NUMERO: 1622A - ANNO: 2015

A P P U N T I

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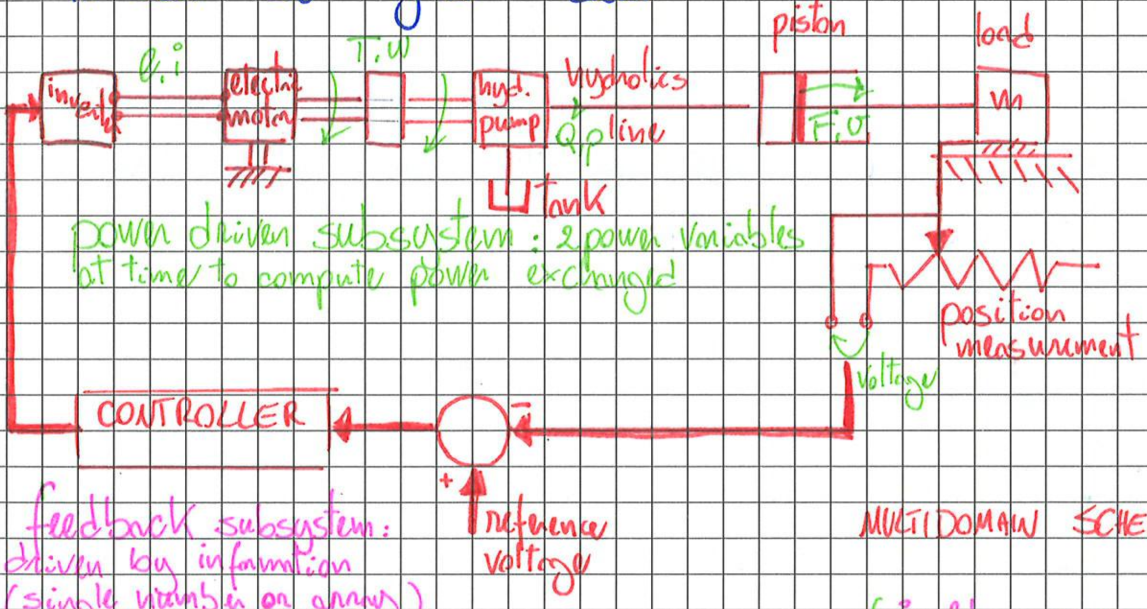
MATERIA: Modelling And Simulation Of Mechatronic System. Prof.Bona_Tonoli

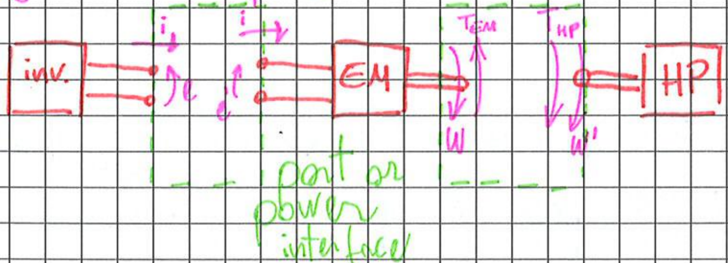
MODELLING AND SIMULATION OF MECHATRONIC SYSTEMS

BOND GRAPH MODEL

Bond graph is an explicit graphical tool for capturing the common energy structure of systems.
By this approach, a physical system can be represented by symbols and lines, identifying the power flow paths.
The elements of the systems are interconnected in an energy conserving way by bonds and junctions.
The derivation of system equations is so systematic that can be algorithmized.



MULTIDOMAIN SCHEME

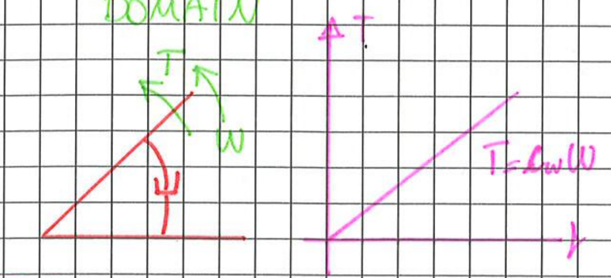


$$\begin{cases} i = i' \\ e = e' \\ \omega = \omega' \\ T_{EM} = T_{HP} \end{cases}$$

power going out of a subsystem is the same entering in the next one

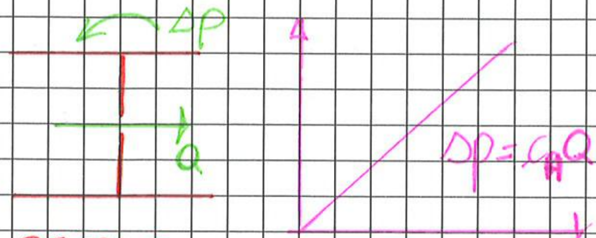
The power exchange can be compute using 2 power variables.
The variable that measures how much something is flowing in time is called FLOW (f), the other measures the strenght of action exchanged and is named EFFORT (e).

ROTATIONAL MECH DOMAIN



TORSIONAL DAMPER

HYDROLICS DOMAIN



RESTRICTION

in general effort and flow are related by a static function ($e = C f$).

The energy in the element is given by:

$$E = \int_{t_0}^{t} P\,dt = \int_{t_0}^{t} C f^2(t)\,dt = \int_{t_0}^{t} \frac{e(t)^2}{C}\,dt$$

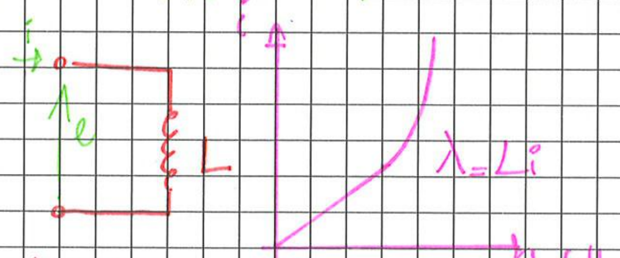
if f as input — if e as input

this time dependence shows that energy spent cannot be given back, but it's dissipated.

R-elements are dissipative components.

I-ELEMENTS

ELECTRIC DOMAIN

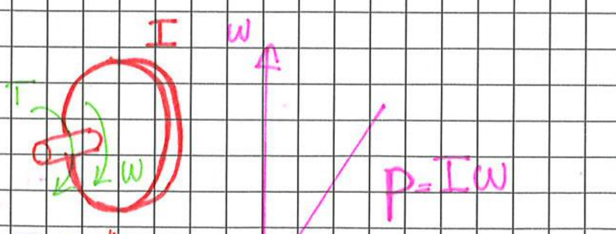


INDUCTOR

LINEAR MECH. DOMAIN

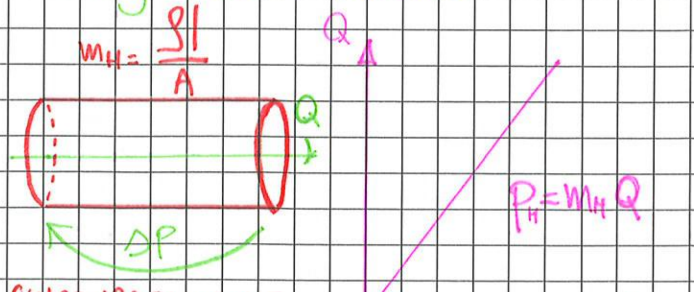
MASS

ROTATIONAL MECH. DOMAIN



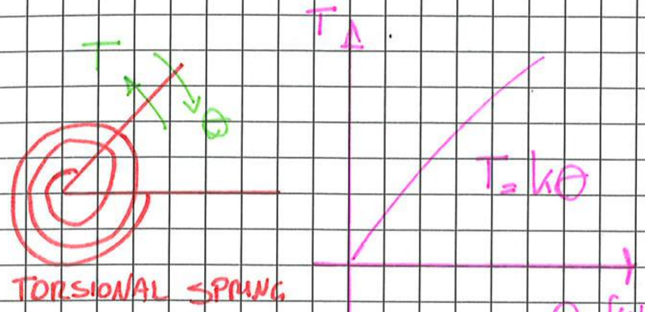
FLYWHEEL

HYDROLICS DOMAIN



UNCOMPRESSIBLE PIPE

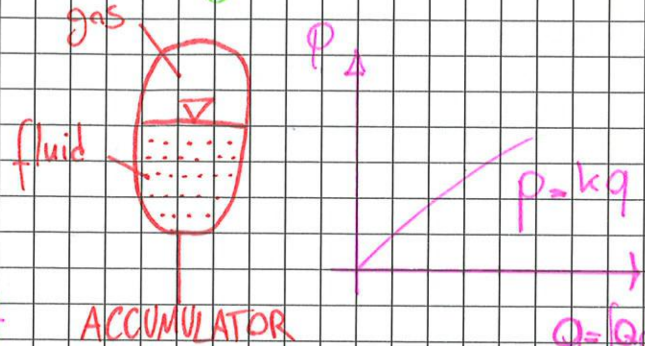
ROTATIONAL MECH. DOMAIN



TORSIONAL SPRING

$\theta = \int \omega \, dt$

HYDROLICS DOMAIN



ACCUMULATOR

$q = \int Q \, dt$

in general the constitutive law relates the effort and the displacement (integral of flow) ($e = kq$ with $q = \int f dt$)

↳ in electrical domain is used $e = \frac{q}{C}$ ← capacitance

The energy is:

$$E = \int_{t_0}^{t} P dt = \int_{t_0}^{t} e f dt = \int_{t_0}^{t} e \frac{dq}{\cancel{dt}} \cancel{dt} = \int_{q_0}^{q} e dq = - \int_{q}^{q_0} e dq$$

C-elements are reversible.

Also C-elements have different causality behaviors:

INTEGRAL CAUSALITY

flow as input

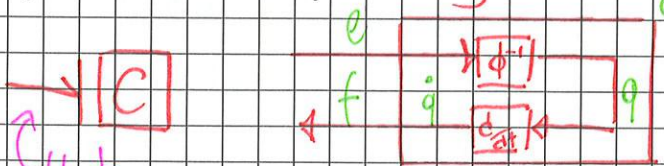
$e = \phi(q)$

$\dot{q} = f \quad q(t=0) = q_0$

state = initial condition

if a compliance recives a flow as input we have an associate state

DERIVATIVE CAUSALITY



effort as input

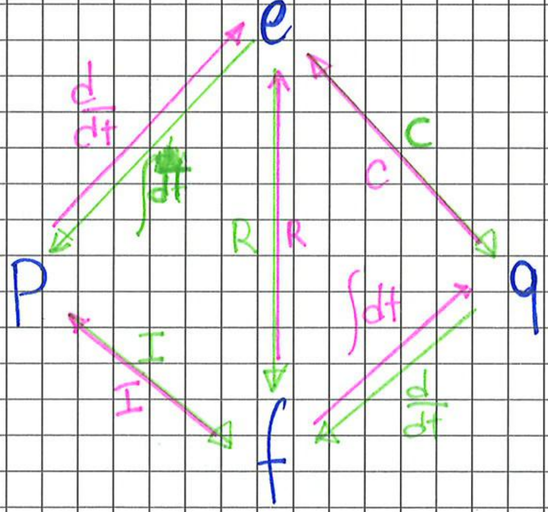
$e = \phi(q)$

$f = \dot{q} = \frac{d}{dt}\left(\phi^{-1}(e)\right)$

if a compliance has an effort as input we get no state

SUMMARY ON SINGLE PORT ELEMENTS

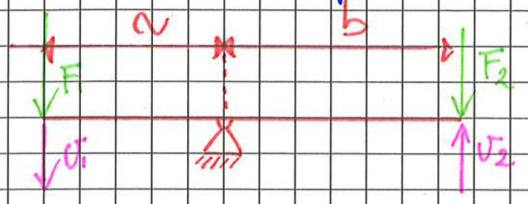
ELEMENT	RELATION	CAUSALITY: INTEGRATIVE	CAUSALITY: DERIVATIVE
R ←	$e = Rf$	NEVER	
I ←	$p = If$ $p = \int e\,dt$	⊣I	⊢I
C ←	$e = kq$ $q = \int f\,dt$	⊢C	⊣C
Se →	$e = e^* \; \forall f$	NEVER	
Sf →	$f = f^* \; \forall e$	NEVER	



2 PORT COMPONENT

TRANSFORMER

Let's take a lever mechanism, where the bar has no mass, inertia and compliance and the pivot/hinge is frictionless.



equilibrium $F_1 a = F_2 b$

$$F_1 = \frac{b}{a} F_2$$

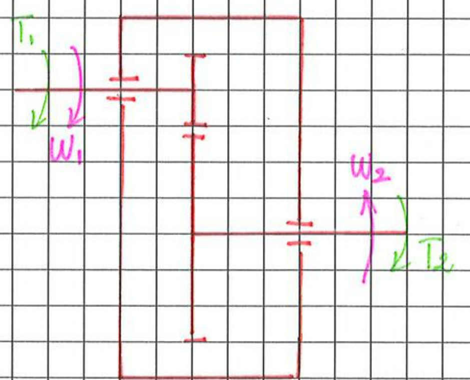
same angular speed $v_1 = \dot{\theta} a \quad v_2 = \dot{\theta} b$

$$v_1 = \frac{a}{b} v_2$$

Since there isn't storage or dissipation of energy the power in input is the same in output.

$$P_{in} = P_{out} \qquad F_1 v_1 = F_2 v_2$$

A gearbox with the same hypothesis behave in the same way:

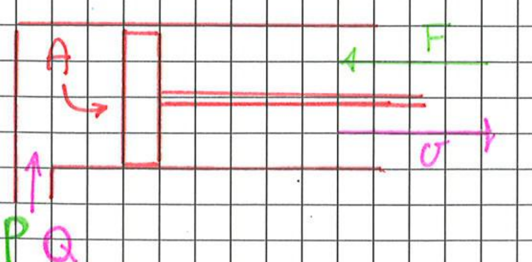


$$\omega_1 = \tau \omega_2$$

$$T_1 \omega_1 = T_2 \omega_2$$

$$T_1 = \frac{1}{\tau} T_2$$

The piston is another example in the hydraulics domain.



$$F = Ap$$

$$Fv = pQ$$

$$v = \frac{1}{A} Q$$

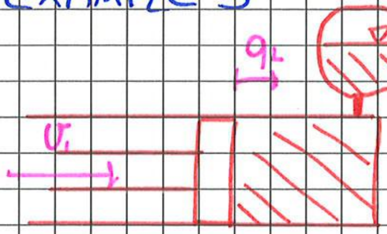
$$\left.\begin{array}{l} F_1 a = F_2 b \rightarrow F_1 = \frac{b}{a} F_2 \\ \frac{b}{a} v_1 = v_2 \end{array}\right\} \text{TRANSFORMER}$$

$F_2 = \beta v_2$ DISSIPATIVE COMPONENT

$$F_1 = F_2 \frac{b}{a} = \frac{b}{a} \beta v_2 = \frac{b}{a} \beta \frac{b}{a} v_1 = \left[\left(\frac{b}{a}\right)^2 \beta\right] v_1$$

equivalent system $\boxed{Se} \rightarrow \boxed{R_{eq}}$

EXAMPLE 3



$\boxed{Sf} \vdash\!\rightarrow \boxed{TF} \vdash\!\rightarrow \boxed{C}$

$F_1 = A p_2$

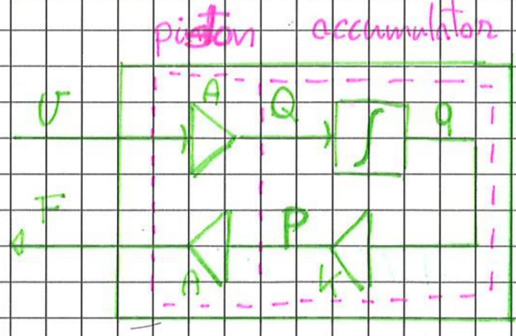
$A v_1 = Q_2$

$p_2 = k q_H = k \int Q_2 dt$

$$F_1 = A p_2 = A k \int Q_2 dt = A^2 k \int v_1 dt = A^2 k q_L$$

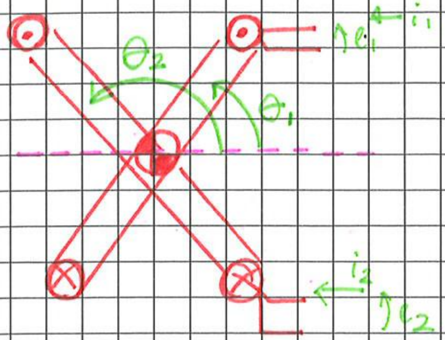
equivalent system $\boxed{Sf} \vdash\!\rightarrow \boxed{C}$

BLOCK DIAGRAM



Note that in a transformer both power can't have the same sign: we need one entering power and one going out.

to have a constant torque regardless θ we add a second coil

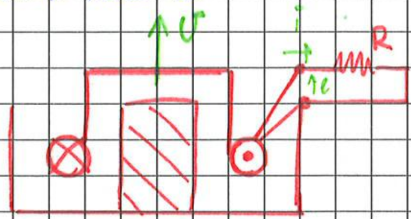


$$T = T_1 + T_2 = i_1 \Lambda \cos\theta_1 + i_2 \Lambda \cos\theta_2 =$$
$$= \Lambda (i_1 \cos\theta_1 + i_2 \cos(\theta_1 + 90°)) =$$
$$= \Lambda (i_1 \cos\theta_1 - i_2 \sin\theta_1)$$

SUMMARY ON 2-PORTS ELEMENTS

COMPONENT	CONSTITUTIVE LAW	CAUSALITY PROPAGATION
TRANSFORMER $\xrightarrow[f_1]{e_1} \boxed{TF} \xrightarrow[f_2]{e_2}$	$f_1 = h f_2$ $e_2 = h e_1$	$\vdash \boxed{TF} \| \!\!-$ $-\!\!\| \boxed{TF} -\!\!\|$
GYRATOR $\xrightarrow[f_1]{e_1} \boxed{GY} \xrightarrow[f_2]{e_2}$	$e_1 = k f_2$ $e_2 = k f_1$	$-\!\!\| \boxed{GY} \| \!\!-$ $\vdash \boxed{GY} -\!\!\|$

EXAMPLE 1



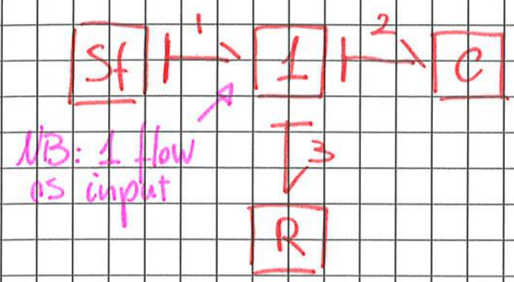
$$\boxed{Sf} \vdash\!\!\rightharpoonup \boxed{GY} \rightharpoonup\!\!| \boxed{R}$$

$$e = K_m v$$
$$i = \frac{1}{R} e = \frac{K_m}{R} v$$
$$F = K_m i = K_m \frac{K_m}{R} v = \left(\frac{K_m^2}{R}\right) v = \beta v$$

the system behaves as a damper, the damping can be tuned using a variable resistence R

1-TYPE, COMMON FLOW JUNCTION

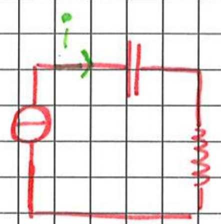
A 1-type juction shares an input flow the same way a 0-type shares an input effort.

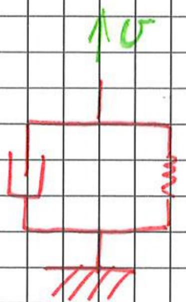


$f_1 = f_2 = f_3$

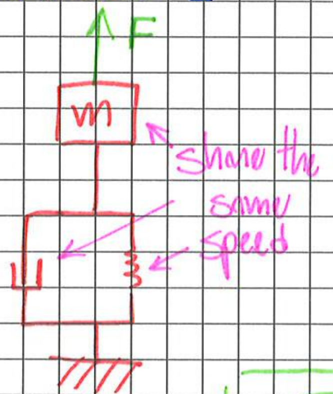
$\cancel{e_1 f_1} - \cancel{e_2 f_2} - \cancel{e_3 f_3} = 0$

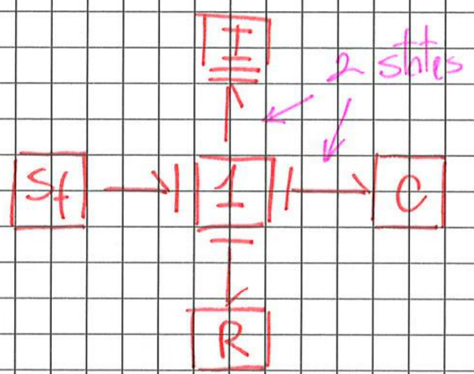
$e_1 - e_2 - e_3 = 0$

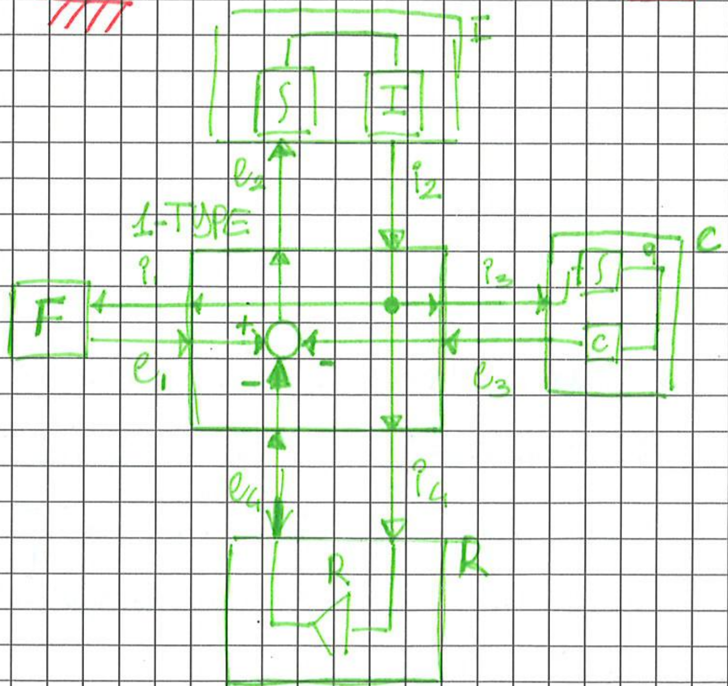




EXAMPLE 1

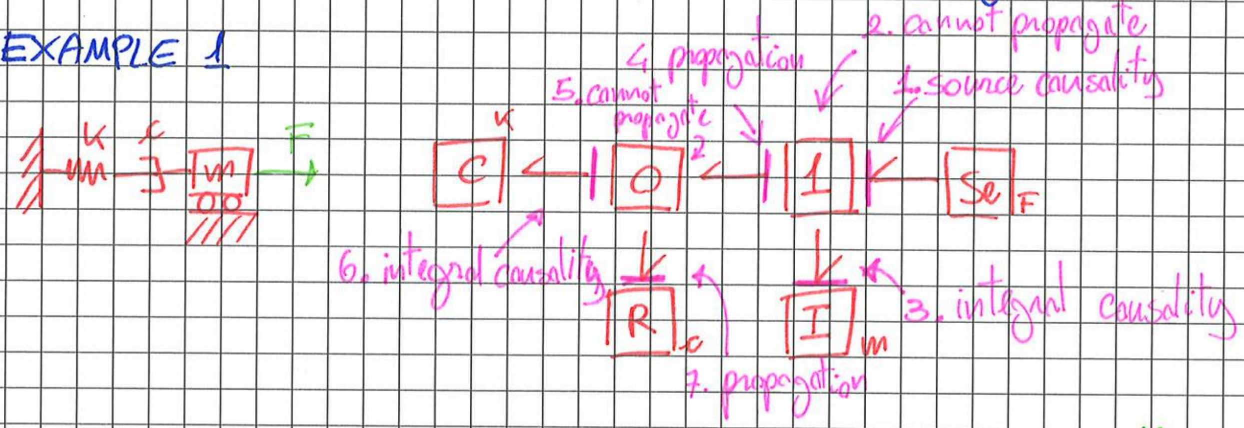




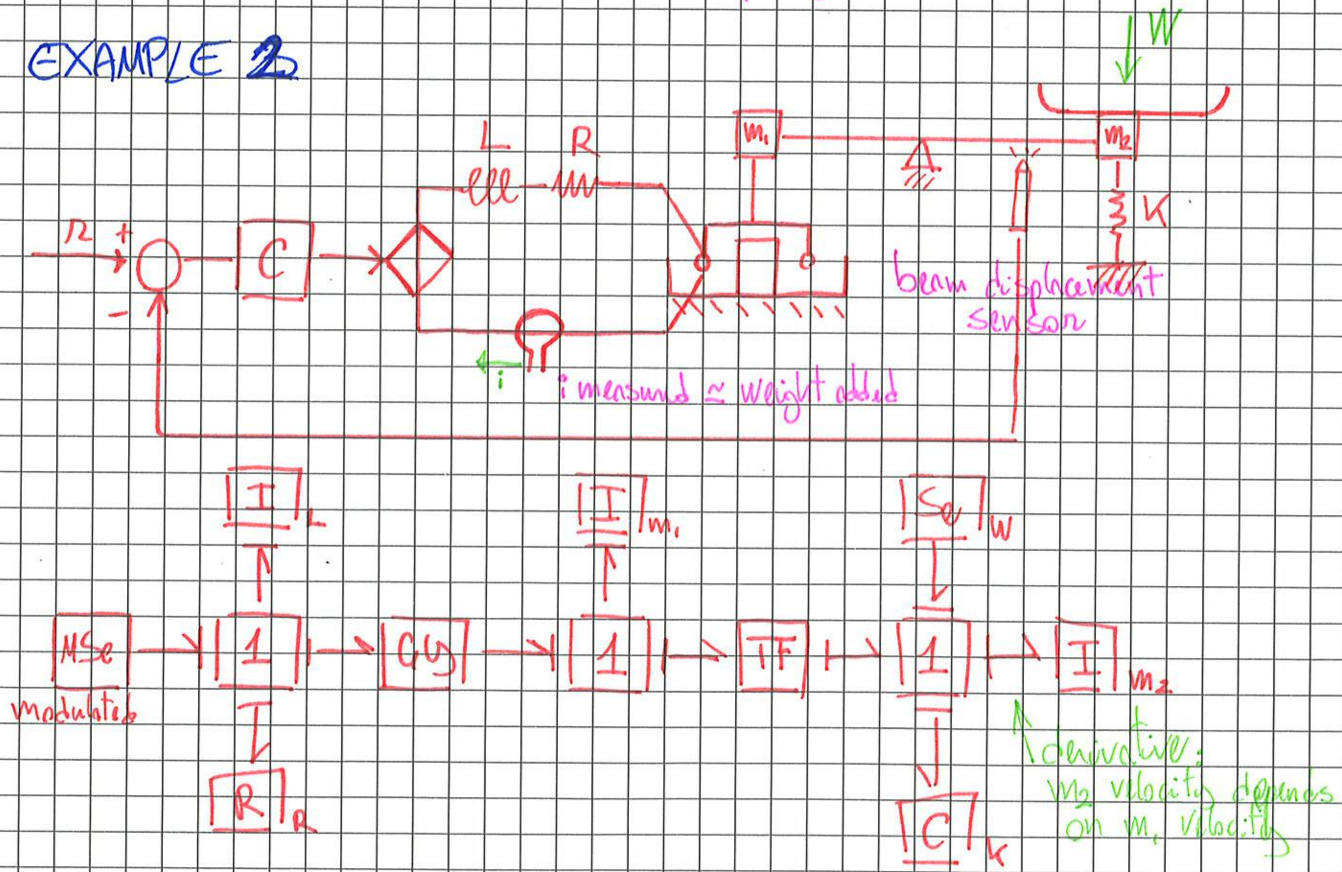


1. Sources causality
2. Propagation as allowed by multi-ports elements
3. Focus on unassigned I and C elements
3.1 Give an integral causality to one unassigned I or C element
3.2 Propagate
3.3 Repeat point 3 until all causality are assigned

EXAMPLE 1

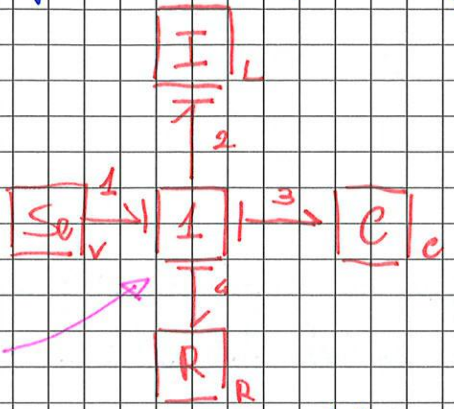


EXAMPLE 2



FROM BOND GRAPH TO STATE EQUATIONS

Let's start from a bond graph model of the system



2 states:

$p_2 = \int e_2 dt$ and $q_3 = \int f_3 dt$

Now to proceed systematically we build a table to summarize our bond graph using general efforts and flows.

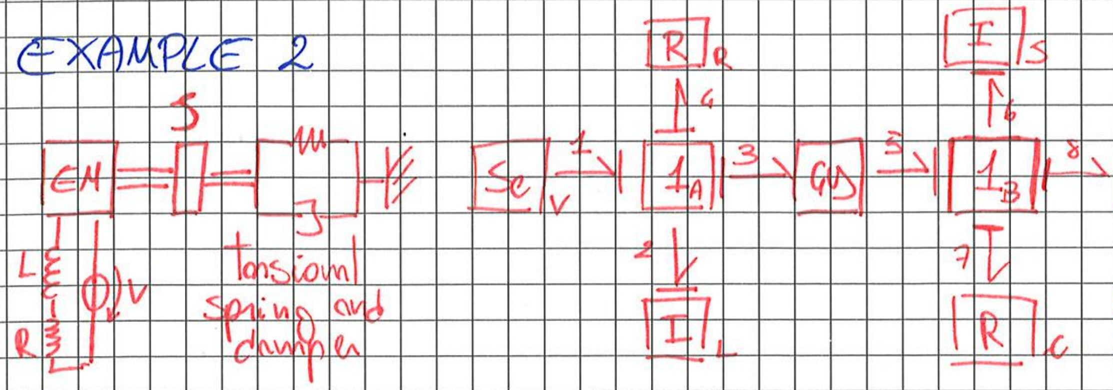
COMPONENT	INPUT	OUTPUT
Se; V	f_1	$e_1 = V$
I; L	state equation $e_2 = \dot{p}_2$	$f_2 = \frac{p_2}{L}$
C; C	state equation $f_3 = \dot{q}_3$	$e_3 = q_3/C$
R; R	f_4	$e_4 = R f_4$
1	f_2 e_1, e_3, e_4	$f_1 = f_3 = f_4 = f_2$ $e_2 = e_1 - e_3 - e_4$

← output power variable and constitutive law

Starting from the state equations in implicit form we have to find them in the explicit form (function of system input and states $\dot{x} = Ax + Bu$) using the relation in the output column:

$$\dot{p}_2 = e_2 = e_1 - e_3 - e_4 = V - \frac{q_3}{C} - R f_4 = V - \frac{q_3}{C} - R f_2 =$$

EXAMPLE 2



3 states p_2, p_6, q_8

ELEMENT	INPUT	OUTPUT
$Se: V$	f_1	$e_1 = V$
$I: L$	$e_2 = \dot{p}_2$	$f_2 = \frac{p_2}{L}$
$R: R$	f_4	$e_4 = R f_4$
GY	f_3	$e_3 = K_m f_5$
	f_5	$e_5 = K_m f_3$
$I: S$	$e_6 = \dot{p}_6$	$f_6 = \frac{p_6}{S}$
$R: C$	f_7	$e_7 = C f_7$
$C: K$	$f_8 = \dot{q}_8$	$e_8 = K q_8$
1_A	f_2	$f_1 = f_3 = f_4 = f_2$
	$e_1; e_3; e_4$	$e_2 = e_1 - e_3 - e_4$
1_B	f_6	$f_5 = f_7 = f_8 = f_6$
	$e_5; e_7; e_8$	$e_6 = e_5 - e_7 - e_8$

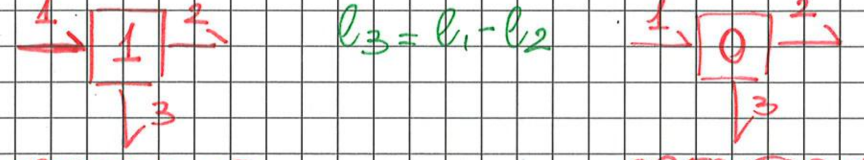
$$\dot{p}_2 = e_2 = e_1 - e_3 - e_4 = V - K_m f_5 - R f_4 = V - K_m f_6 - R f_2 = $$
$$= V - \frac{K_m}{S} p_6 - \frac{R}{L} p_2$$

$$\dot{p}_6 = e_6 = e_5 - e_7 - e_8 = K_m f_3 - C f_7 - K q_8 = K_m f_2 - C f_6 - K q_8 =$$
$$= \frac{K_m}{L} p_2 - \frac{C}{S} p_6 - K q_8$$

$$\dot{q}_8 = f_8 = f_6 = \frac{p_6}{S}$$

SYSTEMATIC BOND GRAPH MODEL

To obtain a systematic way to build up bond graph model is useful to see the junction elements as operators:

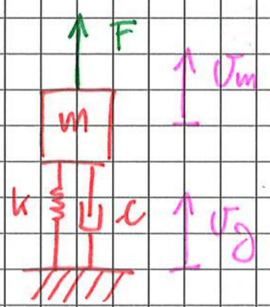


$e_3 = e_1 - e_2$ $\qquad$ $f_3 = f_1 - f_2$

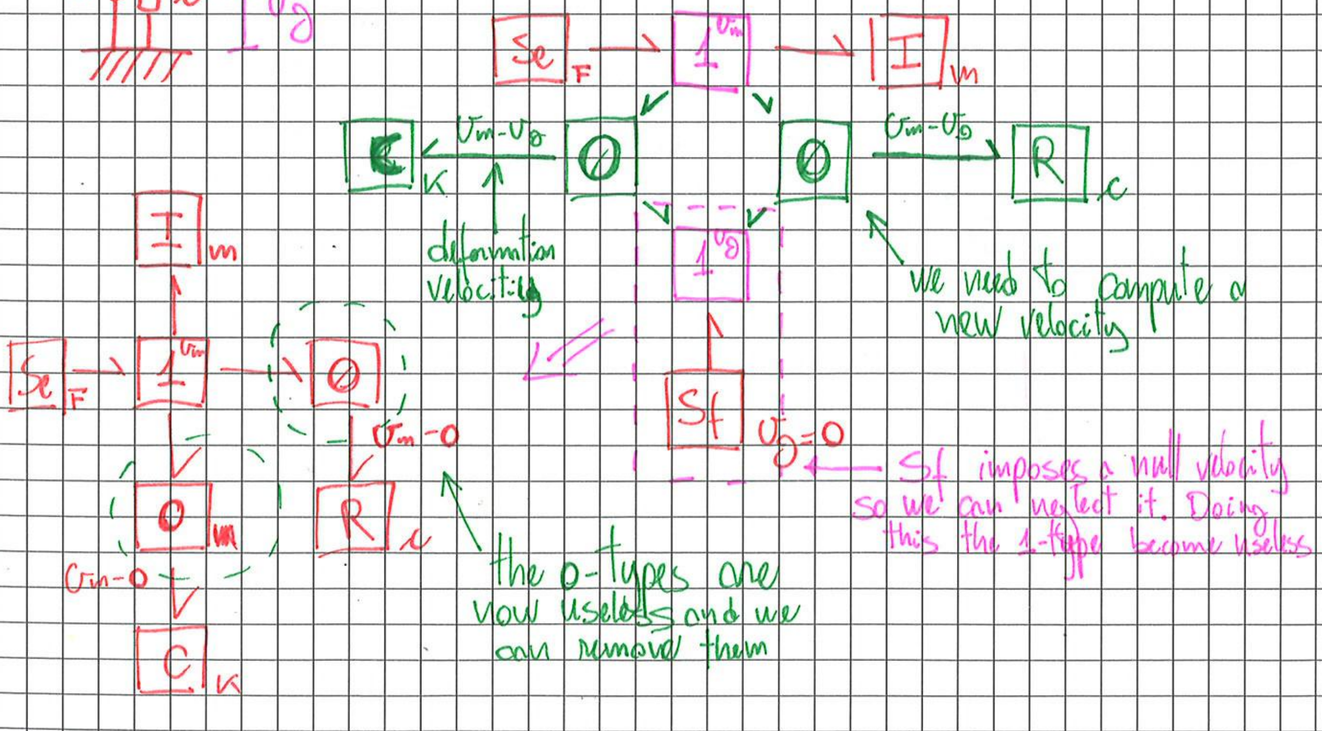
OPERATOR OF EFFORT $\qquad$ OPERATOR OF FLOW

In mechanical system we have to find how many velocities are present and assign a 1-type junction to each of them. Then we can connect the elements of the system and if is needed using the 0-type junction to compute new velocities. At the end we have to try to simplyfy the model.

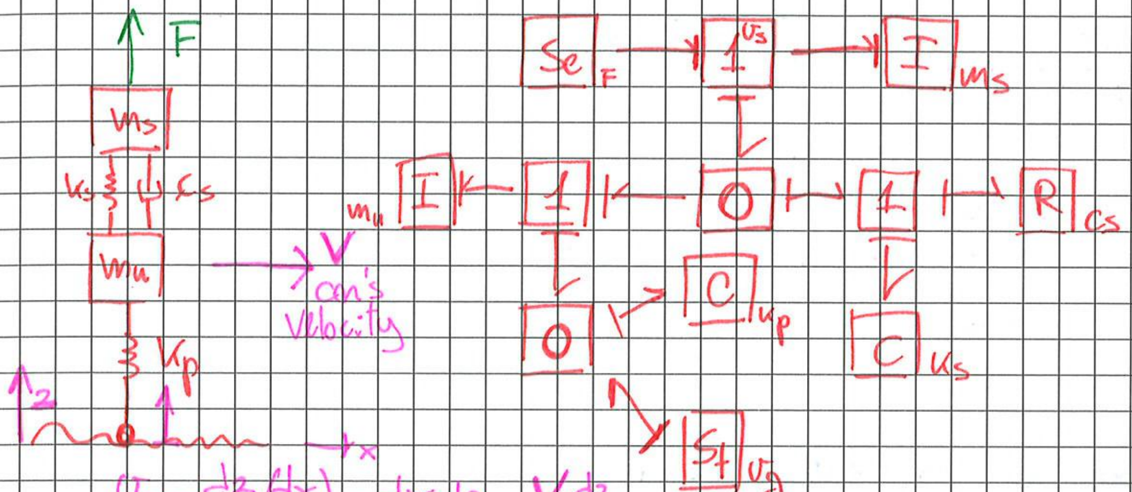
Let's see some examples



We identify 2 velocity: v_{mass}, v_{ground}
↳ we need the corresponding 1-type

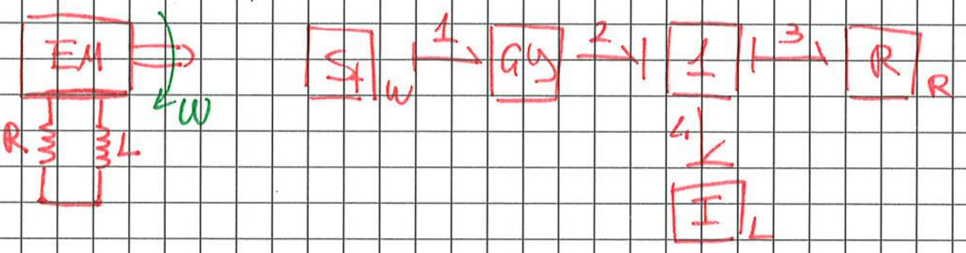


In case of an irregular terrain the velocity of the ground is no more null, so we can't neglect the bottom 0-type junction but the bottom 1-type junction is still useless:



$$v_g = \frac{dz}{dt}\left(\frac{dx}{dx}\right) = \frac{dx}{dt}\frac{dz}{dx} = V\frac{dz}{dx}$$

Before seeing an example of electromagnetic suspension we analize one of its subsystem:



ELEMENT	INPUT	OUTPUT
Sf	e_1	$f_1 = \omega$
GY	f_1	$e_1 = K_m f_2$
	f_2	$e_2 = K_m f_1$
R	f_3	$e_3 = R f_3$
I	$e_4 = \dot{p}_4$	$f_4 = \frac{p_4}{L}$

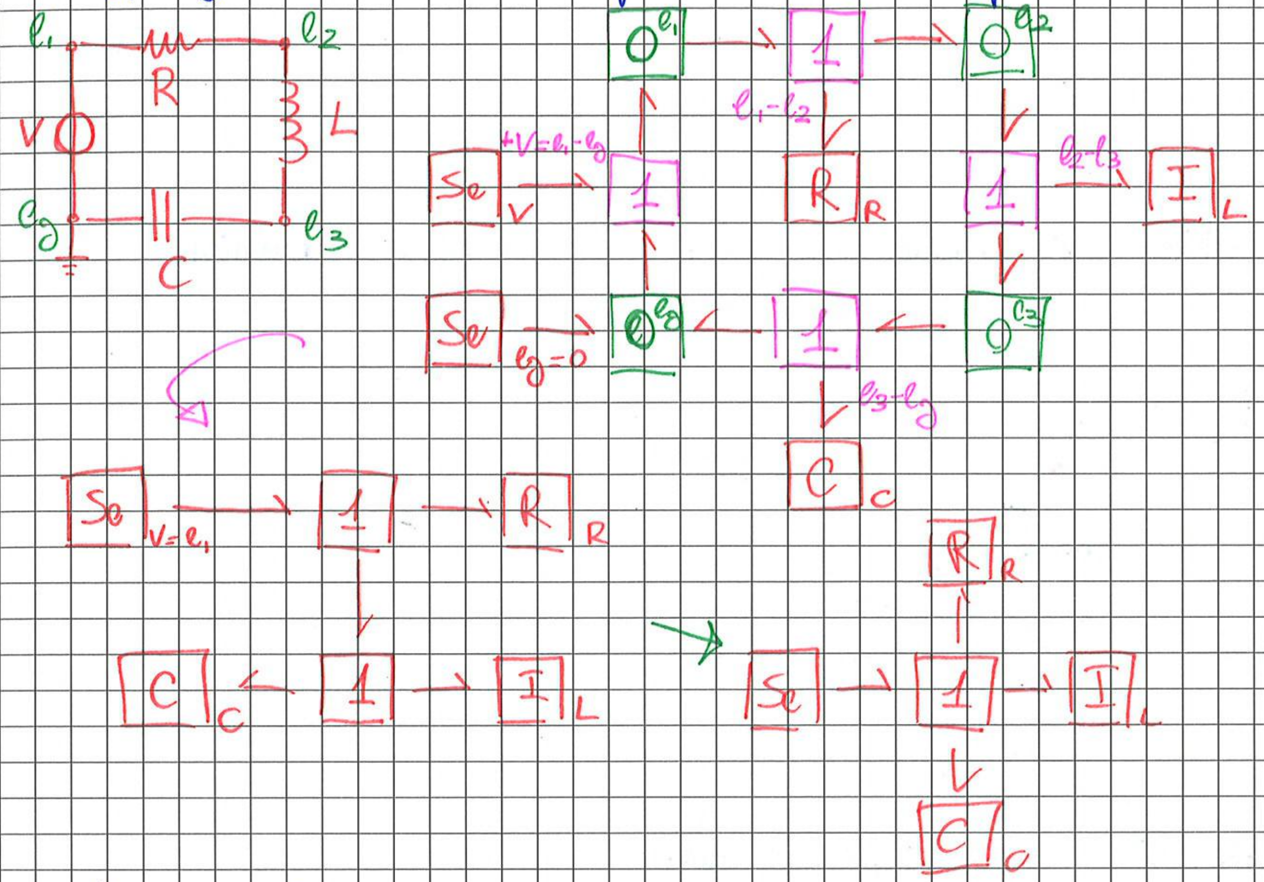
$$\begin{cases} \dot{p}_4 = e_4 = e_2 - e_3 = K_m f_1 - R f_3 = \\ \quad = K_m \omega - \frac{R}{L} p_4 \quad \text{state eq.} \\ T = e_1 = K_m f_2 = \frac{K_m}{L} p_4 \quad \text{output eq.} \end{cases}$$

$$\mathcal{L}(\dot{p}_4) \Rightarrow s p_4 = K \omega - \frac{R}{L} p_4$$

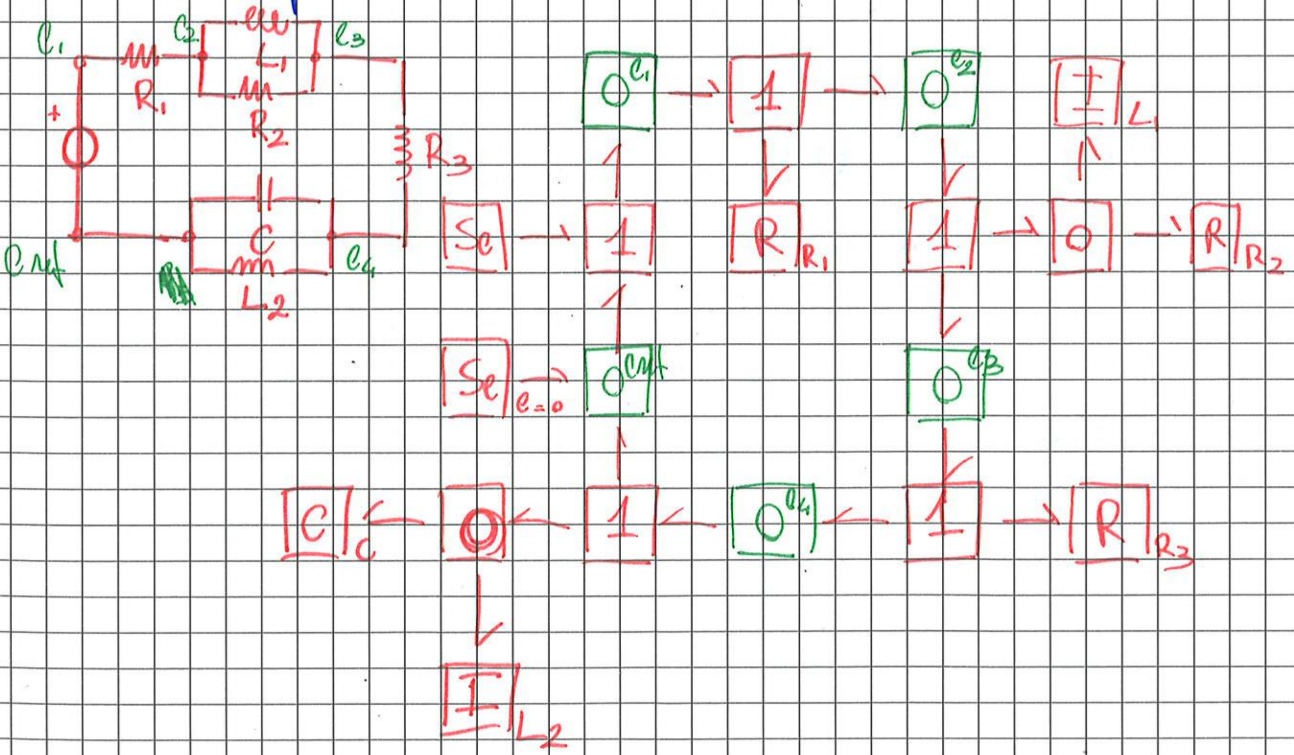
$$\frac{p_4}{\omega} = \frac{K_m}{s + \frac{R}{L}}$$

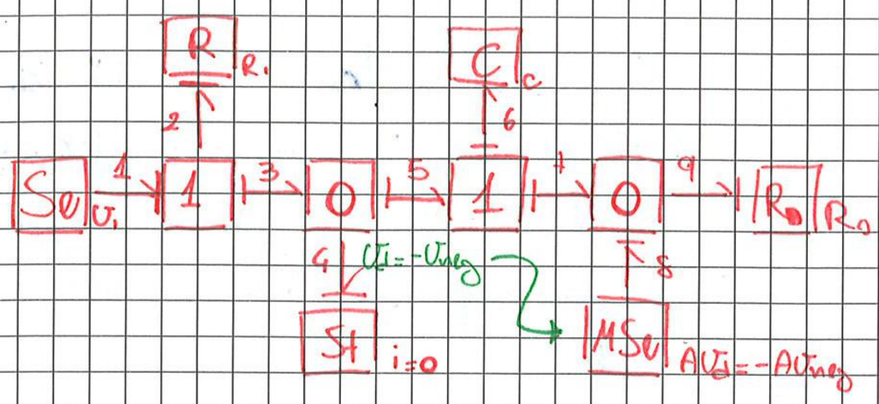
Transfer function $\frac{T}{\omega} = \frac{T}{p_4} \cdot \frac{p_4}{\omega} = \frac{K_m}{L} \, \frac{K_m}{s + \frac{R}{L}} = \frac{K_m^2 / L}{s + R/L}$

In the electrical domain we have to identify the voltages at the nodes, then connecting all the elements using 1-type junctions to compute voltage drop:



Another example:





ELEMENT	INPUT	OUTPUT
$Se;\ v_1$	f_1	$e_1 = v_1$
$R;R_1$	e_2	$f_2 = \frac{e_2}{R_1}$
$Sf:\ i=0$	e_4	$f_4 = 0$
$C:C$	$f_6 = \dot{q}_6$	$e_6 = \frac{q_6}{C}$
MSe	f_8	$e_8 = -Av_{neg}$
$R:R_0$	e_9	$f_9 = \frac{e_9}{R_0}$

$$\dot{q}_6 = f_6 = f_5 = f_3 - \underbrace{f_4}_{0} = f_3 =$$
$$= f_2 = \frac{e_2}{R_1} = \frac{e_1 - e_3}{R_1} =$$
$$= \frac{1}{R_1}(v_1 - e_5)$$

$$e_5 = e_6 + e_7 = \frac{q_6}{C} + e_8 =$$
$$= \frac{q_6}{C} - Av_{neg} = \frac{q_6}{C} - Ae_5$$

$$\Rightarrow e_5 = \frac{1}{C(A+1)} q_6$$

$$\Rightarrow \dot{q}_6 = \frac{1}{R_1}(v_1 - e_5) = \frac{1}{R_1}\left(v_1 - \frac{q_6}{C(A+1)}\right)$$

Using the Laplace transform

$$sq_6 + \frac{1}{RC(A+1)} q_6 = \frac{v_1}{R_1} \qquad q_6\left[s + \frac{1}{RC(A+1)}\right] = \frac{v_1}{R_1}$$

The output equation is the following

$$v_0 = e_9 = e_8 = -Av_{neg} = -Ae_5 = -A \cdot \frac{q_6}{C(A+1)} = -\frac{A}{C(A+1)} \cdot \frac{v_1}{R} \cdot \left[s + \frac{1}{RC(A+1)}\right]^{-1}$$

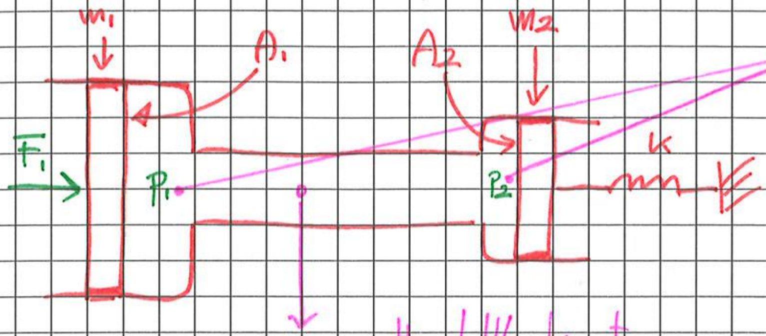
finally we get the transfer function:

$$\frac{v_0}{v_1} = -\frac{A}{\cancel{C(1+A)}} \cdot \frac{1}{\cancel{R_1}} \cdot \frac{\cancel{R_1C(A+1)}}{sR_1C(1+A)+1} = -\frac{A}{sR_1C(1+A)+1} = -\frac{1}{sR_1C}$$

$$A \gg 1$$
$$sR_1CA \gg 1$$

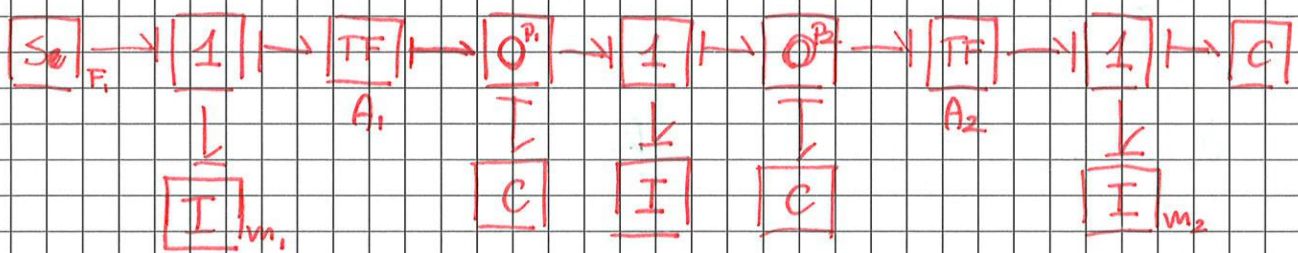
Let's see another example:

these short pipe with large diameter are modeled as hydraulic compliance

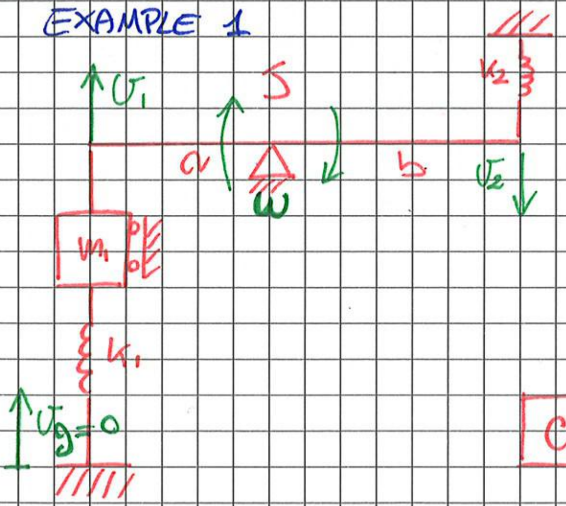


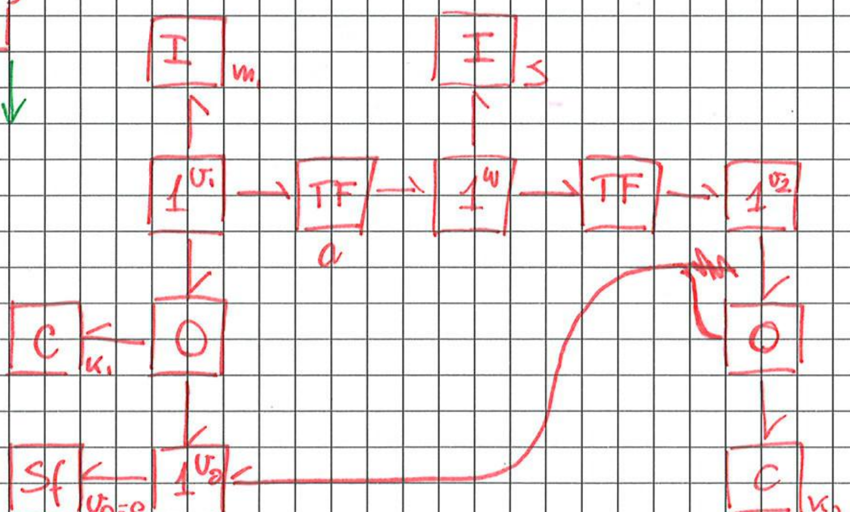
$I_H = \frac{\rho L}{A}$

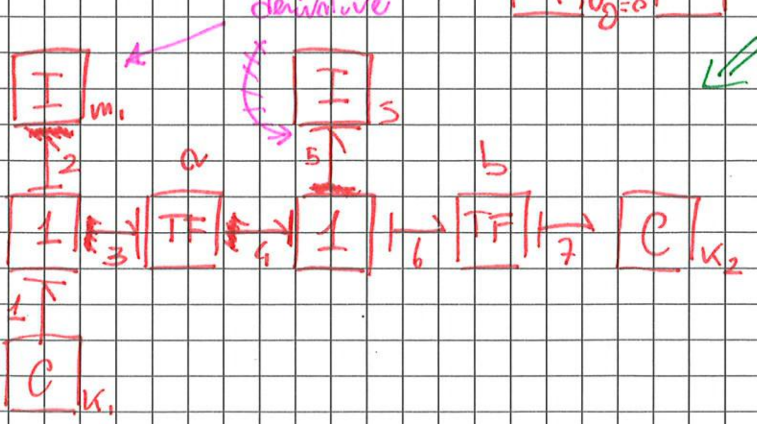
pipe with little diameter and long pipe can be modeled as an I-element due to the large inertia



EXAMPLE 1





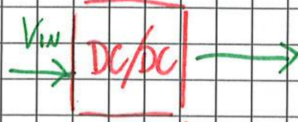


DC/DC CONVERTERS

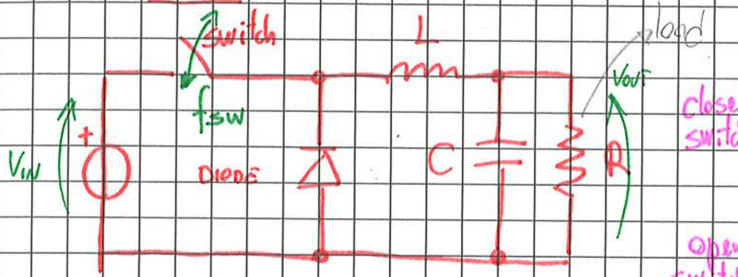
We can use the bond graph approach to model some converters.

BUCK CONVERTER

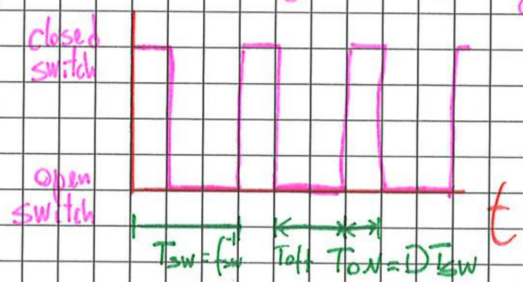
A buck converter gives a lower voltage when supplied. This is its scheme:



$\longrightarrow V_0 = D V_{IN}$ with $D < 1$

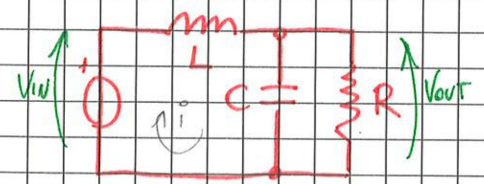


D: duty cycle, percentage of working time on the switch period



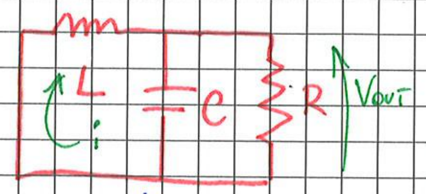
The switch determines 2 different conditions:

- closed, T_{ON}



$$i = \frac{1}{L}\int (V_{IN} - V_{OUT})\,dt = \frac{V_{IN} - V_{OUT}}{L}\,t + i_0$$

- open, T_{OFF}



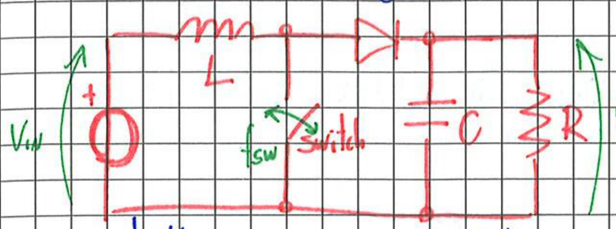
$$i = \frac{1}{L}\int (0 - V_{OUT})\,dt = -\frac{V_{OUT}}{L}\,t + i_1$$

The result is given by superposition.

$\Rightarrow V_0 = D\,V_{IN}$

BOOST CONVERTER

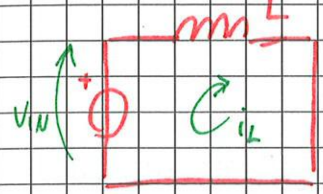
Boost converters give us an higher voltage than the supply.



$V_{OUT} = \frac{V_{IN}}{1-D}$

We still have 2 conditions:

- CLOSED, T_{ON}



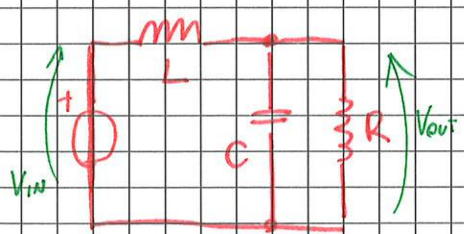
$\boxed{Se} \xrightarrow{V_{IN}} \boxed{I}_L$

$\dot{p}_1 = V_{IN}$

$i_L = \frac{p_1}{L}$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{C} \end{bmatrix}$$

- OPEN, T_{OFF}



We have the same circuit as the buck converter with closed switch. The equations are still the same:

$$A' = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{1}{RC} \end{bmatrix} \quad B' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C' = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{C} \end{bmatrix}$$

Computing the derivative of the energy about the displacement we can get:

$$F_i = \frac{\partial E(q_1, q_2, q_3)}{\partial q_i}$$

Since the energy is a continuous function then is true that:

$$\frac{\partial^2 E}{\partial q_i \partial q_j} = \frac{\partial^2 E}{\partial q_j \partial q_i}$$

Remembering what the derivative of the energy is, we get the Maxwell's reciprocity condition:

$$\frac{\partial^2 E}{\partial q_i \partial q_j} = \frac{\partial}{\partial q_i}\underbrace{\left(\frac{\partial E}{\partial q_j}\right)}_{F_j} = \frac{\partial}{\partial q_j}\underbrace{\left(\frac{\partial E}{\partial q_i}\right)}_{F_i} \Rightarrow \frac{\partial F_j}{\partial q_i} = \frac{\partial F_i}{\partial q_j}$$

Due to this condition we have an important property for the **K** matrix:

$$\mathbf{F} = \mathbf{K}\mathbf{q} \iff \begin{cases} F_1 = K_{11} q_1 + K_{12} q_2 + K_{13} q_3 \\ F_2 = K_{21} q_1 + K_{22} q_2 + K_{23} q_3 \\ F_3 = K_{31} q_1 + K_{32} q_2 + K_{33} q_3 \end{cases}$$

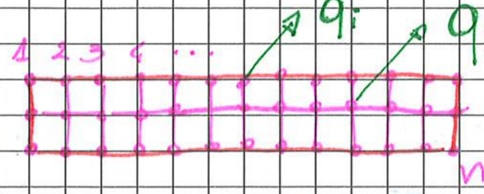
$$\begin{cases} \frac{\partial F_1}{\partial q_2} = K_{12} \\ \frac{\partial F_2}{\partial q_1} = K_{21} \end{cases} \Rightarrow \frac{\partial F_1}{\partial q_2} = \frac{\partial F_2}{\partial q_1} \Rightarrow K_{12} = K_{21}$$

this still applies for every couple of indexes

$\Rightarrow$ **K** is a symmetric matrix

Now we can do the same study for a beam with mass but without stiffness this ~~this~~ time.

structure dividing it in bricks defined by few points called nodes. This approach allow us to study only the nodes behavior instead of the infinite number of points of the structure.

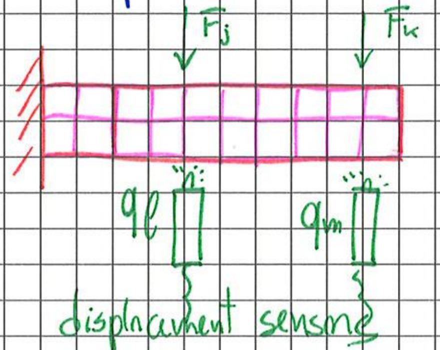


$$\mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix}$$

n = n° of nodes = n° of nodal DOF

more nodes mean more precise model

A structure with this model can be easily see as a multiport element in bond graph:



not all nodes have a force applied so we use a input selection matrix

$$\begin{bmatrix} F_1 \\ \vdots \\ F_j \\ F_k \\ \vdots \\ F_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ F_j \\ F_k \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_j \\ F_k \end{bmatrix} = \mathbf{T}_i \begin{bmatrix} F_j \\ F_k \end{bmatrix}$$

the same apply for the displacement with an output selection matrix:

$$\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_\ell \\ q_m \\ \vdots \\ q_n \end{Bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ q_\ell \\ q_m \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \end{bmatrix}^T \cdot \begin{bmatrix} q_1 \\ \vdots \\ q_\ell \\ q_m \\ \vdots \\ q_n \end{bmatrix}^T = \begin{bmatrix} q_\ell \\ q_m \end{bmatrix}$$

if T_i and T_o select the same nodes we have $T_i = T_o^T$

For each λ_i we have an eigenvector ϕ_i

$$q = a_1 \phi_1 e^{j\omega_1 t} + b_1 \phi_1 e^{-j\omega_1 t} + a_2 \phi_2 e^{j\omega_2 t} + \dots$$

a_i, b_i found imposing initial conditions

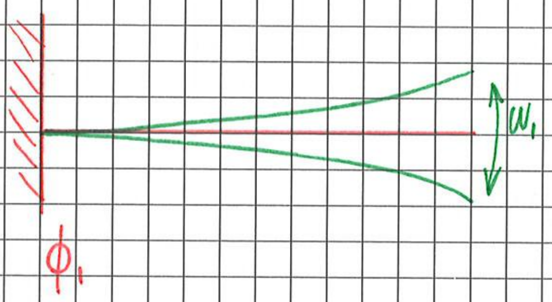
For $t=0$, $q = \phi_1$ and $\dot{q} = 0$:

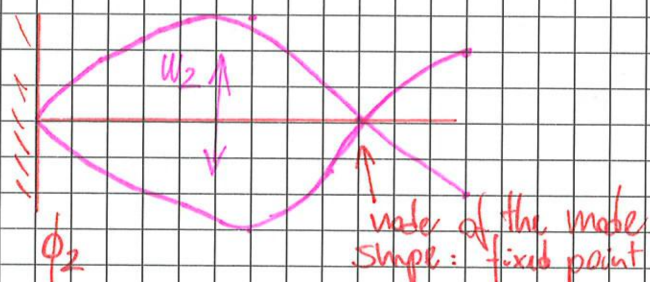
$$\begin{cases} q = \phi_1 = a_1 \phi_1 + b_1 \phi_1 + a_2 \phi_2 + \dots \\ \dot{q} = j\omega_1 a_1 \phi_1 e^{j\omega_1 t} - j\omega_1 b_1 \phi_1 e^{-j\omega_1 t} + \dots \end{cases} \qquad \begin{cases} a_1 = b_1 = \frac{1}{2} \\ a_i = b_i = 0 \quad i > 1 \end{cases}$$

$$q = \left(\phi_1 e^{j\omega_1 t} + \phi_1 e^{-j\omega_1 t}\right)\frac{1}{2}$$

the structure vibrates at the first natural frequency

modes shape: way the structure is deforming when it's vibrating at the corresponding ω





$$q = \frac{1}{2}\phi_1\left(e^{j\omega_1 t} + e^{-j\omega_1 t}\right) = \frac{1}{2}\phi_1(\cos\omega_1 t + j\sin\omega_1 t + \cos\omega_1 t - j\sin\omega_1 t) =$$

$$= \frac{1}{2}\phi_1 2\cos\omega_1 t = \phi\cos\omega_1 t$$

Remembering the following property:

- mass orthogonality

$$\phi_i^T M \phi_j = \begin{cases} 0 & \text{if } i \neq j \\ >0 & \text{if } i = j \end{cases}$$

- stiffness orthogonality

$$\underbrace{\phi_i^T K \phi_j}_{\text{work done by } F_{\phi_j} \text{ on the node } i} = \begin{cases} 0 & \text{if } i \neq j \\ >0 & \text{if } i = j \end{cases}$$

$K\phi_j \rightarrow F_{\phi_j}$

now we can do the modal decoupling:

$$q = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} q_1 + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} q_2 + \dots + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} q_n = \phi_1 \xi_1 + \phi_2 \xi_2 + \dots + \phi_n \xi_n$$

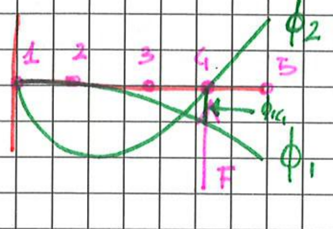
$$C = \alpha M + \beta K$$

$$\phi^T C \phi = \alpha \phi^T M \phi + \beta \phi^T K \phi = \alpha \grave{m}_1 + \beta \grave{k}_1 = \grave{c}_1$$

in the end we get

$$\grave{m}_1 \ddot{\xi} + \grave{c}_1 \dot{\xi} + \grave{k}_1 \xi = \cancel{\phi}\, t_{in} F$$

the selection matrix is given by:



$$t_{in} = \phi^T T = \underbrace{\begin{bmatrix} \phi_1^T \\ \phi_2^T \\ \phi_3^T \\ \phi_4^T \\ \phi_5^T \end{bmatrix}}_{5\times5} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{5\times1} = \begin{bmatrix} \phi_4 \cdot 1 \\ 0 \\ \vdots \end{bmatrix}$$

depend on the modal shape

due to the node even if we apply a force we cannot act on the second mode shape

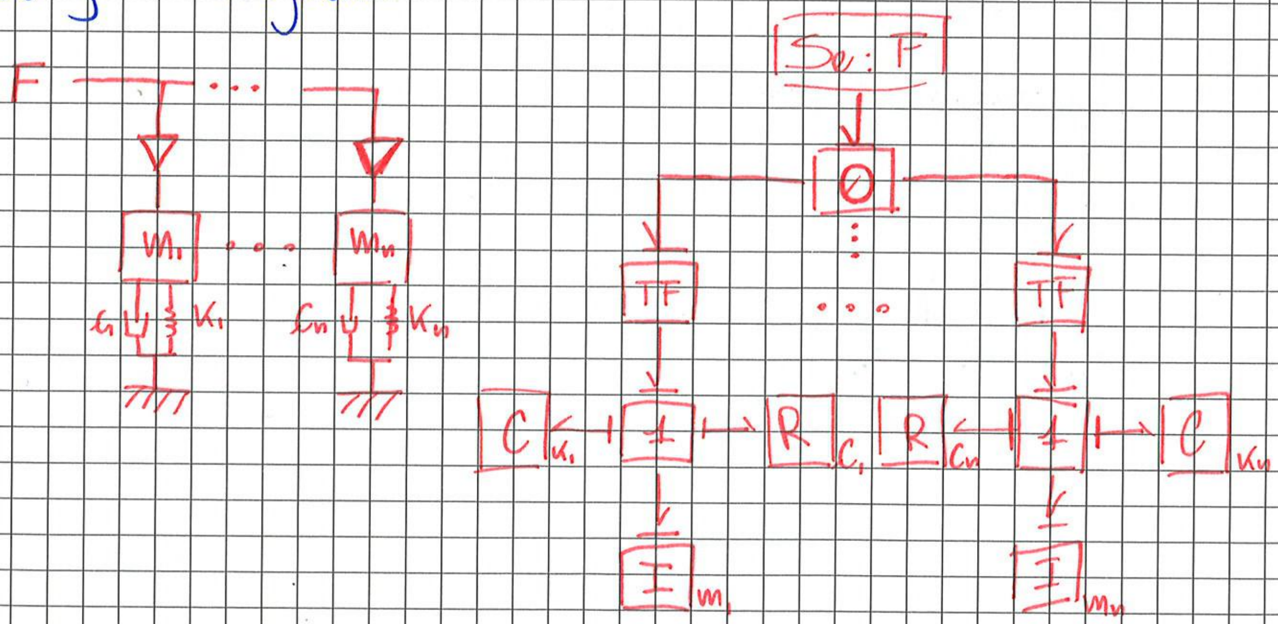
it's also possible to have a selected output:

$$y = T_{OUT}\, q$$

$$\phi\, T_{OUT}\, \phi = t_{OUT}$$

in case of collocation is still valid $t_{IN} = t_{OUT}^T$

Anyway the system behave like a set of mechanical damped oscillator decoupled, where the single force acts differently on every subsystem:



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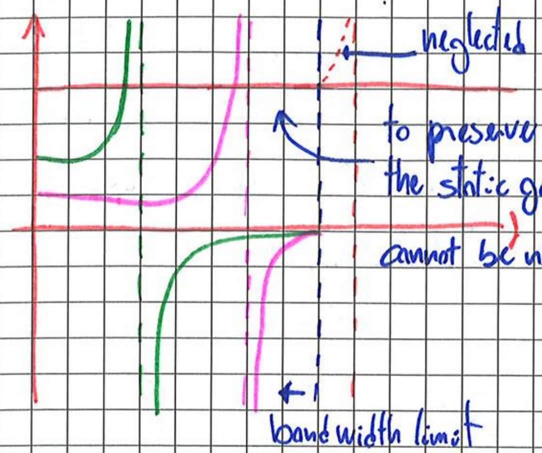
Usually we are interested only in a limited bandwidth so



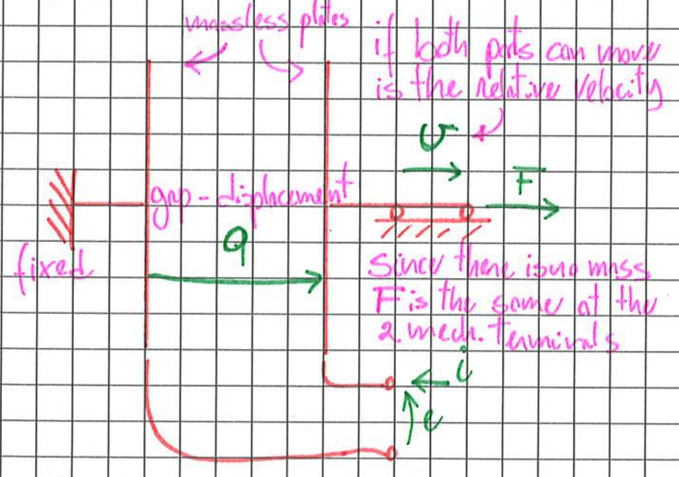
$$\frac{y}{F} = \underbrace{\sum_{1}^{m} \frac{\tau_{out,i}\,\tau_{in,i}}{M_i s^2 + C_i s + K_i}}_{\text{dynamic behavior of relevant resonances}} + \underbrace{\sum_{m+1}^{n} \frac{\tau_{out,i}\,\tau_{in,i}}{K_i}}_{\text{static gain of not relevant resonances}}$$

static gain of not relevant resonances → derivative causality on bond graph

ENERGY STORING SYSTEM

Until now we had seen system where the input power is equal to the output power at any time. Now we will intro-duce some systems that can store energy and the previous statement isn't always valid.

CAPACITATIVE TRANSDUCER



We start the study from the computation of the energy:

$$E = \int_0^t P\,dt = \int_0^t (\underbrace{ei}_{P_{elect}} + \underbrace{Fv}_{P_{mech}})\,dt =$$

$$= \int_0^t \left(e\frac{\overbrace{dQ}^{\text{charge}}}{\cancel{dt}} + F\frac{\overbrace{dq}^{\text{displacement}}}{\cancel{dt}}\right)\cancel{dt} =$$

$$= \int_{Q=Q_0;\,q=q_0}^{Q;\,q} e\,dQ + F\,dq = E(Q,q)$$

← usually $Q_0 = q_0 = 0$

if the energy is known we can compute:

$$F = \frac{\partial E}{\partial q}$$

$$e = \frac{\partial E}{\partial Q}$$

An easy constitutive law is given by

$$Q = C(q)\,e$$

$$C = \frac{S}{q}\,\varepsilon \qquad \varepsilon = \varepsilon_0 \varepsilon_r$$

plates surfaces (S) — gap (q) — gap dielectric constant (ε)

to this follows:

$$E = \frac{1}{2}\frac{Q^2 q}{\varepsilon S}$$

$$F = \frac{1}{2}\frac{Q^2}{\varepsilon S} \quad \leftarrow \text{ideal force transducer with } Q \text{ control}$$

F depends only on Q

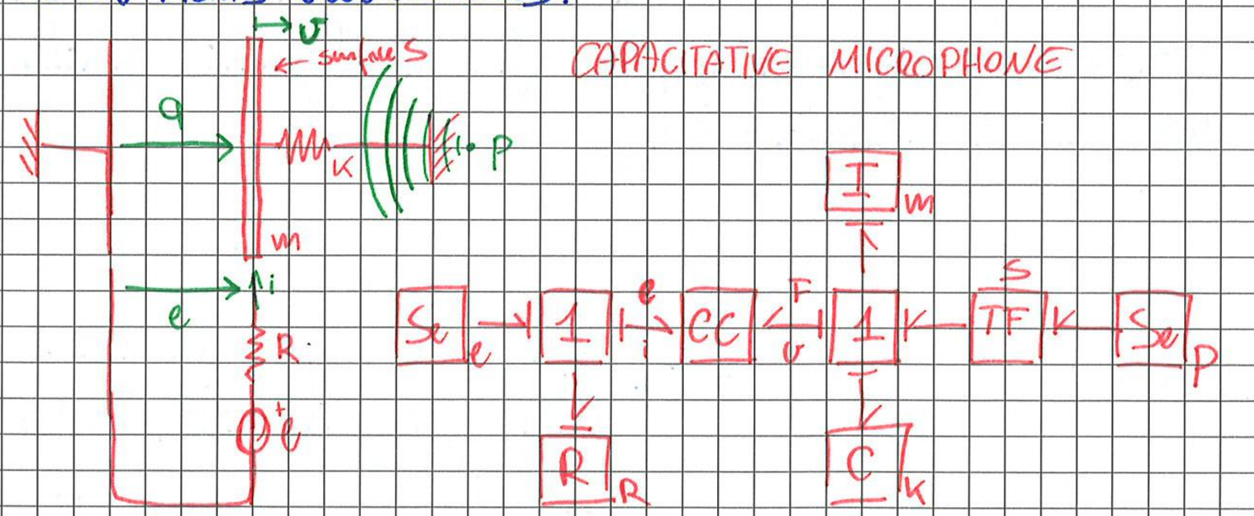
$$e = \frac{Q}{C} = \frac{qQ}{\varepsilon S}$$

easier method

- $$F = \frac{\varepsilon S e^2}{2q^2} \quad \leftarrow F \text{ depends on } q \text{ and } e \text{ with } e \text{ control}$$

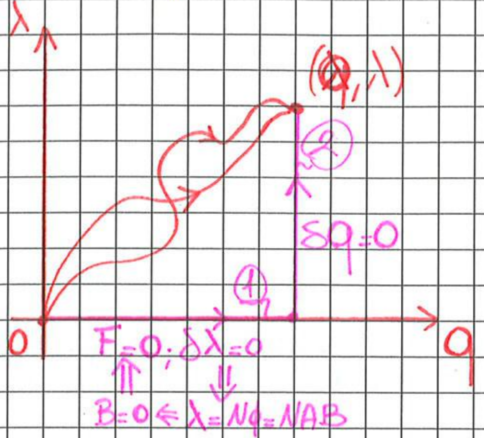
Capacitative transducer are rarely used as actuators due to the small value of ε, so we use them for really small force generation with a fine control.
As sensor they are more used, for example the microphones or the MEMS accelerometers.

CAPACITATIVE MICROPHONE



↳ 4 states

It's important to note the mechanical displacement isn't only the gap between the plates, for example still apply, so we choose a better path for the computations:



$$\mathcal{E} = \int_{①} \underbrace{\cancel{i\,d\lambda}}_{\to 0} + \underbrace{\cancel{F\,dq}}_{\to 0} + \int_{②} i\,d\lambda + \underbrace{\cancel{F\,dq}}_{\to 0} = \int_{\substack{\lambda=0 \\ q=\text{const}}}^{\lambda} i\,d\lambda$$

Introducing the constitutive law:

$$\lambda = L(q)\,i$$

$$\mathcal{E}(\lambda, q) = \int \frac{\lambda}{L(q)}\,d\lambda = \frac{1}{2}\,\frac{\lambda^2}{L(q)}$$

$$F = \frac{\partial \mathcal{E}}{\partial q} = -\frac{1}{2}\,\frac{\lambda^2}{L^2(q)}\,\frac{\partial L}{\partial q}$$ → effort (F) linked with displacement (q)
↳ C - element

$$i = \frac{\partial \mathcal{E}}{\partial \lambda} = \frac{\lambda}{L(q)}$$ → flow (i) linked with momentum (λ)
↳ I - element

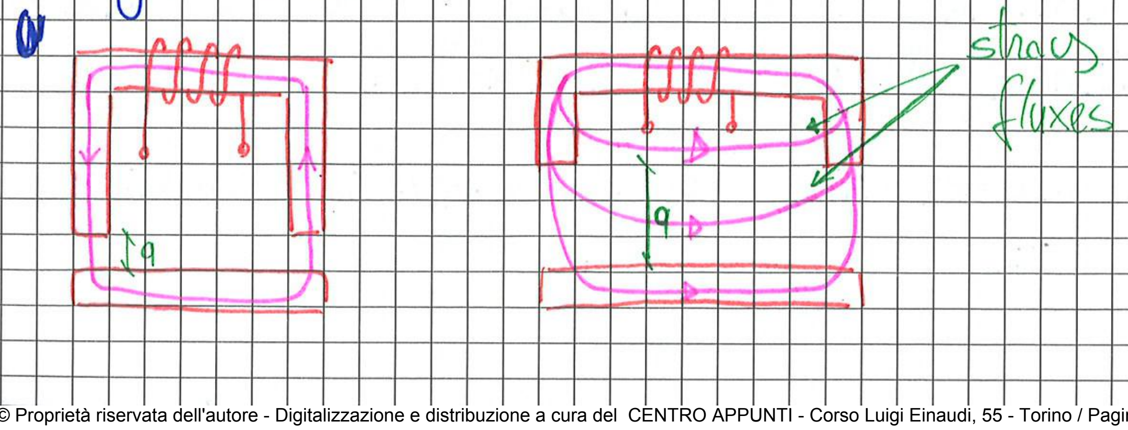
So the bond-graph of the system is:

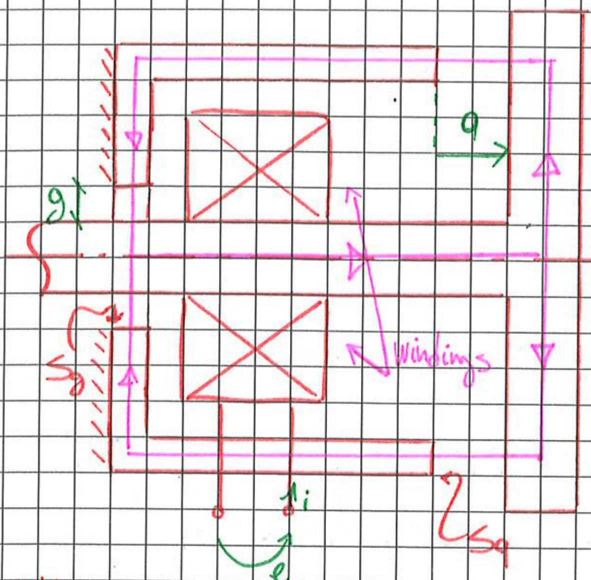
$$\xrightarrow[\lambda]{i} \boxed{IC} \xleftarrow[q]{F}$$

2 potential states

To have a linear constitutive law we need to impose some hypothesis:

- the gap is much smaller than the pole expansion
- $\mu_{Fe} \gg \mu_0$
- no saturation
- no stray flux: all flux lines are confined inside the magnetic circuit.





Hp. $S_g = S_q = S$

$$\mathcal{R} = \frac{l_{Fe}}{\mu_0 \mu_{Fe} S_{Fe}} + \frac{g}{\mu_0 S} + \frac{q}{\mu_0 S} = \simeq \frac{g+q}{\mu_0 S}$$

NB: only 1 time because the flux cross just 1 time

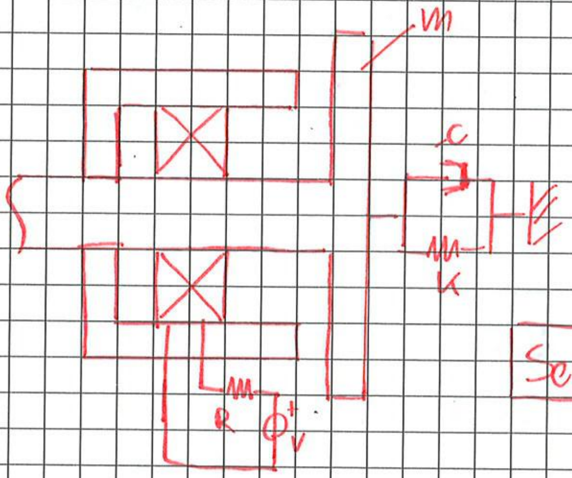
$$\lambda = \frac{N^2 \mu_0 S}{g+q}\, i \quad *$$

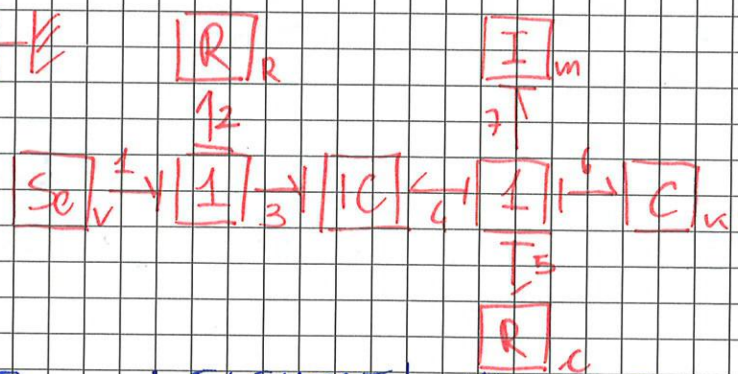
$$E = \frac{\lambda^2}{2L} = \frac{\lambda^2}{2N^2\mu_0 S}(g+q)$$

$$F = \frac{\partial E}{\partial q} = \frac{\lambda^2}{2N^2 \mu_0 S} \quad **$$

EM pressure $\sigma_{EM} = \frac{F}{S} = \frac{\lambda^2}{2N^2\mu_0 S^2} = \left(\frac{\lambda}{NS}\right)^2 \frac{1}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{\phi}{S}\right)^2 = \frac{B_{gap}^2}{2\mu_0}$

We can add some elements and find the state equations:

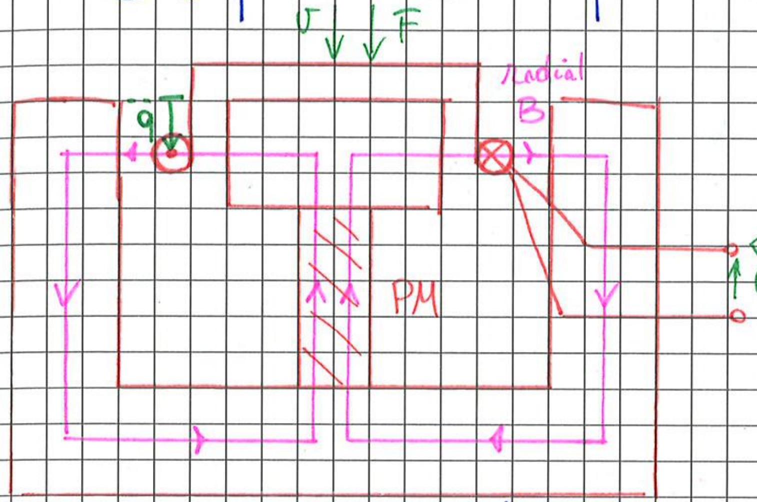




ELEMENT	IN	OUT	ELEMENT	IN	OUT
Se : V	f_1	$e_1 = V$	R : C	f_5	$e_5 = C f_5$
R : R	f_2	$e_2 = R f_2$	C : K	$f_6 = \dot{q}_6$	$e_6 = K q_6$
IC	$e_3 = \dot{p}_3$	$f_3 = \frac{q_4 + g}{\Gamma'} p_3$ *	I : m	$e_7 = \dot{p}_7$	$f_7 = \frac{p_7}{m}$
$\Gamma' = N^2 \mu_0 S$	$f_4 = \dot{q}_4$	$e_4 = \frac{p_3^2}{2\Gamma'}$ **			

VOICE COIL TRANSDUCER

Now we will do a deeper analysis of the voicecoil, introducing the possibility to store energy.
As before we start from the energy:

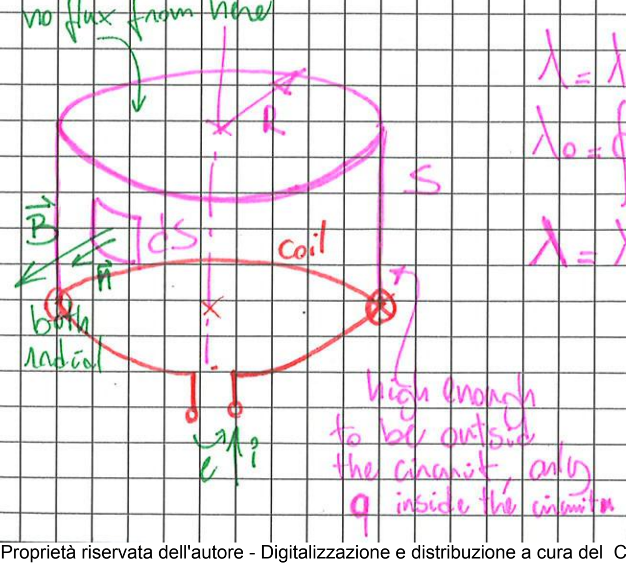


$$E = \int (e i + F v)\,dt = \int \left(i \frac{d\lambda}{dt} + F \frac{dq}{dt} \right) dt = \int i\,d\lambda + F\,dq$$

To find the constitutive law between i and λ we notice that the linked flux is not only produced by the current i but also it's due to the permanent magnet.
When the coil is outside the magnetic flux of the circuit we have only the self inductance of the coil:

$$\lambda = \lambda_{si} = L i$$

When the coil is lowered inside the circuit we need to add the linked flux due to the magnet. To compute the flux we can use any surface with the coil as a border, so we use a cilinder:



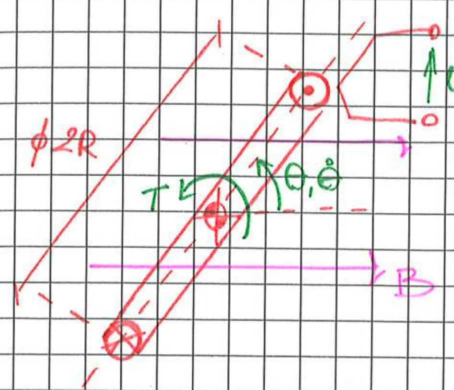
$$\lambda = \lambda_{si} + \lambda_0$$

$$\lambda_0 = \oint_S \vec{B} \cdot \vec{n}\, ds = N \int B\, dS = (NB 2\pi R) q$$

$$\lambda = \lambda_{si} + \lambda_0 = L i + \Lambda q = L i + \lambda_0(q)$$

$$i = \frac{\lambda - \lambda_0(q)}{L}$$

It's also possible to have a rotating coil inside a magnetic field.



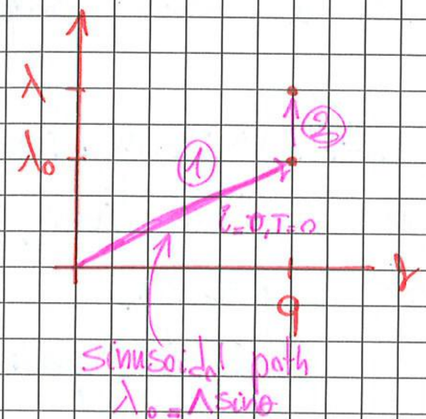
$$E = \int i\,d\lambda + T\,d\theta$$

$$\lambda = \lambda_{si} + \lambda_0$$

$$\lambda_{si} = L i$$

$$\lambda_0 = NBS\sin\theta = NB\pi R^2 \sin\theta = \Lambda \sin\theta$$

$$\lambda = Li + \Lambda\sin\theta \Rightarrow i = \frac{\lambda - \Lambda\sin\theta}{L}$$



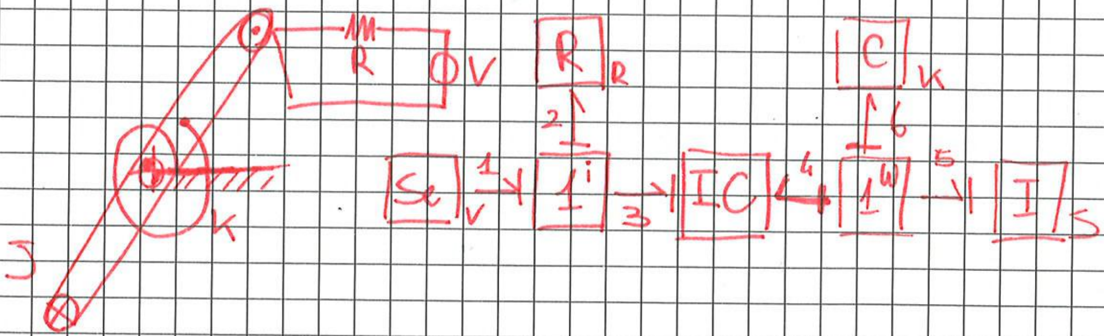
$$E = \int_{②} i\,d\lambda = \int \frac{\lambda - \Lambda\sin\theta}{L}\,dL = \frac{(\lambda - \Lambda\sin\theta)^2}{2L}$$

$$T = \frac{\partial E}{\partial \theta} = \frac{\lambda - \Lambda\sin\theta}{L}(-\Lambda\cos\theta) \quad \text{C-element}$$

$$i = \frac{\lambda - \Lambda\sin\theta}{L} \quad \text{I-element}$$

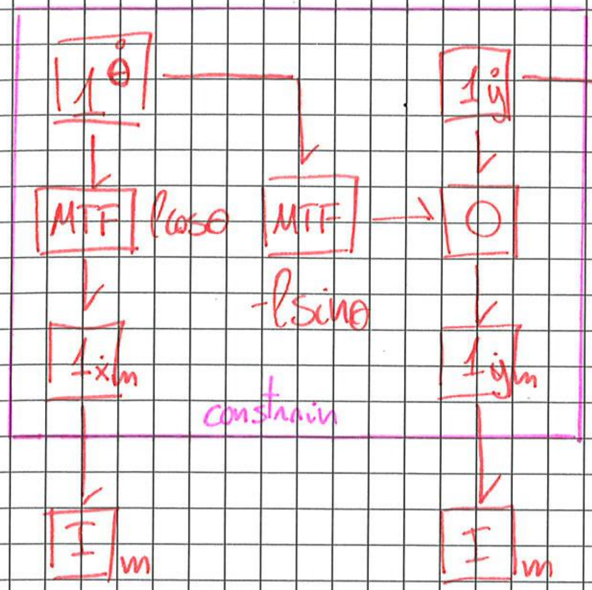
$$\xrightarrow[i]{e} \boxed{IC} \xleftarrow[\omega]{T}$$

This kind of structure was used as analog transducer to measure currents

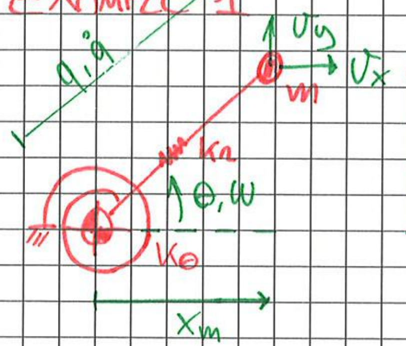


ELEMENT	INPUT	OUTPUT
Se: V	f_1	$e_1 = V$
R: R	f_2	$e_2 = R f_2$
IC	$e_3 = \dot{p}_3$	$f_3 = \frac{p_3 - \Lambda\sin q_4}{L}$
	$f_4 = \dot{q}_4$	$e_4 = -\frac{p_3 - \Lambda\sin q_4}{L}\Lambda\cos q_4$

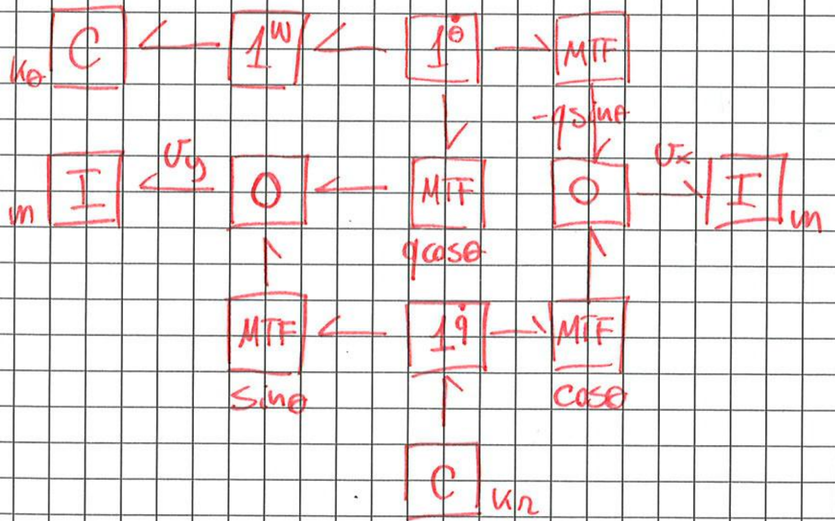
ELEMENT	IN	OUT
I: J	$e_5 = \dot{p}_5$	$f_5 = \frac{p_5}{J}$
C: K	$f_6 = \dot{q}_6$	$e_6 = k q_6$



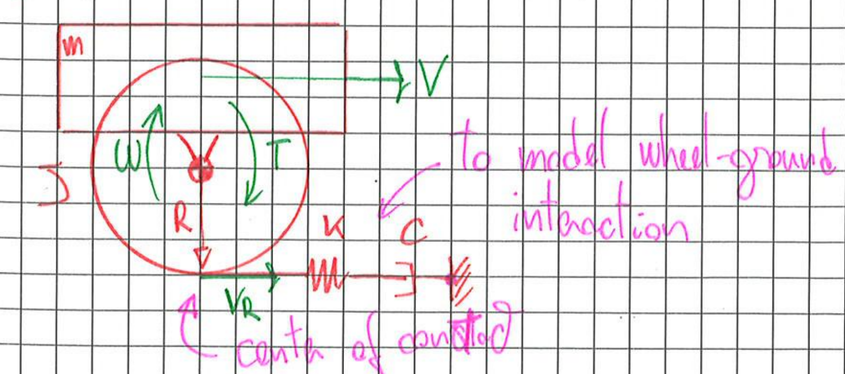
EXAMPLE 1



$$x_m = q\cos\theta$$
$$y_m = q\sin\theta$$
$$v_x = -q\sin\theta\,\dot\theta + \dot q\cos\theta$$
$$v_y = q\dot\theta\cos\theta + \dot q\sin\theta$$

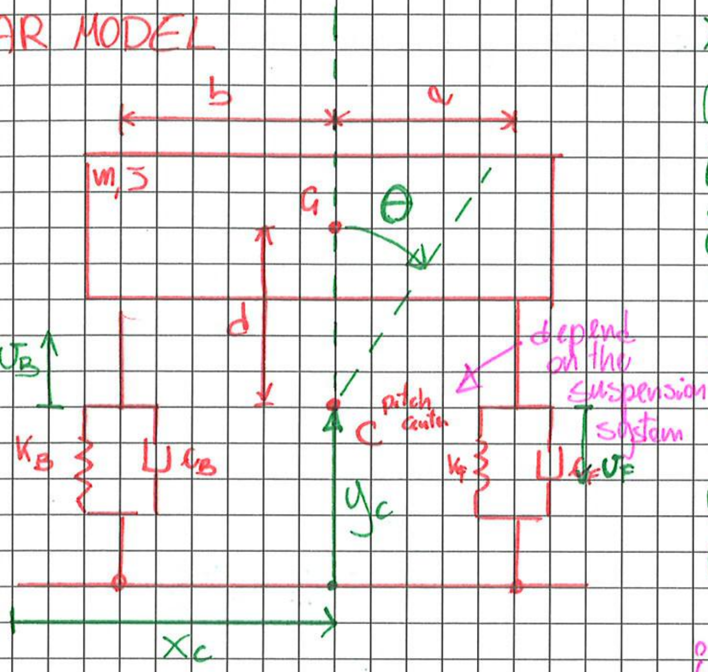


EXAMPLE 2



$$V_R = V - R\omega$$

EXAMPE 4
CAR MODEL



$X_G = X_c + d\sin\theta$

$y_G = y_c + d\cos\theta$

$y_B = y_c + b\sin\theta$

$y_F = y_c - a\sin\theta$

$\dot{X}_G = \dot{X}_c + d\cos\theta\,\dot{\theta}$

$\dot{y}_G = \dot{y}_c - d\dot{\theta}\sin\theta$

$v_B = \dot{y}_c + b\dot{\theta}\cos\theta$

$v_F = \dot{y}_c - a\dot{\theta}\cos\dot{\theta}$

if $\theta \simeq 0$: $\cos\theta \simeq 1$, $\sin\theta \simeq \theta \simeq 0$

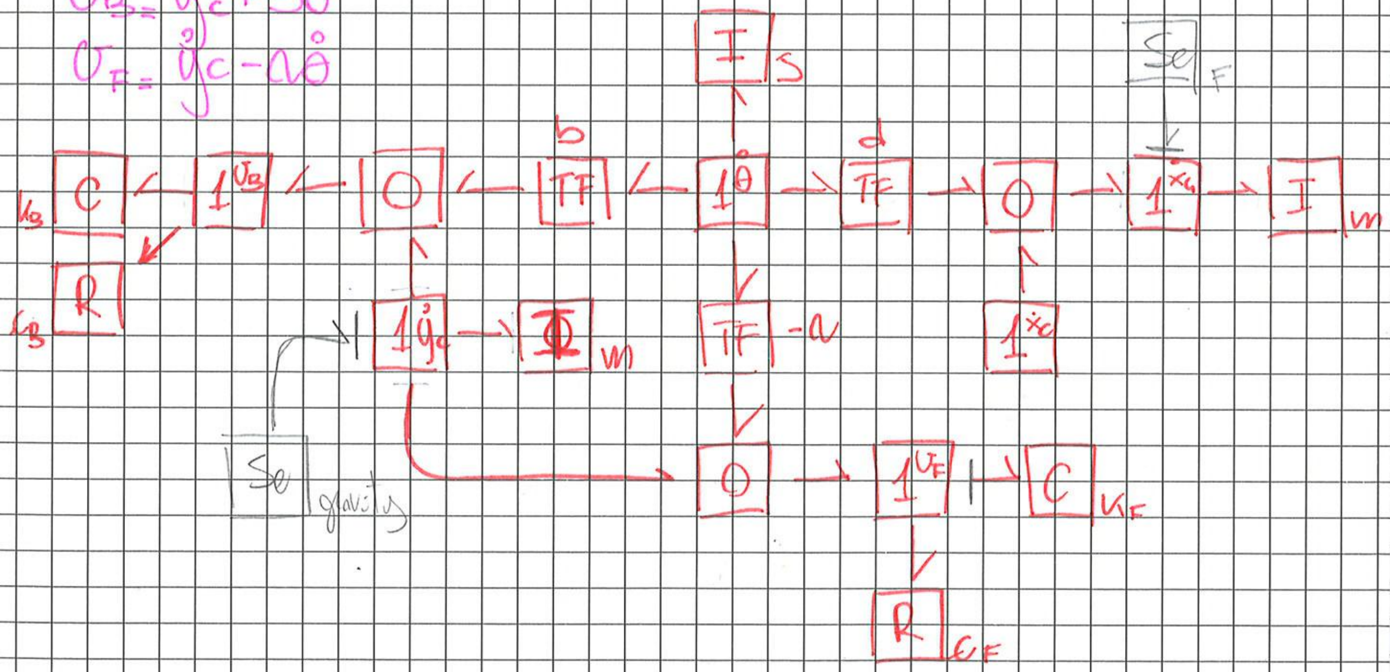
if not all transformers are modulated by θ

$\dot{X}_G = \dot{X}_c + d\dot{\theta}$

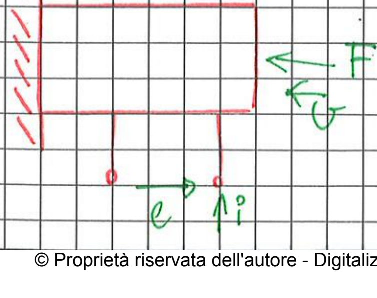
$\dot{y}_G = \dot{y}_c$

$v_B = \dot{y}_c + b\dot{\theta}$

$v_F = \dot{y}_c - a\dot{\theta}$



EXAMPLE 5
Piezo Actuator



$F = k_2 q + \theta e$

$Q = C_p e - \theta q \rightarrow i = C_p \frac{de}{dt} - \theta\dot{q}$

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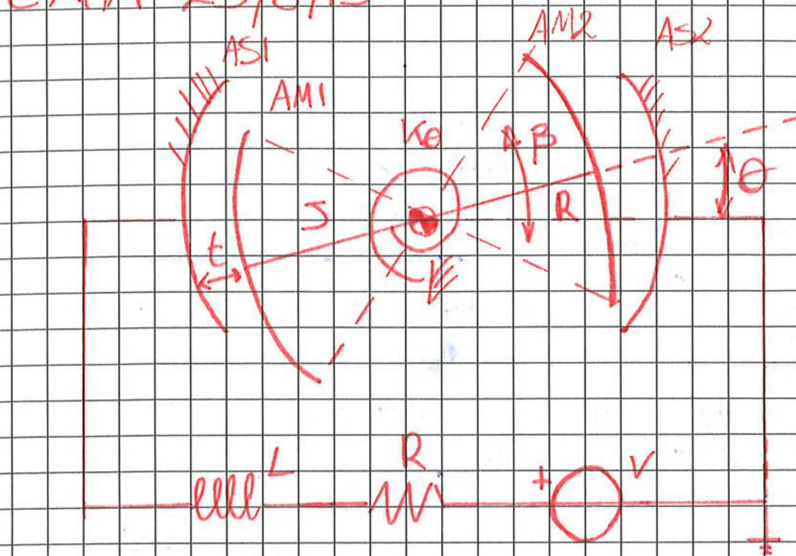
EXAM 25/8/13



1) Model the electrostatic actuator constituted by armatures AS2 and AM2 considering it as a C-field. Write the constitutive equations

$$E = \int_0^t T\omega + e\dot{i} = \int T d\theta + e dq$$

$$\Rightarrow E = 0 + \int_{②} T \underbrace{d\theta}_{\to 0} + e dq = \int_0^q e dq$$

$$q = C(\theta) e \qquad C(\theta) = \varepsilon \frac{S(\theta)}{t} = \varepsilon \frac{l(\beta-\theta)R}{t}$$

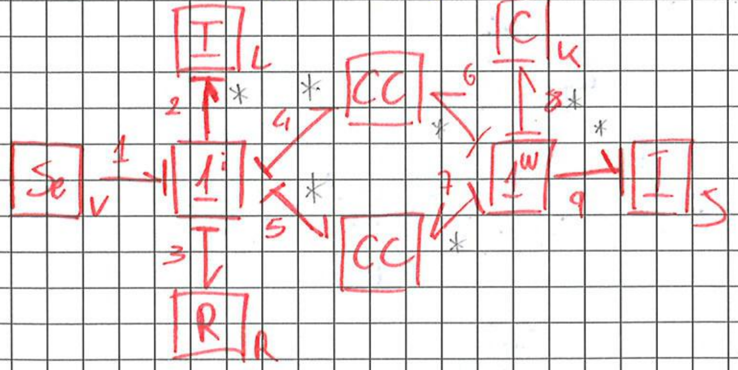
$$E = \int_0^q \frac{t}{\varepsilon l R(\beta-\theta)} q \, dq = \frac{t q^2}{2\varepsilon l R(\beta-\theta)}$$

$$T = \frac{\partial E}{\partial \theta} = -\frac{t q^2}{2\varepsilon l R(\beta-\theta)^2} \to \text{C-element}$$

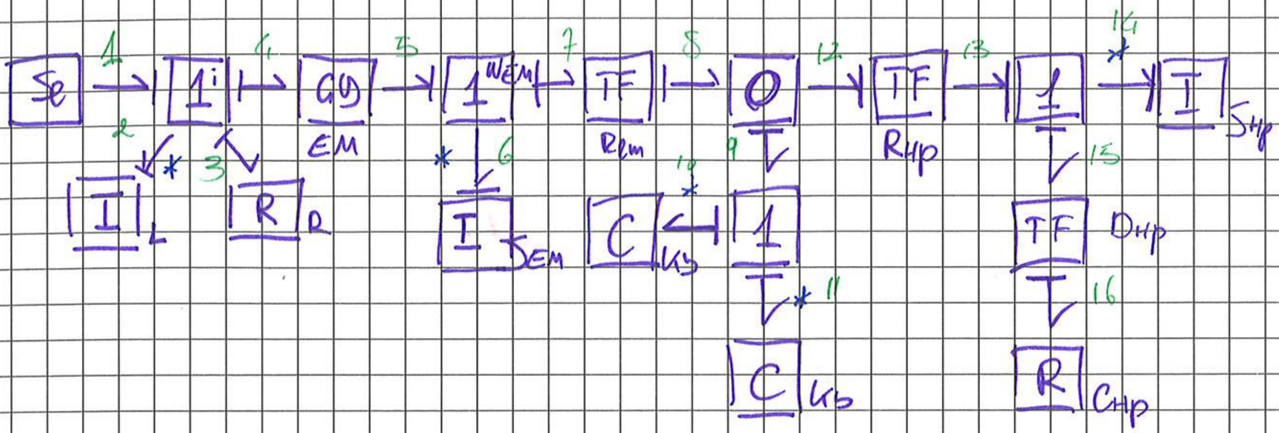
$$\xrightarrow[\omega]{T} \boxed{CC} \xleftarrow[i]{e}$$

$$e = \frac{\partial E}{\partial q} = \frac{t q}{\varepsilon l R(\beta-\theta)} \to \text{C-element}$$

2) Model the system, find the states.



2, 4, 5, 6, 7, 8, 9 states



2) Write the state equations

ELEMENT	INPUT	OUTPUT
Se : V	f_1	$e_1 = V$
I : L	$e_2 = \dot{p}_2$ *	$f_2 = \frac{p_2}{L}$
R : R	f_3	$e_3 = R f_3$
GY	f_4	$e_4 = K_m f_5$
	f_5	$e_5 = K_m f_4$
I : J_{em}	$e_6 = \dot{p}_6$ *	$f_6 = \frac{p_6}{J_{em}}$
TF : R_{em}	f_7	$e_7 = e_8 R_{em}$***
	e_8	$f_8 = f_7 R_{em}$***
C : k_b	$f_{10} = \dot{q}_{10}$ *	$e_{10} = k_b q_{10}$
C : k_b	$f_{11} = \dot{q}_{11}$ *	$e_{11} = k_b q_{11}$
TF : R_{Hp}	e_{12}	$f_{12} = R_{Hp}$*** f_{13}
	f_{13}	$e_{13} = e_{12} R_{Hp}$***
I : J_{Hp}	$f_{14} = \dot{p}_{14}$ *	$f_{14} = \frac{p_{14}}{J_{Hp}}$
TF : D_{Hp}	f_{15} *	e_{15} D e_{16}
	e_{16} *	$f_{16} = D f_{15}$
R : C_{Hp}	f_{16}	$e_{16} = C_{Hp} f_{16}$

$$F = \frac{\partial E}{\partial q} = \frac{1}{2}\Lambda^T \frac{\partial L^{-1}}{\partial q}\Lambda$$

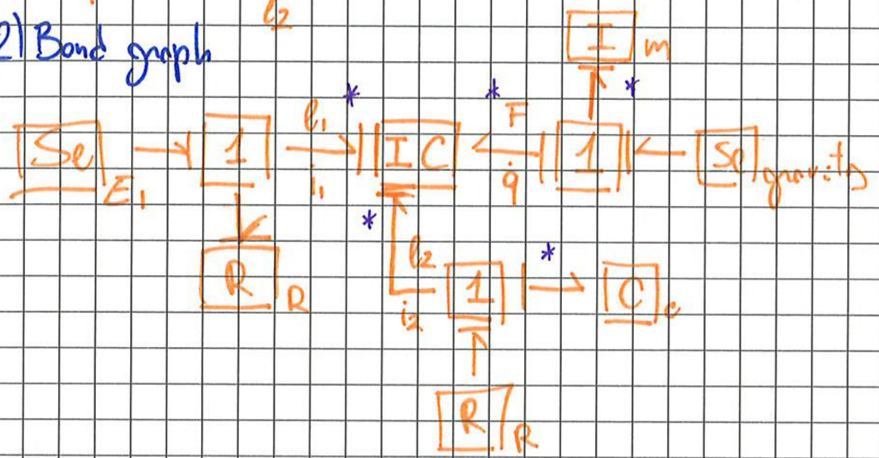
$$\frac{\partial L^{-1}}{\partial q} = \frac{\partial}{\partial q}\left[\frac{1}{L^2-(L+kq)^2}\begin{bmatrix} L & -L+kq \\ -L+kq & L\end{bmatrix}\right] = \frac{\frac{\partial M}{\partial q}}{L^2-M^2}\left\{\frac{2M}{L^2-M^2}\begin{bmatrix} L & -M \\ -M & L\end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0\end{bmatrix}\right\}$$

$\downarrow$ C-element

$$I = \frac{\partial E}{\partial \Lambda} = L^{-1}\Lambda \rightarrow \text{I-element}$$

$$\xrightarrow[\dot q]{F} CI \quad \frac{e_1}{i_1},\ \frac{e_2}{i_2}$$

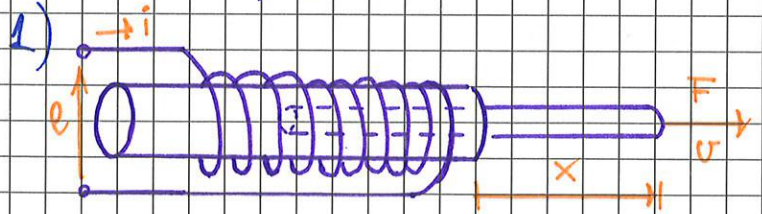
2) Bond graph



5 states

EXAM 23/2/11

1)



$$L(x) = \frac{L_0}{1+(x/d)^2}$$

$$\lambda = L\,i$$

$$\xrightarrow[i]{\lambda = e} IC \xleftarrow[v=\dot x]{F}$$

Determinare le equazioni del componente:

$$E = \int F\,dx + i\,d\lambda \Rightarrow E = \int i\,d\lambda = \int \frac{1+(x/d)^2}{L_0}\lambda\,d\lambda = \frac{1+(x/d)^2}{2L_0}\lambda^2$$

$$i = \frac{\partial E}{\partial \lambda} = \left[\frac{L_0}{1+(x/d)^2}\right]^{-1}\lambda \qquad F = \frac{\partial E}{\partial q} = \frac{\lambda^2 x}{L_0 d}$$

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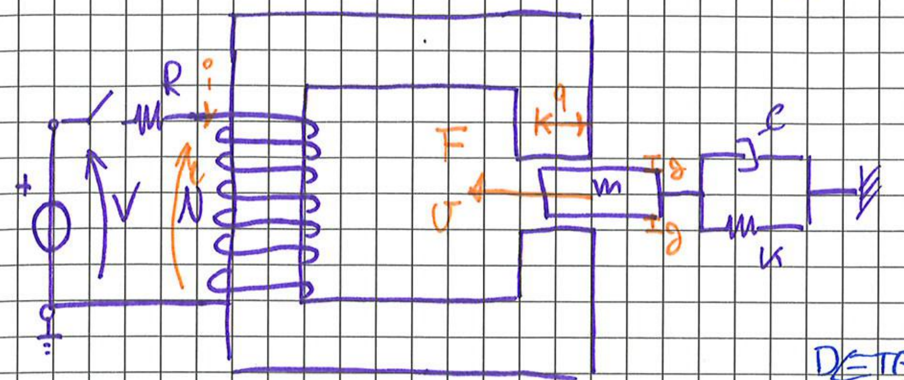
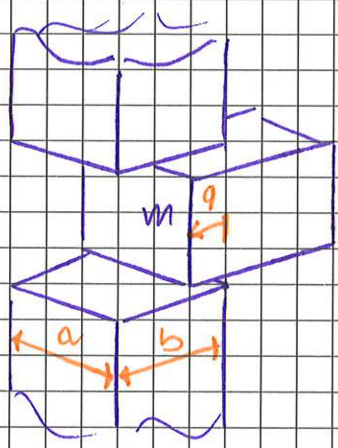
EXERCISE: RELUCTANCE TRANSDUCER





DATA

$N = 200$ $\quad q_0 = 10\ mm$

$R = 2\,\Omega$ $\quad k = 100\ N/mm$

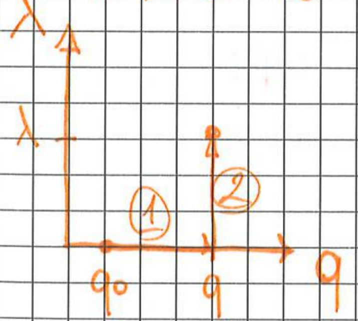
$V = 2\,V$ $\quad c = 1\ Ns/m$

$a = 30\ mm$ $\quad m = 50\ g$

$b = 20\ mm$ $\quad g = 0.50\ mm$

1) Write the transducer's equations

$$E = \int_{0,\,①\to②} F\,dq + i\,d\lambda = \int_{0,\,q\,const}^{\lambda} i\,d\lambda$$



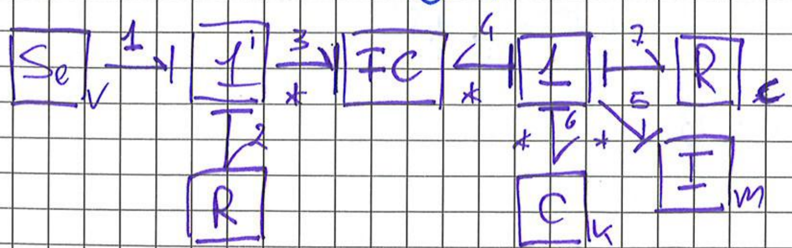
$$\lambda = L i = \frac{N^2}{\mathcal{R}}\, i \qquad \mathcal{R}_{gap} = \frac{\ell}{\mu A} = \frac{2g}{\mu a q} \qquad \mathcal{R}_{Fe} \simeq 0$$

$$i = \lambda \frac{\mathcal{R}}{N^2} = \frac{2g\lambda}{N^2 \mu a q} \rightarrow \text{I-element}$$

$$E = \int \frac{2g\lambda}{N^2 \mu a q}\, d\lambda = \frac{1}{\cancel{2}}\,\cancel{2}\,\frac{g\lambda^2}{N^2 \mu a q}$$

$$F = \frac{\partial E}{\partial q} = -\frac{g\lambda^2}{N^2 \mu a q^2} \rightarrow \text{C-element}$$

2) Realize the bond graph



$\dot{p}_3 = 0$ (static)

1) $V = R \dfrac{2 g p_3}{\mu_0 N^2 a (q_6 + q_0)} = \dfrac{R 2 g p_3}{N^2 \mu_0 a q_6 + N^2 \mu_0 a q_0}$

$V N^2 \mu_0 a q_6 + V N^2 \mu_0 a q_0 = R 2 g p_3$

$p_3 = \left(V N^2 \mu_0 a q_6 + V N^2 \mu_0 a q_0\right) \dfrac{1}{R 2 g}$ (2)

(2) → (1) $g \dfrac{V^2 N^2 \mu_0^2 a^2 q_6^2 + V^2 N^4 \mu_0^2 a^2 q_0^2 + 2 V^2 N^4 \mu_0^2 a^2 q_0 q_6}{R^2 4 g^2 N^2 \mu_0 a K} =$

$= q_6 (q_6^2 + q_0^2 + 2 q_6 q_0)$

$V^2 N^2 \mu_0 a q_6^2 +$

MODELLING AND SIMULATION OF MECHATRONIC SYSTEMS

LAGRANGE APPROACH

PROBLEM 1

A triaxial accelerometer A is mounted on a vehicle V moving on a rough terrain. The accelerometer measures the local vehicle acceleration providing the vector $\underline{a}_A = [a_x \; a_y \; a_z]^T$ given with respect to the reference frame of the accelerometer R_A.

The orientation of R_A wrt the vehicle reference frame R_V is given by the rotation matrix R_A^V. The vehicle orientation wrt the ground station G is given by the matrix R_V^G.

A base station B is required to measure in its reference R_B the vertical component of $\underline{a}$. Its orientation is given by R_B^G

QUESTION 1

Given the following data

$$\underline{a}_A = \begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} \quad R_A^V = \begin{bmatrix} 0.6964 & -0.7151 & 0.0607 \\ 0.6964 & 0.6937 & 0.1839 \\ -0.1736 & -0.0858 & 0.9811 \end{bmatrix} \quad R_V^G = \begin{bmatrix} 0.8659 & 0.3642 & -0.3629 \\ -0.4999 & 0.6062 & -0.6183 \\ -0.0175 & 0.7070 & 0.7070 \end{bmatrix}$$

$$R_B^G = \begin{bmatrix} 0.8659 & -0.3629 & 0.3642 \\ 0.4999 & 0.6185 & -0.6062 \\ -0.0175 & 0.7070 & 0.7070 \end{bmatrix}$$

tell whether the third component of $\underline{a}_B$ (wrt R_B) is longer than the local gravity acceleration $g_B = [0, 0, -3.71]^T$

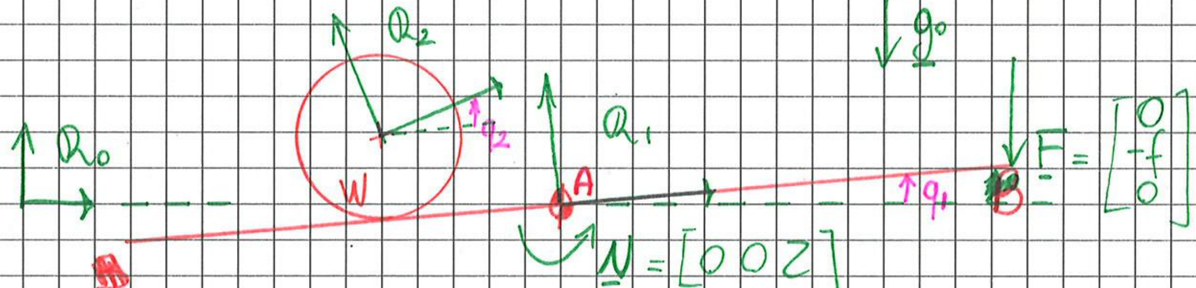
$$\underline{a}_B = R_A^B \, \underline{a}_A$$

PROBLEM 2

The planar system illustrated consists of a bar B and a wheel W moving (no friction, no sliding) along the bar. The bar can rotate around an axis perpendicular to the plane at the pivot point A.
The wheel can rotate around its center and maintains always a contact point with the bar.
No elastic elements are present.

QUESTIONS

1) Define a possible set of generalized coordinates $q_i(t)$. Link q_1 to the bar rotation and q_2 to the wheel motion.



2) Write the homogeneous transformation T_2^0 between R_0 R_2

$$T_1^0 = \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & \frac{L}{2} + \ell_0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline & 0^T & & 1 \end{array}\right] \quad T_2^1 = \left[\begin{array}{ccc|c} C_2 & -S_2 & 0 & -Rq_2 \\ S_2 & C_2 & 0 & R \\ 0 & 0 & 1 & 0 \\ \hline & 0^T & & 1 \end{array}\right]$$

Rotations about same axis (z) are summed

$$T_2^0 = T_1^0 T_2^1 = \left[\begin{array}{ccc|c} \cancel{C_1C_2 + S_1S_2}\ C_{1+2} & -S_{1+2} & 0 & -C_1Rq_2 - S_1R + \frac{L}{2} \\ S_{1+2} & C_{1+2} & 0 & -S_1Rq_2 + C_1R \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$r_A = \left[\frac{L}{2}, 0, 0\right]^T \rightarrow \dot{r}_A = [0\ 0\ 0]^T \quad \|\dot{r}_A\| = 0$$

$$r_B = \left[L c_1\ \ \frac{L}{2} s_1\ \ 0\right]^T \rightarrow \dot{r}_B = \left[-L s_1 \dot{q}_1\ \ \frac{L}{2} c_1 \dot{q}_1\ \ 0\right]^T$$

$$\dot{r}_W = \begin{bmatrix} R(q_2 s_1 \dot{q}_1 - c_1 \dot{q}_1 - c_1 \dot{q}_2) \\ -R(q_2 c_1 \dot{q}_1 + s_1 \dot{q}_1 + s_1 \dot{q}_2) \\ 0 \end{bmatrix}$$

$$\|\dot{r}_W\|^2 = R^2\left((q_2 s_1 \dot{q}_1 - c_1 \dot{q}_1 - c_1 \dot{q}_2)^2 + (q_2 c_1 \dot{q}_1 + s_1 \dot{q}_1 + s_1 \dot{q}_2)^2\right) =$$

$$= R^2\left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 + q_2^2 \dot{q}_1^2\right) = R^2\left((\dot{q}_1 + \dot{q}_2)^2 + q_2^2 \dot{q}_1^2\right)$$

$$\omega_A = [0\ 0\ \dot{q}_1]^T \quad \omega_W = [0\ 0\ \dot{q}_1 + \dot{q}_2]$$

$$K_A^* = \frac{1}{2}\omega_A^T \Gamma_b \omega_A = \frac{1}{2}\dot{q}_1^2 \Gamma_{bz}$$

$$K_W^* = \frac{1}{2} m_W \|\dot{r}_W\|^2 + \frac{1}{2}\omega_W^T \Gamma_W \omega_W = \frac{1}{2} m_W R^2\left((\dot{q}_1 + \dot{q}_2)^2 + q_2^2 \dot{q}_1^2\right) + \frac{1}{2}(\dot{q}_1 + \dot{q}_2)^2 \Gamma_{Wz}$$

$$K_{TOT}^* = K_A^* + K_B^*$$

6) Compute the potential energy

Potential energy is only due to gravitational field

$$P_A = -m_b g_0^T r_A = 0$$

$$P_W = -m_W g_0^T r_W = -m_W [0\ -G\ 0] \begin{bmatrix} R q_2 c_1 - R s_1 + \frac{L}{2} \\ -R q_2 s_1 + R c_1 \\ 0 \end{bmatrix} = m_W G R (c_1 - q_2 s_1)$$

$$P_{TOT} = P_A + P_W = P_W$$

7) Compute the virtual work due to u and F

$$\delta W = u \delta q_1 + F \delta r_B$$

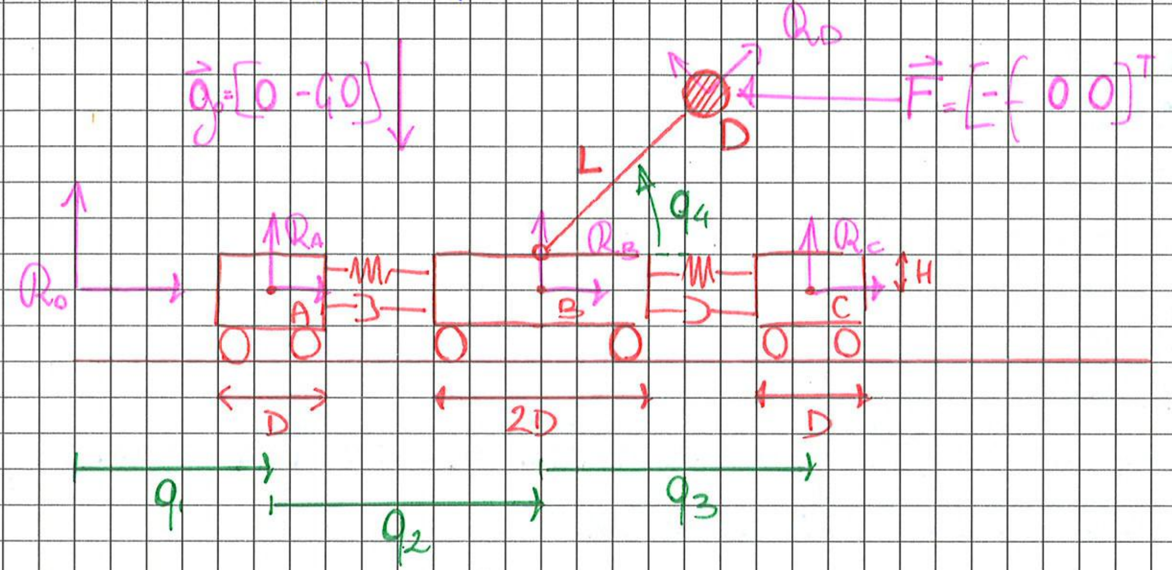
$$\delta r_B \rightarrow d r_B = \dot{r}_B dt = \begin{bmatrix} -L s_1 dq_1 \\ \frac{L}{2} c_1 dq_1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -L s_1 \\ \frac{L}{2} c_1 \\ 0 \end{bmatrix} \delta q_1$$

PROBLEM 3

The dynamical system illustrated consists of 3 carts A, B, C moving along an horizzontal line; they are connected by elastic springs with friction. The second cart (B) transports a massless rotating joint with a massless arm, that supports a solid round mass D at its extremity.

QUESTIONS

1) Define a reference frame attached to the round mass D.



2) Write the homogeneus transformation T^0_D between R_0 and R_D.

$$T^0_D = T^0_B\, T^B_D$$

$$T^0_B = \left[\begin{array}{c|c} I & \begin{matrix} q_1+q_2 \\ 0 \\ 0 \end{matrix} \\ \hline 0^T & 1 \end{array}\right] \qquad T^B_D = \left[\begin{array}{ccc|c} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & H \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} I & \begin{matrix} L \\ 0 \\ 0 \end{matrix} \\ \hline 0^T & 1 \end{array}\right] = \left[\begin{array}{ccc|c} c_4 & -s_4 & 0 & c_4 L \\ s_4 & c_4 & 0 & s_4 L + H \\ 0 & 0 & 1 & \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$T^0_D = \begin{bmatrix} c_4 & -s_4 & 0 & c_4 L + q_1 + q_2 \\ s_4 & c_4 & 0 & s_4 L + H \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \dot{q}_1^2 + \dot{q}_2^2 + L^2 \dot{q}_4^2 + 2\dot{q}_1\dot{q}_2 - 2Ls_4 \dot{q}_4 (\dot{q}_1 + \dot{q}_2) = \|v_D\|^2$$

$$K_D^* = \frac{1}{2}\left(m_D \|v_D\|^2 + I_z \dot{q}_4^2\right) \quad (\dot{q}_1 + \dot{q}_2)^2 + L\dot{q}_4^2 - 2\ldots$$

$$K_{TOT} = \frac{1}{2}\left[m_A \dot{q}_1^2 + m_B(\dot{q}_1+\dot{q}_2)^2 + m_C(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2 + m_D\|v_D\|^2 + I_z \dot{q}_4^2\right]$$

7) Compute the potential energy $\mathcal{P}$

$$\mathcal{P}_{TOT} = \sum_i \left(\frac{1}{2}k_i \Delta l^2\right) + \sum_i \left(-m_i \vec{g}_0^T \vec{r}_i\right) = \frac{1}{2}k q_2^2 + \frac{1}{2}k q_3^2 - m_4 [0 - G\ 0]\left[\cdot\right] =$$

$$= \frac{1}{2}k\left(q_2^2 + q_3^2\right) + m_4 G (Ls_4 + H)$$

8) compute the dissipative function $\mathcal{D}^*$

$$\mathcal{D}_{TOT} = \sum \frac{1}{2}\beta_i \Delta v^2 = \frac{1}{2}\beta \dot{q}_2^2 + \frac{1}{2}\beta \dot{q}_3^2 = \frac{1}{2}\beta\left(\dot{q}_2^2 + \dot{q}_3^2\right)$$

9) compute the virtual work due to F

$$W = (\delta r_D)^T F$$

$$(\delta r_D)^T = (\dot{r}_D dt)^T = (dr_D)^T = [dq_1 + dq_2 - Ls_4 dq_4 \,;\, Lc_4 dq_4 \,;\, 0]$$

$$F = [-f \,;\, 0 \,;\, 0]^T$$

$$\delta W = (\delta q_1 + \delta q_2 - Ls_4 \delta q_4)(-f) = -f\delta q_1 - f\delta q_2 + fLs_4\delta q_4$$

$$\delta W = \sum_i \mathcal{F}_i \delta q_i \Rightarrow \mathcal{F}_1 = \mathcal{F}_2 = -f \quad \mathcal{F}_4 = fLs_4 \quad \mathcal{F}_3 = 0$$

10) Write the 4 second order differential equation using the Lagrange approach:

$$1)\ \frac{d}{dt}\left[m_A\dot{q}_1 + m_B(\dot{q}_1+\dot{q}_2) + m_C(\dot{q}_1+\dot{q}_2+\dot{q}_3) + m_D[\dot{q}_1 + \dot{q}_2 - Ls_4\dot{q}_4]\right] - 0 + 0 + 0 = -f$$

$$m_A\ddot{q}_1 + m_B(\ddot{q}_1+\ddot{q}_2) + m_C(\ddot{q}_1+\ddot{q}_2+\ddot{q}_3) + m_D[\ddot{q}_1 + \ddot{q}_2 - Ls_4\ddot{q}_4 - Lc_4\dot{q}_4^2] = -f$$

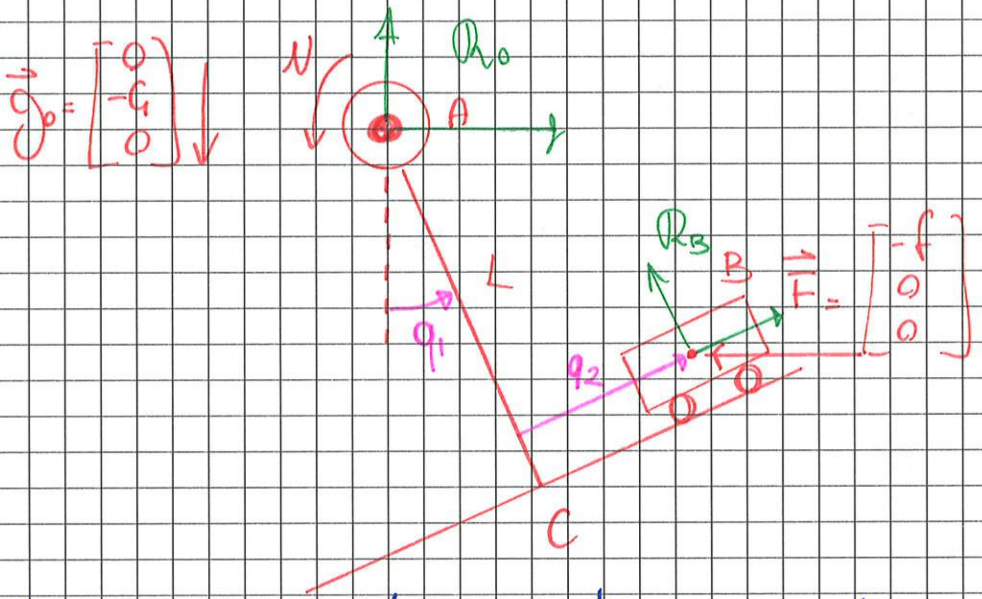
$$\sum_A^D m_i \ddot{q}_1 + \sum_B^D m_i \ddot{q}_2 + m_C\ddot{q}_3 - m_D L s_4 \ddot{q}_4 - m_D L c_4 \dot{q}_4^2 = -f$$

PROBLEM 4

The dynamical system illustrated consists of a rotating joint A, a rigid massless T-shaped support C and a cart B moving on it

QUESTIONS

1) Define the inertial reference frame $\mathcal{R}_0$ and the reference frame $\mathcal{R}_B$ attached to the cart B.



2) Write the homogeneous transformation T_B^0 between $\mathcal{R}_0$ and $\mathcal{R}_B$ using the given generalised coordinates

$$T_B^0 = \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I & \begin{matrix} +q_2 \\ -L \\ 0 \end{matrix} \\ \hline 0^T & 1 \end{array}\right] = \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & +C_1 q_2 + S_1 L \\ S_1 & C_1 & 0 & S_1 q_2 - C_1 L \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$T_B^0 = \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & C_1 q_2 + S_1 L \\ S_1 & C_1 & 0 & S_1 q_2 - C_1 L \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

6) Compute the kinetic co-energy of the system

$$K^* = \frac{1}{2} m_B \|v_B\|^2 + \frac{1}{2}(\Gamma_{2A} + \Gamma_{2B})\dot{q}_1^2$$

$$\|v_B\|^2 = (\dot{q}_2 c_1 - \dot{q}_1 q_2 s_1 + L\dot{q}_1 c_1)^2 + (\dot{q}_2 s_1 + \dot{q}_1 q_2 c_1 + L\dot{q}_1 s_1)^2 =$$

$$= \dot{q}_2^2 c_1^2 + \dot{q}_1^2 q_2^2 s_1^2 + L^2 \dot{q}_1^2 c_1^2 - \cancel{2\dot{q}_1\dot{q}_2 q_2 c_1 s_1} + 2L\dot{q}_1\dot{q}_2 c_1^2 - \cancel{2\dot{q}_1^2 q_2 c_1 s_1 L} +$$

$$+ \dot{q}_2^2 s_1^2 + \dot{q}_1^2 q_2^2 c_1^2 + L^2\dot{q}_1^2 s_1^2 + \cancel{2\dot{q}_1\dot{q}_2 q_2 c_1 s_1} + 2L\dot{q}_1\dot{q}_2 s_1^2 + \cancel{2L\dot{q}_1^2 q_2 c_1 s_1} =$$

$$= \dot{q}_2^2 + \dot{q}_1^2 q_2^2 + L^2\dot{q}_1^2 + 2L\dot{q}_1\dot{q}_2 = (L\dot{q}_1 + \dot{q}_2)^2 + \dot{q}_1^2 q_2^2$$

$$K^* = \frac{1}{2} m_B \left[\dot{q}_2^2 + L^2\dot{q}_1^2 + \dot{q}_1^2 q_2^2 + 2L\dot{q}_1\dot{q}_2\right] + \frac{1}{2}(\Gamma_{A2} + \Gamma_{B2})\dot{q}_1^2 =$$

$$= \frac{1}{2} m_B \left[(L\dot{q}_1 + \dot{q}_2)^2 + \dot{q}_1^2 q_2^2\right] + \frac{1}{2}(\Gamma_{A2} + \Gamma_{B2})\dot{q}_1^2$$

7) Compute the potential energy, where the $P=0$ line cross the origin of R_0

$$P = -m_B g_0^T r_B = -m_B [0 \ -G \ 0] \begin{bmatrix} q_2 c_1 + L s_1 \\ q_2 s_1 - L c_1 \\ 0 \end{bmatrix} =$$

$$= m_B G (s_1 q_2 - L c_1)$$

8) Compute the total work due to F and N

$$W = q_1 N + r_B^T F$$

$$\delta r_B \rightarrow dr_B = \dot{r}_B dt = \begin{bmatrix} dq_2 c_1 - dq_1 q_2 s_1 + L dq_1 c_1 \\ dq_2 s_1 + dq_1 q_2 c_1 + L dq_1 s_1 \\ 0 \end{bmatrix}$$

$$\delta W = \delta q_1 \tau - f c_1 \delta q_2 + f s_1 q_2 \delta q_1 - f L c_1 \delta q_1$$

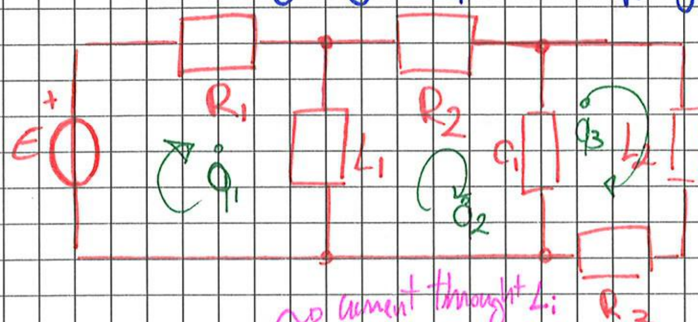
$$\xi_1 = \tau + f(s_1 q_2 - L c_1) \qquad \xi_2 = -f c_1$$

ELECTRICAL SYSTEM PROBLEM 1

The electrical circuit represented is composed by linear constant R, L, C bipoles and electrical ideal generators.

QUESTION 1

Find the Lagrange equations applying the charge approach



no current through L_i

$$C^* = \frac{1}{2}\sum L_i \dot{q}_i^2 = \frac{1}{2}\left[L_1(\dot{q}_1 - \dot{q}_2)^2 + L_2\dot{q}_3^2\right]$$

$$P = \frac{1}{2}\sum \frac{q_i^2}{C_i} = \frac{1}{C_1}(q_2 - q_3)^2$$

integral of the current entering in C_i

$$D = \frac{1}{2}\sum\left[R_i\dot{q}_i^2\right] = \frac{1}{2}\left[R_1\dot{q}_1^2 + R_2\dot{q}_2^2 + R_3\dot{q}_3^2\right]$$

$$\delta W = \sum E_i\delta q_i = E\delta q_1 \Rightarrow \mathcal{F}_1 = E \quad \mathcal{F}_2 = \mathcal{F}_3 = 0$$

1) $\frac{d}{dt}\left[L_1(\dot{q}_1 - \dot{q}_2)\right] - 0 + 0 + R_1\dot{q}_1 = E$

$L_1(\ddot{q}_1 - \ddot{q}_2) + R\dot{q}_1 = E \Rightarrow L_1\frac{d}{dt}(i_1 - i_2) + R i_1 = E$

2) $\frac{d}{dt}\left[-L_1(\dot{q}_1 - \dot{q}_2)\right] - 0 + \frac{1}{C_1}(q_2 - q_3) + R_2\dot{q}_2 = 0$

$-L_1(\ddot{q}_1 - \ddot{q}_2) + \frac{q_2 - q_3}{C_1} + R_2\dot{q}_2 = 0 \Rightarrow -L_1\frac{d}{dt}(i_1 - i_2) + R_2 i_2 + \frac{q_2 - q_3}{C_1} = 0$

3) $\frac{d}{dt}\left[L_2\dot{q}_3\right] + \frac{q_2 - q_3}{C_1}(-1) + R_3\dot{q}_3 = 0$

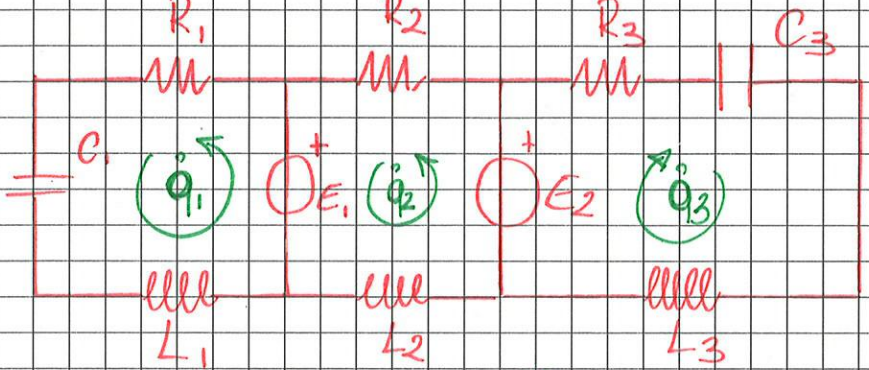
$L\ddot{q}_3 + R_3\dot{q}_3 - \frac{1}{C_1}(q_2 - q_3) = 0 \Rightarrow L\frac{d}{dt}i_3 + R_3 i_3 - \frac{1}{C}(q_2 - q_3) = 0$

ELECTRICAL PROBLEM 2

The electrical circuit in figure is composed by linear constant R, L, C bipoles and by ideal generators.

QUESTION 1

Find the Lagrange equations applying the charge approach.



$$\mathcal{C}^* = \frac{1}{2}\sum_k L_k \dot{i}_k^2 = \frac{1}{2}\left[L_1\dot{q}_1^2 + L_2\dot{q}_2^2 + L_3\dot{q}_3^2\right]$$

$$\mathcal{P} = \frac{1}{2}\sum_k \frac{1}{C_k} Q_k^2 = \frac{1}{2}\left[\frac{q_1^2}{C_1} + \frac{q_3^2}{C_3}\right]$$

$$\mathcal{D} = \frac{1}{2}\sum_k R_k \dot{i}_k^2 = \frac{1}{2}\left[R_1\dot{q}_1^2 + R_2\dot{q}_2^2 + R_3\dot{q}_3^2\right]$$

$$\delta W = \sum \delta i\, E = (\delta q_1 - \delta q_2)E_1 + (\delta q_2 + \delta q_3)E_2 =$$

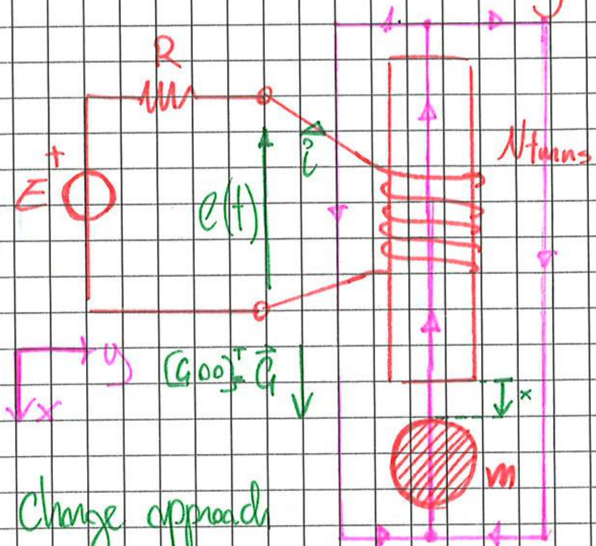
$$= E_1\delta q_1 + (E_2 - E_1)\delta q_2 + E_2\delta q_3 \Rightarrow \mathcal{F}_1 = E_1;\ \mathcal{F}_2 = E_2 - E_1;\ \mathcal{F}_3 = E_2$$

1) $\frac{d}{dt}\left[L\dot{q}_1\right] - 0 + \frac{q_1}{C_1} + R_1\dot{q}_1 = E_1 \quad L\ddot{q}_1 + R_1\dot{q}_1 + \frac{1}{C_1}q_1 = E_1$

2) $\frac{d}{dt}\left[L\dot{q}_2\right] - 0 + 0 + R_2\dot{q}_2 = E_2 - E_1 \Rightarrow L\ddot{q}_2 + R_2\dot{q}_2 = E_2 - E_1$

3) $\frac{d}{dt}\left[L_3\dot{q}_3\right] - 0 + \frac{q_3}{C_3} + R_3\dot{q}_3 = E_2 \Rightarrow L_3\ddot{q}_3 + R_3\dot{q}_3 + \frac{q_3}{C_3} = E_2$

ELECTROMECHANICAL SYSTEM - MAGNETIC SUSPENSION



Change approach

$$\mathcal{R}_{TOT} = \underbrace{\cancel{\mathcal{R}_{Fe}}}_{\simeq 0} + \mathcal{R}_{air} + \mathcal{R}_{gap} = \mathcal{R}_{air} + \mathcal{R}_{gap}$$

$$\mathcal{R}_{TOT} = \frac{l_0}{\mu_0 S} + \frac{x}{\mu_0 S} = \frac{x + l_0}{\mu_0 S}$$

$$N i(t) = \frac{x + l_0}{\mu_0 S} \phi(t)$$

$$\Downarrow$$

$$\phi N = \lambda = \frac{\overbrace{N^2 \mu_0 S}^{K}}{l_0 + x} i(t) = L(x) i(t)$$

$$C^*_{mech} = \frac{1}{2} m \dot{x}^2 \qquad P_{mech} = -m \vec{G}^T \vec{r} = -mGx \qquad D_{mech} = \frac{1}{2} \beta \dot{x}^2$$

$$C^*_{el} = \frac{1}{2} L(x) \dot{q}^2 \qquad P_{el} = \frac{1}{2} \sum_k C_k^{-1} q_k^2 = 0 \qquad D_{el} = \frac{1}{2} R \dot{q}^2$$

$$\mathcal{F}_{mech} = 0$$

$$\mathcal{F}_{el} = E$$

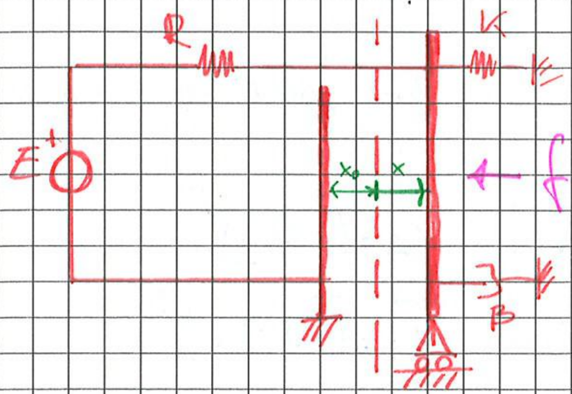
1) x coordinate

$$\frac{d}{dt}\left[m\dot{x}\right] + \frac{K}{2(l_0+x)^2}\dot{q}^2 - mG + \beta\dot{x} = 0 \rightarrow m\ddot{x} + \frac{K}{2(l_0+x)^2}\dot{q}^2 - mG + \beta\dot{x} = 0$$

2) q coordinate

$$\frac{d}{dt}\left[L(x)\dot{q}\right] + R\dot{q} = E \rightarrow L(x)\ddot{q} - \frac{K\dot{x}}{(l_0+x)^2}\dot{q} + R\dot{q} = E$$

ELECTROMECHANICAL SYSTEM - CAPACITATIVE MICROPHONE



$$C(x) = \varepsilon\frac{S}{d} = \frac{\varepsilon S}{x_0+x} = \frac{C_0 x_0}{x_0+x}$$

$$K^*_{mech} = \frac{1}{2} m\dot{x}^2$$

$$K^*_{el} = \frac{1}{2}\sum_i L_i \dot{q}_i^2 = 0$$

$$P_m = \frac{1}{2} k x^2$$

$$P_{el} = \frac{1}{2}\frac{q^2}{C(x)}$$

$$D_m = \frac{1}{2}\beta\dot{x}^2$$

$$D_{el} = \frac{1}{2} R\dot{q}^2$$

$$\mathcal{F}_m = f$$

$$\mathcal{F}_{el} = E$$

$$x)\quad m\ddot{x} + \beta\dot{x} + kx + \frac{q^2}{2C_0x_0} = f$$

$$q)\quad R\dot{q} + \frac{x_0+x}{C_0x_0}q = E$$

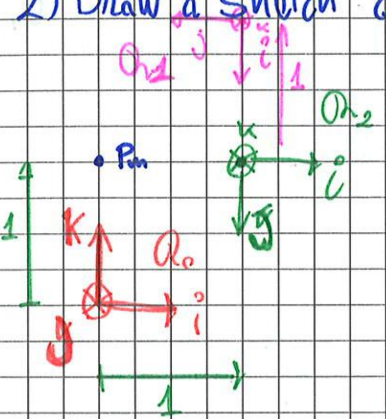
$$T_1^0 = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

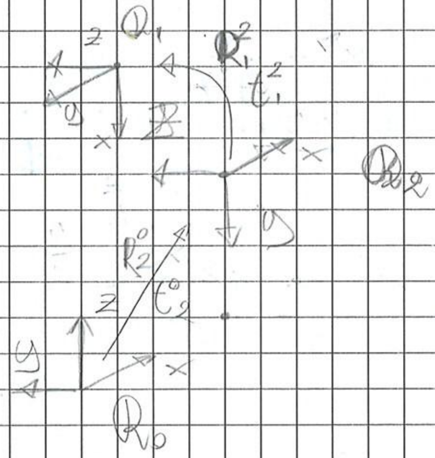
$$p_0 = R_1^0 p_m + t_1^0 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$P = -m\, g_0^T p_0 = -m \begin{bmatrix} 0 & 0 & -G \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = mG = 10$$

$|G| = 10$

2) Draw a sketch of $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2$ and p_m.





QUESTION 2

The planar mechanical system represented consists of a reduction gear composed of 2 rotating gears with their center fixed on the horizzontal line; each gears has radius R_i, mass M_i and a link attached to it, with a point mass at its extremity.
With the angles defined as in figure, assume that when $q_1 = 0$ also $q_2 = 0$.
An elastic and a viscous effect acting on the first link are represented by coefficient k and β.

1) If a torque τ_1 is applied to gear 1 compute the Lagrange equations of the system.

$\mathcal{D} = \frac{1}{2}\beta\dot{q}_1^2$

$\delta W = \delta q_1 \tau_1 \rightarrow \mathcal{F}_1 = \tau_1$

due to kinematic constrain ($q_2 = -\rho q_1$) there is only one Lagrange equation:

1) $\frac{d}{dt}\left[\dot{q}_1\left(I_1 + I_2\rho^2 + m_1L_1^2 + m_2L_2^2\rho^2\right)\right] - 0 + kq + m_1L_1 g s_1 + m_2L_2 g c_{\rho 1}(-\rho) + \beta\dot{q}_1 = \tau_1$

$\left[\left(I_1 + m_1L_1^2\right) + \rho^2\left(I_2 + m_2L_2^2\right)\right]\ddot{q}_1 + \beta\dot{q}_1 + m_1L_1 g \sin(q_1) - m_2L_2\rho\cos(-\rho q_1) = \tau$

2) $q_2 = -\rho q_1$

2) Using the data below, compute the numerical value of the total potential energy when $q_1 = -\frac{\pi}{4}$.

$M_1 = 2\ kg$ $\quad$ $R_2 = 0.2$

$M_2 = 1\ kg$ $\quad$ $L_1 = L_2 = 1\ m$

$m_1 = m_2 = 0.1\ kg$ $\quad$ $k = 1600\ \frac{Nm}{rad}$

$R_1 = 0.4\ m$ $\quad$ $\beta = 10^{-3}\ \frac{Nms}{rad}$

$\rho = \frac{R_1}{R_2} = \frac{0.4}{0.2} = 2$

$P_{TOT} = \frac{1}{2}kq_1^2 + g\left(m_2L_2\sin(-\rho q_1) - m_1L_1\cos q_1\right) =$

$= \frac{1}{2}\cdot 1600\left(\frac{\pi}{4}\right)^2 + 10\left(0.1\cdot 1\cdot\sin\left(\frac{\pi}{2}\right) - 0.1\cdot 1\cdot\cos\left(-\frac{\pi}{4}\right)\right) =$

$= 493.773\ J$