NUMERO: 1624A - ANNO: 2015

A P P U N T I

STUDENTE: Calabrese

MATERIA: Analisi Matematica I + Eserc. Prof.Boteri

→ The set of all subsets of A is called the power set of A
The symbol is $P(A)$

$A = \{a; b; c\}$

Power sets : $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a; b\}$ ... $\{a; b; c\}$
8 subsets in this case

3 elements → 8 subsets $= 2^3$

so that means that if A has n elements, than P(A) has 2^n elements

The COMPLEMENTS

Given a subset $A \subseteq X$, the complement of A (in x) (WE WRITE $\overline{A}$ or $\complement A$ or $\complement_X A$) is the set of x elements that don't belong to A

$\complement_A \{x \in X : x \notin A\}$

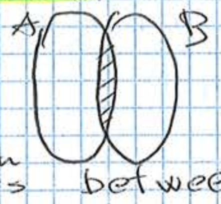
Properties
$\complement_X = \emptyset$ $\complement\emptyset = X$, $\complement(\complement A) = A$

INTERACTION BETWEEN SETS [A - B]

UNION SET: $A \cup B$

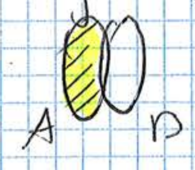
set of all x belonging to A or to B or both

INTERSECTION SET $A \cap B$



The common elements between A and B
when there is not any common element we are talking about DISJOINED SET : $A \cap B = \emptyset$

THE DIFFERENCE SET $A \setminus B$
is The set of elements ~~that~~ which belong to A but not to B



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THE SYMMETRIC DIFFERENCE $A \Delta B$

set of x elements that belong to A and not to B or "vice versa"

we can se the symmetric difference as an union without the intersection

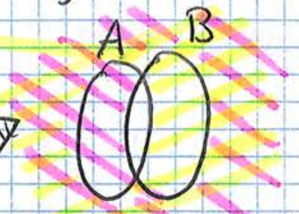
$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

other properties:

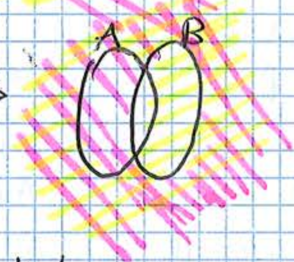
→ $\complement_X A = X \setminus A$

→ De Morgan Laws

$\complement(A \cup B) = (\complement A) \cap (\complement B)$



$\complement(A \cap B) = (\complement A) \cup (\complement B)$



Propositions:

statment could be true or false

$3 < 4$ true mathematical statment

$3 > 4$ false mathematical statment

The negation of a preposition has the following symble $[\neg P]$ not P

if $\neg P$ is true P is false, and false if P is true

P	$\neg P$
T	F
F	T

→ TRUTH TABL

The conjunction of P and Q is symbolised by the following $\wedge$ "and" ⇒ $P \wedge Q$

The conjunction of two preposition is true when both the formulas are true, false in all othe cases

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

De Morgan Laws

The morgan laws allow the expression of conjunctions and disjunction purely in terms of each other via negation

1) The negation of a conjunction is the disjunction of the negations
the negation of a disjunction is the conjunction of the negation

⇒ Not (and B) is the same as (¬) or (¬)
A ∨ B

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

⇒ ¬ (P or Q) is the same as (¬P) ∧ (¬Q)

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$$

(1) → se un elemento non appartiene alla "congiunzione" degli insiemi P e Q [¬(P∧Q)] allora non appartiene o a 'P' [¬P] o non appartiene a 'Q' [¬Q]

Allo stesso modo se P e Q sono affermazioni entrambe non vere se congiunte P vero se anche Q allora una tra P e Q deve essere falsa

(2) → Se un elemento non appartiene all'unione di P e Q [¬(P∨Q)] allora non appartiene né a P (¬P) né a (¬Q)

P	Q	P∪Q	¬(P∪Q)	¬P	¬Q	(¬P)∩(¬Q)
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

⇒ ¬(P ∩ Q) = (¬P) and (¬Q)

$$\complement(A \cup B) = (\complement A) \cap (\complement B)$$

starting from $\exists x, \forall y : p(x,y)$ we wanna see the negation

$$\neg(\exists x, \forall y : p(x,y)) = \forall x\ \exists y : \neg p(x,y)$$

For all people there is not any job that could be done by everybody.

"Ex falso quodlibet sequitur"

THE IMPLICATION

$P \Longrightarrow Q$ (if P then Q)

$P \Rightarrow$ hypothesys

$Q \Rightarrow$ consequence

If P than Q

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

not used just philosopher stuff

→ starting from a false assumption u can arrive to a false or tru conclusion

* $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$

$P \Longleftrightarrow Q$ P $\underbrace{\text{if and only if}}_{\text{iff}}$ Q

We can say that P is a necessary and sufficient condition for Q and vice versa

The simple implication $A \Rightarrow B$ is a deal of subset that means

$$\forall x : (x \in A) \Rightarrow (x \in B)$$

every element of A is exactly an element of B

Otherwise the iff implication $A \Leftrightarrow B$ means

$\forall x : (x \in A) \Leftrightarrow (x \in B)$ $A = B$ $A \subseteq B$ $B \subseteq A$

this implication is not true because it is true that an element belongs to A but it is not sure that an element of B belongs to A

OPERATION : is something that given an x and y belonging to a set, that $x \oplus y$ gives an other element z belonging to the same set

$$\forall x, y \in E : x \oplus y \in E$$

THE SUM

$x + y = y + x \Rightarrow$ COMMUTATIVE PROPERTY

$(x+y)+z = (y+z)+x \Rightarrow$ associative PROPERTY

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
commutative	✓	✓	✓	✓
associative	✓	✓	✓	✓
0 as neutral element	✓	✓	✓	✓
INVERSE	×	✓	✓	✓

$\mathbb{R}$ is the set of numbers that has the property
$\forall x \in \mathbb{R}$ $\exists$ one and only one corresponding point P on the line
in the same way is true the opposite pradicate
$\forall$ point P on the line $\exists$ one and only one $x \in \mathbb{R}$

$\Rightarrow$ FOR ALL POINT THERE IS A NUMBER $\in \mathbb{R}$ AND FOR ALL NUMBERS THERE IS A POINT ON THE LINE!

$\mathbb{R}$ →
- PROPERTIES $+, \cdot, -, \div$
- COMPLETE SET
- THERE IS AN ORDER RELATION IN $\mathbb{R}$

ORDER RELATION

$x, y \in \mathbb{R}$ → $x < y$, $x = y$, $x > y$

given two numbers x y fixing a value for x and than for y u can elways sey which is bigger, smaller or equal.

The order relation of $\mathbb{R}$ interacts with the algebric operations in the following way:

$\forall x, y, z \in \mathbb{R}$: $x \leq y \Leftarrow x + z \leq y + z$

$\forall x, y \in \mathbb{R}$, $\forall z \in \mathbb{R}$, $z > 0$ $x \cdot z \leq y \cdot z$

There are some important subsets of $\mathbb{R}$
we can store that into two groups:
BOUNDED INTERVALS and UNBOUNDED INTERVALS

BOUNDED INTERVALS
$[a; b]$ the set of $x \in \mathbb{R}$ such that $a \leq x \leq b$ $\Rightarrow$ CLOSED INTERVAL
$(a; b]$ the set of $x \in \mathbb{R}$ such that $a < x \leq b$ $\Rightarrow$ HALF OPEN (on the left) INTERVAL
$[a; b)$ the set of $x \in \mathbb{R}$ such that $a \leq x < b$ $\Rightarrow$ HALF OPEN (on the right) INTERVAL
$(a; b)$ The set of $x \in \mathbb{R}$ such that $a < x < b$ $\Rightarrow$ OPEN INTERVAL

$[a; b]$
$(a; b)$

THE UNBOUNDED INTERVALS OR HALFLINES
$\mathbb{R}$ The real line
$(-\infty; a]$ $S = \{\forall x \in \mathbb{R} : x \leq a\}$ unbounded closed interval
$(-\infty; a)$ $S = \{\forall x \in \mathbb{R} : x < a\}$ unbounded open interval
$[a; +\infty)$ $S = \{\forall x \in \mathbb{R} : x \geq a\}$ unbounded closed interval
$(a; +\infty)$ $S = \{\forall x \in \mathbb{R} : x > a\}$ unbounded open interval

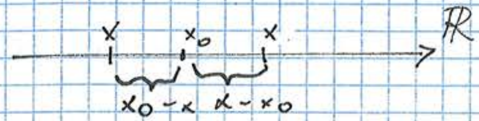
The INTERVAL: it is always a subset of $\mathbb{R}$. All intervals satisfy this property

$\forall x \in I, \forall y \in I, \forall z \in \mathbb{R} : x < z < y \Rightarrow z \in I$

given a point x_0 belonging to $\mathbb{R}$ we can measure the distance between the point x and x_0

x_0 is called **SYMMETRIC POINT**

$$|x - x_0| = \begin{cases} x - x_0 & x \geq x_0 \\ x_0 - x & x < x_0 \end{cases}$$



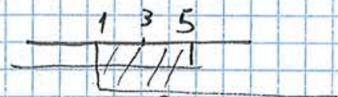
$\{x \in \mathbb{R} : |x - x_0| = a\} = \{x_0 - a ; x_0 + x\}$ => all the xs with the same distance from x

$\{x \in \mathbb{R} : |x - x_0| < a\} = (x_0 - a ; x_0 + x)$ => all the xs with distance smaller than a from the point α

Exp $S = (1 ; 5)$

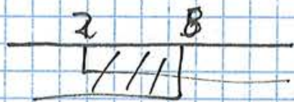
$$|x - \overset{x_0}{3}| \leq \overset{a}{2}$$

$$\begin{cases} x - 3 \leq 2 & x \leq 5 \\ -x + 3 \leq 2 & x > 1 \end{cases}$$



$S = (1 ; 5) = \{x \in \mathbb{R} : |x - 3| < 2\}$

$A = \{x \in \mathbb{R} : |x - 5| \leq 3 = (2 ; 8)$



$B = (-\infty ; 4)$

$A \cup B = (-\infty ; 8)$

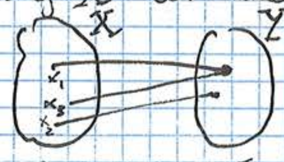
$A \cap B = (2 ; 4)$

$A \setminus B = \{1, 8, 7, 5, 6\}$

$A \Delta B = (-\infty ; 1) \cup \{8, 7, 5, 6\}$

WHAT IS A FUNCTIÒN

Given X and Y not empty sets, A function f defined on X with values in Y is an correspondence associationg to each elements of X at most one $y \in Y$



"at most one" means to each x my law can associate nothing or maximum one element

The set of $x \in X$ to witch f associates an element $\in Y$ is the "domain of f"

Function $f : \text{dom} f \subseteq X \to Y$

domain $\quad \text{dom} f = \{ x \in X : \exists y \in Y \; y = f(x) \}$

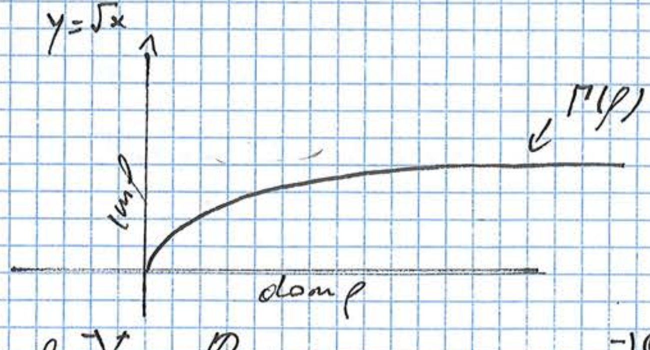
The element $y \in Y$ associated to an element $x \in \text{dom} f$ is called image of x under f and denoted $y = f(x)$
or
$f : x \to f(x)$ $\qquad \text{im} f = \{ y \in Y : \exists x \in X \; y = f(x) \}$

The set of images $y = f(x)$ of all points in the domain constitutes the RANGE OF f, a subset of Y indicated by $\text{im} f$

The graph of f is the subset $\Gamma(f)$ of the Cartesian product $X \times Y$ made of points $(x, f(x))$ when $x \in X$

$$\Gamma(f) = \{ (x, f(x)) \in X \times Y : x \in \text{dom} f \}$$

Ex



$\text{dom} f = X = \mathbb{R}_+$ $\qquad -10 \notin \text{dom} f \qquad 0 \in \text{dom} f$

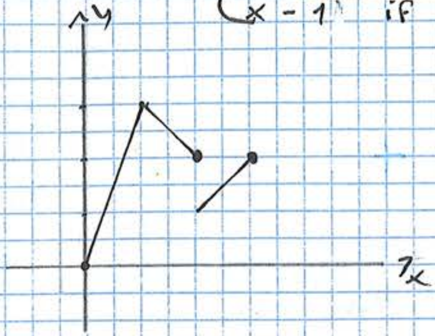
$-3 \notin \text{dom} f$

$\text{im} f = Y = \mathbb{R}_+$

$f : \text{dom} f \subseteq \mathbb{R}_+ \to \mathbb{R}_+$

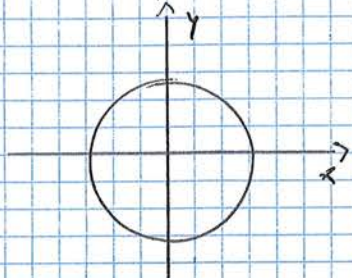
The Piecewise Function

$f: [0; 3] \to \mathbb{R}$, $\quad f(x) = \begin{cases} 3x & \text{if } 0 \le x \le 1 \\ 4 - x & \text{if } 1 < x \le 2 \\ x - 1 & \text{if } 2 < x \le 3 \end{cases}$



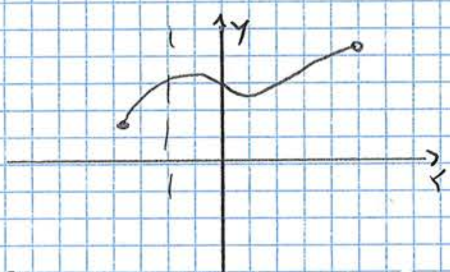
If f is a function $\Rightarrow \Gamma(f) \subseteq \mathbb{R}^2$

$A \subseteq \mathbb{R}$ this is not always the graph a function



Is the circle equation a function?

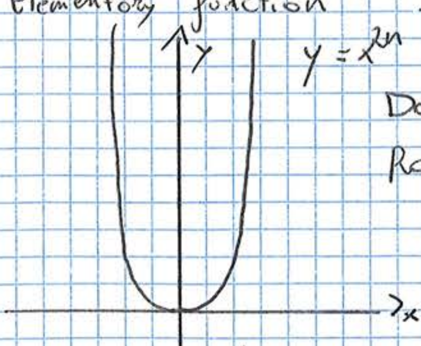
No it is not becouse to every element of the Domine between (-1 and 1) associates 2 element of $y \in Y$



this is a function

to every element of $x \in D$ associates one or noone element of $y \in Y$

Elementary function



$y = x^{2n} \quad n \in \mathbb{Z}$

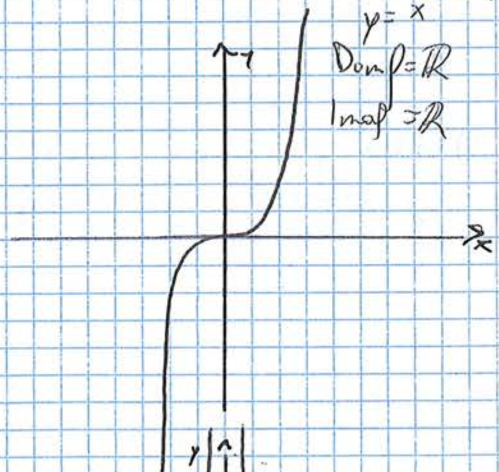
$\mathrm{Dom} f = \mathbb{R}$

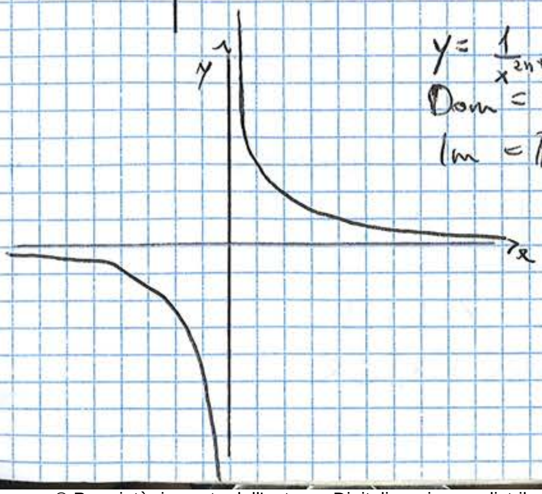
Range - $\mathrm{Imag} f = \mathbb{R}_+$

$y = x^{2n+1}$

$\mathrm{Dom} f = \mathbb{R}$

$\mathrm{Imag} f = \mathbb{R}$

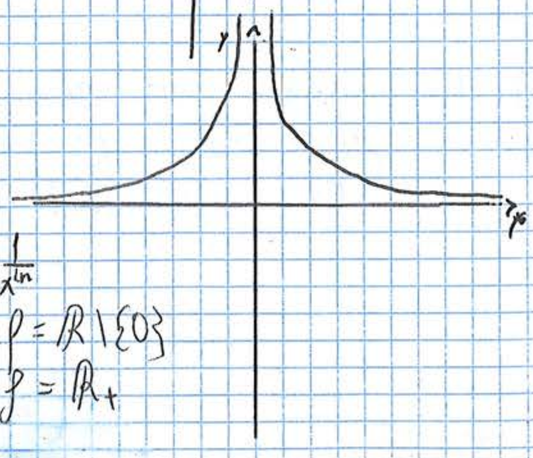




$y = \frac{1}{x^{2n+1}}$

$\mathrm{Dom} = \mathbb{R} \setminus \{0\}$

$\mathrm{Im} = \mathbb{R} \setminus \{0\}$



$y = \frac{1}{x^{2n}}$

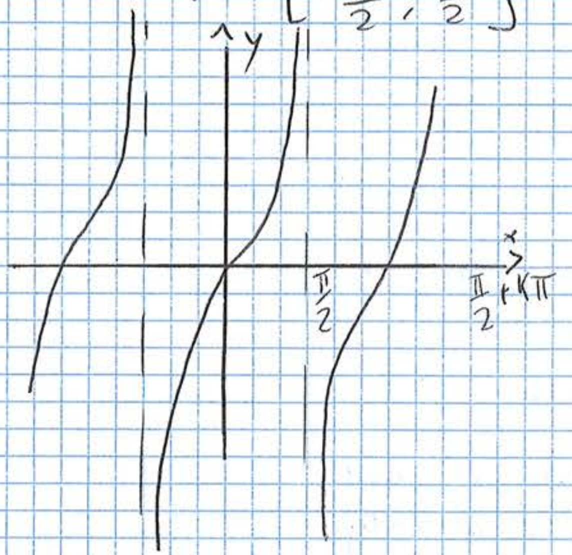
$\mathrm{dom} f = \mathbb{R} \setminus \{0\}$

$\mathrm{Im} f = \mathbb{R}_+$

$y = \operatorname{tg} x$ $\quad$ $\operatorname{Dom} f = \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi \right\}$

$\operatorname{Im} f = \mathbb{R}$

$T = \left[-\frac{\pi}{2} ; \frac{\pi}{2} \right]$

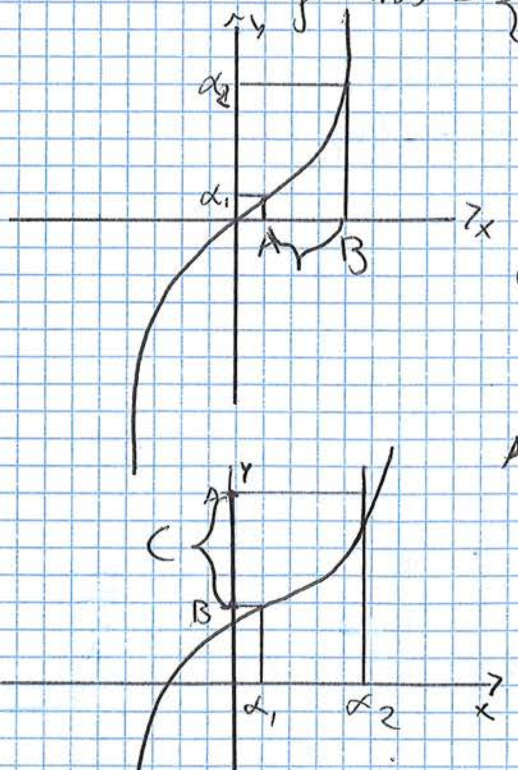


if $A \subseteq X$ the image of A under f is the set

$$f(A) = \{ f(x) : x \in A \}$$

$B \subseteq Y$ Pre image of B under f

$$f^{-1}(B) = \{ x \in \operatorname{dom} f : f(x) \in B)$$



SUBSET C

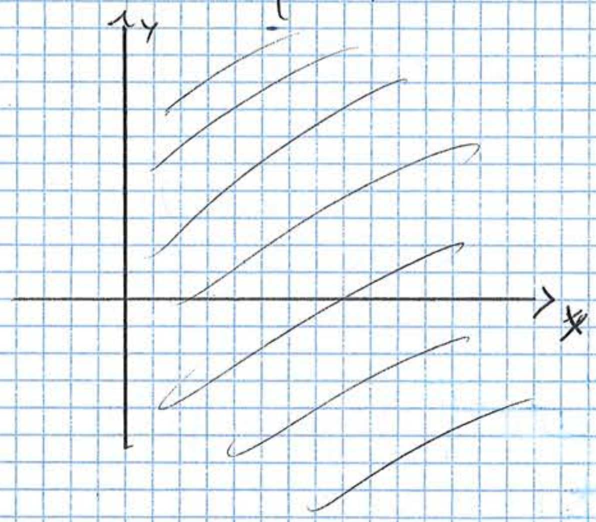
What is the image of the subset C

$$f(C) = \{ f(x) : x \in C \} = (\alpha_2 - \alpha_1]$$

$C \subseteq Y$ What's the pre image of C

$$f^{-1}(C) = \{ x \in D : f(x) \in C \}$$
$$= \{ \alpha_2 - \alpha_1 \}$$

$$A = \{(x,y) \in \mathbb{R}^2 : x > 0\}$$



Mathematical symbols

$\sum$ SUM SIGMA

$$\sum_{n=1}^{3} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{8} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

$$\prod_{n=1}^{3} \frac{1}{n^2} = 1 \cdot \frac{1}{2^2} \cdot \frac{1}{3^2}$$

$$\bigcup_{K=1}^{3} (K, K+1) = (1,2) \cup (2,3) \cup (3,4)$$

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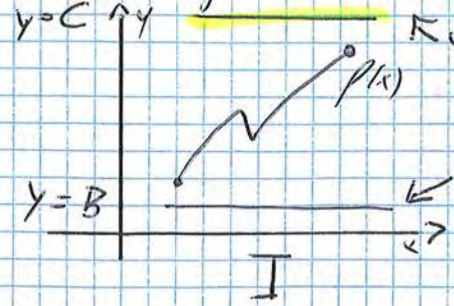
BOUND A FUNCTION

If I take a function defined in the I it can be bounded by belowe or from above



upper bound

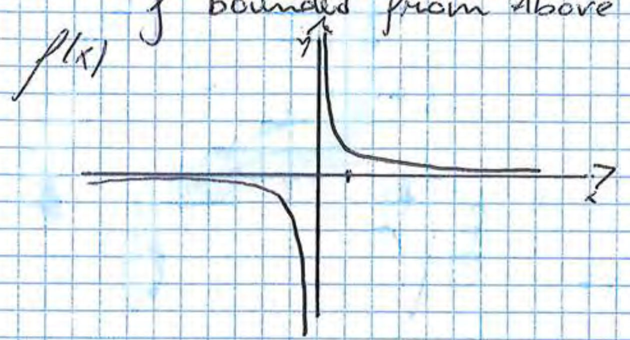
lower bound

if I take a costant function always bigger or smaller then $f(x)$ I can define the bonds of the function

C upper bound for f in $I \Rightarrow \forall x \in I \; f(x) \leq C$ bounded from above

C lower bound for f in $I \Rightarrow \forall x \in I \; f(x) \geq B$ bounded by below

f bounded from Above and by belowe $\Rightarrow f(x)$ is bounded



Dom = (0, 1] bounded by belowe not from above

Dom [1 +∞) bounded from above

Dom [-1, 1] not bounded

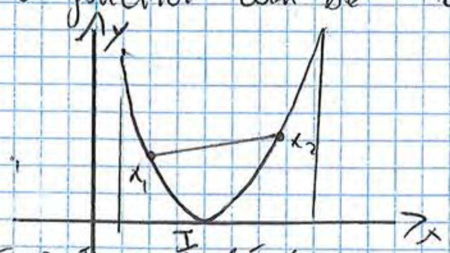
typicall examples of functions bounded by belowe and from above are $\sin x$ $\cos x$ / long

CONVEX AND CONCAVE

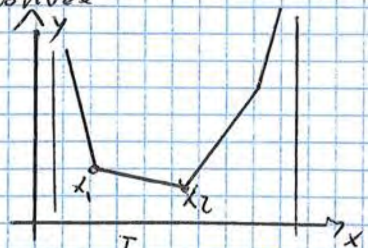
A function defined in an interval I, if the line segment between any two points taken on the graph of the function lies above the graph.

a^x an x^{2n} are typicall examples of convex functions

We can descrive a strictly convex function when there are not two points which describe a line segment conciding with the graph. Instead a function can be only convex



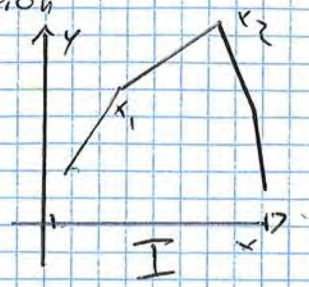
STRICTLY CONVEX



CONVEX

Instead a concave function is definen on an interval I if the line segment lies belowe the graph of the function

STRICTLY CONCAVE



INVERSE FUNCTION

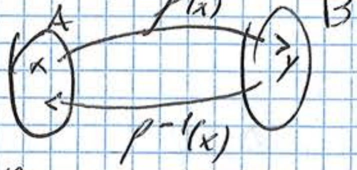
A Bejective corrispondence determinate a function defined on Y, with values on X called inverse function

An inverse function is that function that brings back the value from X to Y and then from Y to X

=> in other words: the inverse function "$f^{-1}(x)$" of $f(x)$ is that function that relates to any $y \in B$ the unique value $x \in A$

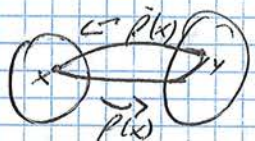
	Dom f	im f
$f(x)$	A	B
$f^{-1}(x)$	B	A

* IF A FUNCTION IS NOT INSECTIVE OR NOT

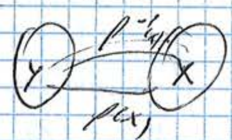


=> Relation between function and inverse function

$$\forall x \in A \Rightarrow f(f^{-1}(x)) = x$$



$$\forall y \in B \Rightarrow f^{-1}(f(y)) = y$$

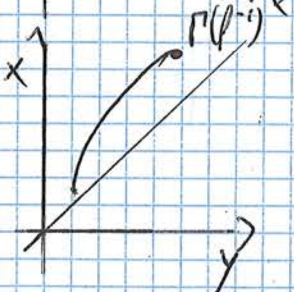


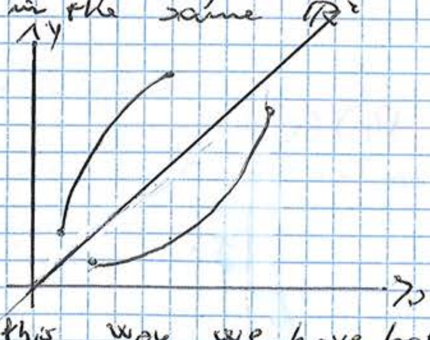
$f(A) = B$
f inj on A

=> if

$\Gamma(f)$ the variable is x
$\Gamma(f^{-1})$ the variable is y

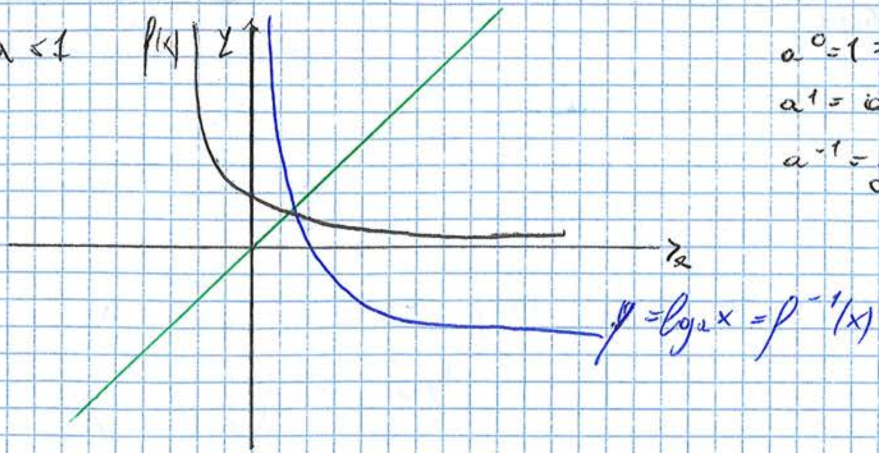
swopping the components we have defined the inverse function in the same $\mathbb{R}^2$





In this way we have both the component on the same $\mathbb{R}^2$ map $x \times y$ by using the simmetric property function $y = x$

3) $0<a<1$



$a^0 = 1 \Rightarrow \log_a 1 = 0$

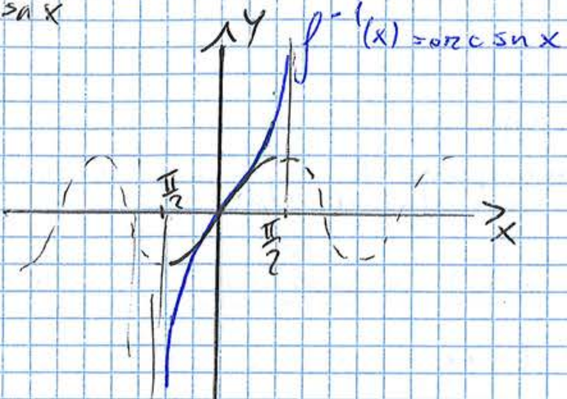
$a^1 = a \Rightarrow \lg_a a = 1$

$a^{-1} = \frac{1}{a} \Rightarrow \lg_a a^{-1} = -1$

$\lg_e e = 1$

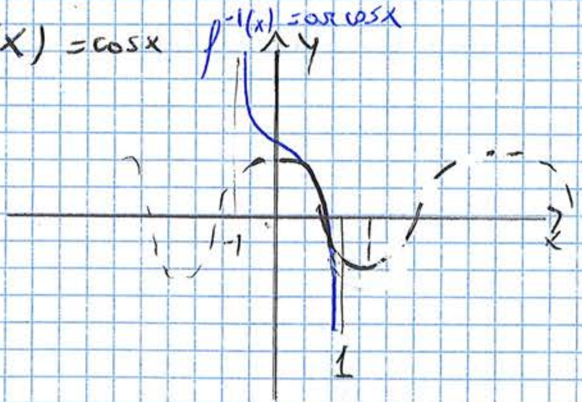
$e \simeq 2,71$

$f(x) = \sin x$



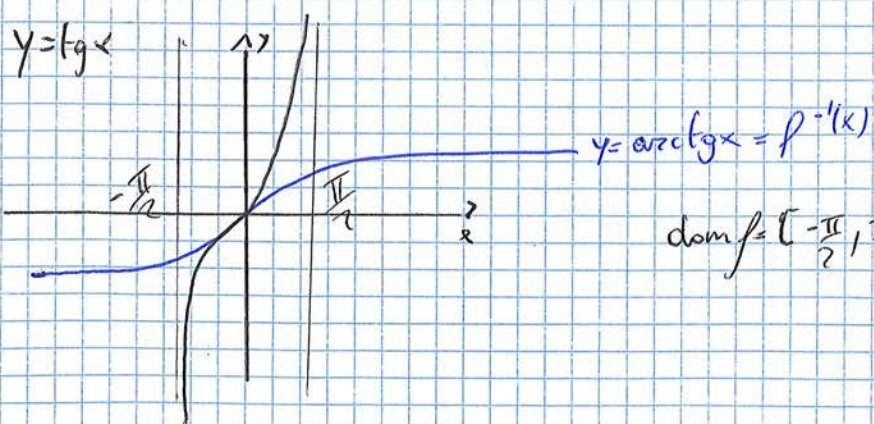
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \underset{\arcsin x}{\overset{\sin x}{\rightleftarrows}} [-1,1]$

$f(x) = \cos x$ $\quad f^{-1}(x) = \arccos x$



$(0, \pi) \underset{\arccos x}{\overset{\cos x}{\rightleftarrows}} [-1,1)$

$y = \operatorname{tg} x$



$\operatorname{dom} f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \underset{\operatorname{arctg} x}{\overset{\operatorname{tg} x}{\rightleftarrows}} \mathbb{R}$

$f \to I \subseteq \operatorname{dom} f$ I interval

$f \to A \subseteq \operatorname{dom} f$ A set

The function f is increasing on I, if given to different x_1 and x_2 so $x_1 \leq x_2$ in I the correspondence $f(x_1)$ and $f(x_2)$ are $f(x_1) \leq f(x_2)$

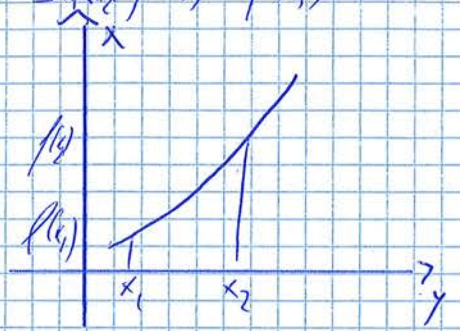
$$\forall \begin{matrix} x_1 \\ x_2 \end{matrix} \in I \; x_1 < x_2 \mid f(x_2) \leq f(x_1)$$

Decreasing function

$$\forall \begin{matrix} x_1 \\ x_2 \end{matrix} \in I \; x_1 < x_2 \mid f(x_2) > f(x_1)$$

Strictly increasing

$\forall x_2, x_1 \in I \; x_1 < x_2 \mid f(x) < f(x_1)$



strictly increasing

Strictly decreasing

$\forall \begin{matrix} x_1 \\ x_2 \end{matrix} \in I \; x_1 < x_2 \mid f(x_1) > f(x_2)$

just increasing

A functione is monotone if on its domain is increasing or decreasing

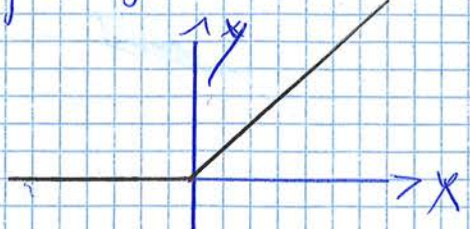
BASIC FUNCTIONS

$$1) \; f(x) = \begin{cases} x & x \in Q \\ -x & x \in R \setminus Q \end{cases}$$

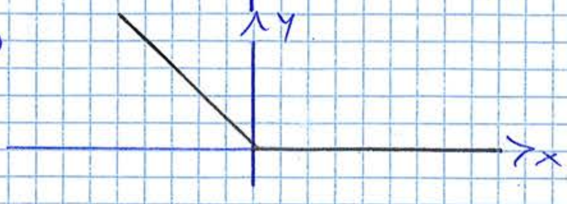
impossible to draw

f is injective but not monotonous

$$2) \; f(x) = x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



$$3) \; f(x) = x^{-1} = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$x^+ + x^- = |x|$

$x^+ - x^- = x$

$$\begin{cases} x^+ + x^- = |x| \\ x^+ - x^- = x \end{cases} + \; = 2x^+ = |x| + x \Rightarrow x^+ = \frac{|x|+x}{2}$$

M[snx]

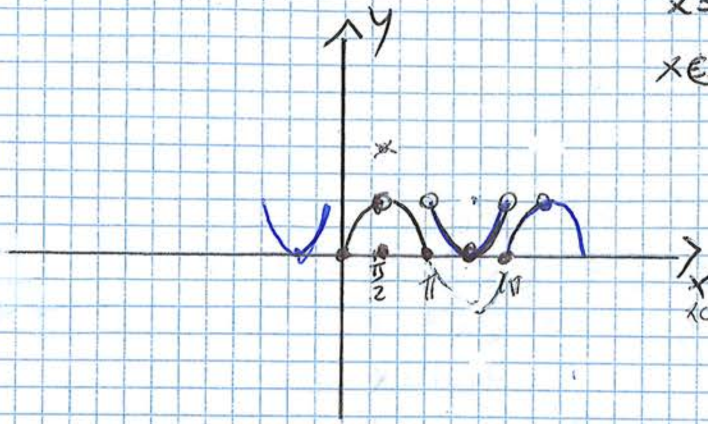
$M(x) = \operatorname{sn}x - [\operatorname{sn}x]$

$x = 0 \Rightarrow 0$

$x \in \left(0, \frac{\pi}{2}\right) \Rightarrow [\operatorname{sn}x] = 0$

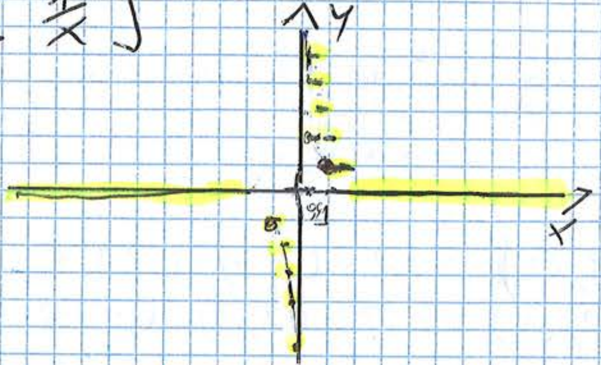
$M(x) = \operatorname{sn}x - 0$

$x = \frac{\pi}{2} \to 2$



$x \in (\pi, 2\pi)\ M(\operatorname{sn}x) = \operatorname{sn}x + 1$

$\left[\frac{1}{x}\right]$



$\boxed{x \in (0,1)} \Rightarrow \frac{1}{x}$

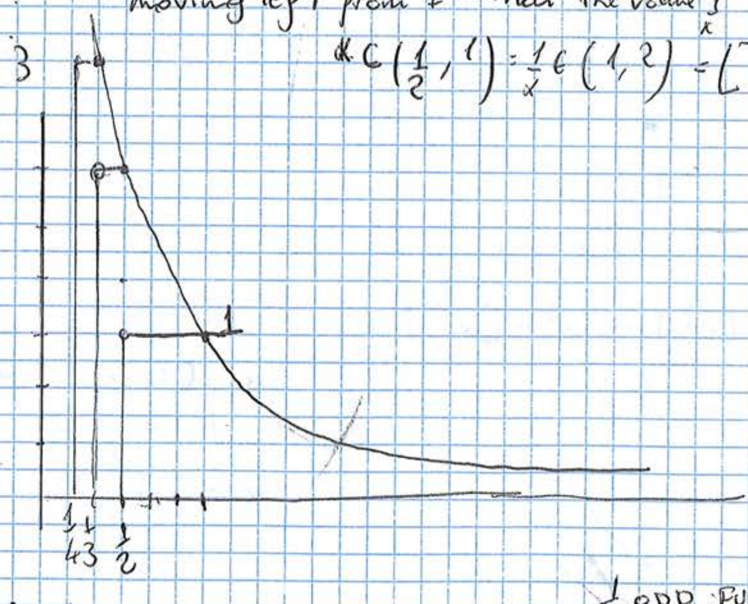
$x \in 1 \Rightarrow \frac{1}{x} = 1$

$x = \frac{1}{2} \quad \frac{1}{x} = 2$

$x \in (1, +\infty)$

moving left from 1 when the value $\frac{1}{x}$ become 1

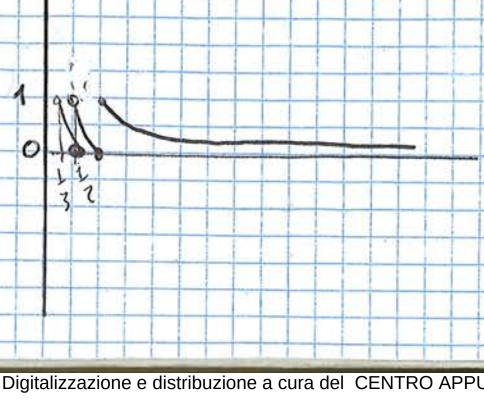
$x \in \left(\frac{1}{2}, 1\right) : \frac{1}{x} \in (1, 2) = [1]$



M(1/x)

$x \in \left(\frac{1}{2}, 1\right) = \frac{1}{x} - 1$

$x \in \left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{x} - 2$



$\frac{1}{x}$ ODD FUNC
BUT
THE INTEGER
PART
IS NOT EVEN OR ODD

LIMITS

First of all we need to say that exist two kind of growing functions. Functions that grow unlimitatelly by x and by y (such as $f(x)=a^x$, $f(x)=\log x$) and the functions that grow to a certain value limit ($y=\arctg x$, $y=\arctg(x)$)

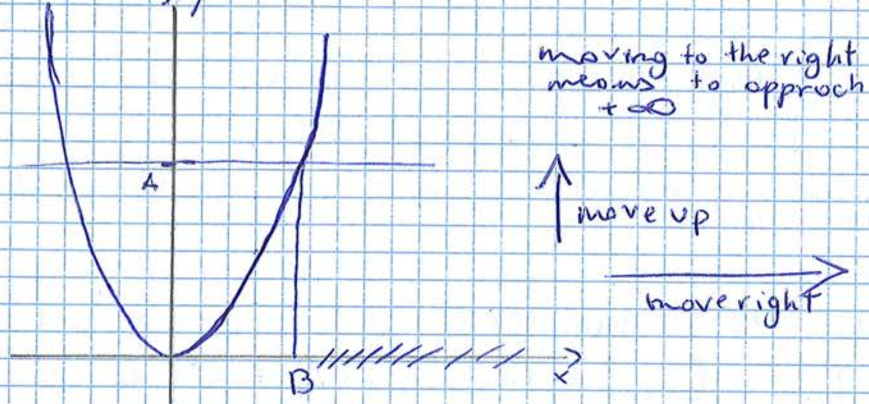
LIMITS AT INFINITY

If we fix a point A on the Y axis, and we fix the corresponding point on the X axis I can se that if I move from B to the right my function is alway bigger than A

A is considered as a Barrier

$$\begin{cases} \text{BARRIER } A \\ \exists B \quad \forall x \geq B \quad f(x) > A \end{cases}$$

=> that means that given an fixed point A the function never goes under the line $y=A$

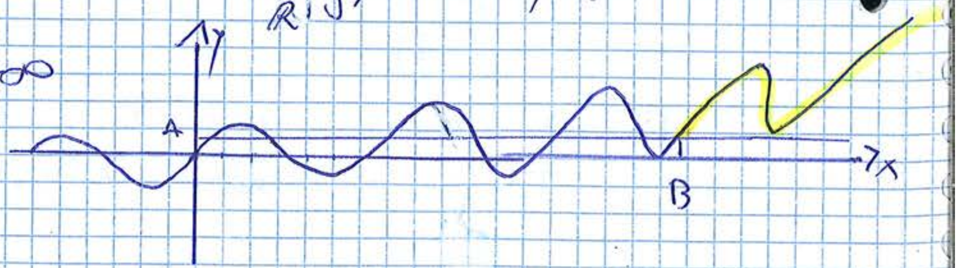


$f : [a, +\infty) \mapsto \mathbb{R}$

$\lim_{x \to +\infty} f(x) = +\infty$

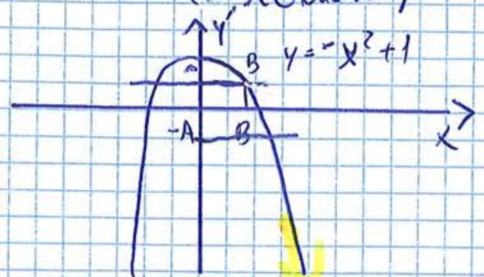
$\forall A > 0 \quad \exists B \geq 0 : \forall x \in \text{dom} f, x > B \Rightarrow f(x) > A$

$\lim_{x \to +\infty} \frac{x}{10} + \sin x = +\infty$



$\lim_{x \to +\infty} f(x) = -\infty$

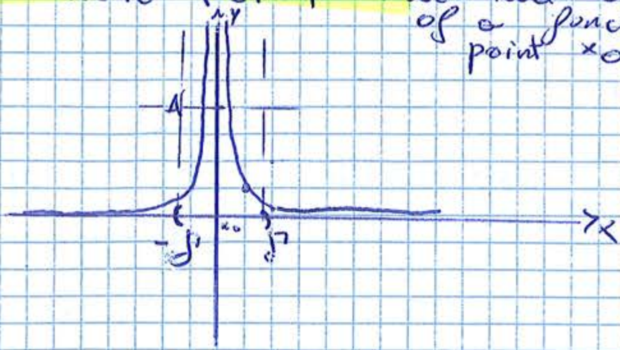
$\forall A > 0 \quad \exists B \geq 0 : \forall x \in \text{dom}, x > B \Rightarrow f(x) < -A$



LIMITS AT REAL POINT

: we need to observe the behaviour of a function in an unknown point x_0

$f(x) = \frac{1}{x^2}$



$\lim_{x \to x_0} f(x) = +\infty$

I need to find a neighbourhood in which my $f(x)$ is always included $(-\delta, \delta)$, with x_0 excluded of radius δ

and given an A on the y axis my function is still included in the neighbourhood above A

Even if I restrict my neighbourhood I will always find a value A such that my function is included in the neighbourhood

$\forall A > 0 \quad \exists \delta > 0$

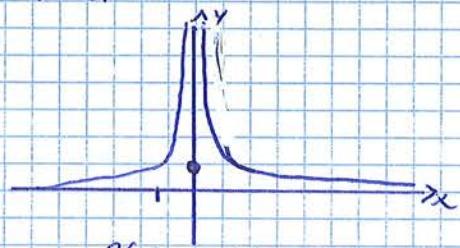
$\forall x : x \in \text{dom} f : 0 < |x - x_0| < \delta \Rightarrow f(x) > A$

↳ excluded $\boxed{x_0}$

when we have a neighbourhood of x_0 we never consider its center

Ex

$$f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

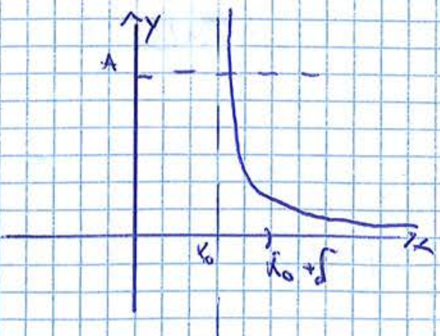


$\lim_{x \to 0} f(x) = 0$

because we consider $0 < |x - x_0| < \delta$
0 exclused

ONE SIDE LIMITS

$\lim_{x \to x^+} f(x) = +\infty$



$\forall A > 0 \ \exists \delta > 0$

$\forall x \ x \in \text{dom} f \quad 0 < |x - x_0| < \delta \Rightarrow f(x) > A$

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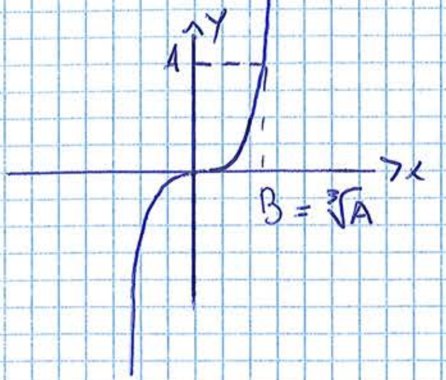
Ex nr 1

$\lim_{x \to +\infty} x^n = +\infty$

$f(x) = x^3 \quad \forall A > 0 \quad \exists B > 0$

$\forall x \in dom f \quad x > B \Rightarrow x^3 > A$
$x \in \mathbb{R}$



$x^3 = A \Rightarrow B = \sqrt[3]{A}$

$B = \sqrt[3]{A}$

=> I have to find the pre-image of A under $f(x) = x^3$

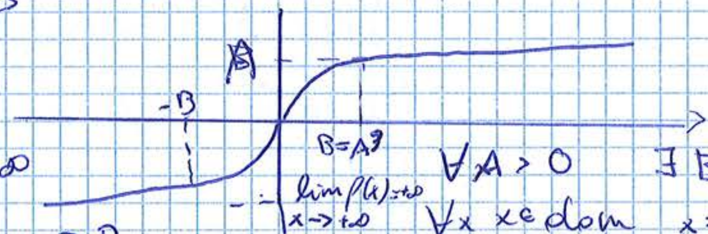
$\Rightarrow \lim_{x \to \pm\infty} x^n \begin{cases} +\infty & \text{EVEN} \\ \pm\infty & \text{ODD} \end{cases}$

=> generally if f(x) is even and u find its right limit for $x \to +\infty$ without any computation u can say that $f(x)$'s left limit is also $+\infty$

Ex: $y = x^2 = \lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$

Ex nr 2

f ODD



$\lim_{x \to -\infty} f(x) = -\infty$

$\lim_{x \to +\infty} f(x) = +\infty$

$\forall A > 0 \quad \exists B \geq 0$

$\forall x \; x \in dom \quad \underline{x > B} \Rightarrow f(x) > A$

$B = A^3$ — A is a positive number so we have found the positive $B = A^3$.

$\forall A > 0 \quad \exists B$

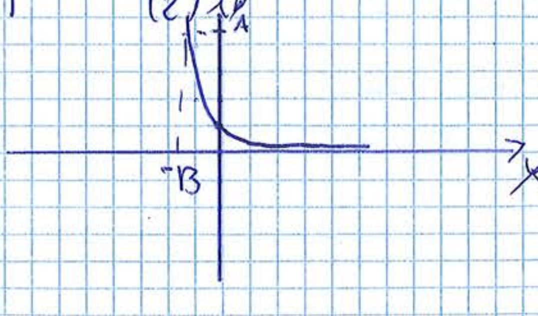
$\forall x \; x \in dom f \quad x < -B \Rightarrow f(x) < -A$

$\underline{B = -\sqrt[3]{A}}$ => Negative value

$\lim_{x \to +\infty} f(x) = -\lim_{x \to -\infty} f(x) =$

Ex nr 3

$f(x) = \left(\frac{1}{2}\right)^x$



$\lim_{x \to -\infty} \left(\frac{1}{2}\right)^x = +\infty$

$\forall A > 0 \quad \exists B \geq 0$

$\forall x \; x \in dom f \quad x < -B \Rightarrow f(x) > A$

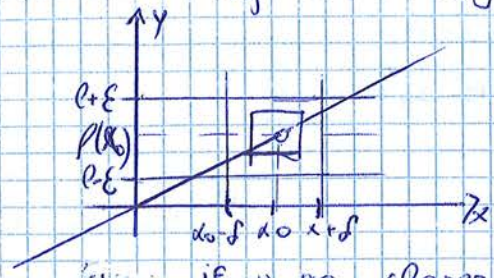
A given

$\left(\frac{1}{2}\right)^x > A \Rightarrow x < \log_{\frac{1}{2}} A$

$x < -\log_2 A$

If I take an $l \in \mathbb{R}$ $f(x)$ defined in $I(x_0) \setminus \{x_0\}$

I have to define a neighbourhood of x_0



if u go closer and closer the point $(x_0, f(x_0))$ ur function is included in this square u can define that

$f(x_0) = l$

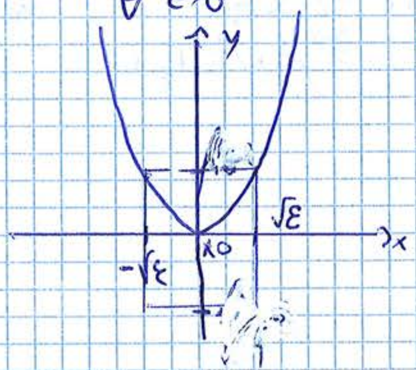
$\forall \varepsilon > 0 \quad \exists \delta \geq 0$

$\forall x \quad x \in dom f \quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$

$\hookrightarrow x_0$ excluded $\quad l - \varepsilon < f(x) < l + \varepsilon$

Ex nr 1 $\lim_{x \to 0} x^2 = 0$ $\quad f(x) = x^2$

$\forall \varepsilon_0$



$\forall \varepsilon > 0 \quad \exists \delta \geq 0$

$\forall x \quad x \in dom f \quad |x| < \delta \Rightarrow |x^2| < \varepsilon$

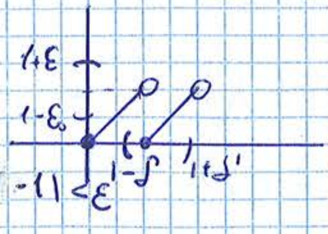
$\delta \leq \sqrt{\varepsilon}$

Any δ smaller or equal to $\sqrt{\varepsilon}$ satisfy this property

Ex nr 2 $\lim_{x \to 1^-} M(x) = 1$

$\forall \varepsilon > 0 \quad \exists \delta \geq 0$

$\forall x \quad x \in dom f \quad |x - x_0| \leq \delta \Rightarrow |M(x) - 1| < \varepsilon$



2) $M(x) - 1 > -\varepsilon$

I am in a negative neighbourhood of 1

$-\varepsilon < M(x) - 1 < \varepsilon$

$-\varepsilon < x - [x] - 1 < \varepsilon$

the mantissa is always < 1

1) so $M(x) - 1 < 0$

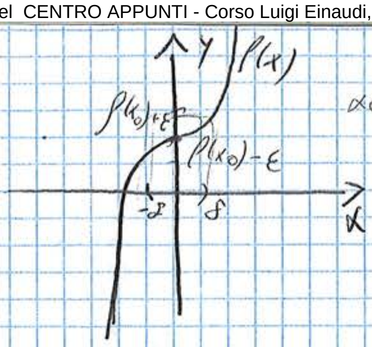
$\underset{\text{negative}}{M(x) - 1} < \underset{\text{positive}}{\varepsilon}$

$-\varepsilon < x - 1 < 1$

$1 - \delta < x - 1 < 1$

$\Rightarrow x > 1 - \varepsilon$

CONTINUOUS FUNCTIONS



1) $f(x) = x^3 + 1$

$\lim_{x \to 0} x^3 + 1 = 1$

$\forall \varepsilon > 0 \quad \exists \delta \geq 0$

$\forall x : x \in dom f \quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\Rightarrow \forall \varepsilon_0 \quad \exists \delta \geq 0$

$\forall x : x \in \mathbb{R} \quad 0 < \underbrace{|x - 0|}_{|x|} < \delta \Rightarrow |x^3 + 1 - 1| < \varepsilon$

$|x^3| < \varepsilon \Rightarrow |x^3| = |x \cdot x \cdot x| = |x| \cdot |x| \cdot |x|$

$|x|^3 < \varepsilon$

$0 < |x| < \delta \Rightarrow |x|^3 < \varepsilon$

OK $\quad \delta \leq \sqrt[3]{\varepsilon}$

Definition

given f defined in $I(x_0)$, if $\exists$ compiut the $\lim_{x \to x_0} f(x) = f(x_0)$ the function is continuous in x_0

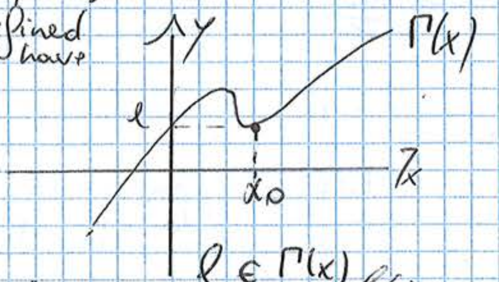
$\Rightarrow$ A function is continuos when the limit of one point is the value of that function of that point

$\forall \varepsilon > 0 \quad \exists \delta > 0$

$\forall x : x \in dom \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

↳ the function is defined in the point x_0 so we have not to write < 0

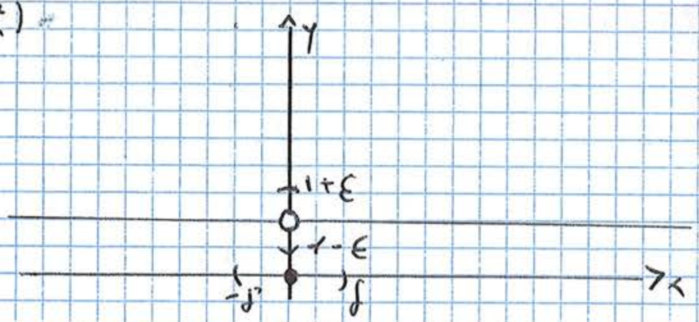
$\Rightarrow \lim_{x \to x_0} f(x) = l = f(x_0)$



$l \in \Gamma(x)$

	$f(x)$	$g(x)$	$g(x) + f(x)$	$k f(x)$	$\frac{f(x)}{g(x)}$
	C	C	C	C	C $g(x) \neq 0$

$f(x) = sing(x^2)$



$\lim_{x \to 0} sing(x^2) = 1$

For all the point of $(-\delta; +\delta)$ excluding its center I have to find the function included between $(1+\varepsilon, 1-\varepsilon)$

$\forall \varepsilon > 0 \quad \exists \delta > 0$
$\forall x : x \in \mathbb{R}, \quad 0 < |x| < \delta \Rightarrow |\underbrace{sign(x^2)}_{1} - 1| < \varepsilon$

$|1 - 1| = 0 < \varepsilon$

it's given $\varepsilon > 0$ so the inequation $sig(x^2) - 1 < \varepsilon$ it's already proved to be true for all ε

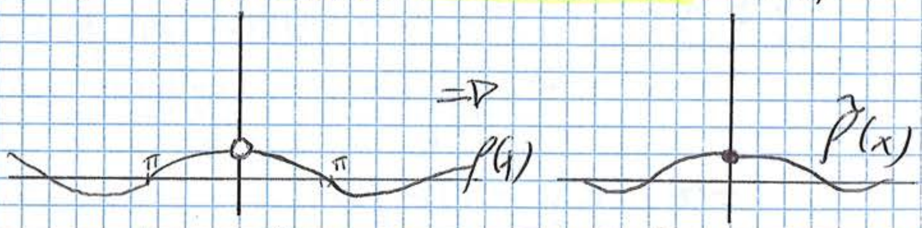
so this limit is proved indipendently from any $\delta > 0$

DEFINITION 2

IF f is defined in $I(x_0)$ and if we have
$\lim_{x \to x_0} f(x) = l \neq f(x_0)$
we say that x_0 is a REMOVABLE SINGULARITY for $f(x)$

) $f(x) = \frac{\sin x}{x}$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$



it's possible to prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and complete my function just adding a point $(0,1)$

DEFINITION

f DEFINED in $I(x_0) \setminus \{x_0\}$
$\lim_{x \to x_0} f(x) = l \in \mathbb{R}$

that's a REMOVABLE SINGULARITY FOR l to witch we can just add a point

so the fuction $f(x)$ becomes the tilde of x ($\tilde{f}(x)$)

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq x_0 \\ l & x = x_0 \end{cases}$$

⇒ also called CONTINUOS PROLUNGATION OF f

PROPERTIES OF LIMITS

GIVEN

$\lim_{x \to \gamma} f(x) = l \in \mathbb{R}$ and $\lim_{x \to \gamma} g(x) = m \in \mathbb{R}$

we have

1) $\lim_{x \to \gamma} f(x) \pm g(x) = l \pm m$

2) $\lim_{x \to \gamma} f(x) \cdot g(x) = l \cdot m$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)} = \frac{l}{m}$ if $m \neq 0$

So if f, g are defined in $I(x_0) \Rightarrow$ $\lim_{x \to x_0} f(x) \pm g(x) = f(x_0) \pm g(x_0)$, $\lim_{x \to x_0} f(x) \cdot g(x)$ are provides $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ $g(x) \neq 0$

the function $h(x) = f(x) + g(x)$ is still continuos at x_0

$\Rightarrow$ if you take 2 continuos functions at the same point x_0 u can do all the possible operation a part from deviding per 0

$\Rightarrow$ so u can operate with continuos function and keep staying in the CONTINUOS FUNCTION SET

f continuous in $A \subseteq dom f$ if f is continuos at x_0 $\forall x_0 \in A$

or

$\forall x_0 \in A$ $f \in C^0(A)$

$\hookrightarrow$ set of All continuos functions on A

Examples

$g(x) = x^n \Rightarrow$ product of n continuos functions

$x^n \in C^0(\mathbb{R})$

$\lim (x) = c \cdot x^n \Rightarrow$ monomial function

2) Poynomial function

$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a x + a_0$

$a_n, a_{n-1} \dots a_n \in \mathbb{R}$

$P_n(x)$ Polynomial function of degree n

$P_n(x) \Rightarrow$ continuous $P_n(x) \in C^0(\mathbb{R})$

THEOREM

Suppose that two functions $f(x)$ and $g(x)$ are equal in a neighbourhood $I(\gamma) \setminus \{\gamma\}$ and that

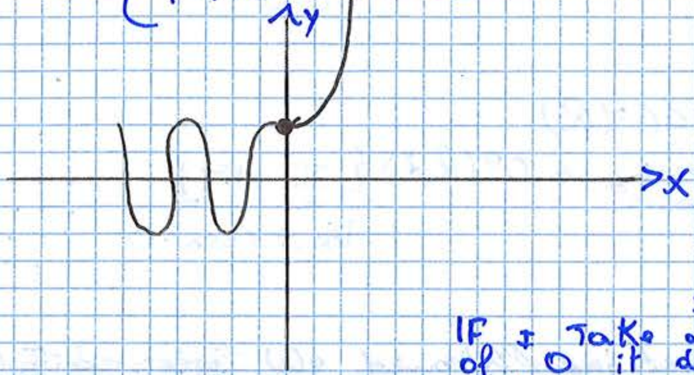
$\lim_{x \to \gamma} f(x) (= \lambda)$ exists.

$\Rightarrow$ Also the limit $\lim_{x \to \gamma} g(x)$ exists and it is λ

"IF 2 Functions are equal in the same neighbour I also their limit is equal"

EXP

$$f(x) = \begin{cases} \cos x & \text{if } x \leq 0 \\ 1 + x^2 & \text{if } x > 0 \end{cases}$$



If $x > 0$
Any $I(x_0)$ I take I find that my function is continuous

If $x < 0$
for any $I(x_0)$ my function is continuous as $\cos x$ is continuous

$x = 0$
If I take a neighbourhood of 0 it doesn't correspond to a single function so I need to distinguish between

a) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 + x^2 = 1$

b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos x = 1$

$\Big\} \lim_{x \to 0} f(x) = 1$

so $f(0) = \cos(0)$

f is continuous at 0

$\Rightarrow f(x) \in C^0(\mathbb{R})$

$f(x) = x^2 \quad x \xrightarrow{+\infty} +\infty$
$g(x) = \frac{1}{x} \quad x \xrightarrow{+\infty} 0$
$\Bigg\} \; f(x)\,g(x) = x \xrightarrow{+\infty} +\infty$

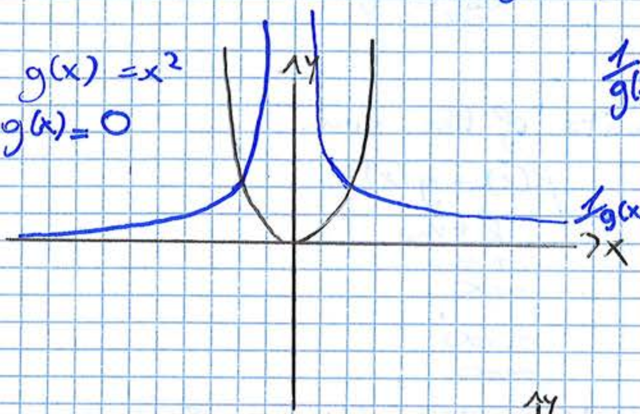
THE RECIPROCAL

$g(x)$	$1/g(x)$
$m \neq 0$	$1/m$
∞	0
0	$\nearrow +\infty \quad g(x) > 0$ $\searrow -\infty \quad g(x) < 0$

$g(x) = x^2$

$\lim_{x \to 0} g(x) = 0$

$\frac{1}{g(x)} = \frac{1}{x^2} \qquad \lim_{x \to 0} \frac{1}{x^2} = +\infty$

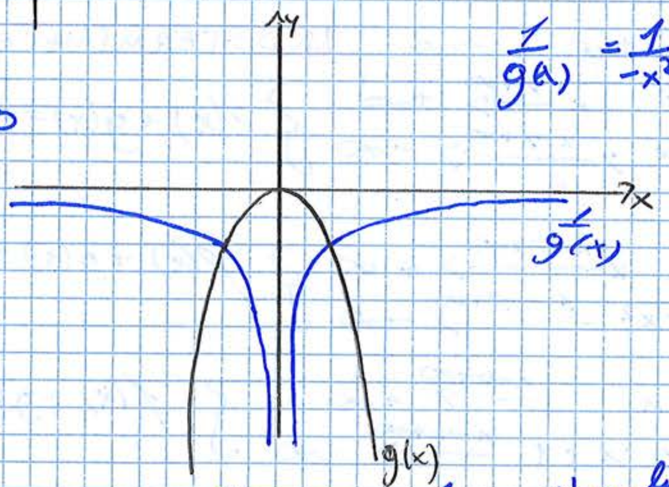


$g(x) = -x^2$

$\lim_{x \to 0} x^2 = 0$

$\frac{1}{g(x)} = \frac{1}{-x^2}$

$\lim_{x \to 0} \frac{1}{-x^2} = -\infty$



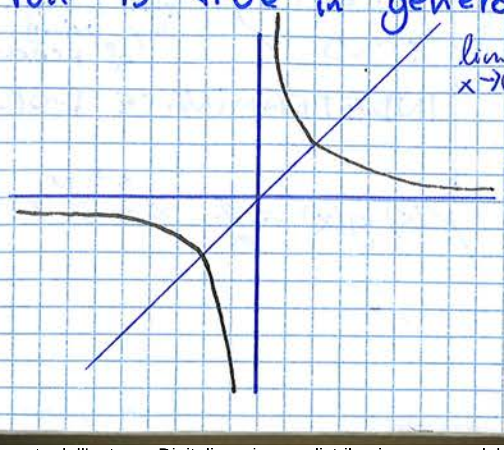
the limit of an inverse function $\frac{1}{g^x}$ when $\lim_{x \to x_0,\ \pm\infty} g(x) = 0$

is $+\infty$ if $g(x) > 0$

$-\infty$ if $g(x) < 0$

This rule is true in general but not always!

Exp



$\lim_{x \to 0} f(x) = 0$

$\lim_{x \to 0} \frac{1}{f(x)} =$ u have to consider separately

$\lim_{x \to x_0^+} \frac{1}{f(x)}$ and $\lim_{x \to x_0^-} \frac{1}{f(x)}$

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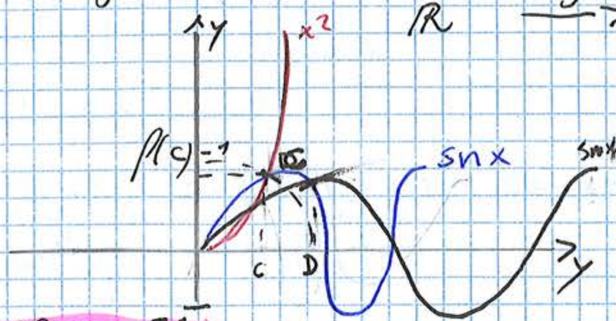
$f(x) = x^2$ $\quad g(f(x)) = \sin x^2$

$g(x) = \sin x$

Dom f $\quad \mathbb{R} \xrightarrow{f} [0; +\infty)$

$\mathbb{R} \xrightarrow{g} [-1; 1]$



Founded D through the translation of $C \in f(x)$ and moving it depending on $f(x)$ and $g(x)$ we can draw the graph of $g(f(x))$

PROPERTIES

- $f(x): I$ injective in I $\quad f(I) = J \Rightarrow g \circ f$ is injective in I
  $g(x): I$ injective in J $\quad \Rightarrow$ INVERTIBLE

- $f(x)$ injective in I
  $g(x)$ injective in $J = f(I)$ $\Rightarrow$ the composed function is invertible in I

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

$$\Rightarrow f^{-1}(\underbrace{g^{-1}(g}_{\text{The identity}}(f(x)))) = f^{-1}(f(x)) = x$$

EXM

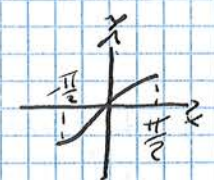
$f(x) = y = x - 6$ $\quad f(g(x)) = \sqrt{x-6} \rightarrow$ defined in $I = [6; +\infty)$

$g(x) = \sqrt{x}$

$g(x) \wedge f(x)$ increasing in I

$\Rightarrow (g \circ f)^{-1} = (f^{-1} \circ g^{-1}) = x^2 + 6$

$h(x) = 2^{\sin x}$ $\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$ $\sin x$ is injective



2^x is always injective

$x \xrightarrow{f} \sin x \xrightarrow{2^x} 2^{\sin x} = z$

$h^{-1}(x) \Rightarrow z = 2^{\sin x}$ $\quad x = \arcsin(\lg_2 z)$

$\sin x = \lg_2 z$

$z \xrightarrow{g^{-1}} \lg_2 z \xrightarrow{f^{-1}} \arcsin \lg_2 z$

Limit of a composition

$\lim_{x \to \gamma} g(f(x)) = \lambda$

1. $\lim_{x \to \gamma} f(x) = \lambda$
2. $y = f(x)$
3. $\lim_{y \to \lambda} g(y) = \mu$
4. $\lim_{x \to \gamma} g(f(x)) = \lim_{x \to \lambda} g(y) = \mu$?

$\lim_{x \to 5} \sqrt{x^2+2} = \sqrt[2]{27}$

$y = x^2+2$

$\lim_{x \to 5} x^2+2 = 27$

$\lim_{y \to 27} \sqrt{y} = \sqrt{27}$

1st case) λ as a number

1) $\text{limit}_{x \to \gamma} f(x) = \ell \in \mathbb{R}$

2) g is defined in a $I(\gamma)$ and continuous at ℓ

$\Rightarrow \lim_{x \to \gamma} g(f(x)) = g(\lim_{x \to \gamma} f(x)) = g(\ell)$

" The limit of g is the g times the limit !! "

Theorem

Suppose that

1) $f(x)$ is defined and continuous at x_0

$\Rightarrow \lim_{x \to x_0} f(x) = \ell = f(x_0)$

2) g is defined in $I(f(x_0))$ and continuous in $I(f(x_0))$

$f(g(x)) \to$

$\Rightarrow \lim_{x \to x_0} g(f(x_0)) = g(f(x_0))$

$\Rightarrow$ the composition of two continuous function is a continuous function !!

$f \to g$

Theorem

1) $\boxed{\lim_{x \to \gamma} f(x)} = +\infty$ or $\lim_{x \to \gamma} f(x) = -\infty$

2) $\lim_{x \to +\infty} g(f(x))$ exists or $\lim_{x \to -\infty} g(f(x))$ exists

$\Rightarrow \lim_{x \to \gamma} g(f(x)) = \lim_{y \to +\infty} g(y)$ or $\lim_{y \to -\infty} g(y)$

Ex

$\lim_{x \to \frac{\pi}{2}^-} e^{-tg x}$

$y = -tg x \qquad \lim_{x \to \frac{\pi}{2}^-} -tang x = +\infty$

$\Rightarrow \lim_{x \to \frac{\pi}{2}^-} e^{-tg x} = \lim_{y \to -\infty} e^{-y} = 0$

$\Rightarrow$ REMARKS:

$\lim_{x \to \gamma} g(f(x)) \to \left. \begin{array}{l} \lim_{x \to \gamma} f(x) = l \\ g \text{ cont at } 0 \end{array} \right\} \to$

$\to \lim_{x \to \gamma} f(x) = +\infty$

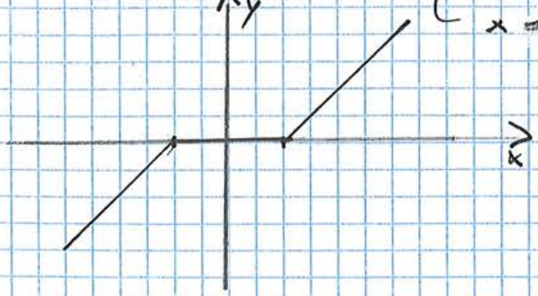
$\lim_{y \to +\infty} g(y)$ ∃

— o —

$\lim_{x \to x_0} f(x) = l \qquad || \qquad \lim_{y \to l} f(x) = m$

$\lim_{x \to x_0} g(f(x)) = m$ in general NO!

$$f(x) = \begin{cases} x+1 & x \le -1 \\ x = 0 & -1 < x < 1 \\ x - 1 & x > 1 \end{cases}$$



TRUE HERE

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Theorem of UNIQUENESS

- suppose that $\lim_{x \to \gamma} f(x)$ exists and it is equal λ
Then this limit is unique

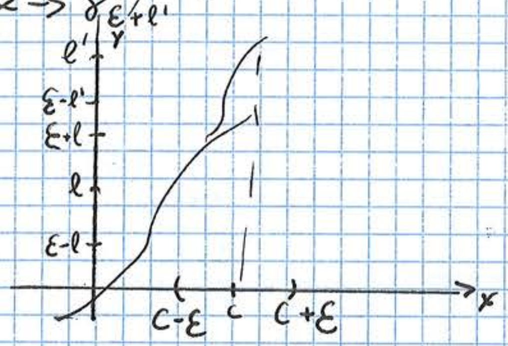
$\Rightarrow$ Then f admits no other limits for $x \to c$

$\Rightarrow$ We assume there exist two limits $\ell \neq \ell'$ and infer a contraddiction

if $\ell \neq \ell'$ we have

$\lim_{x \to \gamma} f(x) = \ell$ such that $I(\ell) \cap I(\ell') = \emptyset$

$\lim_{x \to \gamma} f(x) = \ell'$



we take an ε smaller or equal than the half distance $\ell - \ell'$

$$\varepsilon \leq \frac{1}{2} |\ell - \ell'|$$

than we apply the definition of limit to a define point!

$\forall x : x \in \mathrm{dom} f \quad x \in I(c) \setminus \{c\} \Rightarrow f(x) \in I(\ell)$

$\forall x : x \in \mathrm{dom} f \quad x \in I'(c) \setminus \{c\} \Rightarrow f(x) \in I(\ell')$

$I'(c) \cap I(c) = I''(c)$

$I''(c)$ is a neighbourhood of c in which there exist infinite points belonging to $I(c)$ and $I'(c)$

$\Rightarrow \exists x \in I''(c) \mid f(x) \in I(\ell) \cap I(\ell')$

so the first assumption

$I(\ell) \cap I(\ell')$ is absolutely not true!

$\Rightarrow$ "given a function, given a point may happen that the limit of the function in that point is unique or it doesn't exist"

Theorem signum and limit
- della permanenza del segno"

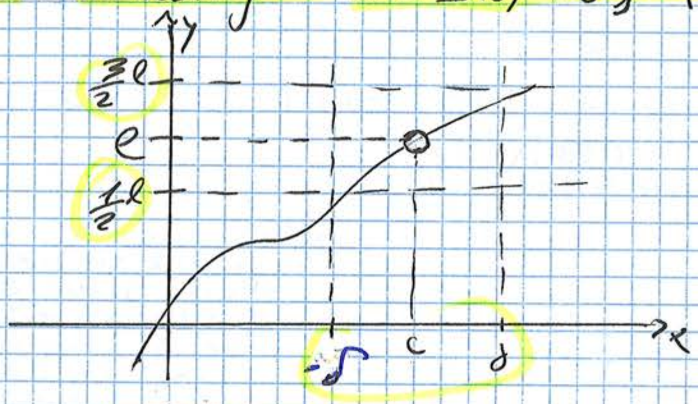
Suppose f admits $\lim_{x \to c} l$ and $l > 0$ or $l = +\infty$, there exists a neighbourhood of c $I(c)$ in which the function is positive. Instead, if my $\lim_{x \to d}$ is m, $m < 0$ or $-\infty$ my function is negative in a neighbourhood of d $I(d)$

PROOF Hp: $\lim_{x \to c} f(x) = \begin{smallmatrix} \nearrow l > 0 \\ \searrow +\infty \end{smallmatrix}$ $\Rightarrow \exists I(c) \setminus \{c\} \mid f(x) \geq 0 \quad \forall x \in I(c) \setminus \{c\}$

l is finite, positive value and we take a neighbourhood of radius $\varepsilon = \frac{l}{2}$. According to the definition, there is a neighbourhood $I(c)$ of c satisfying

$\forall \varepsilon > 0 \quad \exists \delta \geq 0$

$\forall x: x \in \mathrm{dom} f \quad x \in I(c) \setminus \{c\} \mid f(x) \in I(l)$

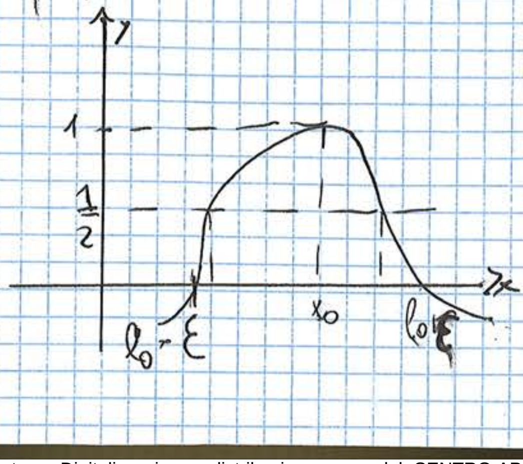


As $I_\varepsilon(l) = \left(\frac{l}{2}; \frac{3}{2}l\right)$ and we got a positive l all values of $f(x)$ are positive

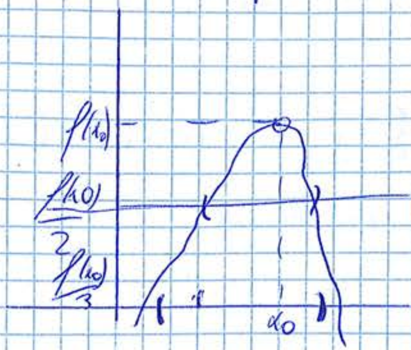
$\Rightarrow$ If, $l = +\infty$ it is sufficient to take a neighbourhood $I_A(+\infty) = (A, +\infty)$ of $+\infty$ $(A > 0)$ and apply the definition of limit

"colorelly"

$\Rightarrow$ if we have a positive l we will always find a smaller and smaller neighbourhood of that point in which the function is positive



IF a have a positive limit there exists a neighbourhood where my function must be positive.



$\lim_{x \to \gamma} f(x) = l > 0$

$\exists I(\gamma) : \forall x \in I(\gamma) \setminus \{\gamma\} \quad f(x) > m$

$\forall m \quad 0 \leq m < l$

Corollary

The Theorem of signum and limit

Assume f admits a limit $x \to c$ and exists a neighbourhood of c $I(c) \setminus \{c\}$ in which my function is ~~strict~~ positive or equal zero $f(x) \geq 0$ then $l \geq 0$ or $+\infty$

$\lim_{x \to c} f(x) =$ exists

$f(x) \geq 0 \quad \forall x \in I(c) / \{c\}$

$\Rightarrow \lim_{x \to c} f(x) = l \begin{cases} +\infty \\ \geq 0 \end{cases}$

PROOF BY CONTRADDICTION $\quad P \Rightarrow Q \Rightarrow$ leads to a contraddiction!

1) $\lim_{x \to c} f(x)$ exists

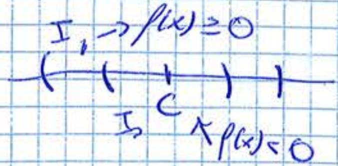
2) $f(x) \geq 0 \quad \forall x \in I(c) \setminus (c)$

3) $\lim_{x \to c} f(x) = l \begin{cases} -\infty \\ < 0 \end{cases}$

↳ then we apply the sign and limit theorem

$\exists I_2(c) \setminus \{c\} \mid f(x) < 0 \quad \forall x \in I_2(c) \setminus \{c\}$

↳ This contraddicts my hypothesis cause it's impossible to exist a neighbourhood $I_3(c) = I_2(c) \cap I(c)$ / in which my function is positive and negative



$I_3(c) = I_2(c) \cap I(c) =$ In the same neighbourhood the negation of the thesis brings me to see that my function is negative and positive in the same time

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II : comparision Theorem to infinite

- f, g defined in $I_1(c) \setminus \{c\}$
- $\lim_{x \to c} f(x) = \lambda$ and $\lim_{x \to c} g(x) = \mu$
- $\exists I_2(\gamma) \; f(x) \leq g(x) \quad \forall x \in I_2(c) \setminus \{c\}$

$$\Rightarrow \lim_{x \to c} f(x) = \lambda \leq \lim_{x \to \gamma} g(x) = \mu$$

$$\lambda \leq \mu$$

If λ is $-\infty$ and μ is $+\infty$ there is nothing to prove.
Otherwise, consider the function $h(x) = g(x) - f(x)$ $\Rightarrow$ this is a positive function by definition in $I(c) \setminus \{c\}$ so $h(x) \geq 0$.
then we have $\lim_{x \to c} h(x) = \lim_{x \to c} g(x) - \lim_{x \to c} f(x) = \mu - \lambda \quad \mu < m$
(ALGEBRA OF LIMITS)
Applying the 1st corollary of the signum and limit we have that

$$\boxed{\mu \leq \lambda}$$

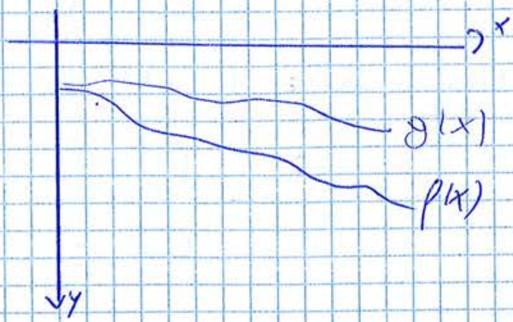
2nd comparision Theorem INFINITE CASE

1) $f(x)$ and $g(x)$ defined in $I(\delta) \setminus \{\delta\}$

$$f(x) \leq g(x)$$

1st case $\Rightarrow \lim_{x \to \delta} f(x) = +\infty \Rightarrow \lim_{x \to \delta} g(x) = +\infty \Rightarrow$

2nd case $\Rightarrow \lim_{x \to \delta} g(x) = -\infty \Rightarrow \lim_{x \to \delta} f(x) = -\infty$



$y = \frac{\sin x}{x}$

$$\lim_{x \to +\infty} \frac{\sin x}{x} = \frac{0}{0}$$

$$\lim_{x \to +\infty} \sin x \cdot \frac{1}{x}$$

$\Rightarrow$ we know

$$-1 \leq \sin x \leq 1$$

we moltiply for x without alterating the inequality $x>0$

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\downarrow \qquad \Downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

then we apply the 2nd comparition theorem

$$\lim_{x \to +\infty} \frac{\sin x}{x} = 0$$

Let us prove the fundamental limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$y = \frac{\sin x}{x} \Rightarrow$ even for $\frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$

$\Rightarrow$ we know that $|\sin x| < |x| \Rightarrow$

for proving $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$

$\Rightarrow \sin x < x \quad \frac{\sin x}{x} < 1$

To find a lower bound in order to apply the squeeze theorem we suppose to have $x < \frac{\pi}{2}$ and the unit circle

$\lim_{x \to \gamma} f(x) = 0 \Rightarrow \lim_{x \to \gamma} |f(x)| = 0$

$\Rightarrow$ that's a composition of functions of
$f(x) = x$ and $g(x) = |x|$

it is true also the opposite assumption

$\lim_{x \to \gamma} |f(x)| = 0 \Rightarrow \lim_{x \to \gamma} f(x) = 0$

$$-|f(x)| < f(x) < |f(x)|$$

$x \to \gamma$: $-|f(x)| \to 0$, $f(x) \to 0$, $|f(x)| \to 0$

$\lim_{x \to +\infty} x - M(x) = +\infty$

we already know

$0 \leq M(x) < 1$

$-1 \leq -M(x) < 0$

$\underbrace{x - 1 \leq -M(x) + x}, < x$

$\underbrace{-M(x) + x}_{\to +\infty} \geq \underbrace{x - 1}_{\to +\infty}$

Applying the 2nd comparision to $+\infty$

$\Rightarrow X \subseteq \mathbb{R}$

X is **bounded** if and only if is bounded by below and from above

Remarks: if X has an upper bound or or lower bound than it has infinitely many upper or lower bounds

$\forall x : x > b$ or $\forall x : x < c$

Ex 1] $X = [0, 3)$ upper bound $b = 10$ or $b = 3$ — — —

lower bound $c = 0$, $c = -1$, $c = -5$ — —

Ex 2] $\mathbb{N} \subseteq \mathbb{R}$ lower bound $c = -1$, $c = 0$

upper bound $\nexists$

Ex 3] $(-\infty, 1)$ $\Rightarrow$ not bounded by below but bounded from above

Ex 4] $X = \{ x \in \mathbb{R} : x = \frac{n}{n+1}, n \in \mathbb{N} \}$

bounded by below $c \in (-\infty, 0]$

bounded from above $b \in [1, +\infty)$

↳ has no maximum

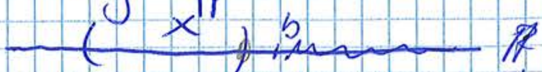
$\min X = 0$

$\max X = \nexists$

X is bounded from above and by below!

• $X \subseteq \mathbb{R}$ X bounded from above

$\Rightarrow$ The set of upper bounds has no max



and is bounded by below having minimum b called:

Least upper bound of X

or

Supremum of X $\Rightarrow \sup X$

Ex 1 $X = (0,3)$ ↳ bounded by below and from above

$(-\infty; a] \Rightarrow \inf X = 0$

$[3, +\infty) \Rightarrow \sup X = 3$

Ex 2

$X = (-\infty; 5]$

Lower bound $\emptyset$ $\inf X = -\infty$

set of upper bounds $[5, +\infty)$ $\sup X = 5$

Ex 3 $\mathbb{N}$

$\inf \mathbb{N} = 0$

$\sup \mathbb{N} = +\infty$

$\mathbb{Z}$

$\inf \mathbb{Z} = -\infty$

$\sup \mathbb{Z} = +\infty$

→ X bounded from above

$s \in \mathbb{R}$: $s = \sup X$

↳ "The ~~last~~ upper bound"

1) $\forall x \in X$ $\underline{x \leq s}$

2) $\forall \varepsilon > 0$ $s - \varepsilon$ is not an upper bound!

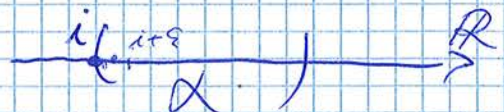
$\exists x \in X$ $x > s - \varepsilon$

→ X bounded by below

$i \in \mathbb{R}$ $i = \inf X$

1) $\forall x \in X$ $x \geq i$

$\forall \varepsilon > 0$ $\exists x \in X$ $x < i + \varepsilon$



→ $X \neq \emptyset$ $x_M = \max X$ is the $\sup X$?

yes

1) $\underline{\forall x \in X}$ $x \leq x_M$

2) $\forall \varepsilon > 0$ $\exists x \in X$ $x > x_M - \varepsilon$

it is x_M itself

$x_M > x_M - \varepsilon$

⇒ if a set has a max then $\max X = \sup X$

⇒ $\min X = \inf X$

SEQUENCES

the sequence is a function defined on $\mathbb{N}$ or in general $\{n \in \mathbb{N} : n \geq n_0\}$

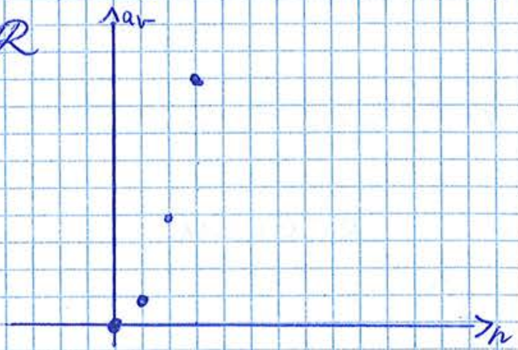
$y = f(x) \quad y = f(n) = f_n$

The sequence $\{a_n\}, \{b_n\}, \{x_n\}$

$\Rightarrow n \longmapsto a_n \in \mathbb{R}$

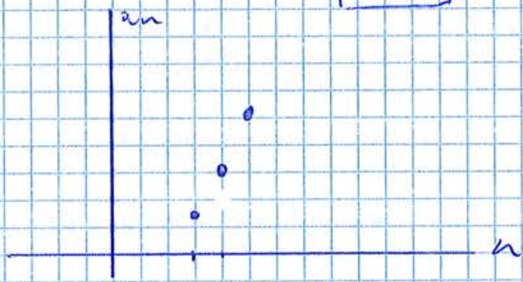
$a_n = n^2$

n	a_n
0	0
1	1
2	4
3	9



The graph of a sequence is an infinite set of points in the 1st and 4th quadrants

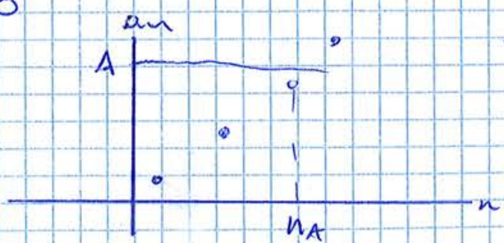
$b_n = \sqrt{n^2 - 7} \qquad n^2 - 7 \geq 0 \quad \boxed{n \geq 3}$



The limit we are interested in is to $+\infty$

$\lim_{n \to +\infty} a_n = \lim_{n \to \infty} a_n$

1) $\lim_{n \to \infty} a_n = +\infty$

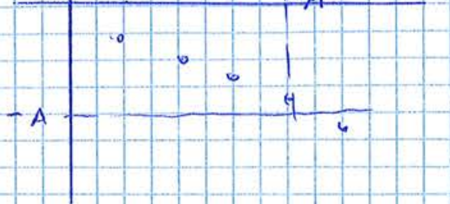


$\forall A > 0 \ \exists n_A \in \mathbb{N}$

$\forall n \ n \geq n_0 \ n > n_A \Rightarrow a_n > A$

$\Rightarrow$ DIVERGENT TO $+\infty$

2) $\lim_{n \to \infty} a_n = -\infty$



$\forall A > 0 \ \exists n_A \in \mathbb{N}$

$\forall n : n \geq n_0 \ n > n_A \Rightarrow a_n < A$

$\Rightarrow$ DIVERGENT TO $-\infty$

We can apply this theorem iff the $f(x)$ is defined

$f(x) = (-1)^x$ is undefined

[$f(x) = a^x$ $a > 0$ ⇒ it is defined also for $a > 0$]

$f(n) = (-1)^n$

2] $a_n = n!$ ⇒ we don't apply the limit of the function when we cannot know the function or is too difficult to study it

$0! = 1! = 1$

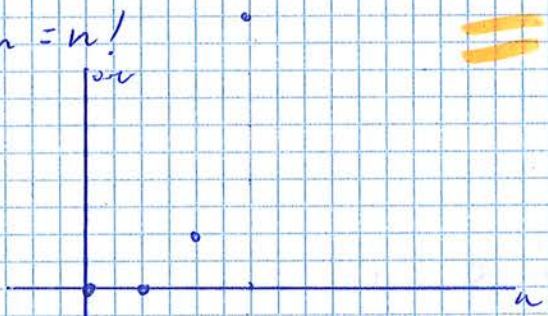
$n! = n \cdot (n-1)!$

$n! = 1 \cdot 2 \cdot 3 \dots n$

factorial is the product of all the integers up to n

⇒ There is a function defined on $\mathbb{R}$ but it is defined on Immaginary set numbers

$a_n = n!$



II comparision theorem (+ INFINITE)

$\lim_{n \to \infty} n! = \infty$

$n! = \underbrace{1 \cdot 2 \cdot 3 \cdot \dots (n-1)}_{n>1} \cdot n$

$n! > n$

$\downarrow +\infty$ $\downarrow +\infty$

Many theorems or sequense are the same of function's theorems

- Uniqueness of limit
- Algebra of limits
- indeterminate forms
- comparision theorem

SIGNUM AND LIMIT THEOREM

$\lim_{x \to \infty} a_n = l > 0 \Rightarrow \exists \bar{n} \; \forall n > \bar{n} \; a_n > 0$

LOCAL BOUNDNESS ⇒ GLOBAL BOUNDNESS THEOREM

$\lim_{n \to \infty} a_n = l \in \mathbb{R}$

$\exists \bar{n} \; \forall n > \bar{n} \; |a_n| < K$

⇒ all the sequence is bounded

$$\lim_{n\to\infty} \frac{1}{n!+2} \Rightarrow 0$$

$$\lim_{n\to\infty} \underbrace{\frac{1}{n!+2}}_{\inf} \cdot \underbrace{(\sin(n!)+3)}_{\text{bounded}} \Rightarrow 0$$

GEOMETRIC SEQUENCE.

A geometric progression or geometric sequence, is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non zero number called the **common ratio**

Ex

2, 6, 18, 54 is a geometric progression with common ratio 3

2 → 6 = 2·3 → 18 = 6·3 → 54 = 18·3

→

Fixed $a \in \mathbb{R}$ $\quad n \longmapsto a^n \quad n \in \mathbb{N}$

property! $\Rightarrow$ $x_{n+1} = a^{n+1}$, $x_n = a^n$ $\Rightarrow$ $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{a^n} = a$

$\Rightarrow$ the ratio between the term and the following term is always $\underline{a}$

In this geometrical sequence we need to distinguish several cases depending on the value of a!

1) $a > 0$
associating a^n to a^x we have

$$\lim_{n\to\infty} a^n = \lim_{x\to\infty} a^x = \begin{cases} +\infty & a > 1 \\ 1 & a = 1 \\ 0 & 0 < a < 1 \end{cases}$$

2) $a = 0$ $\quad n \geq 1$

$$a^n = \{0\}$$

$$\lim_{n\to\infty} a^n = 0$$

3) $-1 < a < 0$

$a = -\frac{1}{2}$ $\quad n = \left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32} \ldots\right\}$

The $\cdot$ of a_n is oscillating between a positive and a negative value

$$\lim_{x\to\infty} a^n = ?$$

$$\underset{\substack{\downarrow \infty \\ 0}}{-|a|^n} < \underset{\substack{\Downarrow \\ 0}}{a^n} \leq \underset{\substack{\downarrow \infty \\ 0}}{|a|^n}$$

$a_n > |a|^n$

$0 < |a| < 1$

COMPOSITIONS OF SEQUENCE: are possible iff the first sequence is $n \to a_n$ $a_n \in \mathbb{N}$
INVERSE FUNCTION: is possible iff the function is defined on $n \mapsto a_n$ $a_n \in \mathbb{N}$

$\Rightarrow$ I can always apply a function to a value

$$n \longmapsto a_n \xrightarrow{f} f(a_n)$$

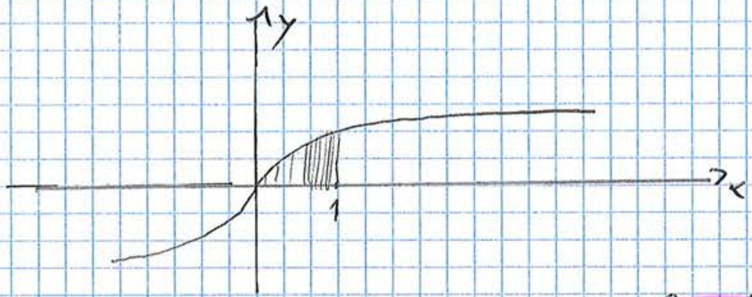
It is not possible if the range of f is differente from the image of n

Ex: $n \to -n^2 \to \lg(-n^2) \Rightarrow$ IMPOSSIBLE

$n > 1$

$$n \longrightarrow \frac{1}{n} \longrightarrow \arctan\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \arctan(0) = 0$$



Theorem

Suppose that $\lim_{n\to\infty} a_n = \lambda \begin{cases} l \in \mathbb{R} \\ \pm\infty \end{cases}$

1) if $\underset{\lambda = l}{l \in \mathbb{R}}$ and g is continuous at l

$$\lim_{n\to\infty} g(a_n) = g\cdot\left(\lim_{n\to\infty} a_n\right) = g(l)$$

2) if $\lambda = \pm\infty$.

$$\lim_{n\to\infty} g(a_n) = \lim_{y\to\lambda} g(y)$$

$$\lim_{n\to\infty} \cos\frac{1}{n^2} = 1$$

$$\begin{cases} \lim_{n\to\infty} \frac{1}{n^2} = 0 \\ \cos x \text{ is continuous at } 0 \end{cases} \Rightarrow \lim_{n\to\infty} \cos\frac{1}{n^2} = \cos\left(\lim_{n\to\infty}\frac{1}{n^2}\right) = 1$$

$$\lim_{n\to\infty} e^{-\sqrt{n}} \quad \begin{cases} \lim_{n\to\infty} -\sqrt{n} = -\infty \\ y = e^x \to \text{continuous} \end{cases} \Rightarrow \lim_{n\to\infty} e^{-\sqrt{n}} = \lim_{y\to-\infty} e^y = 0$$

$y = -\sqrt{n}$

SUFFICIENT CONDITION FOR EXISTENCE OF LIMITS

non monoton sequences => lim ∄

ex

$a_n = (-1)^n$ for $a \leq -1$

=> if I have a monotone sequence or function it goes always or to $\pm\infty$ or to an asymptot, in both case we do find a limit

=> so we can tell there is a strong and deep relation between Monotony and existence of limit

Monotony $\Longrightarrow$ EXIST A LIMIT

$\not\Longleftarrow$

$\lim_{x \to \infty} \frac{\sin x}{x} = 0$

↳ this function is not monotone but there is a limit

=> DEFINITION OF MONOTONOUS INCREASING SEQUENCE

$\forall n \geq n_0$ $\quad a_n \geq a_{n+1}$

DEFINITION OF MONOTONOUS DECRISING SEQUENCE

$a_n \leq a_{n+1}$

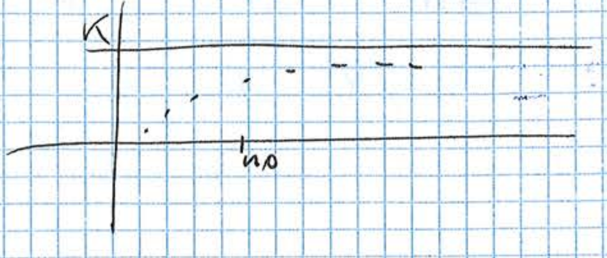
SEQUENCE BOUNDED FROM ABOVE

$\exists K \in \mathbb{R} \quad a_n \leq K \quad \forall n \geq n_0$

SEQUENCE BOUNDED BY BELOW

$\exists K \in \mathbb{R} \quad a_n \geq K \quad \forall n \geq n_0$

$\exists K \in \mathbb{R}$



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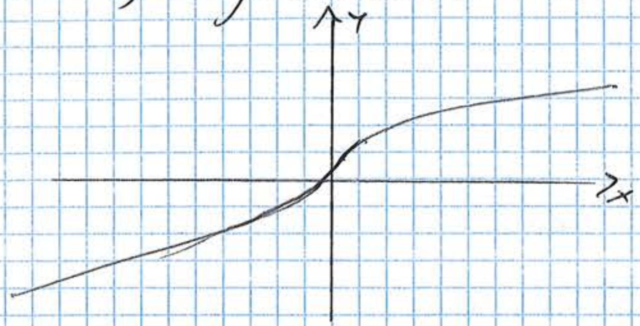
$\Rightarrow$ we do have similar results for functions!

Ex 1) $f(x) = \sqrt[3]{x}$

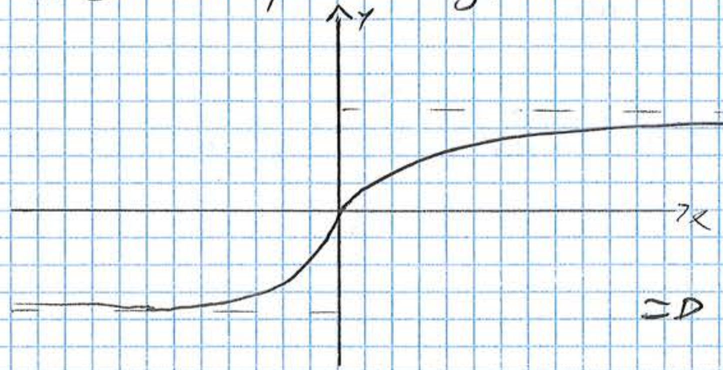


$\exists \lim_{x \to \infty} \sqrt[3]{x} = \sup f(x) = +\infty$

$\exists \lim_{x \to -\infty} \sqrt[3]{x} = \inf f(x) = -\infty$

function unbounded

Ex 2 $y = \arctan x$

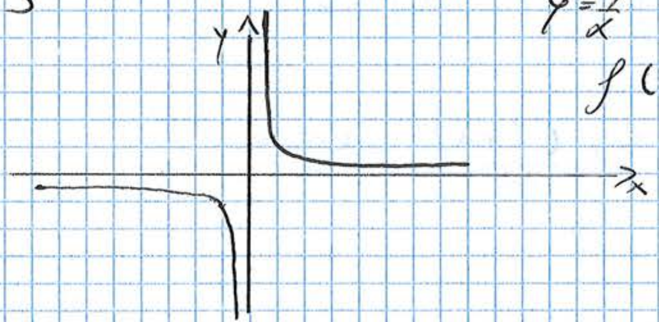


$\exists \lim_{x \to +\infty} \arctan x = \frac{\pi}{2} = \sup f(x)$

$\exists \lim_{x \to -\infty} \arctan x = -\frac{\pi}{2} = \inf f(x)$

$\Rightarrow$ function globally bounded

Ex 3



$y = \frac{1}{x}$ $\quad \lim_{x \to +\infty} \frac{1}{x} = \inf f(x) = 0$

$f\ (0\ +\infty)$ decreasing function

$\lim_{x \to 0^+} \frac{1}{x} = \sup_{x \in (0,1)} \left(\frac{1}{x}\right) = +\infty$

Ex 4

$f(x)$ defined in $(-1, 1)$

$\lim_{x \to -1^+} f(x) = -\infty$ $\quad$ What happens in $x = 0$

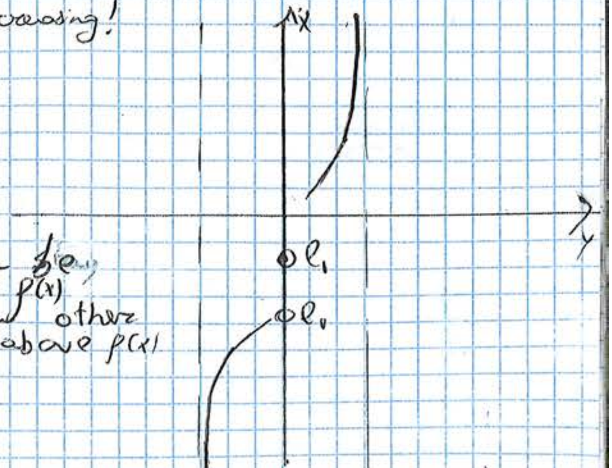
$\forall x \in (-1,1)\ f(x)$ increasing!

$\lim_{x \to 1^-} f(1) = +\infty$

– we study the limit in 0

$\lim_{x \to 0^-} f(x) = \sup_{x \in (-1,0)} f(x) = l_1$

$\hookrightarrow$ it cannot be $+\infty$ since the function is increasing $\Rightarrow$ l_1 can be just $f(x)$ or and other point above $f(x)$



$\lim_{x \to 0^+} f(x) = \inf_{x \in (0,1)} f(x) = l_2$

$\Rightarrow$ if $l_1 = l_2 \Rightarrow$ cont function

if $l_1 \neq l_2 \Rightarrow$ not continuous

DERIVATIVE

f is defined in $I(x_0)$ $\quad f: \text{dom} f \subseteq \mathbb{R} \longmapsto \mathbb{R}$

- increment of the independent variable (increment of x axis)

$$h = \Delta x = x - x_0 \implies x = x_0 + h$$

* h can be a positive or negative quantity depending on x position respect to x_0

- increment of the dependent value (increment of y-axis)

$$\Delta f = f(x) - f(x_0) \implies f(x_0+h) - f(x_0)$$

-> DEFINED $\forall x \in I(x_0) \setminus \{x_0\}$

The ratio $\frac{\Delta f}{\Delta x}$ is called difference quotient of f between x_0 and x

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0+h) - f(x_0)}{x - x_0}$$

This ~~mean~~ represent the absolute increment of the function when x_0 goes to x

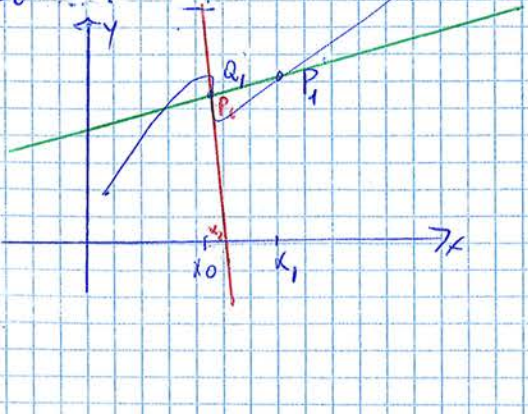
$\frac{\Delta f}{\Delta x}$ -> how much the function has varied while x has increased

What happens when I take a smaller interval of Δx?

We do know that the line passing through the point $P(x, f(x_0))$ and $Q(x_0+h; f(x_0+h))$ is a secant.

$$\frac{f(x) - f(x_0)}{x - x_0} \implies \boxed{\text{secant}}$$

When $x \to x_0$ the two points on the graph are getting closer



so when x_1 goes to x we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \boxed{\text{tangent}}$$

REMARKS!

→ f is continuous at x_0 doesn't imply that f is differentiable at x_0

f cont x_0 $\not\Rightarrow$ f diff at x_0

→ f not cont at x_0 ⇒ f is not differentiable at x_0

⇒ Mantissa, sign not derivable at 0, 1 and so on

Definition of elementary derivatives

1) $f(x) = K$ $\qquad f'(x) = 0$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{K-K}{x-x_0} = \frac{0}{x-x_0} = 0$$

$f(x) = x$

2) $f(x) = x$ $\qquad f'(x) = 1$

$f(x) = x^n$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{x-x_0}{x-x_0} = 1$$

$f(x) = \sin x$

3) $f(x) = \sin x$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{\cancel{x-x_0}\, h} = \lim_{h\to 0} \frac{\sin(x_0+h)-\sin(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{\sin x_0 \cos h + \sin h \cos x_0 - \sin x_0}{h} = \lim_{h\to 0} \frac{\sin x_0(\cos h - 1) + \sin h \cos x_0}{h}$$

$$= \lim_{h\to 0} h\cdot \sin x_0 \underbrace{\frac{(\cos h - 1)}{h^2}}_{\to -\frac{1}{2}} + \lim_{h\to 0} \underbrace{\frac{\sin h}{h}}_{\to 1} \cos x_0 = 0 + \cos x_0 = \cos x_0$$

4) $f(x) = \cos x$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{\cos(x_0+h)-\cos(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{\cos x_0 \cos h - \sin x_0 \sin h - \cos x_0}{h} = \lim_{h\to 0} \frac{h\cdot \cos x_0 (\cos h - 1)}{h^2} \to -\frac{1}{2} = 0$$

$$- \lim_{h\to 0} \frac{\sin x \sin h}{h} \to 1 = 0 - \sin x = \boxed{-\sin x}$$

5) $y = a^x$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{a^{x+h}-a^{x_0}}{h} = \lim_{h\to 0} \frac{a^{x_0}(a^h - 1)}{h} \to \lg a$$

$= a^x \lg a$

6] $f(x) = e^x \longrightarrow f'(x) = e^x$

↳ the derivate is equal to its function (also for $y = 0$)

7) $y = \frac{1}{x^n}$ $\quad \left(\frac{1}{x^n}\right)' = \frac{-n x^{n-1}}{x^{2n}} = -n x^{-n-1}$

Ex

$y = \tan x = \frac{\sin x}{\cos x}$ $\quad y' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$ $\nearrow \frac{1}{\cos^2 x}$ $\searrow 1 + \tan^2 x$

true for $\cos x \neq 0$

$x \neq \frac{\pi}{2} + k\pi$

DERIVATE OF COMPOSITION
CHAIN RULE

$x_0 \xrightarrow{f(x)} f(x_0)$

$y_0 \xrightarrow{g} g(y_0) = g(f(x))$

f differentiable at x_0
g differentiable at y_0 $\Big\} \Rightarrow g \circ f$ is differentiable at x_0

$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

$y = 2^{\sin x}$

$D(2^{\sin x})$ $\quad x \xrightarrow{f} \sin x$

$y \longrightarrow 2^y$

$f(x) = \sin x$ $\quad f'(x) = \cos x$

$g(x) = 2^y$ $\quad g'(x) = 2^y \cdot \lg 2$

$y' = 2^{\sin x} \lg 2 \cdot \cos x$

$D \sin(x^2) = (\cos x^2) \cdot 2x$

$D \sin^2 x = 2 \sin x \cos x$

DERIVATE OF THE INVERSE FUNCTION

$y_0 = f(x_0)$ $\quad f'$

- f is continuous, invertible in $I(x_0)$
- f is diff at x_0

$f'(x) \neq 0$

$\Rightarrow f^{-1}(y)$ is differenziable at $y_0 = f(x_0)$

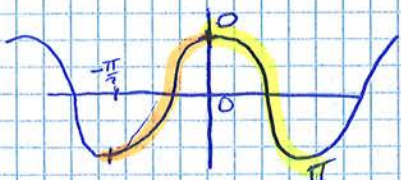
$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$ $\quad f'(3) = 4$

$(f^{-1})'(4) = \frac{1}{f'(3)}$

~~DI~~

$y = \arccos x$

$x = \cos y$



~~cos x def in~~

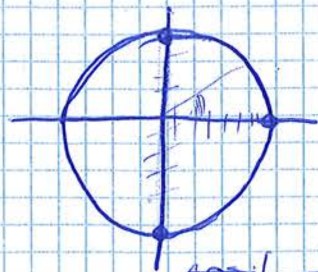
$f^{-1}(\cos x) = [-1; 1]$

dom $\cos x = [0; \pi]$

dom $\arccos y = [-1; 1]$

$f^{-1}(\arccos y) = [0; \pi]$

$$D \arccos x = \frac{1}{D \cos y} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}}$$



$0 < x < \frac{\pi}{2}$ $\quad \boxed{\sin x = \sqrt{1-\cos^2 x}}$

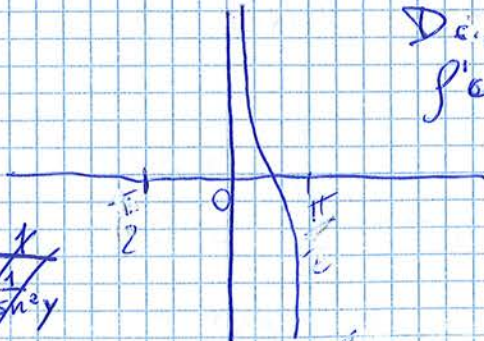
$0 < y < \frac{\pi}{2}$ $\quad \sin y = \sqrt{1-\cos^2 y}$

$$y = \cot g\, x = \frac{\cos x}{\sin x}$$

$$y' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -(1 + \cot g^2 x) \text{ or } -\frac{1}{\sin^2 x}$$

$y = \operatorname{arccotg} x$

$x = \cot g\, y$



$D \cot g x = (0; \pi)$

$f' \cot g x = \mathbb{R}$

$D \operatorname{arccotg} x = \mathbb{R}$

$f' \operatorname{arctg} x = (0; \pi)$

$$D \operatorname{arctg} x = \frac{1}{D \cot g y} = \frac{1}{-\frac{1}{\sin^2 y}}$$

$$= -\frac{1}{1+\cot g^2 y} = -\frac{1}{1+x^2}$$

$f(x) = x + e^x$

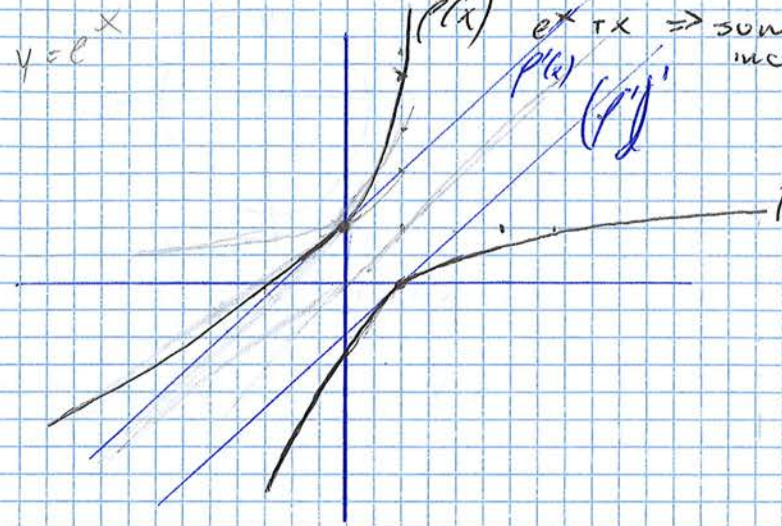
it is invertible?

strictly MONOTONE ⇒ INJECTIVE ⇒ INVERTIBLE

$y = x + e^x$ ⇒ strictly monotone

"as x increases e^x become much higher"

$e^x + x$ ⇒ sum of two strictly increasing functions



$f(0) = 1 \Rightarrow f^{-1}(1) = 0$

$f^{-1}(x)$ exists but it is impossible to define its function

$$(Df^{-1})(y_0) = \frac{1}{f'(x_0)} = \frac{1}{1+e^x}$$

$$h(x) = f(x)^{g(x)}$$

$$\text{dom } f(x)^{g(x)} = \{x \in \mathbb{R} : f(x) > 0\}$$

$$h(x) = f(x)^{g(x)} = e^{g(x)\lg f(x)}$$

$$h'(x) = e^{g(x)\cdot \lg f(x)} \cdot \left[g'(x) \cdot \lg f(x) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x) \right]$$

$$= f(x)^{g(x)} \cdot \left(\lg(f(x))^{g'(x)} + \frac{g(x) \cdot f'(x)}{f(x)} \right)$$

$x \cdot e^x = y_0$

$$D(f^{-1})(y_0) = \frac{1}{f'(x_0)}$$

1. f differentiable at x_0 $\Rightarrow$ f cont at x_0
- f is not continuous at x_0 $\Rightarrow$ f is not differentiable at x_0

f continuous at x_0 →
- f(cont) and indifferentiable at x_0
- f(cont) and ~~not~~ diff at x_0

Points of non differentiability

f diff at x_0 $\Leftrightarrow$ $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ $\exists$ finite

↳ if this limit is not finite we have a point of ~~non~~ non differentiability

$\Rightarrow$ if we consider $\lim_{x \to x_0^+} f'(x)$ and $\lim_{x \to x_0^-} f'(x)$ we ~~[illegible]~~ can have a CUSP or CORNER POINT or a VERTICAL INFLECTION POINT

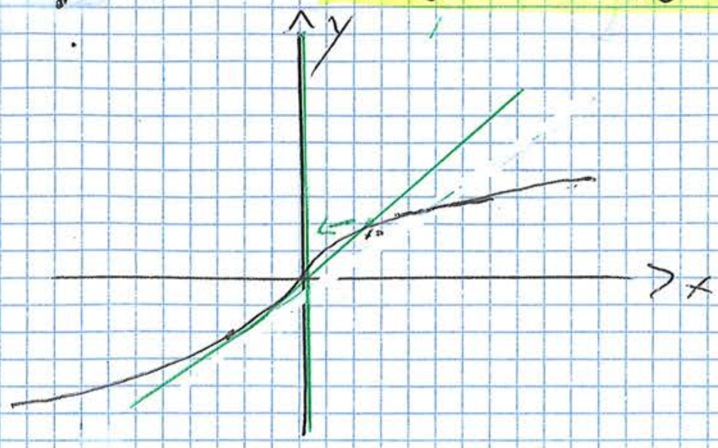
1] $g(x) = \sqrt[3]{x}$

$D'(gx) = \frac{1}{3\sqrt[3]{x^2}}$ Differentiable $\forall x \in \mathbb{R} \setminus \{0\}$

$\lim_{x \to 0^+} \frac{1}{3\sqrt[3]{x^2}} = +\infty = \lim_{x \to 0^-} \frac{1}{3\sqrt[3]{x^2}}$

$\Rightarrow$ VERTICAL INFLECTION POINT

if $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x) = +\infty$ or $-\infty$



$f(x) = |x|$

$x \geq 0$

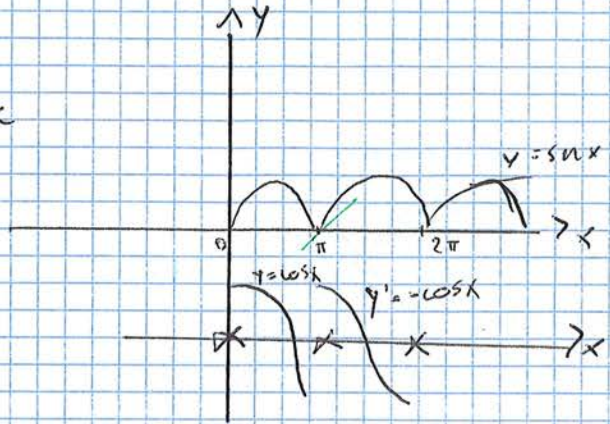
$\lim_{x \to 0^+} \frac{x}{x} = 1 \qquad \lim_{x \to 0^-} \frac{-x}{x} = -1 \Rightarrow$ CORNER POINT

$y' = \sin x$ or $\frac{|x|}{x}$

$f(x) = |\sin x|$

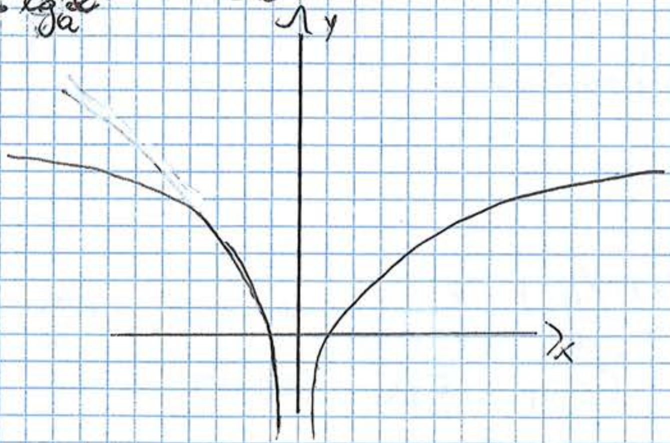
$f'(x) = \frac{|\sin x|}{\sin x} \cdot \cos x$

f is differentiable $\forall x \in \mathbb{R} \neq k\pi$



$h(x) = \lg_a |x| = \begin{cases} \lg_a x & x > 0 \\ \lg_a(-x) & x < 0 \end{cases}$

$h'(x) = \frac{1}{|x|} \cdot \lg_a e$



$D \lg_a|x| = \frac{1}{|x| \lg a} \qquad \forall x \neq 0$

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$$f(x) = \begin{cases} \arctan x + x & x \le 0 \\ \alpha \sin x + \beta \cos x & x > 0 \end{cases}$$

$\alpha, \beta \in \mathbb{R}$

1) $\alpha, \beta \in \mathbb{R}$

2) $f'(x)$? $\Rightarrow$ diff $\forall x \in \mathbb{R}$

1) $\arctan x + x$ continuous $\forall x < 0$

$\alpha \sin x + \beta \cos x$ continuous $\forall x > 0$

$f(0) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \Rightarrow \lim_{x \to 0} f(x)$ exists $= f(0)$

$$\begin{cases} \lim_{x \to 0^+} \arctan x + x & x = 0 \\ \lim_{x \to 0^-} \alpha \underbrace{\sin x}_{\to 0} + \beta \underbrace{\cos x}_{\to 1} = \beta \end{cases} \Bigg] \ \beta = 0$$

$$\Rightarrow f(x) = \begin{cases} \arctan x + x & x \le 0 \\ \alpha \sin x & x > 0 \end{cases} \qquad C^0 \mathbb{R}$$

2) $x < 0$ $\quad \arctan x + x$ is diff $\quad f'(x) = \frac{1}{1+x^2} + 1$

$x > 0$ $\quad \alpha \sin x$ diff $\quad f'(x) = \alpha \cos x$

$$\lim_{x \to 0^-} \frac{\arctan x + x}{x} = \lim_{x \to 0^-} \underbrace{\frac{\arctan x}{x}}_{\to 1} + 1 = 2$$

$$\lim_{x \to 0^+} \frac{\alpha \sin x}{x} = \alpha$$

$\Rightarrow \alpha = 2$

$$\Rightarrow f(x) = \begin{cases} \arctan x + x & \le 0 \\ 2 \sin x & x > 0 \end{cases} \qquad \text{diff } \forall x \in \mathbb{R}$$

$$f(x) = \begin{cases} (x-\beta)^2 - 2 & x \geq 0 \\ \alpha \sin x & x < 0 \end{cases}$$

1) $\alpha, \beta \in \mathbb{R}$

2) $f(x)$? $\Rightarrow$ diff $\forall x \in \mathbb{R}$

1) $(x-\beta)^2 - 2$ continuous $\forall x > 0$

$\alpha \sin x$ continuous $\forall x < 0$

$f(0) = \beta^2 - 2$

$$\begin{cases} \lim_{x \to 0^+} \alpha \sin x = 0 \\ \lim_{x \to 0^-} (x-\beta)^2 - 2 = \beta^2 - 2 \end{cases}$$

$\beta^2 - 2 = 0$

$\beta = \pm\sqrt{2}$

$\beta_1 = \sqrt{2}$ $\qquad$ $\beta_2 = -\sqrt{2}$

$$y = \begin{cases} (x-\sqrt{2})^2 - 2 & x \geq 0 \\ \alpha \sin x & x < 0 \end{cases} \in C^0\mathbb{R} \text{ or } \quad y \begin{cases} (x+\sqrt{2}) - 2 & x \geq 0 \\ \alpha \sin x & x < 0 \end{cases} \in C^0\mathbb{R}$$

1) $x < 0$

$y = \alpha \sin x$ diff $\qquad f'(x) = \alpha \cos x$

$x > 0$ $\quad y = (x-\sqrt{2})^2 - 2$ diff $\quad f'(x) = 2(x-\beta)$

$$\lim_{x \to 0^+} \frac{\alpha \sin x}{x} = \alpha$$

$$\lim_{x \to 0^-} \frac{(x-\sqrt{2})^2 - 2 - [(-\sqrt{2})^2 - 2]}{x} = \lim_{x \to 0^-} \frac{x^2 - 2\sqrt{2}x \cancel{+2} \cancel{-2} \cancel{-2} \cancel{+2}}{x}$$

$$\lim_{x \to 0^-} \frac{\cancel{x}(x - 2\sqrt{2})}{\cancel{x}} = -2\sqrt{2}$$

$$\boxed{\alpha = -2\sqrt{2}}$$

$$y = \begin{cases} (x-\sqrt{2})^2 - 2 & x \geq 0 \\ -2\sqrt{2} \sin x & x < 0 \end{cases}$$

2) $x > 0$ $\beta = -\sqrt{2}$

$x > 0$ $\quad y = (x+\sqrt{2})^2 - 2$ diff $\Rightarrow$ $y' = 2(x+\sqrt{2})$

$x < 0$ $\quad y = \alpha \sin x$ diff $\Rightarrow$ $y' = \alpha \cos x$

$$\lim_{x \to 0^+} \frac{\alpha \sin x}{x} = \alpha \qquad \lim_{x \to 0^-} \frac{(x+\sqrt{2})^2 - 2 - [(\sqrt{2})^2 - 2]}{x}$$

$$\lim_{x \to 0^-} \frac{x^2 + 2\sqrt{2}x \cancel{+2} \cancel{-2}}{x} = 2\sqrt{2}$$

$$y = \begin{cases} (x+\sqrt{2})^2 - 2 & x \geq 0 \\ 2\sqrt{2} \sin x & x < 0 \end{cases} \quad \alpha = 2\sqrt{2}$$

1) $f(x) = e^x$ cont and diff on $\mathbb{R}$ $\Rightarrow e^x \in C^\infty(\mathbb{R})$

$f^{(k)}(x) = e^x$

---

2) $f(x) = a^x$ cont and diff on $\mathbb{R}$ $\Rightarrow a^x \in C^\infty(\mathbb{R})$

$f^{(k)}(x) = a^k \cdot \lg^k a$

$y = \sin x$ con and diff on $\mathbb{R}$

LOOP
$y = \sin x$
$y' = \cos x$
$y'' = -\sin x$
$y''' = -\cos x$
$y^{IV} = \sin x$

all con on $\mathbb{R}$ — $\sin x \in C^\infty(\mathbb{R})$, $\cos x \in C^\infty(\mathbb{R})$

Ex

$D^{79} \sin x = -\cos x = (D^{76} \sin x)''' = \sin'''$

$D^{37} \sin 5x = 5^{37} \cos x = (D^{36} \sin 5x)' = (D^{35} \sin 5x)''$

3) $f(x) = x^n$

$f'(x) = n x^{n-1}$

$f''(x) = n(n-1)x^{n-2}$

$f^n(x) = n(n-1)(n-2)\dots = n!$

$f^{(n+1)}(x) = 0$

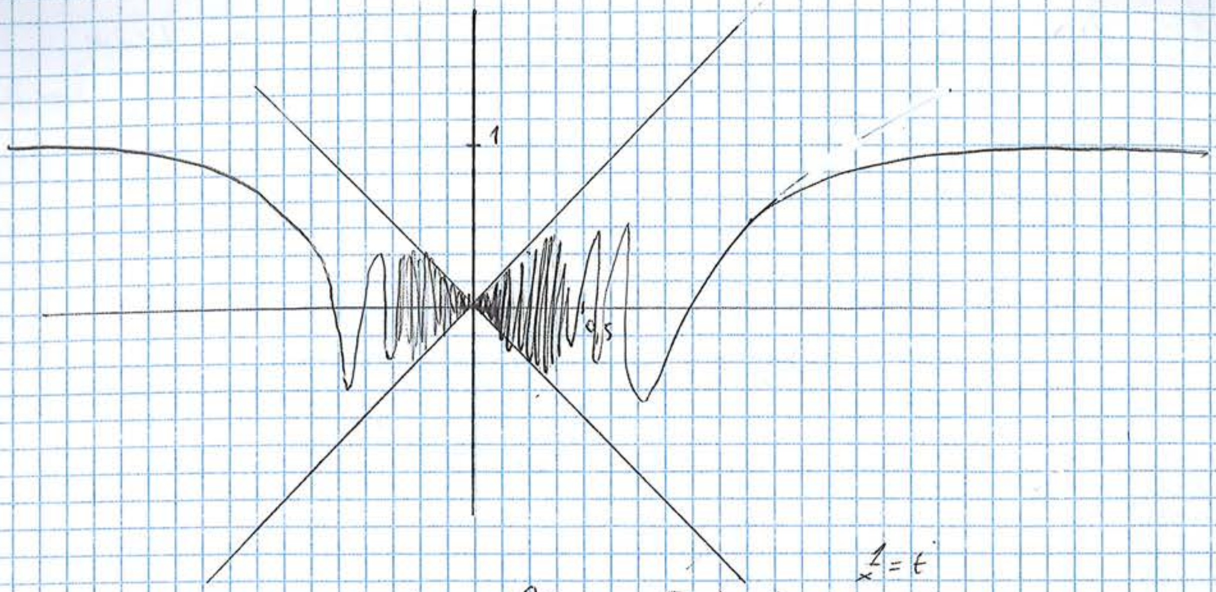
$f^{(n+2)}(x) = 0$

$P_n(x) \in C^\infty(\mathbb{R})$

4) $\dfrac{P_n(x)}{Q_n(x)} \Rightarrow \dfrac{P'(x)\cdot Q_n(x) + P(x)\cdot Q'(x)}{(Q_n(x))^2} \neq 0 \to \dfrac{---}{(Q_n(x))^4}$

~~Pn~~ $\dfrac{P_n(x)}{Q_n(x)}$ is alway derivable when $Q_n \neq 0$

$f(x) = \dfrac{P_n(x)}{Q_n(x)} \in C^\infty(\mathrm{dom}\, f)$



$\frac{1}{x} = t$

$$\lim_{x \to +\infty} x \sin\frac{1}{x} = \lim_{t \to 0} \frac{\sin t}{t} = 1$$

$$f(x) = x^2 \sin\frac{1}{x}$$

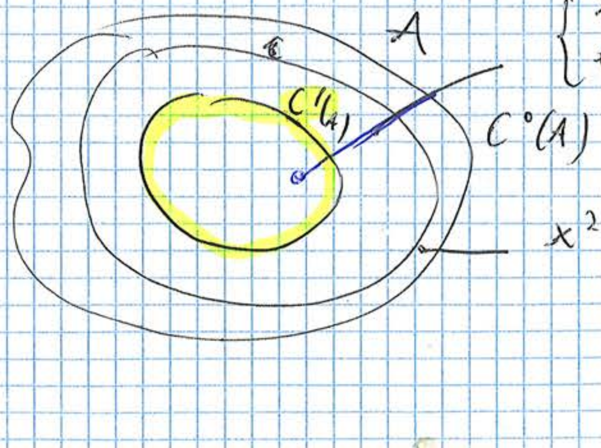
$$\tilde{f}(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\tilde{f}'(x) = \begin{cases} 2x \cdot \sin\frac{1}{x} + (\cancel{x^2} \cdot \cos\frac{1}{x})\left(-\frac{1}{\cancel{x^2}}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\Rightarrow \tilde{f}'(x) =$ [illegible] differentiable $\forall x \in \mathbb{R}$

$$\lim_{x \to 0} \underbrace{2x \sin\frac{1}{x}}_{\to 0} - \cos\frac{1}{x} \quad \nexists$$

$\tilde{f}'(x)$ ~~defi~~ differentiable in $\mathbb{R}$ but not continuous



$$\begin{cases} x^2 \sin\frac{1}{x} \\ 0 \end{cases}$$

$x^2 \sin\frac{1}{x}$ differentiable but not continuous

8) At which point of the graph of $f(x) = 5x^3$ is the tangent line: parallel to $y = 2x$?
orthogonal $y = x + 5$

$f'(x) = 15x^2$

$15x^2 = 2$

$x^2 = \frac{2}{15}$ $\quad x = \pm\sqrt{\frac{2}{15}}$

$y = 5 \sqrt[2]{\left(\frac{2}{15}\right)^3}$

parallel $\left(y - 5\sqrt{\left(\frac{2}{15}\right)^3}\right) = 2\left(x - \sqrt{\frac{2}{15}}\right)$

$x^2 = \frac{1}{30}$ $\quad x = +\sqrt{\frac{1}{30}}$

$y = 5\sqrt{\left(\frac{1}{30}\right)^3}$

orthogonal $\left(y - 5\sqrt{\left(\frac{1}{30}\right)^3}\right) = \frac{1}{2}\left(x - \sqrt{\frac{1}{30}}\right)$

9) Prove that the tangent lines to the graphs of the function $f(x) = x^3$ and $g(x) = \frac{1}{3x}$ at every point where $x \neq 0$, are orthogonal

point $= t$ $\quad x = t$

$f'(x) = 3x^2$

$3t^2$

$f(t) = t^3$ $\quad g(t) = \frac{1}{3t}$

$(y - t^3) = 3t^2(x - t)$ $\quad y = 3t^2x + 4t^3$

$g'(x) = -\frac{1}{3x^2}$

$\left(y - \frac{1}{3t}\right) = -\frac{1}{3t^2}(x - t)$

$y - \cancel{\frac{1}{3t}} = -\frac{1}{3t^2}x + \cancel{\frac{1}{3t}}$

$y = -\frac{1}{3t^2}x$

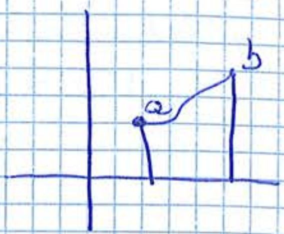
$m_1 = 3t^2$ $\quad m_1 \cdot m_2 = -1$

$m_2 = -\frac{1}{3t^2}$ $\quad \cancel{3t^2} \cdot -\frac{1}{\cancel{3t^2}} = -1$

$\Rightarrow$ orthogonal

Interval of $\mathbb{R}$ → Bounded → OPEN / HAL-OPEN / CLOSED
→ Unbounded

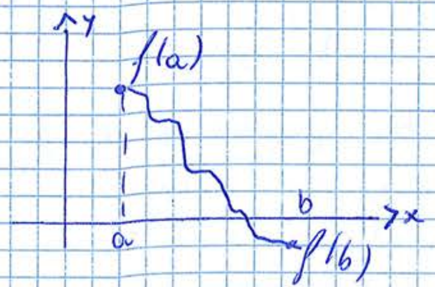
f continuous on an Interval
↳ f continuous on $[a,b]$ $\forall x \in [a,b]$
f right continuous at a
f left continuous at b



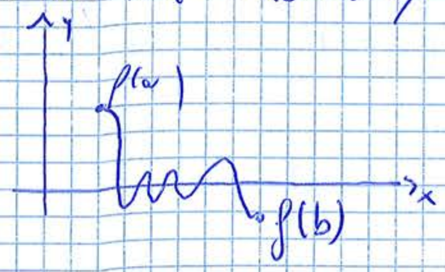
DEF existence of zeros
$x_0 \in dom f$ is zero $f(x)$ if $f(x_0) = 0$

Theorem of existence of zeros

- $\left.\begin{array}{l} f \text{ continuous on } [a;b] \\ f(a) \cdot f(b) < 0 \end{array}\right\} \Rightarrow \exists x_0 \in (a,b) : f(x_0) = 0$



the zero is not necessary unique!



$\left\{\begin{array}{l} \cdot f \text{ continuous on } [a,b] \\ \cdot f \text{ is strictly monotone } [a,b] \\ f(a) \cdot f(b) < 0 \end{array}\right. \Rightarrow \exists! x_0 \in (a;b) : f(x_0) = 0$

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Example

$x - \cos x = 0$

$x = \cos x$

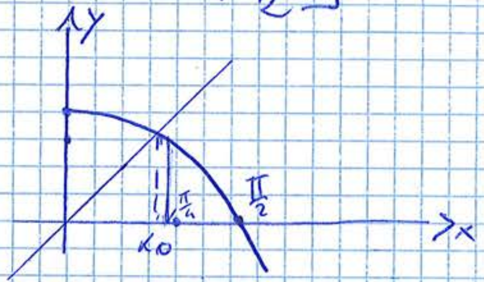
$f(x) = x$ continuous on $\mathbb{R}$ and strictly increasing on $\mathbb{R}$

$g(x) = \cos x$ continuous on $\mathbb{R}$ and strictly decreasing on $[0; \frac{\pi}{2}]$

$f(0) = 0$ $\quad f(\frac{\pi}{2}) = \frac{\pi}{2}$

$g(0) = 1$ $\quad g(\frac{\pi}{2}) = 0$

$\Rightarrow \exists x_0 \in [0; \frac{\pi}{2}] : x - \cos x = 0$



$f(0) = 0$ $\quad f(\frac{\pi}{4}) = \frac{\pi}{4}$ $\quad \frac{\pi}{4} < \frac{1}{\sqrt{2}}$ $\quad (\pi)^2 < (2\sqrt{2})^2$

$g(0) = 0$ $\quad g(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ $\quad \cancel{\frac{\sqrt{2}\pi}{4} < \frac{\sqrt{2}}{2}}$ $\quad \pi^2 < 8$

NO

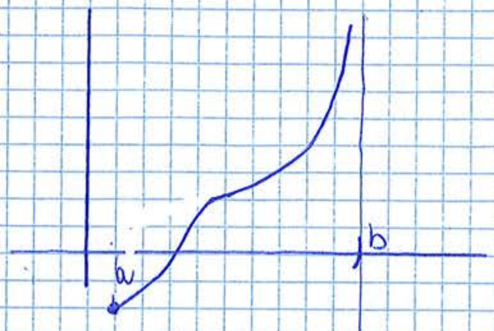
$\frac{\pi}{4} > \frac{\sqrt{2}}{2}$

$\exists x_0 \in [0; \frac{\pi}{4}] : x - \cos x = 0$

$x_0 = 0{,}785$

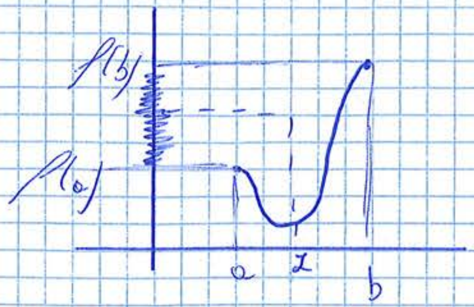
What if $f(x)$ and $g(x)$ are continuous on an open interval

Th] • f cont (a,b), $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ both exist and have different sign $\Rightarrow \exists x_0 \in (a,b) : f(x_0) = 0$



INTERMEDIATE VALUE THEOREM I VERSION

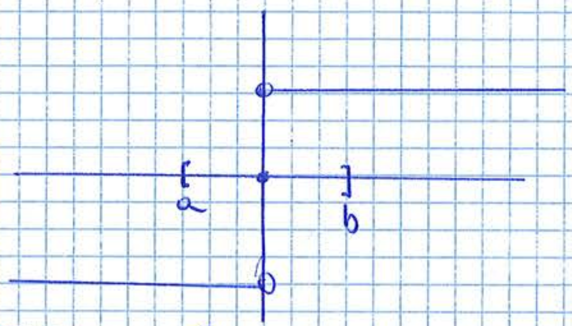
f continuous on $[a,b]$ $\Rightarrow$ f assumes all the values between $f(a)$ and $f(b)$



f assumes all the values on the Y axis

$\forall z \in [f(a); f(b)] \Rightarrow \exists x_0 \in [a;b] \; f(x_0) = z$

if u take a non continuous function this is generally not true



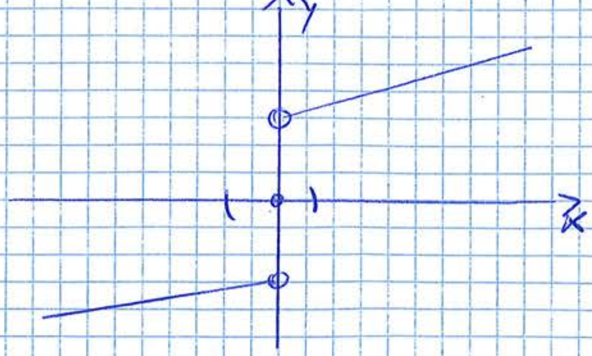
$f(x)$ assumes the values $-1, 0, +1$

$f(x)$ doesn't assume all the intermediate values

[ex $\sqrt{2}, \frac{\pi}{2}$]

Theorem f continuous on an Interval $\Rightarrow f(I)$ is an interval

NOT TRUE FOR NOT CONTINUOUS



$$\begin{cases} x-1 & x \le 0 \\ 0 & x = 0 \\ x+1 & x > 0 \end{cases}$$

dom $f = \mathbb{R}$

Range $(-\infty, -1) \cup \{0\} \cup (1, +\infty)$

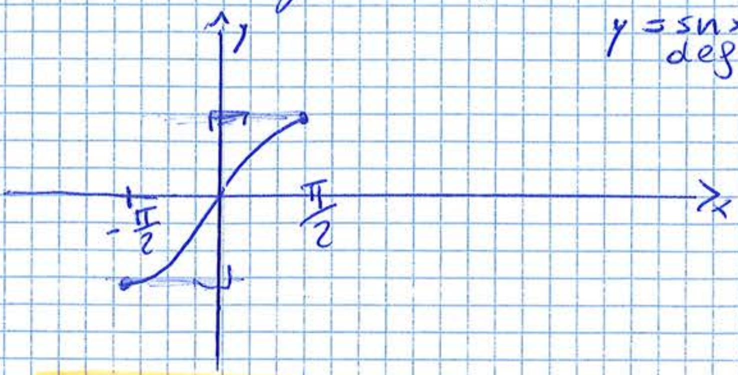
$f(I)$ is not an Interval

$[a,b]$ closed and bounded

$[a,b] \xrightarrow{f\text{ continuous}} f([a,b])$ closed and bounded

$f \in C^0([a;b])$

$y = \sin x$ def $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \rightarrow [-1,1]$



Weistrass Theorem

f continuous in $[a,b]$

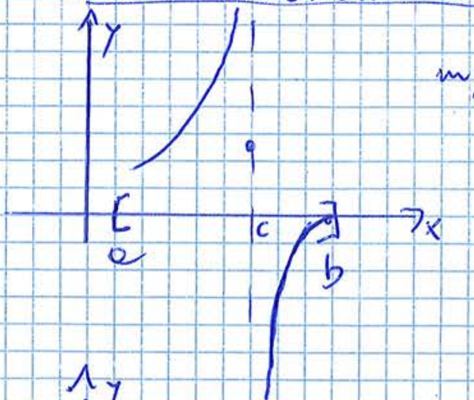
f is bounded on $[a,b]$

f admits the absolute maximum M in $[a,b]$ — $\exists\, x_M \in [a,b] : f(x_M) = M$

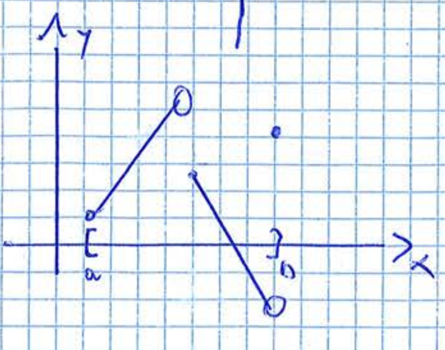
f admits the absolute minimum m in $[a,b]$ — $\exists\, x_m \in [a,b] : f(x_m) = m$

What if f is not continuous but is just defined on $[a,b]$

$\Rightarrow$ the function is not necessary bounded!



my function can have a vertical asimptot for $x = c$ and since it is defined in all $[a,b]$ we have a point corresponding to c



For a not continuous function there is not necessary a Maximum a Minimum

STUDY OF DIFFERENTIABLE FUNCTIONS

f dom f $\quad I \subseteq$ dom f

Definition $\quad x_0 \in$ dom f $\quad$ Absolute maximum point or Global maximum point $\Rightarrow \forall x \in$ dom f $\quad f(x) \leq f(x_0)$

Absolute minimum point or global minimum point $\Rightarrow \forall x_0 \in$ dom f $\quad f(x) \geq f(x_0)$

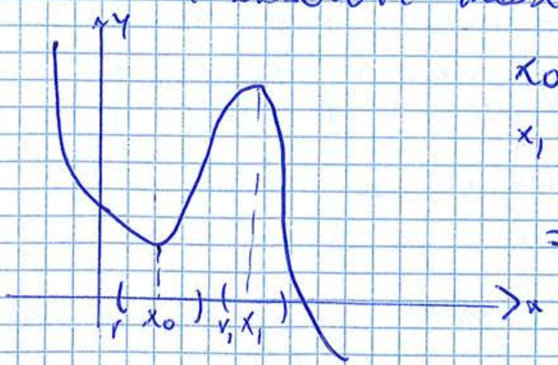
DEFINITION $\quad x_0 \in$ dom f $\quad$ Relative maximum point $\Rightarrow$
$\exists I_r(x_0) \ \forall x : x \in I_r(x_0) \cap \text{dom} f \Rightarrow f(x) \leq f(x_0)$

Relative minimum point $\Rightarrow$
$\exists I_r(x_0) \ \forall x : x \in I_r(x_0) \Rightarrow f(x) \geq f(x_0)$

- in absolute maximum or minimum we have to check the properties for all the point of the domain
- in relative max/min we have to check the properties for the point of the $I_r(x_0)$

REMARKS]
x_0 is an absolute max point or x_0 is an absolute min point $\Rightarrow$ x_0 is also a relative max point, x_0 is also a relative min point

if x_0 is a relative min/max is not necessarly an absolute max/min point



x_0 relative minimum on $I_r(x_0)$
x_1 relative maximum on $I_{r_1}(x_1)$

$\Rightarrow$ no global max/min

Relation between extreme points and stationary points

Fermat's Theorem

$$\left\{\begin{array}{l} f \text{ differentiable at } x_0 \\ x_0 \text{ is an extremum point} \end{array}\right\} \Rightarrow f'(x_0) = 0$$

Is the opposite true??

$$\left\{\begin{array}{l} f \text{ differentiable at } x_0 \\ f'(x) = 0 \end{array}\right. \not\Rightarrow x_0 \text{ is an extremum}$$

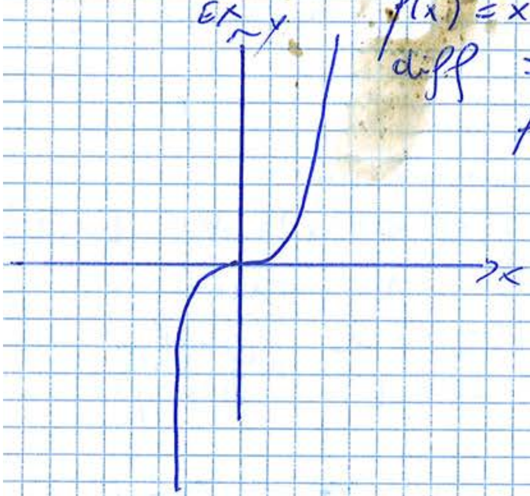
NO!

Ex.

$f(x) = x^3$

diff $\Rightarrow$ $f'(x) = 3x^2$ $\Rightarrow$ 0 is not an extremum point

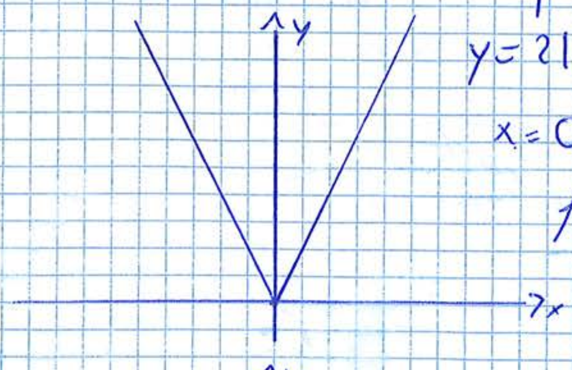
$f'(0) = 0$



What if

x_0 is an extremum point $\Rightarrow$ is $f(x)$ always derivable at x_0

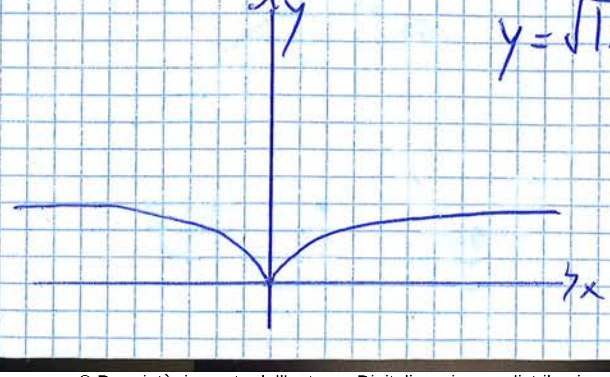
NO!!



$y = |x|$

$x = 0$ global minimum

$f(x)$ not differentiable at $x = 0$



$y = \sqrt{|x|}$

$x = 0$ global minimum

$f(x)$ not differentiable at $x = 0$

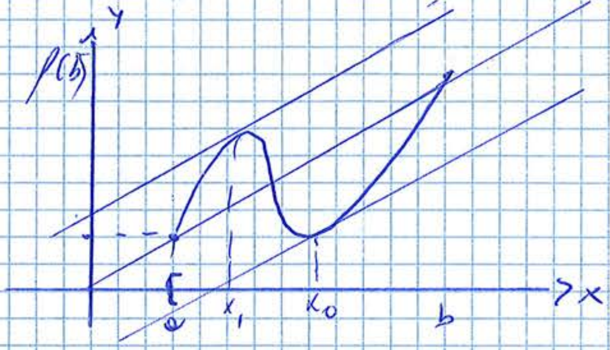
and what if

$$\begin{cases} f \text{ cont } [a,b] \\ f \text{ der } (a,b) \\ \cancel{f(a) = f(b)} \end{cases}$$

Lagrange Theorem (mean value theorem)

↳ born in Turin

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ derivable on } (a,b) \end{cases} \Rightarrow \exists x_0 \; x \in (a,b): \; f'(x_0) = \frac{f(b)-f(a)}{b-a}$$



$\frac{f(b)-f(a)}{b-a}$ slope line joining $(a, f(a))$ and $(b; f(b))$

$f'(x_0)$ = slope of the tangent at x_0

1] x_0 is not unique ($x_0 - x_1$)

2] Rolle's is a particular case of Lagrange's theorem where we have in the hypothesis $f(a)=f(b)$

II Finite increment Formula

- f differentiable on an open interval I

$x_1, x_2 \in I$ $\quad x_1 < x_2$ ($\Rightarrow$ no end points)

$$\exists t \in (x_1, x_2): \underbrace{f(x_2) - f(x_1)}_{\substack{\Delta y \\ \text{increment} \\ \text{of the function}}} = f'(t) \underbrace{(x_2 - x_1)}_{\substack{\Delta x \\ \text{increment} \\ \text{of the variable}}}$$

Law of proportionality

$$\Delta y = (K) \cdot \Delta x$$

Δy increases proportionally to Δx

→ Δf is proportional to Δy

→ The coeff K depends on the points I chose

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Theorem 2

f diff on I open

1) $f'(x) \geq 0 \ \forall x \in I \Rightarrow f$ is increasing on I
   $\leq 0 \qquad f$ is decreasing

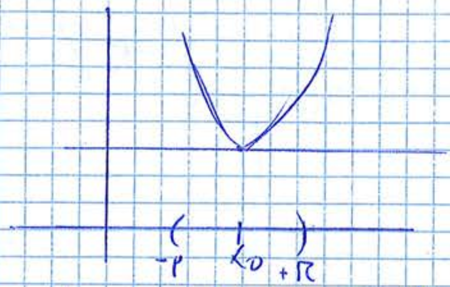
2) $f'(x) > 0 \ \forall x \in I \Rightarrow f$ is strictly increasing on I
   $f'(x) < 0 \ \forall x \in I \Rightarrow f$ is strictly decreasing on I

CLASSIFICATION OF STATIONARY POINTS

f diff on an open interval I
$f'(x_0) = 0$
"To say $f'(x_0)$ is a max or a min

$$\begin{cases} f'(x) \geq 0 & I^-_r(x_0) \\ \Rightarrow \text{increasing function} & \\ f'(x) \leq 0 & I^+_r(x_0) \\ \text{decreasing function} & \end{cases} \Rightarrow x_0 \text{ local maximum point}$$

$$\begin{cases} f'(x) \leq 0 & I^-_r(x_0) \\ (f \text{ decreasing}) & \\ f'(x) \geq 0 & I^+_r(x_0) \\ f \text{ increasing} & \end{cases} \Rightarrow x_0 \text{ is a local minimum point}$$



A RIGHT and LEFT Interval are needed

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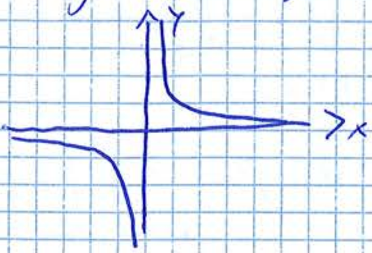
$f(x) = \frac{1}{x}$ $\quad f'(x) = -\frac{1}{x^2} < 0 \Rightarrow f(x)$ strictly decreasing?
NOO!

dom f $\mathbb{R} \setminus \{0\}$

$f'(x) = -\frac{1}{x^2} < 0 \Rightarrow$ strictly decreasing on $(0, +\infty)$

$f'(x) = -\frac{1}{x^2} < 0 \Rightarrow$ strictly decreasing on $(-\infty, 0)$



Cauchy Theorem

f, g continuous on $[a,b]$
f, g differentiable on (a,b) $\Rightarrow \exists x_0 \in (a,b) : \frac{f(b)-f(a)}{b-a} = \frac{f'(x)}{g'(x)}$
$\boxed{g'(x) \neq 0}$

$g(x) = x \quad \exists x_0 \in (a,b) \Rightarrow \frac{f(b)-f(a)}{b-a} = f'(x)$
$\Rightarrow$ Langrange

De L'Hôpital theorem

1) f, g are diff $\boxed{I(\gamma) \setminus \{\gamma\}}$
(true also for $+\infty$, $-\infty$, I^+_γ or I^-_γ)

2) $g'(x) \neq 0$ $\quad \Rightarrow \lim_{x \to \gamma} \frac{f(x)}{g(x)} = \lambda$

3) $\lim_{x \to \gamma} f(x) = \lim_{x \to \gamma} g(x) = 0$

or

$\lim_{x \to \gamma} f(x) = \lim_{x \to \gamma} g(x) = \pm\infty$

4) $\lim_{x \to \gamma} \frac{f'(x)}{g'(x)} = \lambda$

$\Rightarrow \frac{0}{0}$ del'hopital form

$\frac{\infty}{\infty}$ del'hopital form

ORDER OF FUNCTIONS

$\alpha > 0$

$\lim_{x \to +\infty} \frac{e^x}{x^\alpha}$

$\lim_{x \to +\infty} \frac{e^x}{x^x} \Rightarrow$ de l'Hôp $\lim_{x \to +\infty} \frac{e^x}{1} = +\infty$

$\lim_{x \to +\infty} \frac{e^x}{x^2} \Rightarrow$ dHp $\lim_{x \to +\infty} \frac{e^x}{2x} \Rightarrow +\infty$

$\lim_{x \to +\infty} \frac{e^x}{x^n}$ $\lim_{x \to \infty} \frac{e^x}{nx^{n-1}}$ -- $\lim_{x \to \infty} \frac{e^x}{n!} = +\infty$

$e^x > x^\alpha$

$\frac{e^x}{x^{3/2}}$ -- $\frac{e^x}{e^{\frac{3}{2}}} \to \frac{e^x}{\sqrt{x}} \to \frac{e^x}{\alpha^{-1/2}} \Rightarrow +\infty$

$\lim_{x \to +\infty} \frac{e^x}{x^{\sqrt{2}}} = +\infty$

$x^{\sqrt{2}} = x^{1,4\ldots}$

$x < x^{\sqrt{2}} < x^2$

$\frac{1}{x^2} > \frac{1}{x^{\sqrt{2}}} > \frac{1}{x^2}$

$\frac{e^x}{x^2}$ $\boxed{\underbrace{\frac{e^x}{x^{\sqrt{2}}}}_{+\infty} > \underbrace{\frac{e^x}{x^2}}_{+\infty}}$ 2nd comp theorem

$\lim_{x \to +\infty} \frac{e^x}{x^2} = +\infty$ always $+\infty$

$\lim_{x \to +\infty} \frac{e^x}{x^\alpha} = +\infty$

$e^x > x^\alpha$

$\lim_{x \to +\infty} \frac{\ln x}{x^\alpha} = \frac{\infty}{\infty}$ $\alpha > 0$

$\lim_{x \to +\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \lim_{x \to \infty} \frac{1}{\alpha x \cdot x^{\alpha-1}} = \lim_{x \to \infty} \frac{1}{\alpha x^\alpha} = 0$

$\lim_{x \to +\infty} \frac{(\ln x)^\gamma}{x^\alpha} = 0$ $\gamma > 0$, $\alpha > 0$

$\ln x < x^\alpha$

$\lim_{x \to 0^+} \frac{\ln x}{x^\alpha} = -\infty \cdot \frac{1}{x^2} = -\infty \cdot +\infty = -\infty$

$\lim_{x \to 0^+} x^\alpha \cdot \ln x =$
$0 \cdot -\infty$

$\lim_{x \to 0^+} \frac{\lg x}{\left(\frac{1}{x}\right)^\alpha} = \frac{-\infty}{+\infty} = 0$

De l'hopital

$= \lim_{x \to 0^+} \frac{x^\alpha}{\frac{1}{\lg x}} = \lim_{x \to 0^+} \frac{1}{x} : \alpha\left(\frac{1}{x}\right)^{\alpha-1} \cdot \left(-\frac{1}{x^2}\right) = \lim_{x \to 0^+} \frac{1}{\alpha} \cdot (-x^2) \cdot \frac{(x)^{\alpha-1}}{\alpha} = 0$

lg $\frac{x}{\lg x}$

The line

general form $ax+by+c=0$

slope form $y=mx+q$

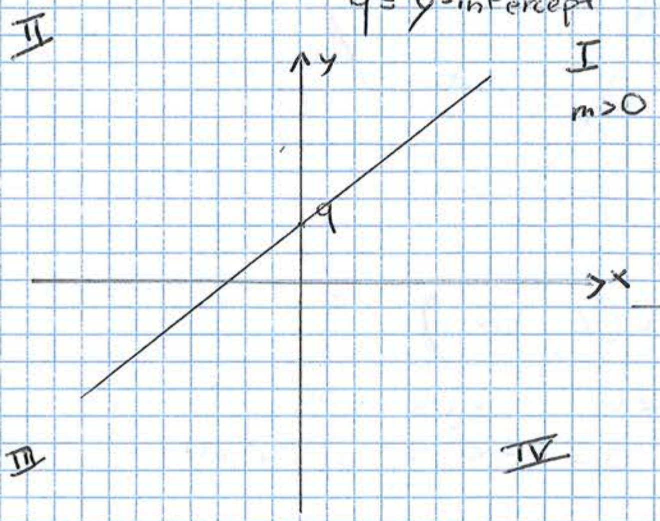
m = slope

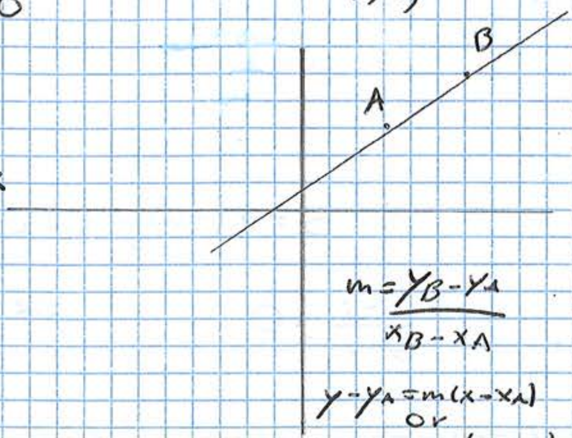
q = y-intercept

having two points we wanna find the general eqation of the line

$A(x_A; y_A)$

$B(x_B; y_B)$





$m = \dfrac{y_B - y_A}{x_B - x_A}$

$y - y_A = m(x - x_A)$
or
$y - y_B = m(x - x_B)$

two lines are parallel if and not if m is equal to m'

r' : $y = mx + q$

r : $y = m'x + q$

$\Rightarrow r \,//\, r' \Longleftrightarrow m = m'$

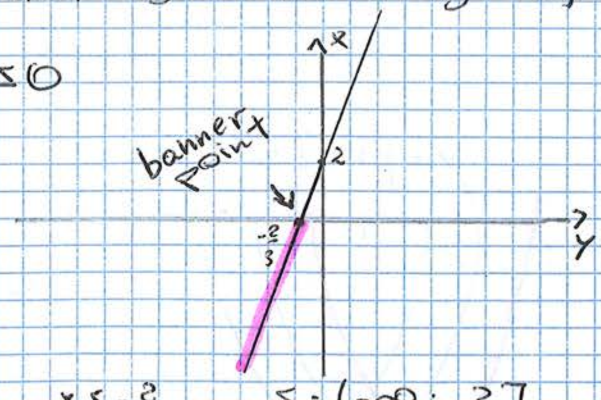
two lines are perpendicular if and not if m times m' gives 1 as a result

$\Rightarrow r \perp r' \Longleftrightarrow m \cdot m' = 1$

Ex 1 $A(1;2)$ $B(-4;5)$

$m = \dfrac{5-2}{-4-1} = -\dfrac{3}{5}$ $y = +5 - \dfrac{3}{5}(x+4)$

Ex 4 $3x + 2 \leq 0$

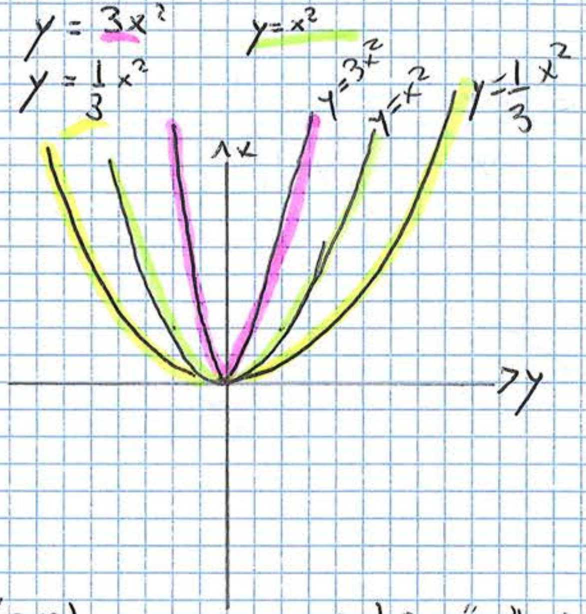


$x \leq -\dfrac{2}{3}$ $S: \left(-\infty; -\dfrac{2}{3}\right]$

2) RESCALING

$y = a \cdot f(x)$

bigger is a greater is the contraction



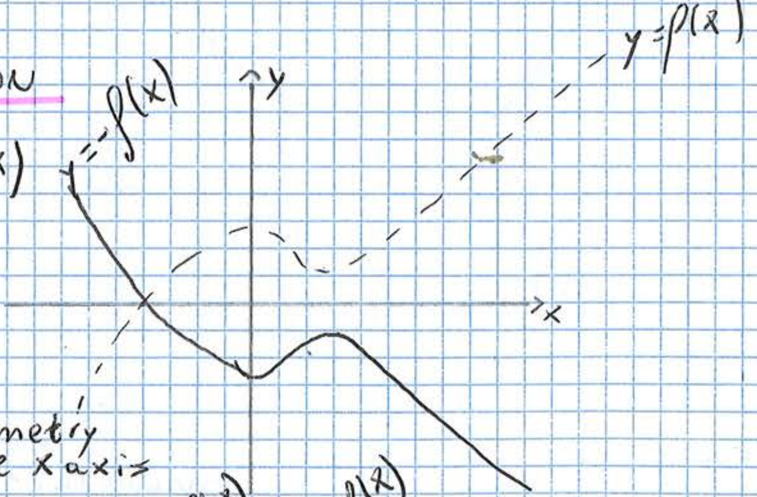
$y = f(ax)$

when "a" is positive there is no distinction with the previous rescaling

$$y = a f(x) = 3x^2 = f(ax) = (\sqrt{3}x)^2$$

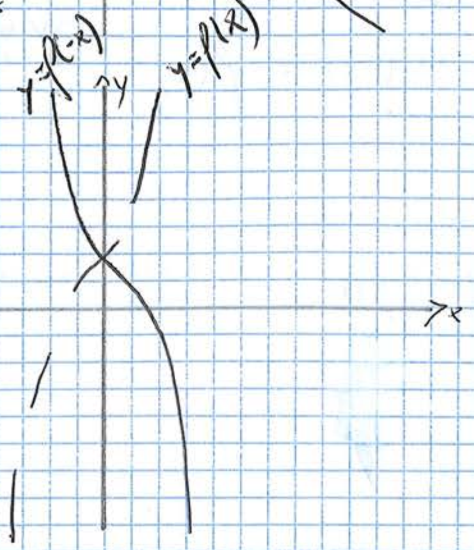
3) REFLECTION

(a) $y = -f(x)$



This is a symmetry respect to the x axis

(b) $y = f(-x)$

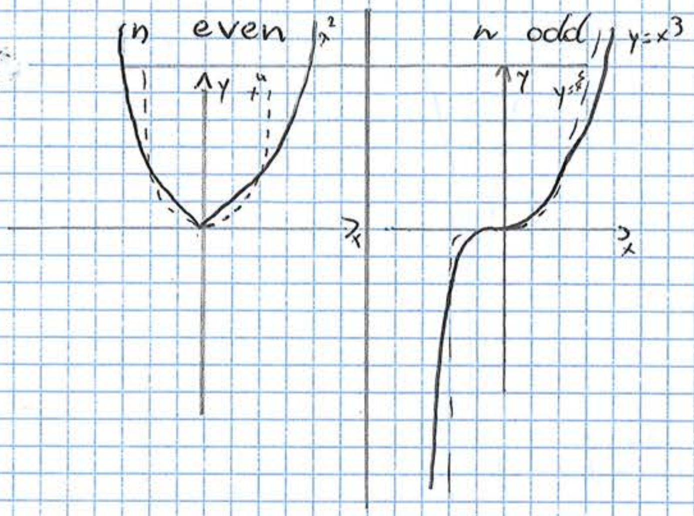


This is a symmetry respect to the y-axis

$y = x^n \qquad n \in \mathbb{N} \qquad n > 1$

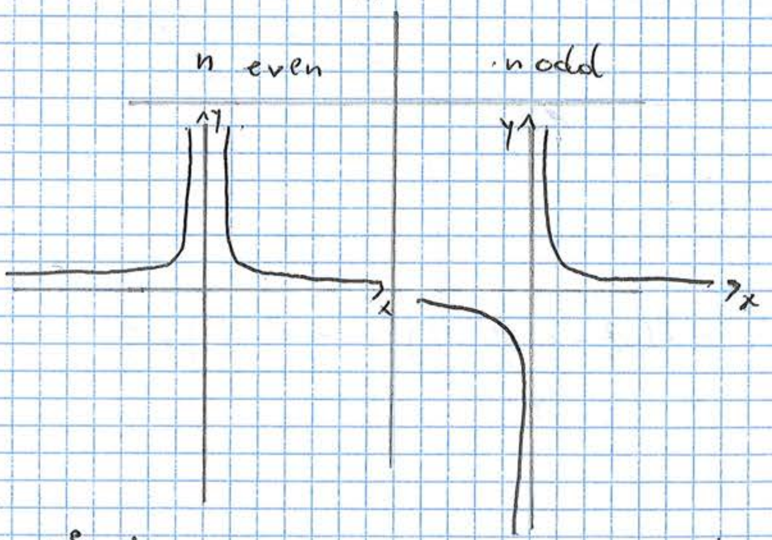
$y = x^3 = f(x)$

$f(-x) = (-x)^3 = -x^3 = -f(x)$



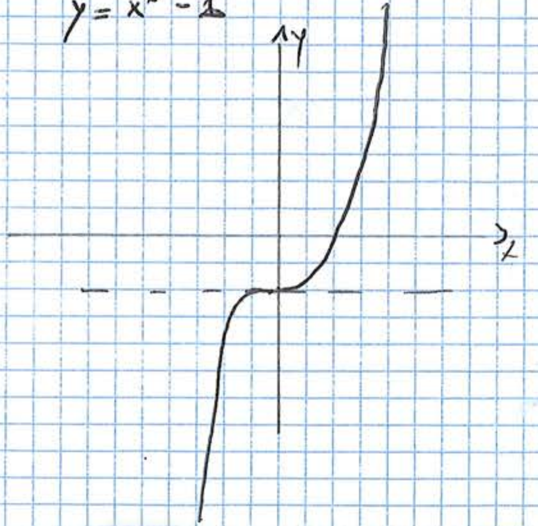
$n < 0$

$y = x^n = \dfrac{1}{x^{-n}}$

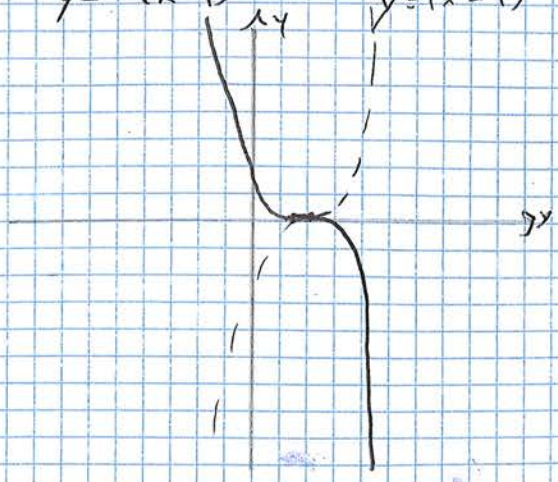


Ex

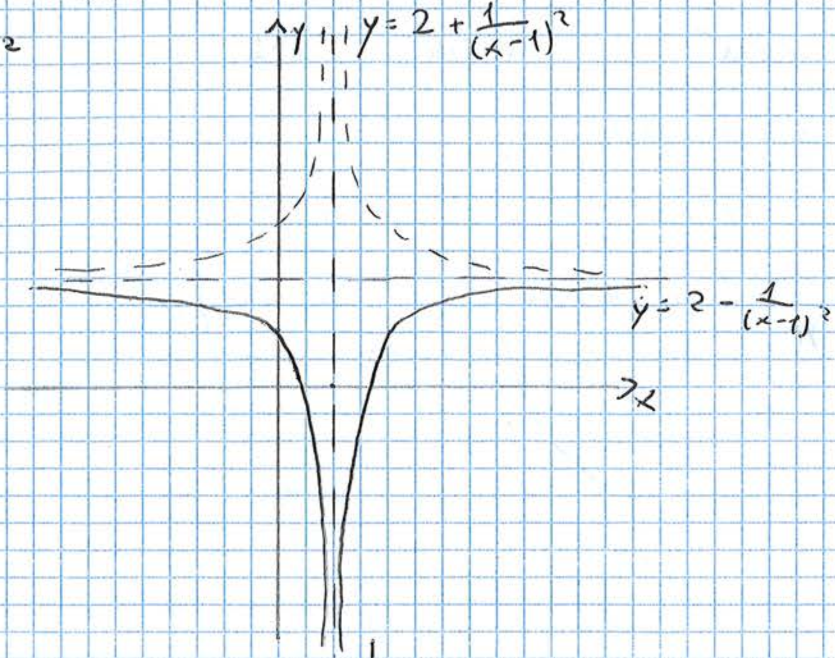
$y = x^3 - 1$



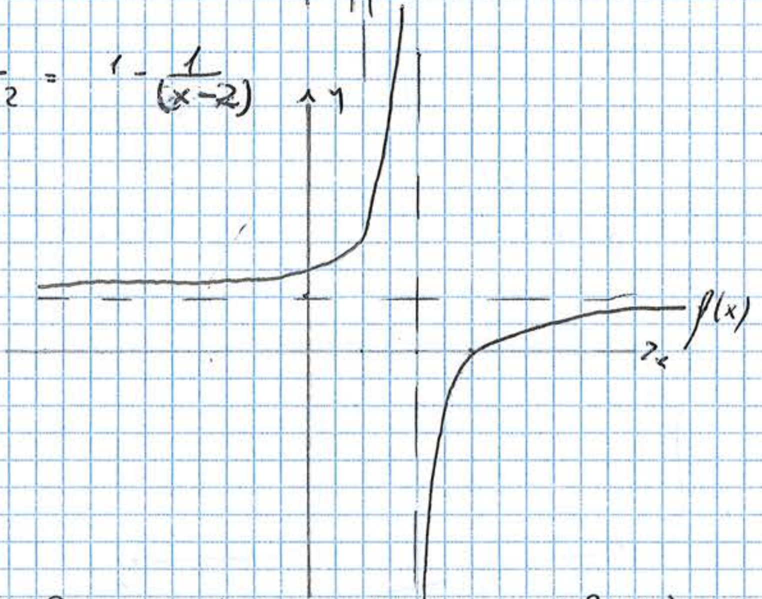
$y = -(x-1)^3 \qquad y = (x-1)^3$



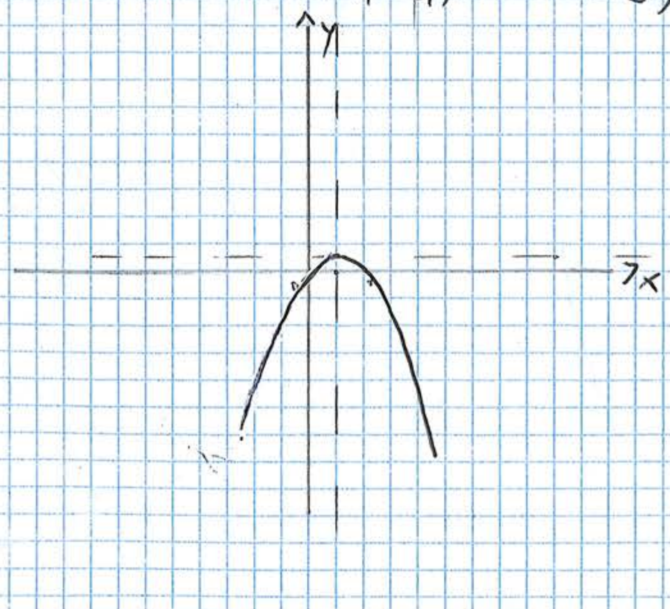
Ex: $g(x) = 2 - \frac{1}{(x-1)^2}$



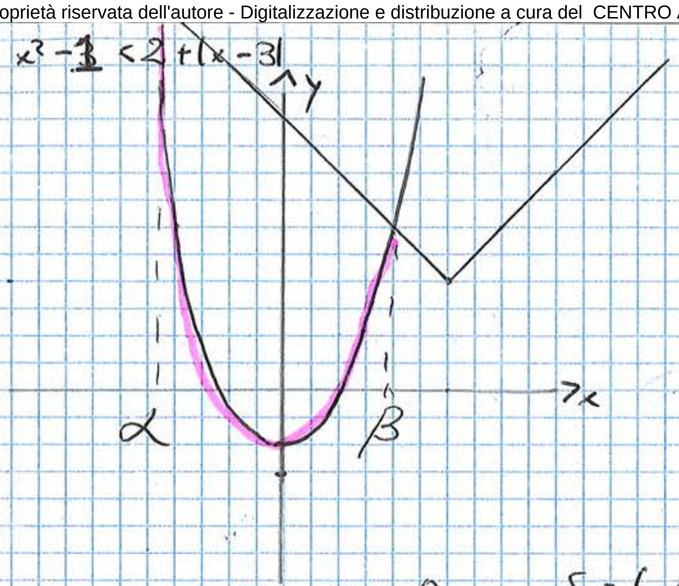
$h(x) = 1 + \frac{1}{-x+2} = 1 - \frac{1}{(x-2)}$



Ex $f(x) = -x^2 + x = -\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) = -\left(x^2 - \frac{1}{2}\right) + \frac{1}{4}$



d) $x^2 - 1 < 2 + |x-3|$



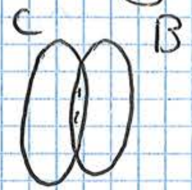
$\alpha < x < \beta \qquad S = (\alpha; \beta)$

Ex we wanna find the intersection set such that

a) $A = \underbrace{\{x \in \mathbb{R} : 0 \le x \le 2\}}_{C} \cap \underbrace{\{x \in \mathbb{R} : x^2 - 3x + 2 = 0\}}_{B}$

$A = [0; 2] \cap \{1; 2\}$

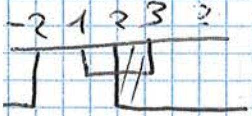
$A = \{1; 2\}$

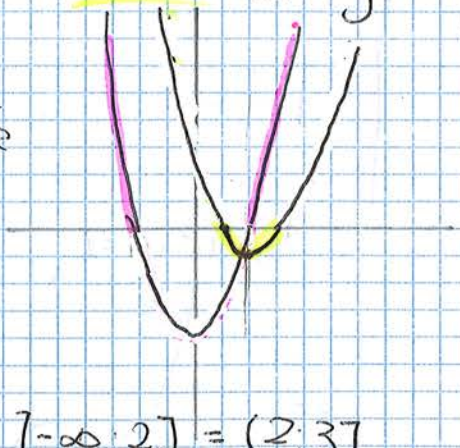


b) $A \{x \in \mathbb{R} / x^2 - 4x + 3 \le 0\} \cap \{x \in \mathbb{R} / x^2 - 4 > 0\}$

$1 \le x \le 3$

$\cap \quad x < -2 \vee x > 2$





$B = [1; 3] \cap]-\infty; 2] = (2; 3]$

c) $A = \underbrace{\{x \in \mathbb{R} \mid 9x^4 - 10x^2 + 1 = 0\}}_{C} \cap \mathbb{N}$

$C = \{-\frac{1}{3}; -1; 1; \frac{1}{3}\} \cap \mathbb{N}$

$A = \{1\}$

NOT Beloved Theorem

f cont in $I(\gamma)\setminus\{\gamma\}$
f diff in $I(\gamma)\setminus\{\gamma\}$ $\Rightarrow \lim_{x\to\gamma} f(x) = l$
$\lim_{x\to\gamma} f'(x) = l$

REMARKS

f cont on $I(\gamma)\setminus\{\gamma\}$
f differentiable $I(\gamma)\setminus\{\gamma\}$ $\cancel{\Rightarrow}$ ~~lim~~ f is not diff at the point
$\nexists \lim_{x\to\gamma} f'(x)$

Ex

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\boxed{f \text{ diff}}$ at x_0 and $f'(0)=0$

But $\lim_{x\to 0} 2x\sin\frac{1}{x} - \cancel{x^2}\cos\frac{1}{x}\cdot\frac{1}{\cancel{x^2}}$ $\nexists$

REMARKS 2

$I(x_0) \longrightarrow I^+(x_0)$
$\lim_{x\to x_0} \longrightarrow \lim_{x\to x_0^+}$

f cont on $I^+(x_0)$
f diff on $I^+(x_0)$ $\Rightarrow \boxed{\lim_{x\to x_0^+} f(x) = l}$
$\lim_{x\to\gamma} f'(x) = l$

REMARKS $\Rightarrow$ SAME FOR INFINITIVE DERIVATE!

$f(x) = \sqrt[3]{\lg x - 1}$ dom f $(0,+\infty)$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt[3]{\lg x - 1} = -\infty$

$\lim_{x\to +\infty} f(x) = +\infty$

$f'(x) = \dfrac{1}{3x\sqrt[3]{(\lg x - 1)^2}}$ $\quad \lg x \neq 1$
$x \neq e$

$f'(x)$ always positive for $(0,e)\cup(e,+\infty)$

MONOTONE strictly INCREASING in $(0,e)\cup(e,+\infty$

What happen in e?

$\lim_{x\to e^\pm} \frac{1}{3x}\cdot\frac{1}{\sqrt[3]{(\lg x-1)^2}} = +\infty$ $\Rightarrow$ VERTICAL TANGENT POINT

$\Rightarrow$ THE FUNCTION is NOT DIFFERENTIABLE IN e

$f(x) = \sqrt{x^2+1}$

$$\lim_{x \to +\infty} \frac{\sqrt{x^2+1}}{x} = \frac{\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}} = 1$$

$$\lim_{x \to +\infty} \sqrt{x^2+1} - x = 0$$

$y = x$ RIGHT OBLIQUE ASYMPTOT

Ex

$f(x) = x + \lg x$

$$\lim_{x \to +\infty} 1 + \frac{\lg x}{x} = 1$$

$$\lim_{x \to +\infty} \cancel{x} + \lg x - \cancel{x} = +\infty \qquad y = x$$

Ex 2 function - linear funct = 0 $\Rightarrow$ asintoto obliquo

$$f(x) = \frac{\ln(5e^{3x} + x^4)}{\lg 5e^{3x}\left(1 + \frac{x^4}{5e^{3x}}\right)}$$

$$f(x) = \lg 5 + 3x + \lg\left(1 + \frac{x^4}{5e^{3x}}\right)$$

$$f(x) - (\lg 5 + 3x) = \lg\left(1 + \frac{x^4}{5e^{3x}}\right)$$

$x \to +\infty$

$\Downarrow$

0

FOR $x \to \infty$

$\Rightarrow f(x) - (\lg 5 + 3x) \Rightarrow 0$

$y = \lg 5 + 3x$ = asintoto obliquo

$$m = \lim_{x \to +\infty} \frac{\ln(5e^{3x} + x^4)}{x} = \lg 5 + 3$$

$$q = \lim_{x \to +\infty} \lg(5e^{3x} + x^4) - x(\lg 5 + 3) = -\infty$$

$y = \lg 5 + 3x.$

$f(x) =$

$$\lim_{x\to 0} \frac{\sin x}{x} = \quad \begin{cases} f = \sin x \\ g = x \end{cases} \qquad \sin x \sim x$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lg(a) \quad \begin{cases} f = a^x - 1 \\ g = x \end{cases}$$

$$\lim_{x\to 0} \frac{a^x - 1}{\lg(a)\cdot x} = 1 \qquad f \sim \lg(a)\, x$$

$$\lim_{x\to 0} \frac{\lg_a(1+x)}{x} = \frac{1}{\lg a}$$

$$\begin{cases} \lg_a(1+x) = f \\ y = x \quad g \end{cases}$$

$$\lim_{x\to 0} \frac{\lg a \cdot \lg_a(1+x)}{x} = 1$$

$$\lg a \cdot \lg_a(1+x) \sim x$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\begin{cases} f = \tan x \\ g = x \end{cases} \qquad \tan x \sim x$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \qquad 1-\cos x \sim x^2 \; ?$$

$$\lim_{x\to 0} \frac{2\cdot(1-\cos x)}{x^2} = 1 \qquad \boxed{2f \sim g}$$

$$\boxed{\cancel{2}(1-\cos x) \sim \frac{1}{2}x^2}$$

$$\lim_{x\to 0} \frac{(1+x)^k - 1}{x} = k$$

$$(1+x)^k - 1 \sim kx$$

$$\lim_{x\to 0} \frac{\arctan x}{x} = 1 \qquad \arctan x \sim x$$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = 1 \qquad \arcsin x \sim 1$$

$$\lim_{x\to 0} t\cdot \lg t = 0 \qquad \lg t = o(t^{-1})$$

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \qquad (1+x)^{\frac{1}{x}} \sim e$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \qquad e^x - 1 \sim x$$

?

$$\lim_{x\to 0} \frac{\lg(x+1)}{x} = 1 \qquad \lg(x+1) \sim x$$

TEST FUNCTION

	infinite	infinitesimal
$x \to x_0$	$\frac{1}{\lvert x-x_0\rvert}$	$\lvert x-x_0\rvert$
$x \to \infty$	$\lvert x\rvert$	$\frac{1}{\lvert x\rvert}$

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→ PRACTICAL USE

in product and quotient substitute a function with an equivalent one or elimination of little os

→ 1) FUNDAMENTAL LIMIT

→ 2) HÔPITAL RULE $\frac{0}{0}$, $\frac{\infty}{\infty}$

→ 3) LANDAU SYMBOL

A) $\lim_{x \to 0} \frac{e^{x^2}-1}{\sin 3x} = \frac{0}{0} \Rightarrow \lim_{x \to 0} \frac{x^2}{3x} = 0$

$e^{x^2}-1 \sim x^2 \quad x \to 0$

$\sin 3x \sim 3x \quad x \to 0$

B) $\lim_{x \to 0} \frac{2\sin^2 x - \ln(1+6x^3)}{x^2 + 2\sin x} = \frac{0}{0}$

1) $\lim_{x \to 0} \frac{x^2}{2\sin x} = \lim_{x \to 0} \frac{x \cdot x}{2\sin x} = 0$

$x^2 = o(2\sin x)$

2) $\lim_{x \to 0} \frac{2\sin^2 x}{-\lg(1+6x^3)} = \lim_{x \to 0} \frac{x^2 \, 2\sin^2 x}{x^2} \cdot -\frac{6x^3}{x^3 6 \lg(1+6x^3)} = \frac{x^2}{x^3} = \infty$

$\lim_{x \to 0} \frac{-\lg(1+6x^3)}{2\sin^2 x} = 0$

$-\lg(1+6x^3) = o(2\sin^2 x)$

$\Rightarrow \lim_{x \to 0} \frac{2\sin^2 x - \ln(1+6x^3)}{x^2 + 2\sin x} = \frac{2\sin^2 x}{2\sin x} = 0$

C) $\lim_{x \to 0} \frac{1-\cos 2x}{\sin^2 3x} = \frac{0}{0}$

$1-\cos 2x \sim 2x^2$

$\sin^2 3x \sim 9x^2$

$\lim_{x \to 0} \frac{1-\cos 2x}{\sin^2 3x} = \frac{2\cancel{x^2}}{9\cancel{x^2}} = \frac{2}{9}$

$\Rightarrow$ We classify all the functions by looking if they are comparable to power functions

$\lg(1+6x^3) \sim 6x^3$ * INF.MAL AT 0

$f(x) = e^{\sqrt{x}} - 1 \sim \sqrt{x}$

$\lim_{x\to 0^+} \frac{e^{\sqrt{x}}-1}{\sqrt{x}} = 1 \Rightarrow e^{\sqrt{x}} - 1 \sim e^{\frac{1}{2}}$

for $x \to 0^+$ if $f(x) \sim \ell x^\alpha$

1) $f(x)$ infinitesimal of order α for $x \to 0$

2) ℓx^α PRINCIPAL PART OF $f(x)$ for $x \to 0$

Ex

$f(x) = e^{3x^4} - 1 \sim 3x^4$

$f(x)$ infmal of order 4 for $x \to 0$

$3x^4$ is the principal part of $f(x)$

$f(x) = \lg(1 + x^{\frac{3}{4}}) \sim x^{\frac{3}{4}}$

INF.MAL OF ORDER $\frac{3}{4}$

Prin. PART $x^{3/4}$

1) $f(x) = e^{-\frac{1}{x}}$ $x \to 0^+$

$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^\alpha} = \lim_{t\to \infty} \frac{e^{-t}}{\left(\frac{1}{t}\right)^\alpha} = \lim_{t\to +\infty} \frac{t^\alpha}{e^t} = 0$

$\frac{1}{x} = t$

$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^\alpha} = 0 \Rightarrow e^{-\frac{1}{x}} = o(x^\alpha)$

2) $f(x) = -\frac{1}{\lg x}$

$\lim_{x\to 0^+} \frac{-\frac{1}{\lg x}}{x^\alpha} = \lim_{t\to\infty} \frac{+\frac{1}{\lg t}}{\left(\frac{1}{t}\right)^\alpha} = \lim_{x\to\infty} \frac{t^\alpha}{\lg t} = \infty \Rightarrow \lg t = o$

$\frac{1}{t} = x$

$\lim_{x\to 0^+} \frac{-\frac{1}{\lg x}}{x^\alpha} = \infty$

$\lim_{x\to 0^+} \frac{x^\alpha}{-\frac{1}{\lg x}} = 0 \Rightarrow$

$x^\alpha = o\left(-\frac{1}{\lg x}\right)$

$$\lim_{x\to 0^+} \frac{(\sin\sqrt{x})\cos x - (\cos\sqrt{x})\sin x}{\sqrt[3]{x}} = \frac{0}{0}$$

$\sin\sqrt{x} \sim \sqrt{x}$ $\qquad (\sin\sqrt{x})\cos x \sim \sqrt{x}$

$\cos x \sim 1$

$\cos\sqrt{x} \sim 1$ $\qquad \underline{(\cos\sqrt{x})\sin x \sim x}$

$\sin x \sim x$

$\Rightarrow \quad (\cos\sqrt{x})\sin x = o\left((\sin\sqrt{x})\cos x\right)$

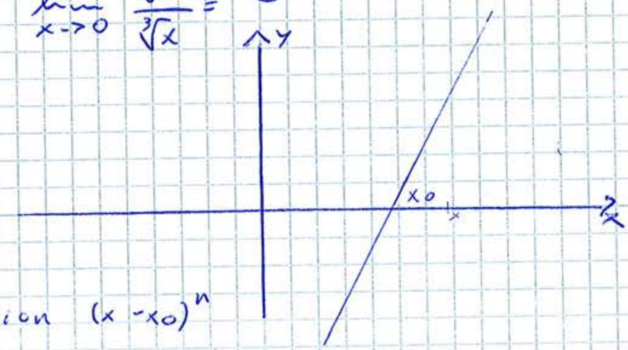
$$\Rightarrow \lim_{x\to 0^+} \frac{\sin\sqrt{x}\cdot\cos x}{\sqrt[3]{x}} = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt[3]{x}} \overbrace{\frac{\sin\sqrt{x}}{\sqrt{x}}}^{\to 1} \cdot \cos x$$

$$= \lim_{x\to 0^+} \sqrt[6]{x}\cdot\cos x = 0$$

$\sin\sqrt{x}\cdot\cos x \sim \sqrt{x}$

$$\Rightarrow \lim_{x\to 0} \frac{\sqrt{x}}{\sqrt[3]{x}} = 0$$

INFINITESIMAL
FOR $x \to x_0 \neq 0$



Function of comparision $(x-x_0)^n$
$n \in \mathbb{N}$

$\alpha > 0 \qquad \alpha \in \mathbb{R} \quad \begin{cases} (x-x_0)^\alpha & x \to x_0^+ \\ (x_0-x)^\alpha & x \to x_0^- \end{cases}$

or

$|x-x_0|^\alpha \qquad x \to x_0$

$\Rightarrow f(x) \sim \ell(x-x_0)^\alpha \qquad x \to x_0^+$

1) f infinitesimal of order α $\quad x \to x_0^+$

2) $\ell(x-x_0)^\alpha$ Principal part of $f(x)$ $x \to x_0^+$

or

$f(x) \sim \ell(x_0-x)^\alpha \qquad x \to x_0^+$

$\Downarrow$

Principal part of $f(x)$ for $x \to x_0^+$

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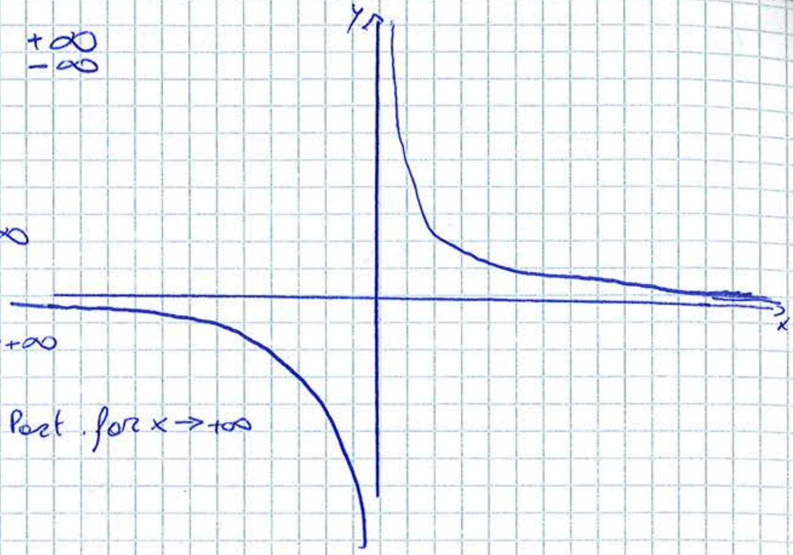
INF.MAL Functions at $\begin{matrix}+\infty\\-\infty\end{matrix}$

$f(x) = \frac{1}{x}$



$f(x) \sim l \cdot \frac{1}{x^\alpha} \quad x \to +\infty$

1) $f(x)$ infinmal order α for $x \to +\infty$

2) $l \cdot \frac{1}{x^\alpha}$ is the Principal Part for $x \to +\infty$

$f(x) = \frac{1}{x}$

$g(x) = \frac{1}{x^3}$

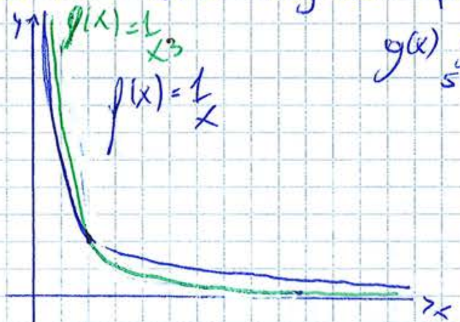
$$\lim_{x\to+\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{1/x}{1/x^3} = \lim_{x\to+\infty} x^2 = +\infty$$

$$\Rightarrow \lim_{x\to+\infty} \frac{g}{f} = 0 \Rightarrow g = o(f)$$

$$\frac{1}{x^3} = o\left(\frac{1}{x}\right)$$

$\Rightarrow$ we cancel the highest power for $x \to +\infty$

$g(x)$ goes to 0 faster so it is negligeble



$f(x) = \frac{\pi}{2} - \arctan x \quad x \to 0$

$$\lim_{x\to+\infty} \frac{\frac{\pi}{2} - \arctan x}{l x^{-\alpha}} = \frac{0}{0} \Rightarrow \exists ! \, l \neq 0, \alpha > 0 \text{ such that } \lim_{x\to+\infty} \frac{\pi/2 - \arctan x}{l \cdot x^{-\alpha}} = 1$$

$\Downarrow \; l \cdot \left(\frac{1}{x}\right)^\alpha$

$$\lim_{x\to+\infty} \frac{\frac{\pi}{2} - \arctan x}{l \cdot x^{-\alpha}} \Rightarrow \text{HÔP} \Rightarrow \lim_{x\to+\infty} \frac{+\frac{1}{1+x^2}}{l \cdot (+\alpha) \cdot x^{-\alpha-1}}$$

$$= \lim_{x\to+\infty} \frac{1}{l \cdot \alpha} \cdot \frac{x^{\boxed{\alpha+1}}}{1+x^2} = 1$$

$\Rightarrow l = 1$

$\alpha = 1$

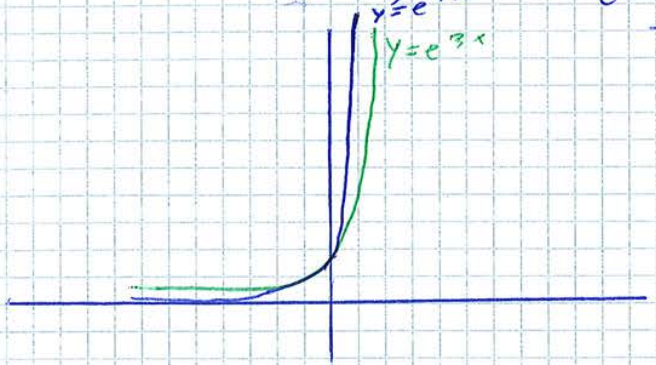
$$\frac{\pi}{2} - \arctan x \sim \frac{1}{x}$$

$f(x) = e^{3x}$

$f(x) = e^{7x}$ $\qquad e^{3x} = o(e^{7x})$

$$\lim_{x \to +\infty} \frac{e^x}{x^\alpha} = +\infty \quad \forall \alpha > 0$$

$$x^\alpha = o(e^x)$$



e^{7x} → goes faster to $+\infty$ so it is negligible

Given a set of functions can we find a function at the power α' such that:

$$f(x)^\alpha < f(x)^{\alpha_I} < f(x)^{\alpha_{II}}$$

NO!

$f(x) = x \ln x$ $\qquad x \sim l x^\alpha$

For $\alpha = 1$

$$\Rightarrow \lim_{x \to +\infty} \frac{\cancel{x} \lg x}{\cancel{x}} = +\infty$$

For $\alpha = 2$

$$\lim_{x \to +\infty} \frac{x \ln x}{x^2} = 0$$

$$\Rightarrow \lim_{x \to +\infty} \frac{x \ln x}{x^{1+\varepsilon}} = \lim_{x \to +\infty} \frac{\cancel{x} \ln x}{\cancel{x} \cdot x^\varepsilon} = 0 \quad \forall \varepsilon > 0$$

$y = x \lg x$ is a function of order superior than 1 so "$1+\varepsilon$"

$\Rightarrow$ No ε exists such that $\lim_{x \to +\infty} \frac{\ln x}{x^\varepsilon} = 1$

& $y = x \lg x$ cannot be classified for $x \to +\infty$

$\frac{x}{\lg x}$ cannot be classified too

$$\lim_{x \to +\infty} \frac{x/\lg x}{l\, x^\alpha} = \lim_{x \to +\infty} \frac{x}{\lg x \cdot l\, x^{\alpha}} = 0 \qquad \forall \alpha \in \emptyset$$

Th] $\lim_{x \to \gamma} f(x) = \ell \Rightarrow f(x) = \ell + o(1)$

"ℓ + a term that goes to 0"

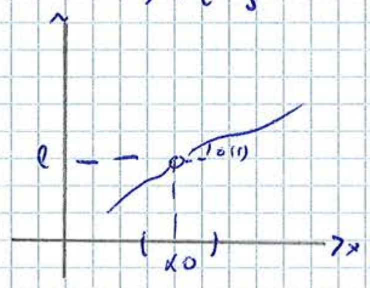
in $I(\gamma) \setminus \{\gamma\}$

PROOF

$\lim_{x \to \gamma} f(x) - \ell = 0$

$\Rightarrow f(x) - \ell = o(1)$

$f(x) = \ell + o(1)$



$\gamma = x_0$ $\quad f$ cont at x_0

$f(x) = f(x_0) + o(1) \quad x \to x_0$

Th] $f \sim g \quad x \to \gamma$

$\Rightarrow f = g + o(g) \quad x \to \gamma$

PROOF

$f \sim g \quad \lim_{x \to \gamma} \frac{f(x)}{g(x)} = 1$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)} - 1 = 0 \Rightarrow \lim_{x \to \gamma} \frac{f(x) - g(x)}{g(x)} = 0$

$f(x) - g(x) = o(g(x))$

$\Rightarrow f(x) = g(x) \quad o(g(x))$

$f = g + o(g)$

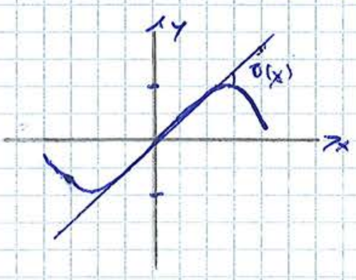
$x \to \gamma$

$\sin x \sim x \quad x \to 0$

in $I(0) \setminus \{0\}$

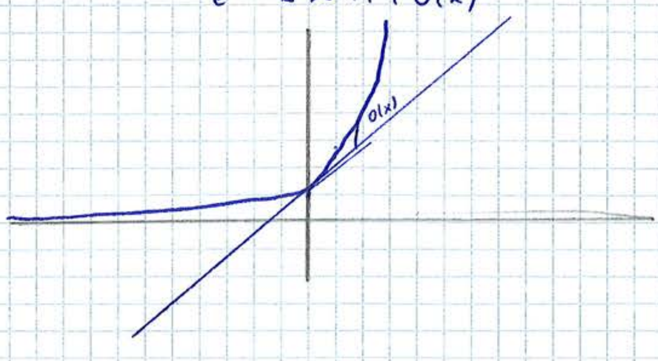
$\sin x = x + o(x)$

$\sin x - x = o(x)$



$e^x - 1 \sim x \Rightarrow e^x - 1 = x + o(x)$

$e^x = x + 1 + o(x)$



$\lim_{x\to 0} \frac{\sin x - \tan x}{x}$ | $\lim_{x\to 0} \frac{\sin x - \tan x}{x^2}$ | $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$

$\lim_{x\to 0} \frac{0-0}{x}$ $\qquad \lim_{x\to 0} \frac{0-0}{x^2}$ $\qquad \lim_{x\to 0} \frac{0-0}{x}$

$\cancel{x} + o(x) - \cancel{x} - o(x)$

$\lim_{x\to 0} \frac{x + o(x) - (x + o(x))}{x}$

$= \lim_{x\to 0} \frac{o(x)}{x} = 0$

$\frac{o(x)}{x^2} \to$ this goes to zero with an esponent $>$ than 1 ?

$\lim_{x\to 0} \frac{x - \sin x}{x^3}$

$\lim_{x\to 0} \frac{x - (x + o(x))}{x^3} = \lim_{x\to 0} \frac{o(x)}{x^3}$ $\quad \sin x = x + o(x)$

$\sin x - x = o(x)$ ORDER > 1

- GUESS ORDER is 2

$\lim_{x\to 0} \frac{\sin x - x}{x^2} \Rightarrow$ HÔP $\Rightarrow \lim_{x\to 0} \frac{\cos x - 1}{2x} = 0$

$\sin x - x = o(x^2)$ ORDER > 0

" $\sin x - x$ " goes faster to 0 than x^2

- GUESS ORDER is 3

$\lim_{x\to 0} \frac{\sin x - x}{x^3} \Rightarrow$ HÔP $= \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = -\frac{1}{6}$

$\sin x - x$ ORDER 3

$\sin x - x \sim -\frac{1}{6}x^3 \quad x \to 0$

$\sin x - x \sim -\frac{1}{6}x^3 + o(x^3)$

$\Rightarrow \sin x - \left(x - \frac{1}{6}x^3\right) = o(x^3)$

we can aproximate $\sin x - x$ infinite times

3] f is three time diff at x_0

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \frac{1}{2} f''(x_0)(x-x_0)^2}{(x-x_0)^3} = \frac{1}{6} f'''(x_0) + o(1)$$

$$\lim_{x \to x_0} \frac{f(x) - [f(x_0) + f'(x_0)(x-x_0)] - [\frac{1}{2} f''(x_0)(x-x_0)^2]}{(x-x_0)^3} = \frac{1}{6} f'''(x_0) + o(1)$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{6} f'''(x_0)(x-x_0)^3 + o(x-x_0)^3$$

4] f is n times diff at x_0

$\exists$ one and only one polynomial "Tf_{n,x_0} «TAYLOR POLYNOMIAL OF DEGREE $\leq n$ at x_0»"

such that

$$f(x) = Tf_{n,x_0}(x) + o((x-x_0)^n)$$

$$Tf_{n,x_0}(x_0) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \dots + \frac{1}{n!} \cdot f^{(n)}(x_0)(x-x_0)^n$$

$Tf_{n,x_0}(x)$ TAYLOR'S FOL

$o((x-x_0)^n)$ REMAINDER IN PEANO'S FORM

When $x_0 = 0$ → MC LAURIN POLYNOMIAL AT the origin

$f(x) = e^x \qquad x_0 = 0$

$f^{(n)}(x) = e^x \qquad f^{n}(0) = 1$

$$e^x = 1 + \underbrace{1 \cdot x}_{f'(x_0)(x-0)} + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \dots \frac{1}{n!} x + o(x^n)$$

$f(x) = a^x \qquad x = 0$

$f^{(n)}(x) = a^x (\ln a)^n \qquad f^{(n)}(0) = (\ln a)^n$

$$a^x = \ln a + (\ln a) x + \frac{1}{2} x^2 \cdot \ln^2 a + \dots \frac{1}{n!} (\lg a)^n x^n + o(x^n)$$

$\boxed{Th}$ if f is differentiable n times at $x_0 = 0$ and it is an even (respectively, odd) function, then the McLaurin polynomial $Tf_{n,0}(x)$ contains only even (resp. odd) powers

f diff n times at $x_0 = 0$

f EVEN (ODD) $\Rightarrow$ McLaurin polynomial of $f(x)$ contains only EVEN (ODD) powers

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{IV}(x) = \cos x \qquad f^{IV}(0) = 1$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \begin{cases} o(x^{2n}) \\ O(x^{2n+1}) \end{cases}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \begin{cases} o(x^{2n+1}) \\ o(x^{2n+2}) \end{cases}$$

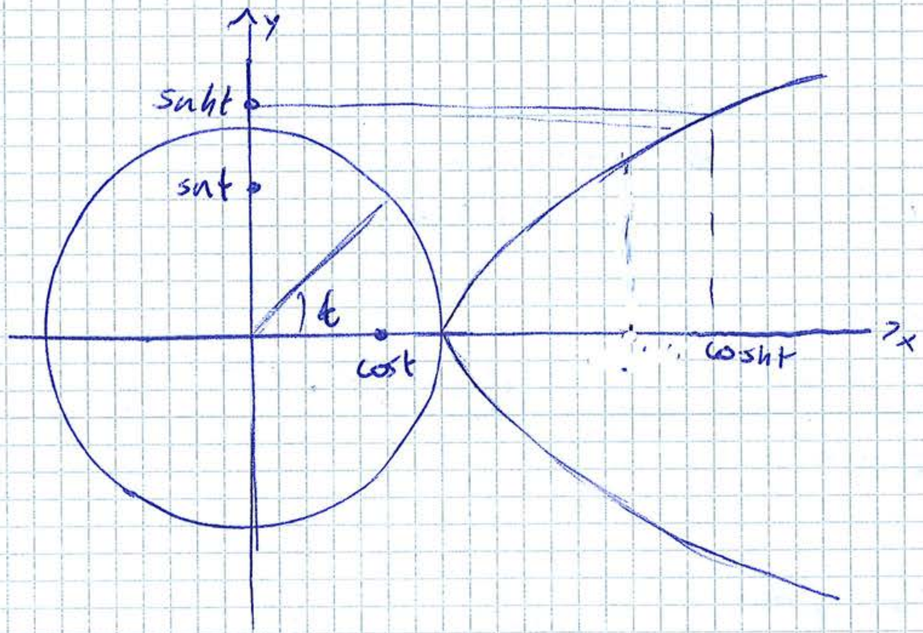
HYPERBOLIC FUNCTIONS

$\cosh x = \frac{e^{x}+e^{-x}}{2}$ $\qquad$ $\sinh x = \frac{e^{x}-e^{-x}}{2}$

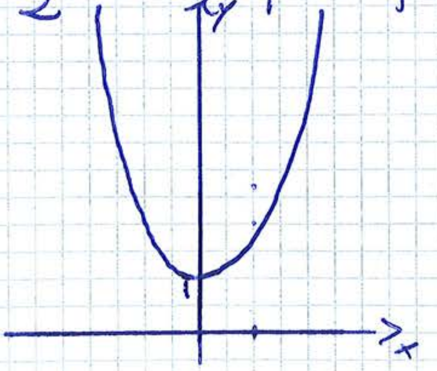
$$\cosh^2 x - \sinh^2 x = \frac{(e^{x}+e^{-x})^2-(e^{x}-e^{-x})^2}{4} = \frac{\cancel{e^{2x}}+\cancel{e^{-2x}}+2+2-\cancel{e^{2x}}-\cancel{e^{-2x}}}{4} = 1$$

$\cosh^2 x - \sinh^2 x = 1$ circle

$\cos^2 x + \sin^2 x = 1$ hyperbolic



$\cosh x = \frac{e^{x}+e^{-x}}{2}$ $\qquad$ $f(x) = f(-x) \Rightarrow$ EVEN FUNCTION



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$f(x) = \sinh x$

$D \sinh x = \cosh x \geq 1$

↳ mon strictly increasing ⇒ injective ⇒ invertible

$\frac{e^x - e^{-x}}{2} = y \qquad e^x = y \pm \sqrt{y^2+1}$

↳ we choose $y \oplus \sqrt{y^2+1}$ because $y - \sqrt{y^2+1}$ is a negative value and $e^x > 0$

$e^x = y + \sqrt{y^2+1}$

⇒

$y = \ln(x + \sqrt{x^2+1})$

$$f(x) = \begin{cases} \sinh 3x & x<0 \\ \sin(\alpha x) + \beta & x \geq 0 \end{cases}$$

1) α, β such that $f \in C^0(\mathbb{R})$

α, β f diff on $\mathbb{R}$

1] f cont $\forall x<0$

f cont $\forall x>0$

$\lim_{x\to 0^-} \sinh 3x = 0$

$\lim_{x \to 0^+} \sin(\alpha x) + \beta = \beta$

$\Rightarrow \boxed{\beta = 0}$

$$\Rightarrow f(x) = \begin{cases} \sinh 3x & x<0 \\ \sin(\alpha x) & x \geq 0 \end{cases}$$

2] $f'(x)$ diff $x<0 \Rightarrow f'(x) = 3\cosh 3x \quad x<0$

$f''(x)$ diff $x>0 \Rightarrow f'(x) = \alpha\cos(\alpha x) \quad x>0$

$\lim_{x\to 0^-} f'(x) = 3$

$\lim_{x\to 0^+} f'(x) = \alpha$

$\boxed{\alpha = 3}$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \dots$$
$$\dots \left[\frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-(n-1))}{n!}\right]x^n + o(x^n)$$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-(n-1))}{n!}$$

$k>2$

$$\binom{\alpha}{n} = \frac{\alpha!}{n!(\alpha-n)!}$$

$$\binom{\alpha}{0} = \binom{\alpha}{\alpha} = 1 \qquad \binom{\alpha}{1} = \binom{\alpha}{\alpha-1} = \alpha$$

TAYLOR

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + o(x^n)$$

$n=3$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \sum_{k=0}^{3} \binom{\frac{1}{2}}{k} x^k + o(x^3)$$

$$= \binom{\frac{1}{2}}{0}x^0 + \binom{\frac{1}{2}}{1}x^1 + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3 + o(x^3)$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \dots$$

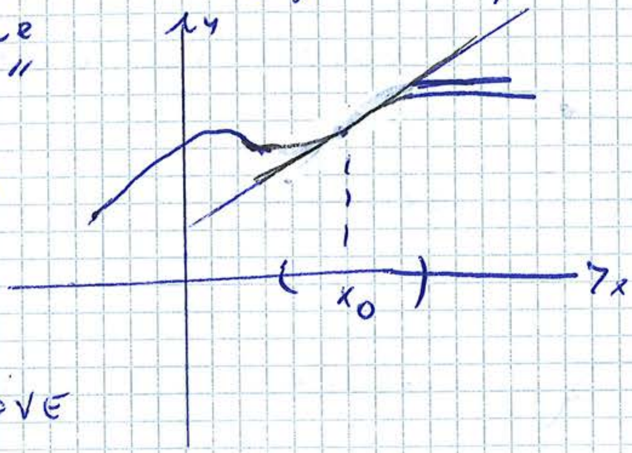
$$-\frac{1}{2}\cdot\frac{1}{2} \qquad \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{3\cdot 2} = 1$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

DEF x_0 is an inflection point if $\exists I_R(x_0) \subseteq dom f$:

$$\forall x \in I_R(x_0) \begin{cases} \text{if } x > x_0 & f(x) \le t(x) \\ \text{if } x < x_0 & f(x) \ge t(x) \end{cases}$$

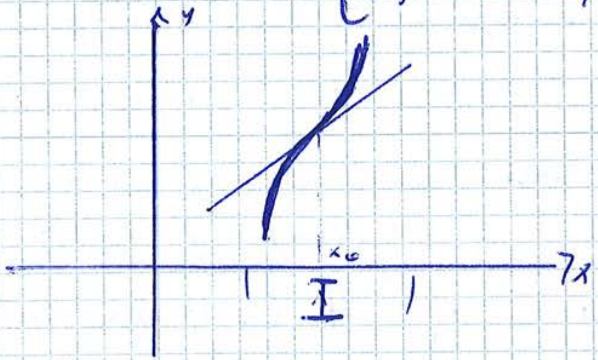
"The tangent line crosses the graph"
=> DESCENDING INFLECTION POINT
↳ FROM ABOVE TO BELOVE



or

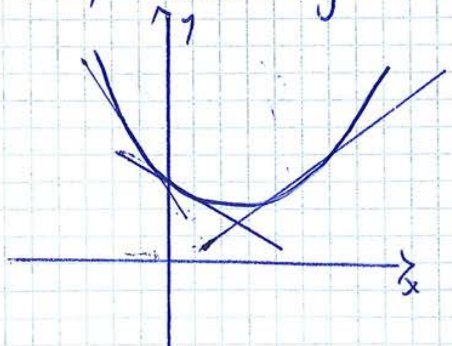
$$\forall x \in I_R(x_0) \begin{cases} \text{if } x > x_0 & f(x) \ge t(x) \\ \text{if } x < x_0 & f(x) \le t(x) \end{cases}$$

=> ASCENDING INFLECTION POINT



Th
- f convex on $I \Leftrightarrow f'$ increasing in I
- f is CONCAVE on $I \longleftrightarrow f'(x)$ is decreasing on I
- f' incre on $I \Leftrightarrow f$ convex in x

-> f' strictly increasing on $I \Rightarrow f$ is strictly convex



* OPPOSITE FOR f' concave => f' decreasing

TAYLOR EXPANTION

$f(x)$ is n times diff at x_0 ➡ $f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + o((x-x_0)^n)$

$a_0 = a_1 = \dots = a_{m-1} = 0 \qquad 1 \leq m < n$

↳ There exist values of the derivative such that $f(x_0) = a_0 = 0$.

$f^{(m-1)}(x_0) = 0$

$\Rightarrow f(x) = a_m(x-x_0)^m + a_{m+1}(x-x_0)^{m+1} + \dots + a_n(x-x_0)^n + o((x-x_0)^n)$

I divide for ➡ $\frac{f(x)}{a_m(x-x_0)^m} = 1 + \frac{a_{m+1}(x-x_0)^{m+1}}{a_m \; (x-x_0)^m} + \dots + a^{n-m}(x-x_0)^{n-m} + o((x-x_0)^{n-m})$

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{a_m(x-x_0)^m} = 1}$$

$\Rightarrow f(x) \sim \underbrace{a_m(x-x_0)^n}_{\text{Principal part or equivalent part}} \qquad x \to x_0$

$f(x) = 1 - \cos(x^2) \qquad x \to 0$

for $x \to 0$ is an infinitesimal function

$\overset{x \to 0}{x^2 \to 0} \quad z \to 0$

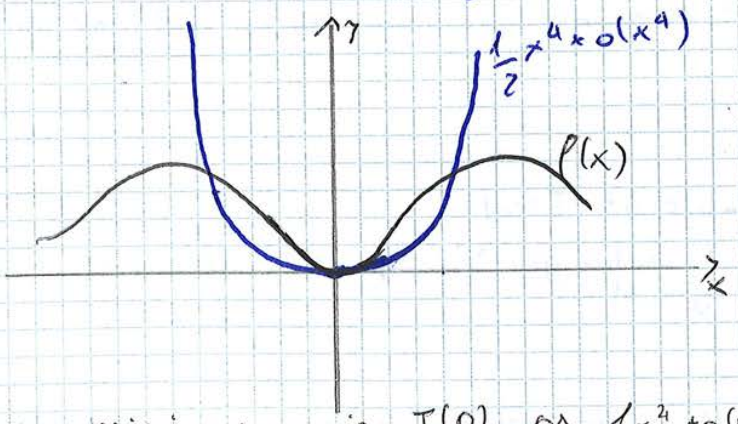
$h(z) = 1 - \cos(z)$

$h(z) = 1 - \left(1 - \frac{1}{2}z^2 + o(z^2)\right) = 1 - 1 + \frac{1}{2}z^2 + o(z^2)$

$h(x)$

$f(x) = 1 - \cos(x^2) = \frac{1}{2}x^4 + o(x^4) \quad$ in $I(0)$

In $I(0)$ my function $f(x) \sim \frac{1}{2}x^4 + o(x^4)$



$f(x)$ as a minimum in $I(0)$ as $\frac{1}{2}x^4 + o(x^4)$ as a minimum in 0!

~~Th.] f diff n times at x_0 n≥3~~

Ex $f(x) = \overset{f(0)}{3} - 4x + 9x^2 + o(x^2) \quad x \to 0$

$f(x)$ is TWICE differentiable at x_0

=> "I CAN DEDUCE MANY THINGS"

$f(0)=3$ the function at the origin is POSITIVE

$f'(0)=-4$ the function is decreasing

$f''(0)=9$ the function is convex

TO ~~ALSO~~ DEDUCE THIS THE BASE IS $f(x)$ continuous

Th

$$\begin{cases} 1) \ f \text{ diff } n \text{ times at } x_0 \quad n \geq 3 \\ 2) \ \exists m \in \mathbb{N} \quad 3 \leq m < n \\ f'''(x_0) = \dots = f^{(m-1)}(x_0) = 0 \\ f^{(m)}(x_0) \neq 0 \end{cases}$$

=> 1) m odd => x_0 is an INFLECTION POINT

2) m even => x_0 is NOT an inflection point

NOW THE QUESTION is?

$f(x) \in C^\infty(\mathbb{R}) \quad f(0)=0 \quad f(x) \neq 0$

$\exists$ an index m such that $f^m(0) \neq 0$?

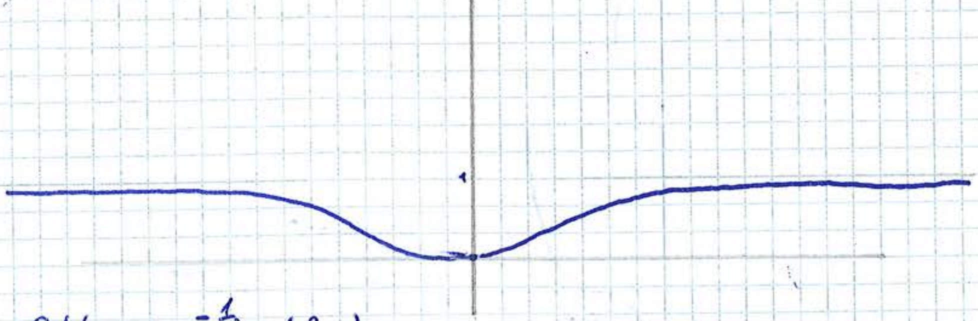
$\boxed{\text{NO}}$

Ex

$$\tilde{f}(x) \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x=0 \end{cases}$$

EVEN FUNCTION

$\tilde{f} \in C^\infty(\mathbb{R})$



$f'(x) = e^{-\frac{1}{x^2}} \cdot \left(\frac{2}{x^3}\right) \quad x \neq 0$

$$\lim_{x \to 0^+} \frac{2}{x^3} e^{-\frac{1}{x^2}} \overset{\frac{1}{x}=t}{=} \lim_{t \to +\infty} 2t^3 e^{-t^2} = \lim_{t \to +\infty} \frac{2t^3}{e^{t^2}} = 0$$

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$$\Rightarrow \frac{\cancel{1} + \cancel{x} + \frac{1}{4}x^2 + \cancel{\frac{1}{6}x^3} + \frac{x^4}{24} + \cancel{1} - \cancel{x} + \frac{1}{4}x^2 - \cancel{\frac{1}{6}x^3} + \frac{x^4}{24} + o(x^4)}{2} = \cosh(x)$$

$$\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad n = 4, \quad x_0 = 0$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + o(x^4)$$

$$e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + o(x^4)$$

$$\sinh(x) = \frac{\cancel{1} + x + \cancel{\frac{1}{2}x^2} + \frac{1}{6}x^3 + \cancel{\frac{1}{24}x^4} + o(x^4) - \cancel{1} + x - \cancel{\frac{1}{2}x^2} + \frac{1}{6}x^3 - \cancel{\frac{1}{24}x^4} + o(x^4)}{2}$$

$$\sinh(x) = x + \frac{1}{6}x^3 + o(x^4)$$

4] PRODUCT

$$f(x) = \underset{\text{EVEN}}{e^{x^2}} \cdot \underset{\text{EVEN}}{\cos x} \qquad f^{(4)}(0),\ f^{5}(0)\ ?$$

$\Downarrow$

$f(x)$ is EVEN
TH EVEN FUNCTION $\Rightarrow$ ONLY EVEN POWERS IN MCLAURIN POLYNOMIAL FORM

$\Downarrow$

$f^{5}(0) = 0$

$f^{(4)}(0) = ?$

I need expansion of order 4!

$$e^{t} = 1 + t + \frac{t^2}{2} \,\|\, + o(t^2)$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

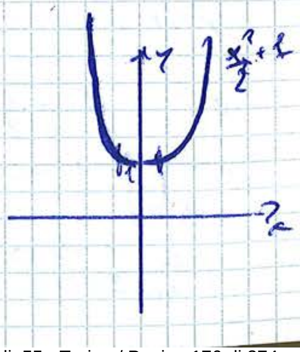
$$f(x) = \left[1 + x^2 + \frac{x^4}{2} + o(x^4)\right]\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right] =$$

$$= 1 + x^2 + \frac{x^4}{24} + o(x^4) + x^2 - \frac{x^4}{24} + o(x^4) + \frac{x^4}{2} + o(x^4)$$

$$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) \Rightarrow f^{4}(0) = \frac{1}{24}$$

$\hookrightarrow$ The beaviour?

$$e^{x^2}\cos x - 1 \sim \frac{x^2}{2} \qquad e^{x^2}\cos x \sim \frac{x}{2} + 1$$



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$n=3$

$$\cos x = \left(1 + x - \frac{x^3}{6} + o(x^3)\right)\left(1 - \frac{1}{2}x^2 + o(x^3)\right)$$

$$= 1 - \frac{1}{2}x^2 + o(x^3) + x - \frac{3}{6}x^3 - \frac{x^3}{6} + o(x^3)$$

$$\boxed{= 1 + x - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{2}{3}x^3 + o(x^3)}$$

$n = 4 \quad I(0)$

$$g(x) = \frac{\sin x}{\cos x}$$

$$(1 - z + z^2)$$

$$= \left(1 + x - \frac{x^3}{6} + o(x^4)\right)\left(1 - (\cos x - 1) + (\cos x - 1)^2 + o(x^4)\right)$$

$$\left(1 + x - \frac{x^3}{6} + o(x^4)\right)\left[1 - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right) + \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right)^2\right]$$

$$\left(1 + x - \frac{x^3}{6} + o(x^4)\right)\left(\cancel{1} - \cancel{1} + \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) + 1 + \frac{6}{24}x^4 - x^2 + \frac{1}{12}x^4\right)$$

$$= \left(1 + x - \frac{x^3}{6} + o(x^4)\right)\left(1 - \frac{1}{2}x^2 + \frac{7}{24}x^4 + o(x^4)\right)$$

$$= 1 - \frac{1}{2}x^2 + \frac{7}{24}x^4 + o(x^4) - x + \frac{3}{6}x^3 + \frac{x^3}{6} + o(x^4)$$

$$g(x) = -1 - x + \frac{2}{3}x^3 + \frac{7}{24}x^4 + o(x^4)$$

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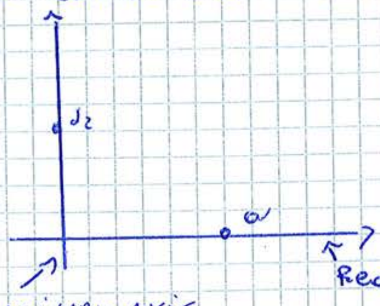
$z = a + i0 = a$ $\quad$ $z = 0 + ib = ib$

↳ called "IMAGINARY NUMBER"

"PURELY IMAGINARY NUMBERS"



PROPERTIES

EQUALITY

$z = a + ib$

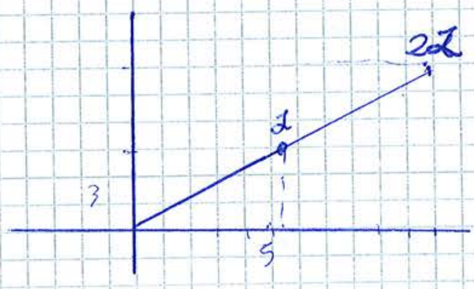
$z' = a' + ib'$

$z = z' \iff \begin{cases} a = a' \\ b = b' \end{cases}$

MULTIPLYING BY REAL NUMBERS

$z = a + ib$ $\quad$ $\alpha z = \alpha a + i\alpha b$

$\alpha = \mathbb{R}$



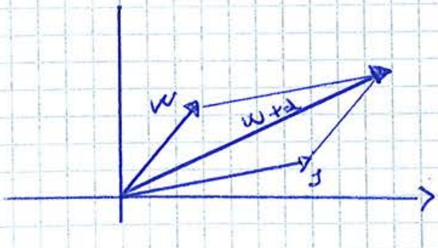
SUM $\quad$ $z = a + ib$ $\quad$ $z + w = a + c + i(b + d)$

$w = c + id$

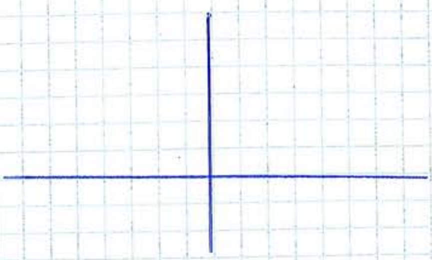
$0 = 0 + i0$ $\quad$ $z + b = \overbrace{a + 0}^{a} + i\overbrace{(b + 0)}^{ib}$

0 plays the same ROLE that plays in Real numbers

$-z = -a - ib$ $\quad$ $z + (-z) = 0$ $\quad$ $-z$ opposite of z



$w + z$



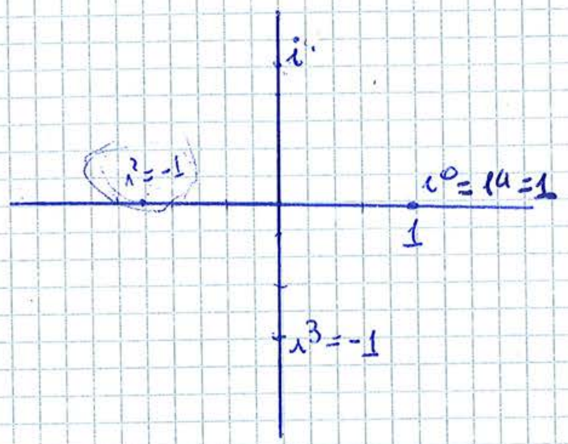
$w - z = w + (-z)$

$i^2 = -1$

$i^3 = i^2 \cdot i = -1$

$i^4 = i^3 \cdot i = 1$

$\Rightarrow i^0 = 1$



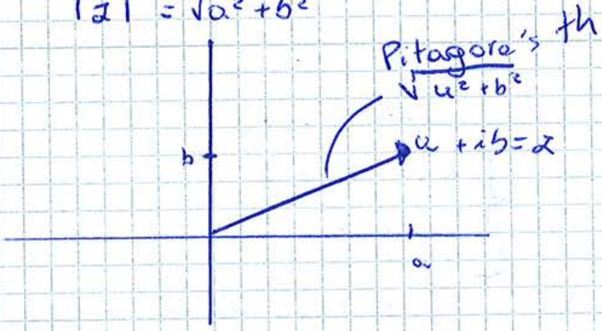
The powers of i just repeat $i^0 = i^5 = i^9$

$i^3 = i^{11}$ ...

$i^{37} = i^{36} \cdot i = i^4 \cdot i = 1 \cdot i = i$

MODULUS OF COMPLEX NUMBERS

$z = a + ib$ $\quad |z| = \sqrt{a^2+b^2}$



$|z|$ = DISTANCE OF z FROM the ORIGIN

$z = a \in \mathbb{R}$ $\quad |z| = |a|$ $\quad$ COMPLEX MODULUS $\equiv$ ABS. VALUES

Properties of the modulus

1) $|z| \geq 0$ $\quad |z| = 0 \Leftrightarrow z = 0$ the origin

2) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

3) $|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ $\Rightarrow$ TRIANGLE INEQUALITY

$$\left|\frac{1-i}{2-i}\right| = \sqrt{\frac{9}{25}+\frac{1}{25}} = \sqrt{\frac{2}{5}}$$

$$|x| = (a^2+b^2)$$

$$\frac{|1-i|}{|2-i|} = \frac{\sqrt{1+1}}{\sqrt{4+1}} = \frac{\sqrt{2}}{\sqrt{5}} = \sqrt{\frac{2}{5}}$$

IS IT POSSIBLE IN D TO DEFINE AN ORDER?

NO IF I WANT AN ORDER RELATION THAT SATISFY THE ORDER RELATION OF R

so if $a \le b \Rightarrow a+c \le b+c$ >NO in D

if $a \le b$, $c>0$ $\Rightarrow ac \le bc$

$\cancel{i > 0}$ "i" is not zero because it is associated to the imaginary part 1

maybe $\cancel{i > 0}$ "i" is positive? because the product of two positive numbers should be positive number but $i \cdot i = -1 < 0$

maybe $\cancel{i < 0}$ i not negative because the product of two negative is a positive

$$\boxed{i \cdot i < 0}$$

Ex

Find the set of $\{z \in \mathbb{C} : z^2 > 1\}$ $\cancel{\forall z \in \mathbb{C}}$ NON SENSE QUESTION

$\{z \in \mathbb{C} : z^2 \neq 1\}$ MEANING FULL QUESTION $\Rightarrow$ U CAN ANSWER

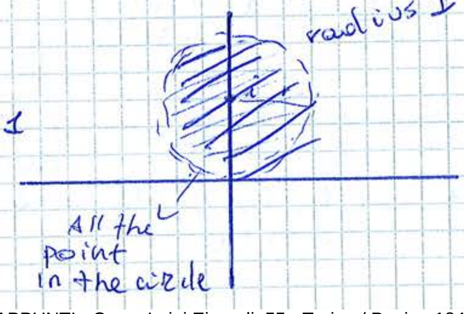
$\{z \in \mathbb{C} : \underbrace{|z|}_{\text{REAL NUMBER}} < |\underbrace{z+1}_{\substack{a+ib \\ a+1}}|\} = \sqrt{a+b^2} < \sqrt{(a+1)^2+b^2}$

$$A = \{z \in \mathbb{C} : |z - i| < 1\}$$

$|z|$ = DISTANCE OF z FROM THE ORIGIN

$|z - z_0|$ DISTANCE OF z FROM z_0

$|z - i| < 1$
$\Downarrow$
the distance between z and 1 < 1



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Ex 5

$A = \{ z \in \mathbb{C} : |z| = |z + 2i| \}$

$z = x + iy \qquad |z| = \sqrt{x^2 + y^2}$

$|z + 2i| = (x + iy) + 2i = x + i(y+2)$

$|z + 2i| = \sqrt{x^2 + (y+2)^2}$

$\sqrt{x^2 + y^2} = \sqrt{x^2 + (y+2)^2}$

$\cancel{x^2} + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} + 4y + 4$

$\boxed{y = -1}$ → line // x axis

$z = x - i$

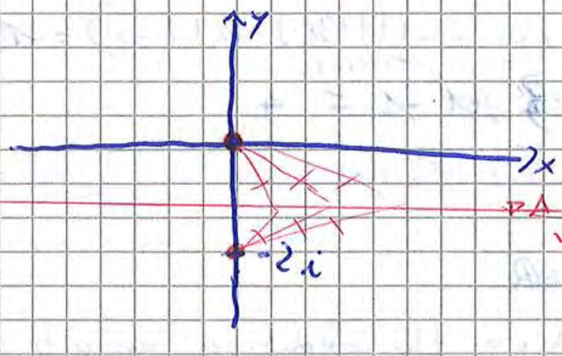
$|z| = |x - i + 2i|$

$|z| = |x + i|$

⇒ $|z|$ = distance between z from the origin

$|z + 2i| = |z - (-2i)|$

$|z - z_0|$ → Dist between z and z_0



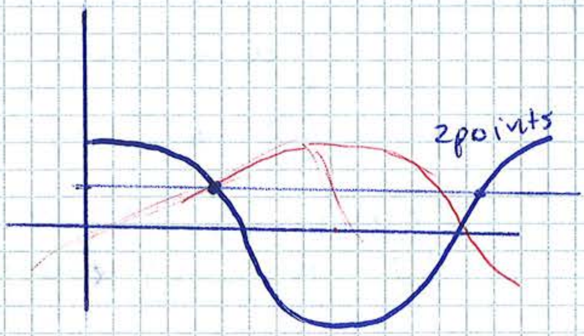
⇒ A is the set of all points in the plane with the same distance from these two points

⇒ The set of points $y = -1$

$$\begin{cases} a = r\cos\vartheta \\ b = r\sin\vartheta \end{cases}$$

Given a, b can i compute r and ϑ

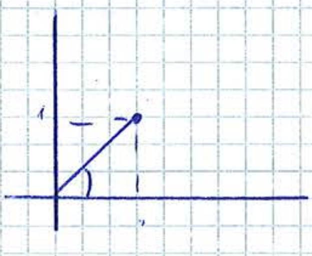
$$\Rightarrow \begin{cases} r = \sqrt{a^2+b^2} \\ \cos\vartheta = \frac{a}{r} = \boxed{\frac{a}{\sqrt{a^2+b^2}}} \end{cases}$$



I need to find

$$\Rightarrow \sin\vartheta = \frac{b}{r} = \frac{b}{\sqrt{a^2+b^2}}$$

$\Rightarrow z = 1 + i \qquad a = b = 1$



$$\begin{cases} r = \sqrt{2} \\ \cos\vartheta = \frac{1}{\sqrt{2}} \\ \sin\vartheta = \frac{1}{\sqrt{2}} \end{cases} \qquad \vartheta = \frac{\pi}{4}$$

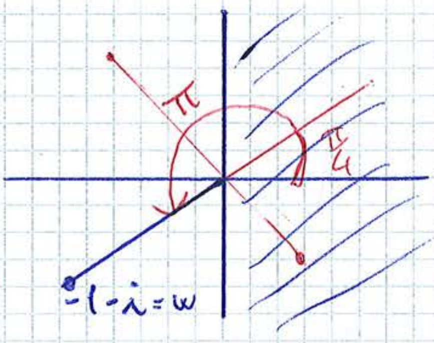
$\Rightarrow \quad a = r\cos\vartheta$

$b = r\sin\vartheta$

$$\tan\vartheta = \frac{b}{a}$$

$$\vartheta = \arctan\frac{b}{a} \Rightarrow \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$$

only if $Re\, z > 0$
↳ Real of $z > 0$



$\vartheta_{\arctan w} = \vartheta_{\arctan 1} = \frac{\pi}{4}$ NO

$\Rightarrow \vartheta_{\arctan w} = \vartheta_{\arctan 1} + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

$\Rightarrow$ $Re\, z < 0$
I add π

$z = x + iy$

$z - 1 = x + i(y-1)$

$|z-1| = \sqrt{x^2 + (y-1)^2} < 1 \Rightarrow x^2 + (y^2-1)^2 < 1$

The circle excluding the line

$$B = \{ z \in \mathbb{C} : Re(z^2) = 0 \}$$

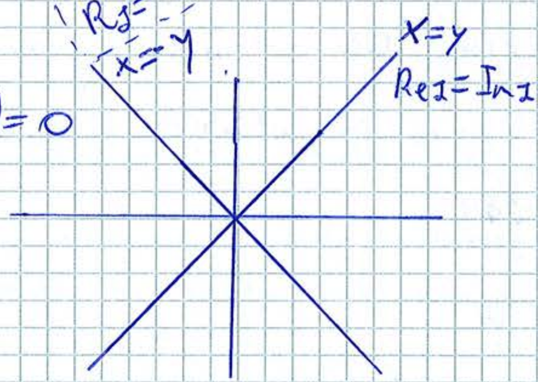
$z = x + iy$

$z^2 = \cancel{x^2 - (y^2)} + 2ixy$

$Re(z^2) = x^2 - y^2$

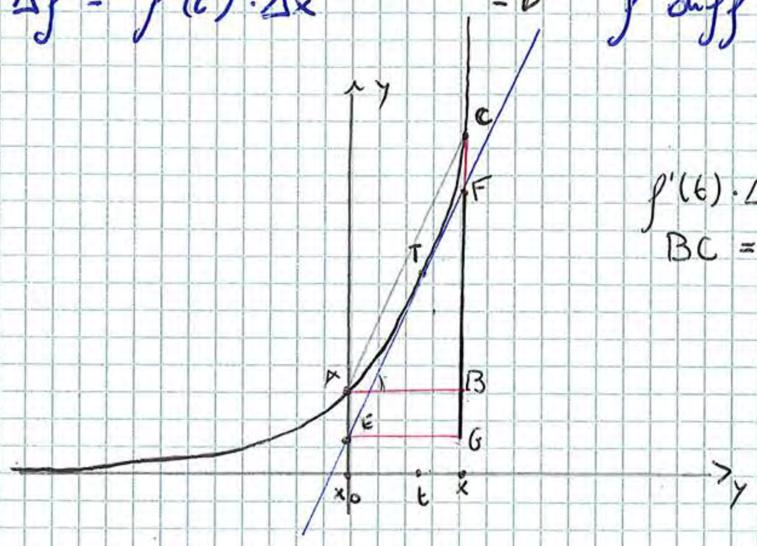
$x^2 - y^2 = 0 \Rightarrow (x-y)(x+y) = 0$

$\boxed{\begin{array}{l} x = y \\ x = -y \end{array}}$



II INCREMENT FORMULA

$\Delta f = f'(t) \cdot \Delta x$ $\Rightarrow$ f diff in I where $x, x_0 \in I$



$$f'(t) = \frac{f(x) - f(x_0)}{x - x_0}$$

$f'(t) \cdot \Delta x = GF = BC$

$BC = \Delta f$ INCREMENT

* Derivabile e differenziabile

⇓ data la funzione $f(x)$ ∃ limite del rapporto incrementale tra x e x_0

⇓ $\Delta f = f'(t) \cdot \Delta x$

II INCREMENT FORMULA

secondo se una funzione è differenziabile è anche derivabile

Differences between I formula and II formula

I Formula	II Formula
• LOCAL "True in a small neighbourhood for $\Delta x \to 0$"	• GLOBAL "True in an interval I where $x, x_0 \in I$"
• COEFFICIENT IS KNOWN $f'(x_0)$ Δf proportional to Δx	• COEFFICIENT IS NOT KNOWN Δf is proportional to Δx but $f'(t)$ is not known
• ERROR TERM $o(x - x_0)$ (Δx)	• NO ERROR TERM
• Differentiability only in a point is needed	• Differentiability on the interval

II formula

$$\Delta y = f'(t) \Delta x$$

$$f(x) - f(x_0) = f'(t)(x - x_0)$$

$$f(x) = \underbrace{f(x_0)}_{T f_{x_0, 0}} + \underbrace{f'(t)(x - x_0)}_{\text{"NEW" ERROR TERM}}$$

TAYLOR FORMULA WITH REMAINDER IN LAGRANGE'S FORM

$f \in C^{n+1}(I(x_0))$ $\quad f(x) = T f_{n, x_0}(x) + \frac{f^{(n+1)}(t)}{(n+1)!}(x - x_0)^{n+1}$

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EX. $f(x) = \sin\sqrt{x}$ $\quad x \geq 0$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + o(z^{2n+1})$$

$\sqrt{x} = z$
$x \to 0$ also $z \to 0$

$$\sin x = \sqrt{x} - \frac{\sqrt{x^3}}{6} + \frac{\sqrt{x^5}}{5!} + o(x^{\frac{5}{2}}) \quad \text{FOR } \boxed{x \to 0^+}$$

$\Rightarrow$ This is not a Taylor exponsion but has some properties of Taylor

$\Rightarrow$ **Asymptotic expansion**

$$f(x) = g_1 + g_2 + g_3 + o(g_3)$$

$$f \sim g_1$$

$g_2 = o(g_1)$ $\quad\Rightarrow$ I DON'T NEED ALWAYS A POLYNOMIAL
$g_3 = o(g_2)$
$g_4 = o(g_3)$

EX:

$f(x) = \sin\frac{1}{x}$ $\quad x \to +\infty$

$$\lim_{x \to +\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{t \to 0} \frac{\sin t}{t} = 1$$

$$\sin\frac{1}{x} \sim_{+\infty} \frac{1}{x} \Rightarrow \sin\frac{1}{x} = \frac{1}{x} + o\left(\frac{1}{x}\right)$$

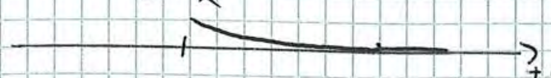
what is $o\left(\frac{1}{x}\right)$

$$\lim_{x \to +\infty} \frac{\sin\frac{1}{x} - \frac{1}{x}}{\frac{1}{x^{\alpha}}} = 1 \text{ and so on}$$

$f(x) = \sin\frac{1}{x}$ for $x \to +\infty$
we have $\frac{1}{x} = t$
$g(x) = \sin t$ $\quad$ as $x \to +\infty$, $t \to 0^+$

the behavior of my function $f(x)$ going to $+\infty$ is equal the behavior of $g(x)$ going to 0^+

sin $\frac{1}{x}$

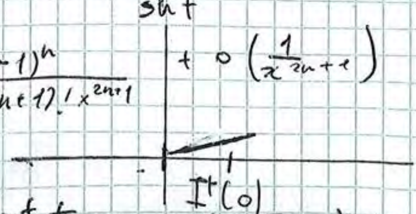


$+\infty$

$$\sin t = t - \frac{t^3}{6} + \frac{t^5}{5!} - \frac{t^7}{7!} + o(t^7) \quad t \to 0^+$$

sin t

$$\Rightarrow \sin\frac{1}{x} = \frac{1}{x} - \frac{1}{6x^3} + \frac{1}{5!x^5} + \dots + \frac{(-1)^n}{(2n+1)!\,x^{2n+1}} + o\left(\frac{1}{x^{2n+2}}\right)$$



$T^+(0)$

$$\sin\frac{1}{x} \sim \frac{1}{x} + o\left(\frac{1}{x}\right)$$

First order with respect to my test function $\frac{1}{x}$

$f(x) = \sin(\sin x)$

$\sin x = t \quad$ os $x \to 0$

$\qquad\qquad\quad t \to 0$

$h(x) = \sin t$

$h(x) = t - \frac{t^3}{6} + o(t^4)$

$$f(x) = \sin x - \frac{(\sin x)^3}{6} + o((\sin x)^4)$$

$$f(x) = x - \frac{x^3}{6} - \frac{\left[x^3 \cancel{+ \frac{x^9}{6^3} + o(x^9)} + \left(x - \frac{x^3}{6} + o(x^3)\right)\cancel{\left(-\frac{x^4}{6} + o(x^4) - o(x^6)\right)}\right]}{6}$$

$$+ o(x^4)$$

$$f(x) = x - \frac{1}{3}x^3 + o(x^4)$$

$f(x) =$

The first one to have an answer was Archimede but nobody cares!

WE HAVE 3 definitions:

1) Definition by CAUCHY ~1820 → Applied to continuous Function
but nobody cares!

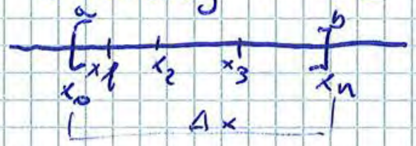
2) Reimann ~ 1850 → Applied to Bounded Function

3) Lebesgue ~ 1900 → Mesurable function
but nobody cares!

1] $[a;b]$ Partition of $f[a;b]$ $\mathcal{P}$ = partition

$\{x_0 = a, x_1, ---- x_n = b\}$

↳ $x_0 = a, x_1, ---- < x_n = b$ increasing order



For $i = 1, --- n$

$[x_{i-1}; x_i]$

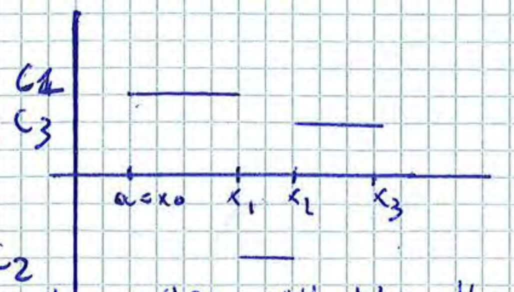
$x_i - x_{i-1} = \Delta x_i$

2] Given $\mathcal{P} = \{x_0, x_1, ---- x_n\}$

if $f(x) = C_i$ ← costant function $\forall x \in (x_{i-1}; x_i)$ $i = 1 --- n$

↳ we say that f is step function if there exist a portion of $[a,b]$ by $\{x_0, x_1, --- x_n\}$ together with $c_1, c_2 --- c_n \in \mathbb{R}$ such that

$f(x) = c_i$ $\forall x \in (x_{i-1}; x_i)$ $i = 1 --- n$



→ It doesn't matter what's the value of $f(x)$ at x_1, x_2 and x_3

* I CANNOT ANY FUCK to the extreme point of the subintervals in fact I defined $\forall x \in (x_{i-1}, x_i)$ with open brackets *

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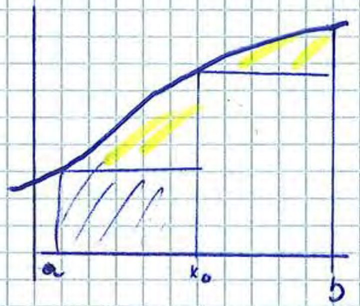
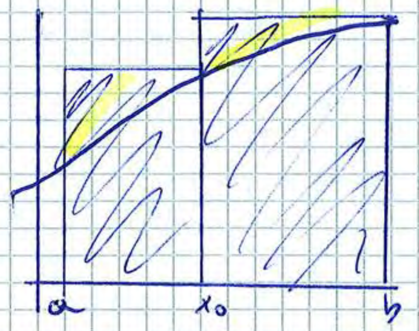
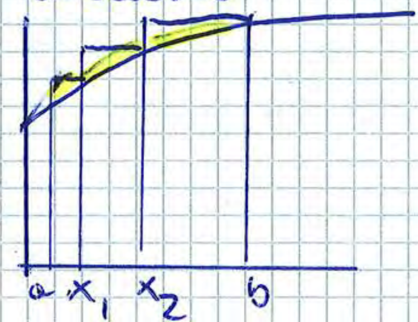




What's the relation between UPPER AND LOWER SUM?

$$s(\mathcal{P}) \le S(\mathcal{P})$$

then if I add $x_0 \in [a;b]$ the lower sum increases and the upper sum decreases



$$\Rightarrow \quad s(\mathcal{P}_1) \le s(\mathcal{P}_1 \cup \mathcal{P}_2) \le S(\mathcal{P}_1 \cup \mathcal{P}_2) \le S(\mathcal{P}_2)$$

if I add a points $S(\mathcal{P}_2)$ decreases

[The set $\{s(\mathcal{P})\}$ when $\mathcal{P}$ varies among All possible portions ~~the set~~ is bounded from above $\Rightarrow \exists s = \sup\{s(\mathcal{P})\}$

[The set $\{S(\mathcal{P})\}$ is bounded by below $\Rightarrow \exists\, \mathcal{S} = \inf\{S(\mathcal{P})\}$

$$\Rightarrow \quad s \le \mathcal{S}$$

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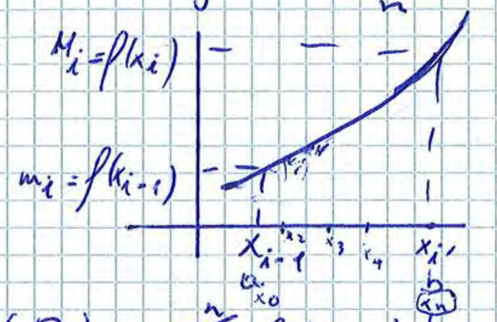
Th] f is integrable on $[a;b] \Longleftrightarrow \forall \varepsilon > 0 \ \exists P_\varepsilon : S(P_\varepsilon) - s(P_\varepsilon) < \varepsilon$

upper sum lower sum

Th] if f is monotone on $[a,b] \Rightarrow f$ is integrable on $[a;b]$

1) I proof f is mon incr

2) I take partition of $[a;b]$ in n intervals of the same lenght $\frac{b-a}{n}$



$$s(P) = \sum_{i=1}^{n} f(x_{i-1}) \frac{b-a}{n} \qquad S(P) = \sum_{i=1}^{n} f(x_i) \underbrace{\frac{b-a}{n}}_{\text{costant}}$$

$$S(P) - s(P) = \frac{b-a}{n} \left[\cancel{f(x_1)} + \cancel{f(x_2)} + \ldots + \cancel{f(x_{n-1})} + f(x_n) - f(x_0) - \cancel{f(x_1)} - \cancel{f(x_2)} - \ldots - \cancel{f(x_{n-1})} \right]$$

$$\Rightarrow \frac{b-a}{n} \left[f(x_n) - f(x_0) \right]$$

$$\frac{b-a}{n} \left[f(b) - f(a) \right]$$

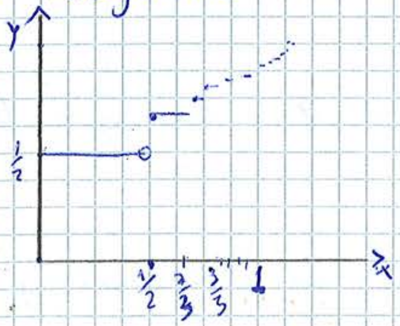
given $\varepsilon > 0$ CHOOSE an n BIG ENOUGH such That

$$\frac{(b-a)}{n} (f(b) - f(a)) < \varepsilon$$

getting bigger this value n becomes smaller $\left[\frac{(b-a)}{n} (f(b) - f(a))\right]$

REMARKS:

Monotone functions can have infinite jumps of discontinuity but are still integrable



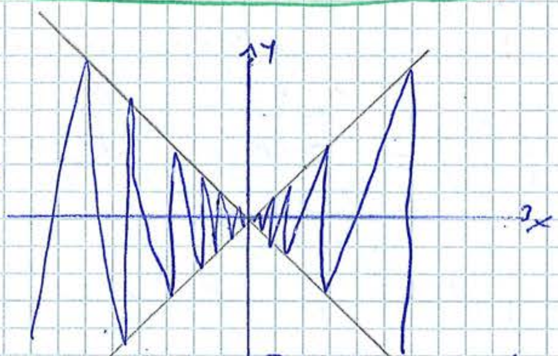
$\Rightarrow$ this function is monotone increasing and is not continuos with ACCUMULATION POINT AT 1
$\Rightarrow$ INTEGRABLE on $[0,1]$

"All continuous function are integrable"

Th] $\boxed{f \text{ continuous on } [a,b] \Rightarrow f \text{ integrable on } [a;b]}$

ex: $f(x) = x^2 \quad [a,b]$

$$\hat{f}(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



UNIFORM CONTINUITY OF A FUNCTION $\Rightarrow$ a function continuous and bounded on $[a,b]$

$\Rightarrow$ we define A PIECEWISE CONTINUOS FUNCTION ON $[a,b]$ if

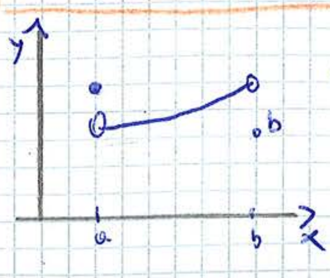
1) f continuous $\forall x \in [a,b]$ except a finite number of points

2) The discontinuity are $\Rightarrow$ REMOVEBLE or $\Rightarrow$ JUMP

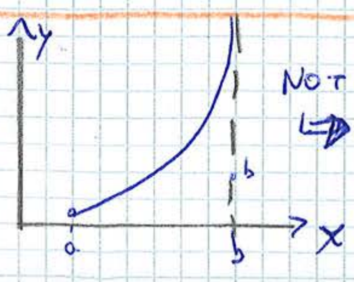
EX: sign x, $M(x)$, $[x]$

Th] f Piecewise continuos in $[a;b] \Rightarrow$ integrable on $[a,b]$

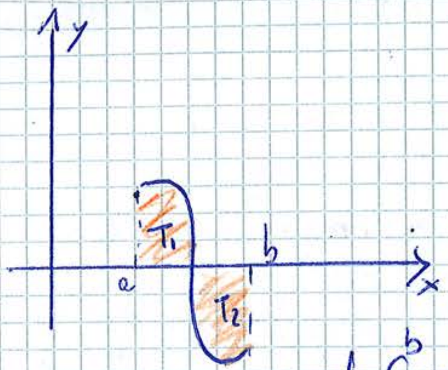
Th] $\left.\begin{array}{l} f \text{ cont on } (a,b) \\ f \text{ bounded on } [a,b] \end{array}\right\} f$ is integrable on $[a,b]$



$f(x)$ is integrable

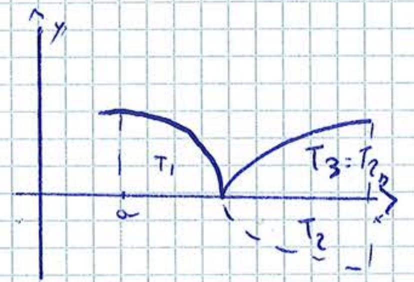


NOT BOUNDED
$\Rightarrow$ NOT INTEGRABLE ON $[a,b]$



$$\int_a^b f(x)\,dx \rightarrow Area(T_1) - Area(T_2)$$

$$\Rightarrow \left| \int_a^b f(x)\,dx \right| = \left| Area(T_1) - Area(T_2) \right|$$



$$\int_a^b |f(x)|\,dx = Area(T_1) + Area(T_{2\equiv 3})$$

$$\Rightarrow \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

$$\left[\begin{array}{l} \text{The area of the} \\ \text{region between} \\ \text{the graph of } f(x) \\ \text{and the x-axis} \end{array} \right] \Longleftrightarrow \int_a^b |f(x)|\,dx$$

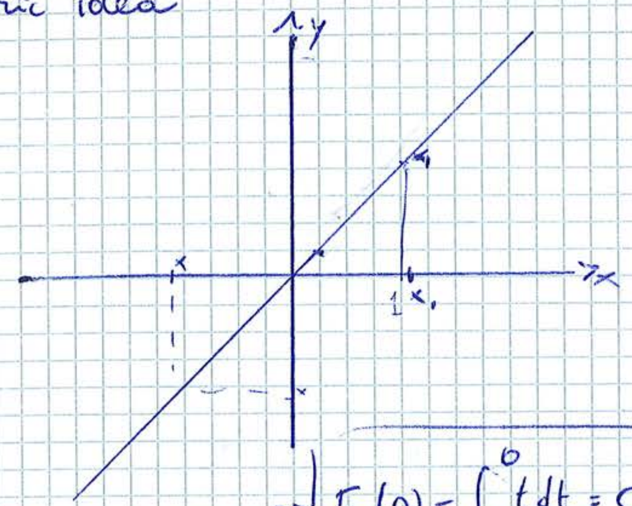
Oriented Integrals

DEFINED $\int_a^b f(x)\,dx$ • $a < b$

• $a = b \Rightarrow \int_b^a f(x)\,dx = 0$

• $a > b$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Geometric idea



$F(x) = \int t\,dt$

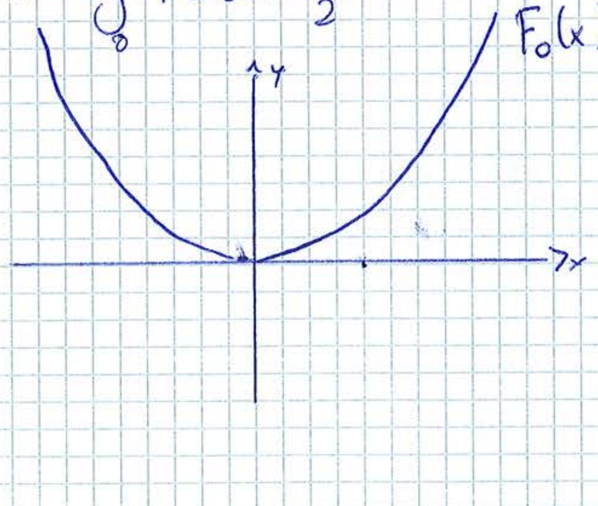
$\Rightarrow \boxed{F_0(0) = \int_0^0 t\,dt = 0}$

$x>0 \quad F_0(x) = \int_0^x t\,dt = \frac{x^2}{2}$

$x<0 \quad F_0(x) = \int_0^x t\,dt = -\int_x^0 t\,dt = -(-\text{Area}\,T) = \frac{x^2}{2}$

$\Rightarrow F_0(x) = \int_0^x t\,dt = \frac{1}{2}x^2$

$F_0(x) = \int t\,dt$



Properties of oriented integrals

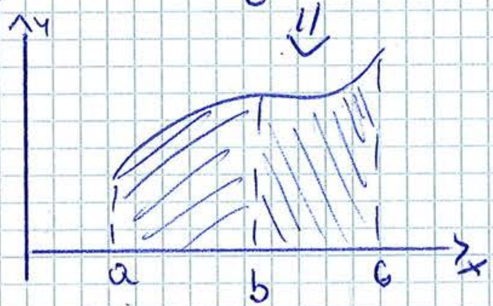
1) LINEARITY

$$\int_a^b (\alpha \cdot f(x) + \beta \cdot g(x))\,dx = \alpha \int_a^b f(x)dx + \beta \int_a^b f(x)dx$$

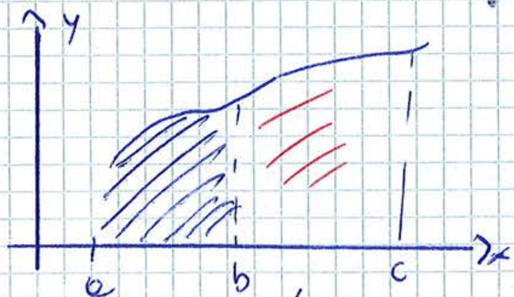
IFF f and g integrable functions

2) f DEFINED IN I, f is integrable in all sub intervals of I
$a, b, c \in I$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



This is true for all position of c



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\Downarrow$$

$$-\int_b^c f(x)dx$$

2) f continuous $\Rightarrow$ we apply Weierstrass's Theorem
"if f cont on bounded on $[a,b]$
it take a maximum and minimum value"

$\Rightarrow \inf f = m \qquad \sup f = M$

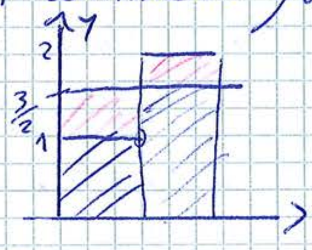
$$m \le m(f;a,b) \le M$$

↳ I Apply The intermediate value theorem (II form)
a continuous function in $[a,b]$ assumes all the points between the minimum and the maximum

$\Rightarrow \exists\ z \in [a,b] : f(z) = m(f;a,b)$

Remarks: The second part in general is not true for non continuous functions

Ex



$\int_0^2 f(x)\,dx = 3$

$m(f;0,2) = \frac{3}{2} \rightarrow$ THIS IS NOT A VALUE TAKEN BY THE FUNCTION

Th] $\cdot f$ defined on an Interval I
$\cdot f$ integrable over $[a,b]$ $\forall [a,b] \subseteq I$ $\Rightarrow F_{x_0}(x)$ is CONTINUOUS on I (INTERVAL)
$\cdot x_0 \in I$

T

Ex $f(x)=x$

$$F_0 x = \frac{x^2}{2} \qquad F_0'(x_0) = \frac{2x}{2} = x$$

$$f(x)=x \longrightarrow F_0(x) = \int_0^x t\,dt = \frac{x^2}{2}$$

$$D$$

Remark

if $I = [a,b]$ or $[a,+\infty)$

The th applied at the end points considering the right/left derivatives

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DEF f defined on an Interval

a function $F(x)$ differentiable on I /

$$F'(x) = f(x)$$

$F(x)$ is called PRIMITIVE of f on the interval I

OR ANTIDERIVATIVE

if the interval is closed or partially closed ~~the~~ we have left/right differentiability

Fundamental Theorem

$f \in C^0(I) \Rightarrow f$ has a primitive that is $F_{x_0}(x) = \int_{x_0}^x f(t)\,dt$

$$F_{x_1}(x) = \int_{x_1}^x f(t)\,dt = \underbrace{\int_{x_1}^{x_0} f(t)\,dt}_{\text{This is a numbe } C} + \int_{x_0}^x f(t)\,dt$$

if we change x_0 we have the primitive + C constant

$$= C + F_{x_0}(x)$$

f has infinitely many primitive according to choice of the starting point of the integration that differ one from the other only by a costant!

Properties of primitive

Th I If f admits a primitive $F(x)$ on the interval I, then any primitive of $f(x)$ on I are in the form

$$F(x) + C \qquad C \in \mathbb{R}$$

"if u know just one primitive, we also know that the other primitive are the one will plus a costan C"

PROOF I 1) $F(x)$ primitive of $f(x)$ on I $\Rightarrow$ $F(x)+c$ primitive of $f(x)$ on I

$$(F(x) + c)' = F'(x) = f(x)$$

2) Suppose that also $G(x)$ is a primitive of f on I such that

$$\Rightarrow G(x) = F(x) + C$$

"how to prove this?"

our hyp $\quad F'(x) = f(x) \qquad \Rightarrow H(x) = G(x) - F(x)$

$\qquad\qquad G'(x) = f(x) \qquad\quad H'(x) = f(x) - f(x) = 0$

$H'(x)$ has zero derivatives

for the Lagrange consequence a function is costant iff its derivate is 0 on I on an Interval I

$\Rightarrow H(x)$ costant

$$H(x) = \boxed{G(x) - F(x) = C}$$

$$\Rightarrow G(x) = F(x) + C$$

$\hookrightarrow$ I interval is absolutely needed

$\Rightarrow$ If $F(x)$ is a primitive of $f(x)$ on I, then the set of all primitive of $f(x)$ on I is

$$\{F(x) + c \,;\, c \in \mathbb{R}\} = \int f(x)\,dx$$

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Th] Torricelli-Barrow Theorem

$\begin{cases} \cdot f(x) \text{ continuous on an interval } I \\ \cdot G \text{ is any primitive of } f(x) \text{ on } I \\ a, b \in I \end{cases} \Rightarrow \int_a^b f(x)dx = G(b) - G(a) \boxed{= [(x)]_a^b}$

PROOF] "I can consider the primitive that vanishes at a

$$G(x) - \underbrace{G(a)}_{G(a)=0} = \int_a^x f(t)\,dt \qquad \forall x \in I$$

$\Rightarrow$ takes $x = b$

$$\boxed{\int_a^b f(t)\,dt = G(b) - G(a)}$$

$\Rightarrow$ Torricelli-Barrow Formula

$x^3 \xrightarrow{D} 3x^2$

The derivative of x^3 is $3x^2$

A primitive of $3x^2$ is x^3 $\Rightarrow$ all other primitives are x^3+c

APPLY

$$\int_0^5 3x^2 dx = [x^3 + c]_0^5 = [x^3]_0^5 = 5^3 - 0 = \boxed{125}$$

$\hookrightarrow c = 0$

Ex $\int \sqrt{\sin x}\,\cos x\,dx = \frac{3}{2}\sqrt{\sin^3} + c$

$g(x) = \sin x$
$g'(x) = \cos x$
$f(y) = \sqrt{y}$
$F(y) = \frac{3}{2}\sqrt[2]{y^3}$

$y = g(x) \Rightarrow \frac{dy}{dx} = g'(x)$

$dy = g'(x)\,dx$

Ex$_2$ $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$\cos x = y$
$-dx \sin x = dy$

$= +\int \frac{-dy}{y} = -\log|y| + c$

$\Rightarrow -\log|\cos x| + C$

2) $\int \frac{3x}{1+x^4}\,dx$

$x^2 = y$
$dx\,2x = dy$
$dx \cdot x = \frac{dy}{2}$

$3\int \frac{x}{1+x^2}dx = \frac{3}{2}\int \frac{dy}{(1+y)^2}$

$= \frac{3}{2}\arctan y + C = \frac{3}{2}\arctan x^2 + C$

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Definite integral

f integrable on $[-a; a]$ $a>0$

1) f odd $\Rightarrow \int_{-a}^{a} f(x)dx = 0$

2) f even $\Rightarrow \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

1) PROOF

if f odd $= \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

$\Rightarrow$ $dx = -dt$

$\Rightarrow \int_{-a}^{0} f(x)dx \overset{x=-t}{=} -\int_{a}^{0} f(-t)dt = -\int_{0}^{a} f(t)dt$

$\rightarrow$ as my function is odd $f(-t) = -f(t)$

Relation between $\int$ integrals and Taylor expansions *Mclaurin

Th] $\begin{cases} h(x) \text{ continuous on } I(0) \\ h(x) = o(x^{\alpha}) \quad x \to 0 \quad \alpha \geq 0 \end{cases}$

$\Rightarrow H(x) = \int_{0}^{x} h(s)ds = o(x^{\alpha+1})$

PROOF $\lim_{x\to 0} \frac{H(x)}{x^{\alpha+1}} = \lim_{x\to 0} \frac{\int_0^x h(s)ds}{x^{\alpha+1}} \overset{H\hat{o}p}{=} \lim_{x\to 0} \frac{h(x)}{(\alpha+1)x^{\alpha}} = 0$

$\Rightarrow \lim_{x\to 0} \frac{H(x)}{x^{\alpha+1}} = 0 \qquad H(x) = o(x^{\alpha+1})$

Improper Integrals

Reinmann Integral ⇒ f BOUNDED on $[a,b]$

I type f defined on $[a,+\infty]$ or $(-\infty; a]$

° f INTEGRABLE on $[a,c]$ $\forall c > a$

(locally reinmann integrable)

$\forall c \geq a$ I can define $\int_a^c f(x)\,dx = F_a(c)$

DEF]

$$\int_a^{+\infty} f(x)dx = \lim_{c\to+\infty}\int_a^c f(x)dx = \lim_{c\to+\infty} F_a(c)$$

$$\lim_{c\to+\infty}\int_a^c f(x)\,dx = \begin{cases} l\in\mathbb{R} & \int_a^{+\infty} f(x)dx \quad \text{CONVERGENT} \\ \pm\infty & \int_a^{+\infty} f(x)dx \quad \text{DIVERGENT} \\ \nexists & \text{INDETERMINATE}\end{cases}$$

1] $$\int_0^{+\infty}\frac{1}{1+x^2}dx = \lim_{c\to+\infty}\int_0^c \frac{1}{1+x^2}dx = \lim_{c\to+\infty}[\arctan x]_0^c$$

$$= \lim_{c\to\infty} \arctan c - \arctan 0 = \frac{\pi}{2} \quad \boxed{\text{CONVERGENT}}$$

2] $$\int_3^{+\infty}\frac{x}{x^2+1}dx = \lim_{c\to+\infty}\frac{1}{2}\int_3^c \frac{2x}{x^2+1}dx = \lim_{c\to+\infty}\frac{1}{2}\left[\lg(x^2+1)\right]_3^c$$

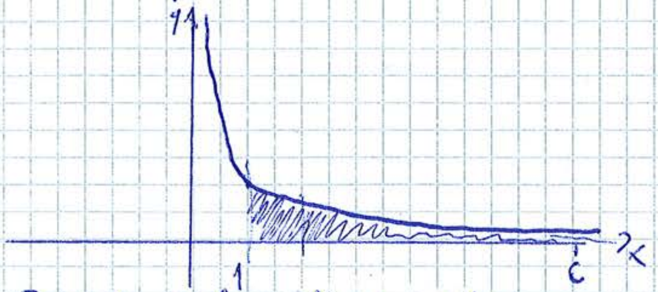
$$\lim_{x\to\infty}\frac{1}{2}\left[\lg(1+c^2) - \lg 10\right] = +\infty \quad \boxed{\text{DIVERGENT}}$$

3] $$\int_0^x \sin x\,dx = \lim_{c\to+\infty}\int_0^c \sin x\,dx = \lim_{c\to+\infty}[-\cos x]_0^c$$

$$= \lim_{c\to+\infty} -\cos c + 1 \quad \nexists \quad \text{INDETERMINATE}$$

2] $\int_1^{+\infty} \frac{1}{x^{\alpha}}\,dx = I \qquad \alpha > 0$

$\alpha = 1$] $I = \lim_{c\to+\infty}\int_1^c \frac{1}{x}\,dx = \lim_{c\to+\infty}\left[\lg x\right]_1^c = \lim_{c\to+\infty} \lg c = +\infty$



FOR $\alpha = 1$, $\int$ is divergent

$\alpha \neq 1$] $I = \lim_{c\to+\infty}\int_1^c \frac{1}{x^2}\,dx = \lim_{c\to+\infty}\left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^c$

$= \lim_{c\to+\infty} \frac{c^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha-1}$

→ $+\infty$ if $-\alpha+1>0$, $\alpha<1$

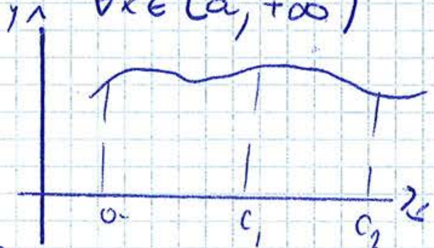
→ "0" $+ \frac{1}{\alpha-1}$ if $\alpha > 1$

$\Rightarrow \int_1^{+\infty}\frac{1}{x^{\alpha}}\,dx$

→ CONVERGENT TO $\frac{1}{\alpha-1}$ if $\alpha > 1$

→ DIVERGENT if $x \leq 1$ (first case)

d) $f(x) \geq 0$, $\forall x \in [a, +\infty)$



$\boxed{c_1 < c_2}$

$F_a(c_2) = \int_a^{c_2} f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx = F_a(c_1) + \underbrace{\int_{c_1}^{c_2} f(x)\,dx}_{\geq 0}$

if $c_1 < c_2$, $F_a(c_2) \geq F_a(c_1)$

Monotone function!!!!!

CONCLUSION] $f(x) \geq 0 \quad \forall x \in [a, +\infty) \Rightarrow F_a(c)$ is monotone increasing on $[a, +\infty)$

$$\int_{\frac{1}{2}}^{+\infty} \frac{\lg x}{x} = \underbrace{\int_{\frac{1}{2}}^{1} \frac{\lg x}{dx}}_{} + \int_{1}^{+\infty} \frac{\lg x}{x} dx$$

Reimann integral
$\Rightarrow$ C | NEGATIVE <0 |

$\frac{\lg x}{x}$ $\frac{1}{x}$

$\hookrightarrow$ WE MUST COMPERE BOTH

$$\Rightarrow \int_{\frac{1}{2}}^{+\infty} \frac{\lg x}{x} dx = \int_{\frac{1}{2}}^{e} \frac{\lg x}{x} dx + \int_{e}^{+\infty} \frac{\log x}{x} dx$$

$\frac{\lg x}{x} \geq \frac{1}{x}$ $\forall x > e$ $x \to +\infty$ $\lg x > 1$

$\frac{\lg x}{x} \geq \frac{1}{x}$

$\Downarrow$

$g(x) \geq f(x)$ $\Rightarrow$ both positive

$g(x) \geq f(x)$
$\int_e^{+\infty} f(x) dx$ DIVERG $\Bigg] \Rightarrow \int_e^{+\infty} \frac{\lg x}{x} dx$ $\boxed{\text{DIVERGENT}}$

EX2] $\int_2^{+\infty} \frac{3+\sin^2 x}{x^2} dx$

* we alway compare our functions to $\frac{1}{x^\alpha}$

$\frac{3}{x^2} \leq \frac{3+\sin^2 x}{x^2} \leq \frac{4}{x^2}$
$\quad f(x) \qquad g(x)$

$\int g(x) dx$ CONVERGENT $\Bigg] \Rightarrow \int \frac{3+\sin^2 x}{x^2} dx$

$\boxed{\text{CONVERGENT}}$

$\int_2^{+\infty} \frac{1}{x \ln x} dx$ $\quad x \lg x > x$ $\quad x \to +\infty$

$\frac{1}{x \lg x} < \frac{1}{x}$

is this usefull for comparision theorem
$\int g(x) dx$ DIVERGENT $\Bigg]$ NO COMP THEOREM because $g(x) \leq \frac{1}{x}$

Asymptotic comparison Theorem

- $f(x) \geq 0 \quad \forall x \in [a, +\infty)$
  f interable $\forall x \in [a, c]$ $c > 0$

1) If There exists $\alpha > 1$ such that

$$f \overset{\cup}{\cap} \frac{1}{x^{\alpha}} \quad \text{or} \quad f(x) = o\left(\frac{1}{x^{\alpha}}\right) \quad x \to +\infty$$

$$\Rightarrow \int_0^{+\infty} f(x)\,dx \quad \text{CONVERGENT}$$

2) If There exists $\beta \leq 1$ such that

$$f \overset{\cup}{\cap} \frac{1}{x^{\beta}} \quad \text{or} \quad \frac{1}{x^{\beta}} = o(f) \quad x \to +\infty$$

$$\Rightarrow \int_0^{+\infty} f(x)\,dx \quad \boxed{\text{DIVERGENT}}$$

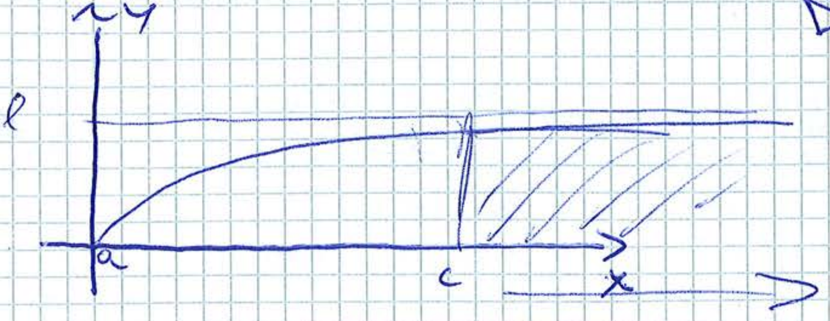
$$\to \int_1^{+\infty} \frac{\arctan x}{x^2+2} \qquad f(x) \geq 0$$

$$\lim_{x \to +\infty} \frac{\frac{\arctan x}{x^2+2}}{\frac{1}{x^{\alpha}}} \quad \alpha = 2 \quad \to \lim_{x \to +\infty} \frac{x^2 \arctan x}{x^2+1} = \frac{\pi}{2}$$

$$\Rightarrow \frac{\arctan x}{x^2+2} \overset{\cup}{\cap} \frac{1}{x^2} \quad \alpha = 2 \quad \boxed{\text{1st case}}$$

$$\Rightarrow \int_1^{+\infty} \frac{\arctan x}{x^2+2}\,dx \quad \boxed{\text{CONVERGENT}}$$

if $\lim_{x \to +\infty} f(x) = \ell$ $\quad \int_a^{+\infty} f(x)dx \Rightarrow$ is always DIVERGENT



This area goes always to $+\infty$

Th] • f integrable on $[a, c]$ $\forall c > a$

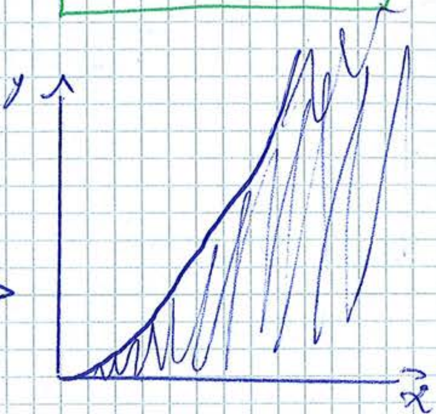
• $\lim_{x \to +\infty} f(x) = \ell \neq 0 \Rightarrow \int_a^{+\infty} f(x)dx$ $\boxed{\text{DIVERGENT}}$

$\hookrightarrow \ell \pm \infty$

Ex

$$\int_0^{+\infty} x^2 dx$$

$\lim_{x \to +\infty} x^2 = +\infty \Rightarrow$ DIVERGENT $\Rightarrow$



$\boxed{\text{PROOF}}$ $\quad \ell \in \mathbb{R} \quad \ell > 0$

$\lim_{x \to \infty} f(x) = \ell$

$\exists \bar{x} : \forall x > \bar{x} \quad f(x) = \frac{\ell}{2}$

$$\int_a^{+\infty} f(x)dx = \underbrace{\int_a^{\bar{x}} f(x)dx}_{\text{Reimann integral} \Rightarrow K} + \underbrace{\int_{\bar{x}}^{+\infty} f(x)dx}_{+\infty}$$

$$\geq K + \int_{\bar{x}}^{+\infty} \frac{\ell}{2} \Rightarrow$$

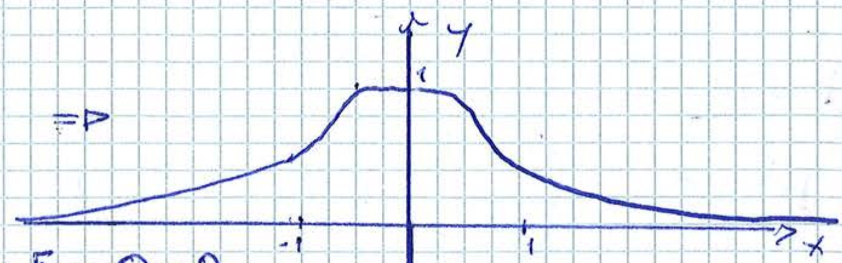
Remark:

The discussion about CONVERGENT / DIVERGENT is not immediately soluble if

$\lim_{x \to +\infty} f(x) = 0$ $\boxed{\text{or}}$ $\lim_{x \to +\infty} f(x) \nexists$

$$F(x) = \int_0^x \frac{1}{1+t^6}\,dt$$

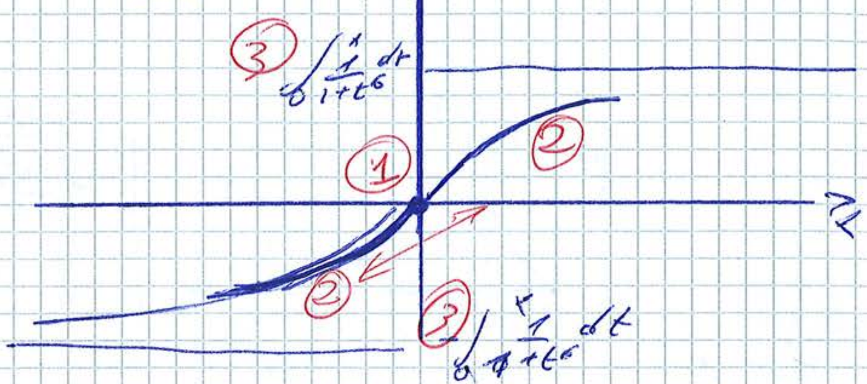
$$F'(x) = \frac{1}{1+x^6} \Rightarrow$$



The integral function F in 0 is 0

$$\int_0^0$$

For positive value my functions is positive



as $F'(x)$ is positive

And is concave

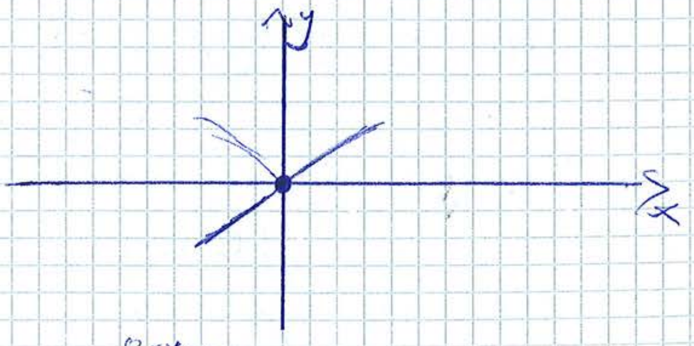
1 as $F'(x)$ is decreasind

② my $F(x)$ is even so the integral function is ODD

③ $\frac{1}{x^6+1} \sim \frac{1}{x^6}$

$6>0 \Rightarrow$ CONVERGENT $\exists$ a limit

$$\Rightarrow F(x) = \int_0^x \frac{t^2(\cos^2 t)}{t^4+1}\,dt$$



$F'(x)$ $\frac{t^2(\cos^2 t)}{t^4+1}$ = always posi

$f'(x)$ = EVEN $\Rightarrow$ ODD

$$\lim_{x\to+\infty} \frac{x^2(\cos^2 t)}{(x^4+1)} \quad \frac{1}{x^2} = \lim_{x\to+\infty} \frac{x^2 \cos^2 x}{x^4+1} = \lim_{x\to+\infty} \cos^2 x \;\nexists \text{ not}$$

$0 \le x \le 1$

we can only say $F(x)$ is increasing

if $f(x) \le 0$

when $f(x)$ is diverging to $+\infty$

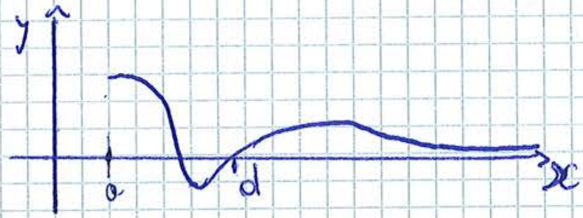
$-f(x)$ is diverging to $-\infty$

IF $F(x)$ CHANGES SIGN

→ two subcases → 1) a finite number of changes

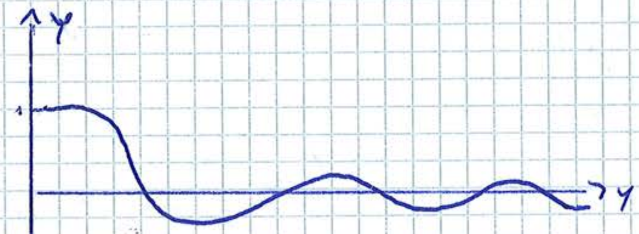
→ 2) infinite number of changes

1]



$$\int_a^{+\infty} f(x)dx = \int_a^{d} f(x)dx + \int_d^{+\infty} f(x)dx$$

2]



DEF | we say that $\int_a^{+\infty} f(x)dx$ Absolutely CONVERGENT if the integral at $+\infty$ $\int_a^{+\infty} |f(x)|dx$ is CONVERGENT

Theorem of the absolute convergence

$$\left|\int_a^{+\infty} f(x)dx\right| \le \int_a^{+\infty} |f(x)|dx$$

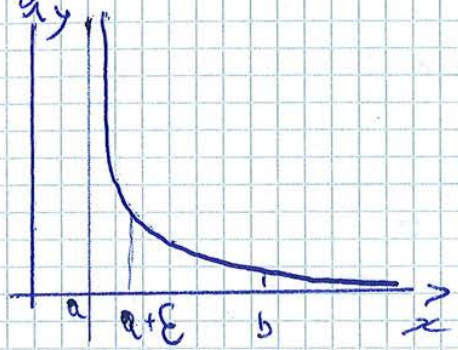
if ⇒ $\boxed{\int_a^{+\infty} |f(x)|dx \text{ is convergent} \Rightarrow \int_a f(x)dx \text{ is convergent}}$

ABSOLUTELY CONVERGENCE ⇒ CONVERGENCE

$\int f(x)$ ⇒ we dunno anything about it. being both positive and negative. But we know that is absolute value is always <u>positive</u>

f bounded on $[a;b]$ $\Rightarrow$ Reimann integral
· f Bounded on $[a, +\infty]$

suppose f unbounded on $(a, b]$



I have to compute the integral $\int_{a+\varepsilon}^{b}$
and go closer and closer to a

I NEEED $\Rightarrow$ f DEFINED on $(a, b]$
· $\forall \varepsilon > 0$ (small enough)
f integrable on $[a+\varepsilon, b]$

$$\Rightarrow \int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^b f(x)\,dx$$

Improper integral of the II type

· Everything true for the first type is the same for II type

1) $\rightarrow$ CON, $\rightarrow$ DIV, $\rightarrow$ $\Rightarrow$ if $f(x) \geq 0$ $\int_a^b f(x)\,dx$ $\rightarrow$ CONV, $\rightarrow$ DIVER

2) COMPARISON THEOREM

3) ASYMPTOTIC COMPARISON THEOREM

CHANGING SIGN $\Rightarrow$ Absolute convergence
$\rightarrow$ Absolute convergence theorem