NUMERO: 1880A - ANNO: 2016

A P P U N T I

STUDENTE: Sagone Matteo

MATERIA: Vibration mechanics - prof. Fasana

VIBRATION MECHANICS

29/09/2014

①

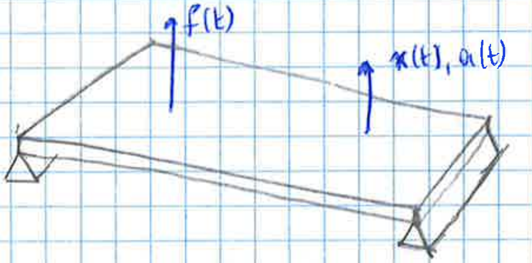
From monday 6/10 → LAIB 1D (next 1.5 hours)

Lab meeting point → near the infirmery.

INTRODUTCION TO MODAL ANALYSIS

A tool to describe a linear vibrating structure as a FUNCTION of:

natural frequencies (ω_n), damping ratios (ξ), mode shapes (ω_{ξ}, ψ_{ξ}) → NATURAL CHARACTERISTICS

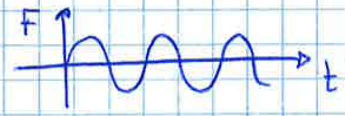


SIMPLY-SUPPORTED FREE PLATE

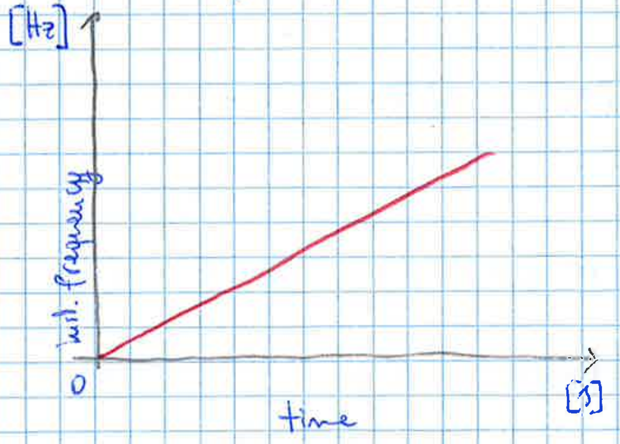
- a force is applied and we get a displacement

If the force varies with time (sine sweep force for example)

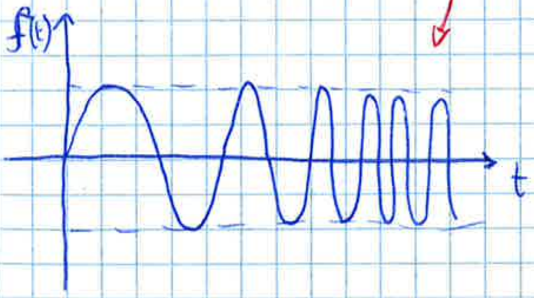
$$f(t) = F_0 \sin(\varphi(t))$$



and $\omega(t) = \frac{d\varphi}{dt}$ → frequency (increases linearly)



the amplitude remains the same, but the frequency increases with time

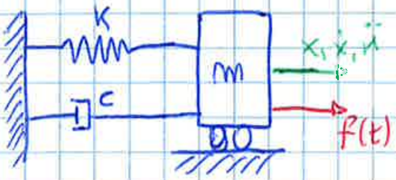


The amplitude of acceleration is not stable, but in certain times can ~~[illegible]~~ get larger.

→ The amplitude of the response increases very much → so does the displacement.

SDOF SYSTEMS single degree of freedom

They can always be assumed as this simple model:



a mass attached to a spring-damper system; the mass can only move in horizontal.

We assume a linear spring (even if it does not exist)

↓

$$\boxed{\leftarrow F_{SPRING} = kx}$$ → LINEARITY

{ K → stiffness
c → damping }

Also the damper is proportional to the displacement $\boxed{\leftarrow F_{DAMPER} = c\dot{x}}$ → " , VISCOUS MODEL

The equation of motion is the following:

$$\boxed{m\ddot{x}+c\dot{x}+kx=f(t)}$$ - SDOF MODEL EQUATION

all the inertia is in the mass m, all the damping in c, all the stiffness in k.

The complete solution consists of FREE RESPONSE ① ($f(t)=0$) and PARTICULAR ② INTEGRAL ($f(t)$)

FREE RESPONSE ①

We do not have an external excitement → $f(t)=0$

$$\Rightarrow \boxed{m\ddot{x}+c\dot{x}+kx=0}$$ HOMOGENEOUS EQ.

and the solution is $x(t)=Ae^{st}$ with $A, s \in \mathbb{C}$ (are complex)

so we can put it in the equation and we get:

$$(ms^2+cs+k)Ae^{st}=0$$

we have two possibilities:

1) $A=0$ → TRIVIAL SOLUTION

2) $ms^2+cs+k=0$ → we moved from a differential to a polynomial II degree eq.

If we substitute s_1 and s_2 we obtain the displacement:

$$\boxed{x(t) = (A\cos \omega_d t + B\sin \omega_d t)e^{-\xi \omega_n t}}$$

↳ HARMONIC FUNCTION { Sine and cosine have got the same freq. The amplitude goes down with ⓣ

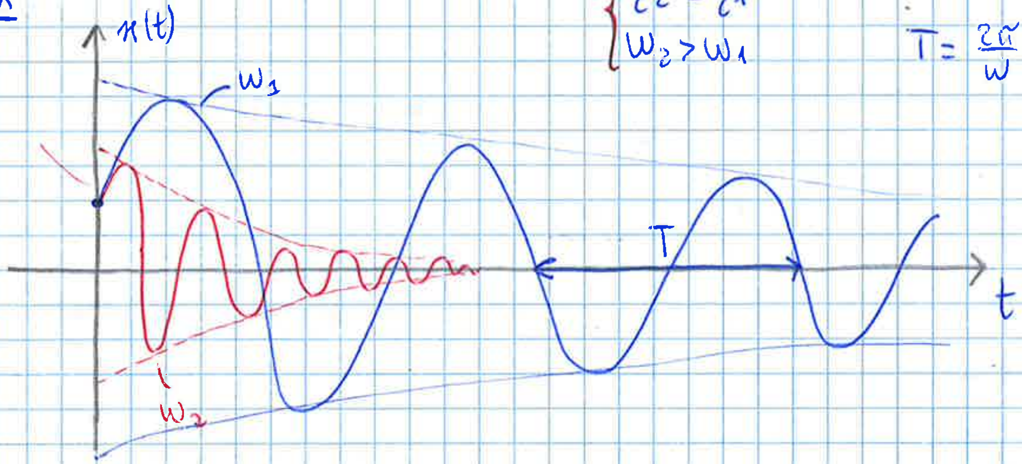
$\boxed{A = x_0}$ $\boxed{B = \frac{v_0 + \xi \omega_n x_0}{\omega_d}}$ { A, B are real, so the solution is real, even if s_1, s_2 are complex

$\boxed{\xi}$ → controls the amplitude when time passes; in a free response we will have always that → harmonic function with a exponential decreasing term.

$\boxed{\omega_n}$ → also the nat. freq. appears in the exponential → it leads the velocity of decreasing

EX

$$\begin{cases}\xi_2 = \xi_1 \\ \omega_2 > \omega_1\end{cases} \qquad T = \frac{2\pi}{\omega} = \frac{1}{f}$$



even if the two systems have the same damping factor, the ω controls the time to go to zero → the period is different, so the decay is different.

So, the solution of the homogeneous equation gives the FREE RESPONSE, the FREE DECAY, the TRANSIENT RESPONSE.

To get the COMPLETE EQUATION we have to look for the PARTICULAR INTEGRAL, called also the STEADY STATE SOLUTION.

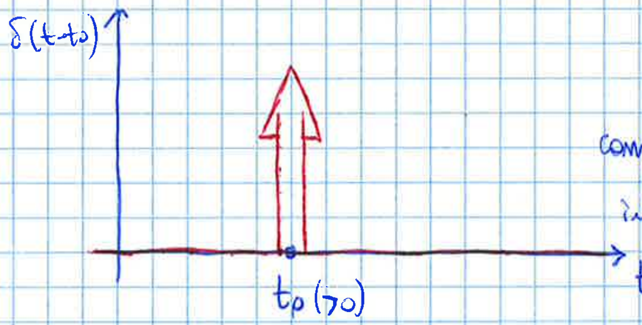
By Determining A_1 and A_2 we get the complete behaviour of the response, the COMPLETE RESPONSE of a forced m, k, c, system.

As the time passes, the transient part goes to zero and the $\frac{f_0}{k}$ only survives.

2) $f(t) \neq$ constant → the general solution is based on what happens when the system is excited by an IMPULSE

IMPULSE

Impulse function / Dirac's delta → $\boxed{\begin{array}{ll} \delta(t-t_0) \to \infty & t = t_0 \\ \delta(t-t_0) = 0 & \forall t \neq t_0 \end{array}}$



computing the integral

$$\begin{cases} \delta(t-t_0) = 0 & \forall t \neq t_0 \quad 1) \\ \delta(t-t_0) \to \infty & t = t_0 \quad 2) \\ \int_{-\infty}^{\infty} \delta(t-t_0)\,dt = ① & \quad 3) \end{cases}$$

→ INTENSITY

Computing the integral we get 1. It is a dimensional 1.

3b) $\int_{-\infty}^{\infty} g(t)\,\delta(t-t_0)\,dt = g(t_0)$ → taking any function the integral is the value of the function in t_0

So if we apply an IMPULSE to an SDOF system, we get:

$$m\ddot{x} + c\dot{x} + kx = f(t) = \underline{\delta(t)}$$

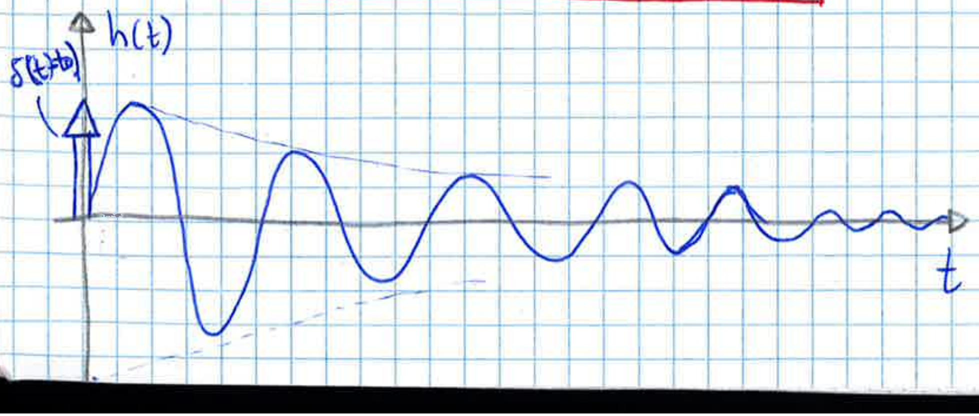
⇓ SOLUTION

$$\boxed{x(t) = h(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t} \sin\omega_d t} \quad (\xi<1)$$

IMPULSE RESPONSE FUNCTION

↓

[any particular system has got his particular h(t); by using this, we can compute any f(t)-RESPONSE]



01/10/2014 ②

The convolution integral can be used anytime to obtain the FRF of a system. The systems we deal with are always $\boxed{\text{linear}}$.

INPUT OUTPUT

$f_1(t) \longrightarrow x_1(t)$

$f_2(t) \longrightarrow x_2(t)$

⇒ given any I which generates an output and the system is LINEAR we get the following:

LINEARITY

$$\boxed{f(t) = a f_1(t) + b f_2(t) \Rightarrow x(t) = a x_1(t) + b x_2(t)}$$

→ a linear combination of INPUTS generates l.c. of the outputs

EX

$f_1(t) = F_1 \cos \Omega_1 t \Rightarrow x_1(t) = A_1 \cos(\Omega_1 t + \varphi_1)$

$f_2(t) = F_2 \sin \Omega_2 t \Rightarrow x_2(t) = A_2 \sin(\Omega_2 t + \varphi_2)$ } assuming that

if the system is linear I can sum these inputs and we get that we can CONCENTRATE ON A CONTRIBUTION AT TIME → before on Ω_1 and x_1 and then Ω_2 and x_2 ⇒ SUPERPOSITION PRINCIPLE IS VALID

↳ we can split a periodic function in a sum of harmonic functions

↓

with FOURIER SERIES

$g(t) = g(t + T_0)$, T_0 = PERIOD

$$\boxed{g(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\Omega_0 t) + b_k \sin(k\Omega_0 t)\right]}$$

where $\boxed{\Omega_0 = \frac{2\pi}{T_0}}$ FUNDAMENTAL FREQUENCY

So any periodic function can be written in that form, in the fourier series form; if the function is not periodic, if the input is not periodic, there is a trick: ⇒ the period $T_0 \to \infty$, goes to infinite, so we can imagine all the function to be periodic; we move from fourier series to fourier transform.

This expression is usually rearranged to have $\omega_n^2 = k/m$, natural frequency, and $\xi = \frac{c}{2m\omega_n}$, damping factor:

comes from outside

$$X(\Omega) = \frac{f_0/k}{1-\frac{\Omega^2}{\omega_n^2}+i2\xi\frac{\Omega}{\omega_n}} \quad ; \quad \text{saying} \quad \boxed{r = \frac{\Omega}{\omega_n}} \quad \text{we get:}$$

property of the system

$$\boxed{X(\Omega) = \frac{f_0/k}{1-r^2+i2\xi r}}$$

F
R
F

It is, as said, a complex function and we can think to write it as a modulus and a phase

$$X(\Omega) = |X|e^{i\varphi} = Ae^{i\varphi} \quad , \text{where} \quad \begin{cases} \to A = \text{MODULUS} \to \text{real} \\ \to \varphi = \text{PHASE} \end{cases}$$

FRF

$$\boxed{A = \frac{f_0/k}{\sqrt{(1-r^2)^2+(2\xi r)^2}}} \qquad \boxed{tg\varphi = \frac{Im[X]}{Re[X]} = \frac{-2\xi r}{1-r^2}}$$

So we can finally say that the output response is:

$$\underline{x(t)} = Ae^{i\varphi}\cdot e^{i\Omega t} = \underline{Ae^{i(\Omega t+\varphi)}}$$

so if the input is $\cos\Omega t$, the output will be the real part of this expression

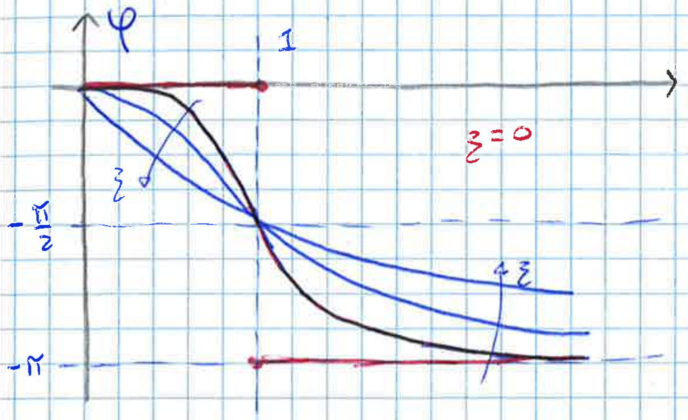
$$f(t) = f_0\cos\Omega t \longrightarrow x(t) = A\cos(\Omega t+\varphi)$$

$$f(t) = f_0\sin\Omega t \longrightarrow x(t) = A\sin(\Omega t+\varphi)$$

where amplitude A and phase φ are given by previous expressions.

So we want to move away faster from resonance zones frequency .

Plotting the PHASE



because the phase is negative, the response has got a delay respect the input force (1)

writing that $\underline{x(t)} = A e^{i(\Omega t+\varphi)} = \underline{A e^{i(\Omega t-\vartheta)}}$ where $\varphi = -\vartheta$

in this case the plotting is twisted into positive angles.

We said $X(\Omega) \overset{①}{=} FRF$; but we may say $FRF \overset{②}{=} \frac{X}{f_0}$ → it's still an FRF

we can use both definitions ;

$\boxed{FRF = \frac{X}{f_0}}$ is a RECEPTANCE

We can talk about velocity either :

even v_0 is an FRF !

$x(t) = X e^{i\Omega t}$

$\dot{x} = i\Omega X e^{i\Omega t} = v_0 e^{i\Omega t}$ where ③ $\underbrace{v_0 = i\Omega X}_{FRF}$

HARMONIC FUNCTIONS ONLY ✓

I can move from displacement to velocity by multiplying for $i\Omega$

④ $\boxed{\frac{v_0}{f_0} = FRF}$ and we call it MOBILITY

⑤ $\ddot{x} = -\Omega^2 X e^{i\Omega t} = a_0 e^{i\Omega t}$ where $\underbrace{a_0 = -\Omega^2 X}_{FRF}$

HARMONIC FUNCTIONS ONLY

I can move from displ. to acceleration by multiplying for $-\Omega^2$

[this works with harmonic forces or sum of harmonics → so pretty much works for everything.]

the equation of motion will be:

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$ stating $\underline{z = x-y}$ relative motion

$$\Rightarrow m(\ddot{y}+\ddot{z}) + c\dot{z} + kz = 0$$

$$\boxed{m\ddot{z} + c\dot{z} + kz = -m\ddot{y}}$$

$\ddot{y}$ is the acceleration of the base, which is what I want to get; assuming the input is harmonic:

$$y = y_0 e^{i\Omega t} \longrightarrow \ddot{y} = -\Omega^2 y_0 e^{i\Omega t}$$

$$\Rightarrow m\ddot{z} + c\dot{z} + kz = \underbrace{m y_0 \Omega^2}_{\text{"}f_0\text{"}} e^{i\Omega t}$$

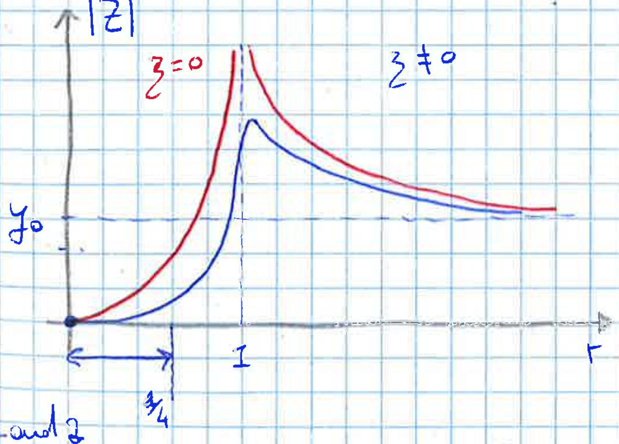
"f_0" → this quantity is a sort of harmonic function

so the response has the same form of $X(\Omega)$:

$$z = \frac{m y_0 \Omega^2 / k}{1 - r^2 + i2\xi r}$$ being $\frac{m}{k} = \frac{1}{\omega_n^2}$, we get:

$$Z(\Omega) = y_0 \frac{r^2}{1 - r^2 + i2\xi r}$$ and it can be plotted as modulus:

$$|z| = y_0 \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$



- we are interested in what happens when $r^2 \ll 1$ $(r < 1/4)$. In this region of FRF the movement of the base is at a frequency much smaller than the nat. frequency of the accelerometer. → $2\xi r$ is negligible and →

and 2

$$1 - r^2 \simeq 1 \Rightarrow |z| \simeq y_0 \frac{r^2}{1} = y_0 \frac{\Omega^2}{\omega_n^2} = \frac{1}{\omega_n^2} (\underline{\underline{y_0 \Omega^2}})$$

↳ ACCELERATION OF BASE

[I want to define an EQUIVALENT VISCOUS DAMPER so that damper produces a work not depending on the frequency:]

$W_H = W_{C_{EQ.}}$ → $a A^2 = C_E A^2 \Omega \pi$

{ a → depends on the material we're using

we'll use it in the eq. of motion ← $\boxed{C_E = \frac{a}{\pi \Omega}}$ EQUIVALENT DAMPER

$m\ddot{x} + c_E\dot{x} + kx = f_0 e^{i\Omega t}$ → $x = X e^{i\Omega t}$

$\ddot{x} = i\Omega X e^{i\Omega t} = i\Omega\, x(t)$

$m\ddot{x} + \frac{a}{\pi \cancel{\Omega}} i \cancel{\Omega} x + kx = f_0 e^{i\Omega t}$

$m\ddot{x} + kx + i\frac{a}{\pi}x = f_0 e^{i\Omega t}$ → $m\ddot{x} + kx\left(1 + \boxed{\frac{a}{k\pi}}\, i\right) = f_0 e^{i\Omega t}$

↳ LOSS FACTOR β or η

{ a depends on the material (small in steels, big in viscoelastic materials)
{ k " " " "

so, finally we get:

$$\boxed{m\ddot{x} + k(1 + i\beta)x = f_0 e^{i\Omega t}}$$ → hysteretical damping model

it has got a "strange spring" formed by a normal spring and another which is responsible of dissipating energy

$\boxed{k(1 + i\beta)}$ → COMPLEX STIFFNESS

→ Re the normal spring

↳ Im dissipating spring

The response will be $x = X e^{i\Omega t}$ with

$$\boxed{X = \frac{f_0}{k - m\Omega^2 + ik\beta} = \frac{f_0/k}{1 - r^2 + i\beta}}$$

→ the imaginary part does not depend on frequency. And that's all.

So the hysteretical damping has this model: 06/10/2014 ③

$$\boxed{m\ddot{x} + k(1+i\eta)x = f_0 e^{i\Omega t}}$$ with $\eta (= \beta)$ → LOSS FACTOR

$$\eta = 2\xi \frac{\Omega}{\omega_n} \Rightarrow \boxed{\eta = 2\xi}$$

[This model is only valid if the input is HARMONIC → we ~~[illegible]~~ must have a CYCLE]

The response was:

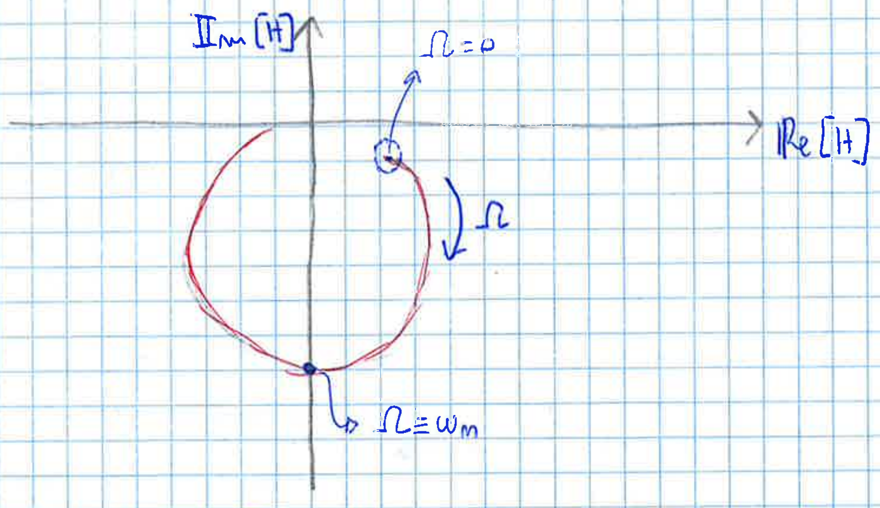
$$x = X e^{i\Omega t} \quad \text{and} \quad \boxed{X = \frac{f_0/k}{1-r^2+i\eta}}$$ → it's one of the FRF of the system

for example we can have as FRF:

$$\frac{X}{\underbrace{f_0/k}_{\text{STATIC DEFORMATION}}} = \frac{1-r^2-i\eta}{(1-r^2)^2+\eta^2} = H(\Omega)$$ → capital H indicates any FRF!

$$\frac{1}{a+ib} \cdot \frac{(a-ib)}{(a-ib)} = \frac{a-ib}{a^2+b^2}$$

We can plot the Re and Im parts of the FRF and we get NYQUIST PLOT

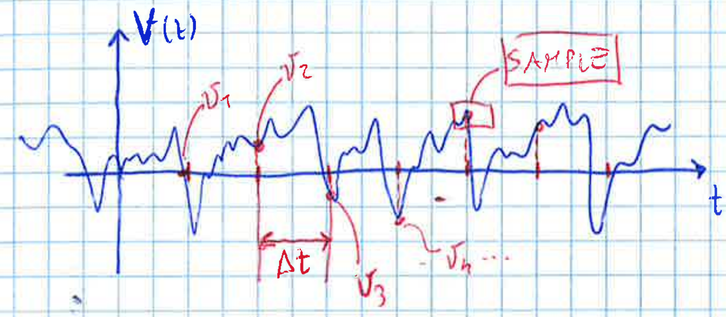


- with $r=0$, Re is +, Im is − ⇒ we start from that point, but ~~[illegible]~~ that point we don't have an hysteretical cycle → we never get there.
- with $r=1$ ⇒ $\Omega = \omega_n$, resonance ⇒ Re $=0$

→ we get a circle and we can prove it is a circle.

ANALOGUE-TO-DIGITAL CONVERSION - ADC

It happens when we move from measurement of the system (phenomena) to the value we get on the display. For example an accelerometer → we get a voltage from the accelerometer through the cable:



we have a continuous function on the cable → but we're no more able or interested in elaborating a continuous function

we want to move to a list of numbers in a row, converting the analog signal to a digital one

↓

in the CONVERTER we have a CLOCK, which is very accurate and divides the time axis in many short Δt elements → SAMPLING PERIOD

we move from a continuous time to a sampled time, a discretized one.

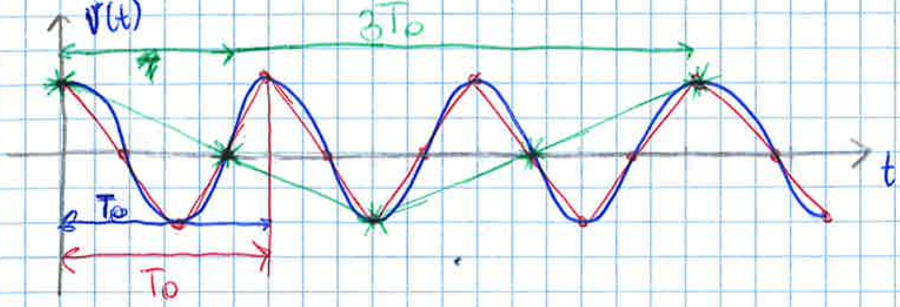
the SAMPLING PERIOD is given in multiples of seconds → can be short or long (1/1000 of a second or just minutes)

The SAMPLING FREQUENCY is $\boxed{f_s = \frac{1}{\Delta t}}$ → [Hz]
→ [SPS] (samples per seconds)

Any ~~points~~ values of the function in the $k\Delta t$ ($k = 1, \dots, n$) points it's a SAMPLE. We move from the continuous function to a digital one and we completely lose any information between consecutive samples

ALIASING

Considering an harmonic function; the voltage:



$f_0 = 1/T_0 = f_{max} = 50 Hz$

I take 4 samples per wave, so I have $f_S = 4 f_0$ by choose

So the sampling period is $\Delta t = \frac{1}{f_S} = \frac{1}{4f_0} = \frac{1}{4} T_0$

the red dots are samples selected according to Shannon's theorem. If I can read both the amplitude and the frequency of my period... I can define, by linear interpolation a function (red) which gives to me frequency and amplitude. I just don't know the cosine function, but I can correctly measure the frequency, because $\boxed{T_0' = T_0}$

↳ I loose the real shape of the function, but I get ~~all~~ both the freq. and ampl.

What happens if I choose an $f_S < 2 f_{MAX}$? I choose to have $\frac{1}{4} + \frac{1}{2}$ of a wave as ~~a~~ sampling period.

⇒ we get a period $3T_0$ → three times a normal period → the digital green signal generated does not represent properly the real function

⇓

ALIASING!! → - I completely miss the frequency:

$$f_{ALIAS} = \frac{1}{3T_0} = \frac{f_0}{3}$$

and so we are not able to read it.

- we only have an ~~useful~~ USELESS list of points with a completely wrong frequency

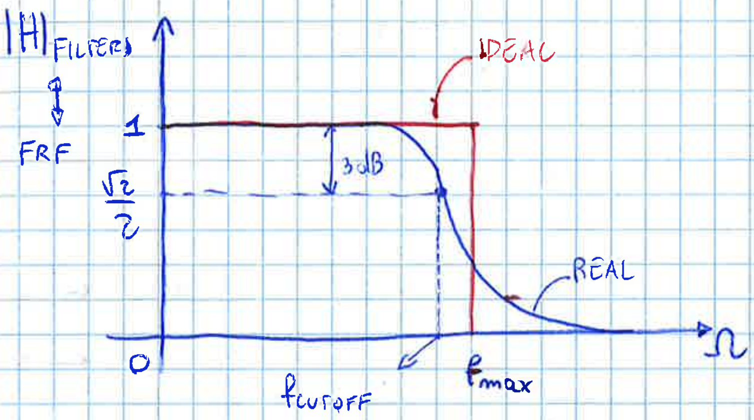
↳ AVOID ALIASING!

LOW-PASS FILTER

It allows the lower level frequencies to pass through without modification; "low" in the frequency domain. The upper fs are theoretically erased

ANTI-ALIASING FILTERS

They cancel high frequencies, so they are analogue-low-pass filters



- So the FRF of an ideal filter, should be like ~~that~~ the red one;
- In practice we get the blue function

So we buy a filter with an appropriate f_{max} (1000 Hz) → in reality we buy a filter with a smooth FRF, not sharped like the red one.

⇒ to characterize the filter, we look at the frequency where FRF has decreased to $\boxed{3dB = \frac{\sqrt{2}}{2} = 0,7}$ than the maximum

↓

so we'll have a CUTOFF FREQUENCY f_{CUTOFF} in practice and not an f_{max}; this implicates a correction in Shannon's Theorem:

$$\boxed{f_s \geq 2{,}56\, f_{CUTOFF}}$$

anti-aliasing filter

⇒ we set the sampling frequency f_s and the ADC automatically introduces a filter so the $f_{CUTOFF} = \frac{f_s}{2{,}56}$

EX

$f_s = 1024$ Hz ⇒ $f_{CT} = 400$ Hz { so we can f.e. read ~~the~~ frequencies lower than 400 Hz, automatically

↓

not 1024 Hz, but 400 Hz!

E → amplitude extention of the signal (±10V)

NBIT → number of bits, depends on the ADC, is an hardware parameter, fixed.
The most diffused value is NBIT = 16, but the best is NBIT = 24

E is usually fixed and depends ~~[illegible]~~ on the hardware we are buying too; is quite common to have E = ±10V => E = 20V → remember!

E = ±1V => E = 2V

E = ±5V => E = 10V

} most diffused amplitude extention

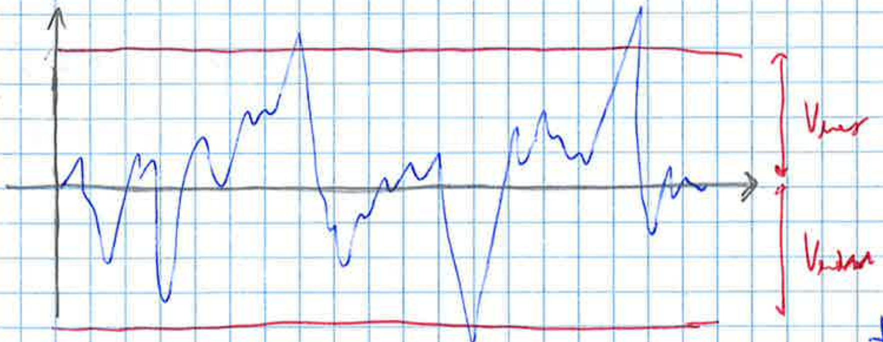
f.e. → E=±10V, NBIT=24 => $\Delta x = \frac{20}{2^{24}-1} = 1{,}19 \cdot 10^{-6}$ V = 0,001 mV
↓
very high resolution

- IF $|x(t)| > X_{MAX}$

If we have, at a certain time, a value which is over the E range → quite tipically we see a RED LIGHT blinking; we get an error called CLIPPING or OVERLOAD

AVOID IT! ← in this case we must stop everything and start measurement again

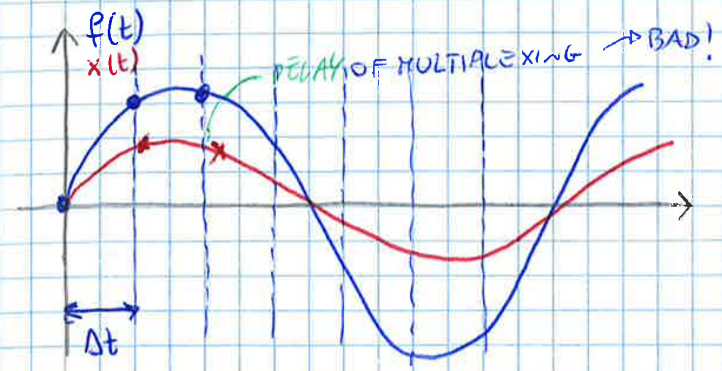
↳ because the overloaded informations collected are false



if I have a signal (random) how can I be sure I'll get values within the limit? I can not. I have to be sure to have large enough limits.

- We just have to modify inputs to stand within the limits. Smarter ADC have a button to adjust the range of measurement
↓
VERY IMPORTANT

If we want to measure many signals at the same time, f.e. :



1 CHANNELS
2
4
8
16

we'll have systems with many channels of measuring; each channel is devoted to measure one of the signals on the field.

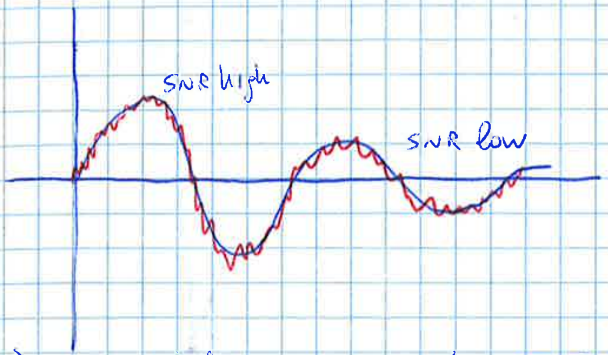
-We fix a sampling period Δt and we want to measure f(t) and x(t). If the system has got MULTIPLEXING function, it measures f(t) and, a little bit late x(t), and so on → we got ~~[illegible]~~ DELAYS, so multiplexing is NOT good. Also, this delay varies between the channels, it's not constant

↓

we loose infos about phase ⇐ MULTIPLEXING IS NOT GOOD FOR VIBRATIONS & ACOUSTIC

$\varphi = \omega t$

If we have some NOISE in the response we get something like this:



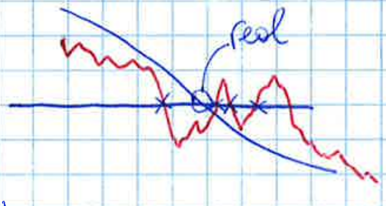
So we can define an SNR,

SIGNAL TO NOISE RATIO

- the real measurements is the red one

in our analysis we have to DELAY the second part, the last part of the signal, because there is an high NOISE, we have much errors there → bad analysis. It is better also to consider a very low ξ (damping ratio), because otherwise we have fast decay and we don't want it (delay the 2nd part, remember, SNR low).

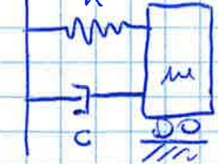
When we have a zero we are not sure where it is:



because of the noise we get many zeros at once

How to select the t_1 point: it's better to select t_1 as max (or min) because the influence of the noise is not so big (SNR high). If we select a point different from max, the importance of the noise compared to the true signal is high there. → SELECT THE MAXIMUM TO APPLY THIS METHOD.

NUMERICAL EXAMPLE



$m = 100\,kg$
$T_d = 2s$
$X_2 = \frac{X_1}{16}$

find K and C

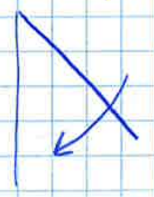
We suppose to know the m; we are only able to extract two of the 3 parameters, so we must have almost one.

We can write $I_O = I_G + ml^2$ (Huygens-Steiner theorem)

[for small oscillations we can assume $\sin\vartheta \simeq \vartheta$]

$\Rightarrow I_O \ddot{\vartheta} + C_0 \,\mathrm{sign}\, \dot{\vartheta} + mgl\vartheta = 0$ PIECEWISE LINEAR $\left\{ \begin{array}{l} \text{lineare} \\ \text{a pezzi} \end{array} \right.$

in $\vartheta(0) = \vartheta_0 > 0 \rightarrow \dot{\vartheta}_0 = 0$



1) FIRST INTERVAL $0 < t < t_1 \rightarrow$ as long as $\dot{\vartheta} < 0$

here $C_0 \,\mathrm{sign}\, \dot{\vartheta} = -C_0$

and we get → $\boxed{I_O \ddot{\vartheta} + mgl\vartheta = C_0}$ in the first period we get a linear function

Response → $\boxed{\vartheta(t) = \frac{C_0}{mgl} + a\cos\omega_n t + b\sin\omega_n t}$ ↘ It only holds when $\dot{\vartheta} < 0$

1) I.C. $\vartheta_0 = \frac{C_0}{mgl} + a \Rightarrow a = \vartheta_0 - \frac{C_0}{mgl}$

2) I.C. $\dot{\vartheta} = \omega_n(-a\sin\omega_n t + b\cos\omega_n t)$

$\dot{\vartheta}_0 = b\omega_n = 0 \Rightarrow b = 0$

$$\Rightarrow \boxed{\vartheta_I(t) = \frac{C_0}{mgl} + \left(\vartheta_0 - \frac{C_0}{mgl}\right)\cos\omega_n t}$$

first solution when $\vartheta_0 > 0$ and $\dot{\vartheta} < 0$

in t_1 $\dot{\vartheta}_I = 0$ so we have to compute this value:

$\dot{\vartheta}_I(t_1) = -a\omega_n \sin\omega_n t_1 = 0 \Rightarrow \omega_n t_1 = \pi \Rightarrow t_1 = \frac{\pi}{\omega_n} = \frac{T_n}{2}$ { t_1 is half the natural period

$$\boxed{\omega_n = \sqrt{\frac{mgl}{I_O}}}$$

{ Square root of zero order derivative divided by ~~second~~ order derivative coefficient

In the viscous decay model was not linear but exponential; here is linear, in the Coulomb friction model the decay is linear.

We have at a point that the restoring force (due to the weight) cannot overcome the static force (coulomb friction) and the motion stops abruptly in the ⊛ zone.

The Coulomb Friction is very common in practice but difficult to describe because the equation is not linear.

There is a reduction of $-\frac{4C_0}{mgl}$ PER CYCLE (we started with ϑ_0)

This response is for SMALL OSCILLATION. But we can compare also for LARGE OSCILLATION → we'll have to integrate the eq. of motion, with some approximation methods (Runge-Kutta, ode45)

ODE (Ordinary Differential Equation) second order:

$$I_0\ddot{\vartheta} + C_0\,\mathrm{sign}\,\dot{\vartheta} + mgl\,\underset{\overset{\downarrow}{\vartheta}}{\sin\vartheta} = 0$$

we can put $\sin\vartheta$ or ϑ so we have 2 ODEs. Using Runge-Kutta routine

$$y_1 = \vartheta \qquad \dot{y}_1 = f(y_1, y_2) = \dot{\vartheta} = y_2 \rightarrow \boxed{\dot{y}_1 = y_2}$$ → passare da 1 eq del II e 2 del I

$$y_2 = \dot{\vartheta} \qquad \dot{y}_2 = g(y_1, y_2)$$

$$\boxed{\dot{y}_2 = \ddot{\vartheta} = -\frac{C_0}{I_0}\,\mathrm{sign}\,y_2 - \frac{mgl}{I_0}\sin y_1 = g(y_1, y_2)}$$

So I have 2 eq. of the first order from one of the second order.

Here the logarithmic decrement method doesn't work → only for viscous damping.

We have to use another method:

↳

In conclusion we have 2 unknown coeffs multiplied ~~too~~ :

$$\underset{1\times 2}{\left[\frac{I_0}{mg\ell} \quad \frac{C_0}{mg\ell}\right]} \underset{2\times N}{[A]} = \underset{1\times N}{[B]}$$

→ so we can extract the first matrix with LEAST SQUARES SOLUTION by PSEUDOINVERSE $\boxed{pinv(A)}$

$[A]\, pinv[A] = I$; so we post-multiply :

$$\left[\frac{I_0}{mg\ell} \quad \frac{C_0}{mg\ell}\right] = [B]\, pinv[A]$$

and we get the first matrix

Pseudoinverse is based on SINGULAR VALUE DECOMPOSITION (SVD) to get displacement we'll have another trend → $V_0 t$ → $\boxed{\text{>>DETREND (x)}}$

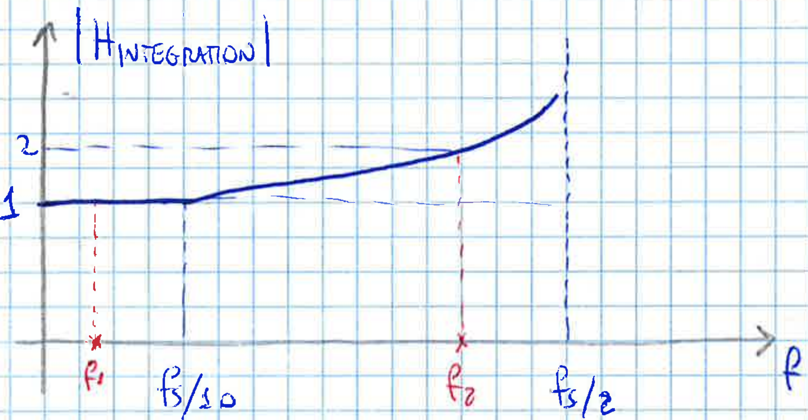
We get a digital series of values, not a continuous; so we have:

$a(t) \Rightarrow a_1, a_2, \dots a_M$ SAMPLES

$\Delta t = \frac{1}{f_s}$ SAMPLING PERIOD

SIMPSON'S RULE

To integrate a function which is not continuous we ~~use~~ this rule → to integrate a series of number:



- any integration rule has got an FRF; we would like to have a perfect integration for any frequency, up to $f_{MAX} = f_s/2$ (SHANNON)

→ we want an integrator which is capable to integrate in the same way at any f_i

but what happens is that we have a "1" value and then an increasing function:

if we have $f < f_s/10$ all the ~~output~~ f_1 will have the same output; otherwise we have amplified values (for $f > f_s/10$)

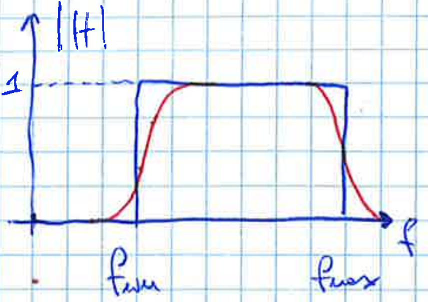
because:

f.e. → $a(f_1) = 10\ m/s^2 \Rightarrow v(f_1) = \frac{10\ m/s^2}{(\cdot\ \omega_1)} \cdot 1$ → $\begin{cases} \ddot{x} = -A\omega^2 \cos\omega t \\ \dot{x} = -A\omega \sin \omega t \\ x = A \cos \omega t \end{cases}$

$v(f_2) = \frac{10\ m/s^2}{\omega_2} \cdot 2$

↳ the output will be incorrectly amplificated by the "2" factor of the Simp. Rule

BAND PASS



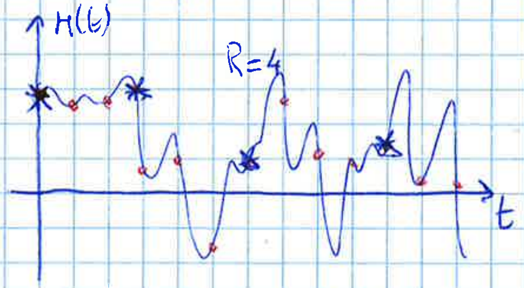
I give to the filter a signal $x(t)$ and I get an output which can be shaped with LP, HP, BP filters.

In matlab I can choose between filters just by setting f_{min} and f_{max}! The only care we have to take is to choose the ORDER of the filter

↳ HIGH ORDER → almost theoretical shape but no-sense results if too high

We will also find in MatLab a command like this:

```
>> DECIMATE (x, R);
```



R = n → it will take 1 sample out of n.

R = 10 → " " " " " " " 10

this is useful when Δt is too short and we don't want to get too many samples. Doing this we increase Δt and so the new $f_s = \frac{1}{R\Delta t}$ will be $\frac{1}{R}$ the original f_s, so we may introduce ALIASING, because it is decreasing! ⇒ Decimate introduces also a LP filter, to be sure of not getting aliasing.

so with these three expression we can build f(t) FOURIER SERIES, which can be written also in this form:

$$\boxed{f(t) = a_0 + \sum_{k=1}^{\infty}\left[C_k \sin(k\Omega_0 t + \varphi_k)\right]}$$

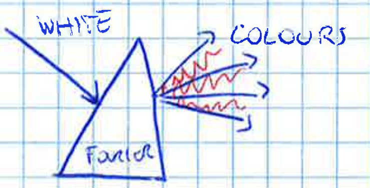
less used, but it shows that this is an harmonic, with AMPLITUDE C_k,

$C_k = \sqrt{a_k^2+b_k^2}$, and PHASE φ_k

The C_k gives the WEIGHT of each harmonic within the signal: if we measure the AC current of Politecnico, all the C_k are zero but the one of 50 Hz frequency.

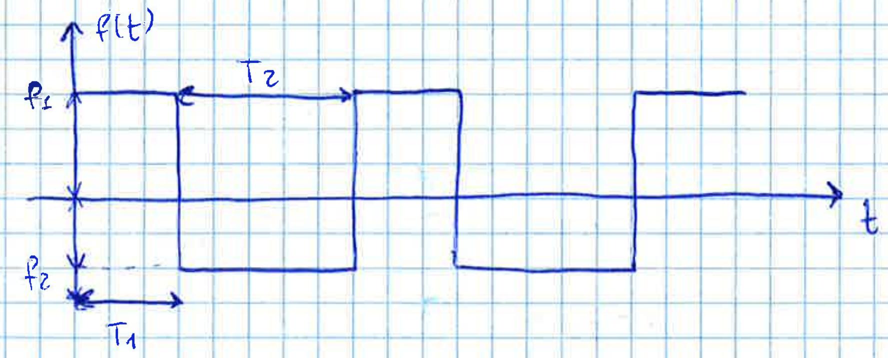
LIKE IN A PRISM

(THE DARK SIDE OF THE MOON)



→ we have one function and Fourier Series decompose it in its harmonic components

Example



this function is periodic

$T_0 = T_1 + T_2$

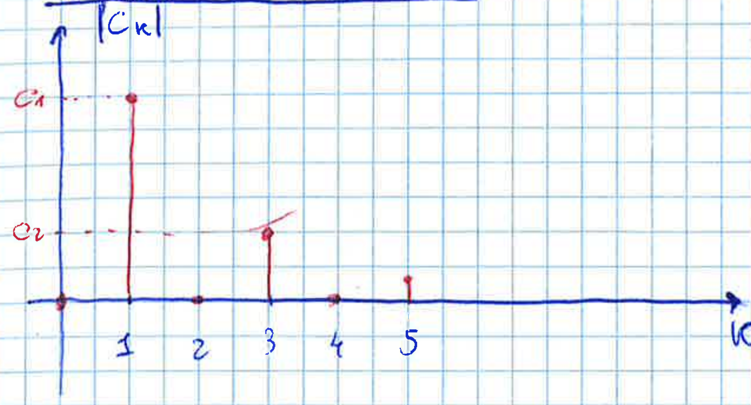
we want to decompose it:

(the Area)

- $a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt = \boxed{\frac{1}{T_0}\left(T_1 f_1 - T_2 f_2\right)}$

$a_k = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(k\Omega_0 t)\,dt = \frac{2}{T_0}\left[\int_0^{T_1} f_1 \cos(k\Omega_0 t)\,dt + \int_{T_1}^{T_0} f_2 \cos(k\Omega_0 t)\,dt\right] =$

SPECTRUM OF THE SIGNAL



every spectral line is separated by the same distance, a certain frequency → $1 \cdot \Omega_0 = \Omega_0 = \Delta\Omega$

→ FREQUENCY RESOLUTION

in this case SQUARE WAVE we get $a_0 = C_0 = 0$, so the first value is null; c_1 is $b_1 = \frac{4F_0}{\pi}$

C_2 is zero, C_3 is three times smaller than $C_1 \Rightarrow C_3 = b_3 = \frac{4F_0}{3\pi}$;

We get a diagram called SPECTRUM, FREQUENCY REPRESENTATION, where each line is a SPECTRAL LINE.

In this case the spectral representation is infinite, we have infinite values of C_k. In reality we'll have spectra which stops at a certain point, a certain maximum frequency. We do not represent all the spectrum of the signal.

EXPONENTIAL REPRESENTATION

According to Euler's formula:

$$\begin{cases} e^{ix} = \cos x + i \operatorname{sen} x \\ e^{-ix} = \cos x - i \operatorname{sen} x \end{cases}$$

$$\boxed{\cos x = \left(e^{ix} + e^{-ix}\right)/2} \quad , \quad \boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i} = -i \frac{e^{ix} - e^{-ix}}{2}}$$

with these two expression, I modify the original one:

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k \frac{e^{ik\Omega_0 t} + e^{-ik\Omega_0 t}}{2} - i b_k \frac{e^{ik\Omega_0 t} - e^{-ik\Omega_0 t}}{2} \right] =$$

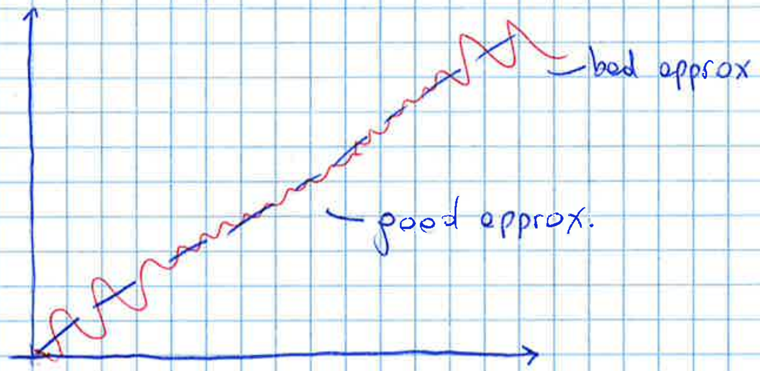
$$= a_0 + \sum_{k=1}^{\infty} \left[\frac{a_k - i b_k}{2} e^{ik\Omega_0 t} + \frac{a_k + i b_k}{2} e^{-ik\Omega_0 t} \right]$$

At a specific k we get → $\Omega_k \leftrightarrow f_k$ SPECTRAL LINE

This representation is the basis of numerical computation FFT - Fast Fourier Transform. We are interested in the SPECTRAL LINES

GIBB'S PHENOMENON → the approximation is good only in the middle ; at the edges we get high oscillation around the real value :

so we need a large number of harmonics to get good approximation



Shannon's theorem does not guarantee the time domain, only the freq. one.

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ik\Omega_0 t} \qquad \text{with} \qquad f_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t) e^{-ik\Omega t} dt$$

FOURIER TRANSFORM

The next step is to move from periodic to generic functions. Fourier series works with CONTINUOUS FUNCTION.

- If the function given is not periodic, we can ~~s~~ imagine that the period (T_0) GOES TO INFINITE

$$\boxed{T_0 \to \infty}$$

in this case f_k goes to zero, so we're not able to use last expression found. We have to rearrange it.

The Fourier Transform is sometimes written in a more compact form:

DIRECT T. $\boxed{F(\Omega) = \mathcal{F}[f(t)]}$ $\boxed{f(t) = \mathcal{F}^{-1}[F(\Omega)]}$ INVERSE T.

where $\mathcal{F}$ is an OPERATOR which indicates we have to compute that integral.

→ WE TRANSFORM THE DOMAIN

PROPERTIES

- Fourier Transform is a LINEAR OPERATOR, which is very useful;

Let's demonstrate that:

$$x_1(t) \Rightarrow X_1(\Omega)$$

[capital letters are for Ω-DOMAIN]

displ.

$$x_2(t) \Rightarrow X_2(\Omega)$$

e.g. → $x(t) = a_1 x_1(t) + a_2 x_2(t) \Rightarrow X(\Omega) = \int_{-\infty}^{\infty} (a_1 x_1 + a_2 x_2)\, e^{-i\Omega t}\, dt =$

$$= a_1 \underbrace{\int_{-\infty}^{\infty} x_1 e^{-i\Omega t} dt}_{\mathcal{F}[x_1] = X_1(\Omega)} + a_2 \underbrace{\int_{-\infty}^{\infty} x_2 e^{-i\Omega t} dt}_{\mathcal{F}[x_2] = X_2(\Omega)} = a_1 X_1(\Omega) + a_2 X_2(\Omega)$$ c.v.d.

- f(t) is real → in t-domain we always measure real signals
F(Ω) can be complex → in Ω-domain we get complex signals often.
} $f(t) \in \mathbb{R}$, $F(\Omega) \in \mathbb{C}$

EVEN RELATIONS

any real function f(t) can be divided in two parts: even & odd;

$$f(t) = e(t) + o(t)$$
$$f(-t) = e(t) - o(t)$$

$\Rightarrow$

$2e = f(t) + f(-t)$ → sum

$2o = f(t) - f(-t)$ → difference

$$F(\Omega) = \int_{-\infty}^{\infty} (e+o)\underbrace{(\cos \Omega t - i \sin \Omega t)}_{e^{-i\Omega t}}\, dt =$$ → we have this for definition of F(Ω)

Example

$x(t) = A\cos \Omega_0 t \qquad \Omega_0 = 2\pi f_0 \qquad f_0 = 50Hz$

Fourier Tranform shoud tell us the function contains only f_0; but it does not respect the condition of existence:

$$\int_{-\infty}^{\infty} |f(t)|\,dt = \infty \rightarrow NO!$$

$\boxed{\text{We can't use F.T. for harmonics.}}$

IMPULSE

In these cases we can use IMPULSE:

because $\delta \neq 0$ only in t_0

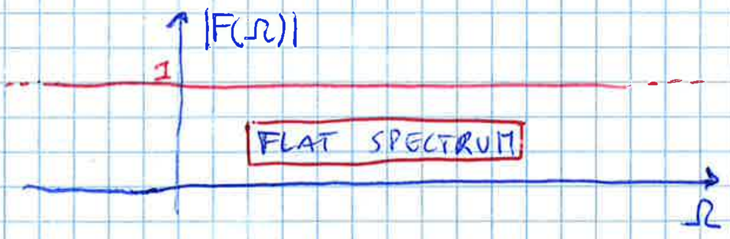
$$\delta(t-t_0) \xrightarrow{+\infty\ \forall t=t_0} \quad \underline{F(\Omega)} = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-i\Omega t}\, dt = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-i\Omega t_0}\, dt =$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

$$= \underline{e^{-i\Omega t_0}}$$, because integral is equal to 1

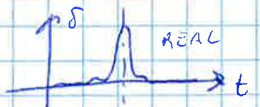
$\boxed{|F(\Omega)| = \cos^2 \Omega t_0 + \sin^2 \Omega t_0 = 1}$ → the modulus is equal to 1.

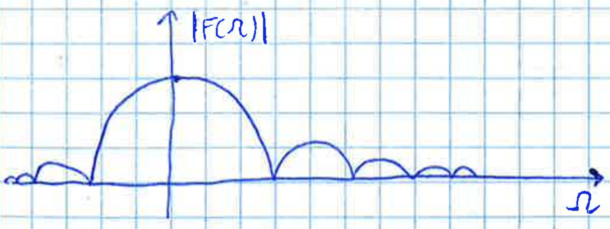
If we plot the spectrum we get 1 everywhere!



every frequency has the same importance

Of course this is not true; the real spectre will not have a flat behaviour but one like this:





[Minor the duration of the impact, flatter the spectrum]

Now we come back to the time domain:

$$\delta(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{e^{-i\Omega t_0}}_{F(\Omega)}\, e^{i\Omega t}\, d\Omega$$

$$= -\frac{x_0}{\pi 2} i \pi \int_{-\infty}^{\infty} \left[e^{-i(\Omega-\Omega_0)t} - e^{-i(\Omega+\Omega_0)t} \right] dt =$$

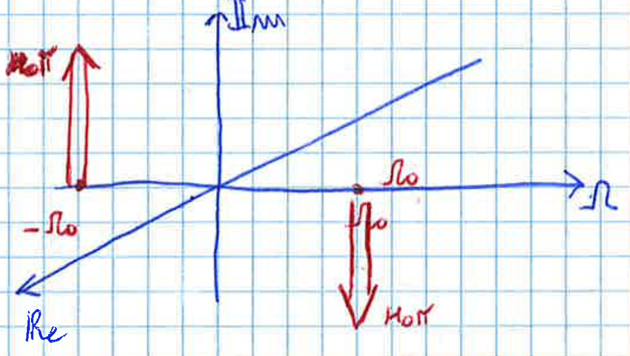
relaying on the previous $\delta(t-t_0)$ expression and multiplying and dividing for π we get:

$$= -i x_0 \pi \, \delta(\Omega-\Omega_0) + i x_0 \pi \, \delta(\Omega+\Omega_0)$$

The starting function is a sine → ODD function; the transform should be ODD and it is! It's imaginary and ODD.

$$\boxed{X(\Omega) = -i x_0 \pi \, \delta(\Omega-\Omega_0) + i x_0 \pi \, \delta(\Omega+\Omega_0)}$$

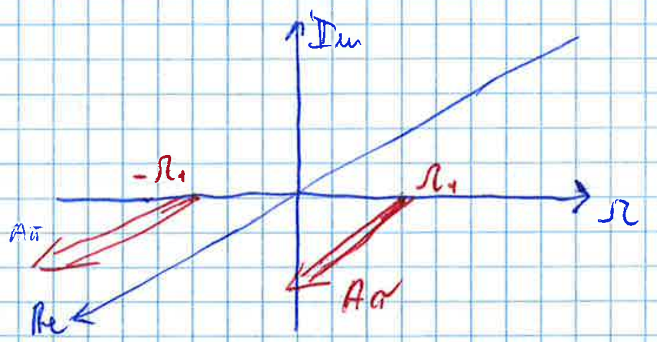
With F.T. tool we introduce NEGATIVE FREQUENCIES → they're not real, but mathematically they are "real", we can have some, they exist.



at$(+\Omega_0)$ we have an impulse there, with negative $x_0\pi$ amplitude

at $-\Omega_0$ viceverse and we're passing to a negative freq.

If we have $x(t) = A \cos \Omega_1 t$ we get two positive impulses at frequencies of the cosinusoid function (even function)



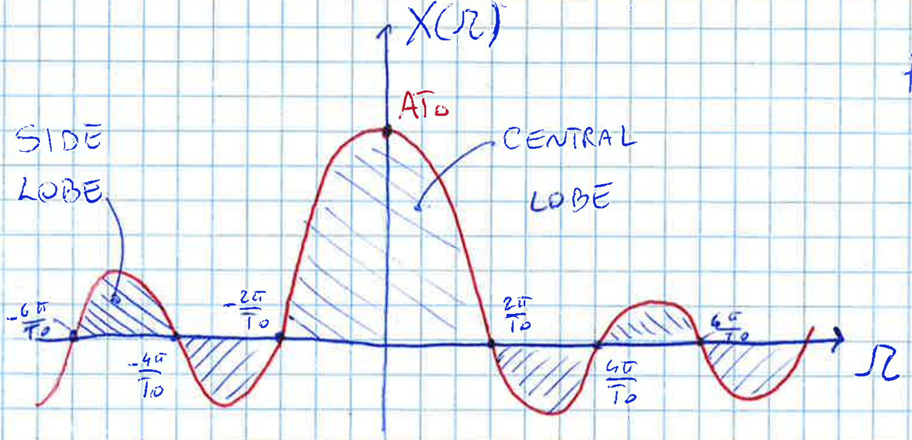
$$\int_{-T_0/2}^{T_0/2} \sin \Omega t \, dt = 0 \rightarrow \text{because Sine is ODD!}$$

$$\Rightarrow X(\Omega) = A \int_{-T_0/2}^{T_0/2} \cos \Omega t \, dt = A \frac{1}{\Omega} \sin \Omega t \Big|_{-T_0/2}^{T_0/2}$$

$$X(\Omega) = \frac{A}{\Omega}\left[\sin\left(\frac{\Omega T_0}{2}\right) - \sin\left(-\frac{\Omega T_0}{2}\right)\right] = \frac{A}{\Omega} 2 \sin\left(\frac{\Omega T_0}{2}\right) \rightarrow \text{because } \sin(\alpha) = -\sin(-\alpha)$$

$$X(\Omega) = \frac{2A T_0}{\Omega T_0} \sin\left(\frac{\Omega T_0}{2}\right) \Rightarrow \boxed{X(\Omega) = A T_0 \frac{\sin(\Omega T_0/2)}{(\Omega T_0/2)}}$$

The frequency domain representation is this one:



the starting function is even and also the X(Ω) because ratio of 2 odd functions

- When $\Omega = 0$, with De L'Hospital, I get $X(\Omega) = AT_0$
- the zeros are where $\sin(\Omega T_0/2) = 0 \Rightarrow \frac{\Omega T_0}{2} = \pm k\pi \Rightarrow \Omega = \pm \frac{2k\pi}{T_0}$

In theory the frequency spectrum is unlimited; but the main contribution comes from CENTRAL LOBE and ~~SOME~~ SIDE LOBES; but they're not negligible

- If T_0 is very very long, $T_0 \rightarrow \infty$ what we get? → A constant function!

The LOBES shrink ~~to~~ zero, we zip everything there → SIDE ones make zero, CENTRAL one sum up and make an impulse in zero → F(Ω) of a const. value.

IT'S A CHECK!

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega)\cdot\left[\int_{-\infty}^{\infty} x\, e^{-i\Omega t}\, dt\right]^{*} d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(\Omega)\cdot X^{*}(\Omega)}\, d\Omega$$

↳ it's the modulus squared

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2\, d\Omega}$$

PARSEVAL'S THEOREM

energy in the time domain

energy in the frequency domain

Being $\Omega = 2\pi f$ we can write:

$$\boxed{\int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{-\infty}^{+\infty} |X(2\pi f)|^2\, df}$$

Parseval's Theorem is useful. Often softwares don't respect Parseval's Theorem (up To Matlab 2010)

↳ it is a way to check if the Fourier Transform computed is ok.

F(Ω) OF CONVOLUTION INTEGRAL

$$a(t) = \int_{-\infty}^{+\infty} b(\tau)\, c(t-\tau)\, d\tau$$ → generic form where

b → impulse response

b → force on the system

but in this case a, b, c are generic functions

MATLAB

>> a = conv(b,c); → the solution is not obtained by numerical methods: Matlab does not compute the integral, but it moves b and c to freq domain, obtaining B(Ω) and C(Ω) and then does B(Ω)·C(Ω). And then does $\mathcal{F}^{-1}$ and gets a(t).

FRF IS F.T. OF AN IMPULSE RESP. FUNCTION

29/10/2014 ⑨

We said that FRF(Ω) is the ratio between output and input, depending both on the frequency:

$$\boxed{FRF(\Omega) = \frac{O(\Omega)}{I(\Omega)}}$$

infact

$$\begin{cases} m\ddot{x}+c\dot{x}+kx = f_0 e^{i\Omega t} \\ x = X e^{i\Omega t} \\ \Rightarrow FRF = \dfrac{X(\Omega)}{f_0} \overset{\text{or}}{=} \dfrac{X(\Omega)}{f(\Omega)} \end{cases}$$

the FRF explains how the output depends on the freq.

$\downarrow$

In this definition, THE INPUT HAS TO BE HARMONIC → so has ~~to~~ be the output.

we said for the convolution integral:

$$H(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau \quad , \quad \text{where}$$

$\downarrow$ OUTPUT

$f(\tau)$ → FORCE (input)

$h(t-\tau)$ → IMPULSE RESPONCE FUNCTION

Now let's apply the Fourier Transform to the conv. integral:

$$X(\Omega) = F(\Omega)\cdot H(\Omega)$$ → because all we said previously;

$$H(\Omega) = \mathfrak{F}[h(t)] = \frac{X(\Omega)}{F(\Omega)} \equiv FRF(\Omega)$$

⇒ So the capital H is given by the ratio ~~of~~ ~~a~~ of a F.T. output over the F.T. of an input ⇒ [the F.T. of the impulse response function is exactly the FRF]

$$\Rightarrow \quad \boxed{\mathfrak{F}[h(t)] \equiv FRF(\Omega)}$$

{ infact we often indicate an FRF with H

So if I am in the position to measure FRF, I can get the Fourier Transform of that function I am measuring. If I apply the inverse F.T. on FRF, I get $h(t)$:

$$\boxed{\mathfrak{F}^{-1}[FRF] = h(t)}$$

$F_k = F(k\Omega_0) \rightarrow$ SPECTRAL LINE, gives the weight of each harmonic

$$\Omega_0 = \frac{2\pi}{T} = \frac{2\pi}{N\Delta t}$$

the generic time will be $t = m\Delta t$, with $0 \le m \le N \rightarrow 0, 1, 2, \dots, m, \dots N$

the variation of time dt will be Δt:

$$F_k = \frac{1}{N\cancel{\Delta t}} \sum_{m=0}^{N-1} f_m e^{-ik\frac{2\pi}{N\cancel{\Delta t}} m\cancel{\Delta t}} \cancel{\Delta t}$$

, because $f(t) = f_m = f(m\Delta t)$

simplifying, we get:

⇒ (DFT)

$$\boxed{F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_m e^{-ik\frac{2\pi m}{N}}}$$

PRACTICAL DEFINITION OF THE FOURIER SERIES

DISCRETE FOURIER TRANSFORM

The frequency is hidden in the k, because $\Omega_k = k\Omega_0 = k\frac{2\pi}{T}$ and this is where the time appears → is not written in the definition

$K = 0, \pm 1, \pm 2, \dots$ → theoretically goes to $\pm\infty$; but we have a limited T.

- let's compute the element $F_{k'}$, where $k' = k + N$

↓

[what happens if we increase k by a number larger than N:]

$$F_{k'} = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-i2\pi(k+N)n/N} = \frac{1}{N} \sum f_m e^{-i2\pi k\frac{m}{N}} \cdot e^{-i2\pi\frac{\cancel{N}m}{\cancel{N}}}$$

the last exponential function → $e^{-i2\pi m} = \cos 2\pi m - i\overset{0}{\cancel{\sin 2\pi m}} =$ ①

$$\Rightarrow F_{k'} = F_{k+N} = \frac{1}{N} \sum f_n e^{-i2\pi kn/N} = \underline{F_k}$$

it is useless to compute many F_k than the samples we have, because they repeat themselves

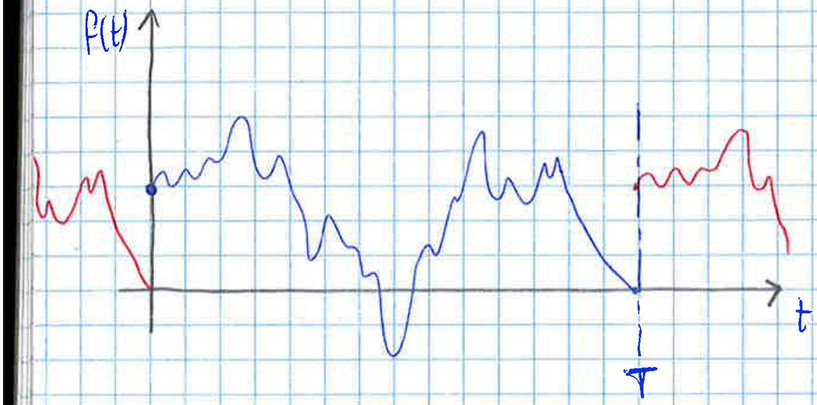
We write, for the FOURIER SERIES:

$$f(t) = \sum_{k=-\infty}^{+\infty} F_k \, e^{ik\Omega_0 t} \qquad F_k = \frac{1}{T}\int_0^T f(t)\, e^{-ik\Omega_0 t}\, dt$$

↳ DFT/FFT

LEAKAGE

This is applicable only for periodic signals; but in labs is quite unlikely to have that kind of function. So how can we apply this tool on non-periodic signals? We introduce an error on the computation:



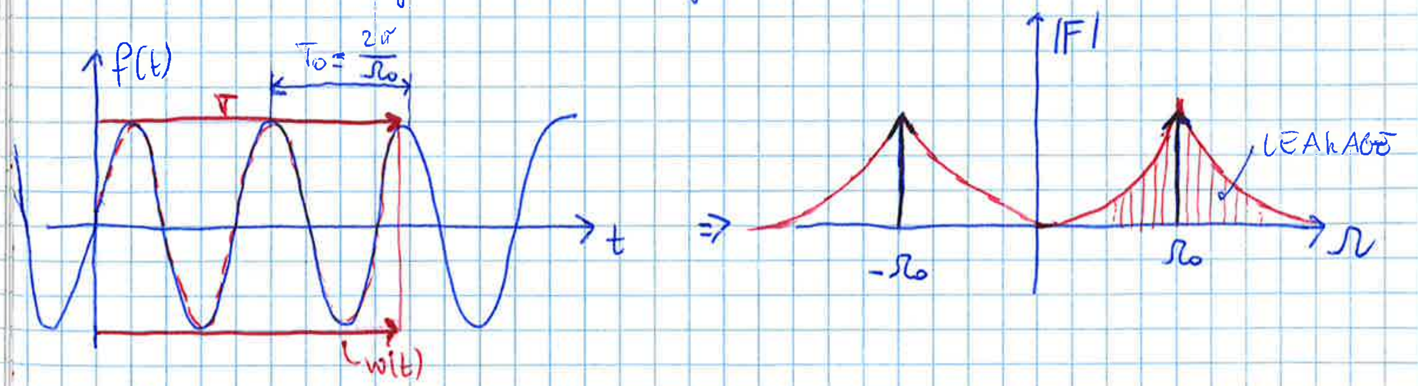
We suppose f(t) replicates after a period T. On the border we don't bother, because of sampling period Δt.

⇓

INTRODUCING AN ERROR, because of this supposition

This error is called LEAKAGE, introduced by passing from time to freq. domain.

It is visible considering an harmonic function:



we already know that F.T. of an harmonic contains only two impulses at Ω_0 and $-\Omega_0$. But if we limit the time duration of the measurement by T, we get a function which is not zero in the proximity of the impulse in Ω-domain ⇒ we don't get a simple spectral lines (red trend). → LEAKAGE → this error creates in this way the DFT can elaborate the signal and we can limit leakage ; but we have to modify the beginning and the final part of the measurement.

There are many specific windows that can be used:

HAMMING, HANN, BLACKMAN, FLAT-TOP, ... → we can find in MatLab

↳ they have many different shapes, almost the same as the previous one.

⇒ we don't want leakage and we accept to modify the data

⟷

We said that in the DFT computation we have $F_{k+N} = F_k$ and $F_{N-k} = F_k^*$

$$\Omega_k = k\Omega_0 \Rightarrow \Omega_{\frac{N}{2}} = \frac{N}{2}\Omega_0 = \frac{N}{2}\frac{2\pi}{T} = \frac{N}{2}\frac{2\pi}{N\Delta t} = 2\pi\frac{1}{2}f_s = (2\pi f_s)\frac{1}{2}$$

and it's what Shannon's theorem says → we can describe frequencies which are half of the sampling frequency.

>> F= fft(f); → Matlab does not ask us for introducing a window; it's up to us. If our f is not periodic, we ~~[illegible]~~ will get leakage.

f= [0 1 2 3 ..]

The m, c, k matrixes are symmetric, so we can write:

$[m] = [m]^T$ $\qquad [c] = [c]^T$ $\qquad [k] = [k]^T$

and they are real, $\in \mathbb{R}$. Moreover $[m]$ and $[k]$ are POSITIVE DEFINITE, which means:

$\{v\}^T [m] \{v\} > 0$ → given any vector $\{v\}$

$\{v\}^T [k] \{v\} \geq 0$ → SEMI DEFINITE POSITIVE, the product can give zero

To get this result we have to use two different methods, based on the damping:

PROPORTIONAL DAMPING

the matrix of damping can be written in this way:

$$\boxed{[c] = \alpha [m] + \beta [k]}$$

it is a special case of damping

MODAL ANALYSYS

We can move to ~~the~~ an uncoupled system of diff. equation. The first step is forgetting about the damping:

$$\boxed{[m]\{\ddot{x}\} + [k]\{x\} = \{0\}}$$ → HOMOGENEOUS PROBLEM

the solution can be written in this form → $\{x(t)\} = \underset{m\times 1}{\{A\}} \cos(\omega t + \varphi)$

vector $\{A\}$ doesn't depend on time; time variation depends only on the harmonic, $\{A\}$ is CONSTANT.

m = number of dof

$\Rightarrow \{\ddot{x}\} = -\omega^2 \{A\} \cos(\omega t + \varphi)$ and we can introduce it into the eq. of motion →

Eigenvectors are usually ordered into a matrix :

$$\begin{bmatrix} \underset{m\times1}{\{\psi_1\}} & \underset{m\times1}{\{\psi_2\}} & \dots & \underset{m\times1}{\{\psi_m\}} \end{bmatrix} = \underset{m\times m}{[\Psi]}$$

MODAL MATRIX

(it's square, real)

it is square matrix, ~~square~~ real but it is not symmetric! $[\Psi] \neq [\Psi]^T$ di solito

1) Compute EIGVAL w_r^2 and EIGVECT $\{\psi_r\}$

03/11/2014 ⑩

So it's difficult to compute eigenvectors and eigenvalues: it's the most difficult part.

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{f(t)\}$$

having $\{x_0\}$, $\{v_0\}$ => $\{x(t)\} = ?$

- $[c] = \alpha[m] + \beta[k]$ → Proportional damping is the basic assumption for modal analysys

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \Rightarrow \{x(t)\} = \{A\}\cos(wt+\varphi)$$ solution of the homogeneous problem

→ This set of equation is formed by coupled equations → we can't solve it

→ We need to use MODAL ANALISYS

We have to assume some hypotesys :

·) Damping proportional to mass and stiffness

From the expression of $\{x(t)\}$ we get the Eigenvalue Problem :

$$([k] - w^2[m])\{A\} = \{0\} \Rightarrow w_r^2 , \underset{m\times1}{\{\psi_r\}}$$

$\boxed{w_1 = 0}$ → means that the displacement of the structure, described by $\{\psi_1\}$ doesn't vary with time => it's a RIGID BODY MOTION

ORTHOGONALITY

After solving the EVP, we take two particular eigenvectors the r~~th~~ and the Sth

$$[k]\{\psi_r\} = \omega_r^2[m]\{\psi_r\} \rightarrow \left[\begin{array}{l} ① \ \underline{\{\psi_s\}^T}[k]\{\psi_r\} = \omega_r^2\underline{\{\psi_s\}^T}[m]\{\psi_s\} \\ ② \ \underline{\{\psi_r\}^T}[k]\{\psi_s\} = \omega_s^2\underline{\{\psi_r\}^T}[m]\{\psi_s\} \end{array}\right.$$

$$[k]\{\psi_s\} = \omega_s^2[m]\{\psi_s\} \rightarrow$$

Transpose one of the two equations → we choose the second

$$① \quad \{\psi_s\}^T[k]\{\psi_r\} = \omega_r^2\{\psi_s\}^T[m]\{\psi_r\}$$

$$②^T \Rightarrow \{\psi_s\}^T[k]^T\{\psi_r\} = \omega_s^2\{\psi_s\}^T[m]^T\{\psi_r\}$$

Matrixes $[k]$ and $[m]$ are real and symmetryc so $[k]^T = [k]$ and $[m]^T = [m]$
we can apply this procedure only if matrixes $[m]$, $[k]$ are symmetryc and if damping is proportional

Now we do $① - ②^T \rightarrow 0 = (\omega_r^2 - \omega_s^2)\{\psi_s\}^T[m]\{\psi_r\}$

IF $r \neq s$

$$\omega_r^2 - \omega_s^2 \neq 0, \text{ so } \underline{\{\psi_s\}^T[m]\{\psi_r\} = 0} \qquad \text{M-ORTHOGONALITY}$$

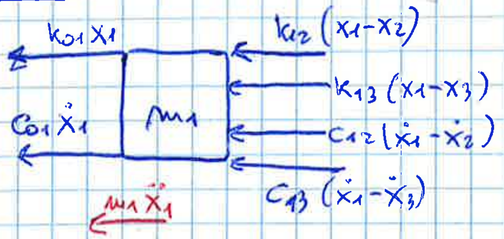
IF $r = s$

$$\omega_r^2 - \omega_s^2 = 0, \text{ so } \underline{\{\psi_s\}^T[m]\{\psi_s\} = M_r > 0} \qquad \text{not zero!}$$

because $[m]$ is positive definite, multiplying it by the same vector on the left and on the right, we obtain a positive, non zero, scalar value:

M_r = MODAL MASS depending on the starting matrix $[m]$ and on the particular eigenvector $\{\psi_r\}$ chosen

MASS m_1

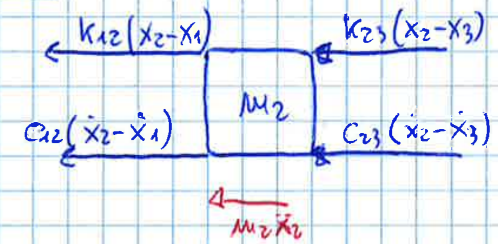


mass m_1 is moving to the right, mass the same well. What would happen if x_3 would be directed to the left?

↳ we would get $k_{13}(x_1+x_3)$ and $c_{13}(\dot{x}_1+\dot{x}_3)$

→ the sign changes depending on signs we decide for displacements

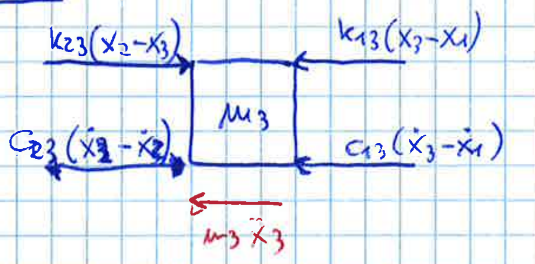
MASS m_2



The arrow for $k_{12}(x_2-x_1)$ is the same, we changed the sign for action reaction principle → NEVER CHANGE BOTH ARROWS AND SIGNS!

for $k_{23}(x_2-x_3)$ we changed the arrow direction, not the expression!

MASS m_3



2) Write down the equations of motion, we choose the right-direction as positive

→) $m_1\ddot{x}_1 + (c_{10}+c_{12}+c_{13})\dot{x}_1 - c_{12}\dot{x}_2 - c_{13}\dot{x}_3 + (k_{10}+k_{12}+k_{13})x_1 - k_{12}x_2 - k_{13}x_3 = 0$

$m_2\ddot{x}_2 - c_{12}\dot{x}_1 + (c_{12}+c_{23})\dot{x}_2 - c_{23}\dot{x}_3 - k_{12}x_1 + (k_{12}+k_{23})x_2 - k_{23}x_3 = 0$

$m_3\ddot{x}_3 - c_{13}\dot{x}_1 - c_{23}\dot{x}_2 + (c_{13}+c_{23})\dot{x}_3 - k_{13}x_1 - k_{23}x_2 + (k_{13}+k_{23})x_3 = 0$

in matrix notation:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_{10}+c_{12}+c_{13} & -c_{12} & -c_{13} \\ -c_{12} & c_{12}+c_{23} & -c_{23} \\ -c_{13} & -c_{23} & c_{13}+c_{23} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} \text{same coeffs as} \\ [c] \text{ matrix. Both} \\ \text{of } [k] \text{ and } [c] \text{ symm.} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \{0\}$$

$$\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

ψ_{21} → indicates the displacement of mass 2 in the eigvect 1 $\{\psi_1\}$

$$\begin{cases} k\psi_{11} - k\psi_{21} = 0 \\ -k\psi_{11} + 2k\psi_{21} - k\psi_{31} = 0 \\ -k\psi_{21} + k\psi_{31} = 0 \end{cases} \Rightarrow \begin{cases} \psi_{11} = \psi_{21} \\ -\psi_{11} + 2\psi_{21} - \psi_{31} = 0 \\ \psi_{31} = \psi_{21} \end{cases}$$

substituting the first and the third expression into the second we get:

$$\begin{cases} \psi_{11} = \psi_{21} \\ 0 = 0 \\ \psi_{31} = \psi_{11} \end{cases}$$

we can't decide the amplitude → we use to assume that the 1st element of the eigenvect is equal to one

There's no way to find which is exactly the amplitude

$$\Rightarrow \{\psi_1\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -\pi \\ -\pi \\ -\pi \end{Bmatrix}$$

we can choose 1 or $-\pi$ → there's no way to define eigenvectors

⇒ Eigenvectors are defined unless a constant value

$$\{\psi_1\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \Rightarrow$$ [all the masses move with the same amplitude]

↳ ! Rigid Motion $\omega_1^2 = 0$!

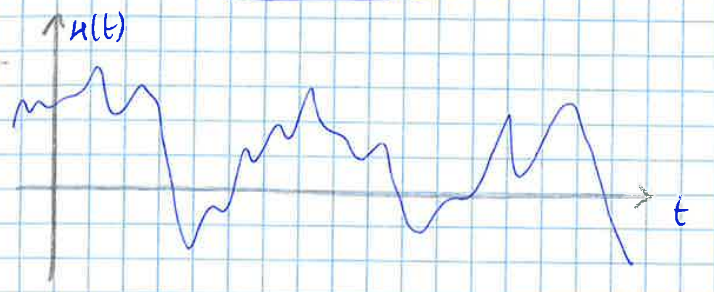
$\underline{\omega = \omega_2}$

$$\left([k] - \omega_2^2 [m]\right)\{\psi_2\} = \{0\} \rightarrow \omega_2^2 = \frac{k}{m}$$ so we'll get:

$$\begin{bmatrix} k - \frac{k}{m}m & -k & 0 \\ -k & 2k - \frac{k}{m}m & -k \\ 0 & -k & k - \frac{k}{m}m \end{bmatrix} \begin{Bmatrix} \psi_{12} \\ \psi_{22} \\ \psi_{32} \end{Bmatrix} = \{0\}$$

RANDOM/CASUAL/STOCHASTIC/NON DETERMINISTIC DATA

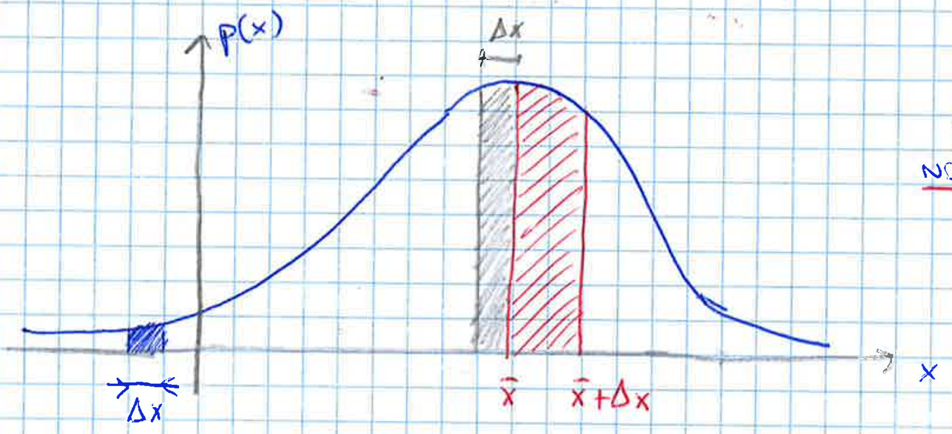
Data that can't be predicted



A random sequence of data is a sequence that, even with a long measurement, does not allow us to predict the ~~sequ~~ following value

→ Ex.: wind force

To describe this kind of data we have to use the PROBABILITY DENSITY FUNCTION



GAUSSIAN OR

NORMAL DISTRIBUTION

Fixed two values $\bar{x}$ and $\bar{x}+\Delta x$ the red area tells us what is the probability that our random function is limited between that two values

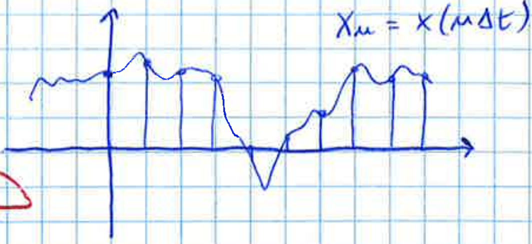
$\bar{x} < x(t) \leq \bar{x}+\Delta x$ → it tells us what is the chance to find a random value between defined limits

Given the same Δx → it's better to have a value in the ~~blue~~ gray area than in the blue one

Gaussian Distribution is defined by → $\boxed{\int_{-\infty}^{+\infty} p(x)\,dx = 1}$

We would like to know what is the possible evolution of functions only using simple parameters

$$\Rightarrow \boxed{\mu_x = \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} X_m} = E[x]$$



$X_m = x(m\Delta t)$

In practice N can't be infinite, but it should be as large as possible

$$\boxed{\sigma_x^2 = \lim_{N\to\infty} \frac{1}{N-1} \sum_{m=1}^{N} (X_m-\mu_x)^2} = E[(x-\mu_x^2)]$$

these are the real quantities that are computed

$$\text{SKEWNESS} = \frac{E[(x-\mu_x)^3]}{\sigma_x^3} \qquad \text{KURTOSIS} = \frac{E[(x-\mu_x)^4]}{\sigma_x^4} - 3$$

They are often used to identify any defects in structures; the -3 depends on the fact that kurtosis (Gauss) ≠ 0

so we have to add -3 in order to have a kurtosis Gaussian equal to zero.

with $w^2 = w_r^2$ we can find the $\{x_0\} = \{\Psi_r\}$, eigenvectors

[So, for every eigenvalue w_r^2 there is an eigenvector $\{\Psi_r\}$]

- $\{\Psi\}_s^T [m] \{\Psi\}_r = \begin{cases} 0 & \text{if } s \neq r \\ m_r & \text{if } s = r \end{cases}$ $\quad r = 1,\dots,N \quad s = 1,\dots,N$

- $\{\Psi\}_s^T [k] \{\Psi\}_r = \begin{cases} 0 & \text{if } s \neq r \\ k_r & \text{if } s = r \end{cases}$

and we can write $\boxed{w_r^2 m_r = k_r}$ $\Rightarrow$ $\left[w_r^2 = \frac{k_r}{m_r} \right]$ natural (modal) frequency

So we have a set of possible moving equation:

$$\{x\} = \{x_0\}\cos(wt+\varphi) = \{\Psi_0\}_r \cos(w_r t + \varphi_r) \qquad r = 1,\dots,N$$

So we can write it: $\boxed{\{x\} = \sum_{r=1}^{N} a_r \{\Psi\}_r \cos(w_r t + \varphi_r)}$ as a linear combination

for every eigen vector we'll have a SCALING FACTOR a_r, and we must calculate it.

The phase angle φ_r depends on every eigenvector.

NORMALIZATION OF EIGEN-VECTORS

In Maths we have:

$\lambda = \lambda_r$
$\bar x = \bar x_r$
$\Uparrow$

$\bar y = A\bar x$ with $\bar x, \bar y \in \mathbb{R}^N$ and $A \in \mathbb{R}^{N,N}$

$\bar y = \lambda \bar x \Rightarrow \lambda \bar x = A \bar x \Rightarrow A\bar x - \lambda \bar x = \bar 0 \Rightarrow [A - \lambda I]\bar x = \bar 0 \Rightarrow |A - \lambda I| = 0$

so I have r different equations $\rightarrow$ $A\bar x_r = \lambda_r \bar x_r$

I can multiply for a scalar quantity $\alpha \in \mathbb{R}$, $\alpha \neq 0$: of these vectors

- $\alpha(A\bar x_r) = \alpha(\lambda_r \bar x_r) \Rightarrow A(\underbrace{\alpha \bar x_r}_{\bar v_r}) = \lambda_r(\underbrace{\alpha \bar x_r}_{\bar v_r})$

these are, again, eigenvectors but SCALED $\rightarrow$ we have CHANGED AMPLITUDE, MODULUS

MODAL EXPANSION THEOREM (vedi Dinamica generale)

If we have a generic vector $\{v\}$, we can write it as a linear combination of eigenvectors:

$$\boxed{\{v\} = \sum_{r=1}^{\tilde{N}} c_r \{\psi\}_r} \quad , \ \{v\} \in \mathbb{R}^N$$

let's multiply on the left side by the $[m]$ matrix:

$$[m]\{v\} = [m]\sum_{r=1}^{\tilde{N}} c_r \{\psi\}_r$$

and again by $\{\psi\}_s^T$, a generic eigenvector (we don't know if $s=r$, $s \neq r$)

$$\{\psi\}_s^T [m] \{v\} = \{\psi\}_s^T [m] \sum_{r=1}^{\tilde{N}} c_r \{\psi\}_r$$

, the $\{\psi\}_s^T$ is not a function of r : neither the $[m]$ matrix

$$\{\psi\}_s^T [m] \{v\} = \sum_{r=1}^{\tilde{N}} c_r \{\psi\}_s^T [m] \{\psi\}_r$$

s is independent from r, we can choose it; for all the value $r \neq s$ the product is equal to zero; if $s=r$ it's different from zero and equal to the modal mass m_s:

$$\{\psi\}_s^T [m] \{v\} = c_s m_s \quad \Rightarrow \quad \left\lceil c_s = \frac{\{\psi\}_s^T [m] \{v\}}{m_s} \right\rfloor$$

if we calculate all this with normalized set of evects, we get:

$$\left\lceil c_s = \frac{\{\phi\}_s^T [m] \{v\}}{1} \right\rfloor \quad \Rightarrow \quad \boxed{\{v\} = \sum_{s=1}^{N} c_s \{\phi\}_s}$$

~~With the unnormalized set~~

$c_r \rightarrow$ modal participation factor

Don't mix the two equations and sets of coefficients dividing by m_r we obtain → $\boxed{\ddot{\eta}_r + \omega_r^2 \eta_r = 0}$ HARMONIC EQUATION

↓

synchronous solution

We can calculate (modal initial conditions in two ways:

1) $\{x(t=t_0)\} = \{x_0\} = [\Psi]\{\eta_0\} \Rightarrow \{\eta_0\} = [\Psi]^{-1}\{x_0\}$

$\{\dot{x}_0\} = \{v_0\} = [\Psi]\{\dot{\eta}_0\} \Rightarrow \{\dot{\eta}_0\} = [\Psi]^{-1}\{v_0\}$

but the calculation of $[\Psi]^{-1}$ is long and boring → not a good idea

2) the other approach allows to not calculate $[\Psi]^{-1}$:

FREE RESPONSE OF UNDAMPED SYSTEM

$\boxed{\eta_r(t) = A_r \cos \omega_r t + B_r \sin \omega_r t} = D_r \sin(\omega_r t - \varphi_r)$ is the solution for any r equation out of N, because they're all harmonic.

$$\Rightarrow \{x(t)\} = \sum_{r=1}^{\tilde{N}} \underbrace{(A_r \cos \omega_r t + B_r \sin \omega_r t)}_{\eta_r(t)} \cdot \{\Psi\}_r$$

→ how can I get A_r, B_r ?

↓

$$\{\Psi\}_s^T [m] \{x(t)\} = \{\Psi\}_s^T [m] \sum_{r=1}^{\tilde{N}} (A_r \cos \omega_r t + B_r \sin \omega_r t) \{\Psi\}_r$$

→ multiplications

↳ constant respect to the sum → can go inside it

$$\{\Psi\}_s^T [m] \{x(t)\} = \sum_{r=1}^{\tilde{N}} (A_r \cos \omega_r t + B_r \sin \omega_r t) \{\Psi\}_s^T [m] \{\Psi\}_r$$

all the products will be equal to zero but when r = s :

$$\boxed{\{\Psi\}_s^T [m] \{x(t)\} = (A_s \cos \omega_s t + B_s \sin \omega_s t)\, m_s} \quad \underline{r = s}$$

$\underline{t = t_0 = 0}$

unknown

$\{\Psi\}_s^T [m] \{x_0\} = \textcircled{A_s}\, m_s$ ↳ $\boxed{A_s = \dfrac{\{\Psi\}_s^T [m] \{x_0\}}{m_s}}$ got it

$a[m]^{-1}[k]\{\psi\}_2 = a\omega_2^2\{\psi\}_2 = a\omega_1^2\{\psi\}_2$ now we sum (I) and (II)

$$[m]^{-1}[k]\underbrace{\left\{\{\psi\}_1 + a\{\psi\}_2\right\}}_{\{v\}} = \omega_1^2\underbrace{\left\{\{\psi\}_1 + a\{\psi\}_2\right\}}_{\{v\}}$$

$$[m]^{-1}[k]\{v\} = \omega_1^2\{v\}$$

so if we have repeated eigenvalues and apparently different evects related to this ones, any linear combination of evects is again a valid eigenvector, whatever I scale the 2 eigenvectors.

30/11/2014 ⑪

We said for the MODAL TRANSFORMATION:

$$\{x\} = \sum_{r=1}^{n}\{\psi_r\}\eta_r = [\psi]\{\eta\}$$ we can write it in these two forms

FREE RESPONSE OF A DAMPED SYSTEM

We suppose initial conditions → $\{x(t=0)\} = \{x_0\}$ ⇒ $\{x(t)\} = ?$

$\{\dot{x}(t=0)\} = \{v_0\}$

I move from the original system to an equivalent set of uncoupled equations:

- $m_r\ddot{\eta}_r + c_r\dot{\eta}_r + k_r\eta_r = 0 \qquad r = 1, \dots, n$

$$\boxed{\eta_r(t) = (A_r\cos\omega_{dr}t + B_r\sin\omega_{dr}t)\,e^{-\xi_r\omega_r t}}$$ for underdamping $\xi_r < 1$

and $\boxed{\omega_{dr} = \omega_r\sqrt{1-\xi_r^2}}$

to compute A_r & B_r we have to use modal transformation, as seen:

$$\{\psi_s\}^T[m]\{x\} = \{\psi_s\}^T[m]\sum_{r=1}^{n}\{\psi_r\}\eta_r$$ and so on ...

we have that $\{X_0\} = [K_D]^{-1}\{f_0\} = [\alpha(\Omega)]\{f_0\}$

the inverse of $[K_D]$ is named RECEPTANCE → $\boxed{[K_D]^{-1} = [\alpha(\Omega)]}$

- we have to find an easy way to define the elements of this matrix;

let's transform the equation of motion in modal coordinates:

$$\{x(t)\} = \sum_{r=1}^{m}\{\psi_r\}\eta_r(t) \rightarrow \{x\} = \sum_{r=1}^{m}\{\psi_r\}\eta_r \quad (x \text{ and } \eta_r \text{ are } t\text{-dependent})$$

$$\Rightarrow [m]\sum_{r=1}^{m}\{\psi_r\}\ddot{\eta}_r + [c]\sum_{r=1}^{m}\{\psi_r\}\ddot{\eta}_r + [k]\sum_{r=1}^{m}\{\psi_r\}\eta_r = \{f_0\}e^{i\Omega t}$$

relaying to the orthogonality of eigenvectors respect to $[m]$ and $[k]$:

$$\{\psi_s\}^T[m]\sum\{\psi_r\}\ddot{\eta}_r + \dots = \{\psi_s\}^T\{f_0\}e^{i\Omega t}$$

every product is zero unless $\overset{\text{when}}{r \equiv s}$ → in this case we get modal matrixes:

- $m_r\ddot{\eta}_r + c_r\dot{\eta}_r + k_r\eta_r = \overbrace{\{\psi_r\}^T\{f_0\}}\,e^{i\Omega t}$ $r = 1, \dots n$

↳ it's like a SDOF system forced by a MODAL FORCE

↓ $\boxed{\{\psi_r\}^T\{f_0\}}$

So, we already know the solution for this:

$\boxed{\eta_r(t) = \eta_{r0}e^{i\Omega t}}$ STEADY STATE $r = 1, 2, \dots, n$

deriving and substituting I get:

$$\boxed{\eta_{r0} = \frac{\{\psi_r\}^T\{f_0\}}{k_r - \Omega^2 m_r + i\Omega c_r}}$$

which is almost the same expression of $X_0 = \dots$

↳ STEADY STATE AMPLITUDE for modal mass m_r

Remember that all the forces are equal to zero except in point q;

$$\boxed{\frac{X_{p0}}{F_{q0}}} = \boxed{\sum_{r=1}^{m} \frac{\psi_{pr}\psi_{qr}}{K_r - \Omega^2 m_r + i\Omega c_r}} = FRF = \underline{RECEPTANCE} = \boxed{\alpha_{pq}(\Omega)}$$

all the elements of this expression are simply defined: we just have to compute multiplications!

But how ~~is~~ α_{pq} is defined? It gives the amplitude of the response when all the forces are zero except in one point:

$$\alpha_{pq}(\Omega) \doteq \frac{X_{0p}}{F_{0q}}, \quad F_{0k} = 0, \quad \forall k \neq q$$

$\{X_p\} = [\alpha]\{F_p\}$ → let's take one element of this expression:

$$X_{p0} = [\alpha_{p1}\ \alpha_{p2} \cdots \alpha_{pm}] \begin{Bmatrix} F_{10} \\ F_{20} \\ \vdots \\ F_{m0} \end{Bmatrix}$$

so if all the elements are zero except F_{q0} we'll get:

$$X_{p0} = \alpha_{pq} F_{q0}, \quad F_{k0} = 0, \ \forall k \neq q$$

⇒ So receptance doesn't have to be computed by inverting $[K_D]$ matrix!

We can also write down this:

$$\boxed{\alpha_{pq} = \sum_{r=1}^{m} \frac{\psi_{pr}\psi_{qr}}{m_r(\omega_r^2 - \Omega^2 + i\Omega 2\xi_r\omega_r)}}$$

we're making clear the presence of eigenvalues here

Sometimes we also find, with NORMALIZED eigenvectors: $\{\psi_r\}/\sqrt{m_r} = \{\phi_r\}$

$$\Rightarrow \boxed{\alpha_{pq} = \sum_{r=1}^{m} \frac{\phi_{pr}\phi_{qr}}{\omega_r^2 - \Omega^2 + i\Omega 2\xi_r\omega_r}}$$

$\boxed{\phi_{pr}\phi_{qr}}$ → MODAL CONSTANT (RESIDUAL)

the two expressions are equal

RECEPTANCE - PROP. DAMPING SYSTEMS

12/11/2014 ⑫

GIORCELLI

$\omega = \Omega$

$$\boxed{\alpha_{ij}(\omega) = \frac{X_{0i}}{F_{0j}}} \qquad f(t) = \{F_0\} e^{i\omega t}$$

we have one angular frequency for all the forcing functions → ω (= Ω fasore)

$$\alpha_{ij}(\omega) = \sum_{r=1}^{m} \frac{\psi_{ir}\psi_{jr}}{\underbrace{m_r(\omega_r^2}_{k_r}-\omega^2)+j2\xi_r m_r\omega_r\omega}$$

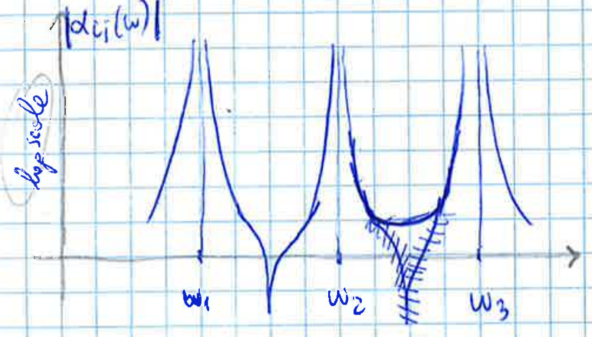
as seen, we can also normalize this expression with $\{\phi_r\} = \frac{1}{\sqrt{m_r}}\{\psi_r\}$

$$\Rightarrow \alpha_{ij}(\omega) = \sum_{r=1}^{m} \frac{\phi_{ir}\phi_{jr}}{\omega_r^2-\omega^2+j2\xi_r\omega_r\omega}$$

→ [the numerator is not function of omega. This happens because the proportional damping]

$\phi_{ir}\phi_{jr}$ → numerator is constant respect to omega and real → RESIDUAL

Plotting the receptance modulus for 3 DOF ($\xi_r = 0$):



$$\alpha_{ij}(\omega) = \frac{\phi_{i1}\phi_{j1}}{\omega_1^2-\omega^2} + \frac{\phi_{i2}\phi_{j2}}{\omega_2^2-\omega^2} + \frac{\phi_{i3}\phi_{j3}}{\omega_3^2-\omega^2}$$

if $\omega_1 \le \omega \le \omega_2$ we have, if both numerator are positive : $\omega << \omega_3$

$(\phi_{i1}\phi_{j1} > 0$ and $\phi_{i2}\phi_{j2} > 0)$

the first term is negative, the second is positive, the 3rd is neglegible;

⇒ there's a point where $|\alpha_{ij}(\omega)| \cong 0$

in a log scale we will have minima → if $\phi_{i3}\phi_{j3} < 0$ → the second minimum is greater than zero

The minimum is not perfecty in the middle

The zeros of the function are called ANTIRESONANCES { in a log scale the zeros are peaks towards the bottom

If j = i

we are calculating the response in the same point where the force is applied

if damping is zero we get:

$\Rightarrow \left| s^2[m] + s[c] + [k] \right| = 0$ → we'll have 2n solutions

because s_r for $r = 1, \ldots, n$!

If $s = s_r \Rightarrow \left[s_r^2 [m] + s_r [c] + [k] \right] \{x_0\} = \{0\}$ ↗ it's a singular matrix in homogenous system ↳ non-trivial solution

$\Rightarrow \{x_0\} = \{\psi_r\} \quad r = 1, \ldots, n$

but if we use s_r eigenvalues and $\{\psi_r\}$ eigenvectors, we'll see that this is not correct:

ⓐ
$$r \Rightarrow \left[s_r^2 [m] + s_r [c] + [k] \right] \{\psi_r\} = \{0\}$$

ⓑ
$$s \Rightarrow \left[s_s^2 [m] + s_s [c] + [k] \right] \{\psi_s\} = \{0\}$$ → using another eigenvalue and eigvector

now I multiply on the left side ~~by~~ the other eigenvector transposed:

ⓐ
$$\{\psi_s\}^T s_r^2 [m] \{\psi_r\} + \{\psi_s\}^T s_r [c] \{\psi_r\} + \{\psi_s\}^T [k] \{\psi_r\} = \{\psi_s\}^T \{0\} = 0$$

ⓑ
$$\{\psi_r\}^T s_s^2 [m] \{\psi_s\} + \{\psi_r\}^T s_s [c] \{\psi_s\} + \{\psi_r\}^T [k] \{\psi_s\} = \{\psi_r\}^T \{0\} = 0$$

now I transpose the second equation, remembering $\left[[A] \cdot [B] \right]^T = [B]^T [A]^T$:

ⓑ^{T}
$$s_s^2 \{\psi_s\}^T [m]^T \{\psi_r\} + s_s \{\psi_s\}^T [c]^T \{\psi_r\} + \{\psi_s\}^T [k]^T \{\psi_r\} = \{0\}$$

Now I ~~do~~ compute the difference ⓐ - ⓑ^{T} ≐ ⓒ : $[m]^T = [m]$ and c, k

ⓒ
$$(s_r^2 - s_s^2) \{\psi_s\}^T [m] \{\psi_r\} + (s_r - s_s) \{\psi_s\}^T [c] \{\psi_r\} + 0 = 0$$

$$(s_r - s_s) \Big((s_r + s_s) \{\psi_s\}^T [m] \{\psi_r\} + \{\psi_s\}^T [c] \{\psi_r\} \Big) = 0$$

if