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Basic Notions

Sets

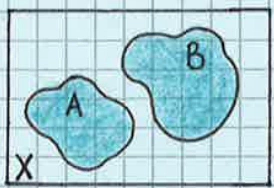
A **set** in math is a collection of well defined and distinct objects, considered as an object in its own right. We shall denote sets by upper case letters X, Y, while for the elements will be used lower case letters x, y... When an element x belongs to the set X one writes $x \in X$, if not $x \notin X$. The sets considered are mainly built starting from sets of numbers.

$\mathbb{N}$ = set of natural numbers $\mathbb{R}$ = set of real numbers

$\mathbb{Z}$ = set of integer numbers $\mathbb{C}$ = set of complex numbers

$\mathbb{Q}$ = set of rational numbers

Let's consider a non-empty set X, called **ambient set**. A **subset** A of X is a set all of whose elements belong to X; so $A \subseteq X$ (A is contained in X) if the subset A is allowed to possibly coincide with X, and $A \subset X$ (A is properly contained in X) if A is a proper subset of X (it doesn't exhaust the whole X).
A subset can be represented by using the so-called **Venn-diagrams**...



... and it can be described by listing the elements of X which belong to it.

$A = \{x, y, \dots, z\}$

The order in which elements appear is not essential. This restricts the use of such notation to subsets with few elements. The notation

$A = \{x \in X \mid p(x)\}$ or $A = \{x \in X : p(x)\}$ will be used, too.

P(x) denotes the **characteristic property** of the elements of the subset, i.e. the condition that is valid for the elements of the subset only. (predicate)
The collection of all subsets of a given set X forms the **power set** of X and it's called $P(X)$. Obviusly $X \in P(X)$. Among the subset of X there is the **empty set** (with no elements inside), denoted by the symbol $\emptyset$, so $\emptyset \in P(X)$. All other subsets of X are proper and non-empty.

Example

$X = \{1, 2, 3\}$ - ambient set.

$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, X\}$

We say that X has **cardinality** 3 ('cause it includes 3 elements) while P(X) has $8 = 2^3$ elements (cardinality 8). In general if a finite set has cardinality n, the power set has cardinality 2^n.

Starting from one or + subsets of X, one can define new subsets taking the complement, which is the simplest set-theoretical operation: if A is a subset of X, the **complement of A** is defined as the subset made of all elements of X not belonging to A.

$CA = \{x \in X : x \notin A\}$. The notation $C_X A$ is sometimes used to underline that complements are taken with respect to the ambient space X.

$CX = \emptyset$ $\quad C\emptyset = X$ $\quad C(CA) = A$

Example If $X = \mathbb{N}$ and A is the subset of even numbers, then CA is the subset of odd numbers.

Given two subsets A and B of X, one defines **intersection** of A and B the subset containing the elements of X that belong to both A and B...

$A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$

... and **union** of A and B the subset made of all the elements that r either in A or in B (non-exclusively)

$A \cup B = \{x \in X : x \in A \text{ or } x \in B\}$

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Predicates: assertions or properties $p(x, \dots)$ that depend upon one or + variables $x, \dots$ (parameters) belonging to suitable sets and which become a formula if variables are fixed.
The aforementioned logic operations can be applied to predicates as well, and give rise to new predicates. This establishes a precise relation among the essential connectives ($\neg, \wedge, \vee$) and the set-theoretical operations of taking complements, intersection and union. In fact, recalling the definition $A = \{x \in X : p(x)\}$ of subset of a given set X, the characteristic property $p(x)$ of the elements of A is a predicate, true precisely for the elements of A. CA is obtained by negating the characteristic property $CA = \{x \in X : \neg p(x)\}$; while intersection and union of A with another subset $B = \{x \in X : q(x)\}$ are described respectively by the conjunction and disjunction of the corresponding charac. properties:
$A \cap B = \{x \in X : p(x) \wedge q(x)\}$ $\quad$ $A \cup B = \{x \in X : p(x) \vee q(x)\}$
The properties of the previous section translate into similar properties enjoyed by the logic operations.

Quantifiers: Given a predicate $p(x)$, with $x \in X$, one is lead to ask if $p(x)$ is true for all elements x, or if there exists at least one element x making $p(x)$ true. When posing such a question we are actually considering the formulas
$\forall x, p(x)$ (for all x, $p(x)$ holds) $\quad$ $\exists x, p(x)$ (there exists at least one x, such that $p(x)$ holds)
The symbol $\forall$ is called universal quantifier, and the symbol $\exists$ is said existential quantifier ($\exists!$ there exists one and only one element - there exists a unique)

* Putting a quantifier in front of a predicate transforms the latter in a formula, whose truth value may be then determined.

$\neg(\forall x, p(x)) \iff \exists x, \neg p(x)$ The negation of $\forall x, p(x)$ is $\exists x, \neg p(x)$

$\neg(\exists x, p(x)) \iff \forall x, \neg p(x)$ The negation of $\exists x, p(x)$ is $\forall x, \neg p(x)$

If a predicate depends upon 2 or + arguments, each of them may be quantified. Yet the order in which quantifiers are written can be essential, but if they are of the same type, can be swapped without modifying the truth value of the formula:

$\forall x \forall y, p(x,y) \iff \forall y \forall x, p(x,y)$ $\qquad$ $\exists x \exists y, p(x,y) \iff \exists y \exists x, p(x,y)$

* On the contrary, exchanging the places of different quantifiers usually leads to different formulas, so one should be very careful when ordering quantifiers.

Example Consider the predicate $p(x,y) = x \geq y$, with $x, y \in \mathbb{N}$. $\forall x \forall y, p(x,y)$ (means given any 2 natural numbers, each one is greater or equal than the other - false statement)
$\forall x \exists y, p(x,y)$ (means given any natural number x, there's a natural number y smaller or equal than x - true statement); $\exists x \forall y, p(x,y)$ (there's a natural number x greater or equal than each natural number - false statement).

Sets of numbers

- **The set $\mathbb{N}$ of natural numbers**
$\mathbb{N} = \{0, 1, 2, \dots\}$ The operations of sum and product, the commutative, associative and distributive properties are defined on $\mathbb{N}$
$\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ set of natural numbers different from 0.
A natural number n is usually represented in base 10 by the expansion $n = c_k 10^k + c_{k-1} 10^{k-1} + \dots + c_1 10 + c_0$ where c_i = natural numbers $0 \to 9$, decimal digits; the expression is unique if $c_k \neq 0$ when $n \neq 0$
$n = (c_k c_{k-1} \dots c_1 c_0)_{10}$ or $n = c_k c_{k-1} \dots c_1 c_0$. Any natural number ≥ 2 may be taken as base (2 = binary base).
Natural numbers can be also represented geometrically as points of a straight.
O = origin (associated to zero), $P \neq O$ (associated to 1). Line $O \to P$ has a positive direction; length OP taken as unit for measurements (we obtain the other points associated to natural numbers)

- **The set $\mathbb{Z}$ of integer numbers**
$\mathbb{Z} : \{0, \pm 1, \pm 2, \dots\} \to$ integers. $\mathbb{N}$ is a subset of $\mathbb{Z}$ consisting of $0, +1, +2 \dots$ called positive numbers (integers); $-1, -2, \dots$ are negative integers. Sum, product and difference are defined in $\mathbb{Z}$. An integer can be represented in decimal base $z = \pm c_k c_{k-1} \dots c_1 c_0$. The geometric picture is a straight of $\mathbb{N}$ extended to the left of the origin.

- **The set $\mathbb{Q}$ of rational numbers**
Set of quotients, or ratios, of two integers (each one), the second of which (denominator) is non-zero.
One can consider the denominator positive, so that a rational number is given by:
$r = z/n$ with $z \in \mathbb{Z}$ and $n \in \mathbb{N}_+$
One may also suppose the fraction is reduced (z, n have no common divisors). In this way $\mathbb{Z}$ is defined as subset of rationals whose denominator is 1. Sum, product, difference and division are defined on $\mathbb{Q}$, so long as the 2° rational is other than 0.
For rational numbers the representation in base 10 is used. $r = \pm c_k c_{k-1} \dots c_1 c_0 . d_1 d_2 \dots$ corresponding to $r = \pm (c_k 10^k + c_{k-1} 10^{k-1} + \dots + c_1 10 + c_0 + d_1 10^{-1} + d_2 10^{-2} + \dots)$

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- **Intervals**: subsets of $\mathbb{R}$ whose elements lie between two fixed numbers.
  a, b real numbers
  $a \leq b$ CLOSED interval with end-points a, b $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ included a, b
  $a < b$ OPEN interval with end-points a, b $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ (also $]a,b[$) not included a, b
  → BOUNDED! a, b = boundary points
  If one includes only one end-point:
  $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$ half-open on the right
  $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$ half-open on the left

  Intervals defined by a single inequality.

  $[a,+\infty) = \{x \in \mathbb{R} : a \leq x\}$ $\quad (a,+\infty) = \{x \in \mathbb{R} : a < x\}$
  $(-\infty,b] = \{x \in \mathbb{R} : x \leq b\}$ $\quad (-\infty,b) = \{x \in \mathbb{R} : x < b\}$
  unbounded

  $\pm\infty$ are symbols used to extend the ordering of the reals with the convention that $x > -\infty$ and $x < +\infty$ $\forall x \in \mathbb{R}$.
  $\mathbb{R} = (-\infty, +\infty)$

* All the other points of an interval are called interior points.

- **Bounded sets**

  - A subset A of $\mathbb{R}$ is called bounded from above if there exists a real number b s.t. $x \leq b$, $\forall x \in A$
    b = an upper bound of A $\quad (-\infty, b] \supseteq X$
  - The set A is bounded from below if there's a real number a s.t. $a \leq x$, $\forall x \in A$
    a = a lower bound of A $\quad [a,+\infty) \supseteq X$
  - A = bounded if it's bounded from above and below. $[a,b] \supseteq X \iff \exists c \geq 0$ s.t. $|x| \leq c$ $\forall x \in X$

Examples

- ARCHIMEDEAN PROPERTY: the set $\mathbb{N}$ is bounded from below, for any real $b > 0$ $\exists$ a natural number $n : n > b$
- The set $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$ is bounded $\to$ $0 \leq \frac{n}{n+1} < 1$ $\forall n \in \mathbb{N}$ $\quad n \leq n+1 \quad \frac{n}{n+1} \neq 1$

- A set $A \subset \mathbb{R}$ admits a maximum if an element $x_M \in A$ exists s.t. $x \leq x_M$, for any $x \in A$
  x_M (unique) = maximum of the set A $\to$ $x_M = \max A$ *
- A set $A \subset \mathbb{R}$ admits a minimum if an element $x_m \in A$ exists s.t. $x \geq x_m$, for any $x \in A$
  x_m (unique) = minimum of the set A $\to$ $x_m = \min A$

* A set admitting a maximum must be bounded from above (MAX = smallest of all possible upper bounds)
The opposite is not true. 1 = smallest upper bound of A: let's show that each real number $r < 1$ is not an upper bound:

$$\frac{n}{n+1} > r = \frac{n+1}{n} < \frac{1}{r} \; ; \; 1 + \frac{1}{n} < \frac{1}{r} \text{ or } \frac{1}{n} < \frac{1-r}{r} \longrightarrow n > \frac{r}{1-r}$$

. 1 = smallest u.b. of A, for $1 \notin A$
there's not $\frac{n}{n+1} = 1$; 1 = supremum or least upper bound of A $(1 = \sup A)$

- $A \subset \mathbb{R}$ bounded from above. Supremum = smallest upper bound of A $(\sup A) = \text{lub}\, X$
- $A \subset \mathbb{R}$ bounded from below. Infimum or greatest lower bound of A = largest lower bound of A $(\inf A)$

$s = \sup A$, two conditions
$x \leq s$ $\forall x \in A$ $\quad s$ = upper bound of A
for any real $r < s$, there's an $x \in A$ with $x > r$ (2) $r < s$ not an upper bound, 'cause s is the smallest one.
It's immediate to see that if a set admits a maximum, this must be the supremum as well.
A not bounded from above $\to$ $\sup A = +\infty$
A not bounded from below $\to$ $\inf A = -\infty$

$$\begin{bmatrix} \text{If } \exists \max A \Rightarrow \max A = \sup A \\ \text{If } \exists \min A \Rightarrow \min A = \inf A \end{bmatrix}$$

- **Completeness of $\mathbb{R}$** can be formalised in different ways
  SEPARABILITY AXIOM: decomposing $\mathbb{R}$ into the union of 2 disjoint subsets C_1, C_2 ((C_1, C_2) is called cut)
  each x of $C_1 \leq x$ of C_2, there exists a (unique) separating element $s \in \mathbb{R}$:
  $x_1 \leq s \leq x_2$ $\quad \forall x_1 \in C_1, \forall x_2 \in C_2$ $\longrightarrow$ $C_1, C_2 \in \mathbb{R}$ s.t. $\forall x \in C_1 \Rightarrow x \leq y$ $\forall y \in C_2$; $C_1 \cup C_2 = \mathbb{R} \Rightarrow \exists!\, s \in \mathbb{R} : x \leq s \leq y$

  NOTION OF SUPREMUM: every bounded set from above admits a supremum in $\mathbb{R}$ = $\exists n \leq$ all u.b. *
  thanks to this property one can prove the existence in $\mathbb{R}$ of $\sqrt{2}$ $\to$ $p (>0)$ s.t. $p^2 = 2$
  Bounded set $B = \{x \in \mathbb{Q} : x^2 < 2\}$ has a $\sup = p$. $p^2 < 2$ cannot occur, p not upper bound for B and neither $p^2 > 2$, p not the least of all u.b. Necessarily $p^2 = 2$; p not rational $\to$ B no u.b.

* The same for infimum

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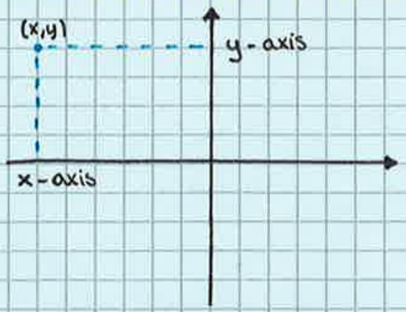
* The most significant example of cartesian product stems from $X = Y = \mathbb{R}$. The set $\mathbb{R}^2$ consists of ordered pairs of real numbers. $\mathbb{R}$ = straight line $\rightarrow$ $\mathbb{R}^2$ = model of the plane.

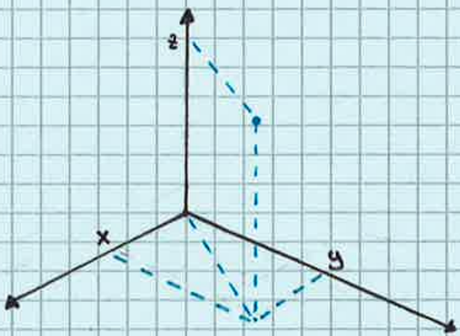


ORTHONORMAL FRAME $(x,y) \in \mathbb{R}^2$ associated to each point P on the plane

The components of the pair are called (cartesian) coordinates of P in the chosen frame.

This notion can be generalised to the product of more sets; n non-empty sets $X_1, X_2, \dots, X_n$ consider n-tuples $(x_1, x_2, \dots, x_n)$ where for every $i = 1,2,\dots,n$ each $x_i \in X_i$. $X_1 \times X_2 \times \dots \times X_n$ = set of all n-tuples.
$X_n = X \times X \times \dots \times X$.

- $\mathbb{R}^3$ set of triples (x,y,z) of real numbers $\rightarrow$ mathematical model of three dimensional space.



$X = \mathbb{R}$ $\quad X^2 = \mathbb{R}^2$

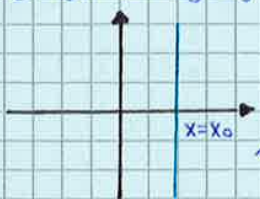
$\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

Relations in the plane

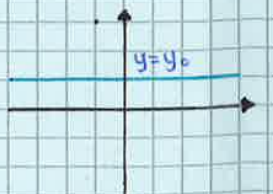
Every non-empty subset R of $\mathbb{R}^2$ defines a relation between real numbers: one says x is R-related to y or x is related to y by R if the ordered pair (x,y) belongs to R. Graph = set of points in the plane whose coordinates belong to R. A relation is commonly defined as the set of pairs (x,y) s.t. x and y satisfy the constraints.

Examples

- $ax + by = c$ a,b costant and not both vanishing defines a straight line. (implicit definition)
$\{(x,y) : ax + by = c\}$

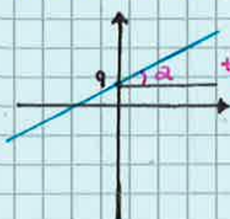


$b = 0$
$R = \{(x_0, y) : y \in \mathbb{R}\}$
// to y-axis

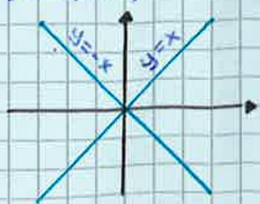


$a = 0$
$R = \{(x, y_0) : x \in \mathbb{R}\}$
// to x-axis

- assuming $a, b \neq 0$ we have $y = mx + q$ with $m = -\frac{a}{b}$ and $q = \frac{c}{b}$; m is called slope of the line, which can be plotted finding two points (two pairs)

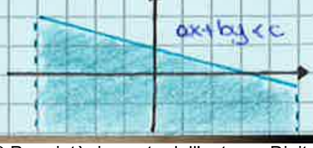


$R = \{(x,y) : y = mx + q\}$



$c = 0$ (or $q = 0$)
the origin belongs to the line
ex. BISECTORS

- $ax + by < c$ inequality which defines one of the half-planes in which the straight line $ax + by = c$ divides the plane. $b > 0$ half plane below the line ($<$ below - $>$ over).



In this case is an open plane (the line is not included) since the inequality is strict.
$\leq$ and $\geq$ define a closet set. $\quad R = \{(x,y) : ax + by < c\}$

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Functions

Definitions and first examples

X, Y two sets. A function f defined on X with values in Y is a correspondence associating to each element $x \in X$ at most one element $y \in Y$. (a function from X to Y). Function = map. The set of $x \in X$ to which f associates an element in Y is the domain of f; it is a subset of X (dom f)

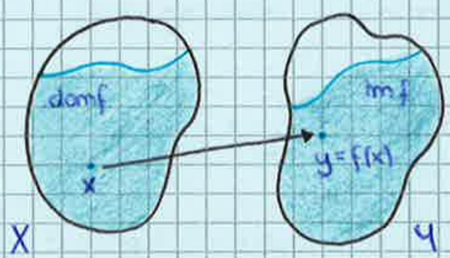
$f: \text{dom} f \subseteq X \to Y$ If dom $f = X$ one writes $f: X \to Y$ (f is defined on X)

The element $y \in Y$ associated to an element $x \in$ dom f is called image of x or under f ($y = f(x)$)

$f: x \mapsto f(x)$ set of images $y = f(x)$ = range of f (im f) Y = cod f

The graph of f is the subset $\Gamma(f)$ of the Cartesian product $X \times Y$ made of pairs $(x, f(x))$ when x varies in the domain of f.

$\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in \text{dom} f\}$



- If $Y = \mathbb{R}$ the function f is said real or real-valued.
  If $X = \mathbb{R}$ the function is of one real variable.
  The graph of a real function is a subset of the Cartesian plane $\mathbb{R}^2$.

* A remarkable special case of map arises when $X = \mathbb{N}$ and the domain contain a set of the type $\{n \in \mathbb{N} : n \geq n_0\}$ for a certain natural number $n_0 \geq 0$. Such a function is called sequence.
see other examples p. 32

$a: \mathbb{N} \to \mathbb{R}$; $a(n) = a_n$; $\{a_n\}_{n \geq 0}$ $\{a_n : n \in \mathbb{N}\}$ $\{\frac{n}{n+1} : n \in \mathbb{N}\}$; $a_n = \frac{n}{n+1} \to f(x) = \frac{x}{x+1}$

Range and pre-image

Let A be a subset of X. The image of A under f is the set of all the images of elements of A.
$\to$ im $f = \{y \in Y \mid \exists x \in X$ with $f(x) = y\}$

$f(A) = \{f(x) : x \in A\} \subseteq \text{im} f$ It's empty $\iff$ A contains no elements of the domain of f.
The image $f(X)$ of the whole set X is the range of f (im f)

Let y be any element of Y; the pre-image of y by f is the set of elements in X whose image is y.

$f^{-1}(y) = \{x \in \text{dom} f : f(x) = y\}$ It's empty precisely when y does not belong to the range of f.
If B is a subset of Y, the pre-image of B under f is defined as the set union of all pre-images of elem. of B.

$f^{-1}(B) = \{x \in \text{dom} f : f(x) \in B\}$

- $A \subseteq f^{-1}(f(A))$ for any subset A of dom f.
- $f(f^{-1}(B)) = B \cap \text{im} f \subseteq B$ for any subset B of Y.

Example
Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Image under f of the interval $A = [1,2]$ is the interval $B = [1,4]$. Yet the pre-image of B under f is the union of the intervals $[-2,-1]$ and $[1,2]$ $\to f^{-1}(B) = \{x \in B : 1 \leq |x| \leq 2\}$

- Let f be a real map and A a subset of dom f. One calls supremum of f on A (or in A), the supremum of the image of A under f.

$\sup_{x \in A} f(x) = \sup f(A) = \sup\{f(x) : x \in A\}$

Then f is bounded from above on A if the set $f(A)$ is bounded from above, or equivalently, if $\sup_{x \in A} f(x) < +\infty$
If $\sup_{x \in A} f(x)$ is finite and belongs to $f(A)$, then it's the maximum of this set.

This number is the maximum value (or simply, maximum) of f on A and is denoted by $\max_{x \in A} f(x)$.
$\max_{x \in A} f(x) = \max\{f(x) : x \in A\}$

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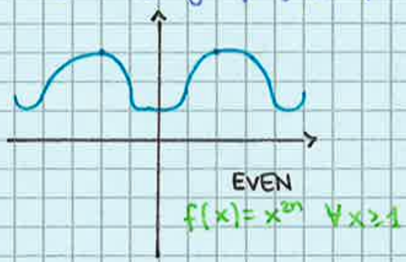
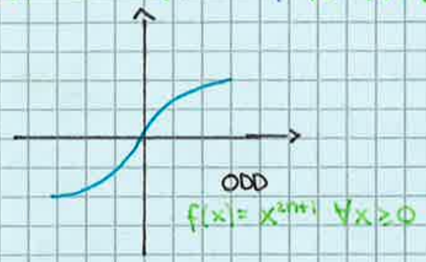
Elementary functions and properties

* Let $f: \text{dom} f \subseteq \mathbb{R} \to \mathbb{R}$ be a map with a symmetric domain with respect to the origin, hence s.t. $x \in \text{dom} f$ forces $-x \in \text{dom} f$ as well. The function f is said **even** if $f(-x) = f(x)\ \forall x \in \text{dom} f$, while it is **odd** if $f(-x) = -f(x)\ \forall x \in \text{dom} f$.
The graph of an even function is symmetric with respect to the y-axis, and that of an odd map symmetric with respect to the origin. If f is odd and defined in the origin, necessarily it must vanish at the origin for $f(0) = -f(0)$.

* A function $f: \text{dom} f \subseteq \mathbb{R} \to \mathbb{R}$ is said **periodic of period p** (with $p > 0$ real) if $\text{dom} f$ is invariant under translations by $\pm p$ (i.e. if $x \pm p \in \text{dom} f, \forall x \in \text{dom} f$) and if $f(x+p) = f(x)$ holds for any $x \in \text{dom} f$)





* In this cases it's possible to study only one function's part cause they're symmetric.

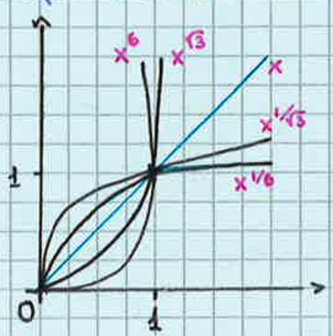
$\to x \in \text{dom} f \iff x \pm p \in \text{dom} f$ and $f(x+p) = f(x)\ \forall x \in \text{dom} f$ THAT'S TRUE!

One easily can see that an f periodic of period p is also periodic for any multiple mp ($m \in \mathbb{N} \setminus \{0\}$) of p. If the smallest period exists, it goes under the name of **minimum period** of the function. A constant map is clearly periodic of any period $p > 0$ and thus has no minimum period.

• Powers

These are functions of the form $y = x^\alpha$. The case $\alpha = 0$ is trivial (constant function $y = x^0 = 1$). Suppose then $\alpha > 0$. For $\alpha = n \in \mathbb{N} \setminus \{0\}$ we find the monomial functions $y = x^n$ defined on $\mathbb{R}$.
n odd $\to$ maps odd, strictly increasing on $\mathbb{R}$ and with range $\mathbb{R}$
n even $\to$ maps even, strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$; range $[0, +\infty)$
Consider now $\alpha > 0$ rational. If $\alpha = 1/m$ where $m \in \mathbb{N} \setminus \{0\}$, we define a function (mth root of x) $y = x^{1/m} = \sqrt[m]{x}$, inverting $y = x^m$.
m odd $\to$ domain $\mathbb{R}$
m even $\to$ domain $[0, +\infty)$
It is strictly increasing and ranges over $\mathbb{R}$ or $[0, +\infty)$, according to whether m is even or odd respectively.
For $\alpha = \frac{n}{m} \in \mathbb{Q}$, $n, m \in \mathbb{N} \setminus \{0\}$ with no common divisors, the function $y = x^{n/m} \to y = (x^n)^{1/m} = \sqrt[m]{x^n}$. As such:
m odd $\to$ domain $\mathbb{R}$
m even $\to$ domain $[0, +\infty)$
It's strictly increasing on $[0, +\infty)$ for any n, m, while if m odd it strictly increases or decreases on $(-\infty, 0]$ according to the parity of n. See examples

Generic function $y = x^\alpha$, $\alpha > 0$ irrational. If a is a non-negative real number we can define a^α with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$ starting from rational exponent powers and exploiting the density of rationals inside $\mathbb{R}$.
If $a > 1 \to a^\alpha = \sup\{a^{n/m} : \frac{n}{m} \leq \alpha\}$
For $0 \leq a \leq 1 \to a^\alpha = \inf\{a^{n/m} : n/m \leq \alpha\}$
So we have $y = x^\alpha$ with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$ defined on $[0, +\infty)$, strictly increasing; range $[0, +\infty)$
If $\alpha < \beta \to 0 < x^\beta < x^\alpha < 1$ for $0 < x < 1$, $1 < x^\alpha < x^\beta$ for $x > 1$



At last, consider the case of $\alpha < 0$, $y = x^\alpha = \frac{1}{x^{-\alpha}}$ by definition. Its domain = domain of $y = x^{-\alpha}$ minus the origin. All maps are strictly decreasing on $(0, +\infty)$, while on $(-\infty, 0)$ we have:
$\alpha = -n/m$, m odd $\to f$ strictly increasing if n even, strictly decreasing with n odd.
In conclusion $\forall \alpha \neq 0$ the inverse of $y = x^\alpha \to y = x^{1/\alpha}$

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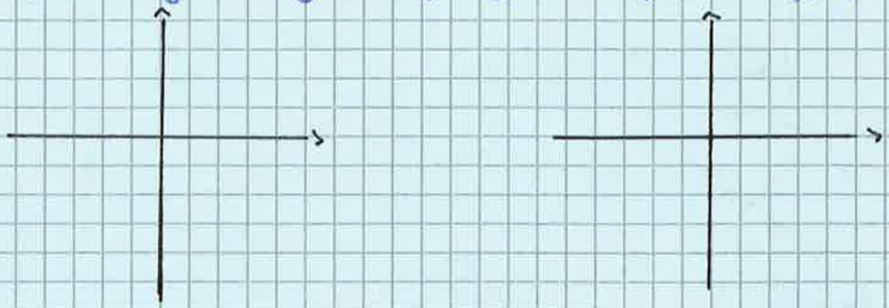
Monotone functions

Let f be a real map of one real variable, and I the domain of f or an interval contained in the domain. The function f is increasing on I if, given elements x_1, x_2 in I with $x_1 < x_2$, one has $f(x_1) \leq f(x_2)$;
in symbols: $\forall x_1, x_2 \in I, \quad x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
The function f is strictly increasing on I if (equivalent to say that $f(x)$ is injective)
$\forall x_1, x_2 \in I, \quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
The definitions of decreasing and strictly decreasing functions on I are obtained from the previous definitions by reversing the inequality between $f(x_1)$ and $f(x_2)$

$\forall x_1, x_2 \in I, \ x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

$\forall x_1, x_2 \in I, \ x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



* The function f is (strictly) monotone on I if it is either (strictly) increasing or (strictly) decreasing on I.
An interval I where f is monotone is said interval of monotonicity of f.

Proposition If f is strictly monotone on its domain, then f is one to one. Proof: to fix ideas, let us suppose f is strictly increasing. Given $x_1, x_2 \in \text{dom} f$ with $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_2 < x_1$. In former case we obtain $f(x_1) < f(x_2)$, hence $f(x_1) \neq f(x_2)$. In the latter case the same conclusion holds by swapping the roles of x_1 and x_2.

- Under the assumption of the above proposition, there exists the inverse function f^{-1} then; it's also strictly monotone and this could be checked in the same way as f

Example
$f: [0, +\infty) \rightarrow [0, +\infty), \ f(x) = x^2$ INVERSE $f^{-1}: [0, +\infty) \rightarrow [0, +\infty), \ f^{-1}(x) = \sqrt{x}$

The logic implication f is strictly monotone on its domain $\Rightarrow$ f is one to one cannot be reversed (a map f may be one-to-one without increasing strictly on its domain.
Example
$f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ is one to one, actually bijective on $\mathbb{R}$, but it's not strictly increasing, nor strictly decreasing

* The sum of functions that are similarly monotone is still a monotone function of the same kind, and turns out to be strictly monotone if one at least of the summands is.
Ex. $f(x) = x^5 + x$ is strictly increasing on $\mathbb{R}$, being the sum of two functions with the same property. It's then invertible.

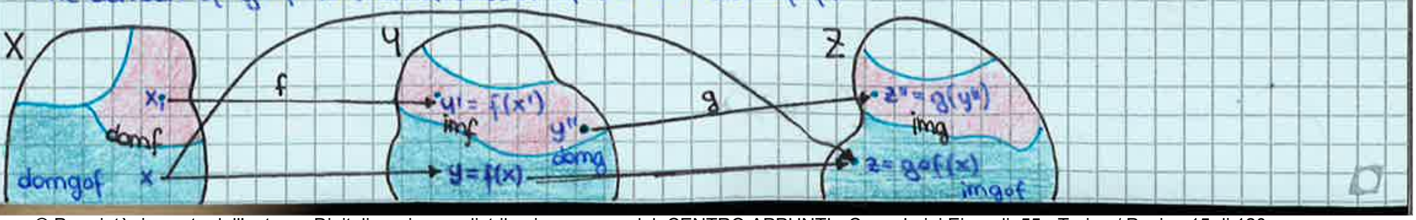
Composition of functions

Let X, Y, Z be sets. Suppose f is a function from X to Y, and g a function from Y to Z. We can manifacture a new function h from X to Z:
$h(x) = g(f(x))$ that is called composition of f and g (or composite map) and it's indicated by the symbol $h = g \circ f$ (g composed with f)

Example
$f(x) = x - 3, \ g(y) = y^2 + 1 \longrightarrow h(x) = g \circ f(x) = (x-3)^2 + 1$

* The domain of the composition $g \circ f$ is determined as follows: in order for x to belong to the domain of $g \circ f$, $f(x)$ must be defined, so x must be $\in$ of the dom$f$; moreover $f(x)$ has to be an element of the domain of g.

$x \in \text{dom}\, g \circ f \iff x \in \text{dom} f$ and $f(x) \in \text{dom}\, g$
The domain of $g \circ f$ is then a subset of the domain of f.



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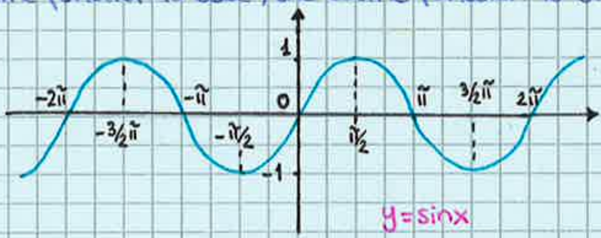
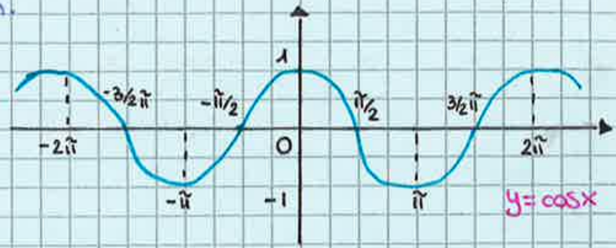
Hence the cosine function $y=\cos x$ and the sine function $y=\sin x$ are defined on $\mathbb{R}$ and assume all values of the interval $[-1,1]$, they are periodic of minimum period 2π.
They satisfy the trigonometric relation $\cos^2 x+\sin^2 x=1$
* The sine function is odd, the cosine function is even.





■ Important values

$\sin x=0$ for $x=k\pi$ $\qquad \cos x=0$ for $x=\pi/2+k\pi$
$\sin x=1$ for $x=\pi/2+2k\pi$ $\qquad \cos x=1$ for $x=2k\pi$
$\sin x=-1$ for $x=-\pi/2+2k\pi$ $\qquad \cos x=-1$ for $x=\pi+2k\pi$

Concerning monotonicity, one has

$$y=\sin x \text{ is } \begin{cases} \text{strictly increasing on } \left[-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi\right] \\ \text{strictly decreasing on } \left[\frac{\pi}{2}+2k\pi, \frac{3}{2}\pi+2k\pi\right] \end{cases}$$

$$y=\cos x \text{ is } \begin{cases} \text{strictly decreasing on } [2k\pi, \pi+2k\pi] \\ \text{strictly increasing on } [\pi+2k\pi, 2\pi+2k\pi] \end{cases}$$

* Addition and subtraction formulas

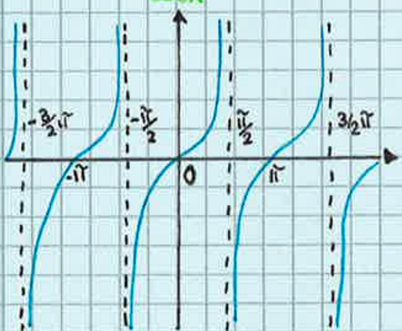
$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$
$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

* Duplication formulas

$\sin 2x=2\sin x\cos x$
$\cos 2x=2\cos^2 x-1$

The tangent function $y=\tan x$ ($y=\mathrm{tg}\,x$) and the cotangent function $y=\cot x$ ($y=\mathrm{cotg}\,x$) are defined by

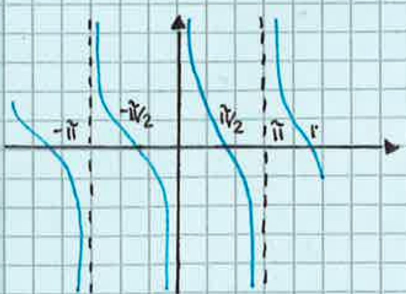
$\tan x=\dfrac{\sin x}{\cos x}$ $\qquad \cot x=\dfrac{\cos x}{\sin x}$ which are periodic of minimum period π and not 2π.



$y=\mathrm{tg}\,x$ defined on $\mathbb{R}\setminus\{\pi/2+k\pi : k\in\mathbb{Z}\}$
strictly increasing on the intervals $(-\pi/2+k\pi, \pi/2+k\pi)$ where it assume every real number as value.

$\tan x$ = y-coordinates of the intersection point $Q(x)$ between the ray from the origin through $P(x)$ and the vertical line containing A.

BOTH ODD FUNCTIONS



$y=\mathrm{cotg}\,x$ defined on $\mathbb{R}\setminus\{k\pi : k\in\mathbb{Z}\}$
strictly decreasing on the intervals $(k\pi, \pi+k\pi)$
on which it assumes every real value

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Limits and continuity I

Neighbourhoods

The process of defining limits and continuity leads to consider real numbers which are close to a certain real number / points on the real line in the proximity of a given point.
Let $x_0 \in \mathbb{R}$ be a point on the real line, and $r > 0$ a real number. We call **neighbourhood** of x_0 of **radius** r the open and bounded interval:

$I_r(x_0) = (x_0 - r, x_0 + r) = \{x \in \mathbb{R} : |x - x_0| < r\}$ $\quad x_0 \in \mathbb{R} \quad r > 0, r \in \mathbb{R}$

By understanding the quantity $|x - x_0|$ as the Euclidean **distance** between the points x_0 and x, we can say that $I_r(x_0)$ consists of the points on the real line whose distance from x_0 is less than r. If we interpret $|x - x_0|$ as the **tolerance** in the approximation of x_0 by x, then $I_r(x_0)$ becomes the set of real numbers approximating x_0 with a better margin of precision than r.

* $I : (a, b)$ $\quad x_0 = \frac{a+b}{2} \longrightarrow I_{\frac{b-a}{2}} = \frac{a+b}{2}$

Varying r in the set of positive real numbers, while maintaining x_0 in $\mathbb{R}$ fixed, we obtain a **family of neighbourhoods** of x_0. Each one is a proper subset of any other in the family that has bigger radius, and in turn it contains all neighbourhoods of lesser radius.
The notion of neighbourhood of a point $x_0 \in \mathbb{R}$ is a particular case of the analogue for a point in the Cartesian product $\mathbb{R}^d$ ($d = 2$ plane, $d = 3$ space)
The upcoming definition of limit and continuity can be stated directly for functions on $\mathbb{R}^d$, by considering functions of one real variable as subcases for $d = 1$

* It's also convenient to include the case where x_0 is one of the points $+\infty$ or $-\infty$.
For any real $a \geq 0$ we call **neighbourhood** of $+\infty$ with **end-point** a the open, unbounded interval
$I_a(+\infty) = (a, +\infty)$
Similarly, a **neighbourhood** of $-\infty$ with **end-point** $-a$ will be defined as
$I_a(-\infty) = (-\infty, -a)$

We shall say that the property $P(x)$ holds in a neighbourhood of a point c ($c = x_0 \in \mathbb{R}$ or $\pm\infty$), if there is a certain neighbourhood of c s.t. for each of its points x, $P(x)$ holds ($P(x)$ holds around c)

$P(x)$ holds in a neigh. of c ($c \in \mathbb{R}$ or $c = \pm\infty$) if $\exists a > 0$ s.t. $P(x)$ is true $\forall x \in I_a(c)$

Example
$f(x) = 2x - 1$ is positive in a neighbourhood of $x_0 = 1 \longrightarrow f(x) > 0 \quad \forall x \in I_{1/2}(1)$

Limit of a sequence

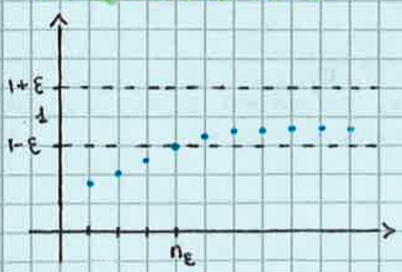
Consider a real sequence $a : n \mapsto a_n$: we are interested in studying the behaviour of the values a_n as n increases.
Examples

table 3.1 p.67

- $a_n = \frac{n}{n+1}$, the values approach 1 as n increases ($1 \in \mathbb{R}$ can be approximated as well as we like by a_n for n sufficiently large: however small we fix $\varepsilon > 0$, from a certain point n_ε onwards all values a_n approximate 1 with a margin smaller than ε.
The condition $|a_n - 1| < \varepsilon$ is equal to $\frac{1}{n+1} < \varepsilon \longrightarrow n+1 > \frac{1}{\varepsilon}$; thus defining $n_\varepsilon = \frac{1}{\varepsilon}$ and taking any natural number $n > n_\varepsilon$, we have:

$n+1 > \frac{1}{\varepsilon} + 1 > \frac{1}{\varepsilon}$, hence $|a_n - 1| < \varepsilon$. In other words, for every $\varepsilon > 0$, there exists an n_ε s.t.

$n > n_\varepsilon \Rightarrow |a_n - 1| < \varepsilon$



- Converge of the sequence $a_n = \frac{n}{n+1}$

$\forall n > n_\varepsilon$ the points (n, a_n) of the graph lie between the horizontal lines $y = 1 - \varepsilon$ and $y = 1 + \varepsilon$.

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- **The number e**

Sequence $a_n = \left(1+\frac{1}{n}\right)^n$; it's possible to prove that it's a strictly increasing sequence ($a_n > 2 = a_1 \ \forall n > 1$) and that it's bounded from above ($a_n < 3 \ \forall n$). The sequence (according to the previous theorem) converges to a limit between 2 and 3, traditionally indicated by the symbol e: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

e = Napier's number or Euler's number, is very important and it's an irrational number ($e = 2.718281828459045\ldots$), one of the most popular bases for exponentials and logarithms

$y = e^x \Rightarrow y = \exp x$; $y = \ln x \Rightarrow$ natural logarithm

Limits of functions; continuity

We want to study the behaviour of the [redacted] function $y = f(x)$ when the variable x approaches to a point $x_0 \in \mathbb{R}$ or one of the points at $\pm\infty$.

Suppose f is defined around $+\infty$, we say that:

* the function f tends to the limit $\ell \in \mathbb{R}$ for x going to $+\infty$ (i.e. $\lim_{x\to+\infty} f(x) = \ell$) if for any real number $\varepsilon > 0$ there is a real $B \geq 0$ s.t. $\forall x \in \operatorname{dom} f,\ x > B \Rightarrow |f(x) - \ell| < \varepsilon$.
This condition requires that:
$\forall I_\varepsilon(\ell), \exists I_B(+\infty): \forall x \in \operatorname{dom} f, x \in I_B(+\infty) \Rightarrow f(x) \in I_\varepsilon(\ell)$

* the function f tends to $+\infty$ for x going to $+\infty$ (i.e. $\lim_{x\to+\infty} f(x) = +\infty$) if for each real $A > 0$ there is a real $B \geq 0$ such that $\forall x \in \operatorname{dom} f,\ x > B \Rightarrow f(x) > A$.

* For functions tending to $-\infty$ one should replace $f(x) > A$ by $f(x) < -A$

$\left[\lim_{x\to+\infty} f(x) = \infty \quad \text{means} \quad \lim_{x\to+\infty} |f(x)| = +\infty\right]$

* Now suppose f is defined around $-\infty$. The previous definitions should be modified to become definitions of limit (L, finite or infinite) for x going to $-\infty$, by changing $x > B$ to $x < -B$, so we have:
$\lim_{x\to-\infty} f(x) = L$

$\left[\lim_{x\to\infty} f(x) = L \quad \text{means that f has limit } L \text{ both for } x \to \pm\infty\right]$

f is defined in a neighbourhood of x_0, but not necessarily at the point x_0 itself.

- Let x_0 be a point in the domain of a function f. This function is called continuous at x_0 if for any $\varepsilon > 0$ there is a $\delta > 0$ s.t. $\forall x \in \operatorname{dom} f,\ x \in I_\delta(x_0) \Rightarrow |f(x) - f(x_0)| < \varepsilon$
In neighbourhood-talk: for any neighbourhood $I_\varepsilon(f(x_0))$ of $f(x_0)$ there exists a neighbourhood $I_\delta(x_0)$ of x_0 s.t. $\forall x \in \operatorname{dom} f,\ x \in I_\delta(x_0) \Rightarrow f(x) \in I_\varepsilon(f(x_0))$

- Let f be a function defined on a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. Then f has limit $\ell \in \mathbb{R}$ (or tends to ℓ or converges to ℓ) for x approaching x_0 (i.e. $\lim_{x\to x_0} f(x) = \ell$) if for any $\varepsilon > 0$ there exists a $\delta > 0$ s.t. $\forall x \in \operatorname{dom} f,\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$
Alternatively: for any given neighbourhood $I_\varepsilon(\ell)$ there is a neighbourhood $I_\delta(x_0)$ s.t. $\forall x \in \operatorname{dom} f,\ x \in I_\delta(x_0) \setminus \{x_0\} \Rightarrow f(x) \in I_\varepsilon(\ell)$.
Examples p. 74-75-76

To have continuity one looks at the values f(x) from the point of view of $f(x_0)$, whereas for limits these f(x) are compared to ℓ, which could be different from $f(x_0)$, provided f is defined in x_0. To test the limit, moreover, the comparison with $x = x_0$ is excluded ($0 < |x - x_0|$ means $x \neq x_0$)
Let f be defined in a neighbourhood of x_0. If f is continuous at x_0, then we have $\ell = f(x_0)$, vice versa if f has limit $\ell = f(x_0)$ for $x \to x_0$ the first definition holds. Thus the continuity of f at x_0 is tantam. to $\lim_{x\to x_0} f(x) = f(x_0)$
In both definitions, one is asked to find at least one positive number δ for which one of them holds. It's not necessary to find the biggest possible δ satisfying the implication, so it will also hold $\forall \delta' < \delta$

- Let f be defined on a neighbourhood of x_0, excluding the point x_0. If f admits limit $\ell \in \mathbb{R}$ for $x \to x_0$ and if: a) f is defined in x_0 but $f(x_0) \neq \ell$ or b) f is not defined in x_0, then we say x_0 is a (point of) removable discontinuity for f. (changing f at x_0, in defining in x_0 – continuous map)
More precisely the function

$\tilde f(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ \ell & \text{if } x = x_0 \end{cases}$ is s.t. $\lim_{x\to x_0} \tilde f(x) = \lim_{x\to x_0} f(x) = \ell = \tilde f(x_0)$ hence it is continuous at x_0.

We have $\tilde g(x) = x$ in a neighbourhood of the origin, while $\tilde h(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ continuous prolongation of $y = \frac{\sin x}{x}$ defined by assigning the value that renders it continuous at the origin.

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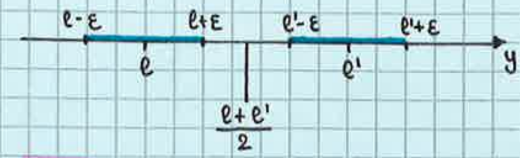
Limits and continuity II

Theorems on limits

- **Uniqueness of the limit (Thm)**
  Suppose f admits (finite or infinite) limit ℓ for $x \to c$. Then f admits no other limit for $x \to c$.



The neighbourhoods of ℓ, ℓ' of radius $\varepsilon \le \frac{1}{2}|\ell-\ell'|$ are disjoint. (i.e. $I_\varepsilon(\ell) \cap I_\varepsilon(\ell') = \emptyset$)

- **PROOF** Suppose that $\exists$ two limits $\ell' \neq \ell$ (proof by contradiction). We consider only the case where ℓ and ℓ' are both finite, the other situation can be easily deduced. Since $\ell' \neq \ell$ we have $I(\ell) \cap I(\ell') = \emptyset$ (considering neighbourhoods of radius ε smaller or equal than half the distance of ℓ and ℓ' $\to$ $\varepsilon \le \frac{1}{2}|\ell - \ell_1|$)
  Taking $I(\ell)$, the hypothesis $\lim_{x\to c} f(x) = \ell$ implies that $\exists\, I(c)$ s.t. $\forall x \in \text{dom} f$, $x \in I(c) \setminus \{c\} \Rightarrow f(x) \in I(\ell)$

  Similarly for $I(\ell')$, $\lim_{x\to c} f(x) = \ell' \Rightarrow \exists\, I'(c)$ s.t. $\forall x \in \text{dom} f$, $x \in I'(c)\setminus\{c\} \Rightarrow f(x) \in I'(\ell')$

  $I(c) \cap I'(c)$ is itself a neighbourhood of c, which contains infinitely many points of dom f since we assume f defined in $I(c)$. Therefore $\bar{x} \in \text{dom} f \to f(\bar{x}) \in I(\ell) \cap I(\ell')$ hence the intervals $I(\ell)$ and $I(\ell')$ do have non-empty intersection.

* The second property concerns the sign of a limit around a point c.

- **Theorem (sign of the limit)**
  Suppose f admits limit ℓ (finite or infinite) for $x \to c$. If $\ell > 0$ or $\ell = +\infty$, $\exists$ a neighbourhood $I(c)$ s.t. f is strictly positive on $I(c)\setminus\{c\}$. A similar assertion holds when $\ell < 0$ or $\ell = -\infty$

- **PROOF** Assume ℓ as finite, positive, and consider the neighbourhood $I_\varepsilon(\ell)$ of radius $\varepsilon = \ell/2 > 0$. According to the definition there is a $I(c)$ s.t. $\forall x \in \text{dom} f$, $x \in I(c)\setminus\{c\} \Rightarrow f(x) \in I_\varepsilon(\ell)$
  As $I_\varepsilon(\ell) = \left(\frac{\ell}{2}, \frac{3\ell}{2}\right) \subset (0, +\infty)$, all values $f(x)$ are positive.
  If $\ell = +\infty$ it suffices to take an $I_A(+\infty)$ $(A>0)$ and use the corr. def. of limit.

- **Corollary**
  Assume f admits limit ℓ (finite or infinite) for $x \to c$. If there is a neighbourhood $I(c)$ s.t. $f(x) \ge 0$ in $I(c)\setminus\{c\}$ then $\ell \ge 0$ or $\ell = +\infty$. A similar assertion holds for a negative limit

  PROOF By contradiction, if $\ell = -\infty$ or $\ell < 0$, the second property would provide a neighbourhood $I'(c)$ s.t. $f(x) < 0$ on $I'(c)\setminus\{c\}$, On the intersection of $I(c)$ and $I'(c)$ we would have both $f(x) < 0$ and $f(x) \ge 0$ that is impossible.

Example Even assuming the stronger inequality $f(x) > 0$ on $I(c)$, we would not be able to exclude ℓ might be zero. For example $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is strictly positive in every neighbourhood of the origin, yet $\lim_{x\to 0} f(x) = 0$

- **Corollary (First comparison theorem)**
  Let a function f have limit ℓ and a function g limit m (ℓ, m finite or not) for $x \to c$. If there is a neighbourhood $I(c)$ s.t. $f(x) \le g(x)$ in $I(c)\setminus\{c\}$, then $\ell \le m$.

  PROOF If $\ell = -\infty$ or $m = +\infty$ there's nothing to prove. Otherwise, consider $h(x) = g(x) - f(x)$. Assuming $h(x) \ge 0$ on $I(c)\setminus\{c\}$. Then we have $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) - \lim_{x\to c} f(x) = m - \ell$

  The previous corollary applied to h forces $m - \ell \ge 0$, hence the claim.

- **Second comparison theorem - finite case or Squeeze rule**
  Let functions f, g and h be given and assume f, h have the same finite limit for $x \to c$, precisely $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = \ell$. If there is a neighbourhood $I(c)$ where the three functions are defined (except at c) and s.t. $f(x) \le g(x) \le h(x)$, $\forall x \in I(c)\setminus\{c\}$ then $\lim_{x\to c} g(x) = \ell$

- **PROOF** We follow the definition of limit for g. Fix a neighbourhood $I_\varepsilon(\ell)$; by the hypothesis $\lim_{x\to c} f(x) = \ell$ we deduce the existence of a neighbourhood $I'(c)$ s.t.
  $\forall x \in \text{dom} f$, $x \in I'(c)\setminus\{c\} \Rightarrow f(x) \in I_\varepsilon(\ell)$
  The condition $f(x) \in I_\varepsilon(\ell)$ can be written as $|f(x) - \ell| < \varepsilon$ or $\ell - \varepsilon < f(x) < \ell + \varepsilon$. Similarly $\lim_{x\to c} h(x) = \ell$ implies $\exists\, I''(c)$ s.t. $\forall x \in \text{dom} h$, $x \in I''(c)\setminus\{c\} \Rightarrow \ell - \varepsilon < h(x) < \ell + \varepsilon$
  Define then $I'''(c) = I(c) \cap I'(c) \cap I''(c)$. On $I'''(c)\setminus\{c\}$ the constraints (*) all hold, hence we have

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Example $\lim_{x\to+\infty} x+\sin x=+\infty$ * we have $x-1\le x+\sin x\ \ \forall x\in\mathbb{R}$ $\quad f(x)=x-1 \to \lim_{x\to+\infty} f(x)=+\infty$ so *

$x\to\infty$ — it's bounded in the same neighbourhood, so the value of the limit is completely determined by the other therm (x)

Algebra of limits.

First though, we must extend arithmetic operations to treat the symbols $+\infty$ and $-\infty$. Let us set:

- $+\infty+s=+\infty$ (if $s\in\mathbb{R}$ or $s=+\infty$) — * All these forms are defined
- $-\infty+s=-\infty$ (if $s\in\mathbb{R}$ or $s=-\infty$)
- $\pm\infty\cdot s=\pm\infty$ (if $s>0$ or $s=+\infty$)
- $\pm\infty\cdot s=\mp\infty$ (if $s<0$ or $s=-\infty$)
- $\frac{\pm\infty}{s}=\pm\infty$ (if $s>0$)
- $\frac{\pm\infty}{s}=\mp\infty$ (if $s<0$)
- $\frac{s}{0}=\infty$ (if $s\in\mathbb{R}\setminus\{0\}$ or $s=\pm\infty$) $=s\cdot\infty$ $\quad \frac{1}{0}=\infty \Rightarrow \frac{s}{0}=s\cdot\infty$ (s cannot be 0)
- $\frac{s}{\pm\infty}=0$ (if $s\in\mathbb{R}$)

Instead, the following expressions are not defined

- $\pm\infty+(\mp\infty)$ • $\pm\infty-(\pm\infty)$ • $\pm\infty\cdot 0$ • $\frac{\pm\infty}{\pm\infty}$ • $\frac{0}{0}$

• **Theorem**

Suppose f admits limit ℓ (finite or infinite) and g admits limit m (finite or infinite) for $x\to c$. Then

- $\lim_{x\to c}(f(x)\pm g(x))=\ell\pm m$
- $\lim_{x\to c}(f(x)g(x))=\ell\cdot m$
- $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\ell}{m}$

Provided the right-hand-side expressions make sense.

(last case $g(x)\neq 0 \in I(c)\setminus\{c\}$)

• **PROOF** We shall prove two relations only. The first is:
$\lim_{x\to c}(f(x)+g(x))=\ell+m$ with ℓ,m finite. Let us fix $\varepsilon>0$ and consider the neighbourhood of ℓ of radius $\varepsilon/2$. By assumption there is a neighbourhood $I'(c)$ of c s.t. $\forall x\in \mathrm{dom} f$, $x\in I'(c)\setminus\{c\}$ $\Rightarrow |f(x)-\ell|<\varepsilon/2$
For the same reason there's also an $I''(c)$ with $\forall x\in\mathrm{dom} g$, $x\in I''(c)\setminus\{c\} \Rightarrow |g(x)-m|<\varepsilon/2$.
Put $I(c)=I'(c)\cap I''(c)$. Then if $x\in\mathrm{dom} f\cap\mathrm{dom} g$ belongs to $I(c)\setminus\{c\}$ $(\cap I(c)\cap I'(c)\setminus\{c\})$ both inequalities hold. The triangle inequality yields
$|f(x)+g(x)-(\ell+m)|=|(f(x)-\ell)+(g(x)-m)|\le|f(x)-\ell|+|g(x)-m|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$ proving the assertion.

The second relation is $\lim_{x\to c}(f(x)g(x))=+\infty$ with $\ell=+\infty$ and $m>0$ finite. For a given real $A>0$ consider the neighbourhood of $+\infty$ with end-point $B=2A/m>0$.
$\exists I'(c)$ s.t. $\forall x\in\mathrm{dom} f$, $x\in I'(c)\setminus\{c\} \Rightarrow f(x)>B=2A/m$
On the other hand, considering the neighbourhood of m of radius $m/2=\varepsilon$
$\exists I''(c)$ s.t. $\forall x\in\mathrm{dom} g$, $x\in I''(c)\setminus\{c\}\Rightarrow |g(x)-m|<m/2=\varepsilon$ $\ \forall\varepsilon>0$ i.e. $m/2<g(x)<3/2\,m$
Set $I(c)=I'(c)\cap I''(c)$. If $x\in\mathrm{dom} f\cap\mathrm{dom} g$ is in $I(c)\setminus\{c\}$ the previous relations will be fullfilled, whence:
$f(x)g(x)>f(x)\frac{m}{2}>B\frac{m}{2}=\frac{2A}{m}\cdot\frac{m}{2}=A$

• **Corollary** If f and g are continuous at a point $x_0\in\mathbb{R}$, then also $f(x)\pm g(x)$, $f(x)\cdot g(x)$ and $f(x)/g(x)$ (with $g(x_0)\neq 0$) are continuous at x_0.

PROOF The condition that f and g are continuous at x_0 is equivalent to $\lim_{x\to x_0}f(x)=f(x_0)$ and $\lim_{x\to x_0}g(x)=g(x_0)$. The previous theorem allows to conclude.

• **Corollary** Rational functions are continuous on their domain. In particular, polynomials are continuous on $\mathbb{R}$

PROOF The constants $y=a$ and the linear function $y=x$ are continuous on $\mathbb{R}$ (p.78). Consequently, maps like $y=ax^n$ with $n\in\mathbb{N}$ are continuous. But then so are polynomials $p(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_n$, being sums of the latter. Rational functions $f(x)=\frac{P(x)}{Q(x)}$ as quotiens of polynomials, inherit the property in their domain (denominator $\neq 0$)

Example

$\lim_{x\to 0}\frac{2x-3\cos x}{5+x\sin x}=\ell$ the substitution of 0 to x produces $\ell=-3/5$ (see others p.98)

$5+x\sin x \to \neq 0$

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- **Substitution theorem**

Suppose a map f admits limit $\lim_{x\to c} f(x) = \ell$ finite or not. Let g be defined on a neighbourhood of ℓ, $I(\ell) \in dom\, g$ (excluding possibly the point ℓ) and s.t.
- if $\ell \in \mathbb{R}$, g is continuous at ℓ
- if $\ell = +\infty$ or $\ell = -\infty \longrightarrow \exists \lim_{y\to\ell} g(y)$, finite or not.

Then the composition $g\circ f$ admits limit for $x\to c$ and $\lim_{x\to c} g(f(x)) = \lim_{y\to\ell} g(y)$

- PROOF see it on p. 102-103

* An alternative condition that yields the same conclusion is the following:
- if $\ell \in \mathbb{R}$ there is a neighbourhood $I(c)$ of c where $f(x) \neq \ell \quad \forall x \neq c$ and the limit $\lim_{y\to\ell} g(y)$ finite or infinite

The proof is analogous.
In case $\ell \in \mathbb{R}$ and g is continuous at ℓ (1° one) then $\lim_{y\to\ell} g(y) = g(\ell)$ so we have:
$\lim_{x\to c} g(f(x)) = g(\lim_{x\to c} f(x))$

- **Corollary** Let f be continuous at x_0 and define $y_0 = f(x_0)$. Let furthermore g be defined around y_0 and continuous at y_0. Then the composite function $g\circ f$ is continuous at x_0.

PROOF $\lim_{x\to x_0} (g\circ f)(x) = g(\lim_{x\to x_0} f(x)) = g(f(x_0)) = (g\circ f)(x_0)$ which is equivalent to the claim.

Examples

- $h(x) = \sin(x^2)$ is continuous on $\mathbb{R}$, being the composition of the continuous functions $f(x) = x^2$, $g(y) = \sin y$
- Let's determine $\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$. Set $f(x) = x^2$ and $g(y) = \begin{cases} \frac{\sin y}{y} & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$ and g is continuous at the origin. Thus
$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{y\to 0} \frac{\sin y}{y} = 1$

SEQUENCE special kind of function, also for it the substitution theorem holds. For a sequence $a: n \mapsto a_n$ the limit is $\lim_{n\to\infty} a^n = \ell$. Namely, under the same assumptions on g $\lim_{n\to\infty} g(a_n) = \lim_{y\to\ell} g(y)$

This result is often used to disprove the existence of a limit, in that it provides a Criterion of non-existence for limits: if two sequences $a: n\mapsto a_n$, $b: n\mapsto b_n$ have the same limit ℓ and $\left|\lim_{n\to\infty} g(a_n) \neq \lim_{n\to\infty} g(b_n)\right|$ then g does not admit limit when its argument tends to ℓ.

Example
We can prove with the aid of the criterion that $y = \sin x$ has no limit when $x \to +\infty$. Define the sequences $a_n = 2n\pi$, $b_n = \pi/2 + 2n\pi$, $n \in \mathbb{N}$ s.t.

$\lim_{n\to\infty} \sin a_n = \lim_{n\to\infty} 0 = 0$ and at the same time $\lim_{n\to\infty} \sin b_n = \lim_{n\to\infty} 1 = 1$

* Similarly, the function $y = \sin\frac{1}{x}$ has neither left nor right limit for $x\to 0$

- $a_n = \frac{1}{2\pi n} \to 0$
$b_n = \frac{1}{2\pi n + \pi/2} \to 0 \quad (b_n = \frac{1}{a_n})$
$\sin(\frac{1}{a_n}) = \sin(2\pi n) = 0$
$\sin(\frac{1}{b_n}) = \sin(2\pi n + \pi/2) = 1$

More fundamental limits

* The following limit holds $\boxed{\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e}$

By manipulating this formula we achieve a series of new fundamental limits.
* The substitution $y = \frac{x}{a}$ with $a \neq 0$ gives $\boxed{\lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\pm\infty}\left(1+\frac{1}{y}\right)^{ay} = \left[\lim_{y\to\pm\infty}\left(1+\frac{1}{y}\right)^y\right]^a = e^a}$
$a \in \mathbb{R}$

* In terms of the variable $y = \frac{1}{x}$ then $\boxed{\lim_{x\to 0}(1+x)^{1/x} = \lim_{y\to\pm\infty}\left(1+\frac{1}{y}\right)^y = e}$

* Continuity of logarithm with (4.11) $\boxed{\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \lim_{x\to 0}\log_a(1+x)^{1/x} = \log_a \lim_{x\to 0}(1+x)^{1/x} = \log_a e = \frac{1}{\log a}}$
for any $a > 0$.

* In particular, taking $a = e \longrightarrow \boxed{\lim_{x\to 0}\frac{\log(1+x)}{x} = 1}$

- Note $a^x - 1 = y$ equivalent to $x = \log_a(1+y)$ $x\to 0$, $y\to 0$ with this substitution...

$\boxed{\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{y}{\log_a(1+y)} = \left[\lim_{y\to 0}\frac{\log_a(1+y)}{y}\right]^{-1} = \log a}$ for any $a > 0$

* Taking $a = e$ produces $\boxed{\lim_{x\to 0}\frac{e^x-1}{x} = 1}$

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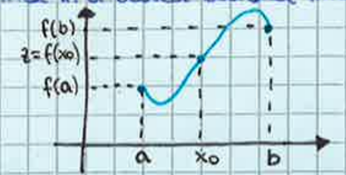
Thm - Intermediate value theorem

If a function f is continuous on the closed and bounded interval $[a,b]$, it assumes all values between $f(a)$ and $f(b)$. The image of a function defined in a closed bounded interval is a closed bounded interval.

$f \in C^0([a,b])$
$f([a,b]) = [f(a), f(b)]$



PROOF $f(a) \neq f(b) \Rightarrow f(a) < f(b)$ (?)
z = arbitrary value s.t. $f(a) < z < f(b)$ and define the map $g(x) = z$. So we have:
$f(a) < g(a)$ and $f(b) > g(b)$. Applying the previous corollary $\rightarrow \exists x_0 \in (a,b) \Rightarrow f(x_0) = g(x_0) = z$
* If $f(a) > f(b)$ we just swap the roles of f and g.

Corollary Let f be continuous on an interval I. The range $f(I)$ of I under f is an interval delimited by $\inf_I f$ and $\sup_I f$. i.e. $f \in C^0([a,b]) \rightarrow f(I) = [\inf_I f, \sup_I f]$

PROOF A subset of $\mathbb{R}$ (S) is an interval $\iff$ it contains the interval $[\alpha,\beta]$ as subset, for any $\alpha < \beta$.
Let then $y_1 < y_2$ be points of $f(I)$. There exist in I two (distinct) pre-images x_1, x_2 i.e. $f(x_1) = y_1$, $f(x_2) = y_2$. If $J \subseteq I$ is the closed interval between x_1, x_2 we need only to apply the Intermediate value theorem to f restricted to J, which yields $[y_1, y_2] \subseteq f(J) \subseteq f(I)$. The range $f(I)$ is then an interval and its end points are $\inf_I f$, $\sup_I f$.
• image of I • different from injectivity • $J = [x_1, x_2]$

* I open / half-open: its image can be an interval of any kind
* I closed bounded interval, its image can only be like this!

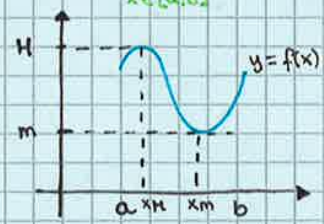
remember that they can also be not members of the interval, they only represent the wall, or they can be infinite. If $\inf_I f \in$ range, f admits a minimum on I (the same for $\sup_I f$).

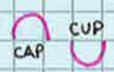
EXAMPLES
$f(x) = \tan x$ $\quad f((-\pi/2, \pi/2)) = (-\infty, +\infty)$ $\quad f([0, \pi/2)) = [0, +\infty)$ → min of f
$g(x) = \arctan x$ $\quad g(I) = (-\pi/2, \pi/2)$ $\quad I = (-\infty, +\infty)$
$h(x) = \sin x$ $\quad I = [0, 2\pi] \rightarrow h(I) = [-1, 1]$

Thm - Weierstrass

A continuous map f on a closed and bounded interval $[a,b]$ is bounded and admits minimum and maximum $m = \min_{x \in [a,b]} f(x)$ and $M = \max_{x \in [a,b]} f(x)$. Consequently $f([a,b]) = [m, M]$.

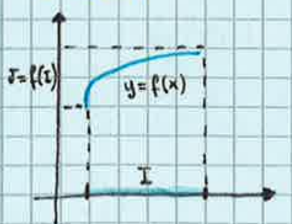
$f \in C^0([a,b])$ $a, b \in \mathbb{R}$
$\Rightarrow f$ is bounded.
It has max, min over $[a,b]$
$x \in [a,b]$
$\Downarrow$
$f([a,b]) = [m, M]$

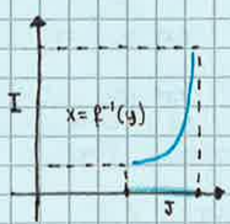




CONSEQUENCES

1. A continuous function f (→ $f \in C^0(I)$) on an interval I is one-to-one if and only if it is strictly monotone (it does not assume the same value twice)
2. Let f (→ $f \in C^0(I)$) be continuous and invertible (→ $\exists f^{-1}$ on I) on an interval I. Then the inverse f^{-1} is continuous on the interval $J = f(I)$.





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In fact we have:

$\lim_{x \to x_0} f(x) = f(x_0) \iff \lim_{x \to x_0} (f(x) - f(x_0)) = 0$

$\iff f(x) - f(x_0) = o(1), \quad x \to x_0$

• Algebra of "little o's"

1- Let us compare the behavior of the monomials x^n as $x \to 0$

$x^n = o(x^m), \quad x \to 0 \iff n > m$

In fact $\lim_{x \to 0} \frac{x^n}{x^m} = \lim_{x \to 0} x^{n-m} = 0 \iff n-m > 0$

Therefore when $x \to 0$, the bigger of two powers of x is negligible

2- Now consider the limit when $x \to \pm\infty$. Proceeding as before we obtain

$x^n = o(x^m), \quad x \to \pm\infty \iff n < m$

So for $x \to \pm\infty$, the lesser power of x is negligible

3- The symbols of Landau allow to simplify algebraic formulas quite a lot when studying limits. Consider for example the limit for $x \to 0$. The following properties hold.

- $o(x^n) \pm o(x^n) = o(x^n)$
- $o(x^n) \pm o(x^m) = o(x^p)$ with $p = \min(n, m)$
- $o(\lambda x^n) = o(x^n)$ for each $\lambda \in \mathbb{R} \setminus \{0\}$
- $\varphi(x) o(x^n) = o(x^n)$ if φ is bounded in a neighbourhood of $x = 0$
- $x^m o(x^n) = o(x^{m+n})$
- $o(x^m) o(x^n) = o(x^{m+n})$
- $[o(x^n)]^k = o(x^{kn})$

• Fundamental limits

The fundamental limits can be reformulated using the symbols of Landau

- $\sin x \sim x \quad x \to 0$
  $\sin x = x + o(x) \quad x \to 0$
- $1 - \cos x \asymp x^2 \quad x \to 0$, precisely, $1 - \cos x \sim \frac{1}{2} x^2 \quad x \to 0$
  $1 - \cos x = \frac{1}{2} x^2 + o(x^2) \quad x \to 0$ or $\cos x = 1 - \frac{1}{2} x^2 + o(x^2) \quad x \to 0$
- $\log(1+x) \sim x \quad x \to 0$, equivalently, $\log x \sim x - 1 \quad x \to 1$
  $\log(1+x) = x + o(x) \quad x \to 0$ or $\log x = x - 1 + o(x-1) \quad x \to 1$
- $e^x - 1 \sim x \quad x \to 0$
  $e^x = 1 + x + o(x) \quad x \to 0$
- $(1+x)^\alpha - 1 \sim \alpha x \quad x \to 0$
  $(1+x)^\alpha = 1 + \alpha x + o(x) \quad x \to 0$

* see Examples p. 127-28

Besides, we shall prove that

- $x^\alpha = o(e^x) \quad x \to +\infty \quad \forall \alpha \in \mathbb{R}$
- $e^x = o(|x|^\alpha) \quad x \to -\infty \quad \forall \alpha \in \mathbb{R}$
- $\log x = o(x^\alpha) \quad x \to +\infty \quad \forall \alpha > 0$
- $\log x = o\left(\frac{1}{x^\alpha}\right) \quad x \to 0^+ \quad \forall \alpha > 0$

• Let's explain now how to use the symbols of Landau for calculating limits. All maps dealt with below are supposed to be defined, and not to vanish, on a neighbourhood of c, except possibly at c.

*** Proposition**

Let us consider the limits $\lim_{x \to c} f(x) g(x)$ and $\lim_{x \to c} f(x)/g(x)$.

Given functions $\tilde{f}$ and $\tilde{g}$ s.t. $\tilde{f} \sim f$ and $\tilde{g} \sim g$ for $x \to c$, then $\lim_{x \to c} f(x) g(x) = \lim_{x \to c} \tilde{f}(x) \tilde{g}(x)$,
$\lim_{x \to c} f(x)/g(x) = \lim_{x \to c} \tilde{f}(x)/\tilde{g}(x)$

PROOF

$$\lim_{x \to c} f(x) g(x) = \lim_{x \to c} \frac{f(x)}{\tilde{f}(x)} \cdot \tilde{f}(x) \cdot \frac{g(x)}{\tilde{g}(x)} \cdot \tilde{g}(x) = \underbrace{\lim_{x \to c} \frac{f(x)}{\tilde{f}(x)}}_{=1} \cdot \underbrace{\lim_{x \to c} \frac{g(x)}{\tilde{g}(x)}}_{=1} \cdot \lim \tilde{f}(x) \tilde{g}(x) = \lim_{x \to c} \tilde{f}(x) \tilde{g}(x)$$

The proof of $f(x)/g(x)$ is compl. analogous.

* This is possible only with product and ratio!

Corollary

Consider the limits $\lim_{x \to c} (f(x) + f_1(x))(g(x) + g_1(x))$ and $\lim_{x \to c} \frac{f(x) + f_1(x)}{g(x) + g_1(x)}$.

If $f_1 = o(f)$ and $g_1 = o(g)$ when $x \to c$, then

$$\lim_{x \to c} (f(x) + f_1(x))(g(x) + g_1(x)) = \lim_{x \to c} f(x) g(x), \quad \lim_{x \to c} \frac{f(x) + f_1(x)}{g(x) + g_1(x)} = \lim \frac{f(x)}{g(x)}$$

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- $\forall \alpha > 0$, $\log x$ infinite of smaller order than $1/x^\alpha$, $x \to 0^+$
- p.131

* we shall express the fact that f is infinitesimal (or infinite) of bigger order than g by saying that f tends to 0 (or ∞) faster than g. This suggests to measure the speed at which an infinitesimal (or infinite) map converges to its limit value. For that purpose let's fix an infinitesimal (or infinite) map φ defined in a $I(c)$ and particularly easy to compute. We shall use it as term of comparison → **infinitesimal (infinite) test function** at c. When the limit behaviour is clear, we refer to φ as test function for brevity. The most common are the following. If $c = x_0 \in \mathbb{R}$, we choose $\varphi(x) = x - x_0$ or $\varphi(x) = |x - x_0|$ as inf. test function (the latter in case we need to consider non integer powers of φ) and $\varphi(x) = 1/x-x_0$ or $\varphi(x) = 1/|x-x_0|$ as infinite test functions. For $c = x_0^+$ ($c = x_0^-$) we will choose infinitesimal $\varphi(x) = x - x_0$ ($\varphi(x) = x_0 - x$) and infinite $\varphi(x) = 1/x-x_0$ ($\varphi(x) = 1/x_0 - x$). For $c = +\infty$ infinitesimal and infinite t.f. are $\varphi(x) = 1/x$ and $\varphi(x) = x$. For $c = -\infty$ we shall take $\varphi(x) = 1/|x|$ and $\varphi(x) = |x|$

* Speed of convergence - depends on how f compares to the powers of the infinitesimal or infinite test function, to be more precise we have the following definition:
Let f be infinitesimal (or infinite) at c. If there exist a real number $\alpha > 0$ s.t. $f \asymp \varphi^\alpha$, $x \to c$, the constant α is called the **order of f at c with respect to the infinitesimal (infinite) test function φ**.

Looking at the previous examples, consider (1): here it's immediate to see that for any $\beta < \alpha$ one has $f = o(\varphi^\beta)$ while $\beta > \alpha$ implies $\varphi^\beta = o(f)$ (similar for infinite maps)
If f has order α at c with respect to φ, there's a $\ell \neq 0 \in \mathbb{R}$ s.t. $\lim_{x \to c} \frac{f(x)}{\varphi^\alpha(x)} = \ell$ i.e.
$f \sim \ell\varphi^\alpha$, $x \to c$ ($f = \ell\varphi^\alpha + o(\ell\varphi^\alpha)$)

- For semplicity we can omit the constant ℓ in the symbol o 'cause $o(\ell\varphi^\alpha) = o(\varphi^\alpha)$

The function $p(x) = \ell\varphi^\alpha(x)$ is called the **principal part of the infinitesimal (infinite) map f at c with respect to the infinitesimal (infinite) test function φ.**

- From the qualitative point of view the behaviour of the function f in a small enough neighbourhood of c coincides with the behaviour of its principal part.

Examples

- $f(x) = \sin x - \tan x$, infinitesimal for $x \to 0$. We can write $\sin x - \operatorname{tg} x = \frac{\sin x(\cos x - 1)}{\cos x} \sim \frac{x(-1/2 x^2)}{1} = -\frac{1}{2}x^3$, $x \to 0$
So $f(x)$ is infinitesimal of order 3 at the origin with respect to $\varphi(x) = x$, principal part $-1/2\, x^3 = p(x)$.

- $f(x) = \sqrt{x+3} - \sqrt{x^2-1}$, infinitesimal for $x \to +\infty$. Rationalising → $f(x) = \frac{4}{x\left(\sqrt{1+3/x^2} + \sqrt{1+1/x^2}\right)}$. $\varphi(x) = 1/x$ then $\lim_{x \to +\infty} f(x)/\varphi(x) = 2$ f = 1° order $p(x) = 2/x$

- $f(x) = \sqrt{9x^5 + 7x^3 - 1}$ infinite when $x \to +\infty$ $\varphi(x) = x$ $\lim_{x \to +\infty} f(x)/x^\alpha = \lim_{x \to +\infty} \frac{x^{5/2}\sqrt{9 + 7/x^2 - 1/x^5}}{x^\alpha}$
$\alpha = 5/2$ $\lim = 3$ f order $5/2$ $p(x) = 3x^{5/2}$

* It may well happen that there's no real number $\alpha > 0$ satisfying $f \asymp \varphi^\alpha$ for $x \to c$. In such a case it is convenient to make a different choice of test function, one more suitable to describe the behaviour of f around c.

Examples

- $f(x) = e^{2x}$, $x \to +\infty$ $x^\alpha = o(e^{2x})$ $\alpha > 0$ (not possible with $\varphi(x) = x$) the exponential map grows too quickly for any polynomial function to keep up with it. If we take $\varphi(x) = e^x$ → order 2

- $f(x) = x\log x$ $x \to 0^+$ ⟶ $f(x) = \frac{\log x}{1/x}$ infinitesimal, $x \to 0^+$ $\varphi(x) = x$ we have
$\lim_{x \to 0^+} \frac{x \log x}{x^\alpha} = \frac{\log x}{x^{\alpha-1}} = \begin{cases} 0 & \alpha < 1 \\ -\infty & \text{otherwise} \end{cases}$ not possible: $|f(x)| = x|\log x|$ goes to zero more slowly than x, yet faster than x^α $\forall \alpha < 1$. Thus it can be used as alternative infinitesimal test map when $x \to 0^+$

Asymptotes

Consider a function f defined in a neighbourhood of $+\infty$ and wish to study the behaviour for $x \to +\infty$.
Remarkable case: f - polynomial of 1st degree. Geometrically this corresponds to the fact that the graph of f will look like a straight line. Precisely, suppose $\exists\, m, q \in \mathbb{R}$ s.t.
$\lim_{x \to +\infty} (f(x) - (mx+q)) = 0$ or, using Landau symbols ⟶ $f(x) = mx + q + o(1)$ $x \to +\infty$.

- We say that the line $g(x) = mx + q$ is a **right asymptote** of the function f. It is called **oblique** if $m \neq 0$, **horizontal** if $m = 0$. In geometrical terms condition tells that the vertical distance $d(x) = |f(x) - g(x)|$ between the graph of f and the asymptote tends to 0 as $x \to +\infty$. The asymptote coefficients can be recovered using limits $m = \lim_{x \to +\infty} \frac{f(x)}{x}$, $q = \lim_{x \to +\infty} (f(x) - mx)$

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Differential calculus

The derivative

Let $f: dom f \subseteq \mathbb{R} \to \mathbb{R}$ be a real function of one real variable, take $x_0 \in dom f$ and suppose f is defined in a neighbourhood $I_r(x_0)$ of x_0. With $x \in I_r(x_0)$, $x \neq x_0$ fixed, denote by $\Delta x = x - x_0$ the (positive or negative) increment of the dependent variable. Note that $x = x_0 + \Delta x$, $f(x) = f(x_0) + \Delta f$

The ratio $\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ is called difference quotient of f between x_0 and x.

In this manner Δf represents the absolute increment of the dependent variable f when passing from x_0 to $x_0 + \Delta x$, whereas the difference quotient detects the rate of increment ($\Delta f / f$ relative increment). Multiplying the difference quotient by 100 we obtain the so-called percentage increment.

* Graphically, the difference quotient between x_0 and a point x_1 around x_0 is the slope of the straight line S passing through $P_0 = (x_0, f(x_0))$ and $P_1 = (x_1, f(x_1)) \in$ graph f; this line is called secant of the graph of f at P_0 and P_1. Putting $\Delta x = x_1 - x_0$ and $\Delta f = f(x_1) - f(x_0)$ the equation of the secant line reads $y = S(x) = f(x_0) + \frac{\Delta f}{\Delta x}(x - x_0) \quad x \in \mathbb{R}$

* Application in physics: the difference quotient $\Delta s / \Delta t$ represents the average velocity of the particle in the given interval of time *

A map f defined on a neighbourhood of $x_0 \in \mathbb{R}$ is called differentiable at x_0 if the limit of the difference quotient $\Delta f / \Delta x$ between x_0 and x exists and is finite, as x approaches x_0. The real number

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$ is called (first) derivative of f at x_0.

- It is also denoted by $y'(x_0)$ — $\frac{df}{dx}(x_0)$ — $Df(x_0)$

* From the geometric point of view $f'(x_0)$ is the slope of the tangent line at $P_0 = (x_0, f(x_0))$ to the graph of f: such line t is obtained as the limiting position of the secant S at P_0 and $P = (x, f(x))$ when P approaches P_0

$$y = t(x) = f(x_0) + f'(x_0)(x - x_0)$$

Let $dom f' = \{x \in dom f : f \text{ is differentiable at } x\}$ and define the function $f' : dom f' \subseteq \mathbb{R} \to \mathbb{R}$, $f' : x \mapsto f'(x)$ mapping $x \in dom f'$ to the value of the derivative of f at x. This map is called (first) derivative of f.

continuity and differentiability

* Let I be a subset of $dom f$. We say that I is differentiable on I (or in I) if f is diff. at each point of I.

Proposition If f is differentiable at x_0 it is also continuous at x_0

PROOF Continuity at x_0 prescribes $\lim_{x \to x_0} f(x) = f(x_0)$ that is $\lim_{x \to x_0} (f(x) - f(x_0)) = 0$*

If f is differentiable at x_0, then $\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0}(x - x_0)$

$= f'(x_0) \cdot 0 = 0$*

Not all continuous maps at a point are differentiable:

EXAMPLE $f(x) = |x|$, continuous at the origin, yet the difference quotient between the origin and a point $x \neq 0$ is $\frac{\Delta f}{\Delta x} = \frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$ so the limit for $x \to 0$ $\nexists$ i.e. f is not differentiable at $x = 0$

Derivatives of elementary functions. Rules of differentiation

1- Consider the affine map $f(x) = ax + b$, $x_0 \in \mathbb{R}$ arbitrary. Then $f'(x_0) = \lim_{\Delta x \to 0} \frac{(a(x_0 + \Delta x) + b) - (ax_0 + b)}{\Delta x} = \lim_{\Delta x \to 0} a = a$ in agreement with the fact that the graph of $f(x)$ is a straight line of slope a. So the derivative of $f(x)$ is the constant map $f'(x) = a$. If f is constant ($a = 0$), its derivative is identically zero.

2- Take $f(x) = x^2$, $x_0 \in \mathbb{R}$. Since $f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} = \lim_{\Delta x \to 0} (2x_0 + \Delta x) = 2x_0$ the derivative of $f(x)$ is the function $f'(x) = 2x$.

3- Let $f(x) = x^n$ with $n \in \mathbb{N}$. The binomial formula yields $f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^n - x_0^n}{\Delta x} =$

$$= \lim_{\Delta x \to 0} \frac{x_0^n + n x_0^{n-1} \Delta x + \sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} (\Delta x)^k - x_0^n}{\Delta x} = \lim_{\Delta x \to 0} \left(n x_0^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} (\Delta x)^{k-1} \right) = n x_0^{n-1}$$ for all $x_0 \in \mathbb{R}$

So the derivative of $f(x)$ is $f'(x) = nx^{n-1}$

4- More generally consider $f(x) = x^a$, $a \in \mathbb{R}$, $x_0 \neq 0 \in dom f$. Then $f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^a - x_0^a}{\Delta x} =$

$= \lim_{\Delta x \to 0} \frac{x_0^a \left[\left(1 + \frac{\Delta x}{x_0}\right)^a - 1 \right]}{\Delta x} = x_0^{a-1} \lim_{\Delta x \to 0} \frac{(1 + \Delta x / x_0)^a - 1}{\Delta x / x_0}$ Substituting $y = \Delta x / x_0$ we obtain a fundamental limit,

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Where differentiability fails

We recall the sign function, for which $D|x| = sign(x)$ $\forall x \neq 0$
The origin is an **isolated point of non-differentiability** for $y = |x|$. For difference quotient at the origin we have that one-sided limits exist and are finite. This fact suggest the introduction of the following notion.

* Suppose f is defined on a right neighbourhood of $x_0 \in \mathbb{R}$. It is called **differentiable on the right** at x_0 if the right limit of the difference quotient $\Delta f / \Delta x$ between x_0 and x exists finite, for $x \to x_0$. The real number $f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ is the **right (or backward) derivative** of f at x_0. Similarly it goes for the **left (or forward) derivative** $f'_-(x_0)$

If f is defined only on a right (or left) neighbourhood of x_0 and is differentiable on the right (left) at x_0, we shall simply say that f is differentiable at x_0, and write $f'(x_0) = f'_+(x_0)$ $(f'_-(x_0))$

Property A map f defined around a point $x_0 \in \mathbb{R}$ is differentiable at x_0 if and only if it is differentiable on both sides at x_0 and the left and right derivatives coincide, in which case
$f'(x_0) = f'_+(x_0) = f'_-(x_0)$

* Instead, if f is differentiable at x_0 on the left and on the right but the two derivatives are different, x_0 is called **corner point** for f. The right derivative represents the slope of the **right tangent** to the graph of f at P_0 (limiting position of the secant). In case the + and - deriv. do not coincide, they form an angle at P_0.
Other interesting cases are when the two limits of difference quotient exist but one at least is not finite: precisely, if just one is infinite, we say that x_0 is a **corner point** for f.
If both limits are infinite and with the same sign, x_0 is a **point with vertical tangent** for f.
If they are infinite and have different signs, x_0 is called a **cusp (point)** of f.

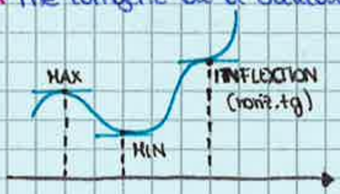
* **Theorem**
Let f be continuous at x_0 and differentiable at all points $x \neq x_0$ in a neighbourhood of x_0. Then f is differentiable at x_0 provided that the limit of $f'(x)$ for $x \to x_0$ exists finite. If so $f'(x_0) = \lim_{x \to x_0} f'(x)$
see example p. 177-78 → see proof p. 200 *
* Do not forget to impose continuity at x_0. **PROOF** after de l'Hôpital

Extrema and critical points

One calls $x_0 \in dom f$ a **relative (or local) maximum point** for f if there is a neighbourhood $I_r(x_0)$ of x_0 s.t. $\forall x \in I_r(x_0) \cap dom f$, $f(x) \leq f(x_0)$
Then $f(x_0)$ is a **relative (or local) maximum of f**. One calls x_0 an **absolute maximum point (or global max point)** for f if $\forall x \in dom f$, $f(x) \leq f(x_0)$ and $f(x_0)$ becomes the **(absolute) maximum** of f. In either case, the maximum is said **strict** if $f(x) < f(x_0)$ when $x \neq x_0$.

* Exchanging the symbols $\leq$ with $\geq$ one obtains the definitions of **relative and absolute minimum point**. A minimum or maximum point shall be referred to generically as an **extremum(point)** of f
Example
$y = \arcsin x$ $dom f: [-1,1]$
$x_0 = 1$ strict absolute maximum point $f(1) = \pi/2 = \max f > f(x)$ $\forall x \in dom f - \{1\}$
$x_1 = -1$ strict absolute minimum point $f(-1) = -\pi/2$
The function is not differentiable in $-1, 1$ cause it goes to ∞ $\left(\frac{1}{\sqrt{1-x^2}}\right)$, so there's no derivative in these points.

It might be useful to look for the points where the first derivative vanishes. A **critical point (or stationary point)** of f is a point x_0 at which f is differentiable with derivative $f'(x_0) = 0$
* The tangent at a critical point is horizontal.



* **Theorem - Fermat**
Suppose f is defined in a full neighbourhood of a point x_0 and differentiable at x_0. If x_0 is an extremum point, then it is critical for f, i.e. $f'(x_0) = 0$

$f(x): A \to \mathbb{R}$
1) continuous
2) differentiable at x_0
3) x_0 = extremum point $\in A$ (relative min or max)

$\Rightarrow y'(x_0) = 0$ (x_0 critical for f)

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PROOF Introduce an auxiliary map $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$ defined on $[a,b]$. It is continuous on $[a,b]$ and differentiable on (a,b), as difference of f and an affine map, which is differentiable on all of $\mathbb{R}$. Note $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$; it is easily seen that $g(a) = f(a)$, $g(b) = f(a)$ so Rolle's Theorem applies to g, with the consequence that there is a point $x_0 \in (a,b)$ satisfying $g'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a} = 0 \longrightarrow$ this is exactly the Lagrange thm.

* At each Lagrange point, the tangent to the graph of f is parallel to the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.

Example

$f(x) = 1 + x + \sqrt{1-x^2}$ continuous on its domain $[-1,1]$ as composite of elementary continuous functions. It is also different. on $(-1,1)$, not at the end-points, in fact: $f'(x) = 1 - \frac{x}{\sqrt{1-x^2}}$. Thus f fulfils the Lagrange thm and must admit a Lagrange point in $(-1,1)$. Now we have:

$$1 = \frac{f(1)-f(-1)}{1-(-1)} = f'(x_0) = 1 - \frac{x_0}{\sqrt{1-x_0^2}} \qquad \text{satisfied by } x_0 = 0$$

First and second finite increment formulas

- We shall discuss a couple of useful relations to represent how a function varies when passing from one point to another of its domain. Let's begin by assuming f is differentiable at x_0. By definition $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ that is to say $\lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right) = \lim_{x\to x_0} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{x-x_0} = 0$

Using Landau symbols becomes:

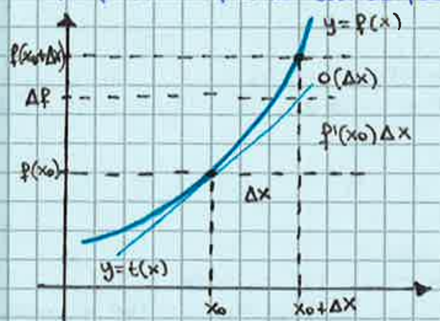
$f(x) - f(x_0) - f'(x_0)(x-x_0) = o(x-x_0)$, $x \to x_0$

An equivalent formulation is:

$f(x) - f(x_0) = f'(x_0)(x-x_0) + o(x-x_0)$, $x\to x_0$ or $\Delta f = f'(x_0)\Delta x + o(\Delta x)$, $\Delta x \to 0$

by putting $\Delta x = x - x_0$, $\Delta f = f(x) - f(x_0)$

Those two equations are equivalent writings of the so-called first formula of the finite increment, the geometric interpretation of which can be found in the following figure.



It tells that if $f'(x_0) \neq 0$ the increment Δf, corresponding to a change Δx, is proportional to Δx itself, if one disregards an infinitesimal which is negligible with respect to Δx. For Δx small enough, in practice, Δf can be treated as $f'(x_0)(\Delta x)$.

- Now take f continuous on an interval $I \subset \mathbb{R}$ and differentiable on the interior points. Fix $x_1 < x_2$ in I and note that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore f, restricted to $[x_1, x_2]$ satisfies the Mean Value Thm, so there's $\bar{x} \in (x_1, x_2)$ s.t. $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\bar{x})$ that is, a point $\bar{x} \in (x_1, x_2)$ with $f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1)$

that is the second formula of the finite increment. It has to be noted that the point $\bar{x}$ depends upon the choice of x_1 and x_2, albeit this dependency is in general not explicit. The formula's relevance derives from the possibility of gaining information about the increment $f(x_2) - f(x_1)$ from the behaviour of f' on the interval $[x_1, x_2]$

* 2nd formula may be used to describe the local behavior of a map in the neighbourhood of a certain x_0 with more precision than that permitted by the 1st formula.

Suppose f is continuous at x_0 and differentiable around x_0 except at the point itself. If x is a point in the neighbourhood of x_0, 2nd formula can be applied to the interval bounded by x_0 and x, to the effect that $\Delta f = f'(\bar{x})\Delta x$ where $\bar{x}$ lies between x_0 and x. This alternative formulation expresses the increment of the dependent variable Δf as if it were a multiple of Δx; the proportionality coefficient (the derivative evaluated at a point near x_0) depends upon Δx (and on x_0) besides being usually not known.

A further application of 2nd formula is described in the next result.

Property

A function defined on a real interval I and everywhere differentiable is constant on I if and only if its first derivative vanishes identically.

PROOF Let f be the map. Suppose first f is constant, therefore for every $x_0 \in I$ the difference quotient $\frac{f(x)-f(x_0)}{x-x_0}$ with $x \in I$, $x \neq x_0$ is zero. Then $f'(x_0) = 0$ by definition of derivative. Vice versa

suppose f has zero derivative on I and let us prove that f is constant on I. This would be equivalent to demanding $f(x_1) = f(x_2)$ $\forall x_1, x_2 \in I$. Take $x_1, x_2 \in I$ and use 2nd formula on f. For a suitable $\bar{x}$ between x_1, x_2 we have $f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1) = 0 \longrightarrow f(x_1) = f(x_2)$

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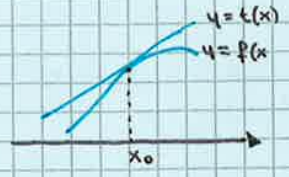
Convexity and inflection points

Let f be differentiable at the point x_0 of the domain. As customary, we indicate by $y=t(x)=f(x_0)+f'(x_0)(x-x_0)$, the equation of the tangent to the graph of f at x_0.

* The map f is **convex at** x_0 if there is a neighbourhood $I_r(x_0)\subseteq \operatorname{dom} f$ s.t. $\forall x\in I_r(x_0)$ $f(x)\geq t(x)$;
f is **strictly convex** if $f(x)>t(x)$, $\forall x\neq x_0$

- The definitions for **concave** and **strictly concave** functions are alike (just change $\geq, >$ to $\leq, <$)

* Geometrically, a map is convex at a point if around that point the graph lies above the tangent line, concave if its graph is below the tangent.



* A differentiable map f on an interval I is **convex on** I if it is convex at each point of I.

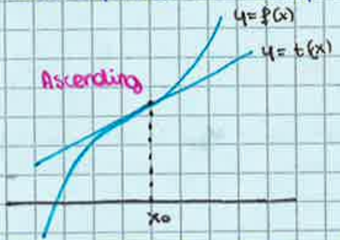
For understanding convexity, inflection points play a role reminiscent of extremum points for the study of monotone functions.

* The point x_0 is an **inflection point** for f if there is a neighbourhood $I_r(x_0)\subseteq \operatorname{dom} f$ where one of the following conditions holds:

either

- $\forall x\in I_r(x_0)$, $\begin{cases} \text{if } x<x_0, & f(x)\leq t(x) \\ \text{if } x>x_0, & f(x)\geq t(x)\end{cases}$
- $\forall x\in I_r(x_0)$, $\begin{cases} \text{if } x<x_0, & f(x)\geq t(x) \\ \text{if } x>x_0, & f(x)\leq t(x)\end{cases}$

In the former case we speak of an **ascending** inflection, in the latter the inflection is **descending**.



* In the plane, the graph of f "cuts through" the inflectional tangent at an inflection point.

- The analysis of convexity and inflections of a function is helped a great deal by the next results.

- **Theorem**

Given a differentiable map f on the interval I

a) If f is convex on I, then f' is increasing on I

b1) If f' is increasing on I, then f is convex on I

b2) If f' is strictly increasing on I, then f is strictly convex on I

- **Corollary**

If f is differentiable twice on I, then

a) f convex on I implies $f''\geq 0$ $\forall x\in I$

b1) $f''\geq 0$, $\forall x\in I$ implies f convex on I

b2) $f''>0$, $\forall x\in I$ implies f strictly convex on I

PROOF This follows directly from the previous theorem by applying the theorem of monotone maps to the function f'.

* There is a second formulation for this; namely: under the same hypothesis, the following formulas are true:
$f''(x)\geq 0$, $\forall x\in I \iff f$ is convex on I and $f''(x)>0$, $\forall x\in I \implies f$ is strictly convex on I
Here, as in the characterization of monotone functions, the last implication has no reverse.

- **Corollary**

Let f be twice differentiable around x_0.

a) If x_0 is an inflection point, then $f''(x_0)=0$

b) Assume $f''(x_0)=0$. If f'' changes sign when crossing x_0, then x_0 is an inflection point (ascending if $f''(x)\leq 0$ at the left of x_0 and $f''(x)\geq 0$ at its right, descending otherwise). If f'' does not change sign, x_0 is not an inflection point.

- The PROOF relies on Taylor's formula

Example p. 192

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Taylor expansions and applications

* The Taylor expansion of a function around a real point x_0 is the representation of the map as sum of a polynomial of a certain degree and an infinitesimal function of order bigger than the degree

Taylor formulas

We wish to tackle the problem of approximating a function f, around a given point $x_0 \in \mathbb{R}$, by polynomials of increasingly higher degree.
We begin by assuming f be continuous at x_0. Introducing the constant polynomial (degree zero)
$Tf_{0,x_0}(x) = f(x_0)$, $\forall x \in \mathbb{R}$ $\longrightarrow f(x) = Tf_{0,x_0}(x) + o(1)$, $x \to x_0$
Put in different terms, we may approximate f around x_0 using a zero degree polynomial, in such a way that the difference $f(x) - Tf_{0,x_0}(x)$, called **error of approximation** or **remainder**, is infinitesimal at x_0. The above relation is the **first instance of Taylor's formula**.

Suppose now f is not only continuous but also differentiable at x_0: then the first formula of the finite increment holds. By defining the polynomial in x of degree one
$Tf_{1,x_0}(x) = f(x_0) + f'(x_0)(x-x_0)$, whose graph is the tangent line to f at x_0, relation reads
(1st formula of finite increment) $f(x) = Tf_{1,x_0}(x) + o(x-x_0)$, $x \to x_0$.
This is another **Taylor formula**: it says that a differentiable map at x_0 can be locally approximated by a linear function, with an error of approximation that not only tends to 0 as $x \to x_0$, but is infinite. of order bigger than one

In case f is differentiable in a neighbourhood of x_0, except perhaps at x_0, the 2nd formula of the finite increment is available: putting $x_1 = x_0$, $x_2 = x$ we write the latter as $f(x) = Tf_{0,x_0}(x) + f'(\bar{x})(x-x_0)$ where $\bar{x}$ denotes a suitable point between x_0 and x. This is a Taylor polynomial as well and the remainder is called **Lagrange's remainder**. Instead, in the previous two formulas we call it **Peano's remainder**.

* Now that we have approximated f with polynomials of degrees 0 and 1, as $x \to x_0$, and made errors $o(1) = o((x-x_0)^0)$ or $o(x-x_0)$ respectively, the natural question is whether it is possible to approximate the function by a quadratic polynomial, with an error $o((x-x_0)^2)$ as $x \to x_0$. Equivalently, we seek for a real number a such that

• $f(x) = f(x_0) + f'(x_0)(x-x_0) + a(x-x_0)^2 + o((x-x_0)^2)$, $x \to x_0$

This means

$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - a(x-x_0)^2}{(x-x_0)^2} = 0$ ... By de l'Hôpital thm, such limit holds if

$\lim_{x \to x_0} \frac{f'(x) - f'(x_0) - 2a(x-x_0)}{2(x-x_0)} = 0$ i.e. $\lim_{x \to x_0} \left(\frac{1}{2} \frac{f'(x) - f'(x_0)}{x-x_0} - a \right) = 0$ or $\frac{1}{2} \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x-x_0} = a$

$\longrightarrow a = \frac{1}{2} f''(x_0)$. • Is valid when the right-hand-side limit exists and is finite (when f is twice differentiable at x_0. If so $\to a = \frac{1}{2} f''(x_0)$. In this way we have obtained the Taylor formula (with Peano's remainder)

$f(x) = Tf_{2,x_0}(x) + o((x-x_0)^2)$, $x \to x_0$ where $Tf_{2,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$ is the Taylor polynomial of f at x_0 with degree 2.

* The recipe just described can be iterated, and leads to polynomial approximations of increasing order. The final result is the content of the next theorem.

• **Theorem**
Let $n \geq 0$ and f be n times differentiable at x_0. Then the Taylor formula holds
$f(x) = Tf_{n,x_0}(x) + o((x-x_0)^n)$, $x \to x_0$ where $Tf_{n,x_0}(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$

$= f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$

* The term $Tf_{n,x_0}(x)$ is the **Taylor polynomial** of f at x_0 **of order (or degree)** n, while $o((x-x_0)^n)$ is the **Peano's remainder of order n**. The representation of f is called **Taylor expansion** of f at x_0 of order n, with remainder in Peano's form.
Under stronger hypotheses on f we may furnish a preciser formula for the remainder.

• **Theorem**
Let $n \geq 0$ and f differentiable n times at x_0, with continuous nth derivatives, be given; suppose f is differentiable $n+1$ times around x_0. Then the Taylor formula $f(x) = Tf_{n,x_0}(x) + \frac{1}{(n+1)!} f^{(n+1)}(\bar{x})(x-x_0)^{n+1}$ holds, for a suitable $\bar{x}$ between x_0 and x.

* This remainder is said **Lagrange's remainder of order**, and the previous one is the Taylor expansion of f at x_0 of order n with Lagrange's remainder.

* A Taylor expansion centred at the origin ($x_0 = 0$) is sometimes called **Maclaurin expansion**. A useful relation to simplify the computation of a Maclaurin expansion goes as follows.

Property
The Maclaurin polynomial of an even (respectively odd) map involves only even (odd) powers of the independent variable.

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Power functions

Consider the family of maps $f(x) = (1+x)^\alpha$ for arbitrary $\alpha \in \mathbb{R}$. We have
$f'(x) = \alpha(x+1)^{\alpha-1}$ — $f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$ — $f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$
From the general relation $f^{(k)}(x) = \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k}$ we get
$f(0) = 1$, $\quad \frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ for $k \geq 1$

At this point becomes convenient to extend the notion of binomial coefficient and allow α to be any real number by putting $\binom{\alpha}{0} = 1$, $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ for $k \geq 1$

Maclaurin's expansion to order n is thus

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \dots + \binom{\alpha}{n}x^n + o(x^n) = \sum_{k=0}^{n}\binom{\alpha}{k}x^k + o(x^n)$$

* When $\alpha = -1$ we have

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^n x^n + o(x^n) = \sum_{k=0}^{n}(-1)^k x^k + o(x^n)$$

* Choosing $\alpha = 1/2$ the expansion of $f(x) = \sqrt{1+x}$ arrested to the third order is

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

Operations on Taylor expansions

Consider the situation where a map f has a complicated analytic expression, that involves several elementary functions; it might not be that simple to find its Taylor expansion using the definition, because computing derivatives at a point up to a certain order n is no straightforward task. But with the expansions of the elementary functions at our avail, a more convenient strategy may be to start from these and combine them suitably to arrive at f. This approach is indeed justified by the following result.

Proposition Let $f:(a,b) \to \mathbb{R}$ be n times differentiable at $x_0 \in (a,b)$. If there exists a polynomial P_n of degree $\leq n$, s.t. $f(x) = P_n(x) + o((x-x_0)^n)$ for $x \to x_0$, then P_n is the Taylor polynomial $T_n = T_{f_n, x_0}$ of order n for the map f at x_0.

PROOF This formula is equivalent to $P_n(x) = f(x) + \varphi(x)$, with $\varphi(x) = o((x-x_0)^n)$ for $x \to x_0$.
On the other hand, Taylor's formula for f at x_0 reads
• $T_n(x) = f(x) + \psi(x)$ with $\psi(x) = o((x-x_0)^n)$
Therefore • $P_n(x) - T_n(x) = \varphi(x) - \psi(x) = o((x-x_0)^n)$ but the difference $P_n(x) - T_n(x)$ is a polynomial of degree lesser or equal than n, hence it may be written as $P_n(x) - T_n(x) = \sum_{k=0}^{n} c_k(x-x_0)^k$
The claim is that all coefficients c_k vanish. Suppose, by contradiction, there are some non-zero c_k, and let m be the smallest index between 0 and n such that $c_m \neq 0$. Then
$P_n(x) - T_n(x) = \sum_{k=m}^{n} c_k(x-x_0)^k$ so $\frac{P_n(x) - T_n(x)}{(x-x_0)^m} = c_m + \sum_{k=m+1}^{n} c_k(x-x_0)^{k-m}$ by factoring out $(x-x_0)^m$.
Taking the limit for $x \to x_0$ and recalling • we obtain $0 = c_m$, in contrast with the assumption.

* The proposition guarantees that however we arrive at an expression like • (in a mathematically correct way) this must be exactly the Taylor expansion of order n for f at x_0.

* For simplicity we shall assume henceforth $x_0 = 0$. This is always possible by a change of the variables, $x \to t = x - x_0$
Let now $f(x) = a_0 + a_1 x + \dots + a_n x^n + o(x^n) = p_n(x) + o(x^n)$ and $g(x) = b_0 + b_1 x + \dots + b_n x^n + o(x^n) = q_n(x) + o(x^n)$ be the Maclaurin expansions of the maps f and g.

Sums

$f(x) \pm g(x) = [p_n(x) + o(x^n)] \pm [q_n(x) + o(x^n)] = [p_n(x) \pm q_n(x)] + [o(x^n) \pm o(x^n)] = p_n(x) \pm q_n(x) + o(x^n)$
The expansion of a sum is the sum of the expansions involved.

Examples

• $e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^{2n+2}}{(2n+2)!} + o(x^{2n+2})$ changing x to $-x$ gives $e^{-x} = 1 - x + \frac{x^2}{2} - \dots + \frac{x^{2n+2}}{(2n+2)!} + o(x^{2n+2})$

• $\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$

• $\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$

Note that the expansion of f and g have the same monomial terms up to the exponent n, these all cancel out in the difference $f-g$. In order to find the first non-zero coefficient in the expansion of $f-g$ one has to look at an expansion of f and g of order $n' > n$. In general it is not possible to predict what the minimum n' will be, so one must proceed case by case.

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Local behaviour of a function

The knowledge of the Taylor expansion of f to order two around a point x_0, $f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + o((x-x_0)^2)$, $x \to x_0$ allows us to deduce that $f(x_0) = a_0$, $f'(x_0) = a_1$, $f''(x_0) = 2a_2$.
Suppose f is differentiable twice with continuity around x_0. By the Thm - Sign of the limit, the signs of a_0, a_1, a_2 (when $\neq 0$) coincide with the signs of $f(x)$, $f'(x)$, $f''(x)$, respectively, in a neighbourhood of x_0. This fact permits, in particular, to detect local monotonicity and convexity, because of theorem 6.26 b2) and corollary 6.37 b2)

Nature of critical points

Let x_0 be a critical point for f, which is assumed differentiable around x_0. By corollary 6.27, different signs of f' at the left and right of x_0 mean that the point is an extremum; if the sign is the same instead, x_0 is an inflection point with horizontal tangent.
When f possesses higher derivatives at x_0, in alternative to the sign of f' around x_0 we can understand what sort of critical point x_0 is by looking at the first non-zero derivative of f evaluated at the point. In fact,

• Theorem

Let f be differentiable $n \geq 2$ times at x_0 and suppose $f'(x_0) = \dots = f^{(m-1)}(x_0) = 0$, $f^{(m)}(x_0) \neq 0$ for some m $2 \leq m \leq n$.
i) when m is even, x_0 is an extremum, namely a maximum if $f^{(m)}(x_0) < 0$; a minimum if $f^{(m)}(x_0) > 0$
ii) when m is odd, x_0 is an inflection point with horizontal tangent; more precisely the inflection is descending if $f^{(m)}(x_0) < 0$, ascending if $f^{(m)}(x_0) > 0$

PROOF

Compare $f(x)$ and $f(x_0)$ around x_0. We know that $f(x) - f(x_0) = \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m + o((x-x_0)^m)$
But $o((x-x_0)^m) = (x-x_0)^m o(1)$, so
$f(x) - f(x_0) = (x-x_0)^m \left[\frac{f^{(m)}(x_0)}{m!} + h(x) \right]$ for a suitable $h(x)$, infinitesimal when $x \to x_0$. Therefore, in a small neighbourhood of x_0, the term in square brackets has the same sign as $f^{(m)}(x_0)$, hence the sign of $f(x) - f(x_0)$, in that same neighbourhood, is determined by $f^{(m)}(x_0)$ and $(x-x_0)^m$. Examining all sign possibilities proves the claim.

Points of inflection

Consider a twice differentiable f around x_0. By Taylor's formulas we can decide whether x_0 is an inflection point for f. PROOF Corollary 6.38 p.246

* Suppose, from now on, that $f''(x_0) = 0$ and f admits derivatives higher than the second. Instead of considering the sign of f'' around x_0, we may study the point x_0 by means of the first nonzero derivative of order > 2 evaluated at x_0.

• Theorem

Let f be n times differentiable ($n \geq 3$) at x_0, with $f''(x_0) = \dots = f^{(m-1)}(x_0) = 0$, $f^{(m)}(x_0) \neq 0$ for some m $3 \leq m \leq n$.
i) when m is odd, x_0 is an inflection point: descending if $f^{(m)}(x_0) < 0$, ascending if $f^{(m)}(x_0) > 0$.
ii) when m is even, x_0 is not an inflection for f.

PROOF

Just like in the previous Thm, we obtain $f(x) - t(x) = (x-x_0)^m \left[\frac{f^{(m)}(x_0)}{m!} + h(x) \right]$ where $h(x)$ is a suitable infinites. function for $x \to x_0$. The claim follows from a sign argument concerning the right-hand side.

SVILUPPO DI TAYLOR - punti critici

- Per essere un punto critico, $f'(x_0) = 0$

$f''(x_0) > 0 \longrightarrow$ punto di minimo

$f''(x_0) < 0 \longrightarrow$ punto di massimo

$f''(x_0) = 0 \longrightarrow$ punto di flesso

$\downarrow$

$f'''(x_0) > 0$ ascendente

$f'''(x_0) < 0$ discendente

$f'''(x_0) = 0$

$\downarrow$

$f^{IV}(x_0) = 0$

non c'è flesso!

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* The table of derivatives of the main elementary functions can be at this point read backwards, as a list of primitives:

- $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1)$
- $\int \frac{1}{x} dx = \log|x| + c \quad$ (for $x>0$ or $x<0$)
- $\int \sin x\, dx = -\cos x + c$
- $\int \cos x\, dx = \sin x + c$
- $\int e^x dx = e^x + c$
- $\int \frac{1}{1+x^2} dx = \arctan x + c$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$
- $\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos x + c$

- See examples

Rules of indefinite integration

The integrals of the elementary functions are important for the determination of other indefinite integrals. The rules below provide basic tools for handling integrals.

- **Theorem - Linearity of the integral**

Suppose $f(x), g(x)$ are integrable functions on the interval I. For any $\alpha, \beta \in \mathbb{R}$ the map $\alpha f(x) + \beta g(x)$ is still integrable on I, and $\int (\alpha f(x) + \beta g(x)) dx = \alpha \int f(x) dx + \beta \int g(x) dx$

PROOF Suppose $F(x)$ is a primitive of $f(x)$ and $G(x)$ a primitive of $g(x)$. By linearity of the derivative
$(\alpha F(x) + \beta G(x))' = \alpha F'(x) + \beta G'(x) = \alpha f(x) + \beta g(x) \quad , \forall x \in I$
This means $\alpha F(x) + \beta G(x)$ is a primitive of $\alpha f(x) + \beta g(x)$ on I.

* The above property says that one can integrate a sum one summand at a time, and pull multiplicative constants out of the integral sign.

- **Theorem - Integration by parts**

Let $f(x), g(x)$ be differentiable over I. If the map $f'(x) g(x)$ is integrable on I, then so is $f(x) g'(x)$ and $\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$

PROOF Let $H(x)$ be any primitive of $f'(x) g(x)$ on I. Applying the formula, we have:

$$[f(x) g(x) - H(x)]' = (f(x) g(x))' - H'(x) = f'(x) g(x) + f(x) g'(x) - f'(x) g(x) = f(x) g'(x)$$

Therefore the map $f(x) g(x) - H(x)$ is a primitive of $f(x) g'(x)$, exactly what the previous thm claims.

* In practice, one integrates a product of functions by identifying first one factor with $f(x)$ and the other with $g'(x)$; then one determines a primitive $g(x)$ of $g'(x)$ and, at last, one finds the primitive of $f'(x)g(x)$ and uses that theorem.

- See examples

- **Theorem - Integration by substitution**

Let $f(y)$ be integrable on the interval J and $F(y)$ a primitive. Suppose $\varphi(x)$ is a differentiable function from I to J. Then the map $f(\varphi(x))\varphi'(x)$ is integrable on I and $\int f(\varphi(x)) \varphi'(x) dx = F(\varphi(x)) + c$ **1**
which is usually stated in the less formal yet simpler way $\int f(\varphi(x))\varphi'(x) dx = \int f(y) dy$ **2**

PROOF Using the formula for differentiating a composite map gives

$$\frac{d}{dx} F(\varphi(x)) = \frac{dF}{dy}(\varphi(x)) \frac{d\varphi}{dx}(x) = f(\varphi(x)) \varphi'(x)$$

Thus $F(\varphi(x))$ integrates $f(\varphi(x))\varphi'(x)$, i.e., the theorem is proven.

* The correct meaning of **2** is expressed by **1**: the integral on the left is found by integrating f with respect to y and then substituting to y the function $\varphi(x)$, so that the right-hand side too depends on the variable x.

Examples p.307-08-09-10

The following useful relation $\int \frac{\varphi'(x)}{\varphi(x)} dx = \log|\varphi(x)| + c$ descends from **2** by $f(y) = \frac{1}{y}$

All instances had one common feature: the maps f were built from a finite number of elementary functions by algebraic operations and compositions, and so were the primitives F. In such a case, one says that f is **integrable by elementary methods**. Unfortunately though, not all functions arising this way are integrable by elementary methods.

Examples

$f(x) = e^{-x^2} \qquad f(x) = \frac{\sin x}{x}$

The problem of finding an explicit primitive for a given function is highly nontrivial. A large class of maps which are integrable by elementary methods is that of rational functions.

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Note that many functions $f(x)$ that are not rational in the variable x can be transformed, by an appropriate change $t = \varphi(x)$, into a rational map in the new variable t. Special cases thereof include:

1 f is a rational function of $\sqrt[p]{x-a}$ for some integer p and a real. Then one gets
$t = \sqrt[p]{x-a}$ whence $x = a + t^p$ and $dx = pt^{p-1}\,dt$.

2 f is a rational in e^{ax} for some real $a \neq 0$. The substitution
$t = e^{ax}$ gives $x = \frac{1}{a}\log t$ and $dx = \frac{1}{at}\,dt$

3 f is rational in $\sin x$ and/or $\cos x$. In this case $\longrightarrow$ $t = \tan x/2$ together with the identities:
$\sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$ does the job, because then $x = 2\arctan t$, hence $dx = \frac{2}{1+t^2}\,dt$

4 If f is rational in $\sin^2 x$, $\cos^2 x$, $\tan x$, it is more convenient to set $t = \tan x$ and use
$\sin^2 x = \frac{t^2}{1+t^2}$ $\cos^2 x = \frac{1}{1+t^2}$

EXAMPLES

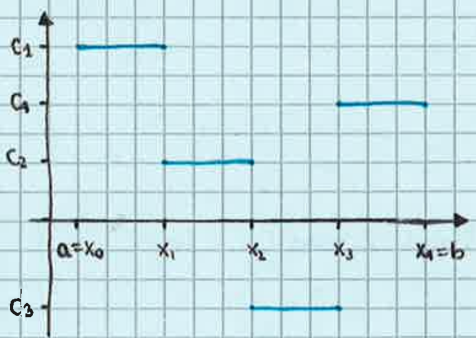
Definite integrals

Let us consider a bounded map f defined on a bounded and closed interval $I = [a,b] \subset \mathbb{R}$. One suggestively calls **trapezoidal region of f over the interval** $[a,b]$, denoted by $T(f;a,b)$, the part of plane enclosed within the interval $[a,b]$, the vertical lines passing through the end-points a,b and the graph of f.
$T(f;a,b) = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ 0 \leq y \leq f(x) \text{ or } f(x) \leq y \leq 0\}$ (y depends on the sign of $f(x)$)
Under suitable assumptions on f one can associate to the trapezoidal region of f over $[a,b]$ a number, the definite integral of f over $[a,b]$. In case f is positive, this number is indeed the area of the region. In particular, when the region is simple, the definite integral returns one of the classical formulas of elementary geometry.
The many notions of definite integral depend on what is demanded of the integrand.

- A map $f: [a,b] \longrightarrow \mathbb{R}$ is **piecewise continuous** when it is continuous everywhere except at a finite nr. of points, at which the discontinuity is either removable or a jump.

The Riemann integral

Throughout the section f will indicate a bounded map on $[a,b]$. Let's start integrating some el. functions.
Choose $n+1$ points of $[a,b]$ (not necessarily uniformly spread)
$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ They induce a partition of $[a,b]$ into sub-intervals $I_k = [x_{k-1}, x_k]$, $k = 1, \ldots, n$
Dividing further one of the I_k we obtain a finer partition (refinement of the initial partition)
Step functions are constant on each subinterval of a partition of $[a,b]$



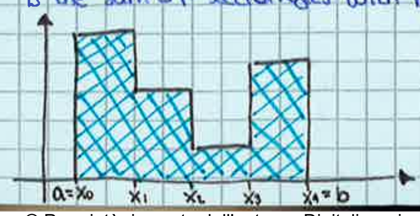
* A map $f: [a,b] \longrightarrow \mathbb{R}$ is a **step function** if there exist a partition of $[a,b]$ by $\{x_0, x_1, \ldots, x_n\}$ together with constants $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that
$f(x) = c_k \quad \forall x \in (x_{k-1}, x_k) \quad k = 1, \ldots, n$

We say that the partition is **adapted** to f if f is constant on each interval (x_{k-1}, x_k). Refinements of adapted partitions are still adapted. In particular if f and g are step functions on $[a,b]$, it is always possible to manufacture a partition that is adapted to both maps just by taking the union of the points of two part. adapted to f and g respectively. $S([a,b])$ = set of step functions on $[a,b]$.

Let $f \in S([a,b])$ and $\{x_0, x_1, \ldots, x_n\}$ be an adapted partition. Call c_k the constant value of f on (x_{k-1}, x_k). Then the number $\int_I f = \sum_{k=1}^{n} c_k (x_k - x_{k-1})$ is called **definite integral** of f on $I = [a,b]$

* A few remarks are necessary
1 The definition is independent of the chosen partition. In particular, if $f(x) = c$ is constant on $[a,b]$, $\int f = c(b-a)$
2 Redefining f at a finite number of places leaves the integral unchanged; in particular, the definite integral does not depend upon the values of f at points of discontinuity.

In case f is positive on I, the number $\int f$ is the area of the trapezoidal region of f over I: the latter is the sum of rectangles with base $x_k - x_{k-1}$ and height c_k.



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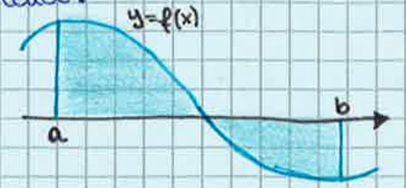
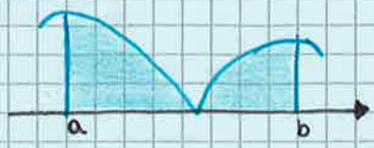
Properties of definite integrals

* Recall $\int_a^b f(x)\,dx$ is a number, depending only on f and the interval $[a,b]$; it certainly depends upon no variable. The letter x is a "virtual variable", and as such may be substituted by one's own referred letter.





* If $f \in \mathcal{R}([a,b])$ is positive $\rightarrow$ the definite integral expresses the area of the trapezoidal region of f over $[a,b]$. For negative f the same holds provided one changes sign to the value. When f has no fixed sign, the integral measures the difference of the positive regions (above the x-axis) and the negative regions (below it), so the area between f and the horizontal axis is also the integral of the map $|f|$

Area of $T(f;a,b) = \int_a^b |f(x)|\,dx$ This is due to the symmetrising effect of the absolute value.

Let us generalise the definite integral. Take $f \in \mathcal{R}([a,b])$. For $a \le c < d \le b$, set
$\int_d^c f(x)\,dx = -\int_c^d f(x)\,dx$ and $\int_c^c f(x)\,dx = 0$

The symbol $\int_c^d f(x)dx$ is now defined whichever limits c and d we consider in the integrability domain $[a,b]$. The following properties descend immediately from the definition.

• **Theorem** Let f and g be integrable on a bounded interval I of the real line.

i) (Additivity with respect to the domain of integration) For any $a,b,c \in I$,
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

ii) (Linearity) For any $a,b \in I$ and $\alpha,\beta \in \mathbb{R}$.
$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$$

iii) (Positivity) Let $a,b \in I$, with $a<b$. If $f \ge 0$ on $[a,b]$ then $\int_a^b f(x)\,dx \ge 0$
If f is additionally continuous, equality holds if and only if f is the zero map.

iv) (Monotonicity) Let $a,b \in I$, $a<b$. If $f \le g$ in $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

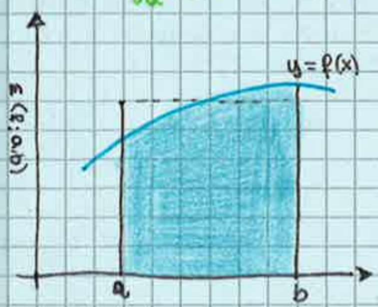
v) (Upper and lower bounds) Let $a,b \in I$, $a<b$. Then $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Integral mean value

The definite integral of an integrable map f over the usual interval $[a,b]$ furnishes a way of approximating the function's behaviour by a constant.

* By (integral) mean value (sometimes integral average) of f over the interval $[a,b]$ one understands the number $m(f;a,b) = \frac{1}{b-a}\int_a^b f(x)dx$

The geometric meaning is clear when f is positive on $[a,b]$ for an equivalent version of the mean value reads $\int_a^b f(x)\,dx = (b-a)\,m(f;a,b)$



In this case $T(f;a,b)$ equals the area of the rectangle with base $[a,b]$ and having the integral average as height.

• **Theorem - Mean Value Theorem**
Let f be integrable over $[a,b]$. The integral mean of f over $[a,b]$ satisfies $\inf_{x\in[a,b]} f(x) \le m(f;a,b) \le \sup_{x\in[a,b]} f(x)$
If moreover f is continuous on $[a,b]$, there is at least one $z \in [a,b]$ s.t.
$m(f;a,b) = f(z)$

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* **Corollary** Let f be continuous on $[a,b]$ and G any primitive of f on that interval. Then
$$\int_a^b f(x)\,dx = G(b) - G(a)$$

PROOF Denoting F_a the integral map vanishing at a, one has $\int_a^b f(x)\,dx = F_a(b)$.
The previous corollary proves the claim once we put $x_0 = a$, $x = b$

* Very often the difference $G(b) - G(a)$ is written as $[G(x)]_a^b$ or $G(x)\Big|_a^b$

- There is a generalisation of the Fundamental theorem of integral calculus to piecewise-continuous maps, which goes like this. If f is piecewise continuous on all closed and bounded subintervals of I, then any integral function F on I is continuous on I, it is differentiable at all points where f is continuous, and $F'(x) = f(x)$.
Jump discontinuities for f inside I correspond to corner points for F.
The integral F is called then a **generalised primitive** of f on I.

Now we present an integral representation of a differentiable map, which turns out to be useful in many circumstances.

* **Corollary** Given f differentiable on I with continuous first derivative, and any $x_0 \in I$
$$f(x) = f(x_0) + \int_{x_0}^{x} f'(s)\,ds, \quad \forall x \in I$$

PROOF Obviously f is a primitive of its own derivative, so the previous corollary (9.24) gives
$\int_{x_0}^{x} f'(s)\,ds = f(x) - f(x_0)$ whence the result follows.

- As an application we can justify the MacLaurin expansions of $f(x) = \arcsin x$ and $f(x) = \arctan x$.
First though, a technical lemma.

- **Lemma**
If φ is a continuous map around 0 s.t. $\varphi(x) = o(x^a)$ for $x \to 0$, and $a \geq 0$, then the primitive $\psi(x) = \int_0^x \varphi(s)\,ds$ satisfies $\psi(x) = o(x^{a+1})$ as $x \to 0$. This can be written as $\int_0^x o(s^a)\,ds = o(x^{a+1})$ for $x \to 0$

PROOF From de l'Hôpital's Theorem
$$\lim_{x\to 0} \frac{\psi(x)}{x^{a+1}} = \lim \frac{\psi'(x)}{(a+1)x^a} = \frac{1}{a+1} \lim_{x\to 0} \frac{\varphi(x)}{x^a} = 0$$

So now take $f(x) = \arctan x$. As its derivative reads $f'(x) = \frac{1}{1+x^2}$, previous corollary allows us to write
$$\arctan x = \int_0^x \frac{1}{1+s^2}\,ds$$
The Maclaurin expansion of $f'(s)$, obtained changing $x = s^2$, reads:
$$\frac{1}{1+s^2} = 1 - s^2 + s^4 - \ldots + (-1)^m s^{2m} + o(s^{2m+1}) = \sum_{k=0}^{m} (-1)^k s^{2k} + o(s^{2m+1})$$
Term by term integration yields Maclaurin expansion for $f(x)$:
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1} + o(x^{2m+2}) = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2m+2})$$
As for the inverse sine, write
$$f(x) = \arcsin x = \int_0^x \frac{1}{\sqrt{1-s^2}}\,ds \qquad \text{Fixing } a = -\tfrac{1}{2} \text{ and changing } x = -s^2$$
$$\frac{1}{\sqrt{1-s^2}} = 1 + \frac{1}{2}s^2 + \frac{3}{8}s^4 + \ldots + \left|\binom{-1/2}{m}\right| s^{2m} + o(s^{2m+1}) = \sum_{k=0}^{m} \left|\binom{-1/2}{m}\right| s^{2k} + o(s^{2m+1})$$
Integrating term by term yields the expansion:
$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots + \left|\binom{-1/2}{m}\right| \frac{x^{2m+1}}{2m+1} + o(x^{2m+2}) = \sum_{k=0}^{m} \left|\binom{-1/2}{k}\right| \frac{x^{2k+1}}{2k+1} + o(x^{2m+2})$$

Rules of definite integration

The Fundamental Theorem of integral calculus and the rules that apply to indefinite integrals furnish similar results for definite integrals.

- **Theorem - Integration by parts**
Let f and g be differentiable with continuity on $[a,b]$. Then $\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$

PROOF If $H(x)$ denotes any primitive of $f'(x)g(x)$ on $[a,b]$, the known result on integration by parts prescribes that $f(x)g(x) - H(x)$ is a primitive of $f(x)g'(x)$.
Thus we have $\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - [H(x)]_a^b$. It then suffices to use (9.24) on the map $f'(x)g(x)$.

- **Theorem - Integration by substitution**
Let $f(y)$ be continuous on $[a,b]$. Take a map $\varphi(x)$ from $[\alpha,\beta]$ to $[a,b]$, differentiable with continuity.
Then $\int_\alpha^\beta f(\varphi(x))\varphi'(x)\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(y)\,dy$ 1
If φ bijects $[\alpha,\beta]$ onto $[a,b]$, this formula may be written as $\int_a^b f(y)\,dy = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(x))\varphi'(x)\,dx$ 2

PROOF Let $F(y)$ be a primitive of $f(y)$ on $[a,b]$. Formula (1) follows from (9.4) and the prev. corollary.
When φ is bijective, the two formulas are equivalent for $a = \varphi(\alpha)$, $b = \varphi(\beta)$ if φ is strictly increasing, and $a = \varphi(\beta)$, $b = \varphi(\alpha)$ if strictly decreasing.

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Integral calculus II

Improper integrals

Hitherto integrals have been defined for bounded maps over closed bounded intervals of the real line. However, several application induce one to consider unbounded intervals quite often, or functions tending to infinity. To cover such cases the notion of integral must be extended by means of limits.

Unbounded domains of integration

Let $\mathcal{R}_{loc}([a,+\infty))$ be the set of maps defined on the ray $[a,+\infty)$ and integrable on every closed and bounded subinterval $[a,c]$ of the domain. Taking $f\in\mathcal{R}_{loc}([a,+\infty))$ we can introduce the integral function $F(c)=\int_a^c f(x)\,dx$ on $[a,+\infty)$. The natural question to answer concerns its behaviour when $c\to+\infty$.

* Let $f\in\mathcal{R}_{loc}([a,+\infty))$, we (formally) set $\int_a^{+\infty} f(x)\,dx = \lim_{c\to+\infty}\int_a^c f(x)\,dx$.
The symbol on the left is said **improper integral** of f on $[a,+\infty)$

i) If the limit exists and is finite, we say that the map f is integrable over $[a,+\infty)$, or equivalently, that its improper **integral converges**.

ii) If the limit exists but its infinite, we say that the improper **integral of f diverges**.

iii) If the limit does not exist, we say that the improper **integral** is **indeterminate**.

- The class of integrable maps over $[a,+\infty)$ will be indicated $\mathcal{R}([a,+\infty))$. Visualising the improper integral of a positive function is easy. Note first that the following holds.

Proposition Let $f\in\mathcal{R}_{loc}([a,+\infty))$ be s.t. $f(x)\geq 0$, $\forall x\in[a,+\infty)$. Then the integral map $F(c)$ is increasing on $[a,+\infty)$.

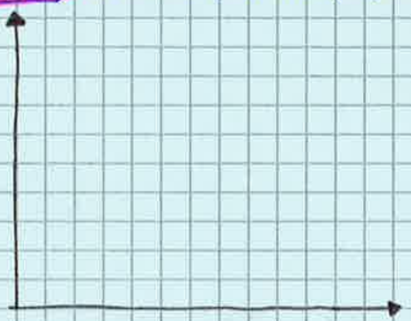
PROOF Take $c_1,c_2\in[a,+\infty)$ with $c_1<c_2$. By the property of additivity of the domain of integration,

$$F(c_2)=\int_a^{c_2} f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx = F(c_1)+\int_{c_1}^{c_2} f(x)\,dx \quad (\mathrm{I})$$

last integral is ≥ 0 by iii), therefore $F(c_2)\geq F(c_1)$

Corollary The improper integral of a positive map belonging to $\mathcal{R}_{loc}([a,+\infty))$ is either convergent or divergent to $+\infty$.

- **PROOF** Descends from the proposition



Going back to the geometric picture, we can say that the improper integral of a positive function represents the area of the trapezoidal region over $[a,+\infty)$. This region is unbounded and can be viewed as the limit, for $c\to\infty$, of the regions defined over the subintervals $[a,c]$. The area of the trapezoidal region over the entire domain of integration $[a,+\infty)$ is finite if the improper integral converges, and one says that the area is infinite when the integral is divergent.

Examples

- $f(x)=1/x^a$, $a>0$ over $[1,+\infty)$

$$\int_1^c \frac{1}{x^a}\,dx = \begin{cases} \left.\frac{x^{1-a}}{1-a}\right|_1^c & \text{if } a\neq 1 \\ \left.\log x\right|_1^c & \text{if } a=1\end{cases} = \begin{cases} \frac{c^{1-a}-1}{1-a} & \text{if } a\neq 1 \\ \log c & \text{if } a=1\end{cases}$$

$$a\neq 1 \longrightarrow \int_1^{+\infty}\frac{1}{x^a}\,dx = \lim_{c\to+\infty}\frac{c^{1-a}-1}{1-a} = \begin{cases}\frac{1}{a-1} & \text{if } a>1\\ +\infty & \text{if } a<1\end{cases}$$

$a=1 \longrightarrow \int_1^{+\infty}\frac{1}{x}\,dx = \lim_{c\to+\infty}\log c = +\infty$ The integral behaves in the same manner whichever the lower limit of integration $a>0$.

Therefore $\int_a^{+\infty}\frac{1}{x^a}\,dx \begin{cases}\text{converges} & a>1\\ \text{diverges} & \text{if } a\leq 1\end{cases}$

- Set $f(x)=\cos x \longrightarrow F(c)=\int_0^c \cos x\,dx = \sin c$ does not admit limit for $c\to+\infty$, hence $\int_0^{+\infty}\cos x\,dx$ is indeterm

Improper integrals inherit some features of definite integrals; if f,g belong to $\mathcal{R}([a,+\infty))$:

i) for any $c>a$

$$\int_a^{+\infty} f(x)\,dx = \int_a^c f(x)\,dx + \int_c^{+\infty} f(x)\,dx$$

iii) $f\geq 0$ on $[a,+\infty)$

$$\int_a^{+\infty} f(x)\,dx \geq 0$$

ii) for any $\alpha,\beta\in\mathbb{R}$

$$\int_a^{+\infty}(\alpha f(x)+\beta g(x))\,dx = \alpha\int_a^{+\infty} f(x)\,dx + \beta\int_a^{+\infty} g(x)\,dx$$

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Unbounded integrands

Consider the set $\mathcal{R}_{loc}([a,+b))$ of functions defined on the bounded interval $[a,b)$ and integrable over each closed subinterval $[a,c]$, $a<c<b$. If $f \in \mathcal{R}_{loc}([a,b))$ the integral function $F(c)=\int_a^c f(x)\,dx$ is thus defined over $[a,b)$. We wish to study the limiting behaviour of such, for $c \to b^-$.

* Set $f \in \mathcal{R}_{loc}([a,b))$ and define, formally, $\int_a^b f(x)\,dx = \lim_{x\to b^-}\int_a^c f(x)\,dx$ as before, the left-hand side is called **improper integral** of f over $[a,b)$.

i) If the limit exists and is finite, one says f is (improperly) integrable on $[a,b)$, or that its **improper integral converges**.

ii) If the limit exists but infinite, one says that the **improper integral** of f is **divergent**

iii) If the limit does not exist, one says that the **improper integral** is **indeterminate**.

- As usual, integrable functions over $[a,b)$ shall be denoted by $\mathcal{R}([a,b))$. If a map is bounded and integrable on $[a,b]$, it is also integrable on $[a,b)$ in the above sense. Its improper integral coincides with the definite integral. Indeed, letting $M=\sup_{x\in[a,b]}|f(x)|$, we have $\left|\int_a^b f(x)\,dx - \int_a^c f(x)\,dx\right| = \left|\int_c^b f(x)\,dx\right| \le \int_c^b |f(x)|\,dx \le M(b-c)$.

We obtain $\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$ for $c\to b^-$, this because the symbol is the same for def. and improper integrals.

Example

- $f(x)=\dfrac{1}{(b-x)^\alpha}$ with $\alpha>0 \longrightarrow \int_a^c \dfrac{1}{(b-x)^\alpha}dx = \begin{cases} \dfrac{(b-x)^{1-\alpha}}{\alpha-1}\Big|_a^c & \text{if } \alpha\neq 1 \\ -\log(b-x)\Big|_a^c & \text{if } \alpha=1\end{cases} = \begin{cases} \dfrac{(b-c)^{1-\alpha}-(b-a)^{1-\alpha}}{\alpha-1} & \text{if } \alpha\neq 1 \\ \log\dfrac{b-a}{b-c} & \text{if } \alpha=1\end{cases}$

when $\alpha\neq 1$

$$\int_a^b \frac{1}{(b-x)^\alpha}dx = \lim_{c\to b^-}\frac{(b-c)^{1-\alpha}-(b-a)^{1-\alpha}}{\alpha-1} = \begin{cases}\dfrac{(b-a)^{1-\alpha}}{1-\alpha} & \text{if } \alpha<1 \\ +\infty & \text{if } \alpha>1\end{cases}$$

For $\alpha=1$

$$\int_a^b \frac{1}{b-x}dx = \lim_{c\to b^-}\log\frac{b-a}{b-c} = +\infty$$

Therefore $\int_a^b \dfrac{1}{(b-x)^\alpha}dx \begin{cases}\text{converges} & \text{if } \alpha<1\\ \text{diverges} & \text{if } \alpha\geq 1\end{cases}$

- **Theorem - Comparison Test**

Let $f,g \in \mathcal{R}_{loc}([a,b))$ be s.t. $0\le f(x)\le g(x)$ for any $x\in[a,b)$. Then $0\le\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

In particular

i) if the integral of g converges, the integral of f converges

ii) if the integral of f diverges, the integral of g diverges.

- **Theorem - Asymptotic comparison test**

If $f\in\mathcal{R}_{loc}([a,b))$ is infinite of order α for $x\to b^-$ with respect to $\varphi(x)=\dfrac{1}{b-x}$, then

i) if $\alpha<1$, $f\in\mathcal{R}([a,b))$

ii) if $\alpha\ge 1$, $\int_a^b f(x)\,dx$ diverges.

* Integrals over $(a,b]$ are defined similarly $\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$

see examples

More improper integrals

Suppose we want to integrate a map with finitely many discontinuities in an interval I, bounded or not. Subdivide I into a finite number of intervals I_j, $j=1,\dots,n$, s.t. the restricted maps falls into one of the cases examined so far.

Then formally define $\int_I f(x)\,dx = \sum_{j=1}^{n}\int_{I_j} f(x)\,dx$. One says that the **improper integral** of f on I **converges** if the integrals on the right all converge. It is not so hard to verify that the improper integral's behaviour and its value, if convergent, are independent of the chosen partition of I.

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The argument can assume infinitely many values, all differing by integer multiples of 2π. One calls **principal value** of $\arg z$ ($\operatorname{Arg} z$) the unique value θ of $\arg z$ s.t. $-\pi < \theta \leq \pi$.

- Two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are equal $\Leftrightarrow$ $r_1 = r_2$ and θ_1, θ_2 differ by an integer multiple of 2π. This representation is useful to multiply complex numbers.

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ $\rightarrow$ addition formulas for trigonometric functions

$\rightarrow z_1 z_2 = r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)] = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

Therefore $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

This identity is false when using Arg.

* The so-called **exponential form** is also useful. To define it, let us extend the exponential function to the case where the exponent is purely imaginary, by putting $e^{i\theta} = \cos\theta + i\sin\theta$ for any $\theta \in \mathbb{R}$. Such a relation is called **Euler formula**, and can be actually proved within the theory of series over the complex numbers. We shall take it as definition without further mention. $\rightarrow z = re^{i\theta}$ exp form of z. The complex conjugate is $\bar{z} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + \sin(-\theta))) = re^{-i\theta}$

The product of $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ is $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Thus, the moduli are multiplied and the arguments added. Division gives, with $r_1 = r_2 = 1$ $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ $e^{i\theta} e^{-i\theta} = 1$ in particular. The reciprocal of $z = re^{i\theta} \neq 0$ is $z^{-1} = \frac{1}{r} e^{-i\theta}$

Combining this formula with the product $\rightarrow$ the ratio of z_1 and z_2 is $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Powers and nth roots

For any $n \in \mathbb{Z}$, we have $z^n = r^n e^{in\theta}$

For $r = 1$, this is the so-called **De Moivre's formula** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Thanks to 1st formula we can calculate nth roots of a complex number. Fix $n \geq 1$ and a complex number $w = \rho e^{i\varphi}$, to determine the numbers $z = re^{i\theta}$ st. $z^n = w$ $\longrightarrow$ $z^n = r^n e^{in\theta} = \rho e^{i\varphi} = w$ $\longrightarrow$

$\begin{cases} r^n = \rho \\ n\theta = \varphi + 2k\pi, \ k \in \mathbb{Z} \end{cases}$, hence $\begin{cases} r = \sqrt[n]{\rho} \\ \theta = \frac{\varphi + 2k\pi}{n}, \ k \in \mathbb{Z} \end{cases}$ The expression of θ does not necessarily give the principal values of the roots' arguments. Nevertheless, as sine and cosine are periodic, we have n distinct solutions: $z_k = \sqrt[n]{\rho}\, e^{i\frac{\varphi + 2k\pi}{n}} = \sqrt[n]{\rho}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right)$, $k = 0, 1, \dots, n-1$ to the problem.

These points lie on the circle centred at the origin with radius $\sqrt[n]{\rho}$; they are precisely the vertices of a regular polygon of n sides.

see example

We can define the exp. of arbitrary complex numbers $z = x + iy \longrightarrow e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$

We can also verify that the fundamental relation $e^{z_1+z_2} = e^{z_1} e^{z_2}$ is still valid for complex numbers.

In addition $\rightarrow |e^z| = e^{\operatorname{Re} z} > 0$, $\arg e^z = \operatorname{Im} z$

The first tells that $e^z \neq 0$ $\forall z \in \mathbb{C}$. The periodicity of trigonometric functions implies $e^{z + 2k\pi i} = e^z$ $\forall k \in \mathbb{Z}$

Algebraic Equations

We will show that the quadratic equation with real coefficients $az^2 + bz + c = 0$ admits two complex-conjugate solutions in case of $\Delta < 0$. We can suppose $a > 0$. Inspired by a square of a binomial we write

$$0 = z^2 + \frac{b}{a}z + \frac{c}{a} = \left(z^2 + 2\frac{b}{2a}z + \frac{b^2}{4a^2}\right) + \frac{c}{a} - \frac{b^2}{4a^2} \longrightarrow \left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2} < 0$$

Therefore $z + \frac{b}{2a} = \pm i\frac{\sqrt{-\Delta}}{2a} \longrightarrow z = \frac{-b \pm i\sqrt{-\Delta}}{2a}$

We write $z = \frac{-b \pm i\sqrt{\Delta}}{2a}$ in analogy to the case $\Delta \geq 0$

The procedure can be applied when $a \neq 0$, b, c are complex numbers, too. $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are the 2 solutions. 3rd and 4th degree equations have 3-4 solutions respectively, obtained by algebraic operations.

The Fundamental Thm of Algebra warrants that every equation $p(z) = 0$ (p = polyn. of degree n with real or complex coefficients) admits n solutions in $\mathbb{C}$.

- **Theorem**

Let $p(z) = a_n z^n + \dots + a_1 z + a_0$ with $a \neq 0$, be a polynomial of degree n with coefficients $a_k \in \mathbb{C}$, $0 \leq k \leq n$. There exist $m \leq n$ distinct complex numbers $z_1, \dots, z_m$ and m non-zero natural numbers $\mu_1, \dots, \mu_m$ with $\mu_1 + \dots + \mu_m = n$, s.t. $p(z)$ factorises as $p(z) = a_n(z - z_1)^{\mu_1} \dots (z - z_m)^{\mu_m}$

z_k = roots of $p(z)$ (solutions); the exp μ_k = multiplicity of roots z_k. It is opportune to remark that if the coefficients of p are real and if z_0 is a complex root, then also $\bar{z}_0$ is a root of p. In fact, taking conjugates of $p(z_0) = 0$ we obtain $0 = \bar{0} = \overline{p(z_0)} = \bar{a}_n \bar{z}_0^n + \dots + \bar{a}_1 \bar{z}_0 + \bar{a}_0 = a_n \bar{z}_0^n + \dots + a_1 \bar{z}_0 + a_0 = p(\bar{z}_0)$

$p(x)$ is divisible by $(z - z_0)(z - \bar{z}_0)$, a quadratic polynomial with real coefficients.

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Ordinary differential equations

A large part of natural phenomena can be described by a mathematical model, a collection of relations involving a function and its derivatives. An example is the uniformly accelerated motion:
$\frac{d^2s}{dt^2} = g$ where $s = s(t)$ = motion in function of time g = acceleration
Another example is the radioactive decay (the rate of disintegration of a radioactive substance in time is proportional to the quantity of matter:
$\frac{dy}{dt} = -Ky$ where $y = y(t)$ = mass of the element $K>0$ decay constant.

General definitions

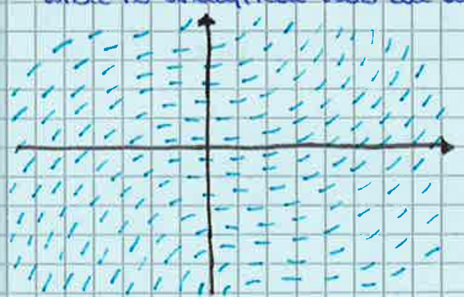
By an ordinary differential equation (ODE), one understands a relation among an independent real variable (x) and unknown function $y = y(x)$ and its derivatives $y^{(k)}$ up to a specified order n. It is indicated by $\mathcal{F}(x, y, y', \dots y^{(n)}) = 0$ where $\mathcal{F}$ is a real map depending on $n+2$ real variables. The differential equation has order n, if n is the highest order of differentiation. A solution of the ODE over a real interval I is a function $y: I \to \mathbb{R}$ differentiable n times on I, s.t.
$\mathcal{F}(x, y(x), y'(x), \dots y^{(n)}(x)) = 0, \ \forall x \in I$

* It happens many times that the highest derivative $y^{(n)}$ can be expressed in terms of x and the remaining derivatives explicitly, $y^{(n)} = f(x, y, \dots, y^{(n-1)})$ with f a real function of $n+1$ variables. If so, the diff. equation is written in normal form.
* A differential equation is said autonomous if $\mathcal{F}$ (or f) does not depend on the variable x.

First order differential equations

Let f be a real-valued map defined on a subset of $\mathbb{R}^2$. A solution to the equation $y' = f(x, y)$ over an interval I of $\mathbb{R}$ is a differentiable map $y = y(x)$ s.t. $y'(x) = f(x, y(x))$ for any $x \in I$. The graph of a solution to this equation is called integral curve of the differential equation.
The eq. also admits a geometric interpretation. For each point (x,y) in the domain of f, $f(x,y)$ is the slope of the tangent to the integral curve containing (x,y) (assuming that the curve exists) so equation is represented by a field of directions in the plane.

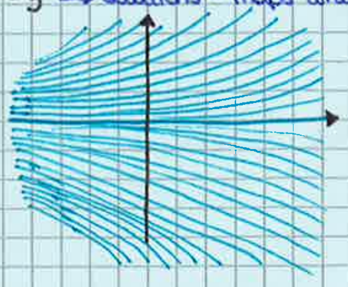
If we start to move from $(x,y) = (x_0, y_0)$ along the straight line with slope $f(x_0, y_0)$ (the tangent), we reach a point (x_1, y_1) near the integral curve passing through (x_0, y_0). From there we can advance and then reach (x_2, y_2) nearby the curve and so on, building a polygonal path. This is called explicit Euler method $\to$ simplest numerical procedure for approximating the solution of a diff. equation where no analytical tools are available.



* If f depends on x and not on y, we have $y' = f(x)$; assuming f continuous on I, the solutions are $y(x) = F(x) + C$, with F particular primitive and C a constant. So, when f does not depend upon y, it has infinitely many distinct solutions, depending on one constant. Any chosen integral curve is the vertical translate of another.

Under fairly general hypotheses, one can prove that $y' = f(x,y)$ admits a one-parameter family of distinct solutions, depending on an arbitrary constant of integration C. Solutions: $y = y(x, C)$ with C varying on $I \in \mathbb{R}$. This expression is the general integral of the eq., while any solution associated to a particular choiche of C is called particular integral.

Example
$y' = y \to$ solutions = maps which are their first derivative. The exponential $y(x) = e^x$ enjoys this property.



Since differentiating is a linear operation, any function $y(x) = Ce^x$, $C \in \mathbb{R}$ has this feature. But there are no other maps doing the same, so we can conclude that the solutions to $y' = y$ belong to family
$y(x, C) = Ce^x$, $C \in \mathbb{R}$

To get hold of a particular integral, one should select a value of the constant of integration. A way to do so is to ask that the solution assume a specific value at a point x fixed in advance. We impose $y(x_0, C) = y_0$ (x_0, y_0 are given) corresponding to the geometric constraints that the integral

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Linear equations

A differential equation akin to $y'+a(x)y=b(x)$ with a,b continuous on I, is called **linear**, cause the function $f(x,y)=-a(x)y+b(x)$ is a linear polynomial in y with coefficients in the variable x. This eq. is said **homogeneous** if the source term vanishes, $b(x)=0$, **non-homogeneous** otherwise.

* We begin by solving the hom. case:
$y'=-a(x)y$ (sep. variables) $\longrightarrow g(x)=-a(x),\ h(y)=y$, $y(x)=0$ is a solution.
Excluding this, we can write:
$\int \frac{1}{y}\,dy=-\int a(x)\,dx$, if $A(x)=$ primitive of $a(x) \longrightarrow \int a(x)\,dx=A(x)+C,\ C\in\mathbb{R}$ then
$\log|y|=-A(x)-C$ or $|y(x)|=e^{-C}e^{-A(x)} \longrightarrow y(x)=\pm Ke^{-A(x)}$ with $K=e^{C}>0$

- $y(x)=0$ is included if $K=0$. The solutions are $y(x)=Ke^{-A(x)},\ K\in\mathbb{R}$

* Now consider the case where $b\neq 0$. We use the **method of variation of parameters** (search for solut. of the form $y(x)=K(x)e^{-A(x)}$, $K(x)=f$ of x unknown; it exists since $e^{-A(x)}>0$)
We obtain $K'(x)e^{-A(x)}+K(x)e^{-A(x)}(-a(x))+a(x)Ke^{-A(x)}=b(x)$ or $K'(x)=e^{A(x)}b(x)$
Calling $B(x)$ the primitive of $e^{A(x)}b(x)$
$\int e^{A(x)}b(x)\,dx=B(x)+C,\ C\in\mathbb{R} \longrightarrow K(x)=B(x)+C \longrightarrow$ solutions $y(x)=e^{-A(x)}(B(x)+C)$

* The integral is often found in the form
$y(x)=e^{-\int a(x)dx}\int e^{\int a(x)\,dx}b(x)\,dx$ (integration TWICE, in succession)

* If we are asked to solve the init. value problem
$\begin{cases} y'+a(x)y=b(x) & \text{on } I \\ y(x_0)=y_0 & x_0\in I,\ y_0\in\mathbb{R}\end{cases}$ we might want to choose, as primitive for $a(x)$, the one vanishing at $x_0 \longrightarrow A(x)=\int_{x_0}^{x}a(s)\,ds$. The same for $B(x)=\int_{x_0}^{x}e^{\int_{x_0}^{t}a(s)\,ds}b(t)\,dt$
Substituting, we obtain $y(x_0)=C$, the solution will satisfy $C=y_0$, namely
$y(x)=e^{-\int_{x_0}^{x}a(s)ds}\left(y_0+\int_{x_0}^{x}e^{\int_{x_0}^{t}a(s)ds}b(t)\,dt\right)$

Second order equations reducible to first order

Suppose an equation of 2nd order does not contain the variable y explicitly, that is $y''=f(y',x)$. Then the substitution $z=y'$ transforms it into a first order equation. $z'=f(z,x)$ in the unknown $z=z(x)$. If the latter has general solution $z(x;C_1)$, we can recover the integrals by solving $y'=z$, hence by finding the primitives of $z(x,C_1)$. This will generate a new constant of integration C_2. The general solution will have the form
$y(x;C_1,C_2)=\int z(x,C_1)\,dx=Z(x;C_1)+C_2$ where $Z(x;C_1)$ is a particular primitive of $z(x,C_1)$
[Example]

Linear second order equations with constant coefficients

A linear equation of order 2 with constant coefficients has the form $y''+ay'+by=g$ where a,b are real constants and $g=g(x)$ is a continuous map. We shall prove that the general integral can be computed without too big an effort in case $g=0$, hence when the equation is **homogeneous** and we'll show how to find the explicit solutions when g is a product of exponentials, algebraic polynomials, sine and cosine type func. or sum of these.
To study the previous equation, we let the map $y=y(x)$ be complex-valued, for convenience. The function $y:I\subseteq\mathbb{R}\to\mathbb{C}$ is (n times) differentiable if $y_r=\operatorname{Re}y:I\to\mathbb{R}$ and $y_i=\operatorname{Im}y:I\to\mathbb{R}$ are (n times) differentiable, in which case $y^{(n)}(x)=y_r^{(n)}(x)+iy_i^{(n)}(x)$.

* A special case of this situation goes as follows. Let $\lambda=\lambda_r+i\lambda_i\in\mathbb{C}$ be an arbitrary complex number. We consider the complex map of one real variable $x\mapsto e^{\lambda x}=e^{\lambda_r x}(\cos\lambda_i x+i\sin\lambda_i(x))$, then $\longrightarrow$ $d/dx\cdot e^{\lambda x}=\lambda e^{\lambda x}$ precisely as if λ were real. In fact:

$$\frac{d}{dx}e^{\lambda x}=\frac{d}{dx}(e^{\lambda_r x}\cos\lambda_i x)+i\frac{d}{dx}(e^{\lambda_r x}\sin\lambda_i x)=\lambda_r e^{\lambda_r x}\cos\lambda_i x-\lambda_i e^{\lambda_r x}\sin\lambda_i x+i(\lambda_r e^{\lambda_r x}\sin\lambda_i x+$$
$$+\lambda_i e^{\lambda_r x}\cos\lambda_i x)=\lambda_r e^{\lambda x}(\cos\lambda_i x+i\sin\lambda_i x)+i\lambda_i e^{\lambda_r x}(\cos\lambda_i x+i\lambda_i\sin\lambda_i x)=(\lambda_r+i\lambda_i)e^{\lambda x}=\lambda e^{\lambda x}$$

Let us indicate by $\mathcal{L}y=y''+ay'+by$ the left-hand side of the considered equation. Differentiating is a linear operation, so $\longrightarrow \mathcal{L}(\alpha y+\beta z)=\alpha\mathcal{L}y+\beta\mathcal{L}z$ for any $\alpha,\beta\in\mathbb{R}$ and any twice-diff. real functions $y=y(x)$, $z=z(x)$. Furthermore, the result holds also for $\alpha,\beta\in\mathbb{C}$ and $y=y(x)$, $z=z(x)$ complex-valued. This sort of linearity of the differential equation will be crucial in the study.
Now, let's begin with the homogeneous case $\mathcal{L}y=y''+ay'+by=0$ and denote by $\chi(\lambda)=\lambda^2+a\lambda+b$ the **characteristic polynomial** of the differential equation, obtained by replacing kth derivatives by the power λ^k for every $k\geq 0$. Looking for a solution of the form $y(x)=e^{\lambda x}$ for a suitable λ, we have

* $\mathcal{L}(e^{\lambda x})=\lambda^2e^{\lambda x}+a\lambda e^{\lambda x}+be^{\lambda x}=\chi(\lambda)e^{\lambda x}$ and the eq. holds if and only if λ is a root of the **characteristic equation** $\longrightarrow \lambda^2+a\lambda+b=0$

* When the discriminant $\Delta=a^2-4b$ is non-zero, there are two distinct roots $\lambda_1,\lambda_2 \longrightarrow$ correspond 2 distinct solutions $y_1(x)=e^{\lambda_1 x}$ and $y_2(x)=e^{\lambda_2 x}$; roots and relative solutions are real if $\Delta>0$, complex-conjugate if $\Delta<0$.

* When $\Delta=0$, there's a double root λ, hence one solution $y_1(x)=e^{\lambda x}$. Multiplicity two implies $\chi'(\lambda)=0$, setting $y_2(x)=xe^{\lambda x}$, we have $y_2'(x)=(1+\lambda x)e^{\lambda x}$, $y_2''(x)=(2\lambda+\lambda^2x)e^{\lambda x}$

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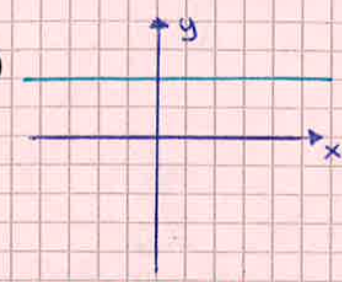
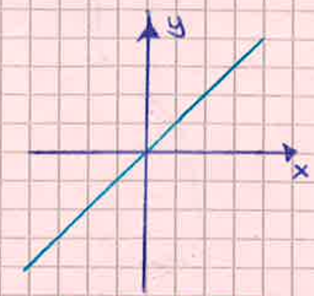
GRAPHS of the main FUNCTIONS

ALGEBRAIC FUNCTIONS



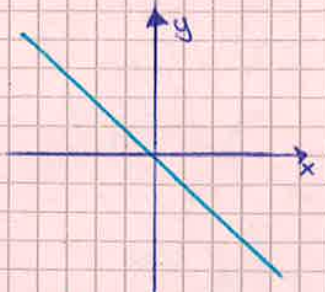
constant function
(straight line // to x-axis)

$y=c$ $A=\mathbb{R}$ $B=\{c\}$
$I=\emptyset$



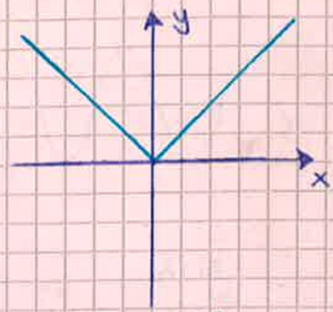
function of 1st degree
bisector I-III quarters

$y=x$ $A=\mathbb{R}$ $B=\mathbb{R}$
$I=\mathbb{R}$



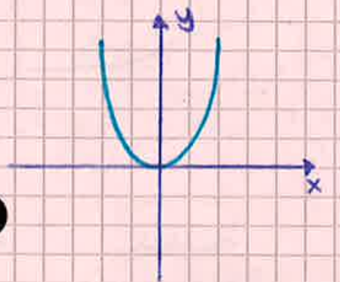
function of 1° degree
bisector II-IV quarters

$y=-x$ $A=\mathbb{R}$ $B=\mathbb{R}$
$I=\mathbb{R}$



function of 1st degree
(absolute value)

$y=|x|$ $A=\mathbb{R}$ $B=[0,+\infty)$
$I=[0,+\infty)$

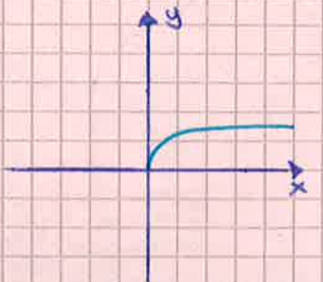


function of 2nd degree
parabola

$y=x^2$ $A=\mathbb{R}$ $B=[0,+\infty)$
$I=[0,+\infty)$

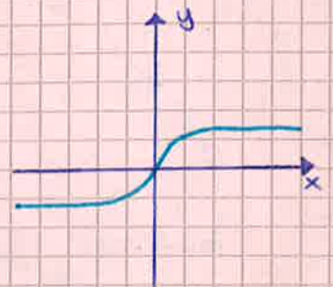
function of 3rd degree

$y=x^3$ $A=\mathbb{R}$ $B=\mathbb{R}$
$I=\mathbb{R}$



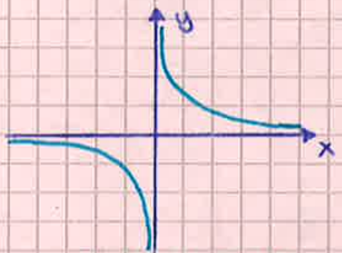
square root

$y=\sqrt{x}$ $A=[0,+\infty)$ $B=[0,+\infty)$
$I=[0,+\infty)$



cubic root

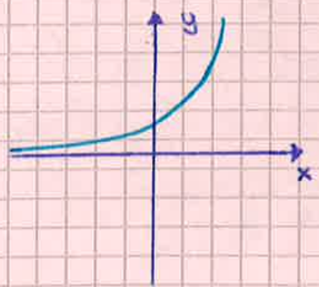
$y=\sqrt[3]{x}$ $A=\mathbb{R}$ $B=\mathbb{R}$
$I=\mathbb{R}$



equilateral hyperbole

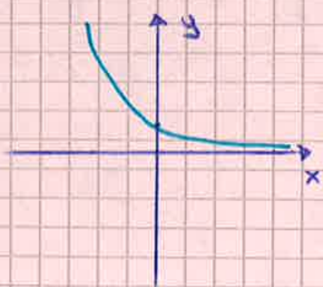
$y=\frac{1}{x}$ $A=\mathbb{R}-\{0\}$ $B=\mathbb{R}-\{0\}$
$I=\mathbb{R}-\{0\}$

EXPONENTIAL FUNCTIONS AND LOGARITHM



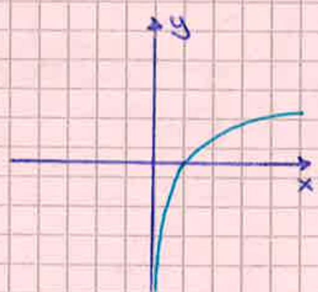
exp. function with $a>1$

$y=a^x$, $a>1$ $A=\mathbb{R}$
$B=(0,+\infty)$ $I=\mathbb{R}$



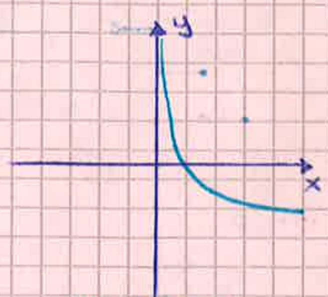
exp. function with $a<1$

$y=a^x$, $0<a<1$ $A=\mathbb{R}$
$B=(0,+\infty)$ $I=\mathbb{R}$



log. with $a>1$

$y=\log_a x$, $a>1$ $A=(0,+\infty)$
$B=\mathbb{R}$ $I=(0,+\infty)$



log. with $a<1$

$y=\log_a x$, $0<a<1$
$A=(0,+\infty)$ $B=\mathbb{R}$
$I=(0,+\infty)$

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FUNCTIONS

- SURJECTIVE (onto) → $im f = Y$, each $y \in Y$ is the image of one element $x \in X$ at least
- INJECTIVE (one to one) → every $y \in Y$ is the image of a unique element $x \in dom f$. → INVERTIBLE
- BIJECTIVE → bij. correspondence between X and Y: the two sets have the same no. of elements, each $x \in X$ is associated to a unique $y \in Y$ and viceversa.

* MONOTONICITY (STRICT) implies INJECTIVITY, but not vice versa.

- EVEN - simmetric w.r.t. the y-axis $f(x) = f(-x)$
- ODD - simmetric w.r.t. the origin $f(-x) = -f(x)$

POWERS

$y = x^a$

$a = n$:
- n odd - map odd, stric. increasing on $\mathbb{R}$, range $\mathbb{R}$
- n even - map even, stric. increasing on $[0,+\infty)$, str. decr. on $(-\infty,0]$, range $[0,+\infty)$

$a > 0$ rational $= 1/m \rightarrow x^{1/m} = \sqrt[m]{x}$:
- m odd dom. $\mathbb{R}$ range $\mathbb{R}$
- m even dom $[0,+\infty)$ range $[0,+\infty)$

stric. increasing

$a = \frac{n}{m} \rightarrow x^{n/m} = \sqrt[m]{x^n}$:
- m odd - dom $\mathbb{R}$
- m even - dom $[0,+\infty)$

stric. increasing on $[0,+\infty)$ for any n,m
stric. incr. or decr. on $(-\infty,0]$ for m odd according to the parity of n.

$a < 0 \rightarrow \frac{1}{x^{-a}}$ decreasing on $(0,+\infty)$

on $(-\infty,0) \rightarrow a = -n/m$ < m odd incr. if even, decreasing n odd

SEQUENCES

Monotone sequence: (increasing $= \forall n \geq n_0$, $a_n \leq a_{n+1}$)

1) If it's bounded from above, then it converges to the supremum ℓ of its image ($\lim_{x\to+\infty} a_n = \ell$)
2) If it's not bounded from above, it diverges to $+\infty$

* In case it's decreasing, the assertions modify in the obvious way.

- Thm - Existence of zeroes
- Thm - Intermediate value thm
- Thm - Weierstrass

LANDAU

$f = O(g)$ with $\lim_{x\to c} \frac{f(x)}{g(x)} = \ell$ (finite)

1) ℓ finite or non-zero → $f \asymp g$ → $f \sim \ell g$
- $\ell = 1$ → $f \sim g$
- $\ell = 0$ → $f = o(g)$

2) ℓ non finite → $\lim_{x\to c} \frac{g(x)}{f(x)} = \frac{1}{\ell} = 0$ $g = o(f)$

INFINITE f,g	INFINITESIMAL f,g
$f \asymp g$ → same order	$f \asymp g$ → same order
$f = o(g)$ → f smaller order	$f = o(g)$ → f bigger order than g
$g = o(f)$ → f bigger order	$g = o(f)$ → f smaller order than g
non comparable	non comparable

INFINITESIMAL (INFINITE)
TEST FUNCTION
used to measure the speed at which the map tends to 0 (or ∞)

- $c = x_0^+$ (x_0^-)
  infinitesimal $= x - x_0$ ($x_0 - x$)
  infinite $= 1/x-x_0$ ($1/x_0-x$)
- $c = \pm\infty$
  infinitesimal $= 1/x$ ($1/|x|$)
  infinite $= x$ ($|x|$)

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SUCCESSIONI

Una successione è una legge che associa ad ogni elemento di $\mathbb{N}$ un numero reale, cioè una funzione reale definita su $\mathbb{N}$:

$f: \mathbb{N} \to \mathbb{R}$ $\quad f(n) = a_n \quad n \mapsto a_n$

Si denota con $\{a_n\}_{n\in\mathbb{N}}$ $\quad \{a_n\} \quad a_n \quad n \mapsto a_n$

Spesso le successioni sono definite da un certo n_0 in poi, cioè il loro dominio è del tipo $\{n\in\mathbb{N} \mid n \geq n_0\}$
In tal caso si scrive

$\{a_n\}_{n\geq n_0}$

Successioni limitate

Una successione $\{a_n\}$ si dice:
- limitata inferiormente se esiste $m \in \mathbb{R}$ tale che, per ogni n, $a_n \geq m$
- limitata superiormente se esiste $M \in \mathbb{R}$ tale che, per ogni n, $a_n \leq M$
- limitata se esistono $m, M \in \mathbb{R}$ tali che, per ogni n, $m \leq a_n \leq M$

L'operazione di limite consente di studiare il comportamento dei numeri a_n quando n diventa sempre più grande.

Limiti di successioni

Una successione $\{a_n\}$ possiede definitivamente una proprietà se esiste $N \in \mathbb{N}$ tale che a_n soddisfa quella proprietà per ogni $n \geq N$

Successioni convergenti

Una successione si dice convergente se esiste un numero $\ell \in \mathbb{R}$ con questa proprietà: qualunque sia $\varepsilon > 0$, risulta definitivamente

$|a_n - \ell| \leq \varepsilon$, i.e., per ogni $\varepsilon > 0$ esiste $N \in \mathbb{N}$ tale che $|a_n - \ell| \leq \varepsilon \quad \forall n \geq N$

Limite di una successione

Quindi, se una successione è convergente ad essa è associato un numero ℓ. Si prova che ℓ È UNICO
Sia $\{a_n\}$ una successione convergente. Il numero reale ℓ che compare nella definizione precedente si chiama limite della successione $\{a_n\}$.
Si scrive $\lim_{n\to+\infty} a_n = \ell$ oppure $a_n \to \ell$ per $n \to +\infty$.

Si noti che, dalle proprietà del valore assoluto, la disuguaglianza $|a_n - \ell| \leq \varepsilon$ equivale a $\ell - \varepsilon \leq a_n \leq \ell + \varepsilon$
Dunque la definizione di convergenza significa che, fissata una striscia orizzontale $[\ell-\varepsilon, \ell+\varepsilon]$ "comunque stretta", da un certo indice in poi i punti della successione non escono più da quella striscia

Da questa osservazione risulta che OGNI SUCCESSIONE CONVERGENTE È LIMITATA

Successioni divergenti

Sia $\{a_n\}$ una successione
Si dice che $\{a_n\}$ diverge a $+\infty$ se per ogni $M > 0$ si ha $a_n \geq M$ definitivamente e si scrive $\lim_{n\to+\infty} a_n = +\infty$
Si dice che $\{a_n\}$ diverge a $-\infty$ se per ogni $M > 0$ si ha $a_n \leq -M$ definitivamente e si scrive
$\lim_{n\to+\infty} a_n = -\infty$

Ricorda che $\pm\infty$ non sono numeri; l'insieme dei numeri reali $\mathbb{R}$ con l'aggiunta di due elementi si indica con $\mathbb{R}^*$
L'operazione di limite ha completamente significato se ambientata in $\mathbb{R}^*$: il lim. di una successione, se esiste, è un elemento di $\mathbb{R}^*$.
Esistono successioni che non sono né convergenti né divergenti (x esempio $\{(-1)^n\}$) Tali successioni si dicono irregolari o indeterminate. Per esse l'operazione di limite non è definita.

Insiemi non limitati

È comodo adottare la convenzione usata per i limiti anche per il sup e inf di insiemi
Sia $E \subseteq \mathbb{R}$:
Se E non è limitato superiormente si dice che $\sup E = +\infty$
Se E non è limitato inferiormente si dice che $\inf E = -\infty$

Infinitesimi e infiniti

Una successione $\{a_n\}$ si dice infinitesima se $\lim_{n\to+\infty} a_n = 0$

Una successione si dice infinita se $\lim_{n\to+\infty} a_n = \pm\infty$

Gli infinitesimi (infiniti) non sono numeri ma quantità variabili che tendono a diventare infinitamente piccoli (grandi)

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Confronto tra infiniti

Se $\{a_n\}$ e $\{b_n\}$ sono due infiniti, si dice che

- $\{a_n\}$ è un infinito di ordine inferiore a $\{b_n\}$ se $\lim_{n\to+\infty} \frac{a_n}{b_n} = 0$
- $\{a_n\}$ e $\{b_n\}$ sono infiniti dello stesso ordine se $\lim_{n\to+\infty} \frac{a_n}{b_n} = \ell \in \mathbb{R}\setminus\{0\}$
- $\{a_n\}$ è un infinito di ordine superiore a $\{b_n\}$ se $\lim_{n\to+\infty} \frac{a_n}{b_n} = \pm\infty$
- $\{a_n\}$ e $\{b_n\}$ non sono confrontabili se il limite del loro rapporto non esiste.

Confronto tra infinitesimi

Se $\{a_n\}$ e $\{b_n\}$ sono due infinitesimi, si dice che

- $\{a_n\}$ è un infinitesimo di ordine superiore a $\{b_n\}$ se $\lim_{n\to+\infty} \frac{a_n}{b_n} = 0$
- $\{a_n\}$ e $\{b_n\}$ sono infinitesimi dello stesso ordine se $\lim_{n\to+\infty} a_n/b_n = \ell \in \mathbb{R}\setminus\{0\}$
- $\{a_n\}$ è un infinitesimo di ordine inferiore a $\{b_n\}$ se $\lim_{n\to\pm\infty} \frac{a_n}{b_n} = \pm\infty$
- $\{a_n\}$ e $\{b_n\}$ non sono confrontabili se il limite del loro rapporto non esiste

Successioni asintotiche

Siano $\{a_n\}$ e $\{b_n\}$ due successioni. Se $\lim_{n\to+\infty} \frac{a_n}{b_n} = 1$, si dice che le due succ. sono asintotiche ($a_n \sim b_n$)

Proprietà delle successioni asintotiche

- Se $a_n \sim b_n$, allora $\{a_n\}$ e $\{b_n\}$ hanno lo stesso comportamento: o convergono allo stesso limite o divergono o entrambe non hanno limite.
- Se $a_n \sim b_n \sim \dots \sim c_n$, allora $a_n \sim c_n$
- Se $a_n \sim a'_n$, $b_n \sim b'_n$, $c_n \sim c'_n$, allora $\frac{a_n b_n}{c_n} \sim \frac{a'_n b'_n}{c'_n}$

Inoltre $a_n \sim b_n \iff a_n = b_n \cdot c_n$ con $c_n \to 1$

* Esempio di successioni che non sono asintotiche a $\{n^\alpha\}$ per nessun $\alpha > 0$:
Per ogni $a > 1$, $\alpha > 0$ si ha

$$\lim_{n\to+\infty} \frac{\log_a n}{n^\alpha} = 0 \qquad \lim_{n\to+\infty} \frac{n^\alpha}{a^n} = 0$$

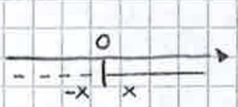
Questi limiti descrivono la velocità con cui i logaritmi (base >1), le potenze, gli esponenziali (base >1) vanno all'∞:

- i logaritmi più lentamente di qualsiasi potenza.
- le potenze più lentamente di qualsiasi esponenziale.

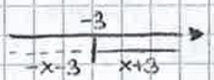
EQUAZIONI CON PIÚ VALORI ASSOLUTI

Es. $|x| - 2|x+3| = 0$

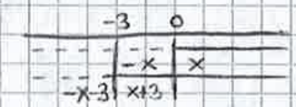
- $x \geq 0 \longrightarrow x$
  $x < 0 \longrightarrow -x$



- $x+3 \geq 0$, $x \geq -3 \longrightarrow x+3$
  $x+3 < 0$, $x < -3 \longrightarrow -x-3$



$\Longrightarrow$



TRE FORME

$$\begin{cases} -x-2(-x-3) & x<-3 \\ -x-2(x+3) & -3 \leq x < 0 \\ x-2(x+3) & x \geq 0 \end{cases}$$

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✓ 11. Consider the Cauchy problems:

$$1)\ \begin{cases} xy' = y \\ y(2) = 2 \end{cases} \qquad 2)\ \begin{cases} xy' = y \\ y(1) = 0 \end{cases}$$

(a) $y = 0$ is the solution of problem 1)

(b) problem 1) admits constant solutions

X (c) problem 2) has one and only one solution, which is $y(x) = 0$

(d) $y = x$ is a solution of problem 2)

(e) the solution of problem 1) is $y = Cx,\ C \in \mathbb{R}$

✓ 12. Consider the equation $xy' = y$. Then one can say that

(a) the equation has no constant solutions

(b) $y = Cx,\ C \in \mathbb{R}\backslash\{0\}$ is the general integral

X (c) $y = 0$ is a particular integral

(d) $y = -x + 2$ is a particular integral

(e) $y = x - 1$ is a particular integral

13. Consider the equation $y'' + 9y = \cos 2x$. Then

(a) The general integral of the homogeneous associated equation is $y = C_1 e^{3x} + C_2 e^{-3x} + \frac{1}{5}\cos 2x$

(b) The general integral of the homogeneous associated equation is $y = C_1 \cos 2x + C_2 \sin 2x$

(c) The general integral of the homogeneous associated equation is $y = C_1 + C_2 e^{-9x}$

$\lambda^2 + 9 = 0 \rightarrow \lambda = \pm 3i$

$\sigma = 0 \quad \omega = 3$

(d) A particular integral of the complete equation is $y_p(x) = \frac{1}{5}\sin 2x$

X (e) The general integral of the complete equation is $y = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{5}\cos 2x$

✓ 14. Consider $y'' + 9y = \cos 3x$. Then

(a) The forcing term is not resonant

X (b) A particular integral of the complete equation is of the form $y_p(x) = x(\alpha \cos 3x + \beta \sin 3x)$

(c) The general integral of the complete equation is $y(x) = \alpha \cos 3x$

(d) The solutions are periodic

(e) The general integral of the complete equation is $y(x) = x(C_1 \cos 3x + C_2 \sin 3x)$

✓ 15. Consider the equation $x'' - 2x' + 5x = g(t)$.

(a) The general integral of the homogeneous associated equation is $x_o(t) = C_1 \cos 2t + C_2 \sin 2t$

X (b) If $g(t) = e^t \cos 2t$ then the particular integrals are of the form $x_p(t) = t\,e^t(a \cos 2t + b \sin 2t)$

(c) If $g(t) = \cos 2t$ then the particular integrals are of the form $x_p(t) = t(a \cos 2t + b \sin 2t)$

(d) If $g(t) = (2t - 1)e^t$ then a particular integral is of the form $x_p(t) = a(2t - 1)e^t$

(e) If $g(t) = e^t \sin 3t$ then the forcing term is resonant

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11. The McLaurin polynomial of order 6 of the function $f(x) = e^{\cos x^3}$ is

(a) $1 + \frac{1}{2}x^6$
(b) $e - \frac{e}{2}x^6$
(c) $2 + \frac{1}{2}x^6$
(d) $1 - \frac{e}{2}x^6$
(e) $1 + \frac{1}{6}x^6$

12. The Taylor expansion of $f(x)$ for $x \to 2$ is $f(x) = 4 - 3(x-2)^6 + o((x-2)^6)$. Then

(a) $x = 2$ is an inflection point for f
(b) $x = 0$ is a maximum point for f
(c) $x = 0$ is a minimum point for f
(d) $x = 2$ is a minimum point for f
(e) $x = 2$ is a maximum point for f

13. The function $f : [0,3] \to \mathbf{R}$ is continuous and decreasing. Then we can say that

(a) $f([0,3]) = [f(0), f(3)]$
(b) $f((0,3)) = (f(3), f(0))$
(c) $f((0,3]) = (f(3), f(0)]$
(d) $f([0,3]) = [f(3), f(0)]$
(e) all other statements are false

14. The function $f : \mathbf{R} \to \mathbf{R}$ is differentiable and $f(0) = f(1) = 0$. Setting $g(x) = f(x)^4$, then we have that

(a) the derivative $g'(x)$ has at least three zeroes
(b) the derivative $g'(x)$ has two (and only two) zeroes
(c) the derivative $g'(x)$ has three (and only three) zeroes
(d) the derivative $g'(x)$ han no zeroes
(e) the derivative $g'(x)$ has at least four zeroes

15. The principal part (with respect to $\varphi(x) = x$) for $x \to 0^+$ of $f(x) = 3x + \sqrt{4x^2 + 2x^3}$ is

(a) $5x$
(b) $x^{3/2}$
(c) $\sqrt{2}x^{3/2}$
(d) $3x$
(e) $3x + o(x)$

16. A primitive of the function $f(x) = \dfrac{3x}{2x^2+2}$ is

(a) $\frac{3}{4}\log(x^2+1)$
(b) $\frac{3}{4}\arctan x$
(c) $3\log(x^2+1)$
(d) $\frac{3}{4}\arctan(x^2+1)$
(e) all other statements are false

17. One of the following statements is true. Which one?

(a) if f is integrable on $[a,b]$, then $\exists c \in [a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$
(b) if f continuous in $[a,b]$, then $\exists c \in [a,b]$ such that $f'(c) = \frac{1}{b-a}\int_a^b f(x)dx$
(c) if f continuous in $[a,b]$, then $\exists c \in [a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$
(d) if f continuous in $[a,b]$, then $\exists c \in [a,b]$ such that $f(c) = \int_a^b f(x)dx$
(e) if f is integrable on $[a,b]$, then $\exists c \in [a,b]$ tsuch that $f(c) = \int_a^b f(x)dx$

18. If $F(x) = \int_0^x t^2\cosh(t^2)\,dx$ then

(a) F is increasing on $(0,+\infty)$ and decreasing on $(-\infty,0)$
(b) F is increasing on $\mathbf{R}$
(c) F has a minimum in 0
(d) F has a maximum in 0
(e) all other statements are false

19. The function f is continuous on $[0,+\infty)$ and $f(x) \leq 0$ for all $x \geq 0$. The the improper integral
$$\int_0^{+\infty} f(x)\,dx$$

(a) is indeterminate
(b) diverges to $-\infty$
(c) converges to a negative number
(d) is either convergent or divergent to $-\infty$
(e) all other statements are false

20. The differential equation $y'' - y = 0$

(a) has at least one solution which is unbounded on $(0,+\infty)$
(b) has no solution bounded on $(0,+\infty)$
(c) has no solution unbounded on $(0,+\infty)$
(d) has at least one solution that changes it sign infinitely many times
(e) has only positive solutions

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6) Set $f(x) = \log(x-2-\sqrt{x^2+1})$ Then:

a) $dom(f) = \mathbb{R}_+$

b) $dom(f) = (2, +\infty)$

X c) f is never defined

d) $dom(f) = \mathbb{R}$

e) $dom(f) = (-\infty, 3/4)$

7) Suppose that $A \subseteq \mathbb{R}$, $\sup A = 2$ and $\inf A = 0$. Then

a) $2 \in A$

b) there exists $x \in A$ such that $0<x<2$

X c) there exists $x \in A$ such that $x > 1$

d) $A = (0,2)$

e) either $0 \in A$ or $2 \in A$

8) Set $A = \{x = 3 + \frac{3}{n}, n \in \mathbb{N} \setminus \{0\}\} \cap \{x \in \mathbb{R} : x > 2\}$. Then

a) $\sup(A) = +\infty$

X b) A has maximum

c) A has minimum

d) $A = \{x \in \mathbb{R} : x > 2\}$

e) A is not bounded

9) Take the set $A = \{x = 3 - \frac{3}{n}, n \in \mathbb{N} \setminus \{0\}\} \cap \{x \in \mathbb{R}: 0<x<4\}$. Then

X a) A is bounded

b) A has maximum

c) A has minimum

d) $\sup(A) = 4$

e) $\sup(A) \neq 3$

10) Consider the set $K = \{x \in \mathbb{R} : |6-7x| \leq 5\}$. Then:

X a) $K = [1/7, 11/7]$

b) $K = (\frac{1}{7}, \frac{11}{7})$

c) $K = (-\infty, 0) \cup (\frac{1}{7}, \frac{11}{7})$

d) $K = (1/7, +\infty)$

e) $K = \begin{cases} 6-7x \leq 5 & \text{if } x \geq 0 \\ 7x-6 \leq 5 & \text{if } x < 0 \end{cases}$

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6) Set $f(x) = x^2$ and $g(x) = \sqrt{x}$, then

× a) $g(f(x)) = |x|$, $\forall x \in \mathbb{R}$

b) $f(g(x)) = x$

c) $g(f(x)) = f(g(x))$, $\forall x \in \mathbb{R}$

d) $f(g(-1)) = -1$

e) $g(f(x)) = x$, $\forall x \in \mathbb{R}$

7) Take $f(x) = \sin x$, $g(x) = H(x)$; then

a) $(f \circ g)(x)$ is periodic of period 2π

× b) $(g \circ f)(x)$ is periodic of period 2π

c) $Im(f \circ g) = [-1,1]$

d) $Dom(f \circ g) = (0, +\infty)$

e) $Im(f \circ g) = [0,1]$

8) Take $f(x) = H(x)$ and $g(x) = \cos x$, then

a) $Im(f \circ g) = [0, \cos 1]$

× b) $Im(g \circ f) = (\cos 1, 1]$

c) $g \circ f$ is periodic of period 2π

d) $f \circ g$ is periodic of period 1

e) $Im(f \circ g) = [\cos 1, 0]$

9) Given the functions $f(x) = \sin x$, $g(x) = [x]$, $h(x) = sign(x)$, then

a) $h \circ f$ is not periodic

× b) $h \circ g \circ f$ and $g \circ f$ coincide $\forall x \in \mathbb{R}$

c) $h \circ g \circ f$ is not periodic

d) $(h \circ g \circ f)(\pi) = -1$

e) $Im(h \circ g \circ f) = \{0, -1\}$

10) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -\frac{1}{x}$, then

a) f is onto

b) $f(-2) < f(-1)$ and $f(1) < f(2) \Rightarrow f$ is strictly increasing in $\mathbb{R}$

c) since f is one to one then it's strictly monotone

d) f is monotone

× e) f is one to one and not monotone in its domain

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Exercise - 4

TEST

1) The limit $\lim_{x\to+\infty} \frac{4x-1}{\sin x} \cdot \sin\frac{1}{x}$ is

a) 4

b) $+\infty$

X c) $\nexists$

d) 0

e) 1

2) $f(x) = (1+\sin x)\,x$, then

X a) f has infinite zeroes

b) $\lim_{x\to-\infty} f(x) = -\infty$

c) $\operatorname{im}(f) = (0,+\infty)$

d) $\lim_{x\to+\infty} f(x) = 0$

e) $\lim_{x\to+\infty} f(x)$ exists

3) If $f: \mathbb{R} \to \mathbb{R}$ is s.t. $\forall \varepsilon > 0\ \exists k > 0 : \forall x \in (7-k, 7) \cap \operatorname{dom} f,\ f(x) < -\varepsilon$, then:

a) $\lim_{x\to 7} f(x) = 0$

b) $\lim_{x\to 7^-} f(x) = 0$

X c) $\lim_{x\to 7^-} f(x) = -\infty$

d) $\lim_{x\to+\infty} f(x) = 0$

e) $\lim_{x\to 7} f(x) = -\infty$

4) $\lim_{x\to 0} \left(x \sin\frac{\pi}{x} + x^2 \cos\frac{1}{x} + e^{x^2} \right)$ is

X a) 1

b) 0

c) $\nexists$

d) $+\infty$

e) -1

5) $\lim_{x\to 0} \frac{1}{\sin(\pi \cos x)}$ is

X a) $+\infty$

b) $-\infty$

c) $\nexists$

d) 0

e) 1

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Exercise – 5

TEST

1) The domain of the function $f(x) = \log(-x^2 - 6x - 5)$ is

a) $\emptyset$

b) $(-\infty, -5) \cup (-1, +\infty)$

X c) $(-5, -1)$

d) $(0, +\infty)$

e) $(1, 5)$

2) Given the function $f(x) = \log(x^2 + x + 1)$ which of the following statements is FALSE?

a) $\text{dom} f = \mathbb{R}$

X b) $\text{dom} f = (0, +\infty)$

c) $f(x) = 0 \iff x = -1 \text{ e } x = 0$

d) $f(x) > 0 \quad \forall x \in (-\infty, -1) \cup (0, +\infty)$

e) f is not injective

3) Given $A = \left\{ a_n = (-1)^n \, 7 \overbrace{\frac{2^n}{1+2^n}}^{\substack{1 \\ \shortparallel}}, n \in \mathbb{N} \right\}$

for $x \to \infty$, so the function tends to 7, the other term is changing sign ($-$ if n = odd, $+$ if n = even)

$(-1)^n \, 7 < \frac{2^n}{1+2^n} < 7$

a) $\min A = -7$; $\max A = 7$

b) $\min A = -\frac{14}{3}$, $\sup A = 7$

X c) $\inf A = -7$, $\sup A = 7$ they are not max or min cause you never arrive to these numbers

d) $\min A = -14/3$, $\max A = 7$

e) a_n converges to 7

4) $\lim_{n \to +\infty} \frac{7n^2 + n}{n + \sin(n!)} \cdot \frac{n^n \, n!}{(n+2)^n (n+1)!}$

a) 7

b) $2e^{-1}$

c) $+\infty$

X d) $7e^{-2}$

e) $7e^2$

5) $\lim_{x \to 0^+} x^2 \log(x + x^2)$

a) $+\infty$

b) 1

X c) 0

d) does not exist

e) 2

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11) For $x \to 3$ the function $f(x) = 1 - \cos(x-3)^2$

a) has the same order of infinitesimal as $(x-3)^2$

X b) has order of infinitesimal bigger than $(x-3)^2$

c) has order of infinitesimal smaller than $(x-3)^2$

d) has order of infinite bigger than $(x-3)^2$

e) has order of infinitesimal not comparable with $(x-3)^2$

12) Let $f(x) = e^{5x^2} - 1$. Then, for $x \to 0$

X a) $f(x) = o(x)$

b) $f(x) = o(x^2)$

c) $f(x) = o(x^3)$

d) $f(x) = o(x^4)$

e) f is not comparable with x

13) Let f be a strictly increasing function which admits $y = 10$ as a horizontal asymptote, then

a) f has one zero

b) f has more than one zero

c) if $f(-2) > 0$, then f has no zeroes

X d) f has at most one zero

e) none of the previous answers

14) Which is the right statement?

a) $\pi^x \sim e^x$ for $x \to +\infty$

b) $\pi^x = o(e^x)$ for $x \to +\infty$

X c) $e^x = o(\pi^x)$ for $x \to +\infty$ $\quad \pi > e \quad \pi^x / e^x$ no. > 1

d) $\pi^x \asymp e^x$ for $x \to +\infty$

e) $\pi^{-x} = o(e^x)$ for $x \to -\infty$

15) Let $f(x)$ and $g(x)$ be two functions defined in a neighbourhood of x_0, where $f \sim g$, then:

a) $e^{f(x)} \sim e^{g(x)}$ for $x \to x_0$

b) $e^{f(x)} \sim o(e^{g(x)})$, for $x \to x_0$

c) $\sqrt{f(x)} \sim \sqrt{g(x)}$, for $x \to x_0$

X d) $\sqrt[3]{f(x)} \sim \sqrt[3]{g(x)}$ for $x \to x_0$

e) if $f(x) = x^2$ and $g(x) = x^2 + \sin x$, then $5^{f(x)} \sim 5^{g(x)}$ for $x \to +\infty$

16) Which is the false statement?

a) $e^{2\sin x} \sim e^{\tan x}$ $x \to 0$

b) $e^{x\sin x} \sim e^{1-\cos x}$ $x \to 0$

X c) $e^{\sin x + 1} \sim e^{\tan x}$ $x \to 0$

d) $\ln(1+\tan^2 x) = o(\sin x)$, $x \to 0$

e) $-x \sim (\sqrt{e^{\sin^2 x} - 1})$ $x \to 0^-$

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4) The derivative of $f(x) = e^{\sin\frac{1}{(x-1)^2}}$ is

X a) $f'(x) = \frac{2}{(1-x)^3} \cos \frac{1}{(x-1)^2} \, e^{\sin\frac{1}{(x-1)^2}}$

b) $f'(x) = 2(x-1)\, e^{\sin 1/(x-1)^2} \cos(x-1)^2$

c) $f'(x) = 2e^x \cos x (x-1)$

d) $f'(x) = -2(x-1)^3 e^{\sin 1/(x-1)^2} \cos 1/(x-1)^2$

e) $f'(x) = e^{\sin\frac{1}{(x-1)^2}} + \cos 1/(x-1)^2 - 2\, 1/(x-1)^3$

5) The function $f(x) = \begin{cases} 4a\arctan x & x<1 \\ 2ax+b & x \geq 1 \end{cases}$ is diff. in $\mathbb{R}$ for

a) $a=\pi$, $b=2$

b) $a=1$, $b=2-\pi$

c) $a=1$, $\forall b$

X d) $a=1$, $b=\pi-2$

e) $a=\pi$, $b=2+\pi$

6) Let $f(x) = e^x |x-2\pi| \sin x$. Which the false statement?

a) $f(x)$ is diff. at $x=0$

b) $\nexists \lim_{x\to+\infty} f(x)$

c) $f(x)$ is continuous

d) $\lim_{x\to-\infty} f(x) = 0$

X e) $f(x)$ is not differentiable at $x=2\pi$

7) Suppose that $f: [-2,3] \subseteq \mathbb{R} \to \mathbb{R}$, is continuous and such that $f(-2)=-3$, $f(3)=6$. Which of the following statements is not necessarily true?

a) $f([-2,3])$ is a closed and bounded interval

X b) $f([-2,3]) = [-3,6]$

c) the equation $f(x)=\lambda$ has at least one solution if $-1 \leq \lambda \leq 4$

d) $f(x)$ has at least one zero in $(-2,3)$

e) there exist at least one point $x \in (-2,3)$ s.t. $f(x) = e+\pi$

8) The function $f(x) = \frac{1}{|x^2-9|}$

a) Is differentiable in $[0, 3\sqrt{2}]$

b) is not differentiable at three points

X c) has first derivative which vanishes at one point in $[-1,1]$

d) satisfies the hypothesis of Rolle Thm in $[-3,3]$

e) does not satisfy the hypothesis of Lagrange theorem in $[0,2]$

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14) Given $f(x) = \sqrt[3]{(x-2)^2}$, which is the false statement?

a) $f(x)$ does not satisfy the Hypothesis of the Rolle Theorem in $[0,4]$

b) f is not differentiable at $x=2$

c) f has a cusp at $x=2$

d) $f(x)$ satisfies the hypothesis of the Lagrange thm in $[2,4]$

X e) f has a point with vertical tangent at $x=2$

EXERCISES - 7

TEST

1) Which of the following functions coincide with $f(x) = x^{x^x}$?

a) $f(x) = x^{x^2}$

X b) $f(x) = x^{(x^x)}$

c) $e^{x^2 \log x} = f(x)$

d) $f(x) = x^{3x}$

e) $f(x) = x^{x^3}$

2) Which of the following properties is not satisfied by $f(x) = (x^x)^x$?

a) $f(x) = x^{x^2}$

X b) $\operatorname{im} f = (0, +\infty)$

c) $\operatorname{dom} f = (0, +\infty)$

d) $f(x) = e^{x^2 \log x}$

e) f has a right continuous prolongation at $x=0$

3) The derivative of $f(x) = (x^x)^x$ is

a) $f'(x) = x^2 x^{x^2-1}$

b) $f'(x) = 2x (x^x)^x \log x$

c) $f'(x) = 2(x^x)^x$

X d) $f'(x) = x^{x^2+1} (2\log x + 1)$

e) $f'(x) = x^{x^2} (2\log x + 1)$

4) The derivative of $f(x) = x^{x^x}$ is

a) $f'(x) = x^x x^{x^x - 1}$

b) $f'(x) = x^{x^x} x^{-x^x}$

c) $f'(x) = x^{x^x} (x^x \log x (\log x + 1) + x^x)$

d) $f'(x) = x^{x^x + x - 1} (\log x (\log x + 1) + 1)$

X e) $f'(x) = x^{x^x + x - 1} (x \log x (\log x + 1) + 1)$

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11) $\lim_{x \to +\infty} \frac{\log\left(\frac{3}{x^2} - \frac{4}{x^6}\right)}{\log 2x} =$

a) 1

b) $+\infty$

c) $\nexists$

d) $\log 2$

X e) -2

13) The sequence $a_n = (3\sin(n\pi) + \cos(n\pi))^{n^2-3n+5}$

X a) is bounded

b) is increasing

c) is decreasing

d) has finite limit for $n \to +\infty$

e) has infinite limit for $n \to +\infty$

14) Let $f: I = [-3,10] \to \mathbb{R}$ be a differentiable function on I and let $x_0 \in I$. Which is the correct statement?

a) if x_0 is a minimum point for f then $f'(x_0) = 0$

b) if x_0 is a maximum point for f then $f'(x_0) = 0$

X c) if $x_0 \in [-2,9]$ is a minimum point for f, then $f'(x_0) = 0$

d) if $f(x_0)$ is a maximum for f then $f'(x_0) = 0$

e) if x_0 is a maximum point for f then $f'(x_0) = 0$, $f''(x_0) < 0$

15) Given the function $f: [-2,3) \to \mathbb{R}$, $f(x) = |x^2 - 1|$, which is the false statement?

a) f admits a supremum, but not a global maximum

b) f has global ~~maximum~~ minimum

c) $x = 1$ is a global minimum point

d) 0 is the global minimum of f

X e) 8 is the global maximum of f

16) Let $f(x) = \arcsin x$. Which is the false statement?

a) $x = 1$ is a global maximum point for f

X b) $f'(1) = 0$

c) $\pi/2$ is the maximum of f

d) $\exists f^{-1}(3/2)$

e) $f(-e/3) = -f(e/3)$

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EXERCISES - 8

TEST

1) Consider the sequences $a_n = \frac{(n+2)! - n!}{(n+1)!}$ and $b_n = \frac{(n+3)! - n!}{(n+1)!}$

a) $a_n \sim b_n$, per $n \to +\infty$

b) a_n and b_n are both convergent

X c) $a_n \sim n$ and $b_n \sim n^2$, for $n \to +\infty$

d) $a_n \asymp b_n$, for $n \to +\infty$

e) $\lim_{n \to +\infty} a_n = n+2$

2) Set $f(x) = 2x + \log x$; the tangent line to the graph of the inverse function f^{-1}, at the point $x_0 = f^{-1}(2)$, is:

a) $y-2 = \frac{5}{2}(x-2)$

b) $y-1 = \frac{1}{3}(x-2)$

X c) $y = 3(x-1)+2$

d) $y = \frac{1}{3}(x+1)$

e) $y = \frac{1}{3}(x-2)$

3) The derivative of the function $g(x) = \log^2(f^3(x)+2)$ is

a) $g'(x) = 2\log(f^3(x)+2)\ \frac{f'(x)}{f^3(x)+2}$

b) $g'(x) = 6\log x \cdot f^2(x) \cdot f'(x) + 2$

c) $g'(x) = 2\log(f^3(x)+2) + \log^2(f^3(x)+2)\,3f^2(x)$

d) $g'(x) = 2\log x\,(f^3(x)+2) + 3f^2(x)\log(f^3(x)+2)$

X e) $g'(x) = 2\log(f^3(x)+2) \cdot \frac{3f^2(x) \cdot f'(x)}{f^3(x)+2}$

4) Take $g(x) = \log^2(f^3(x)+2)$ and suppose that $f(1) = 2$ and $f'(1) = 5$. Then $g'(1)$ is

a) $3/50$

X b) $12\log 10$

c) $1/10$

d) $3\log 2$

e) $12/5$

5) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous and differentiable on $[-2,3]$
suppose that $f(-2) = -1$, $f(3) = 4$, $f'(-1) = 3$, $f'(-2) = 2$. Which is the false statement?

a) If $f(x)$ is strictly increasing then the tangent line to the graph of the inverse function $f^{-1}(x)$ at its point $x_0 = -1$ is $x+1 = 2(y+2)$

b) If $f(x)$ is strictly increasing on $[-2,3]$, then $f([-2,3]) = [-1,4]$

c) There exists x_1 in $(-2,3)$ such that $f(x_1) = 0$

d) There exists $x_0 \in (-2,3)$ such that $f'(x_0) = 1$

X e) If f is injective and f^{-1} is its inverse, then $(f^{-1})'(-1) = 3$

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12) If the MacLaurin expansion of a function $f \in C^{(\infty)}(\mathbb{R})$ is $f(x) = -\frac{2}{3}x^2 + 3x^4 + o(x^4)$, which of the following statement is not true in general?

a) f is infinitesimal for $x \to 0$

X b) f is even

c) $x=0$ is a critical point for f

d) $f''(0) = -4/3$

e) f is negative in a neighbourhood of $x=0$

13) If the MacLaurin expansion of a function $f \in C^{(\infty)}(\mathbb{R})$ is $f(x) = 3 - 2x^2 + 5x^4 + o(x^4)$, which of the following statement is not true in general?

a) $f(0) = 3$

b) f is positive in a neighbourhood of $x=0$

c) $x=0$ is a local maximum for f

d) the tangent line to the graph of f at $A=(0,3)$ is $y=3$

X e) $f^{(4)}(0) = 5$

14) If the MacLaurin expansion of a function $f \in C^{(\infty)}(\mathbb{R})$ is $f(x) = -2x^3 + 5x^5 + o(x^6)$ which of the following statement is not true in general?

a) $f^{(6)}(0) = 0$

b) f changes sign in a neighbourhood of 0

c) $f^{(3)}(0) = -12$

X d) $x=0$ is a local maximum for f

e) $f^{(4)}(0) = 0$

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