NUMERO: 1677A - ANNO: 2015

A P P U N T I

STUDENTE: Ottina

MATERIA: Metodi Numerici in Ingegneria Geotecnica 2014-15. Prof.Barla

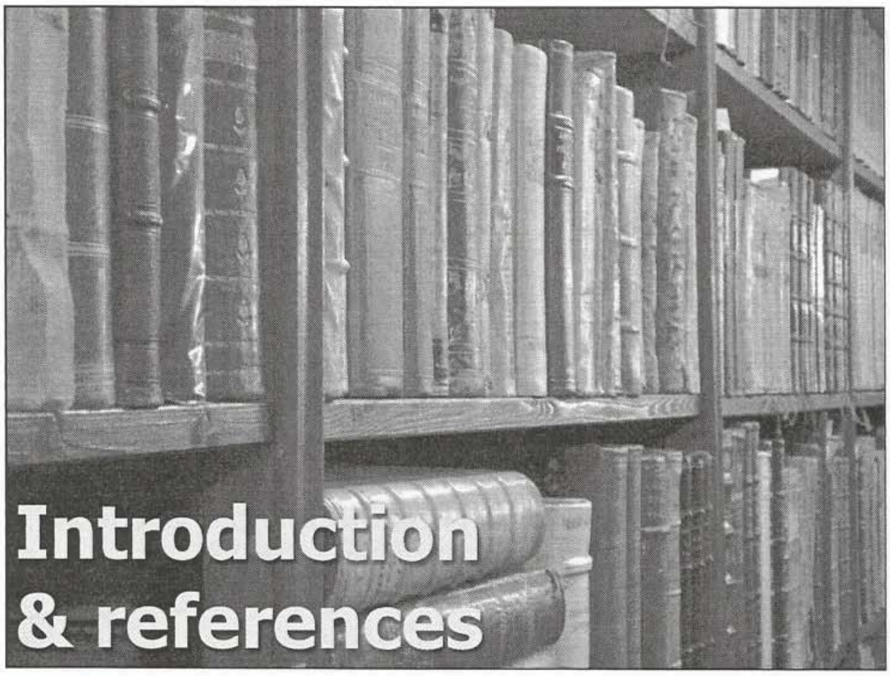
NUMERICAL METHODS IN GEOTECHNICAL ENGINEERING

LECTURE 1
Introduction

Marco Barla
Dipartimento di Ingegneria Strutturale, Edile e Geotecnica

EXPECTATIONS

WHAT ARE MY EXPECTATIONS FOR THE NUMGE COURSE?



TEACHERS

Marco Barla

Consulting hours:
Wed 10-11 a.m.
☎ 4824
💻 marco.barla@polito.it

Guido Musso

Consulting hours:
Thu 10-11 a.m.
☎ 4837
💻 guido.musso@polito.it

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 5 di 752

PROGRAMME

LECTURES	
1 & 2	Introduction and refresh
3 to 9 + 16 to 19	The Finite Element Method (FEM)
10 to 15	Constitutive modeling of geomaterials
20 to 23	Other Numerical methods
24 to 34	Case studies
SEMINAR LECTURES	
1 to 5	Extension of FEM formulation to flow with water
6 to 8	Multi anchored retaining walls
APPLICATIONS	
1 to 12	Design of a tunnel by FEM

See detailed programme on the web site.

TEACHING MATERIAL

Slides of each lecture will be available on the WEB site together with some selected reading material.

Additional reference:

- Elementi di meccanica e ingegneria delle rocce by Barla, Celid, 2010

BASI OF FEM

- The Finite Element Method: its basis and fundamentals by O. C. Zienkiewicz, R. L. Taylor, J.Z. Zhu

PRATICO - SOLO TERRENI, NO ROCCE

Finite element analysis in geotechnical engineering - theory by David M Potts & Lidija Zdravkovic, Thomas Telford 1999

THEORY

APPLICATIONS (MUSSO)

Geotechnical modelling by David Muir Wood, Spon Press 2004

LECTURE 1 : INTRODUCTION

17) The different about GEOTHECNICAL Model and GEOLOGIC model is:

- GEOLOGIC MODEL it's for the HISTORY of ROCKS;
- GEOTHECNICAL MODEL it's QUANTIFY the PROPRIETY of the rocks and geomaterials because there aren't KNOWN a priori and we don't known the propriety of the rocks.

Geothecnical need to investigate and make a laboratory tests (or in site) to know them for build a geothecnical model.
Investigations it's a different methods to design methods.
People that investigations or design are different and never speak but their have a different interacts between the roules.

18) Have save regulations: "legge quadro dei lavori pubblici, NTC 2008.."

There are 3 different levels of design and for a level we have a different investigations that depend of level.

The parameters are corrects?

INVESTIGATIONS

DESIGN PARAMETERS

MONITOR

The parametry was correct?

yes NO, go back

Data acquisation: of monitory
search of improve that.
Go back and revise the previous steps.

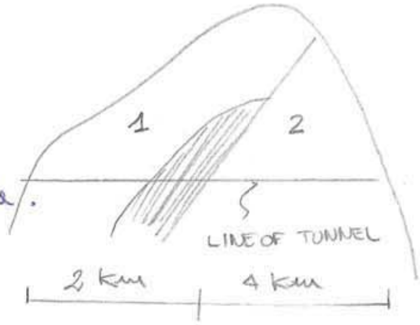
Measure everydays?
It is important i control the different of displacement of millimeters $\Delta\delta$ = mm.

- DEFINISCO I LIMITI
- PROGETTO UN PIANO DI MONITORAGGIO
- DEFINISCO delle AZIONI da fare DOPO che ho PASSATO I LIMITI

↳ defined during the project.

29) DESIGN THE SINGLE ELEMENTS.

All different lines and support's tunnel
We used ones depends of rock condition.
used construction rock different.
because we have different conditions.



We have the possibility of finding a different scheme when we dig (SCAVARE) the tunnel when you advance.
We define the design for every situations while the tunnel go on, step by step, we correcting the geothecnical model.

34) STRESS ANALYSIS

The analitical methods is for simplified geometry of the tunnel.
We used continuum or discontinuum?
The choice of continuum or discontinuum is the geothecnical model (in funzione del rev).

The definition of the geometry, rock's propriety it's a problem to have a different approach.

In the rock mass it's clear this different, not more in the soil: used a Numerical methods.

Soil

- The CONTINUM depends of the scale of problem
- The discontinuum numerical methods

GEOTECHNICAL ENGINEERING

GEOTECHNICAL ENGINEERING draws not only on relevant scientific and engineering fundamentals, but also on public restraints, construction practice, and risk management.

(Morgenstern 2000)

Geology
Engineering Geology
Rock Mechanics
Soil Mechanics
Hydrogeology
Public Policy
Structural Mechanics
GEOTECHNICAL ENGINEERING
Contract Law
Numerical Analysis
Risk Management
For Geotechnical
Materials
Mechanical Engineering (TBM)
Ground Movements
Geochemestry
Construction
Site Exploration

GEOTECHNICAL ENGINEERING DESIGN METHODS → PROJECTS

- The logic process which, starting with idealisation of a civil, environmental structure... interacting with soil and/or rock mass (ground) takes to its construction, forms the design process.
- This requires one to adopt design analysis methods which most often imply the use of numerical methods. Our class will be devoted to the use and understanding of these methods.
- A very important step is the schematic and simplified representation of the engineering problem at hand. In most cases this implies the definition of an appropriate "model" to describe the engineering structure, by keeping well in mind the design methods to be used.

There are some methods and we use Numerical methods for build and model.

GEOTECHNICAL DESIGN IN ITALY

Yesterday >>> NORMATIVA ITALIANA (Decreto del Ministero delle Infrastrutture 11 MARZO 1988)

Today >>> EUROCODE 7

NTC 2008

Need to valuate a security to compare the resistence forces (R) with the solecitations (S, active forces) and have a number : $F_S \geq 1{,}3$ or more.

DM 1988 (oldies...)

Safety is evaluated by means of global safety factors, comparing the resisting forces R to the soliciting ones S:

$$\boxed{\frac{R}{S} \geq (FS)_{adm}}$$

Required safety coefficients:	
Bearing capacity of direct foundations:	3
Limit capacity for single pile:	2,5 (2 if loading tests are perfomed on piles)
Walls on direct foundations:	1.5 (toppling) 1,3 (horizontal traslation) 2 (bearing capacity) 1,3 (global stability)
Slope global stability:	1,3

For structures (foundations) a normal situations.
Example ALLUVIONAL SOIL
We have a different soil and divided in a big zones.

EUROCODE 7

Geotechnical Category 2 should include conventional types of structure and foundation with no exceptional risk or difficult soil or loading conditions.

Design for structures in Geotechnical Category 2 should normally include quantitative geotechnical data and analysis to ensure that the fundamental requirements are satisfied.

Routine procedures for field and laboratory testing and for design and execution may be used for Geotechnical Category 2 designs.

The following are example of conventional structures or parts of structures complying with Geotechnical Category 2:

spread foundations, raft foundations, pile foundations, walls and other structures retaining or supporting soil or water, excavations, bridge piers and abutments, embankments and earthworks, ground anchors and other tie-back systems, tunnels in hard, non fractured rock and not subjected to special water tightness or other requirements.

EUROCODE 7

Geotechnical Category 3 should include structures or parts of structures which fall outside the limits of Geotechnical Categories 1 and 2 .

Geotechnical Category 3 should normally include alternative provisions and rules to those in the EC7 standard.

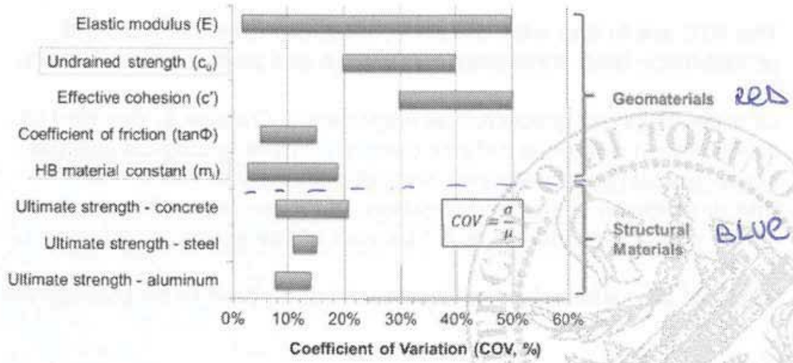
Geotechnical Category 3 includes the following examples:

very large or unusual structures, structures involving abnormal risks, or unusual or exceptionally difficult ground or loading conditions, structures in highly seismic areas, structures in areas of probable site instability or persistent ground movements that require separate investigations or special measures.

DAMN foundations : use a numerical methods.

THE OBSERVATIONAL METHOD

Uncertainty and variability of parameters may be quite high for geo-materials.



27

THE OBSERVATIONAL METHOD

The following requirements shall be set before construction is started:

A • the limits of behaviour which are acceptable shall be established; the range of possible behaviour shall be assessed and it shall be shown that there is an acceptable probability that the actual behaviour will be within the acceptable limits.

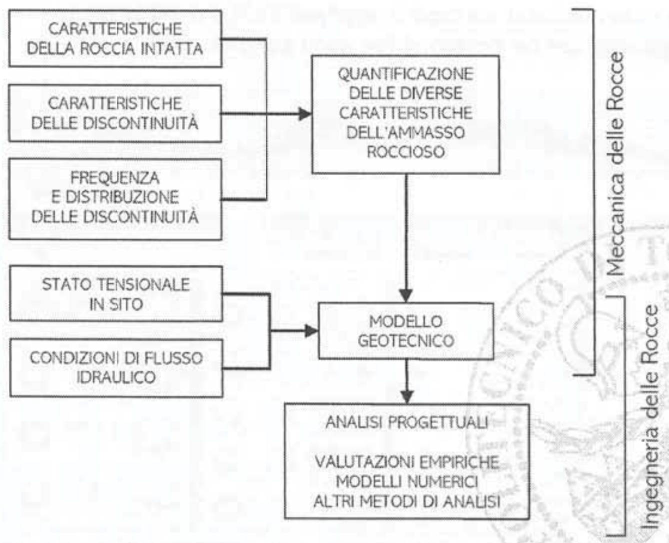
B • a monitoring plan shall be devised which will reveal whether the actual behaviour lies within the acceptable limits. The monitoring shall make this clear at a sufficiently early stage and with sufficiently short intervals to allow contingency actions to be undertaken successfully.

C • the response time of the instruments and the procedures for analysing the results shall be sufficiently rapid in relation to the possible evolution of the system.

D • a plan of contingency actions shall be devised which may be adopted if the monitoring reveals behaviours outside acceptable limits.

Define the characteristics of the discontinuities:
- SPACING
- ORIENTATIONS } geometry.
- PRESENCE OF WATER

GEOTECHNICAL MODEL



The reality is complicated but we need to use a SCHEMATIC RAPPRESENTATION, not the real but a simplify methods that contain all geotechnical parameters.

GEOTECHNICAL MODEL

It is a schematic representation of reality able to describe the fundamentals aspects of the behaviour of a soil or a rock mass.

It is obtained combining together all the information provided by the geotechnical characterisation process.

CONTINUUM vs DISCONTINUUM

EQUIVALENT CONTINUUM MODEL, the rock mass is treated as a continuum with equal in all directions input data for the strength and deformability properties, which define a given constitutive relation for the medium: elastic, elasto-plastic, etc.

DISCONTINUUM MODEL, the rock mass is represented as a discontinuum and most of the attention is devoted to the characterisation of the rock elements and the rock joints/discontinuities. The modelling approach consists in considering the blocky nature of the system being analysed. Each block may interact with the neighbouring blocks trough the joints.

We aren't physical but engineers and we need a practical model: we need a lot of time and lot of cost to obtain the right model but we used the PRACTICAL MODEL.
Isn't perfect but ADEQUATE OF CHARACTERIZATION.

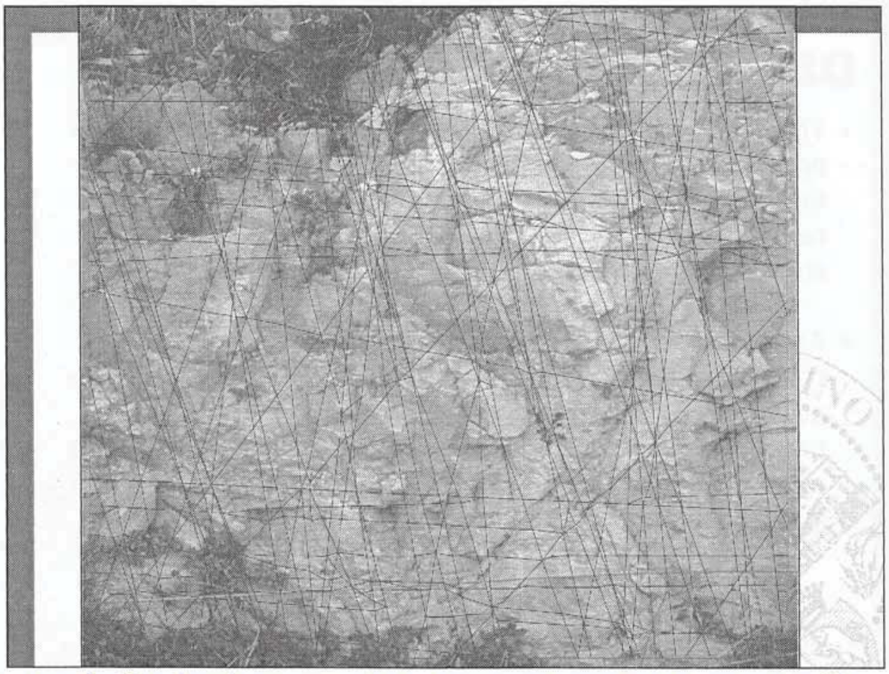
GEOTECHNICAL MODEL

- At microscale all materials are discrete systems. However, representing the microscopic components individually is intractable mathematically and unnecessary in practice.
- The model does not have to be complete and perfect: it only has to be adequate for the purpose.

(*Jing 2003*)

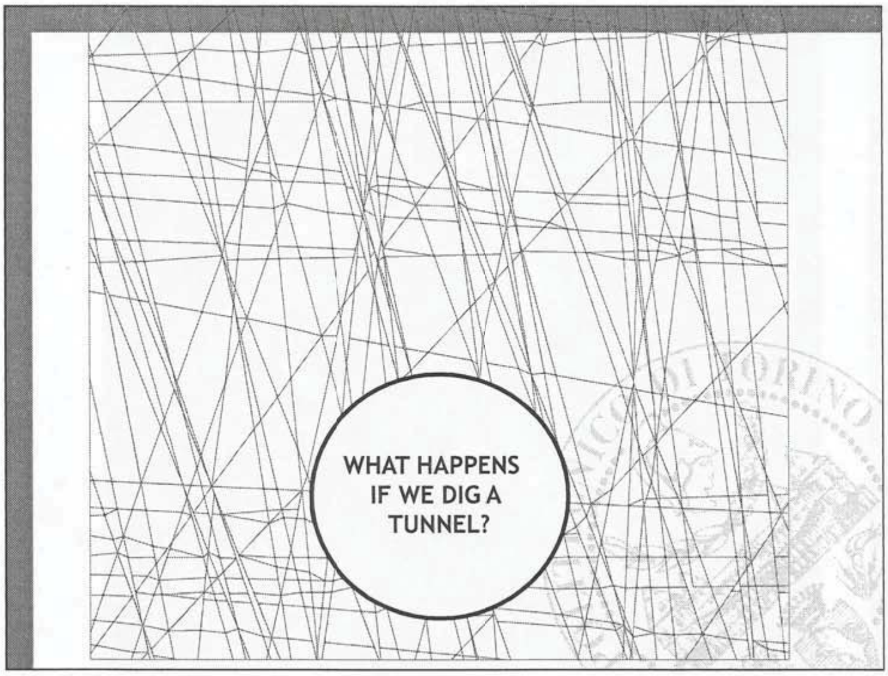
We choose the characterization and the model depends on the scope of the work.

This is the right rappresentation of the discontinuities but isn't simplified.



This is a real model but:
- HOW IS THE INTERACTIONS?
- WHAT IS THE SOLID RAPPRESENTATIVE?

Have different strain and deformability with different propriety.



40

If I apply the perturbations of the rock mass, we can do the design process?

NUMERICAL METHODS IN GEOTECHNICAL ENGINEERING

LECTURE 2
Stresses and strains

Marco Barla
Dipartimento di Ingegneria Strutturale, Edile e Geotecnica

OUTLINE

- Matrix algebra
- Stress and strain
- Solution of a boundary value problem in Continuum Mechanics

a^T = transpose change rows and coloumns.

MATRIX ALGEBRA

The transpose: $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$ $A^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \end{bmatrix}$

$m \times p$ = rows time coloumns.

Matrix multiplication:

$$\underset{m\times p}{C} = \underset{m\times n}{A}\ \underset{n\times p}{B} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \\ A_{31}B_{11} + A_{32}B_{21} & A_{31}B_{12} + A_{32}B_{22} \end{bmatrix}$$

3×2

THREE TIME TWO

MATRIX ALGEBRA

The identity matrix or unit matrix of size n is the n x n square matrix with ones on the main diagonal and zeros elsewhere. Some examples of unit matrices are:

$$I_1 = [1], I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

A zero matrix or null matrix is a matrix with all its entries being zero. Some examples of zero matrices are:

$$0_{1,1} = [0], 0_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

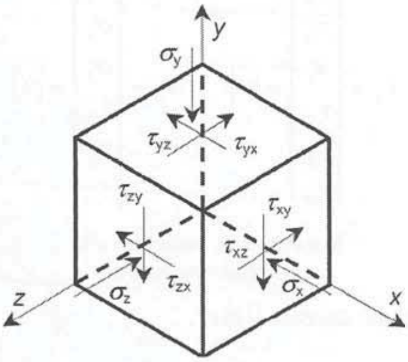
One generic element of solid it's composed by STRESS TENSOR with 6 indipendent components:

- σ_{xyz} 3 NORMAL STRESS
- τ_{xyz} 3 SHEARE STRESS

} the matrix is symmetric

STRESS TENSOR

The state of stress at a point in a continuous medium is defined by using a 2nd rank symmetric tensor with 6 independent components ($\sigma_{ij}=\sigma_{ji}$)



- *In matrix notations:*

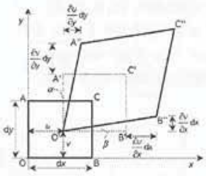
$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ sym & & \sigma_{33} \end{bmatrix}$$

- *In engineering notations:*

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

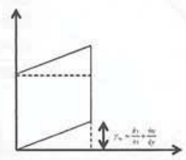
STRAIN TENSOR

The state of strain at a point in a continuous medium is defined by using a 2nd rank symmetric tensor with 6 independent components ($\varepsilon_{ij}=\varepsilon_{ji}$)



Shear strain tensor is the average of two strains

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$



Engineer shear strain is the total shear strain

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

- The diagonal components give the longitudinal strain in the x_i direction
- The out of diagonal components are half the angular shear of fibers which are initially orthogonal ($2\varepsilon_{ij}=\gamma_{ij}$)

- *In matrix notations:*

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{11} & \varepsilon_{23} \\ sym & & \varepsilon_{33} \end{bmatrix}$$

- *In engineering notations:*

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{bmatrix}$$

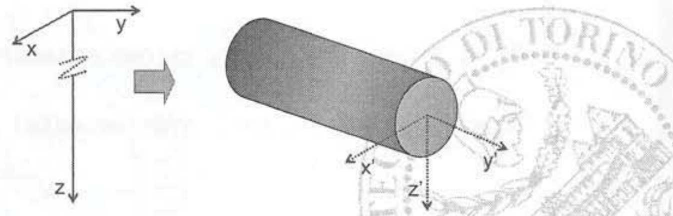
The STRAIN TENSOR is more complicated:

- ε = LONGITUDINAL STRAIN in 3 different directions;
- γ = SHEARE STRAIN in the FULL ANGLE.

To pass with the different reference systems.

TENSORIAL EQUATIONS

$$\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ & \sigma_{y'} & \tau_{y'z'} \\ \text{sym} & & \sigma_{z'} \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ \text{sym} & & \sigma_z \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$



STRESS AND STRAIN INVARIANTS

During the transformation from one reference system to the other, the following quantities remain INVARIANT

For σ_{ij} (i.e. for the six stress components)

$I_\sigma = I_{1\sigma} = \sigma_x + \sigma_y + \sigma_z$ FIRST INVARIANT AVERAGE STRESS

$II_\sigma = I_{2\sigma} = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau^2_{xy} - \tau^2_{yz} - \tau^2_{zx}$ SECOND INVARIANT

$III_\sigma = I_{3\sigma} = \sigma_x\sigma_y\sigma_z - \sigma_x\tau^2_{yz} - \sigma_y\tau^2_{zx} - \sigma_z\tau^2_{xy} + 2\,\tau_{xy}\tau_{yz}\tau_{zx}$ THIRD INVARIANT

NOTE:

$\boxed{\sigma^3 - I_\sigma\sigma^2 + II_\sigma\,\sigma - III_\sigma = 0}$ ➲ Principal stresses: $\sigma_1, \sigma_2, \sigma_3$

What is the difference about $\sigma_{1,2,3}$ to σ_{xyz}?
σ_{123} it's a stress acting (agisce) in a plane that haven't a shear stress.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 33 di 752

ISOTROPIC & DEVIATORIC STRESS

The reasoning above holds true also for the stress tensor as follows:

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \underbrace{\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}}_{\text{Isotropic tensor (Volume change)}} + \underbrace{\begin{bmatrix} \sigma_x - p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - p \end{bmatrix}}_{\text{Deviator (Shape change)}}$$

⇒ Isotropic Stress $\boxed{p = \dfrac{(\sigma_x + \sigma_y + \sigma_z)}{3} = \dfrac{I_{1\sigma}}{3}}$

ISOTROPIC & DEVIATORIC STRAIN

Invariants $J_{1\varepsilon}$, $J_{2\varepsilon}$, $J_{3\varepsilon}$ of the deviatoric strain:

$$J_{1\varepsilon} = (\varepsilon_x - \frac{I_\varepsilon}{3}) + (\varepsilon_y - \frac{I_\varepsilon}{3}) + (\varepsilon_z - \frac{I_\varepsilon}{3}) = I_\varepsilon - I_\varepsilon = 0$$

$$J_{2\varepsilon} = \frac{I^2_\varepsilon}{3} - II_\varepsilon = \frac{1}{6} [(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2]$$

$$J_{3\varepsilon} = I_\varepsilon - 1/3\, II_\varepsilon\, I_\varepsilon + 2/3\, I_\varepsilon$$

FILE 2 : STRESSES AND STRAINS

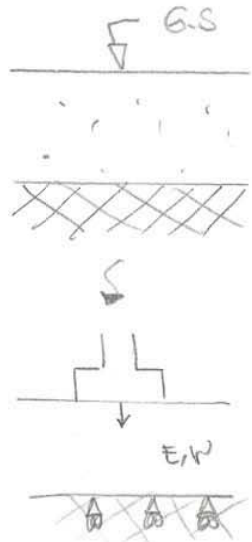
23) BOUNDARY VALUE PROBLEMS

We have a region with different equations and with a set of ADDITIONAL RESTRAINTS : BOUNDARY CONDITIONS (B.C).

"Sul bordo conosci il risultato delle equazioni differenziali".

The B.C. needs to simulate the REAL IMPACTS (fenomeni reali) and to build the limit to obtain a real design.

EXAMPLE DIFFERENT MATERIALS

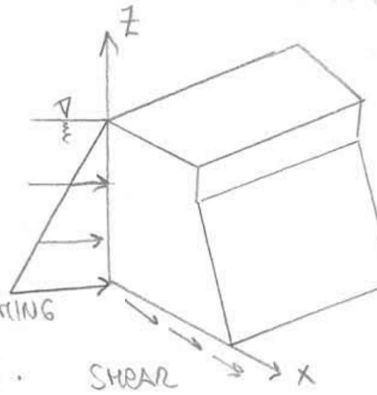


When i build a foundations i need to arrive in a bed rock (it's never move with apply the foundations): it's this MY B.C.

Compute the displacement or find (E, ν), it's the right solution of equations and satisfied the B.C.

33) PLANE STRAIN

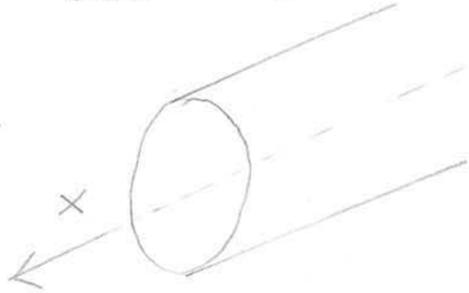
DUMP = DIGA



To simplify the problem i used (x,z) because we have different forces on (x,z) axes.

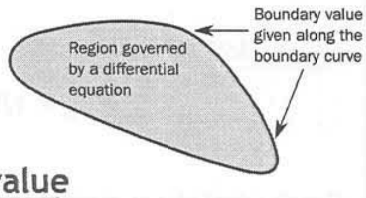
It's too simplify to considered the axial direction of the tunnel.

The real it's 3D but we used the 2D PLANE STRAIN to simplify.



BOUNDARY VALUE PROBLEMS

FIELD = CAMPO



In mathematics, in the field of differential equations, a boundary value problem is a differential equation together with a set of additional restraints, called the boundary conditions.
A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

23

SOLUTION OF B.V.P. IN CONTINUUM MECHANICS

To solve a boundary value problem in Continuum Mechanics, the following conditions need to hold true:
DEVONO VALERE

- equilibrium equations
- compatibility equations
- stress-strain law ("costitutive law")
- boundary conditions (both force and displacement)

Each of these conditions is considered separately, by keeping in mind the developments in the following with reference to Numerical Analysis/Numerical Modelling.

MATHEMATICAL COMPATIBILITY

The problem can be studied by paying attention to the definition of the strain components, by assuming small strain theory and a compression positive sign convention. As well known, the strains ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{xz} can be written in terms of the displacement components u, v, w as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} \;;\; \varepsilon_y = \frac{\partial v}{\partial y} \;;\; \varepsilon_z = \frac{\partial w}{\partial z} \;;$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \;;\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \;;$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \;;$$

As the strains are a function of only three displacements, they are not independent. It can be shown mathematically that for a compatible displacement field to exist, all the above components of strain and their derivatives must exist (are bounded) and be continuous to at least the second order. The displacement field must satisfy any specified displacements or restraints imposed on the boundary.

MATHEMATICAL COMPATIBILITY

The compatibility equations are as follows: 6 equations

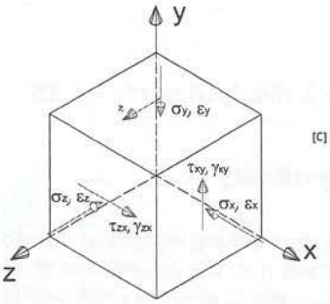
$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{2\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad \frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{2\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \qquad \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{2\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$[C] = [D]^{-1}$ ELASTIC CONSTANT MATRIX

3D PROBLEMS



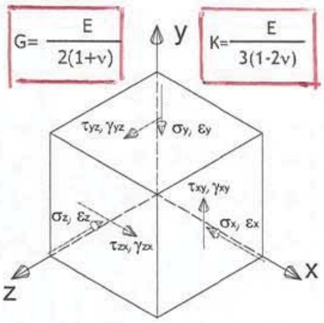
$$[C] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

The constitutive law above can be re-written, in order to obtain the stress tensor from the strain tensor. To this purpose we need to write the inverse of [D], i.e. $[D]^{-1} = [C]$. This is the most often used matrix in our course.

G = SHEAR MODULUS for the variation of volume
K = BULK MODULUS for deviatoric

3D PROBLEMS

$$\boxed{G = \frac{E}{2(1+\nu)}} \qquad \boxed{K = \frac{E}{3(1-2\nu)}}$$



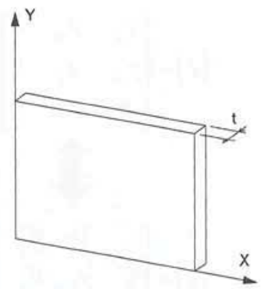
$$[C] = \begin{bmatrix} K+4/3\,G & K-2/3\,G & K-2/3\,G & 0 & 0 & 0 \\ & K+4/3\,G & K-2/3\,G & 0 & 0 & 0 \\ & & K+4/3\,G & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & \text{sym} & & & G & 0 \\ & & & & & G \end{bmatrix}$$

The above constitutive law can be re-written by introducing the shear modulus G and the bulk modulus K: this is particularly useful whenever we need to write the volumetric and deviatoric response separately, which is often the case.

Normally we used (E, ν) but to divided deviatoric and isotropic matrix we used (G, K) and used in a numerical softwares.

In a plane stress we have a simplify equations

PLANE STRESS



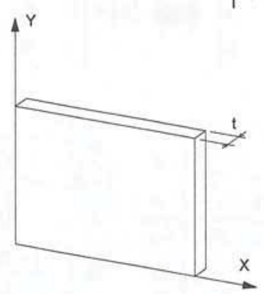
$$[\sigma]=\begin{bmatrix}\sigma_x & \tau_{xy} & 0\\ \tau_{yx} & \sigma_y & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\Updownarrow$$

$$[\sigma]=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

PLANE STRESS 2D

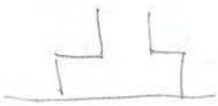
$$[\sigma]=\begin{bmatrix}\sigma_x & \tau_{xy} & 0\\ \tau_{yx} & \sigma_y & 0\\ & 0 & 0\end{bmatrix} \Leftrightarrow [\sigma]=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & 0\end{bmatrix}$$



$$[c]=\frac{E}{(1-\upsilon^2)}\begin{bmatrix}1 & \upsilon & 0\\ \upsilon & 1 & 0\\ 0 & 0 & \frac{1-\upsilon}{2}\end{bmatrix}$$

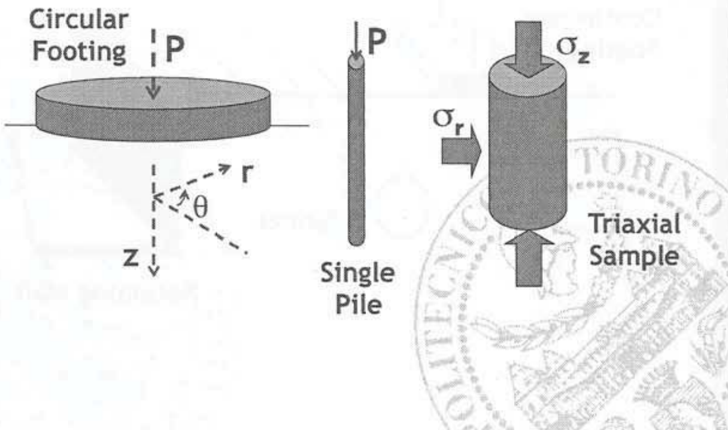
This condition is rarely applied in the solution of Geotechnical Engineering problems

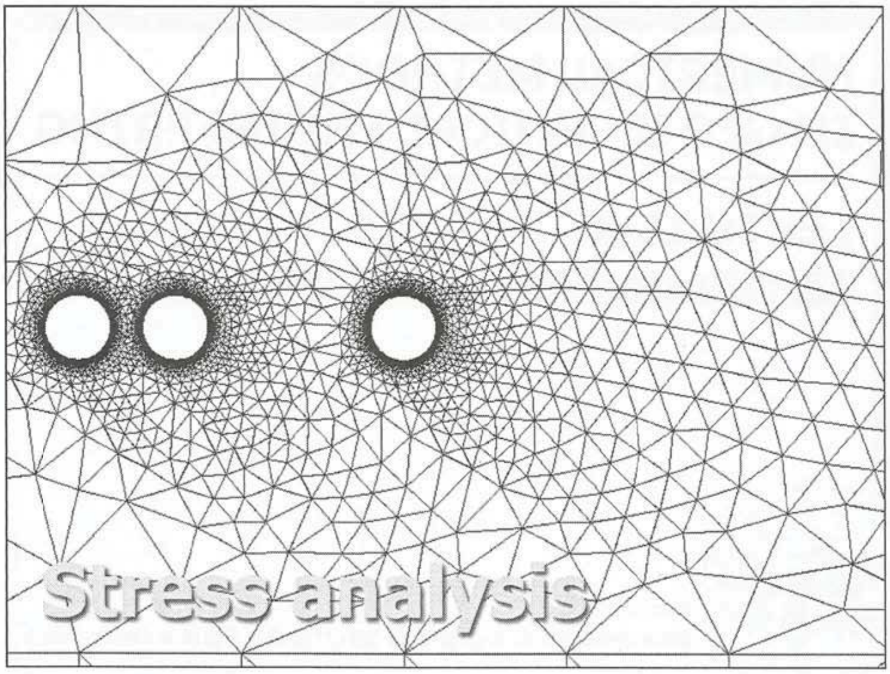
We often use plain condition.



this is the plain strain

EXAMPLES OF AXI SYMMETRY





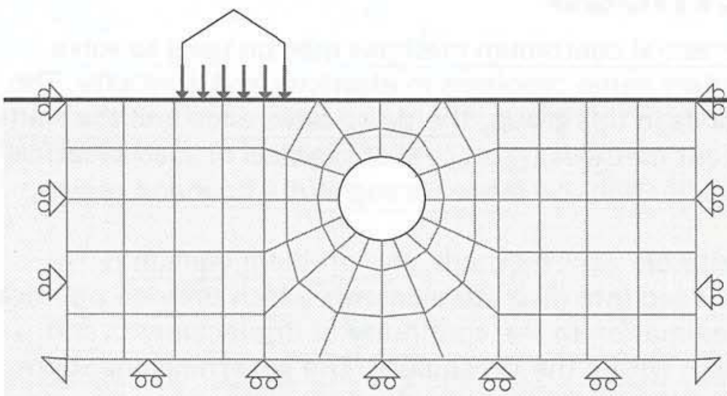
STRESS ANALYSIS

In engineering mechanics, Stress Analysis consists of resolving a problem into its simple elements, representing the problem by tractable equations, and solving those equations. Analytical Methods are generally regarded as being those which produce closed form or pseudo-closed form solutions.

Only in a few cases Analytical Solutions can be found to geotechnical engineering problems of practical concern. The boundary conditions are not easy to describe, the governing partial differential equations are non linear, the problem domain is non homogeneous, or the constitutive equations for soil/rock are non linear or otherwise insufficiently simple mathematically. In these cases, approximate solutions may be found using computer-based Numerical Methods.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 49 di 752

THE PRINCIPLES OF FEM



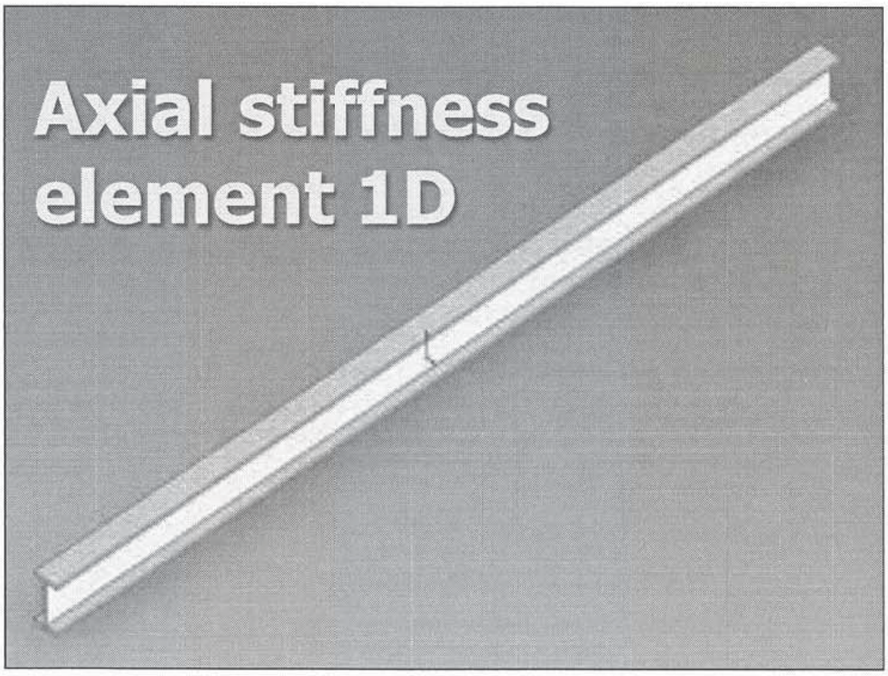
STEPS

The finite element method involves the following steps:

4. Global equations
   Combine element equations to form global equations.

5. Boundary conditions
   Formulate boundary conditions and modify global equations.

6. Solve the global equations
   To obtain the displacements at all the nodes, from which secondary quantities such as stresses and strains are evaluated.



STEP 2 – PRIMARY VARIABLE APPROXIMATION

DISPLACEMENTS

The displacement $u(x)$ is selected as the primary variable. Its components are assumed to have a simple polynomial form, where the order of the polynomial depends on the number of nodes in the element.

$$u(x) = \alpha_o + \alpha_1 x$$

$$u(x) = [\,1 \quad x\,]\begin{bmatrix}\alpha_o \\ \alpha_1\end{bmatrix} \Rightarrow \boxed{u(x) = [\phi]^T[\alpha]}$$

Writing it for the two nodes i, j we obtain:

$$u(0) = u_i = [\,1 \quad 0\,]\begin{bmatrix}\alpha_o \\ \alpha_1\end{bmatrix}$$

$$u(L) = u_j = [\,1 \quad L\,]\begin{bmatrix}\alpha_o \\ \alpha_1\end{bmatrix}$$

STEP 2 – PRIMARY VARIABLE APPROXIMATION

DISPLACEMENTS

The vector of the nodes displacements is:

$$\begin{bmatrix}u_i \\ u_j\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 1 & L\end{bmatrix}\begin{bmatrix}\alpha_o \\ \alpha_1\end{bmatrix} \qquad \boxed{[u]_e = [A][\alpha]}$$

Solving for a we have: $[\alpha] = [A]^{-1}[u]_e$

$$u(x) = [\phi]^T[\alpha] = [\phi]^T [A]^{-1} [u]_e = [H]^T [u]_e$$

where [H] is known as the matrix of *SHAPE FUNCTIONS*

$$\boxed{[H]^T = [1 \quad x]\begin{bmatrix}1 & 0 \\ -\dfrac{1}{L} & \dfrac{1}{L}\end{bmatrix}}$$

finally:

$$\boxed{u(x) = [(1- x/L) \quad x/L]\begin{bmatrix}u_i \\ u_j\end{bmatrix}}$$

[H] CONTAINS THE GEOMETRY OF THE PROBLEMS

STEP 3 – ELEMENT EQUATIONS
STRAIN AND STRESSES

Applying the principle of Virtual Work:

virtual strain — virtual nodal displacements

$$\underbrace{\int_{V_e} [\bar{\varepsilon}]_e^T [\sigma]_e \, dV_e}_{\text{Internal virtual work}} = \underbrace{X_i \bar{u}_i + X_j \bar{u}_j}_{\text{External virtual work}}$$

Strain energy = Virtual work due to nodal forces X_i and X_j

$$\int_{V_e} [\bar{\varepsilon}]_e^T [\sigma]_e \, dV_e = \begin{bmatrix} \bar{u}_i & \bar{u}_j \end{bmatrix} \begin{bmatrix} X_i \\ X_j \end{bmatrix} = [\bar{u}]_e^T [X]_e$$

STEP 3 – ELEMENT EQUATIONS
STRAIN AND STRESSES

Internal virtual work can be written as:

$[\sigma] = [C]\,[\varepsilon]$

$[\varepsilon] = [B]\,[u]_e$

$$\int_{V_e} [\bar{\varepsilon}]_e^T [\sigma]_e \, dV_e = \int_{V_e} [B]^T [\bar{u}]_e^T [C][B][u]_e \, dV_e$$

$$= [\bar{u}]_e^T \left(\int_{V_e} [B]^T [C][B] dV_e \right) [u]_e$$

From which:

$$[\bar{u}]_e^T [X]_e = [\bar{u}]_e^T \left(\int_{V_e} [B]^T [C][B] dV_e \right) [u]_e$$

$$\boxed{[X]_e = [k]_e [u]_e}$$

And finally:

S(x)= cross section

$$[k]_e = \int_{V_e} [B]^T [C][B] dV_e = \int_0^L S(x) [B]^T [C][B] dx$$

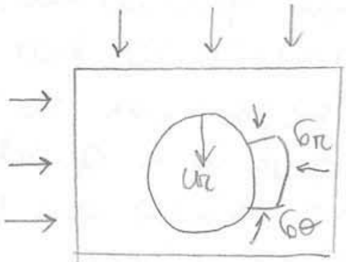
THE FINITE ELEMENT METHOD

FILE 3

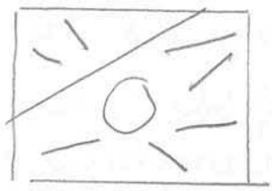
4) STRESS ANALYSIS

The closed form solution (soluzione in forma chiusa) is a solution of Analytical Methods but the Analytical Methods and A. Solution isn't available for all real solution: we used a NUMERICAL METHODS.

TUNNEL: THE PLANE WITH THE HOLE.



I suppose it's an omogeneous materials with elastic parameters (E, ν) or (G, K) and used a numerical solution.
If i used an Analytical solution i obtain the RADIAL displacement (U_r).

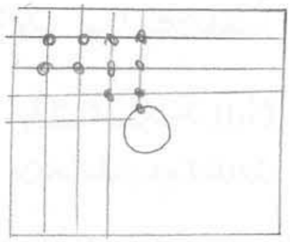


If is a simplify situation, i use an analytical methods:

- CILE — ANALYTICAL δ_i
- ELPLA — NUMERICAL δ_i

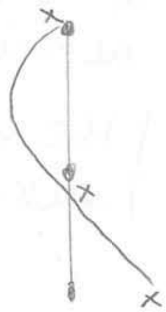
For solve analytical problem we used a NUMERICAL METHODS but not the opposite.

6) DIFFERENTIAL CONTINUUM METHODS



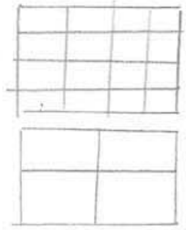
The finite element methods (FEM) discretize the domain in a number of elements and discretize them.

The **nodes** elements it's an element to interact 2 different elements (appartengono a 2 elementi separati).
The nodes are and have an approximate solution of the equations.



10) STEPS OF DEFINE ELEMENT METHODS

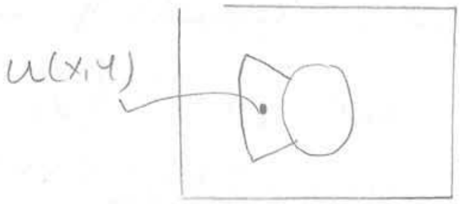
1) ELEMENT DISCRETIZATION



It's more important because the dimension of elements influence the results: are important for the solution.

2) PRIMARE VARIABILE SELECTIVE

In Geotechnical we used the displacement $u(x,y)$.



3) ELEMENT EQUATIONS

4) GLOBAL EQUATIONS

Build and combine the global equations.

5) BOUNDARY CONDITIONS

6) SOLVE THE GLOBAL EQUATIONS

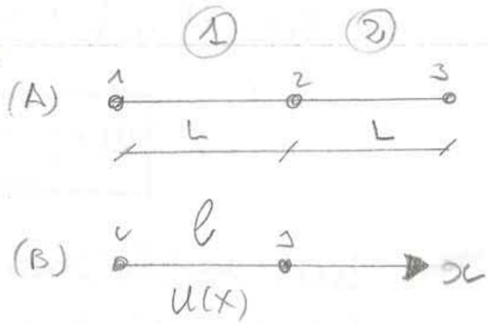
Find the displacement for all nodes.

13) PROBLEM

The axis is the direction of x.

u_1^*, u_3^* is two constants.

14) STEP 1: DISCRETIZE THE ELEMENTS



In 1D used the line when 1,2,3 it's the node name.

Define the element mesh (B)

15) STEP 2: PRIMARY VARIABLE APPROXIMATION

$u(x)$ it's the primary variable. (spostamento all'interno dell'elemento)

Two nodes (i, j) = two components

$$\boxed{u(x) = \alpha_0 + \alpha_1 x}$$

$$[A] = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \rightarrow \det[A] = L - 0 = L$$

$$[A]^{-1} = \frac{1}{L}\begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

$$[H] = [\phi]^T[A]^{-1} = [1 \quad x]\begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} = \left[(1-\frac{x}{L}) : \frac{x}{L}\right]$$

$$\boxed{u(x) = \left[(1-\frac{x}{L}) \quad \frac{x}{L}\right]\begin{bmatrix} u_i \\ u_j \end{bmatrix}}$$

In $u(x)$ compare (appears) the displacements of the nodes.

17)

STEP 3 : ELEMENT EQUATIONS

The axial strain is given by

$$\varepsilon_x = \frac{du(x)}{dx} = \frac{d}{dx}[H]^T[u]_e = \frac{d}{dx}[\phi]^T[A]^{-1}[u]_e$$

For the generic element:

$$\boxed{[\varepsilon] = [\phi']^T[A]^{-1}[u]_e}$$

- $[B]$ = ELEMENT STRAIN MATRIX

$$[B] = [\phi']^T[A]^{-1} = [0 \quad 1]\begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

When $[\phi]^T = [1 \quad x] \rightarrow [\phi']^T = [0 \quad 1]$

$$\boxed{[\varepsilon] = [B][u]_e}$$

Introducing the constitutive law:

$$\boxed{[\sigma] = [C][\varepsilon]} \qquad [C] \Rightarrow E$$

$$\sigma = E \cdot [B]\underbrace{[u]_e}_{\text{for the NODES}}$$

$$[X]_e = \left(\int_{V_e} [B]^T [C][B] \, dV_e \right) [U]_e$$

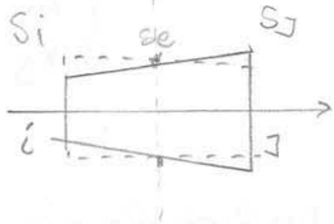
$$\boxed{[X]_e = [K]_e [U]_e}$$

The k-matrix it's related on C-matrix and obtain the elastic modules.

$$[k]_e = \int_{V_e} [B]^T [C][B] \, dV_e = \int_0^L S(x) [B]^T [C][B] \, dx$$

where $\int_0^L S(x) \, dx$ for the middle section (sezione media)

21) For the axial stiffness element:



$$S(x) \sim \frac{1}{2} (S_i + S_J) = S_e$$

where $[C] = E$, $[B] = \left[-\frac{1}{L} \;\; \frac{1}{L} \right]$

$$[k]_e = \int_0^L S_e [B]^T E [B] \, dx =$$

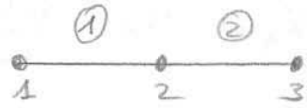
$$= S_e E \int_0^L \begin{bmatrix} -1/L \\ 1/L \end{bmatrix} \left[-\frac{1}{L} \;\; \frac{1}{L} \right] dx$$

$$\boxed{[K]_e = \frac{S_e E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

4 components for the stiffness matrix.

K_{22} have a contribution of element 1,2, both element.
Written the stiffness matrix for the finite elements.

8) STEP 5 : BOUNDARY CONDITIONS



The boundary conditions was the value of displacement of first and last node:

$$u_1 = u_1^* \qquad u_3 = u_3^*$$

u_2 it's UNKNOWN but X_2 it's KNOWN (X_1, X_3 UNKNOWN)
With the boundary conditions we have a right solutions of the problem and can be solve it.
We assume some HYPOTHESIS (the B.C) in finite element method.

The B.C. was
1) $u_1 = u_1^*$
2) $u_3 = u_3^*$
3) $X_2 = 0$ FOR SYMMETRY

$$\begin{vmatrix} X_1 = ? \\ X_2 = 0 \\ X_3 = ? \end{vmatrix} = \frac{E}{L} \begin{Vmatrix} S_1 & -S_1 & 0 \\ -S_1 & S_1+S_2 & -S_2 \\ 0 & -S_2 & S_2 \end{Vmatrix} \begin{vmatrix} u_1^* \\ u_2 \\ u_3^* \end{vmatrix}$$

On the second equations:

$$X_2 = 0 = -\frac{S_1 E}{L} u_1^* + \frac{S_1 E}{L} u_2 + \frac{S_2 E}{L} u_2 - \frac{S_2 E}{L} u_3^*$$

Where u_2 it's unknown, and simplify $\frac{E}{L}$:

$$0 = -S_1 u_1^* + S_1 u_2 + S_2 u_2 - S_2 u_3^*$$

$$u_2 (S_1 + S_2) = S_1 u_1^* + S_2 u_3^*$$

$$\boxed{u_2 = \frac{S_1 u_1^* + S_2 u_3^*}{S_1 + S_2}}$$

u_2 now it's known because contains only constants.

18

The strain it's constant and this is realistic because element rexist without have gradients of stress in the element.

When you discretize, you search to have a small element to have a more right solutions.

The solution is approximated because should be exist a gradient of displacement in the element, for the node 1 and node 2.

© Proprietà riservata dell'autore - Digitalizzazione e distribuzione a cura del CENTRO APPUNTI - Corso Luigi Einaudi, 55 - Torino / Pagina 69 di 752

Axial stiffness element 1D

...continue

STEP 4 - GLOBAL STIFFNESS MATRIX

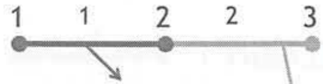
The next step in the formulation of the finite element equations is the assembly of the separate element equilibrium equations into a set of global equations by the direct stiffness method.

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ & K_{22} & K_{23} \\ \text{sym} & & K_{33} \end{bmatrix}}_{\text{Global Stiffness Matrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Equilibrium Equations for the Assembly

$$\boxed{[X] = [K][u]}$$

STEP 4 - GLOBAL STIFFNESS MATRIX



$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{S_1E}{L} & -\frac{S_1E}{L} & 0 \\ -\frac{S_1E}{L} & \frac{S_1E}{L} + \frac{S_2E}{L} & -\frac{S_2E}{L} \\ 0 & -\frac{S_2E}{L} & \frac{S_2E}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$[X] = [K][u]$$

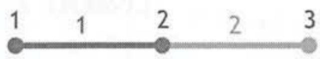
7

STEP 5 - BOUNDARY CONDITIONS

$u_1 = u_1^*$ KNOWN $X_1 =$ UNKNOWN

$u_2 =$ UNKNOWN $X_2 =$ KNOWN

$u_3 = u_3^*$ KNOWN $X_3 =$ UNKNOWN



$$\begin{bmatrix} X_2=0 \\ X_1=? \\ X_3=? \end{bmatrix} = E/L \begin{bmatrix} S_1 + S_2 & -S_1 & -S_2 \\ -S_1 & S_1 & 0 \\ -S_2 & 0 & S_2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_1^* \\ u_3^* \end{bmatrix}$$

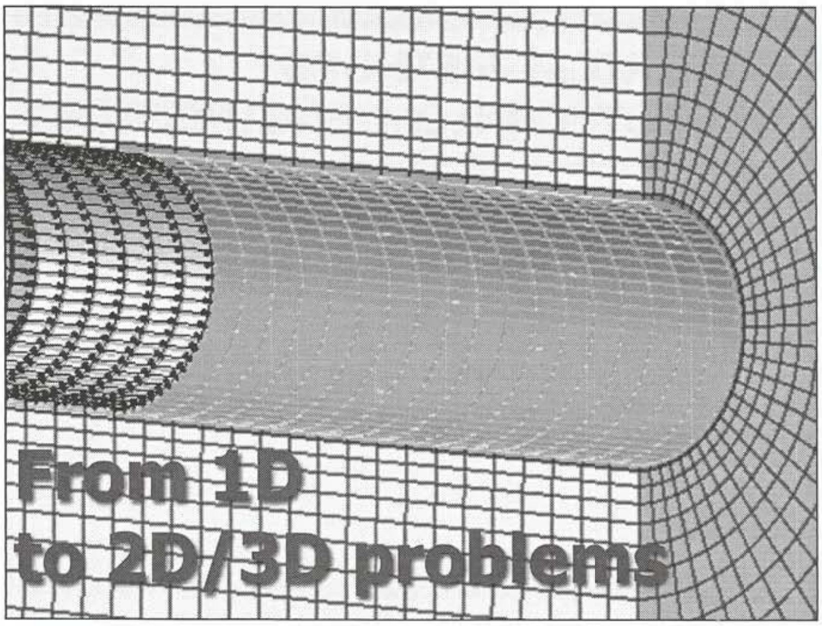
This methods can developped on PC.

SUMMARY

The essential features of the Finite Element Method (FEM) have been described by taking the axial stiffness element ("beam" element) as case example. The procedure shown and the different equations derived (Steps 1-6) apply in general.
In summary we have:

1. Element discretisation (FEM mesh)
2. Displacement Approximation — $u(x)=[H]^T [u]_e$
3.
   - Computation of *Strains and Stresses* — $[\varepsilon]=[B] [u]_e$
   - Computation of *Element Stiffness* — $[k]_e= \int_V [B]^T [C][B]dV$
4. Global Equilibrium Equations — $[X] = [K][u]$
5. Boundary conditions
6. Computation of strains and stresses — $[\varepsilon]=[B] [u]_e$, $[\sigma]=[C][B][u]_e$

3D finite element mesh (TUNNEL)
We have 3D mesh, quadrilateral element, not linear.



STEPS

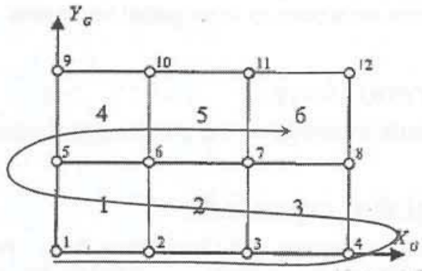
The finite element method involves the following steps:

1. **Element Discretisation**
   Modelling the geometry of the problem by assemblage of small regions named finite elements.
2. **Primary variable approximation**
   A primary variable must be selected as well as how it should vary over a finite element. In geotechnical engineering it is usual to adopt displacements as the primary variable.
3. **Element equations**
   Use of an appropriate variational principle to derive element equations.

(1)

ELEMENT DISCRETISATION

In order to refer to the complete finite element mesh, the elements and nodes must be numbered in systematic manner.



WARNING

We'll see later that the way we number the mesh influences the computation process.

ELEMENT DISCRETISATION

When constructing the Finite Element mesh the following should be considered:

- The geometry of the boundary value problem must be approximated as accurately as possible and the discretization process need be performed with great care.
- If there are curved boundaries (such as in the case of tunnels) or curved material interfaces, higher order elements, with mid-side nodes should be used.
- Mesh design may also be influenced by geometric discontinuities or the applied boundary conditions. These can be modeled by placing nodes at the discontinuity points.

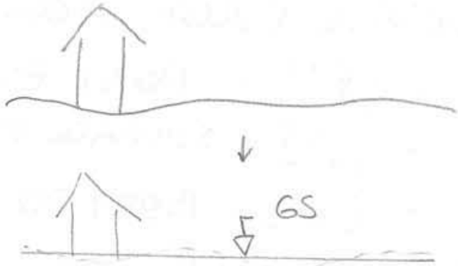
"In base a come NUMERO gli elementi influenzo il tempo computazionale e cambia la matrice di rigidezza: cambiamo le parti con gli zeri e devo cercare di trovare la forma migliore per NUMERARE LA MESH, aspetto importante."

- DENSER ELEMENT

Find to have small element when higher stress gradient are expected (PREVISTO).
It's better have small elements than BIG elements because influence the results of the problem.

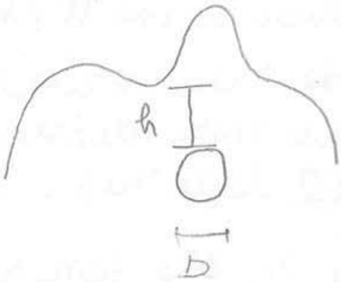
EXAMPLES

1) Use a LINE and not a real surface (GS = GROUND SURFACE):
It's the boundary condition for the geometry of the problem.

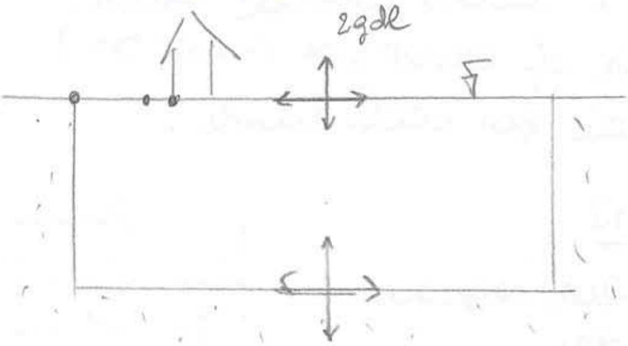


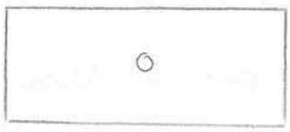
In this case: $h >> 2D$?

- yes: DEEP TUNNEL, used a line
- NO: SHALLOW TUNNEL, used a different mesh and the construction depends of the case.

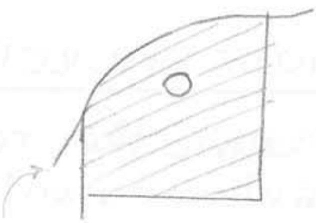


In this situation:



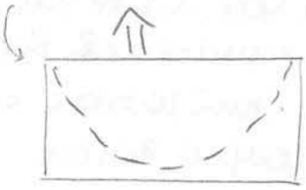


DEEP TUNNEL



SURFACE TUNNEL

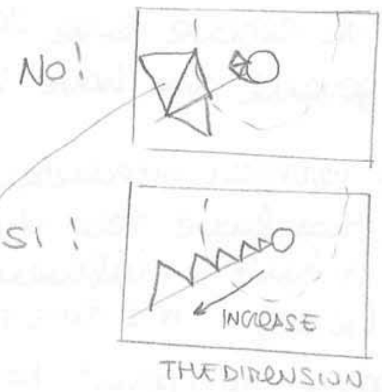
i used CURVED BOUNDARY?
The dimension of mesh influence the boundary.



SHALLOW TUNNEL

In (A) - zone we have a small elements but far of (A) we not needs small elements: i need a different dimension of elements.

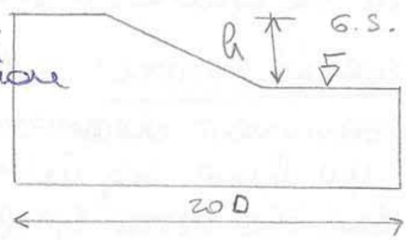
less density mesh



It's important to have a gradually increasing of the density and dimension of mesh to have a gradually variation.

2) GARANTIRE LA STABILITA' ROCK SLOPE

The Ground Surface it's the BOUNDARY CONDITION and we defined the dimension of the problem.



How to decide where put the limit of the boundary?

- 10 D depends of problem;
- Use the height of ROCK SLOPE (pendio in roccia).

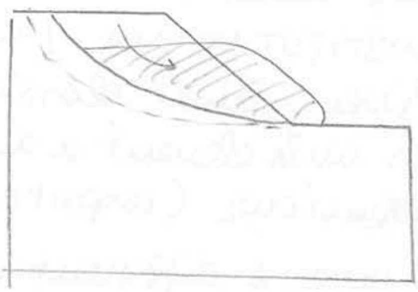
The B.C. of the boundary not influence the results of stability of rock slope: 20 D it's where put the boundary but we didn't say what is the boundary condition used (da usare).

In general rule we considered 10 h of rock slope. During the exam, if we used 20 h or 15 ÷ 21 h it's the same.

If $h = 19{,}7\ m \rightarrow 21 h \sim 400\ m$

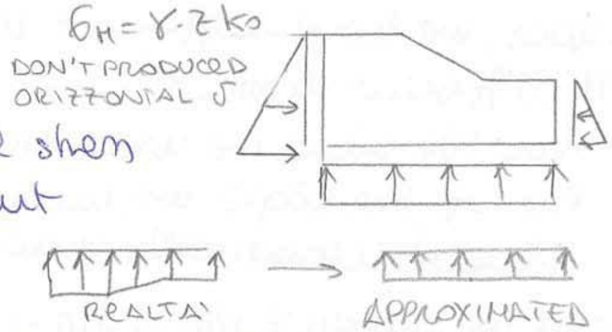
How can discretize the region with mesh element? Used triangular mesh element?

If we used an equivalent continuum, we considered joint and rock over rock mass.



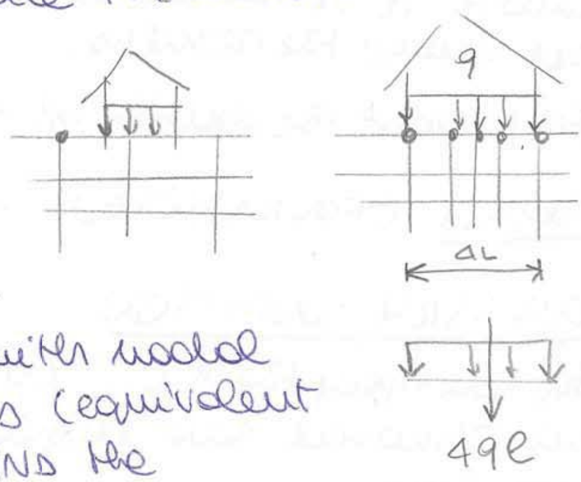
The gravity it's the reason that produced the instability of the slope. $\sigma_H = \gamma \cdot z k_0$

So, if we combined the initial stress and external stress to obtain zero displacement on the boundary.



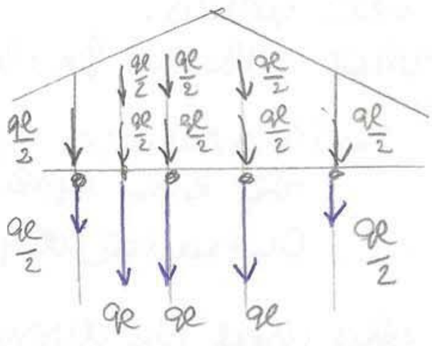
And if $k_0 = 0,5$?
What's the value of vertical stress?
Depends of height of gravel, BUT, because we are so far of the slope, we apply the CONSTANT LOAD because THIS ZONE isn't influenced by slope and this load not influence the result.

EXAMPLE: HOUSE



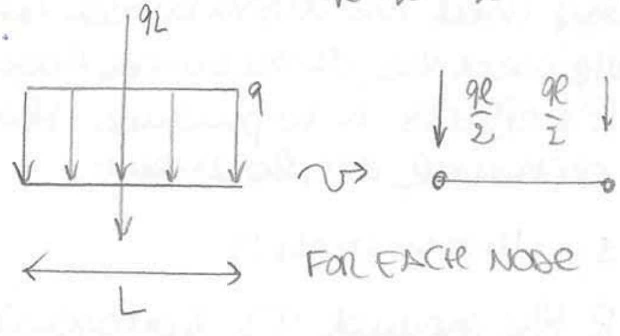
If we have the loads apply NOT all on the nodes, we increase the density of mesh.

We have distributed loads with nodal equations and nodal forces (equivalent nodal forces): NOW we FIND the equivalent nodal forces.



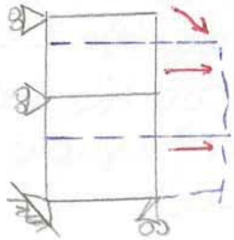
With the division on the forces on the element, we obtain the value of equivalent nodal forces, that isn't the same value (non hanno lo stesso valore).

If you used a triangular load, the computation was more difficult.



Come si deformerà il contorno per effetto dell'imposizione delle B.C.?

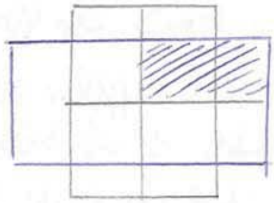
How to deform the boundary as a result of the imposition of B.C.?



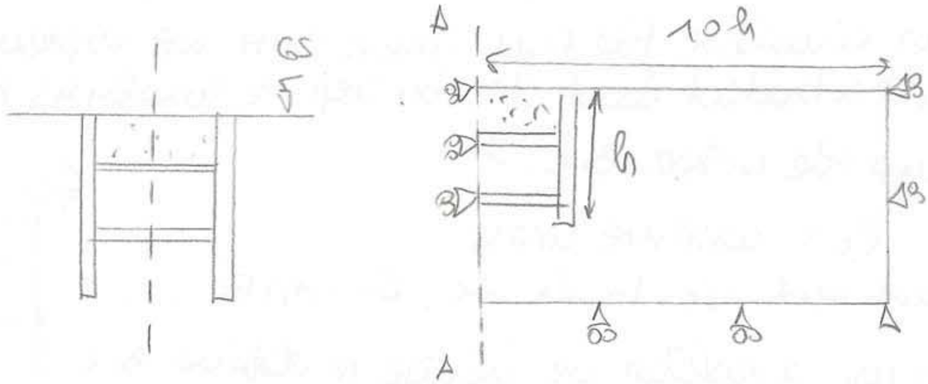
→ : this vector it's the VECTOR OF DISPLACEMENTS OF THE NODES;

- - - : DEFORMED CONFIGURATION of a quarter element.

I see only a quarter of problem but with a quarter of solution i find the GLOBAL DISPLACEMENT and GLOBAL STRAIN.



EXAMPLE



For build a model we can cut in 2 parts.

A : A it's the axes of symmetry and we haven't a displacements.

We see that the equations was change in 2D/3D element

- FIRST UNKNOWN : THE DISPLACEMENT
- SECOND UNKNOWN : STRAINS AND STRESSES

(2)

DISPLACEMENT APPROXIMATION

In the displacement based finite element method the primary unknown quantity is the displacement field which varies over the problem domain. Strains and stresses are treated as secondary quantities which can be found from the displacement field once it has been determined. The main approximation of the finite element method is to assume a particular form for the way these displacement components vary over the domain under study. Clearly, this assumed variation must satisfy the conditions of compatibility.

$$\boxed{[u(x,y,z)]=[H(x,y,z)]\ [u]_e}$$

displacement of element — Element Shape Function — displacement of the nodes

If we use this function for the elements in the mesh, we obtain a FEM approximation of the equilibrium equations. To this end, we apply the Principle of Virtual Work.

DISPLACEMENT APPROXIMATION

The essential feature of the element wise approximation is that the variation of the unknown displacements within an element is expressed as a simple function of the displacements at the nodes. The problem of determining the displacement field throughout the finite element mesh is, therefore, reduced to determining the displacement components at a finite number of nodes. These nodal displacements are referred to as the unknown degrees of freedom. For two dimensional plane strain problems there are two degrees of freedom at each node: the u and v displacements in the x and y coordinate directions respectively.

u = in the axial direction } 2D } 3D
v = in the vertical direction
w =

ELEMENT EQUATIONS

Again we write:

$$[F]_e = [k]_e\,[u]_e$$

Where:

$$[k]_e = \int_{V_e} [B]_e^T\,[C]_e\,[B]_e\,dV_e$$ Element stiffness matrix

[R] = EXTERNAL FORCES, vectors of forces and it was composed by 5 forces:

- F^V VOLUME FORCES
- F^S DISTRIBUTED LOAD
- F^i BODY FORCE
- $F^{\varepsilon,\sigma}$ INITIAL STRESS AND STRAIN FORCES

GLOBAL EQUATIONS (4)

The next stage in the formulation of the finite element equations is the assembly of the separate element equilibrium equations into a set of global equations:

$$[R] = [K]\,[u]$$

The Global stiffness matrix is obtained by summing the element stiffness matrices:

$$[K] = \Sigma_e \int_{V_e} [B]_e^T\,[C]_e\,[B]_e\,dV_e$$ Global stiffness matrix

The vector of the forces at the nodes is:

$$[R] = [F]^V + [F]^S + [F]^{\sigma}{}_i + [F]^{\varepsilon}{}_i + [F]^i$$

If local axes are defined for the elements, it is necessary to transform the element stiffness matrices and the load boundary conditions prior to assembling the global system of equations.

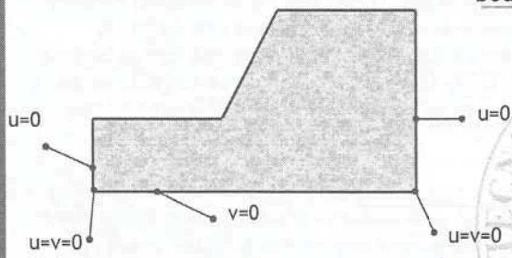
$F^{\sigma,\varepsilon}$ normally was zero for ours applications.

We composed each forces for each elements.

BOUNDARY CONDITIONS

2 - displacement boundary conditions

The displacements on the boundaries are known

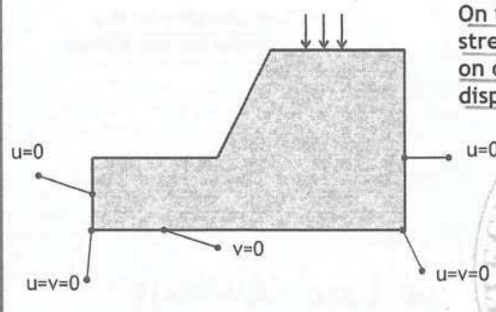


Value of displacement along boundary

BOUNDARY CONDITIONS

3 - loading/displacement conditions

On the surface boundary the stresses are known and on other boundaries the displacements are given



(6)

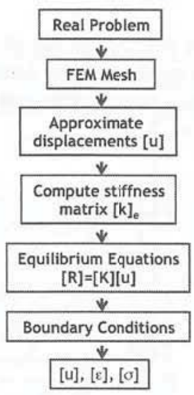
SOLVE THE GLOBAL EQUATIONS

Once the global stiffness matrix has been established and the boundary conditions added, a large system of simultaneous equations is obtained (N equations with N unknowns, where N = n•l, with n=number of nodes in the mesh and l=number of unknowns per node). There are several different mathematical techniques for solving large systems of equations:

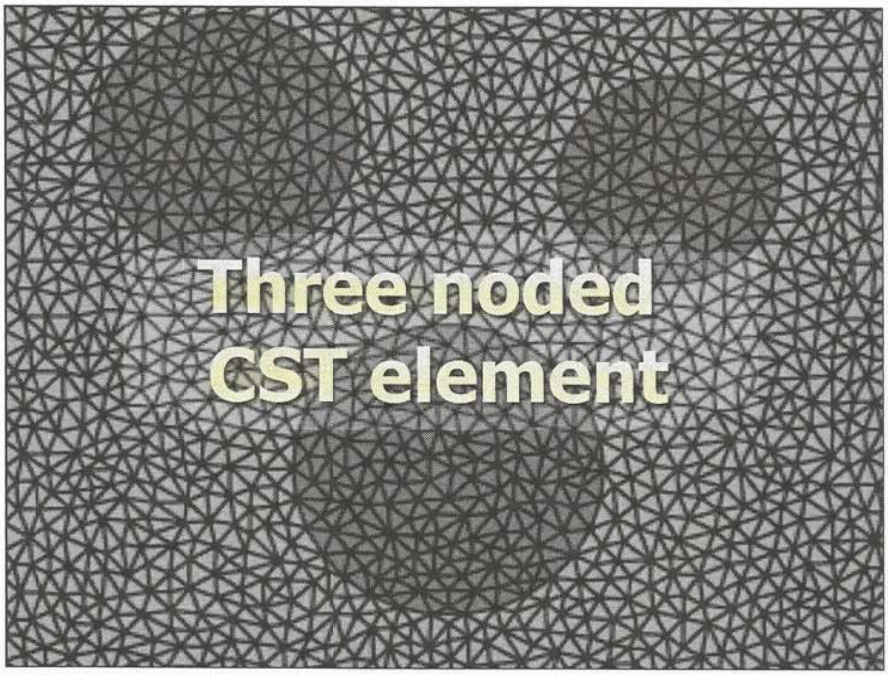
- direct methods (e.g.: Gaussian elimination; most finite element programs adopt this method)
- iterative methods (e.g.: the conjugate gradient method which is shown to be more effective for solving 3D problems)

Sometimes we have 1 unknown for each nodes (anche la rotazione ai nodi ...).

FLOW DIAGRAM FOR FEM



Flow Diagram for a typical FEM Code



STEPS

The finite element method involves the following steps:

1. **Element Discretisation**
   Modelling the geometry of the problem by assemblage of small regions named finite elements.
2. **Primary variable approximation**
   A primary variable must be selected as well as how it should vary over a finite element. In geotechnical engineering it is usual to adopt displacements as the primary variable.
3. **Element equations**
   Use of an appropriate variational principle to derive element equations.