NUMERO: 1856A - ANNO: 2016

A P P U N T I

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MATERIA: Statistica (Esercitazioni) prof. Vicario

Statistica

I testi degli esercizi sono presenti sul portale della didattica: si tratta delle 14 esercitazioni svolte in aula

Esercitazione 1

06/10/2014
FOGLIO N°1

Es 1

50 biglietti $\leftrightarrow$ 50 premi
7 piacciono a Chiara
5 " " Marco
1 piace a entrambi

C = "il premio ricevuto piace a Chiara"
M = "il premio ricevuto piace a Marco"

SPAZIO UNIFORME $\rightarrow$ EVENTI EQUIPROBABILI

1) $P(C) = \dfrac{\text{casi favorevoli}}{\text{casi possibili}} = \dfrac{\text{card } C}{\text{card } S} = \dfrac{7}{50}$

2) $P(M) = \dfrac{\text{card } M}{\text{card } S} = \dfrac{5}{50} = \dfrac{1}{10}$

3) $P(M \cap C) = \dfrac{\text{card}(M \cap C)}{\text{card } S} = \dfrac{1}{50}$

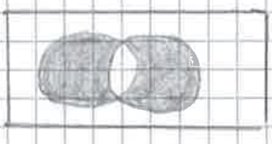
4) $P(M \cup C) = P(M) + P(C) - P(M \cap C) = \dfrac{5}{50} + \dfrac{7}{50} - \dfrac{1}{50} = \dfrac{11}{50}$

probabilità che piaccia ad almeno uno dei due

5) $P(M^c \cap C^c) = P((M \cup C)^c)$

$= 1 - P(M \cup C) = 1 - \dfrac{11}{50} = \dfrac{39}{50}$

6) $P((M \cap C^c) \cup (C \cap M^c))$



$= P(M \cup C) - P(M \cap C) = \dfrac{11}{50} - \dfrac{1}{50} = \dfrac{10}{50} = \dfrac{1}{5}$

b) $A =$ "esca un n° dispari"

$= \{1, 3, 5\}$

$P(A) = P(\{1\}) + P(\{3\}) + P(\{5\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

c) $B =$ "esca un n° > 3"

$= \{4, 5, 6\}$

$P(B) = P(\{4\}) + P(\{5\}) + P(\{6\})$

$= \frac{1}{4} + \frac{1}{8} + \frac{1}{4} = \frac{5}{8}$

Es. 4

$S = \{(i, j, k) : i, j, k \in \{1 \dots 6\}\}$

$Q = P(S)$

P uniforme 6 possibilità per il 1° dado, 6 possibilità per il 2° dado, 6 possibilità per il 3° dado.

$P(\{(i, j, k)\}) = \frac{1}{216}$

cardinalità di $S = 6^3 = 216$

a) M = massimo punteggio

$A =$ "$M = 6$" $= \{(6, 6, 6)\}$

$P(A) = \frac{1}{216}$

b) $B =$ "$(M > 1)$" $= \{(i, j, k) : i, j, k \in (2, \dots, 6)$

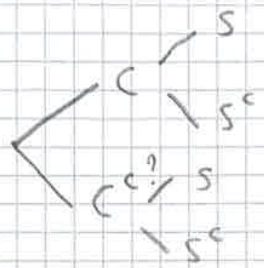
card $B = 5 \times 5 \times 5 = 5^3 = 125$

$\frac{\text{card } B}{\text{card } S} = \frac{5^3}{6^3} = \left(\frac{5}{6}\right)^3$

c) = "$M = 1$" $= B^c$

$P(C) = 1 - P(B) = 1 - \frac{125}{216} = \frac{91}{216}$

$$P(S|C^c) \neq 1 - P(S|C) \longleftarrow$$ Ricorda che:



Formula delle probabilità totali

$$P(S) = P(S|C)\,P(C) + \underbrace{P(S|C^c)}_{?}\,P(C^c)$$

$$P(S|C^c)\,P(C^c) = P(S) - P(S|C)\,P(C)$$

$$P(S|C^c) = \frac{P(S) - P(S|C)\,P(C)}{P(C^c)} = \frac{0{,}98 - 0{,}96 \cdot 0{,}2}{0{,}8} = 0{,}985$$

Es. 3

E = "lo studente risponde esattamente"

C = "lo studente conosce la risposta"

Indichiamo $p = P(C)$

$P(E|C) = 1$

$P(E|C^c) = \frac{1}{3}$ (sceglie a caso)

Formula delle probabilità totale

$$P(E) = P(E|C)\,P(C) + P(E|C^c)\,P(C^c)$$

$$= 1 \cdot p + \frac{1}{3} \cdot (1-p) = \frac{2}{3}p + \frac{1}{3}$$

$P(C|E) = ?$

Bayes

$$P(C|E) = \frac{P(E|C)\,P(C)}{P(E)} = \frac{1 \cdot p}{\frac{2}{3}p + \frac{1}{3}} = \frac{3p}{2p+1}$$

$$= \mathbb{P}(C_1) + \mathbb{P}(C_2 | A_1)\,\mathbb{P}(A_1) + \mathbb{P}(C_2 | B_1)\,\mathbb{P}(B_1) = ?$$

$$\mathbb{P}(C_2 | A_1) = \mathbb{P}(C_2 | B_1) = \frac{1}{2}$$

$\mathbb{P}(A_1)$? $\quad \mathbb{P}(B_1)$? $\quad \mathbb{P}(C_1)$?

2ª parte:

A_1 = " la somma è ≤ 5 "

B_1 = " la somma è ≥ 10 "

C_1 = " la somma è tra 5 e 10, esclusi "

$$S' = \{(1,1), (1,2) \dots (6,6)\}$$

$\mathbb{P}$ uniforme $\quad \mathbb{P}(\{i,j\}) = \dfrac{1}{\text{card } S}$

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

card $S' = 36$

$$A_1 = \{\underbrace{(1,1)}_{2}, \underbrace{(1,2), (2,1)}_{3}, \underbrace{(1,3)(2,2),(3,1)}_{4}, \underbrace{(1,4),(2,3),(3,2),(4,1)}_{5}\}$$

card $A_1 = 10$

$$B_1 = \{\underbrace{(4,6), (5,5), (6,4)}_{10}, \underbrace{(5,6), (6,5)}_{11}, \underbrace{(6,6)}_{12}\}$$

card $B_1 = 6$

$$\mathbb{P}(A_1) = \frac{\text{card } A_1}{\text{card } S} = \frac{10}{36} = \frac{5}{18}$$

$$\mathbb{P}(B_1) = \frac{\text{card } B_1}{\text{card } S} = \frac{6}{36} = \frac{1}{6}$$

$$\mathbb{P}(C_1) = \mathbb{P}((A_1 \cup B_1)^c) = \mathbb{P}(A_1^c \cap B_1^c) = 1 - \mathbb{P}(A_1 \cup B_1) = 1 - \mathbb{P}(A) - \mathbb{P}(B_1)$$

$$= (1-2p)^3 + (1-2p)(1-p) - (1-2p)^3(1-p)$$

$$= (1-2p)\left[(1-2p)^2 + 1-p - (1-2p)^2(1-p)\right]$$

$$= (1-2p)\left[1-p+(1-2p)^2 p\right] \cancel{+p(1-2p)^2}$$

$$= (1-2p)\left[1-\cancel{p}+\cancel{p}+4p^3-4p^2\right] =$$

$$= (1-2p)\left[1-4p^2+4p^3\right]$$

2° metodo

$$P(ok) = P\left(C_1 \cap \left((C_2 \cap C_3) \cup C_4\right)\right)$$

$$= P(C_1)\, P\left((C_2 \cap C_3) \cup C_4\right) =$$

$$= P(C_1)\left[P(C_2 \cap C_3) + P(C_4) - P(C_2 \cap C_3 \cap C_4)\right]$$

$$= P(C_1)\left[P(C_2)P(C_3) + P(C_4) - P(C_2)P(C_3)P(C_4)\right]$$

$$= (1-2p)\left[(1-2p)^2 + 1-p - (1-2p)^2(1-p)\right] =$$

$$= \ldots = (1-2p)\left[1-4p^2+4p^3\right]$$

3° metodo

Serie tra 2 e 3:

2	3	
1	1	$\rightarrow (1-2p)^2$
1	0	
0	1	
0	0	

parallelo tra 2 3, e 4

2 3	4
1	1
1	0
0	1
0	0

U_i = " estraggo dall'urna i "

a) $P(B_J)$? $J = 1, 2, 3$

b) $P(B_3 \mid B_1 \cap B_2)$?

c) $P(U_1 \mid B_1 \cap B_2)$?

a) $\{U_1, U_2\}$ partizione dell'evento certo (incompatibili ed esaustivi)

$U_2 = U_1^c$ Formula delle probabilità totali

$$P(B_J) = P(B_J \mid U_1) \cdot P(U_1) + P(B_J \mid U_2) P(U_2)$$

$P(U_1) = P(T) = 0,4$ $\quad$ $P(U_2) = P(C) = 0,6$

C'è reinserimento

$$P(B_1 \mid U_1) = P(B_2 \mid U_1) = P(B_3 \mid U_1) = \frac{3}{5}$$

$$P(B_1 \mid U_2) = P(B_2 \mid U_2) = P(B_3 \mid U_2) = \frac{2}{5}$$

$\forall J = 1, 2, 3$

$$P(B_J) = \frac{3}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{3}{5} = \frac{6}{25} + \frac{6}{25} = \frac{12}{25} = \underline{0,48}$$

b) $$P(B_3 \mid B_1 \cap B_2) = \frac{P(B_1 \cap B_2 \cap B_3)}{P(B_1 \cap B_2)}$$

$$P(B_1 \cap B_2) \underset{\text{prob. totale}}{=} P(B_1 \cap B_2 \mid U_1) P(U_1) + P(B_1 \cap B_2 \mid U_2) P(U_2)$$

per reinserimento

$$P(B_1 \cap B_2 \mid U_1) = P(B_1 \mid U_1) P(B_2 \mid U_1)$$

$$P(B_1 \cap B_2 \mid U_2) = P(B_1 \mid U_2) P(B_2 \mid U_2)$$

$$\Rightarrow P(B_1 \cap B_2) = P(U_1) P(B_1 \mid U_1) P(B_2 \mid U_1) + P(U_2) P(B_1 \mid U_2) P(B_2 \mid U_2)$$

a) $\mathbb{P}(A \cup B^c \cup C^c) \leq \mathbb{P}(A) + \mathbb{P}(B^c) + \mathbb{P}(C^c)$

$= \frac{1}{10} + \left(1 - \frac{8}{10}\right) + \left(1 - \frac{9}{10}\right) = \frac{4}{10} = \frac{2}{5} < \frac{3}{5}$ sì

b) $\mathbb{P}(A \cup B^c) = \mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A \cap B^c)$

A, B indipendenti $\Rightarrow A, B^c$ indipendenti

$\mathbb{P}(A \cup B^c) = \mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A)\,\mathbb{P}(B^c)$

$= \frac{1}{10} + \frac{2}{10} - \frac{1}{10} \cdot \frac{2}{10} = \frac{3}{10} - \frac{2}{100} = \frac{28}{100} = 0{,}28$

$\mathbb{P}(A \cup B^c \cup C^c) = \mathbb{P}(A) + \mathbb{P}(B^c) + \mathbb{P}(C^c) - \mathbb{P}(A \cap B^c) - \mathbb{P}(A \cap C^c) - \mathbb{P}(B^c \cap C^c)$

$+ \mathbb{P}(A \cap B^c \cap C^c)$ $= \rightarrow$ indipendenza

$= \mathbb{P}(A) + \mathbb{P}(B^c) + \mathbb{P}(C^c) - \mathbb{P}(A)\mathbb{P}(B^c) - \mathbb{P}(A)\mathbb{P}(C^c) - \mathbb{P}(B^c)\mathbb{P}(C^c) + \mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C^c)$

$\left(\frac{1}{10} + \frac{2}{10} + \frac{1}{10}\right) - \left(\frac{1}{10} \cdot \frac{2}{10} + \frac{1}{10} \cdot \frac{1}{10} + \frac{2}{10} \cdot \frac{1}{10}\right) + \frac{1}{10} \cdot \frac{2}{10} \cdot \frac{1}{10} = \frac{4}{10} - \frac{5}{100} + \frac{2}{1000} =$

$= \frac{44}{125} = 0,$

oppure: $\mathbb{P}(A \cup B^c \cup C^c) = \mathbb{P}((A^c \cap B \cap C)^c) =$

$= 1 - \mathbb{P}(A^c \cap B \cap C) = 1 - \mathbb{P}(A^c)\mathbb{P}(B)\mathbb{P}(C) = \frac{44}{125}$

Trasmissioni successive sono indipendenti

b) Con quale probabilità si effettuano 3 trasmissioni senza errori?

E_i = "si commette errore nella trasmissione i-esima"

$\forall i = 1,2,3$

$$P(E_1^c \cap E_2^c \cap E_3^c) \overset{\text{indipendenza}}{=}$$

$$= P(E_1^c)\,P(E_2^c)\,P(E_3^c) =$$

$$= (1 - P(E_1))(1 - P(E_2))(1 - P(E_3))$$

$$= (1 - P(E))^3 = (1 - 0{,}058)^3 = \underline{0{,}8359}$$

c) $A = 00$ $B = 01$
$C = 10$ $D = 11$

Con quale probabilità c'è errore, sapendo che si trasmette A?

T_0^i = "si trasmette 0 alla trasmissione i-esima"

T_1^i = "si trasmette 1 alla trasmissione i-esima"

R_0^i = "si riceve 0 alla i-esima trasmissione"

R_1^i = " " " 1 " " " "

T_A = "si trasmette A"

R_A = "si riceve A"

$T_A = T_0^1 \cap T_0^2$

$R_A = R_0^1 \cap R_0^2$

Sia $\tilde{E}$ = "si commette un errore"

$$P(\tilde{E}|T_A) = 1 - P(\tilde{E}^c|T_A) = 1 - P(R_A|T_A) = 1 - P(R_0^1 \cap R_0^2 | T_0^1 \cap T_0^2)$$

$$= 1 - \frac{P((R_0^1 \cap R_0^2) \cap (T_0^1 \cap T_0^2))}{P(T_0^1 \cap T_0^2)} = 1 - \frac{P((R_0^1 \cap T_0^1) \cap (R_0^2 \cap T_0^2))}{P(T_0^1 \cap T_0^2)} \overset{\text{indipendenza}}{=}$$

$$= 1 - \frac{P(R_0^1 \cap T_0^1)\,P(R_0^2 \cap T_0^2)}{P(T_0^1)\,P(T_0^2)} = 1 - P(R_0^1|T_0^1)\,P(R_0^2|T_0^2)$$

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$$E[X] = \sum_{k=1}^{6} k\, f_X(k) =$$

$$= 1 \times 0{,}153 + 2 \times 0{,}153 + \dots + 6 \times 0{,}233 = 3{,}7$$

Se i dadi fossero tutti omogenei $f_X(k) = \frac{1}{6} \quad \forall k = 1 \dots 6$

$$E[X] = \sum_{k=1}^{6} k \cdot \frac{1}{6} = 3{,}5$$

$$Var(X) = E[X^2] - E[X]^2$$

Ricorda: $\left(E[X^2] \neq E[X]^2 \right)$

$$E[X^2] = \sum_{k=1}^{6} k^2 f_X(k) =$$

$$= 1^2 \times 0{,}153 + 2^2 \times 0{,}153 + \dots 6^2 \times 0{,}153 = 16{,}83$$

$$Var(X) = 16{,}83 - (3{,}7)^2 = 3{,}14$$

$$\sqrt{Var(X)} = \sqrt{3{,}14} = 1{,}77$$

b) $P(X \leq 2) \qquad P(X \geq 5)$

$$P(X \leq 2) = P(X=1) + P(X=2)$$

$$= f_X(1) + f_X(2) = 0{,}307$$

$$\frac{23}{150} + \frac{23}{150} = \frac{46}{150}$$

$$P(X \geq 5) = P(X=5) + P(X=6)$$

$$= f_X(5) + f_X(6) = \frac{23}{150} + \frac{7}{30} = 0{,}387$$

$f_X(6) = \mathbb{P}(\{(2,4),(4,2)\}) = \frac{1}{5}$

$f_X(7) = \mathbb{P}(\{(3,4),(4,3)\}) = \frac{1}{5}$

$f_X(8) = \mathbb{P}(\{(4,4)\}) = \frac{1}{10}$

c) $\mathbb{P}(X \leq x) = F_X(x) \quad \forall x \in \mathbb{R}$ — Esercitazione 5

$= \sum_{k \in X(S)} f_X(k)$

se $x < 3$ $F_X(x) = 0$

se $3 \leq x < 4$ $F_X(x) = f_X(3) = \frac{1}{10}$

se $4 \leq x < 5$ $F_X(x) = f_X(3) + f_X(4) = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$

se $5 \leq x < 6$ $F_X(x) = f_X(3) + f_X(4) + f_X(5) = \frac{1}{10} + \frac{1}{10} + \frac{3}{10} = \frac{1}{2}$

se $6 \leq x < 7$ $F_X(x) = f_X(3) + f_X(4) + f_X(5) + f_X(6) = \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$

se $7 \leq x < 8$ $F_X(x) = f_X(3) + \dots + f_X(7) = \frac{7}{10} + \frac{1}{5} = \frac{9}{10}$

se $x = 8$ $F_X(x) = 1$

d) $\mathbb{E}(X) = \sum_{k \in X(S)} k\, f_X(k) = 3f_X(3) + 4f_X(4) + \dots + 8f_X(8) = \frac{56}{10} = 5{,}6$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$\mathbb{E}[X^2] = \sum_{k \in (S)} k^2 f_X(k) = 3^2 f_X(3) + 4^2 f_X(4) + \dots + 8^2 f_X(8) = \frac{334}{10} = 33{,}5$

$Var(X) = 33.4 - (5{,}6)^2 = 2{,}04$

e) $P(X \geq 7) = f_X(7) + f_X(8) = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}$

Osservazione

$P(X \geq 7) = 1 - \mathbb{P}(X < 7)$

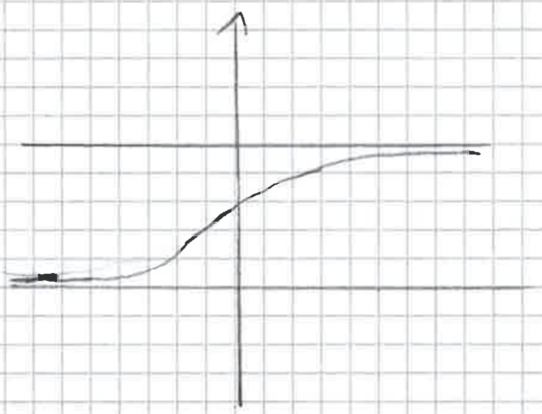
$= 1 - \mathbb{P}(X \leq 6) = 1 - F_X(6) = 1 - \frac{7}{10} = \frac{3}{10}$

$$F_X(x) = F_X(0) + \int_0^x \frac{1}{2} e^{-t}\, dt$$

$$= \frac{1}{2} + \left(-\frac{1}{2} e^{-t}\right)_0^x =$$

$$= \frac{1}{2} + \left(-\frac{1}{2} e^{-x} + \frac{1}{2}\right) = 1 - \frac{1}{2} e^{-x}$$

$$F_X(x) = \begin{cases} \frac{1}{2} e^{x} & \text{se } x \le 0 \\ 1 - \frac{1}{2} e^{-x} & \text{se } x > 0 \end{cases}$$



c) $$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx = \int_{-\infty}^{+\infty} \frac{1}{2} x e^{-|x|}\, dx = 0$$

$$\text{var}(x) = E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = \int_{-\infty}^{+\infty} \frac{1}{2} x^2 e^{-|x|}\, dx = \int_0^{+\infty} x^2 e^{-x}\, dx =$$

$= \dots = 2$ (fare una doppia integrazione per parti)

d) $$E[2X-1] \underset{\text{linearità}}{=} 2\underbrace{E[X]}_{=0} - 1 = -1$$

$$\text{var}(2X-1) = 4\underbrace{\text{var}(X)}_{=2} = 8$$

ES. 4

$\mu_X = E[X] = 1 \qquad \text{var}(X) = \sigma_X^2 = 4$

Determinare un minorante per $P(-3 < X < 5)$

$P(-3 < X < 5) \ge ?$

$P(X \leq x) \geq \frac{1}{2}$ verificata per $x \geq 2$

$P(X \geq x) \geq \frac{1}{2}$ verificata per $x \leq 2$

Entrambe sono verificate in 2 $\Rightarrow$ mediana = 2

b) $n = 5$ $\quad p = \frac{1}{2}$

	0	1	2	3	4	5
$f_X(x)$	1/32	5/32	10/32	10/32	5/32	1/32
$P(X \leq x)$	1/32	6/32	$\boxed{10/32}$			
$P(X \geq x)$	1	31/32	26/32	$\boxed{16/32}$		

$P(X \leq x) \geq \frac{1}{2}$ è verificata per $x \geq 2$

$P(X \geq x) \geq \frac{1}{2}$ è verificata per $x \leq 3$

entrambe per $2 \leq x \leq 3$ $\Rightarrow$ mediana = 2

c) $n = 2$ $\quad P = 9/10$ $\qquad f_X(k) = \binom{2}{k}\left(\frac{1}{10}\right)^k\left(\frac{1}{10}\right)^{n-k}$

	0	1	2
$P_X(x)$	1/100	18/100	81/100
$P(X \leq x)$	1/100	19/100	1
$P(X \geq x)$	1	99/100	81

$\begin{cases} x \geq 2 \\ \forall x \end{cases}$ $\quad med(X) = 2$

Y = n° di confezioni "difettose" tra quelle fornite

$Y \sim B(f, q)$

q = probabilità che una confezione sia "difettosa"

$= \mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2) = 1 - 0{,}9619 = \underline{\underline{0{,}0381}}$

$\mathbb{P}(\text{accettata}) =$

$= 1 - \mathbb{P}(\text{rifiutate}) =$

$= 1 - \mathbb{P}(Y \geq 2) = \mathbb{P}(Y < 2)$

$= \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) =$

$= \binom{150}{0}(0{,}0381)^0(0{,}9619)^{150} + \binom{150}{1}(0{,}0381)^1(0{,}9619)^{149} =$

$= (0{,}9619)^{150} + 150 \cdot 0{,}0381 \cdot (0{,}9619)^{149} \cong 0{,}0205$

$f \geq 100$ $\quad$ $q \leq 0{,}05$

$\|$ $\quad$ $\|$

150 $\quad$ $0{,}0381$

Poisson

Approssimiamo Y con $Y' \sim P(\lambda)$

$\lambda = fq = 150 \cdot 0{,}0381 = 5{,}715$

$\mathbb{P}(Y < 2) \simeq \mathbb{P}(Y' < 2) =$

$= \mathbb{P}(Y' = 0) + \mathbb{P}(Y' = 1)$

$Y' \sim \mathbb{P}(\lambda)$ $\quad \forall k \in \mathbb{N}$

$\mathbb{P}(Y = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

$= e^{-\lambda}\dfrac{\lambda^0}{0!} + e^{-\lambda}\dfrac{\lambda^1}{1!} =$

$= e^{-\lambda}(1 + \lambda) = e^{-5{,}715} \cdot 6{,}715 \cong 0{,}0221$

2) $c = 20$ $\quad$ $f = 20$

$\mathbb{P}(X \leq 2) = 0{,}67693$

$= 0,9909$

4) $s = 20$

$X \sim J(20, 2, 4)$

$\mathbb{P}(X \leq 1) = \underline{0,9684}$

$\frac{n}{N} = \frac{4}{20} = \underline{0,2 > 0,1}$

<u>No BINOMIALE</u>

se $X' \sim B(n, p)$ $\quad n = 20 \quad p = \frac{2}{20} = \frac{1}{10}$

$\mathbb{P}(X' \leq 1) = \boxed{0,9477}$

ES. 3

3 R
4 B
5 N

estrazioni con reinserimento

X_n = n° di palline nere estratte, facendo n estrazioni

successo = pallina nera

$X_n \sim B(n, p)$

$p = \frac{5}{12}$

$\mathbb{P}(X_5 = 3) = \binom{5}{3}\left(\frac{5}{12}\right)^3\left(\frac{7}{12}\right)^2 = 0,246$

Y = n° di palline estratte per avere la nera

= n° di prove per avere il 1° successo

$Y' = Y - 1$ = n° di insuccessi prima del 1° successo

$Y' \sim G(p)$ geometrica ?

$$\mathbb{P}\left(\left|\frac{X_n}{n} - \mathbb{E}\left[\frac{X_n}{n}\right]\right| < \varepsilon\right) \geq 1 - \frac{var\left(\frac{X_n}{n}\right)}{\varepsilon^2}$$

$$\mathbb{P}\left(\left|\frac{X_n}{n} - p\right| < \varepsilon\right) \geq 1 - \frac{p(1-p)}{n\varepsilon^2} \geq 0{,}9$$

$$1 - \frac{p(1-p)}{n\varepsilon^2} \geq 0{,}9$$

$$\frac{p(1-p)}{n\varepsilon^2} \leq 0{,}1$$

$$n \geq \frac{p(1-p)}{0{,}1 \cdot \varepsilon^2}$$

$$\varepsilon = 0{,}01 \Rightarrow n \geq \frac{\frac{5}{12} \cdot \frac{7}{12}}{0{,}1 \cdot (0{,}01)^2} = 24305{,}5$$

$$n \geq 24306$$

4) $\varepsilon = 0{,}1 \Rightarrow n \geq \dfrac{\frac{5}{12} \cdot \frac{7}{12}}{(0{,}1)^3} = 243{,}05$

$$\boxed{n \geq 244}$$

ES. 4

A = "il bicchiere proviene dalla fabbrica A"
B = "il bicchiere proviene dalla fabbrica B"

$\mathbb{P}(A) = 0{,}6$

$\mathbb{P}(B) = 0{,}4$

$B = A^c$

R = "il bicchiere si rompe"

$\mathbb{P}(R|A) = 0{,}02$

$\mathbb{P}(R|B) = 0{,}01$

3) $$\mathbb{P}(A|U) = \frac{\mathbb{P}(U|A)\,\mathbb{P}(A)}{\mathbb{P}(U)}$$

$$= \frac{0{,}7024 \cdot 0{,}6}{0{,}6026} = 0{,}6994$$

FOGLIO N° 5

ES. 1

F1 = "il prodotto arriva dal primo fornitore"

F2 = "il prodotto arriva dal secondo fornitore"

$\mathbb{P}(F1) = 0{,}7 \quad \mathbb{P}(F2) = 0{,}3 \quad F2 = (F1)^c$

X_1 = caratteristica del prodotto proveniente da F1

X_2 = ------------- proveniente da F2

$X_1 \sim U([a,b]) \quad \mu_1 = 29{,}9 \quad \sigma_1^2 = 0{,}09$?

$X_2 \sim U([29{,}5,\ 30{,}5])$

$$\begin{cases} \mu_1 = \dfrac{a+b}{2} = 29{,}9 \\ \sigma_1^2 = \dfrac{(b-a)^2}{12} = 0{,}09 \end{cases}$$

Risultati

$\Rightarrow a = 29{,}38 \ , \ b = 30{,}42$

$X_1 \sim U([29{,}38\ ,\ 30{,}42])$

$$\mathbb{P}(29{,}6 \le X_1 \le 30{,}4) = \frac{30{,}4 - 29{,}6}{30{,}42 - 29{,}38}$$

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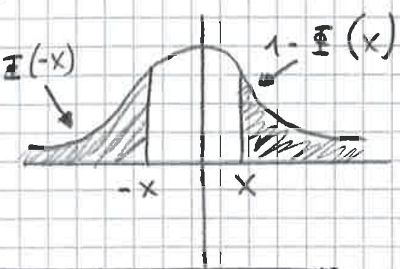
$= \mathbb{P}(-0{,}5 \leq Z \leq 1)$

dove $Z = \dfrac{X_1 - \mu_1}{\sigma_1} \sim \mathcal{N}(0,1)$

$= \Phi(1) - \Phi(-0{,}5)$

$\Phi(x) = \mathbb{P}(Z \leq x)$

Φ - cumulata normale



perché $\Phi(x) = \mathbb{P}(X \leq x)$

$$\boxed{\Phi(-x) = 1 - \Phi(x)}$$

$= \Phi(1) - (1 - \Phi(0{,}5))$

$= \Phi(1) + \Phi(0{,}5) - 1 =$

$= 0{,}8413 + 0{,}6915 - 1 = 0{,}5328$

2) $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

$\mathbb{P}(X_2 \leq 2{,}1) = 0{,}8413$

$\mathbb{P}(X_2 \geq 1{,}8) = 0{,}9772$

μ_2? σ_2^2?

$$\begin{cases} 0{,}8413 = \mathbb{P}(X_2 \leq 2{,}1) = \mathbb{P}\left(Z \leq \dfrac{2{,}1 - \mu_2}{\sigma_2}\right) \\ 0{,}9772 = \mathbb{P}(X_2 \geq 1{,}8) = \mathbb{P}\left(Z \geq \dfrac{1{,}8 - \mu_2}{\sigma_2}\right) \end{cases}$$

$$\begin{cases} \Phi\left(\dfrac{2{,}1 - \mu_2}{\sigma_2}\right) = 0{,}8413 \\ 1 - \Phi\left(\dfrac{1{,}8 - \mu_2}{\sigma_2}\right) = \Phi\left(\dfrac{\mu_2 - 1{,}8}{\sigma_2}\right) = 0{,}9772 \end{cases}$$

2) $\mathbb{P}\left(X>68 \mid X>\mu+\frac{1}{2}\sigma\right)=?$

determiniamo σ

$$0{,}84=\mathbb{P}(X>59{,}5)=\mathbb{P}\left(\frac{X-\mu}{\sigma}>\frac{59{,}5-65}{\sigma}\right)$$

$$=\mathbb{P}\left(Z>-\frac{5{,}5}{\sigma}\right)=1-\Phi\left(-\frac{5{,}5}{\sigma}\right)$$

$Z\sim\mathcal{N}(0,1)$

$$=\Phi\left(\frac{5{,}5}{\sigma}\right)=0{,}84$$

il valore deve essere $>\frac{1}{2}$ Ricorda!

$$\boxed{0{,}2=1-\Phi(x)=\cancel{\Phi(-x)}\Rightarrow\Phi(x)=1-0{,}2=0{,}8}$$

$$\Rightarrow\frac{5{,}5}{\sigma}=0{,}995$$

$$\Rightarrow\sigma=\frac{5{,}5}{0{,}995}=5{,}5276$$

$\sigma^2=30.5548$

$X\sim\mathcal{N}(65,30.5448)$

<u>Nota</u>: $\mu+\frac{1}{2}\sigma=65+\frac{1}{2}\cdot5{,}5276=67{,}7638<68$

$$\mathbb{P}\left(X>68\mid X>\mu+\frac{1}{2}\sigma\right)=$$

$$=\frac{\mathbb{P}\left((X>68)\cap\left(X>\mu+\frac{1}{2}\sigma\right)\right)}{\mathbb{P}\left(X>\mu+\frac{1}{2}\sigma\right)}$$

$$=\frac{\mathbb{P}(X>68)}{\mathbb{P}\left(X>\mu+\frac{1}{2}\sigma\right)}=\frac{\mathbb{P}\left(Z>\frac{68-65}{5{,}5276}\right)}{\mathbb{P}(Z>0{,}5)}=\frac{1-\Phi(0{,}54)}{1-\Phi(0{,}5)}=\frac{1-0{,}7054}{1-0{,}6915}=0{,}9549$$

3) $\mathbb{P}(63{,}5<X<66{,}5)$

$$=\mathbb{P}\left(\frac{63{,}5-65}{5{,}5276}<Z<\frac{66{,}5-65}{5{,}5276}\right)$$

$$=\mathbb{P}(-0{,}94<Z<0{,}94)$$

$= 1 - \Phi(3) = 1 - 0{,}9987 = 0{,}0013$

$\mathbb{P}(S) = \mathbb{P}(S|B)\,\mathbb{P}(B) = \mathbb{P}(S \cap B)$

$= 0{,}0013 \cdot 0{,}60 = \underline{0{,}00078}$

2) I = "la quantità di liquido è < 1"

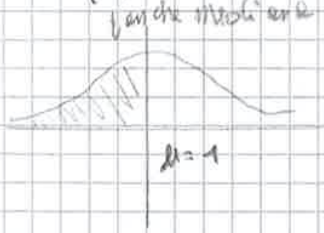
$$\mathbb{P}(A|I) \overset{\text{Bayes}}{=} \frac{\mathbb{P}(I|A)\,\mathbb{P}(A)}{\mathbb{P}(I)}$$

$$\mathbb{P}(I|A) = \mathbb{P}(X_A < 1) = \frac{1}{2} = \Big(\underbrace{1-0{,}95}_{\text{base}} \cdot \underbrace{\frac{1}{1{,}05-0{,}95}}_{\text{altezza}} = \frac{0{,}5}{1} = 0{,}5\Big)$$

$\mathbb{P}(I|B) = \mathbb{P}(X_B < 1)$

$X_B \sim \mathcal{N}(1, (0{,}03)^2)$

anche mediana



$$\Rightarrow \mathbb{P}(I) = \mathbb{P}(I|A)\,\mathbb{P}(A) + \mathbb{P}(I|B)\,\mathbb{P}(B)$$

$$= \frac{1}{2}\mathbb{P}(A) + \frac{1}{2}\mathbb{P}(B)$$

$$= \frac{1}{2}\underbrace{\big(\mathbb{P}(A) + \mathbb{P}(B)\big)}_{1} = \frac{1}{2}$$

$$\Rightarrow \mathbb{P}(A|I) = \frac{\mathbb{P}(I|A)\,\mathbb{P}(A)}{\mathbb{P}(I)} = \frac{\cancel{\frac{1}{2}} \cdot 0{,}4}{\cancel{\frac{1}{2}}} = 0{,}4$$

3) Y = n° di flaconi con < 1 litro di liquido, contenuti nella scatola

$Y \sim B(n,p) \quad n = 10 \quad p = \mathbb{P}(I) = \frac{1}{2}$

$$\mathbb{P}(Y=3) = \binom{10}{3} p^3 (1-p)^{10-3} = \frac{10!}{3!\,7!}\left(\frac{1}{2}\right)^{10} = 0{,}117$$

$$f_X(3) = f_X(4) = f_X(5) = \frac{1}{8}$$

$$\sum_{k=0}^{5} f_X(k) = 1$$

$$Y = g(X) = (X-2)^2$$

Densità discreta di y

$$Y(S) = \{0, 1, 4, 9\}$$

se $X=0 \rightarrow Y=4$

se $X=1 \rightarrow Y=1$

se $X=2 \rightarrow Y=0$

se $X=3 \rightarrow Y=1$

se $X=4 \rightarrow Y=4$

se $X=5 \rightarrow Y=9$

tutti valori di x che vanno in 0

$$f_Y(0) = \sum_{k: g(k)=0} f_X(k) = f_X(2) = \frac{1}{16}$$

$$f_Y(1) = \sum_{k: g(k)=1} f_X(k) = f_X(1) + f_X(3) = \frac{1}{16} + \frac{1}{8} = \frac{3}{16}$$

$$f_Y(4) = \sum_{k: g(k)=4} f_X(k) = f_X(0) + f_X(4) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

$$f_Y(9) = \sum_{k: g(k)=9} f_X(k) = f_X(5) = \frac{1}{8}$$

$E[Y]$?

più modi:

① $\mathbb{E}[Y] = \sum_k k f_Y(k)$

$= 0 \cdot f_Y(0) + 1 \cdot f_Y(1) + 4 f_Y(4) + 9 \cdot f_Y(9)$

② $\mathbb{E}[Y] = \mathbb{E}[(X-2)^2]$

$$= \sum_{k=0}^{5} (k-2)^2 f_X(k) = (0-2)^2 \cdot f_X(0) + \ldots\ldots$$

$$= e^y \frac{\theta}{(e^y)^{\theta+1}} = \cancel{e^y} \frac{\theta}{e^{y\theta + y}} = \frac{\theta}{e^{y\theta}}$$

$$\forall y \in D_y$$

$$f_Y(y) = \begin{cases} \theta e^{-\theta y} & \text{se } y \geq 0 \\ 0 & \text{altrove} \end{cases}$$

$$Y \sim \exp(\theta)$$

ALTRI METODI:

1) $F_Y(y) = P(Y \leq y) = P(\ln X \leq y) = P(X \leq e^y) = F_X(e^y) \quad \forall y \in \mathbb{R}$

$$= \int_{-\infty}^{e^y} f_X(x)\,dx = 0 \quad \text{se } e^y < 1 \text{ ovvero se } y < 0$$

se $y \geq 0$

$$F_Y(y) = \int_{-\infty}^{e^y} f_X(x)\,dx = \int_1^{e^y} f_X(x)\,dx$$

$$= \int_1^{e^y} \theta x^{-\theta-1}\,dx =$$

$$= \left[\cancel{\theta} \frac{x^{-\theta}}{-\cancel{\theta}} \right]_1^{e^y} = 1 - (e^y)^{-\theta} = 1 - e^{-\theta y}$$

$$F_Y(y) = \begin{cases} 1 - e^{-\theta y}, & y \geq 0 \\ 0 & \text{altrove} \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \theta e^{-\theta y} & y \geq 0 \\ 0 & \text{altrove} \end{cases}$$

$$Y \sim \exp(\theta)$$

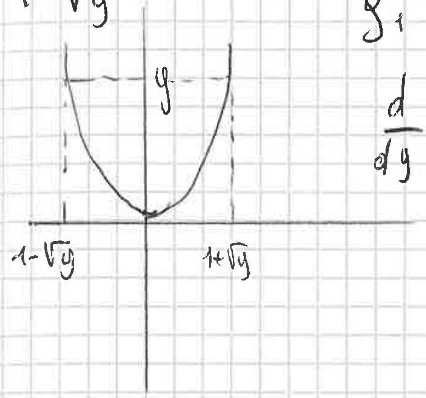
$\boxed{(x-1)^2 = y \Rightarrow x = 1 \pm \sqrt{y}}$

$\Rightarrow x - 1 = -\sqrt{y}$

$x = 1 - \sqrt{y}$

$g_1^{-1}(y) = 1 - \sqrt{y}$

$\frac{d}{dy} g_1^{-1}(y) = -\frac{1}{2\sqrt{y}}$



$D_X^2 = (1, 5)$

$g_2(x) = (x-1)^2 \Rightarrow g_2^{-1}(y) = 1 + \sqrt{y}$

$\frac{d}{dy} g_2^{-1}(y) = +\frac{1}{2\sqrt{y}}$

$f_Y(y) = f_X(g_1^{-1}(y)) \left| \frac{d}{dy} g_1^{-1}(y) \right| + f_X(g_2^{-1}(y)) \left| \frac{d}{dy} g_2^{-1}(y) \right|$

$\forall y \in (0, 16) = D_g \setminus \{0\}$

$f_Y(y) = f_X(1-\sqrt{y}) \left| -\frac{1}{2\sqrt{y}} \right| + f_X(1+\sqrt{y}) \left| +\frac{1}{2\sqrt{y}} \right|$

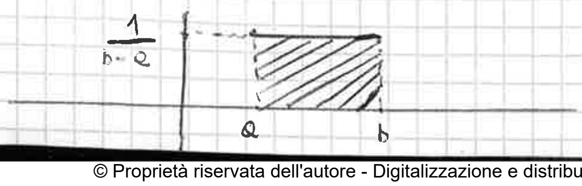
$= \frac{1}{2\sqrt{y}} \left(f_X(1-\sqrt{y}) + f_X(1+\sqrt{y}) \right)$

$f_X(x) = \begin{cases} \frac{1}{5-(-3)} = \frac{1}{8} & \text{se } -3 < x < 5 \\ 0 & \text{altrove} \end{cases}$

$\forall y \in (0, 16)$

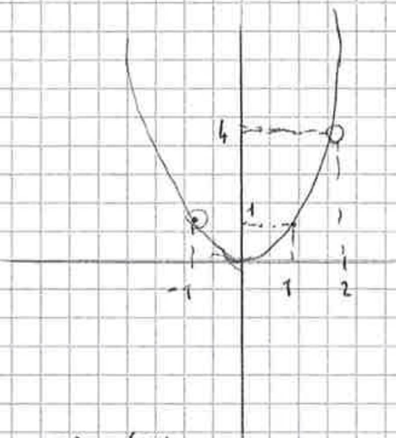
$f_Y(y) = \frac{1}{2\sqrt{y}} \left(\frac{1}{8} + \frac{1}{8} \right) = \frac{1}{8\sqrt{y}}$

se $y \notin (0, 16)$ $\quad f_Y(y) = 0$



ES. 4

$$f_X(x) = \begin{cases} \frac{2}{9}(x+1) & \text{se } -1 < x < 2 \\ 0 & \text{altrove} \end{cases}$$



$y = x^2$

$g(x) = x^2$

$D_X = (-1, 2)$

$D_Y = [0, 4)$

Se $y \in [1, 4)$ la controimmagine di y contiene un punto solo in $[1, 2)$

se $y \in (0, 1)$ la controimmagine contiene due punti, uno in $(-1, 0)$, l'altro in $(0, 1)$

0 non ci interessa.

1° ramo di parabola

$g_1(x) = x^2 \Rightarrow g_1^{-1}(y) = -\sqrt{y}$

$$\frac{d}{dy} g_1^{-1}(y) = -\frac{1}{2\sqrt{y}}$$

2° ramo di parabola

$g_2(x) = x^2 \Rightarrow g_2^{-1}(y) = +\sqrt{y}$

$$\frac{d}{dy} g_2^{-1}(y) = +\frac{1}{2\sqrt{y}}$$

Se $y \in [1, 4)$

allora $g_2^{-1}(y) \in [1, 2)$

$$f_Y(y) = f_X(g_2^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = f_X(\sqrt{y}) \left| +\frac{1}{2\sqrt{y}} \right|$$

Esercitazione 10

ES.5

$X \sim \mathcal{N}(2,4)$ $\quad \rightarrow \mu = 2$

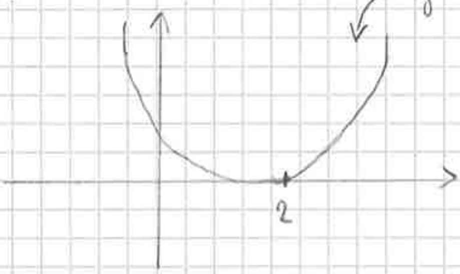
$Y = \left(\frac{X}{2} - 1\right)^2$ $\quad \rightarrow \sigma^2 = 4$

f_Y ? $\quad \mathbb{E}[Y]$?

$D_X = \mathbb{R}$ ← insieme in cui f di densità $\neq 0$

$g(x) = \left(\frac{X}{2} - 1\right)^2$

g non è invertibile



$D_Y = g(D_X) = [0, +\infty)$

$\{0\}$ è un punto isolato $\Rightarrow$ lo escludiamo

Poniamo $D_Y = (0, +\infty)$

$D_X = (-\infty, 2) \cup \{2\} \cup (2, +\infty)$

↳ trascuriamo

Consideriamo g_1 la restrizione di g a

$g_1(x) = \left(\frac{x}{2} - 1\right)^2 = y$

$\Rightarrow \frac{x}{2} - 1 = \sqrt{y} \Rightarrow x = 2 - 2\sqrt{y}$

$g_1^{-1}(y) = 2 - 2\sqrt{y}$

$\Rightarrow \frac{d}{dy} g_1^{-1}(y) = -\frac{2}{2\sqrt{y}} = \frac{1}{\sqrt{y}}$

Consideriamo g_2 la restrizione di g a $D_X^2 = (2, +\infty)$

$g_2(x) = \left(\frac{x}{2} - 1\right)^2 = y \quad \Rightarrow \frac{x}{2} - 1 = \sqrt{y} \Rightarrow x = 2 + 2\sqrt{y}$

$g_2^{-1}(y) = 2 + 2\sqrt{y} \quad \forall y \in (0, +\infty) = D_Y$

$\cdot)\ \mathbb{E}[y] = \mathbb{E}\left[\left(\frac{x}{2}-1\right)^2\right] = \mathbb{E}\left[\frac{x^2}{4}+1-x\right] = \frac{1}{4}\mathbb{E}[x^2]+1-\mathbb{E}[x] =$

$= \frac{1}{4}\left(var(x)+\mathbb{E}[x]^2\right)+1-\mathbb{E}[x] =$

$= \frac{1}{4}(4+4)+1-2 = 2+1-2 = 1$

$\cdot)\ \mathbb{E}[y] = \mathbb{E}\left[\left(\frac{x}{2}-1\right)^2\right] = var\left(\frac{x}{2}\right)$

$\mathbb{E}[x]=2$

$\mathbb{E}\left[\frac{x}{2}\right]=1$

$\mathbb{E}\left[\left(\frac{x}{2}\right)^2\right]+1-\mathbb{E}[x] \Rightarrow \mathbb{E}\left[\left(\frac{x}{2}\right)^2\right]-1 = \mathbb{E}\left[\left(\frac{x}{2}\right)^2\right]-\left(\mathbb{E}\left[\frac{x}{2}\right]\right)^2 = var\left(\frac{x}{2}\right)$

$\cdot)\ \mathbb{E}[y] = \mathbb{E}\left[\left(\frac{x}{2}-1\right)^2\right] = \mathbb{E}\left[\left(\frac{x-2}{2}\right)^2\right] = \mathbb{E}[Z^2] = var(Z) = 1$

$X \sim \mathcal{N}(2,4)$

$\Rightarrow \frac{x-2}{2} \sim \mathcal{N}(0,1)$

FOGLIO 7

ES 1

$N_A(l)$ = n° di difetti di tipo A in l metri di filato

$N_B(l)$ = ... di tipo B . . . .

$N_A(l)$ e $N_B(l)$ sono indipendenti

λ_A: n° medio di difetti di tipo A per metro

$\lambda_A = \frac{0,25}{100} = 0,0025 \qquad N_A(1) \sim P(\lambda_A)$

$N_A(l) \sim P(\lambda_A\, l)$

λ_B = n° medio di difetti di tipo B per metro

$\lambda_B = \frac{0,5}{125} = 0,004$

$N_B(1) \sim P(\lambda_B)$

$N_B(l) \sim P(\lambda_B \cdot l)$

3) $\ell = 400$

$N_A(400) \sim P(\lambda_A \cdot 400) = P(1)$

$N_B(400) \sim P(\lambda_B \cdot 400) = P(1{,}6)$

$N(400) \sim P(\lambda \cdot 400) = P(2{,}6)$

$$\mathbb{P}\left(N_A(400) = 1 \mid N(400) = 3\right)$$

$$= \frac{\mathbb{P}\left(N_A(400) = 1,\ N(400) = 3\right)}{\mathbb{P}\left(N(400) = 3\right)} = \frac{\mathbb{P}\left(N_A(400) = 1,\ N_B(400) = 2\right)}{\mathbb{P}\left(N(400) = 3\right)} =$$

$$= \frac{\left(\cancel{e^{-1}} \frac{1^1}{1!}\right)\left(\cancel{e^{-1,6}} \frac{(1{,}6)^2}{2!}\right)}{\left(\cancel{e^{-2,6}} \frac{(2{,}6)^3}{3!}\right)}$$

$$= \frac{1^1}{1!} \frac{(1{,}6)^2}{2!} \cdot \frac{3!}{(2{,}6)^3} \cong 0{,}4370$$

$$= \frac{3!}{2!\,1!} \frac{1^1 \cdot (1{,}6)^2}{(2{,}6)^3} = \binom{3}{1}\left(\frac{1}{2{,}6}\right)^1\left(1 - \frac{1}{2{,}6}\right)^2 = \mathbb{P}(X = 1)$$

dove $X \sim B(n, p)$

$n = 3$, $p = \frac{1}{2{,}6} = \frac{\lambda_A \cdot 400}{\lambda \cdot 400} = \frac{\lambda_A}{\lambda}$ n° di difetti di tipo A tra i tre difetti totale

$N_C(t)$ = n° di camper che transitano in un tempo t

$N_A(t)$ = n° di altri veicoli che transitano in un tempo t

$\lambda_C = 0,2$ $\quad N_C(t)$ Poisson

$N_A(t)$ indipendente da $N_C(t)$

$N(t) = N_C(t) + N_A(t)$

$N_A(t)$ = Poisson con $\lambda_A = \lambda - \lambda_C = 2,8$

$$\mathbb{P}\left(N_C(10) = 1 \mid N(10) = 10\right) =$$

$$= \frac{\mathbb{P}\left(N_C(10) = 1\right)\mathbb{P}\left(N_A(10) = 9\right)}{\mathbb{P}\left(N(10) = 10\right)}$$

$$= \frac{\left(\cancel{e^{-2}} \cdot \frac{2^1}{1!}\right)\left(\cancel{e^{-28}} \cdot \frac{28^9}{9!}\right)}{\cancel{e^{-30}} \frac{(30)^{10}}{10!}} = 0,3583$$

$N_C(10) \sim P(2)$
$N_A(10) \sim P(28)$
$N(10) \sim P(30)$

$t = \frac{1}{2}$

$N\left(\frac{1}{2}\right) \sim P(1,5)$

$$\mathbb{P}\left(N\left(\tfrac{1}{2}\right) = 0\right) = e^{-1,5}\frac{(1,5)^0}{0!} = e^{-1,5} = 0,2231$$

T = tempo tra due veicoli

$T \sim \exp(\lambda)$

$$\mathbb{P}\left(T > \tfrac{1}{2}\right) = 1 - F_T\left(\tfrac{1}{2}\right) = 1 - \left(1 - e^{-\lambda t}\right) = e^{-\lambda t} = e^{-3 \cdot \frac{1}{2}} = e^{-1,5}$$

2) $\mathbb{P}\left(N\left(\frac{1}{3}\right) \geq 1 \;\middle|\; N(1) = 2\right)$

$= 1 - \mathbb{P}\left(N\left(\frac{1}{3}\right) = 0 \;\middle|\; N(1) = 2\right)$

$$= 1 - \frac{\mathbb{P}\left(N\left(\frac{1}{3}\right)=0\right) \mathbb{P}\left(N\left(\frac{2}{3}\right)=2\right)}{\mathbb{P}(N(1)=2)} =$$

$$= 1 - \frac{\left(\cancel{e^{-5/3}} \cdot \frac{\left(\frac{5}{3}\right)^0}{0!}\right)\left(\cancel{e^{-10/3}} \cdot \frac{\left(\frac{10}{3}\right)^2}{\cancel{2!}}\right)}{\cancel{e^{-5}} \; \frac{5^2}{\cancel{2!}}}$$

$$= 1 - \frac{\left(\frac{10}{3}\right)^2}{5^2} = 1 - \frac{4}{9} = \frac{5}{9}$$

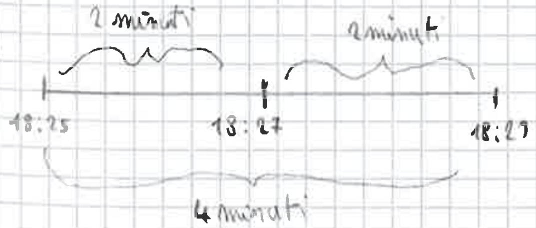
ES 4

$N(t_1, t_2)$ = n° di volte in cui la porta si apre tra t_1 e t_2

$N(t_1, t_2) \sim \mathcal{P}(\alpha(t_2 - t_1))$

α = 1,5 = n° medio di aperture al minuto

1)



$$\mathbb{P}\left(N(18{:}27,\ 18{:}29) = 3 \;\middle|\; N(18{:}25,\ 18{:}29) = 5\right)$$

$$= \frac{\mathbb{P}\left(N(18{:}27, 18{:}29) = 3\right) \mathbb{P}\left(N(18{:}25, 18{:}27) = 2\right)}{\mathbb{P}\left(N(18{:}25, 18{:}29) = 5\right)}$$

$= *$

$N(18{:}25, 18{:}27) \sim \mathcal{P}(2\alpha) = \mathcal{P}(3)$

$\overline{X}_{12} \sim N\left(\mu, \frac{\sigma^2}{12}\right)$ è ancora una variabile gaussiana perché combinazione lineare di variabili gaussiane indipendenti.

$P(63{,}5 < \overline{X}_{12} < 66{,}5) = ?$

$$P(63{,}5 < \overline{X}_{12} < 66{,}5) \overset{\text{standardizziamo}}{=} P\left(\frac{63{,}5-65}{5{,}5276/\sqrt{12}} < \frac{\overline{X}_{12}-\mu}{\sigma/\sqrt{12}} < \frac{66{,}5-65}{5{,}5276/\sqrt{12}}\right) = P(-0{,}94 < Z < 0{,}94)$$

$$= P(-0{,}94 < Z < 0{,}94) = \Phi(0{,}94) - \Phi(-0{,}94) = \Phi(0{,}94) - (1 - \Phi(0{,}94)) = 2\Phi(0{,}94) - 1 =$$

$$= 2 \cdot 0{,}8264 - 1 = 0{,}6528$$

ES. 2

(Es 2 del foglio 5)

X_1: diametro di una rondella della 1ª scatola

$X_1 = N(\mu_1, \sigma_1^2)$, $\mu_1 = 2{,}1$ $\sigma_1^2 = 0{,}04$

$X_1^1 \dots X_{10}^1$ campione di rondelle della 1ª scatola (diametri)

$$\overline{X}_{10}^1 = \frac{1}{10}\sum_{i=1}^{10} X_i^1 \sim N\left(\mu_1, \frac{\sigma_1^2}{10}\right)$$

X_2 = diametro di una rondella della 2ª scatola

$X_2 \sim N(\mu_2, \sigma_2^2)$, $\mu_2 = 2$ $\sigma_2^2 = 0{,}01$

$X_1^2 \dots X_{12}^2$ campione di rondelle della 2ª scatola (diametri)

$$\overline{X}_{12}^2 = \frac{1}{12}\sum_{i=1}^{12} X_i^2 \sim N\left(\mu_2, \frac{\sigma_2^2}{12}\right)$$

$\overline{X}_{10}^1$ = diametro medio delle 10 rondelle della prima scatola

$\overline{X}_{12}^2$ = diametro medio delle 12 rondelle della seconda scatola

$$P(\overline{X}_{10}^1 < \overline{X}_{12}^2) = P(\overline{X}_{10}^1 - \overline{X}_{12}^2 < 0) = ?$$

Supponiamo che i diametri della 1ª scatola siano indipendenti da quelli della seconda scatola.
$\overline{X}_{10}^1$ e $\overline{X}_{12}^2$ sono indipendenti.

$\overline{X}_{10}^1 - \overline{X}_{12}^2 \sim N(\mu, \sigma^2)$ variabile gaussiana

$\mu = \mu_1 - \mu_2 = 2{,}1 - 2 = 0{,}1$

$$\sigma^2 = \frac{\sigma_1^2}{10} + \frac{\sigma_2^2}{12} = \frac{0{,}04}{10} + \frac{0{,}01}{12} = 0{,}00483 \Rightarrow \sigma = 0{,}0695$$

$$P(\overline{X}_{10}^1 < \overline{X}_{12}^2) = P(\overline{X}_{10}^1 - \overline{X}_{12}^2 < 0) =$$

$= \Phi(0{,}39) = 0{,}6517$

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ES. 4

$X_1 \sim Exp(1)$ $\quad$ $E[X_1] = 1$ $\quad$ $var(X_1) = 1$

$X_2 \sim Exp(2)$ $\quad$ $E[X_2] = 1/2$ $\quad$ $var(X_2) = 1/4$

$X_3 \sim Exp(4)$ $\quad$ $E[X_3] = 1/4$ $\quad$ $var(X_3) = 1/16$

mutuamente non correlate $\quad cov(X_i, X_j) = 0 \quad \forall i \neq j$

1) $Y = X_1 + X_2 - X_3$ $\quad E[Y] = ? \quad var[Y] = ?$

$$E[Y] = E[X_1 + X_2 - X_3] = E[X_1] + E[X_2] - E[X_3] = 1 + \frac{1}{2} - \frac{1}{4} = \frac{5}{4}$$

$$var(Y) = var(X_1 + X_2 - X_3) =$$

$$= var(X_1) + var(X_2) + var(X_3) - 2\cancel{cov(X_1, X_3)} - 2cov(X_2, X_3) + 2\cancel{cov(X_1, X_2)}$$

$$= var(X_1) + var(X_2) + var(X_3) = 1 + \frac{1}{4} + \frac{1}{16} = \frac{21}{16}$$

2) $U = 2X_1 - X_3$

$V = X_1 + X_2 + X_3$

Sono non correlate?

$$cov(U, V) = \quad ?$$

$$= cov(2X_1 - X_3, X_1 + X_2 + X_3)$$

$$= 2cov(X_1, X_1) + 2\cancel{cov(X_1, X_2)} + 2\cancel{cov(X_1, X_3)} - \cancel{cov(X_3, X_1)} - \cancel{cov(X_3, X_2)} - cov(X_3, X_3)$$

$$= 2var(X_1) - var(X_3)$$

$$= 2 \cdot 1 - \frac{1}{16} = \frac{31}{16} \neq 0$$

U e V sono correlate

$$\rho(U, V) = \frac{cov(U, V)}{\sqrt{var(U) \cdot var(V)}}$$

$$var(U) = var(2X_1 - X_3) = 4var(X_1) + var(X_3) - 4\cancel{cov(X_1, X_3)}$$

$$= 4 \cdot 1 + \frac{1}{16} = \frac{65}{16}$$

$$cov(X_1, X_2) = \rho(X_1, X_2)\sqrt{var(X_1)\ var(X_2)}$$

$$= 0{,}4\sqrt{1 \cdot 1/4} = 0{,}2$$

$$cov(X_1, X_3) = \rho(X_1, X_3)\sqrt{var(X_1)\ var(X_3)} = 0{,}4\sqrt{1 \cdot \frac{1}{16}} = 0{,}1$$

$$cov(X_2, X_3) = \rho(X_2, X_3)\sqrt{var(X_2)\ var(X_3)} = 0{,}4\sqrt{\frac{1}{4} \cdot \frac{1}{16}} = 0{,}05$$

$$var(Y) = \frac{21}{16} + 2 \cdot 0{,}2 - 2 \cdot 0{,}1 - 2 \cdot 0{,}05 = \frac{113}{80}$$

ES. 5

$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ indipendenti

1) $\mu = \mu_1 = \mu_2 = \mu_3 = 60$

$\sigma^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 15$

$$Y = X_1 + X_2 + X_3$$

$$Y \sim \mathcal{N}(\mu_1 + \mu_2 + \mu_3,\ \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

$$Y \sim \mathcal{N}(\underbrace{3\mu}_{180},\ \underbrace{3\sigma^2}_{45})$$

$$Y \sim \mathcal{N}(180, 45)$$

$P(Y \leq 200) =$ $Z \sim \mathcal{N}(0,1)$

$$= P\left(Z \leq \frac{200-180}{\sqrt{45}}\right) =$$

$$= \Phi(2{,}98) = 0{,}9986$$

$$P(150 \leq Y \leq 200) =$$

$$= P\left(\frac{150-180}{\sqrt{45}} \leq Z \leq \frac{200-180}{\sqrt{45}}\right) = P(-4{,}47 \leq Z \leq 2{,}98) = \Phi(2{,}98) - \underbrace{\Phi(-4{,}47)}_{0}$$

$\mathbb{P}(-10 \leq T \leq 5) =$

$= \mathbb{P}\left(\frac{-10-0}{\sqrt{\frac{45}{2}}} \leq Z \leq \frac{5-0}{\sqrt{\frac{45}{2}}}\right)$

$= \Phi(1{,}05) - \Phi(-2{,}11) =$

$= \Phi(1{,}05) - (1 - \Phi(2{,}11))$

$= \Phi(1{,}05) + \Phi(2{,}11) - 1$

$= 0{,}8531 + 0{,}9826 - 1 = \underline{0{,}8357}$

4) $\mu_1 = 40 \qquad \mu_2 = 50 \qquad \mu_3 = 60$

$\sigma_1^2 = 10 \quad \sigma_2^2 = 12 \quad \sigma_3^2 = 14$

$Y = X_1 + X_2 + X_3 \qquad , \qquad Y \sim \mathcal{N}(\mu_1 + \mu_2 + \mu_3,\ \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$

$Y \sim \mathcal{N}(150, 36)$

$\mathbb{P}(T \leq 160) = \mathbb{P}\left(Z \leq \frac{160-150}{6}\right) = \mathbb{P}\left(Z \leq \frac{5}{3}\right) = \Phi(1{,}67) = 0{,}9525$

$U = X_1 + X_2 - 2X_3$

$\mathbb{P}(X_1 + X_2 \geq 2X_3) = \mathbb{P}(X_1 + X_2 - 2X_3 \geq 0)$

$U \sim \mathcal{N}(\mu_1 + \mu_2 - 2\mu_3,\ \sigma_1^2 + \sigma_2^2 + 4\sigma_3^2)$

$U \sim \mathcal{N}(-30, 78)$

$\mathbb{P}(U \geq 0) = \mathbb{P}\left(Z \geq \frac{0-(-30)}{\sqrt{78}}\right) = \mathbb{P}\left(Z \geq \frac{30}{\sqrt{78}}\right) = 1 - \Phi(3{,}40) = 1 - 0{,}9997 =$

$0{,}0003 \simeq 0$

Esercitazione 13

FOGLIO N° 9

ES 1

Siano $X_1 \dots X_n \underset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$

↳ indipendenti identicamente distribuite

Cerchiamo n tale che

$$P(|\overline{X}_n - \mu| < 0{,}05\sigma) \geq 0{,}95$$

Ricordiamo che $\overline{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$

$$P(|\overline{X}_n - \mu| \leq 0{,}05\sigma) = P\left(\left|\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}\right| \leq \frac{0{,}05\sigma}{\sigma/\sqrt{n}}\right) \underset{Z \sim \mathcal{N}(0,1)}{=} P(|Z| \leq 0{,}05\sqrt{n}) =$$

$$= P(-0{,}05\sqrt{n} \leq Z \leq +0{,}05\sqrt{n}) = \Phi(0{,}05\sqrt{n}) - \Phi(-0{,}05\sqrt{n})$$

$$= \Phi(0{,}05\sqrt{n}) - [1 - \Phi(0{,}05\sqrt{n})]$$

$$= \boxed{2\Phi(0{,}05\sqrt{n}) - 1 \geq 0{,}95}$$

$$\Phi(0{,}05\sqrt{n}) \geq \frac{1{,}95}{2} = 0{,}975$$

$$0{,}05\sqrt{n} \geq \overset{\text{quantile}}{z_{0{,}975}} = \Phi^{-1}(0{,}975)$$

$$0{,}05\sqrt{n} \geq 1{,}96$$

$$\sqrt{n} \geq \frac{1{,}96}{0{,}05} \Rightarrow n \geq \left(\frac{1{,}96}{0{,}05}\right)^2 = (39{,}2)^2$$

$$n \geq 1536{,}64$$

$$n \geq 1537$$

Se non conosciamo la distribuzione di $X_1 \dots X_n$?

$$E[\overline{X}_n] = \mu \qquad var(\overline{X}_n) = \frac{\sigma^2}{n}$$

Lvi@??

$$P(|\overline{X}_n - \mu| \leq 0{,}05\sigma) = P\Big(|\overline{X}_n - E[\overline{X}_n]| \leq \underbrace{(0{,}05\sqrt{n})}_{t}\underbrace{\sqrt{var(X_n)}}_{\sigma/\sqrt{n}}\Big)$$

$= \left(\frac{3}{2}+4\theta\right)^n \prod_{i=1}^{n} x_i^{4\theta+\frac{1}{2}}$

Consideriamo $l(\theta; x_1 \dots x_n) = \log L(\theta; x_1 \dots x_n)$

$l(\theta) = \log\left(\left(\frac{3}{2}+4\theta\right)^n \prod_{i=1}^{n} x_i^{4\theta+\frac{1}{2}}\right)$

$= n \log\left(\frac{3}{2}+4\theta\right) + \sum_{i=1}^{n} \log\left(x_i^{4\theta+\frac{1}{2}}\right) = n \log\left(\frac{3}{2}+4\theta\right) + \left(4\theta+\frac{1}{2}\right)\sum_{i=1}^{n} \log x_i$

$$\frac{dl}{d\theta}(\theta) = n \cdot \frac{1}{\frac{3}{2}+4\theta} \cdot 4 + 4\sum_{i=1}^{n} \log x_i = \frac{4n}{\frac{3}{2}+4\theta} + 4\sum_{i=1}^{n} \log x_i = 0$$

$$\frac{n}{\frac{3}{2}+4\theta} = -\sum_{i=1}^{n} \log x_i \Rightarrow \frac{3}{2}+4\theta = -\frac{n}{\sum_{i=1}^{n} \log x_i}$$

$$\theta = \frac{1}{4}\left(-\frac{n}{\sum_{i=1}^{n} \log x_i} - \frac{3}{2}\right) = -\frac{3}{8} - \frac{n}{4\sum_{i=1}^{n} \log x_i}$$

→ punto di estremo relativo

Consideriamo

$$\frac{d^2 l}{d\theta^2}(\theta) = 4n\left(-\frac{1}{\left(\frac{3}{2}+4\theta\right)^2}\right) \cdot 4 = -\frac{16n}{\left(\frac{3}{2}+4\theta\right)^2} < 0$$

La funzione l è concava

Il punto che ho trovato è un punto di massimo relativo

Cosa succede ai bordi?

$\theta > -\frac{3}{8} \Rightarrow \theta \in \left(-\frac{3}{8}, +\infty\right)$

Osserviamo che $-\frac{3}{8} - \frac{n}{4\sum \log x_i} > -\frac{3}{8}$

$x_i \in (0,1)$

$\log x_i < 0$

$$\lim_{\theta \to -\frac{3}{8}} l(\theta) = \lim_{\theta \to -\frac{3}{8}} \left[\underbrace{n \log\left(\frac{3}{2}+4\theta\right)}_{\to -\infty} + \underbrace{\left(4\theta+\frac{1}{2}\right)\sum_{i=1}^{n} \log x_i}_{\to -1 \cdot (<0)}\right] = -\infty$$

?

$$\lim_{\theta \to +\infty} l(\theta) = \lim_{\theta \to +\infty} \left[n \log\underbrace{\left(\frac{3}{2}+4\theta\right)}_{\to +\infty} + \underbrace{\left(4\theta+\frac{1}{2}\right)}_{+\infty}\underbrace{\sum_{i=1}^{n} \log x_i}_{<0}\right] = -\infty$$

va più lentamente

$-\infty$

Il punto trovato è un punto di massimo assoluto

Lo stimatore di massima verosimiglianza è $\Theta^{MV} = \Theta^{MLE} = -\frac{3}{8} - \frac{n}{4\sum_{i=1}^{n} \log X_i}$

Una stima è

$$\hat{\theta} = -\frac{3}{8} - \frac{n}{4\sum_{i=1}^{n} \log x_i}$$

ES. 5

$X_1 \dots X_n$ campione

$$f(x, \theta) = \begin{cases} \frac{1}{\sqrt{\theta}} e^{-\frac{x}{\sqrt{\theta}}} & \text{se } x > 0 \\ 0 & \text{altrimenti} \end{cases}$$

$\theta > 0$

Θ^{MV} ? $\quad \Theta^{MM}$?

Sono corretti?

Stimatore di max verosimiglianza

$$L(\theta; x_1 \dots x_n) = \prod_{i=1}^{n} f(x_i; \theta)$$

Siano $x_1 \dots x_n > 0$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{\theta}} e^{-\frac{x_i}{\sqrt{\theta}}}$$

$$= \left(\frac{1}{\sqrt{\theta}}\right)^n \prod_{i=1}^{n} e^{-\frac{x_i}{\sqrt{\theta}}}$$

Consideriamo

$$l(\theta) = \log L(\theta) = \log\left(\theta^{-\frac{n}{2}} \prod_{i=1}^{n} e^{-\frac{x_i}{\sqrt{\theta}}}\right)$$

$$= -\frac{n}{2}\log\theta + \sum_{i=1}^{n}\left(-\frac{x_i}{\sqrt{\theta}}\right)$$

$$= -\frac{n}{2}\log\theta - \frac{1}{\sqrt{\theta}}\sum_{i=1}^{n} x_i$$

$$\frac{d}{d\theta} l(\theta) = -\frac{n}{2\theta} + \frac{1}{2}\theta^{-\frac{1}{2}-1}\sum_{i=1}^{n} x_i =$$

$$= -\frac{n}{2\theta} + \frac{1}{2\theta^{3/2}}\sum_{i=1}^{n} x_i =$$

$$= -\frac{n}{2\theta} + \frac{1}{2\theta\sqrt{\theta}}\sum_{i=1}^{n} x_i =$$

$$= \frac{1}{2\theta}\left(-n + \frac{1}{\sqrt{\theta}}\sum_{i=1}^{n} x_i\right) = 0$$

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$\Rightarrow$ punto di max assoluto per ℓ $\Rightarrow$ anche per L

Stimatore

$\Theta^{MV} = \overline{X}_n^2$

Stima

$\hat{\theta} = \overline{x}_n^2$

Stimatore con il metodo dei momenti

$$\mu_1' = \mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, f(x;\theta)\, dx = \int_0^{+\infty} x \frac{1}{\sqrt{\theta}} e^{-\frac{x}{\sqrt{\theta}}}\, dx$$

X distribuite secondo $f(x;\theta)$

$X \sim Exp\left(\frac{1}{\sqrt{\theta}}\right) \Rightarrow \mathbb{E}[X] = \sqrt{\theta}$

$\mu_1' = \sqrt{\theta}$

$\underset{\theta>0}{\Rightarrow} \theta = (\mu_1')^2$

$\Rightarrow \hat{\theta} = (\overline{x}_n)^2 \Rightarrow \Theta^{MM} = (M_1')^2 = \overline{X}_n^2$

È corretto (non distorto)?

$\mathbb{E}[\overline{X}_n^2] = \theta$???

$\mathbb{E}[\overline{X}_n^2] = var(\overline{X}_n) + \mathbb{E}[\overline{X}_n]^2$

$= \frac{\sigma^2}{n} + \mu^2$

$\mu = \mathbb{E}[X] = \sqrt{\theta}$

$\sigma^2 = var[X] = \theta$

$\Rightarrow \mathbb{E}[\overline{X}_n^2] = \frac{\theta}{n} + \theta = \frac{n+1}{n}\theta \neq \theta$

Non è corretto

$$E[\Theta^{MV}] = \frac{n}{n+1}\theta \neq \theta$$

però se consideriamo

$$T_n = \frac{n+1}{n}\Theta^{MV} = \frac{n+1}{n}\max_{1\le i\le n} x_i$$

ES. 7

$X_1 \dots X_n$ campioni

$$E[X_i] = \mu$$

$$\mathrm{Var}(X_i) = 2$$

$$T_1(n) = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n$$

$$T_2(n) = \frac{2}{n(n+1)}\sum_{i=1}^{n} i\, X_i$$

$$T_3(n) = \sum_{i=1}^{n} (-1)^i X_i$$

$$T_4(n) = \frac{X_1 + \sum_{i=2}^{n-1} (-1)^i X_i + X_n}{2 + \sum_{i=2}^{n-1} (-1)^i}$$

stimatori di μ

a) Corretti?

$$E[T_1(n)] = \mu \quad \text{sì perché } T_1 = \overline{X}_n$$

RICORDARE

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$E[T_2(n)] = E\left[\frac{2}{n(n+1)}\sum_{i=1}^{n} i\, X_i\right] =$$

$$= \frac{2}{n(n+1)}\sum_{i=1}^{n} i\, E[X_i] = \frac{2\mu}{n(n+1)}\sum_{i=1}^{n} i$$

$$= \frac{2\mu}{n(n+1)} \cdot \frac{n(n+1)}{2} = \mu \quad \text{sì}$$

Se $T_i(n)$ è corretto

$$MSE_{T_i(n)} = var(T_i(n))$$

$T_i(n)$ è consistente se

$$MSE_{T_i(n)} \xrightarrow[n \to \infty]{} 0$$

$$MSE_{T_2(n)} \xrightarrow[n \to +\infty]{} 0 \quad \text{SÌ} \quad ?$$

$$MSE_{T_2(n)} \underset{\text{corretto}}{=} var(T_2(n))$$

$$= var\left(\frac{2}{n(n+1)} \sum_{i=1}^{n} i X_i\right) \underset{\text{INDIP}}{=} \frac{4}{n^2(n+1)^2} \sum_{i=1}^{n} i^2 \, var(X_i)$$

$$= \frac{8}{n^2(n+1)^2} \sum_{i=1}^{n} i^2 \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{8}{n^2(n+1)^2} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{4}{3} \frac{2n+1}{n(n+1)} \xrightarrow[n \to +\infty]{} 0 \quad \text{SÌ}$$

$$MSE_{T_3(n)} = var(T_3(n)) + \left(E[T_3(n)] - \mu\right)^2$$

$$var(T_3(n)) = var\left(\sum_{i=1}^{n} (-1)^i X_i\right) = \sum_{i=1}^{n} \left((-1)^i\right)^2 var(X_i) = \sum_{i=1}^{n} 2 = 2n$$

$$\left(E[T_3(n)] - \mu\right)^2 = \begin{cases} (0-\mu)^2 = \mu^2 & n \text{ pari} \\ (-\mu-\mu)^2 = 4\mu^2 & n \text{ dispari} \end{cases}$$

$$MSE_{T_3(n)} = \begin{cases} 2n + \mu^2 & n \text{ pari} \\ 2n + 4\mu^2 & n \text{ dispari} \end{cases} \quad n \to \infty, \to 0 \; ?$$

= NO

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Esercizio svolto in aula (output, test di ipotesi)

Si consideri il seguente output riferito a due insiemi di valori che sono le misurazioni dei diametri del cranio (in cm) di due campioni indipendenti di scimpanzé:

Descriptive statistics: A; B

Variable	Total Count	Mean	StDev	Variance
A	23	17,796	3,504	12,279
B	18	18,239	2,972	8,832

Assumendo che le misure dei diametri dei crani delle due popolazioni di scimpanzé seguano una distribuzione normale, testare l'ipotesi nulla che non vi è differenza tra le medie delle due popolazioni (assumere un livello di significatività pari al 5%).

X_A = variabile casuale che indica diametro della prima popolazione $\quad X_A \sim N(\mu_1, \sigma_1^2)$

X_B = variabile casuale che indica diametro della seconda popolazione $\quad X_B \sim N(\mu_2, \sigma_2^2)$

$H_0: \mu_1 - \mu_2 = 0 \qquad (\overset{\text{differenza}}{d} = 0)$

$H_A: \mu_1 - \mu_2 \neq 0 \qquad (\mu_1 \neq \mu_2)$

$n_1 = 23 \qquad \overline{X}_1 = 17,796 \qquad s_1^2 = 12,279$

$n_2 = 18 \qquad \overline{X}_2 = 18,239 \qquad s_2^2 = 8,832$

$H_0: \sigma_1^2 = \sigma_2^2 \qquad \left(\frac{\sigma_1^2}{\sigma_2^2} = 1\right)$

$H_A: \sigma_1^2 \neq \sigma_2^2$

$\alpha = 5\%$

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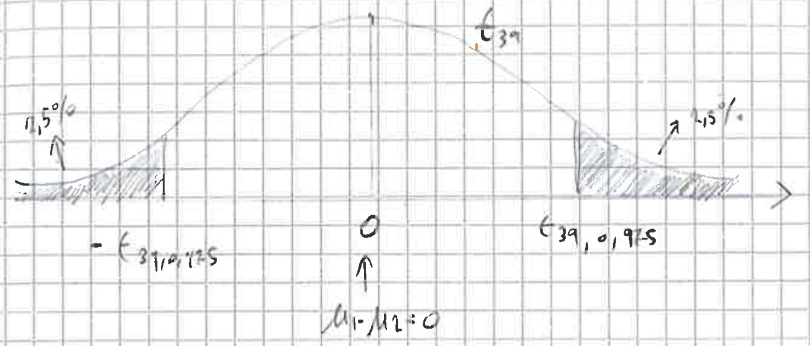


$$t_{39,\ 0,975} \begin{cases} \to \cong z_{0,975} = 1,96 \\ \to \cong t_{40,\ 0,975} = 2,021 \\ \to = 2,0227 \end{cases}$$

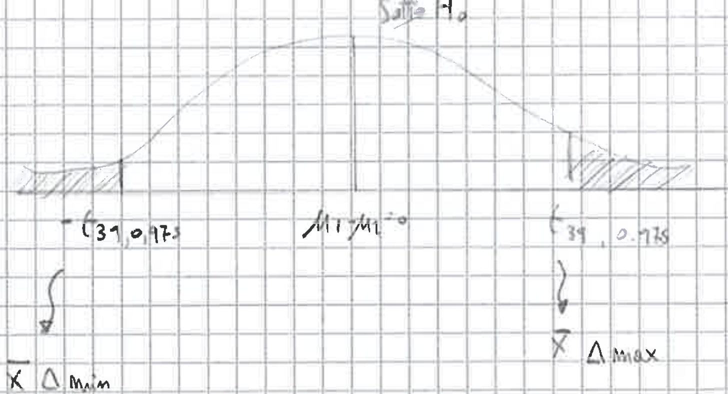
CI $\times$ $\mu_1 - \mu_2$ $\qquad \sigma_1^2 = \sigma_2^2 = \sigma^2$

$$l_i = (\bar{X}_1 - \bar{X}_2) \mp t_{n_1+n_2-2,\ 1-\alpha/2} \sqrt{S^2_{pool} \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

$$l_i = -0,443 \mp t_{39,\ 0,975} \sqrt{10,78 \left(\frac{1}{23} + \frac{1}{18}\right)}$$

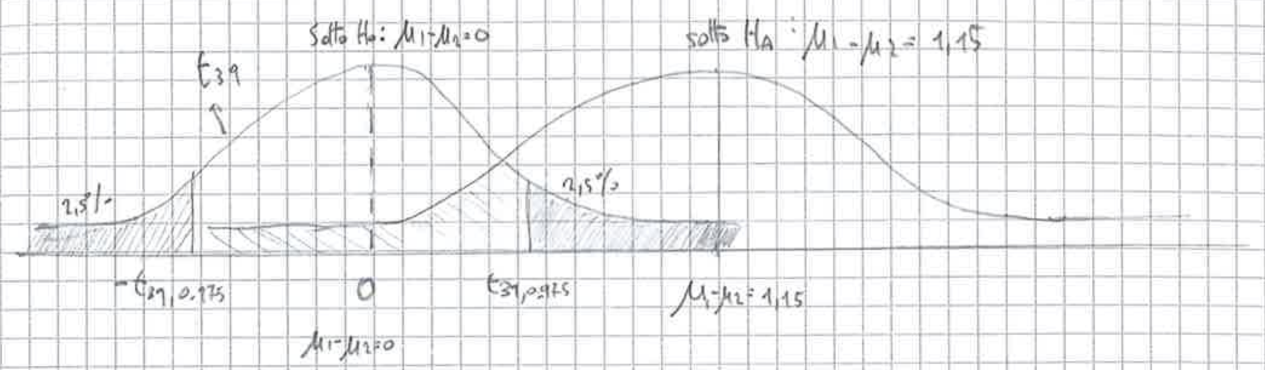
$$\cong t_{40,\ 0,975} = 2,021$$

$(-2,53 ,\ 1,645)$



$$\bar{X}_{\Delta max}: \quad t_{39,\ 0,975} = \frac{\bar{X}_{\Delta max} - 0}{\sqrt{S^2_{pool}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$\bar{X}_{\Delta max} = \overset{\mu_1-\mu_2}{0} + t_{39,\ 0,975} \sqrt{S^2_{pool}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$



$$\beta\ H_0: \mu_1-\mu_2=1,15 = P\left[\bar{X}_{\Delta min} < \bar{X}_1-\bar{X}_2 < \bar{X}_{\Delta max} \mid \text{essendo vera } H_A: \mu_1-\mu_2=1,15\right] = (*)$$

$$\underset{\substack{\| \\ -2,09}}{\bar{X}_{\Delta min}} : \quad -t_{39,\,0.975} = \frac{X_{\Delta min}-0}{\sqrt{10,78\left(\frac{1}{23}+\frac{1}{18}\right)}}$$

$$\underset{\substack{\| \\ 2,09}}{\bar{X}_{\Delta max}}$$

$$(*) = P\left[\frac{-2,09-1,15}{\sqrt{10,78\left(\frac{1}{23}+\frac{1}{18}\right)}} < T_{39} \le \frac{2,09-1,15}{\sqrt{10,78\left(\frac{1}{23}+\frac{1}{18}\right)}}\right]$$

$$= P\left[-3,13 < T_{39} \le 0,90\right] = P\left[T_{39} \le 0,90\right] - P\left[T \le -3,13\right] \cong P\left[Z \le 0,9\right] - 0$$

$$\cong 81\%$$

→ cercare sulle tavole 0,975 e 6 ($6 = n-1$)

$$= 2,447 \cdot \frac{0,23}{\sqrt{7}} = 0,213$$

⇓

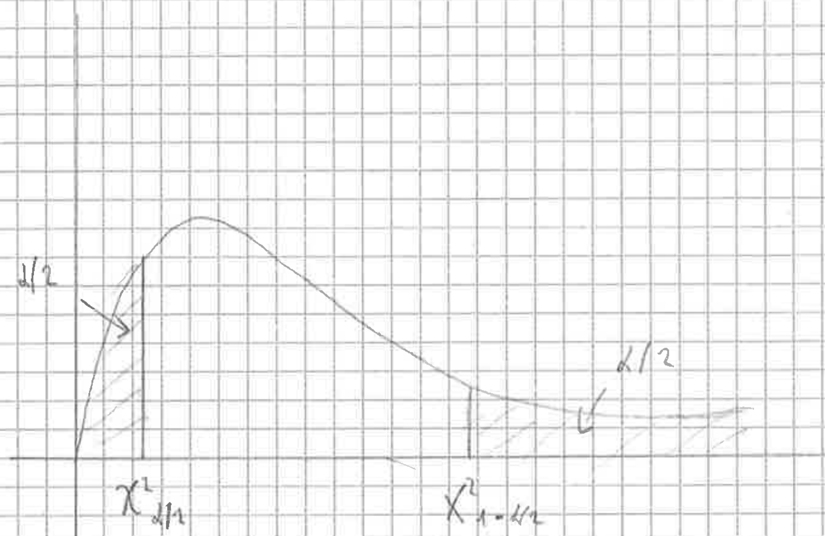
$$(l_i, l_s) = (2,03 - 0,213 \;,\; 2,03 + 0,213)$$

$$= (1,82 \;,\; 2,24)$$

b) Usiamo la statistica

$$V_n = \frac{S_n^2 (n-1)}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$1 - \alpha = \mathbb{P}\left(\chi^2_{n-1, \alpha/2} < V < \chi_{n-1, 1-\alpha/2}\right)$$



$$(L_i, L_s) = \left(\frac{S_n^2 (n-1)}{\chi^2_{n-1, 1-\alpha/2}} \;,\; \frac{S_n^2 (n-1)}{\chi^2_{n-1, \alpha/2}}\right)$$

fatte le misure

$$(l_i, l_s) = \left(\frac{s^2 (n-1)}{\chi^2_{n-1, 1-\frac{\alpha}{2}}} \;,\; \frac{s^2 (n-1)}{\chi^2_{n-1, \frac{\alpha}{2}}}\right)$$

$$= \left(\frac{s^2 (n-1)}{\chi^2_{6, 0.975}} \;,\; \frac{s^2 (n-1)}{\chi^2_{6, 0.025}}\right)$$

$$= \left(\frac{0,0524 \cdot 6}{14,449} \;,\; \frac{0,0524 \cdot 6}{1,237}\right) = (0,022 \;,\; 0,254)$$

$1 - \frac{\alpha}{2} = 0,925 \Rightarrow \alpha = 0,15$

$\boxed{1 - \alpha = 0,85}$

c) Determinare n^* t.c.

$2 I_\alpha \leq 2,5$

$1 - \alpha = 0,98$

n = numerosità del campione utilizzato per stimare la varianza

$I_\alpha = t_{n-1, 1-\alpha/2} \cdot \frac{D}{\sqrt{n^*}}$

$1,25 \geq I_\alpha$

$= t_{15, 0,99} \cdot \frac{\sqrt{10,433}}{\sqrt{n^*}}$

$1 - \alpha = 0,98$
$\alpha = 0,02$
$1 - \frac{\alpha}{2} = 0,99$

$\sqrt{n^*} \geq \frac{2,602 \cdot \sqrt{10,433}}{1,25} = 6,7236$

$\sqrt{n^*} \geq (6,7236) \Rightarrow n \geq 45,18 \Rightarrow n \geq 46$

ES. 3

$X_1 \dots X_n \underset{iid}{\sim} b(p)$

$n = 540$

p incognita $\quad \hat{p} = \bar{X}_n = \frac{248}{540} = \frac{62}{135}$

Intervallo di fiducia per p

$P_n = \bar{X}_n$ è approssimabile con $\mathcal{N}\left(p, \frac{p(1-p)}{n}\right)$

$$\Rightarrow n = \left(\frac{1,96^2 \cdot 0,52 \cdot 0,48}{0,04^2} = 599,29 \right.$$

$n = 600$

ES. 5

$n_A = 10 \quad \overline{X}_A = 55 \quad D_A^2 = 1,96 \quad (D_A = 1,4)$

$n_B = 10 \quad \overline{X}_B = 53 \quad D_B^2 = 2,25 \quad (D_B = 1,5)$

$X_1^A \dots X_{n_A}^A \underset{iid}{\sim} \mathcal{N}(\mu_A, \sigma_A^2)$

$X_1^B \dots X_{n_B}^B \underset{iid}{\sim} \mathcal{N}(\mu_B, \sigma_B^2)$

$\mu_A, \mu_B, \sigma_A^2, \sigma_B^2$ incognite

Intervallo al 95% per $\mu_A - \mu_B$

σ_A^2, σ_B^2 incognite

Possiamo ipotizzare che $\sigma_A^2 = \sigma_B^2 = \sigma^2$?

Usiamo la statistica

$$F = \frac{S_A'^2 / \sigma_A^2}{S_B'^2 / \sigma_B^2} = \frac{S_A'^2}{S_B'^2} \cdot \frac{\sigma_B^2}{\sigma_A^2}$$

$F \sim \text{Fisher}(n_A - 1, n_B - 1)$

intervallo di fiducia per $\frac{\sigma_B^2}{\sigma_A^2}$ è dato da:

$$(l_i, l_s) = \left(\frac{S_B'^2}{S_A'^2} f_{n_A-1, n_B-1, \frac{\alpha}{2}} , \frac{S_B'^2}{S_A'^2} f_{n_A-1, n_B-1, 1-\frac{\alpha}{2}} \right)$$

Consideriamo la statistica $T = \dfrac{(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)}{S_{pool}\sqrt{\dfrac{1}{n_A} + \dfrac{1}{n_B}}}$

$$T \sim t(n_A + n_B - 2)$$

$$(L_i, L_s) = \left((\bar{X}_A - \bar{X}_B) - t_{n_A+n_B-2,\,1-\alpha/2} \cdot S_{pool}\sqrt{\frac{1}{n_A}+\frac{1}{n_B}} \,,\, (\bar{X}_A - \bar{X}_B) + t_{n_A+n_B-2,\,1-\alpha/2} \cdot S_{pool}\sqrt{\frac{1}{n_A}+\frac{1}{n_B}} \right)$$

$$I_\alpha = t_{n_A+n_B-2,\,1-\alpha/2}\, S_{pool}\sqrt{\frac{1}{n_A}+\frac{1}{n_B}}$$

$$= t_{n_A+n_B-2,\,1-\alpha/2}\left(\frac{S_A^2(n_A-1)+S_B^2(n_B-1)}{n_A+n_B-2}\right)\sqrt{\frac{1}{n_A}+\frac{1}{n_B}}$$

$$n_A = n_B = 10 = n$$

$$= t_{2(n-1),\,1-\alpha/2}\sqrt{\frac{\cancel{(n-1)}(S_A^2+S_B^2)}{\cancel{2}\,\cancel{(n-1)}}}\sqrt{\frac{\cancel{2}}{n}}$$

$$= t_{2(n-1),\,1-\alpha/2}\sqrt{\frac{S_A^2+S_B^2}{n}} = t_{18,\,0.975}\sqrt{\frac{S_A^2+S_B^2}{10}}$$

$$(l_i, l_s) = \left((\bar{X}_A - \bar{X}_B) - t_{18,\,0.975}\sqrt{\frac{S_A^2+S_B^2}{10}} \,,\, (\bar{X}_A - \bar{X}_B) + t_{18,\,0.975}\sqrt{\frac{S_A^2+S_B^2}{10}} \right)$$

$$= \left(2 - 2{,}101\sqrt{\frac{1{,}96+2{,}25}{10}} \,,\, 2 + 2{,}101\sqrt{\frac{1{,}96+2{,}25}{10}}\right) = (0{,}637\ ;\ 3{,}363)$$

$$\not\ni 0$$

NO

Foglio 11

ES 1

$X_1 \ldots\ldots X_n \underset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$

μ incognita

$\sigma^2 = 4$ <u>nota</u>

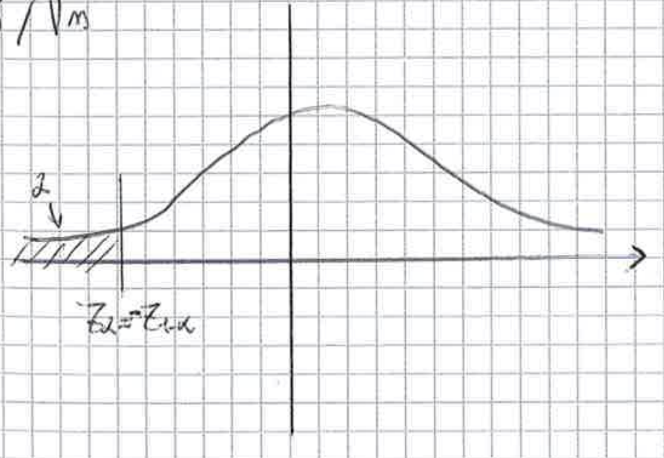
$$\begin{cases} H_0 : \mu = 10 \\ H_A : \mu < 10 \end{cases}$$

$\alpha = 0{,}05$

$0{,}05 = \alpha = \mathbb{P}(\text{rifiuto } H_0 \mid H_0 \text{ vera})$

Consideriamo la statistica

$$Z = \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \sim \mathcal{N}(0,1)$$



$$\alpha = \mathbb{P}(Z < -z_{1-\alpha}) = \mathbb{P}\left(\bar{X}_n < \mu - z_{1-\alpha} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

$$= \mathbb{P}\left(\bar{X}_n < \mu_0 - z_{1-\alpha} \cdot \frac{\sigma}{\sqrt{n}} \,\middle|\, H_0 \text{ vera}\right)$$

costruito con la $\hat{p}$ che ho [illegible]

$= \mathbb{P}\left(\bar{X}_n > 10 - 1{,}645 \frac{2}{\sqrt{n}} \,\Big|\, \mu = 7\right) \rightarrow = \mathbb{P}\left(\bar{X} > X_{min} \,|\, \mu = 7\right)$

Sotto H_A $\quad \bar{X}_n \sim \mathcal{N}\left(7, \frac{4}{n}\right)$

standardizzazione della variabile componente

$0{,}05 = \mathbb{P}\left(\frac{\bar{X}_n - 7}{2/\sqrt{n}} > \frac{X_{min} - 7}{2/\sqrt{n}}\right)$

$= \mathbb{P}\left(Z > \frac{10 - 3{,}29/\sqrt{n} - 7}{2/\sqrt{n}}\right) = \mathbb{P}\left(Z > \left(3 - \frac{3{,}29}{\sqrt{n}}\right) \cdot \frac{\sqrt{n}}{2}\right)$

$= \mathbb{P}\left(Z > 1{,}5\sqrt{n} - 1{,}645\right)$

$= 1 - \Phi\left(1{,}5\sqrt{n} - 1{,}645\right)$

$\Rightarrow \Phi\left(1{,}5\sqrt{n} - 1{,}645\right) = 0{,}95$

$\Rightarrow 1{,}5\sqrt{n} - 1{,}645 = z_{0,95} = 1{,}645$

$\Rightarrow \sqrt{n} = \frac{1{,}645 + 1{,}645}{1{,}5} = 2{,}193 \Rightarrow n = 4{,}8107 \Rightarrow \boxed{n \geq 5}$

2ª parte:

$\begin{cases} H_0: \mu = 10 \\ H_A: \mu \neq 10 \end{cases}$

$X_{min} = \mu_0 - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$

$= 10 - 1{,}96 \cdot \frac{2}{\sqrt{n}} = 10 - \frac{3{,}92}{\sqrt{n}}$

$\alpha = 0{,}05$

$1 - \alpha/2 = 0{,}975$

$X_{max} = \mu_0 + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} = 10 + \frac{3{,}92}{\sqrt{n}}$

$0{,}05 = \beta = \mathbb{P}(\text{accetto } H_0 \,|\, H_A \text{ vera})$

$= \mathbb{P}\left(X_{min} < \bar{X}_n < X_{max} \,|\, \mu = 7\right)$

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