ f differentiable at x_0

There are point where f is continuos and NOT differentiable at this points

PROOF

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \in \mathbb{R} \quad \text{known}$$

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \text{To be proven}$$

$$\lim_{x \to x_0} f(x) - f(x_0) = 0$$

$$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) =$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0$$

c.v.d.

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4b PROOF

$$\left(\frac{f}{g}\right)'(x_0) = \left(f\,\frac{1}{g}\right)'(x_0) = f'(x_0)\cdot\frac{1}{g(x_0)} + f(x_0)\left(-\frac{g'(x_0)}{(g(x_0))^2}\right) =$$

$$= \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{(g(x_0))^2}$$

C.V.d.

DERIVATIVE OF ELEMENTARY FUNC.

1 $f(x) = c \rightsquigarrow f'(x) = 0$

$\text{dom } f = \mathbb{R} \qquad x_0 \in \mathbb{R}$

$$\lim_{x\to x_0} \frac{c-c}{x-x_0} = 0$$

always 0 except at x_0

2 $f(x) = x \rightsquigarrow f'(x) = 1$

$$\lim_{x\to x_0} \frac{x-x_0}{x-x_0} = 1$$

always 1 except at x_0

3 $f(x) = x^2 \rightsquigarrow f'(x) = 2x_0$

$$\lim_{x\to x_0} \frac{x^2-x_0^2}{x-x_0} = \lim_{x\to x_0} \frac{(x-x_0)(x+x_0)}{x-x_0} = 2x_0$$

$\Rightarrow (x^n)' = n x^{n-1} \qquad n \geq 2$

Now I can compute the derivatives of polynomials and rational function

4 $f(x) = \sin x \rightsquigarrow f'(x) = \cos x \qquad \forall x \in \mathbb{R}$

$$\lim_{x\to 0} \frac{\sin x - \sin 0}{x-0} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{h\to 0} \frac{\sin(x_0+h)-\sin x_0}{h} = \lim_{h\to 0} \frac{\sin x_0\cos h + \cos x_0 \sin h - \sin x_0}{h} =$$

$$= \lim_{h\to 0} \frac{\sin x_0(\cos h - 1)}{h} + \cos x_0 \frac{\sin h}{h} = \sin x_0 \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x_0 \lim_{h\to 0} \frac{\sin h}{h} =$$

($\lim \frac{\cos h - 1}{h} \to 0$, $\lim \frac{\sin h}{h} \to 1$)

$$= \cos x_0$$

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DERIVATIVE OF THE INVERSE

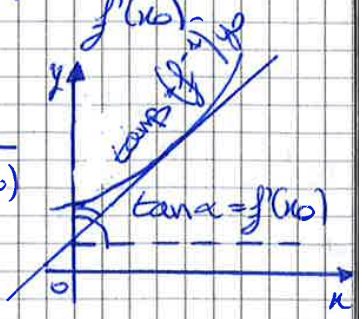
f continuous and invertible on $I(x_0)$

f differentiable at x_0 and $f'(x_0) \neq 0$

$\rightarrow$ $f^{-1}(y)$ is differentiable at $y = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

PROOF

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \xrightarrow{x \to x_0} \frac{1}{f'(x_0)}$$



$y = \tan x = f(x) \quad x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \rightarrow y_0 = \tan x_0$

$y' = 1 + \tan^2 x$

$x = \arctan y = f^{-1}(y)$

$$(\arctan y)'(y_0) = \frac{1}{(\tan)'(x_0)} = \frac{1}{1 + \tan^2 x_0} = \frac{1}{1 + y_0^2}$$

$\rightarrow$ $f(x) = \arctan x \rightarrow f'(x) = \frac{1}{1 + x^2}$

$y = a^x = f(x) \quad x \in \mathbb{R} \rightarrow y_0 = a^{x_0}$

$x = \log_a y \quad y \in (0; +\infty) \rightarrow x_0 = \log_a y_0$

$$(\log_a)'(y_0) = \frac{1}{(a^x)'} = \frac{1}{a^{x_0} \ln a} = \frac{1}{y_0 \ln a}$$

ex

$f(x) = x + e^x \quad \text{dom } f = \mathbb{R} \quad \text{range} = \mathbb{R}$

f invertible

$f(0) = 1 \rightarrow f(x_0) = y_0$

but we can not write $f(x)^{-1}$ formula

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{1 + e^x} = \frac{1}{2}$$

$1 + e^x$ when $x = 0$

$$(f^{-1})'(3) = \frac{1}{f'(x_0)} \qquad f(x_0) = 3 \rightarrow x_0 + e^{x_0} = 3$$

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$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = l_1 \in \mathbb{R}$ right derivative $f'_+(x_0)$

$\lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = l_2 \in \mathbb{R}$ left derivative $f'_-(x_0)$

→ corner point

TH

f differentiable on (a,a) or $\mathbb{R}$

f EVEN → f' ODD (or f ODD → f' EVEN)

PROOF

$f(x) = f(-x) \quad \forall x \in \text{dom} f$ EVEN

$f'(x) = -f'(-x)$ → f ODD

C.V.d

HIGHER ORDER DERIVATIVES

f differentiable on $I(x_0)$

↓

f' defined on $I(x_0)$

↓

$\frac{f'(x)-f'(x_0)}{x-x_0} \quad \forall x \in I(x_0) \setminus \{x_0\}$

$\lim_{x \to x_0} \frac{f'(x)-f'(x_0)}{x-x_0}$ exists and it is finite $f''(x_0)$

DEF

f' defined on $I(x_0) \setminus \{x_0\}$

if $\lim_{x \to x_0} \frac{f'(x)-f'(x_0)}{x-x_0} = f''(x_0)$ twice differentiable at x_0

→ symbols: $f''(x_0)$, $y''(x_0)$, $D^2 f(x_0)$, $\frac{d^2 f}{dx^2}(x_0)$

Same idea for $f'''(x)$, $f^{(IV)}(x)$, $f^{(V)}$

f continuous on $A \subseteq \text{dom} f$ → $f \in C^0(A)$

f continuous on $A \subseteq \text{dom} f$
f differentiable on $A \subseteq \text{dom} f$
f' continuous on $A \subseteq \text{dom} f$
→ $f \in C^1(A)$

function differentiable with continuous derivatives

f cont, f diff, f' cont, f' diff., f'' cont → $f \in C^2(A)$

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$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$P_n^{(n)}(x) = a_n \cdot n!$ $\qquad P_n^{(n+k)}(x) = 0 \quad \forall k \geq 1$

➡ $P_n(x) \in C^\infty(\mathbb{R})$

➡ $f(x) = \dfrac{P_n(x)}{Q_n(x)} \in C^\infty(\text{dom} f)$

PROOF

$$f'(x) = \frac{Q_n(x) P_n'(x) - P_n(x) Q_n'(x)}{(Q_n(x))^2} \in C^0(\text{dom} f)$$

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TH Fermat's theorem

f differentiable at x_0 and x_0 local extremum point for f $\Rightarrow f'(x_0)=0$

f differentiable

x_0 extremum $\Rightarrow x_0$ is a stationary point

NO the opposite!

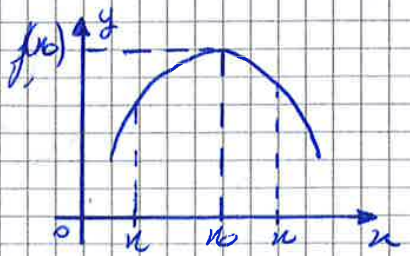
A point x_0 such that $f'(x)=0 \rightarrow$ is not necessary an extremum point

ex

$f(x)=x^3$

$f'(x)=3x^2$ $\qquad f'(0)=0$

but $x=0$ is not a max or a min



PROOF

Suppose x_0 local max

$x > x_0$

$\frac{f(x)-f(x_0)}{x-x_0} \leq 0$ ($f(x)-f(x_0) \leq 0$, $x-x_0 > 0$)

$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = f'_+(x_0) \leq 0$

This limit exists since f is diff. at x_0 and is ≤ 0 (corollary to sign and limit)

$x < x_0$

$\frac{f(x)-f(x_0)}{x-x_0} \geq 0$ ($f(x)-f(x_0) \leq 0$, $x-x_0 < 0$)

$\lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = f'_-(x_0) \geq 0$

Since f differentiable at x_0

$$0 \geq f'_+(x_0) = f'(x_0) = f'_-(x_0) \geq 0$$

$$0 \leq f'(x_0) \leq 0 \rightarrow f'(x_0)=0$$

C.V.d.

f continuous on an interval I

Find the extrema of f on I

1) points where f is differentiable and $f'(x)=0$
2) points where f is Not differentiable
3) end points of I (if it is close)

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TH Lagrange's theorem (mean value theorem)

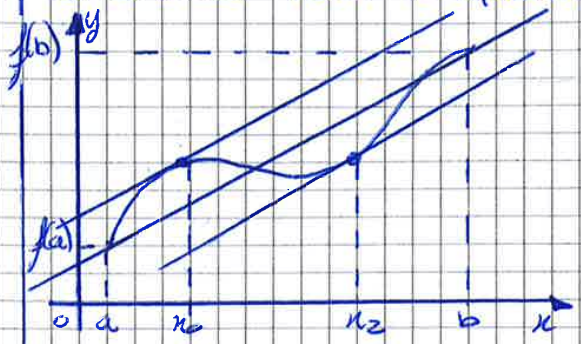
f continuous on $[a;b]$
f differentiable on $(a;b)$
$\rightarrow \exists x_0 \in (a;b) : f'(x_0) = \dfrac{f(b)-f(a)}{b-a}$

x_0 Lagrange's point for f on $[a;b]$

Remark

Rolle is a particular case of Lagrange's th. (with the additional assumption $f(a)=f(b)$)

existence NOT uniqueness



$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

The tangent at x_0 is // to the line joining $(a;f(a))$ and $(b;f(b))$

PROOF

$$g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$$

$g(a) = f(a) - f(a) = 0$

$g(b) = f(b) - f(b) = 0$

$g(x)$ continuous on $[a;b]$
$g(x)$ differentiable on $(a;b)$
$g(a) = g(b)$
$\rightarrow$ Rolle theorem $\rightarrow \exists x_0 \in (a;b) : g'(x_0) = 0$

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$g'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a} = 0 \rightarrow f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

C.V.d.

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f strictly increasing on I $\Rightarrow$ $f'(x) > 0 \quad \forall x \in \mathbb{R}$

ex

$f(x) = x^3$ str. incr.

$f'(x) = 3x^2 \quad f'(x) \geq 0$

TH

f differentiable on an open interval I

(A) $f'(x) \geq 0 \ \forall x \in I \Rightarrow f$ increasing on I

(B) $f'(x) > 0 \ \forall x \in I \Rightarrow f$ strictly increasing on I

PROOF

(A) II formula finite increment

$x_1, x_2 \in I, \ x_1 < x_2$

$f(x_2) - f(x_1) = f'(\xi) \cdot (x_2 - x_1) \Rightarrow f(x_2) \geq f(x_1) \Rightarrow f$ increasing on I c.v.d.

(≥ 0, > 0)

(B) $f(x_2) - f(x_1) = f'(\xi) \cdot (x_2 - x_1) \Rightarrow f(x_2) > f(x_1) \Rightarrow f$ strictly increasing on I c.v.d.

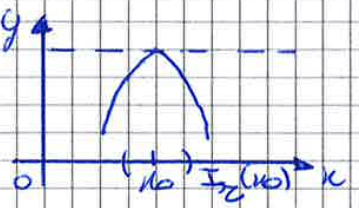
(> 0, > 0)

Classification of stationary points

f differentiable on an open interval I

$x_0 \in I$

$f'(x_0) = 0$



$\begin{cases} f'(x) \geq 0 \text{ on } I_\varepsilon^-(x_0) \\ f'(x) \leq 0 \text{ on } I_\varepsilon^+(x_0) \end{cases}$ $\Rightarrow x_0$ is a local maximum point

$\begin{cases} f'(x) \leq 0 \text{ on } I_\varepsilon^-(x_0) \\ f'(x) \geq 0 \text{ on } I_\varepsilon^+(x_0) \end{cases}$ $\Rightarrow x_0$ is a local minimum point

$\Rightarrow$ sign of $f'(x)$ in the neighbourhood

We need to consider an interval

ex

$f(x) = \frac{1}{x} \qquad f'(x) = -\frac{1}{x^2} < 0 \ \forall x \in \mathbb{R} - \{0\} \not\Rightarrow f(x)$ strictly decreasing on $\mathbb{R} - \{0\}$

but it is not true in a set

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ex

$$\lim_{x\to+\infty} \frac{2x+\sin x}{3x-\cos x} = \left[\frac{\infty}{\infty}\right] = \frac{2}{3}$$

$g'(x) = 3+\sin x \neq 0$ — De l'Hôpital → $\lim_{x\to+\infty} \frac{2+\cos x}{3-\sin x} \not\exists$

f, g differentiable

I prove that that limit $\not\exists$

$a_n \to +\infty \quad a_n = 2n\pi \Rightarrow \frac{f(a_n)}{g(a_n)} = 1$

$b_n \to +\infty \quad b_n = (2n+1)\pi \Rightarrow \frac{f(a_n)}{g(a_n)} = 1/3$ → $\lim_{x\to+\infty} \frac{2+\cos x}{3-\sin x} \not\exists$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{2}{3} \neq \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} \not\exists$$

TH Derivative at a point and limit of the derivative

f continuous on $I(x_0)$
f differentiable on $I(x_0) - \{x_0\}$
$\lim_{x\to x_0} f'(x) = \ell \in \mathbb{R}$
→ f differentiable at x_0 and $f'(x_0) = \ell$

PROOF

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \left[\frac{0}{0}\right] \overset{\text{De l'Hôpital}}{=} \lim_{x\to x_0} \frac{f'(x)}{1} = \ell$$

$g(x) = x - x_0 \quad g'(x) = 1$

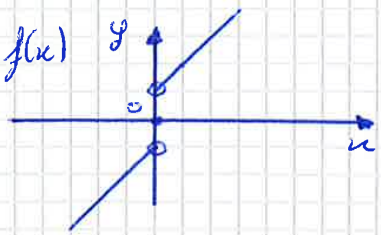
c.v.d.

Remark

Continuity is essential

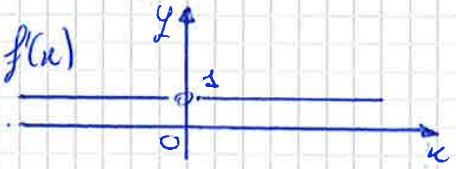
ex

$$f(x) = \begin{cases} x-1 & x<0 \\ 0 & x=0 \\ x+1 & x>0 \end{cases}$$



at $x_0 = 0$ → f NON continuous → f NON differentiable

$$f'(x) = \begin{cases} 1 & x>0 \\ 1 & x<0 \end{cases} \qquad \lim_{x\to 0} f'(x) = 1$$



But if I use the theorem in a wrong way

f continuous on $I(x_0)$
f differentiable on $I(x_0) - \{x_0\}$
$\lim_{x\to x_0} f'(x) \not\exists$
→ NO → f NON differentiable on x_0

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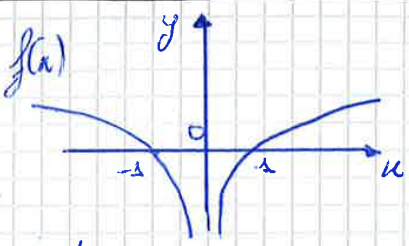
ex

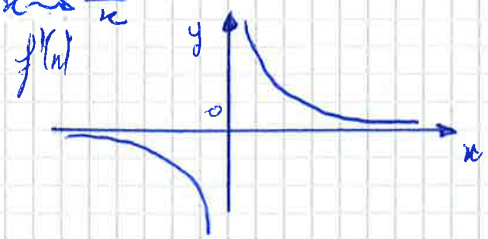
$f(x) = \log_a|x| \quad x \neq 0$

$x > 0 \Rightarrow f'(x) = \frac{1}{x \ln a}$ $\quad \ln x \to \frac{1}{x}$

$x < 0 \Rightarrow f'(x) = \frac{-1}{-x \ln a} = \frac{1}{x \ln a}$ $\quad \ln x \to \frac{1}{x}$

$D \ln|x| = \frac{1}{x} \quad \forall x \neq 0$



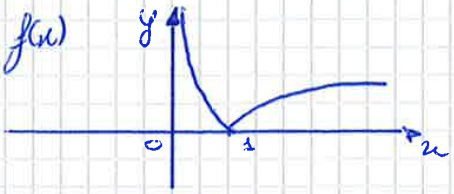


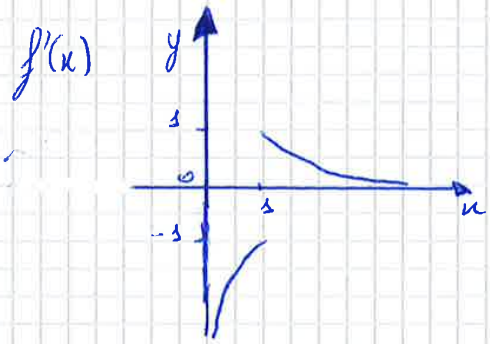
ex

$f(x) = |\ln x|$

$$f(x) = \begin{cases} \ln x & x \geq 1 \\ -\ln x & 0 < x < 1 \end{cases}$$

$$f'(x) = \begin{cases} 1/x & x \geq 1 \\ -1/x & 0 < x < 1 \end{cases}$$





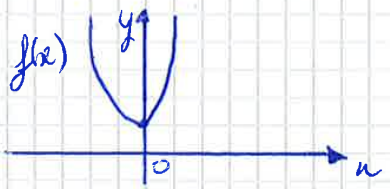
$x = 1$ corner point

$\lim_{x \to 1^+} \frac{\log x}{x-1} \overset{x-1=t}{=\!=} \lim_{x \to 1^+} \frac{\log(1+t)}{t} = 1$

$\lim_{x \to 1^-} \frac{\log x}{x-1} = -1$

ex

$$f(x) = \begin{cases} 1+x^2 & x \leq 0 \\ e^x & x > 0 \end{cases}$$



$f(x) \in C^0(\mathbb{R})$: $f(x)$ continuous $\forall x \neq 0$

$\lim_{x \to 0^-} 1+x^2 = 1 = f(0) = \lim_{x \to 0^+} e^x \Rightarrow \lim_{x \to 0} f(x) = 1 \to f$ continuous at $x = 0$

$$f'(x) = \begin{cases} 2x & x < 0 \\ e^x & x > 0 \end{cases}$$

$x = 0$ corner point

① pro theorem

$\lim_{x \to 0^-} 2x = 0 = f'_-(0)$

$\lim_{x \to 0^+} e^x = 1 = f'_+(0)$

② difference quotient

$\lim_{x \to 0^-} \frac{1+x^2-1}{x} = 0$

$\lim_{x \to 0^+} \frac{e^x - 1}{x} = 1$

f NOT differentiable at $x = 0$

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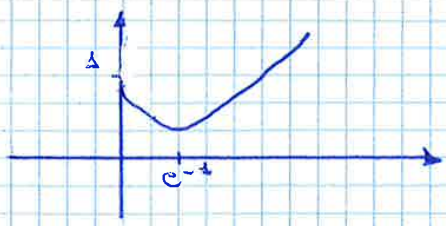
$h'(x) > 0 \rightarrow x > e^{-1}$

$h'(x) < 0 \rightarrow x < e^{-1}$

$f(e^{-1}) = e^{e^{-1}} = e^{1/e}$



① $\lim_{x \to 0^+} \dfrac{e^{x \ln x} - 1}{x - 0}$ difference quotient

OK

② Theorem about $\lim_{x \to 0} f'(x)$

$\lim_{x \to 0^+} e^{x \ln x} (\ln x + 1) = -\infty \Rightarrow x = 0$ vertical tangent line from right

Number of zeros of $f(x)$

1) $f(x) = e^x + x^3 \quad [-3;3]$

ⓐ one pos. zero ⓑ one neg. zero

ⓒ two zeros ⓓ no zeros ⓔ ....

$f(x)$ strictly increasing on $\mathbb{R}$

$f(-3) = e^{-3} + (-3)^3 < 0$

$f(3) = e^3 + 27 > 0$

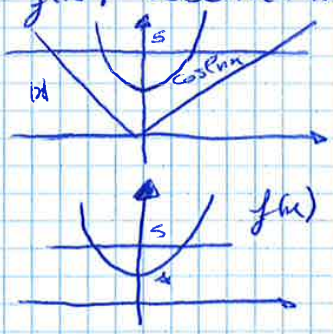
$\Rightarrow \exists$ one zero

$f(0) = 1 + 0$

$\rightarrow$ ⓑ

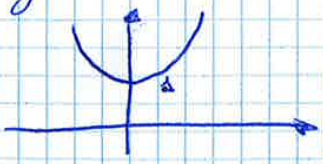
2) $f(x) = \cos \ln x + |x|$ how many solution at $f(x) = 5$

$f(x)$ strictly increasing $\quad ; +\infty)$



$\rightarrow$ 2

3) $f(x) = \cos \ln x + |x| \quad f(x) = \lambda \quad \lambda \in \mathbb{R}$



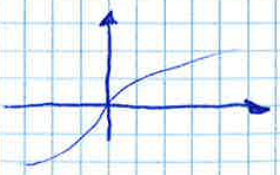
$\lambda < 1 \rightarrow$ no sol.

$\lambda = 1 \rightarrow 1$ sol.

$\lambda > 1 \rightarrow 2$ sol.

a) $f(x) = x^2 \arctan x$

$\forall a \in \mathbb{R}$ the eq. $f(x) = a$ has exactly one sol.



$f(x)$ ODD

$\lim_{x \to +\infty} x^2 \arctan x = +\infty$

$\rightarrow f(x)$ monotone increasing function

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LOCAL COMPARISON OF FUNCTIONS

Before: what is the limit?

Now: how a function "reaches" the limit? → comparing 2 functions

Landau's symbols

Ipotesi

f, g defined on $I(\gamma) \setminus \{\gamma\}$

$g(x) \neq 0$ $I(\gamma) \setminus \{\gamma\}$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)}$

DEF 1 $f \asymp g$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)} = \ell \neq 0$ → f and g are of the SAME order for $x \to \gamma$

↳ $f \asymp g$

DEF 2 $f \sim g$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)} = 1$ → f EQUIVALENT to g for $x \to \gamma$

↳ $f \sim g$

DEF 3 $f = o(g)$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)} = 0$ → f NEGLIGIBLE with respect to g for $x \to \gamma$

↳ $f = o(g)$ (f little-o of g)

DEF 4 $g = o(f)$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)} = \infty$ → $\lim_{x \to \gamma} \frac{g(x)}{f(x)} = 0$ → $g = o(f)$ for $x \to \gamma$

DEF 5

$\lim_{x \to \gamma} \frac{f(x)}{g(x)} \nexists$ → f, g NON COMPARABLE for $x \to \gamma$

ex

$\lim_{x \to 0} \frac{\sin x}{x} = 1 \iff \sin x \sim x \ (x \to 0)$

(also $\tan x$, $\arcsin x$, $\arctan x$)

ex

$\lim_{x \to +\infty} \frac{\sin x}{x} = 0 \iff \sin x = o(x) \ (x \to +\infty)$

DEF 6 $f = O(g)$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)} = \ell \in \mathbb{R}$ → f CONTROLLED by g for $x \to \gamma$

↳ $f = O(g)$ (f BIG-O of g)

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$\sim$ $f,g,h \neq 0$ on $I(\gamma)\setminus\{\gamma\}$

1) $f \sim f$

2) $f \sim g \Rightarrow g \sim f$

3) $f \sim g,\ g \sim h \Rightarrow f \sim h$

PROOF

$f/g \to 1,\ g/h \to 1 \Rightarrow f/h \to 1$ to be proven

$\to f/g \cdot g/h = f/h = 1$

$\asymp$ $f,g,h \neq 0$ on $I(\gamma)\setminus\{\gamma\}$

1) $f \asymp g$

2) $f \asymp g \to g \asymp f$

3) $f \asymp g,\ g \asymp h \Rightarrow f \asymp h$

o $f,g,h \neq 0$ on $I(\gamma)\setminus\{\gamma\}$

1) $f = o(f)$ NO

2) $f = o(g) \to g = o(f)$ NO

3) $f = o(g),\ g = o(\ ) \Rightarrow f = o(h)$

TH 1

$f \sim f_1 \quad (x \to \gamma)$

$g \sim g_1 \quad (x \to \gamma)$

$\Rightarrow \lim_{x\to\gamma} f(x)g(x) = \lim_{x\to\gamma} f_1(x)g_1(x)$

$\lim_{x\to\gamma} \frac{f(x)}{g(x)} = \lim_{x\to\gamma} \frac{f_1(x)}{g_1(x)}$

in products (or quotients) we can substitute factors (or numerator and denominator) with equivalent functions

PROOF

$\lim_{x\to\gamma} f/g = \lim_{x\to\gamma} \frac{f}{f_1}\frac{f_1}{g_1}\frac{g_1}{g} = \lim_{x\to\gamma} \frac{f_1}{g_1}$

($\frac{f}{f_1} \to 1$, $\frac{g_1}{g} \to 1$)

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$x \to 0^+$ Infinitesimal at 0

$\varphi(x) = x^\alpha \quad \alpha > 0$

ex

$f(x) = x^5$
$g(x) = x^3$
$\lim_{x\to 0} \frac{x^3}{x^5} = +\infty \Rightarrow x^5 = o(x^3)$

→ When comparing 2 powers we must neglect the highest power

ex

$f(x) = x^2 + 3x^3 + x^5 \sim x^2$
$f(x) = \sqrt{x^1} + x \sim \sqrt{x^1}$

DEF

If $\lim_{x\to 0} \frac{f(x)}{cx^\alpha} = 1 \quad c \neq 0$

1) $f(x)$ INFINITESIMAL (for $x \to 0$) of order α

2) the function cx^α is called the PRINCIPAL PART of $f(x)$ $(x \to 0)$

ex

$f(x) = \log(1+6x^5) \sim 6x^5$ → principal part of $f(x)$ $(x \to 0)$

$\lim_{x\to 0} \frac{\log(1+6x^5)}{6x^5} = 1$ → f infinitesimal of order 5

$\log(1+z) \sim z \quad (z \to 0) \quad z = 6x^5$

ex

$\lim_{x\to 0} \frac{3\sin^3 x + \log(1+6x^5)}{x^2 + \tan x} = \lim_{x\to 0} \frac{3x^3 + 6x^5}{x^2 + x} = 0$

ex

$\lim_{x\to 0} \frac{\sin^2(\sin(\ln x)) - \sin(\ln^3(\sin x))}{x} = \lim_{x\to 0} \frac{x^2 - x^3}{x} = \lim_{x\to 0} x = 0$

$\sin(\ln z) \sim z$
$\sin z \sim z$
$\sin(\sin(\ln x)) \sim \sin(\ln x) \sim x$
$\sin^2(\sin(\ln x)) \sim x^2$

$-\frac{1}{\log} \quad x^\alpha \quad e^{-\frac{1}{x}}$

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$x \to 0^-$

substitute $|x|$ and then is the same

$x \to x_0 \neq 0$

substitute $x - x_0$

ex

$f(x) = \ln x \quad (x \to 1)$

$$\lim_{x\to 1} \frac{\ln x}{x-1} \overset{x-1=t}{=\!=} \lim_{t\to 0} \frac{\ln(1+t)}{t} = 1$$

$\to \ln x \ (x \to 1) \sim x-1$ Infinitesimal of order 1 and principal part is $x-1$ (for $x \to 1$)

$x \to +\infty$

$\varphi(x) = \frac{1}{x}$

If $f \sim l \frac{1}{x^\alpha}$ Infinitesimal order α and principal part $l \frac{1}{x^\alpha}$

$\frac{1}{\log^\gamma x}$ $\quad \frac{1}{x^\alpha}$ $\quad e^{-\beta x}$ $\quad \beta > 0, \gamma > 0$

ex

$f(x) = \frac{\pi}{2} - \arctan x \quad (x \to +\infty)$

$$\lim_{x\to+\infty} \frac{\frac{\pi}{2} - \arctan x}{\frac{1}{x}} = \left[\frac{0}{0}\right] = \lim_{x\to+\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{x^2}{1+x^2} = 1$$

$\to \frac{\pi}{2} - \arctan x \sim \frac{1}{x}$ Infinitesimal of order 1 and principal part is $\frac{1}{x}$ (for $x \to +\infty$)

ex

$f(x) = \frac{1}{x} - \sin\frac{1}{x} \quad (x \to +\infty)$

$$\lim_{x\to+\infty} \frac{\frac{1}{x} - \sin\frac{1}{x}}{\frac{1}{x}} = \lim_{\frac{1}{x}\to 0} \frac{\frac{1}{x}}{\frac{1}{x}} - \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 0 \qquad f(x) = o\left(\frac{1}{x}\right)$$

$$\lim_{x\to+\infty} \frac{\frac{1}{x} - \sin\frac{1}{x}}{\frac{1}{x^2}} = \lim_{t\to 0} \frac{t - \sin t}{t^2} = \lim_{t\to 0} \frac{1-\cos t}{2t} = 0 \qquad f(x) = o\left(\frac{1}{x^2}\right)$$

$$\lim_{x\to+\infty} \frac{\frac{1}{x} - \sin\frac{1}{x}}{\frac{1}{x^3}} = \lim_{t\to 0} \frac{t - \sin t}{t^3} = \lim_{t\to 0} \frac{1-\cos t}{3t^2} = \frac{1}{6} \qquad f(x) \sim \frac{1}{6x^3}$$

$\to f(x)$ infinitesimal of order 3 and principal part is $\frac{1}{6x^3}$ (for $x \to +\infty$)

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PROOF Lhôt $\ln x = o(x^\alpha)$

$$\lim_{x\to+\infty} \frac{\ln x}{x^\alpha} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \lim_{x\to+\infty} \frac{1}{\alpha x^\alpha} = 0$$

c.v.d.

$x \to 0^+$

$\varphi(x) = \frac{1}{x}$ $\quad f \sim c\frac{1}{x^\alpha}$ $\quad f$ infinite order α, principal part is $c/\frac{1}{x^\alpha}$

$x \to 0^-$

$\varphi(x) = \frac{1}{|x|}$

$x \to x_0 \neq 0$

$\varphi(x) = \frac{1}{x - x_0}$

Recap

	$x=0$	$x_0 \neq 0$	$x \to +\infty$
INFINITESIMAL	$\varphi(x) = x$	$\varphi(x) = x - x_0$	$\varphi(x) = \frac{1}{x}$
INFINITE	$\varphi(x) = \frac{1}{x}$	$\varphi(x) = \frac{1}{x - x_0}$	$\varphi(x) = x$

Finding p.p. of $f(x)$ for $x \to \gamma$

1) infinite or infinitesimal?
2) choose the $\varphi(x)$
3) $\lim \frac{f(x)}{c(\varphi(x))^\alpha}$ and check if you can find c and α in order to have limit $= 1$

a) IF yes → { a) f infinitesimal [infinite] order α ; b) $c(\varphi(x))^\alpha$ p.p. of $f(x)$

ex

$f(x) = \frac{1}{\cos x}$ $\quad x \to \frac{\pi}{2}^-$

1) infinite for $x \to \frac{\pi}{2}^-$
2) $\varphi(x) = \frac{1}{x - \frac{\pi}{2}}$
3) $\lim_{x\to\pi/2^-} \frac{\frac{1}{\cos x}}{c\left(\frac{1}{x-\frac{\pi}{2}}\right)^\alpha} \overset{?}{=} 1$ $\quad \frac{1}{\cos x} \sim -\frac{1}{x - \frac{\pi}{2}}$

a) infinite order 1
p.p. $-\frac{1}{x - \pi/2}$

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ex

$f(x) = x - \sqrt[3]{x^3+x^4}$ $\quad x \to 0$ infinitesimal

$$\lim_{x\to 0} \frac{\sqrt[3]{x^3+x^4}}{x} = \lim_{x\to 0} \sqrt[3]{\frac{x^3+x^4}{x^3}} = \lim_{x\to 0} \sqrt[3]{1+x} = 1$$

$\sqrt[3]{x^3+x^4} \sim x \rightarrow \sqrt[3]{x^3+x^4} = x + o(x)$

$f(x) = x - (x + o(x))$ — pp. of its self

CANCELLATION OF THE PRINCIPAL PARTS

$\sqrt[3]{x^3+x^4} = \sqrt[3]{x^3(1+x)} = x\sqrt[3]{1+x} = x\left(1 + \frac{1}{3}x + o(x)\right)$

$f(x) = x - \left(x + \frac{1}{3}x^2 + o(x^2)\right) = -\frac{1}{3}x^2 + o(x^2)$

INF.MAL ORDER 2 and P.P. $-\frac{1}{3}x^2$

ex

$f(x) = \sin x - x =$ $\quad x \to 0$ infinitesimal

$= x + o(x) - x$

(I) Taylor's formula with PEANO's remainder

0] f continuous at x_0 : $f(x) = f(x_0) + o(1)$ $\quad x \to x_0$

1] f differentiable at x_0:

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \in \mathbb{R} \qquad x \to x_0$$

$$\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) + o(1) \rightarrow f(x) - f(x_0) = f'(x_0)(x-x_0) + o(x-x_0)$$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{tangent line, polynomial degree } \le 1} + \underbrace{o(x-x_0)}_{\text{ERROR term}}$$

2] f twice differentiable at x_0 : $(\exists f''(x_0))$

$f(x) - \left[f(x_0) + f'(x_0)(x-x_0)\right] = o(x-x_0)$ → order > 1

$$\lim_{x\to x_0} \frac{f(x) - [\dots]}{(x-x_0)^2} \overset{\text{l'Hôp}}{=} \lim_{x\to x_0} \frac{f'(x) - f'(x_0)}{2(x-x_0)} = \frac{1}{2}f''(x_0)$$

$$\frac{f(x) - [\dots]}{(x-x_0)^2} = \frac{1}{2}f''(x_0) + o(1)$$

$$f(x) - [\dots] = \frac{1}{2}f''(x_0)(x-x_0)^2 + o(x-x_0)^2$$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2}_{\text{parabola (or osculating parabola)}} + o((x-x_0)^2)$$

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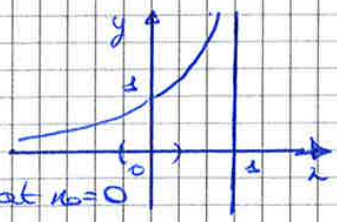
ex

$f(x) = \frac{1}{1-x}$ $\quad x_0 = 0$



f differentiable ∞ many times at $x_0 = 0$

$f(x) = (1-x)^{-1}$	1
$f'(x) = +(1-x)^{-2}$	1
$f''(x) = 2(1-x)^{-3}$	$2!$
$f'''(x) = 2 \cdot 3 (1-x)^{-4}$	$3!$
$f^{(n)}(x) = n!(1-x)^{-(n+1)}$	$n!$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + o(x^n)$$ → Taylor polynomial, all coefficients are 1

ex

$f(x) = 5 - 2x + 3x^2 + x^4 \quad x_0 = 0$

- $T_{0,0}(x) = 5 \rightsquigarrow f(x) = 5 + o(1)$ (0 = order)
- $T_{0,1}(x) = 5 - 2x \rightsquigarrow f(x) = 5 - 2x + o(x)$
  $f'(x) = -2 + 6x + 4x^3 \quad f'(0) = -2$
- $T_{0,2}(x) = 5 - 2x + 3x^2 \rightsquigarrow f(x) = 5 - 2x + 3x^2 + o(x^2)$
  $f''(x) = 6 + 12x^2 \quad f''(0) = 6$
- $T_{0,3}(x) = 5 - 2x + 3x^2 \rightsquigarrow f(x) = 5 - 2x + 3x^2 + o(x^3)$
- $T_{0,4}(x) = 5 - 2x + 3x^2 + x^4 \rightsquigarrow f(x) = 5 - 2x + 3x^2 + x^4$ → no error term
  $f^{(iv)}(0) = 24$

→ The MC LAURIN polynomial of order $\geq n$ of polynomial of degree n is the polynomial itself with zero error

f differentiable at x_0 → $f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$

$f(x) = f(x_0) + \lambda(x - x_0) + o(x - x_0) \quad (x \to x_0)$ → f differentiable at x_0 and $f'(x_0) = \lambda$

TH

f differentiable at x_0 ⟷ $f(x) = f(x_0) + \lambda(x - x_0) + o(x - x_0)$ with $f'(x_0) = \lambda$

PROOF

$f(x) = f(x_0) + \lambda(x - x_0) + o(x - x_0) \rightsquigarrow f(x) - f(x_0) = \lambda(x - x_0) + o(x - x_0)$

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ex.

$\sin 1$

↳ 1 rad $\simeq 50°$

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120}$ $\quad x=1 \rightarrow \quad \sin 1 = 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} = 0{,}8417$

ERROR TERM: $\left| \frac{-\sin t}{720} 1^6 \right| \leq \frac{1}{720} = 0{,}0013 \rightarrow$ so 0,84 is correct for sure

↳ $0{,}8417 \pm 0{,}0013$

TH and **PROOF**

f differentiable n-times at x_0

$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + o((x-x_0)^n)$

$a_0 = a_1 = \dots = a_{m-1} = 0$

$a_m \neq 0$

$f(x) = a_m(x-x_0)^m + a_{m+1}(x-x_0)^{m+1} + \dots + a_{n-m}(x-x_0)^n + o((x-x_0)^n)$

$$\frac{f(x)}{a_m(x-x_0)^m} = 1 + \dots (x-x_0) + \dots (x-x_0)^2 + \dots + \dots (x-x_0)^{n-m} + o((x-x_0)^{n-m})$$

$$\lim_{x \to x_0} \frac{f(x)}{a_m(x-x_0)^m} = 1 \rightarrow f \sim a_m(x-x_0)^m \rightarrow \text{infinitesimal order } m \text{, principal part } a_m(x-x_0)^m$$

ex.

$f(x) = x - \sin x \quad (x \to 0)$

$f(x) = x - \left(x - \frac{1}{6}x^3 + o(x^3)\right) = \frac{1}{6}x^3 + o(x^3)$

$f(x)$ infinitesimal order 3 $(x \to 0)$ and p.p. $f(x)$ $\frac{1}{6}x^3$

ex.

$f(x) = e^x - \cos x \quad (x \to 0)$

$f(x) = (1 + x + o(x)) - \left(1 - \frac{x^2}{2} + o(x^2)\right) = x + o(x) + \frac{x^2}{2} + o(x^2) = x + o(x)$

$f(x)$ infinitesimal order 1 $(x \to 0)$ and pp. $f(x)$ x

ex.

f differentiable 3 times at $x=0$

$f(x) = 3x^2 + 5x^3 + o(x^3) \quad (x \to 0)$

1) $x=0$ local minimum? Yes

2) $f'''(0)$?

3) $f^{(4)}(0)$? → I can not say anything for order >3 → IMPOSSIBLE.

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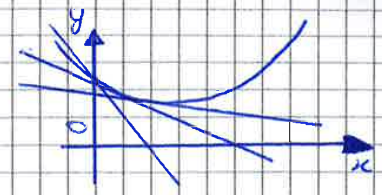
TH

f CONVEX on I $\leftrightarrow$ f' INCREASING ON I



TH

f' STRICTLY INCREASING ON I $\rightarrow$ f STRICTLY CONVEX

TH

f twice differentiable on I

f CONVEX on I $\leftrightarrow$ $f''(x) \geq 0 \ \forall x \in I$

$f''(x) > 0 \ \forall x \in I \rightarrow f$ STRICTLY CONVEX

TH

f twice differentiable at x_0

1) x_0 INFLECTION POINT $\rightarrow f''(x_0) = 0$

2) $f''(x_0) = 0$ and f'' changes sign when crossing $x_0 \rightarrow x_0$ INFLECTION POINT

Remark

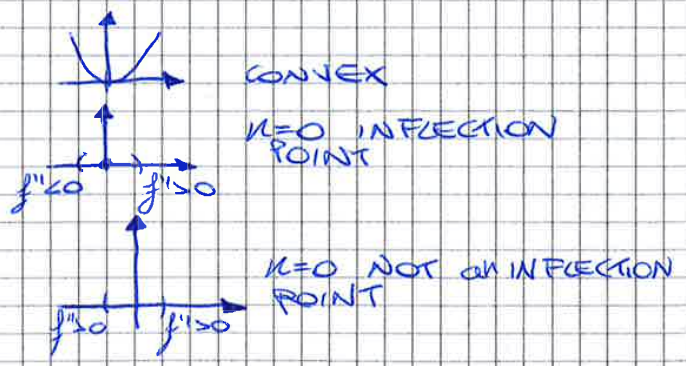
if $f''(x) = 0$ and f'' does NOT changes sign when crossing $x_0 \rightarrow x_0$ IS NOT an inflection POINT

ex

$f(x) = x^2 \quad f'(x) = 2x \quad f''(x) = 2$

$f(x) = x^3 \quad f'(x) = 3x^2 \quad f''(x) = 6x$

$f(x) = x^4 \quad f'(x) = 4x^3 \quad f''(x) = 12x^2$



MAX/MIN $f'(x_0) = 0$

1) 1st method and then study the sign of f' in $I(x_0)$

2) 2nd method

TH

f differentiable n times at x_0

$\exists m \in \mathbb{N} \quad 2 \leq m \leq n$

$f'(x_0) = f''(x_0) = \dots = f^{(m-1)}(x_0) = 0$

$f^{(m)}(x_0) \neq 0$

1) IF m EVEN
   - $f^{(m)}(x_0) \leq 0 \rightarrow x_0$ LOCAL MAX
   - $f^{(m)}(x_0) \geq 0 \rightarrow x_0$ LOCAL min
2) IF m ODD $\rightarrow x_0$ INFLECTION POINT with horizontal tangent

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ex

$1-\cos(x^3+2x^2) \sim 2x^4 \quad x\to 0$

$1-\cos z \sim z^2/2 \quad z\to 0$

$1-\cos(x^3+2x) \sim \frac{1}{2}(x^3+2x)^2$

$1-\cos(x^3+2x) \sim \frac{1}{2}(x^6+4x^4+4x^3)$

(x^6: negligible with respect to $4x^4$; $4x^3$: negligible with respect to $4x^4$)

$1-\cos(x^3+2x) \sim 2x^4$

ex

$f(x)=x^3 \quad g(x)=\sqrt[3]{x^8-7x^2}$ compare when $x\to+\infty$

$$\lim_{x\to+\infty}\frac{\sqrt[3]{x^8-7x^2}}{x^3}=\lim_{x\to+\infty}\sqrt[3]{\frac{x^8-7x^2}{x^9}}=0 \rightsquigarrow \sqrt[3]{x^8-7x^2}=o(x^3)$$

ex

$f(x)=\sqrt{x-3} \quad g(x)=\sqrt{x}-\sqrt{3} \quad h(x)=\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{3}} \quad x\to 3^+$

$f(x)=(x-3)^{1/2}$ infinitesimal order $1/2$

$g(x)$ try

$$\lim_{x\to3^+}\frac{\sqrt{x}-\sqrt{3}}{x-3}=\lim_{x\to3^+}\frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}=\frac{1}{2\sqrt{3}}$$

$\rightsquigarrow g(x)=\sqrt{x}-\sqrt{3}$ infinitesimal order 1 and pp. $x-3$

$g(x)=\sqrt{x}-\sqrt{3} \sim \frac{1}{2\sqrt{3}}(x-3)$

$h(x)$ same as $g(x)$

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Given a partition $\{x_0, \dots x_n\}$

$m_i = \inf_{(x_{i-1}, x_i)} f(x)$ $\quad M_i = \sup_{(x_{i-1}, x_i)} f(x)$

Lower sum $s(P) = \sum_{i=1}^{n} m_i \Delta x_i$

Upper sum $S(P) = \sum_{i=1}^{n} M_i \Delta x_i$

Properties

1) $s(P) \leq S(P)$

2) $s(P_1) \leq s(P_1 \cup P_2) \leq S(P_1 \cup P_2) \leq S(P_2)$

[The set $\{s(P)\}$ when P varies among all possible partitions is bounded above [or below]]

→ so for completeness of $\mathbb{R}$ → $s = \sup\{s(P)\}$ $[S = \inf\{S(P)\}]$

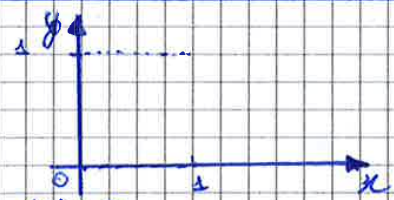
IN GENERAL $s \leq S$

DEF

IF $s = S$ we say that f is (RIEMANN) integrable on $[a;b]$ and $s = S = \int_a^b f(x)\,dx$

ex Dirichlet function

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$$



$\inf f(x) = 0$ $\quad \forall$ subinterval $m_i = 0$

$\sup f(x) = 1$ $\quad \forall$ subinterval $M_i = 1$

Subdivision P

$s(P) = 0\Delta x_1 + 0\Delta x_2 + \dots + 0\Delta x_n = 0$

$S(P) = 1\Delta x_1 + 1\Delta x_2 + \dots + 1\Delta x_n = 1$

$s = \sup s(P) = 0$

$S = \inf S(P) = 1$

→ $s \neq S$ → $f(x)$ NON integrable

TH

f bounded on $[a;b]$

f is integrable on $[a;b]$ ⟷ $\forall \varepsilon > 0$ $\exists P_\varepsilon$: $S(P_\varepsilon) - s(P_\varepsilon) < \varepsilon$

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DEF

f PIECEWISE CONTINUOS FUNCTION on $(a;b)$

1) f continuos $\forall x \in [a;b]$ except at most a finite number of points
2) the discontinuity are removable points or jump

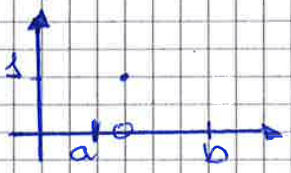
Properties of REIMANN integral

IF f, g integrable on $[a;b]$ then

1) f integrable on $[c;d]$ for all $[c;d] \subseteq [a;b]$

2) $f(x) \geq 0 \ \forall x \in [a;b] \rightarrow \int_a^b f(x)dx \geq 0$

3) $f(x) \geq 0 \ \forall x \in [a;b]$, f continuos on $[a;b]$ $\rightarrow \int_a^b f(x)dx = 0 \longleftrightarrow f(x) = 0$ on $[a;b]$



NON continuos but $\int_a^b f(x)dx \geq 0$

4) MONOTONICITY property
$f(x) \leq g(x) \ \forall x \in [a;b] \rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$

5) $\rightarrow |f|$ is integrable
$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\,dx$

Remark

we said f integrable $\rightarrow |f|$ integrable BUT $|f|$ integrable $\not\rightarrow f$ integr.

ex

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \qquad |f(x)| = 1$$

NON integrable — integrable

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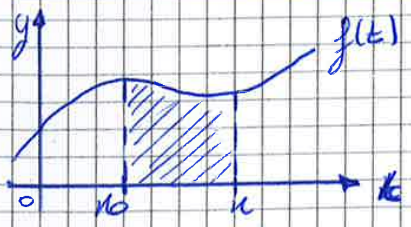
DEF

f defined on an interval I
f INTEGRABLE $\forall [a;b] \subseteq I$
$x_0, x \in I$

→ **INTEGRAL FUNCTION**

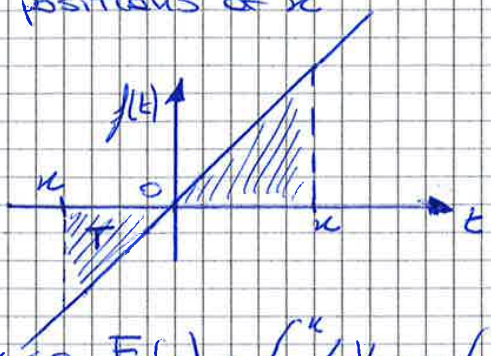
$$F_{x_0}(x) = F(x) = \int_{x_0}^{x} f(t)\,dt$$

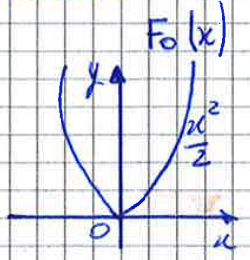
integral from a fixed point to a moving one



x_0 FIXED and x variable

I suppose to move x for all possible positions of x





ex

$f(t) = t$

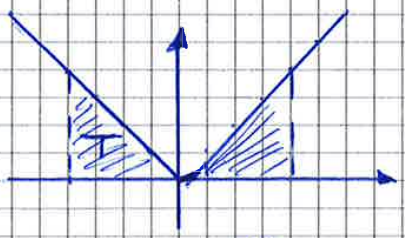
$$\int_0^x t\,dt = F_0(x)$$

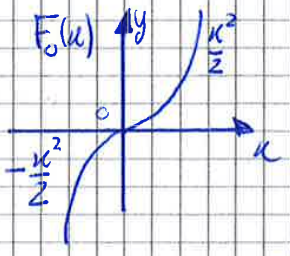
$F_0(0) = 0$

$x > 0$ $F_0(x) = \frac{x^2}{2}$

$x < 0$ $F_0(x) = \int_0^x t\,dt = -\int_x^0 t\,dt = -(-\text{area}(T)) = \text{area}(T) = \frac{x^2}{2}$

$f(t) = t \rightarrow F_0(x) = \frac{x^2}{2}$





ex

$f(t) = |t|$

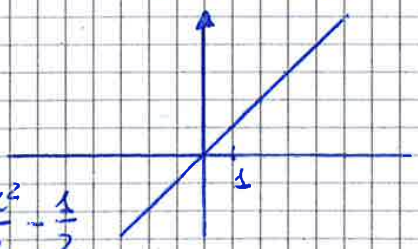
$F_0(0) = 0$

$x > 0$ $F_0(x) = \frac{x^2}{2}$

$x < 0$ $F_0(x) = \int_0^x |t|\,dt = -\int_x^0 |t|\,dt = -\text{area}(T) = \frac{-x^2}{2}$

ex

$f(t) = t$ $\quad F_1(x) = \int_1^x t\,dt$



$F_1(1) = 0$

$x > 1$ $F_1(x) = \frac{(x+1)(x-1)}{2} = \frac{x^2}{2} - \frac{1}{2}$

$0 \le x < 1$ $F_1(x) = -\frac{(x+1)(1-x)}{2} = \frac{x^2}{2} - \frac{1}{2}$

$x < 0$ $F_1(x) = \int_1^x t\,dt = -\int_x^1 t\,dt = -\left(\int_x^0 t\,dt + \int_0^1 t\,dt\right) = -\left(-\frac{x^2}{2} + \frac{1}{2}\right) = \frac{x^2}{2} - \frac{1}{2}$

$$\rightarrow f(t) = \begin{cases} \int_0^x t\,dt = \frac{x^2}{2} \\ \int_1^x t\,dt = \frac{x^2}{2} - \frac{1}{2} \end{cases}$$

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PROOF (2)

applying Weierstrass theorem → $\overbrace{m}^{\text{abs. min}} \leq m(f;a,b) \leq \overbrace{M}^{\text{abs. max}}$

applying Intermediate value theorem (II version) → $\exists c \in [a;b] \; f(c) = m(f;a,b)$ c.v.d.

TH

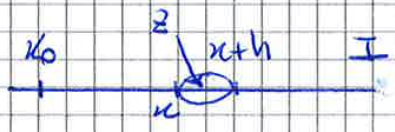
f defined on an interval I
f integrable on $[a;b]$ $\forall [a;b] \in I$ → $F_{x_0}(x)$ continuous on I

TH Fundamental theorem of integral calculus

f continuous on I
$x_0 \in I$ → $F_{x_0}(x)$ differentiable $\forall x \in I$ and $F'_{x_0}(x) = f(x)$

PROOF

$\dfrac{F_{x_0}(x+h) - F(x)}{h}$



$$\frac{1}{h}\left(\int_{x_0}^{x+h} f(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right) = \frac{1}{h}\left(\int_{x_0}^{x} f(t)\,dt + \int_{x}^{x+h} f(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right) =$$

$$= \frac{\int_{x}^{x+h} f(t)\,dt}{h} = m(f;x,x+h) = f(z(h))$$

$$\lim_{h\to 0} \frac{F_{x_0}(x+h) - F(x)}{h} = \lim_{h\to 0} f(z(h)) = \quad \rightarrow f \text{ is continuous}$$
$$\rightarrow \text{it's like a composition}$$

$$= f\left(\lim_{h\to 0} z(h)\right) = \quad x \leq z(h) \leq x+h \quad [\text{if } h<0 \Rightarrow \text{the opposite}]$$

applying 2 policemen theorem: $x \leq z(h) \leq x+h$, $h \to 0$, $x \to x$, $x \to x$

$$= f(x)$$

c.v.d.

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PART 2 I prove:

Suppose that $G(x)$ is a primitive of f on $I \rightarrow G(x) = F(x) + c$

$F'(x) = f(x)$ $\quad G'(x) = f(x)$

I define $H(x) = G(x) - F(x)$

$H'(x) = (G'(x) - F(x))' = G'(x) - F'(x) = f(x) - f(x) = 0$

applying Lagrange theorem $\rightarrow H(x) = c$ (constant)

$G(x) - F(x) = c$

$G(x) = F(x) + c$

c.v.d.

IF $f(x)$ is [A] primitive of $f(x)$ on I the set of all primitives of $f(x)$ on I is

$$\{F(x) + c \,;\, c \in \mathbb{R}\} = \int f(x)\,dx$$

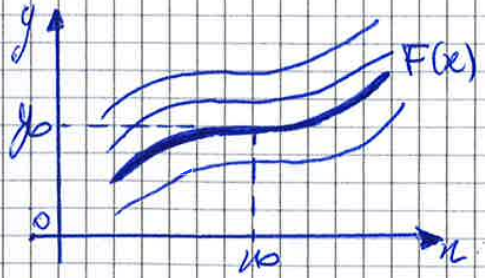
INDEFINITE INTEGRAL OF f (on I)

RECAP

$\int_a^b f(x)\,dx$ RIEMANN (oriented) integral DEFINITE INTEGRAL → a number

$\int_{x_0}^{x} f(t)\,dt$ INTEGRAL FUNCTION → FUNCTION

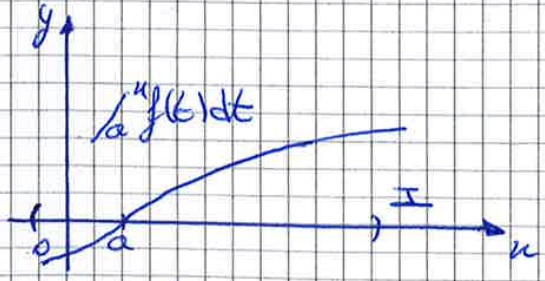
$\int f(x)\,dx$ INDEFINITE INTEGRAL → set of infinitely many functions



$F(x)$ is a primitive of f on I

among the infinite primitives there is one and only one whose graph passes trough (x_0, y_0)

i.e. $F(x_0) = y_0$



f continuous on I → FUND. TH.

$F_{x_0}(x)$ is a primitive → among them the integral function $\int_a^x f(t)\,dt$ vanishes at $x = a$

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