NUMERO: 1620A - ANNO: 2015

A P P U N T I

STUDENTE: Botta

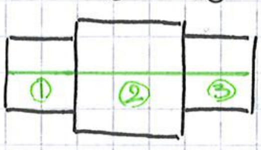
MATERIA: Fondamenti Meccanica Strutturale. Prof.Curà

MODELLI

Un modello serve per rappresentare la realtà in modo semplificato, utilizzando le semplificazioni che più interessano.

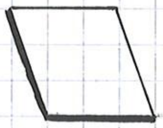
esempi:

- modello a trave: rappresenta gli oggetti lineari con una linea.

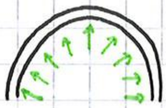


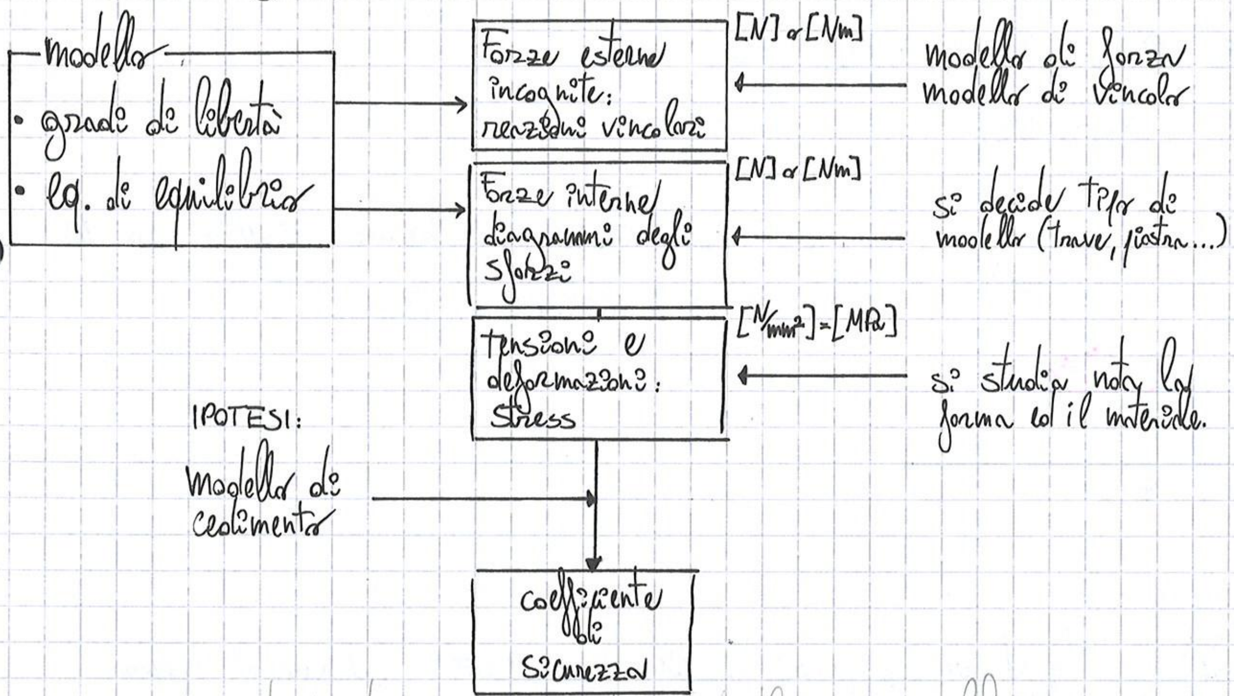
- modello continuo: le caratteristiche dell'oggetto sono identiche all'interno del modello.

- modello a piastra:



- modello a guscio:





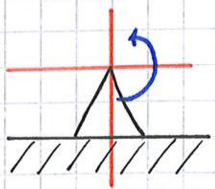
Principio di Saint Venant: Se un sistema di forze in equilibrio agisce su una parte della superficie di un solido omogeneo isotropo ed elastico lineare i suoi effetti si attenuano allontanandosi dalla superficie.

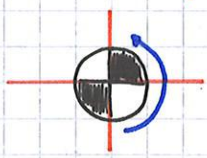
MODELLI DI VINCOLO

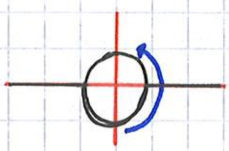
- Incastro: movimenti impediti: x, y, θ_z

$m = 3$

- Cerniera: movimenti impediti: x, y movimenti consentiti: θ_z $m = 2$



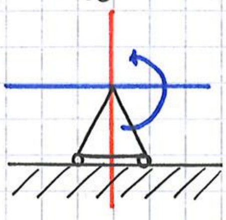




Cerniera fissa o esterna

Cerniera interna

- Appoggio semplice o carrello:



movimenti impediti: y movimenti consentiti: x, θ_z

$m = 1$

esempi:

$n = 3$ $m = 3$ $l = n - m = 0 \Rightarrow$ sistema isostatico.

$n = 3$ $m = 2$ $l = n - m = 1 \Rightarrow$ sistema ipostatico.

$n = 3$ $m = 3+1$ $l = n - m = -1$ (grado di iperstaticità) $\Rightarrow$ sistema iperstatico.

~~Il numero di equazioni~~

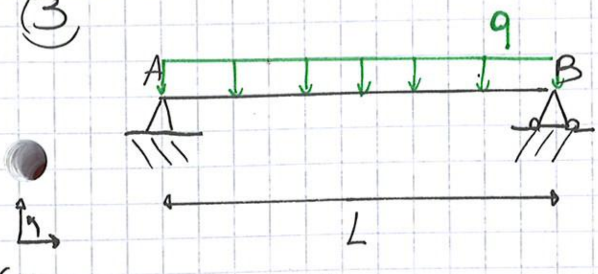
EQUAZIONI DI EQUILIBRIO

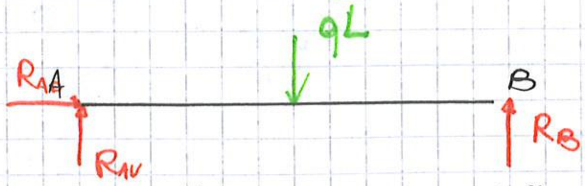
Il numero di equazioni di equilibrio è pari a n.

m equazioni servono per il calcolo delle reazioni vincolari.

l equazioni servono per determinare l'equazione del moto

③

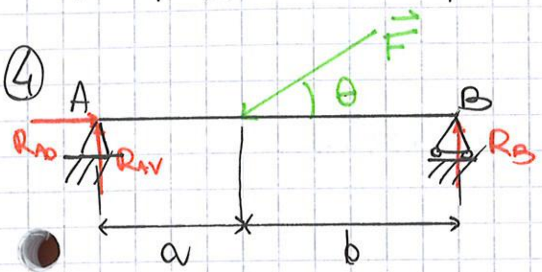




$$\begin{cases} R_{AO} = 0 \\ R_{AV} + R_B - qL = 0 \\ R_B \cancel{L} - q\cancel{L}\frac{L}{2} = 0 \end{cases} \qquad \begin{cases} R_{AV} = q\frac{L}{2} \\ R_B = q\frac{L}{2} \end{cases}$$

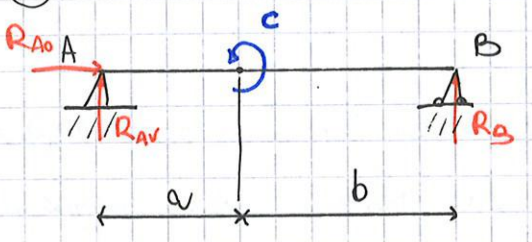
qL è applicato a metà della superficie caricata da q.
Il sistema è simmetrico, quindi il carico sarà diviso equamente.

④



$$\begin{cases} R_{AO} - F\cos\theta = 0 \\ R_{AV} + R_B - F\sin\theta = 0 \\ R_B L - F\sin\theta\, a = 0 \end{cases} \qquad \begin{cases} R_{AO} = F\cos\theta \\ R_{AV} = \frac{F\sin\theta\, b}{L} \\ R_B = \frac{F\sin\theta\, a}{L} \end{cases}$$

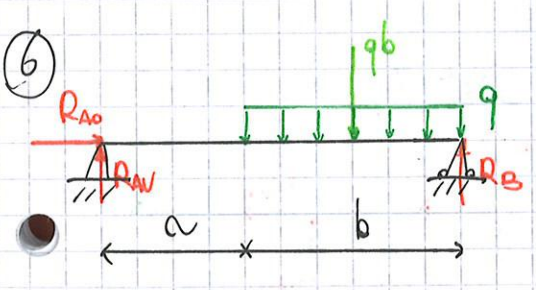
⑤



$$\begin{cases} R_{AO} = 0 \\ R_{AV} + R_B = 0 \\ C + R_B L = 0 \end{cases} \qquad \begin{cases} R_{AV} = \frac{C}{L} \\ R_B = -\frac{C}{L} \end{cases}$$

Le reazioni non dipendono dalla posizione di C.

⑥



$$\begin{cases} R_{AO} = 0 \\ R_{AV} + R_B - qb = 0 \\ R_B L - qb\left(a + \frac{b}{2}\right) = 0 \end{cases} \qquad \begin{cases} R_{AV} = \frac{qb^2}{2L} \\ R_B = \frac{qb}{L}\left(a + \frac{b}{2}\right) \end{cases}$$

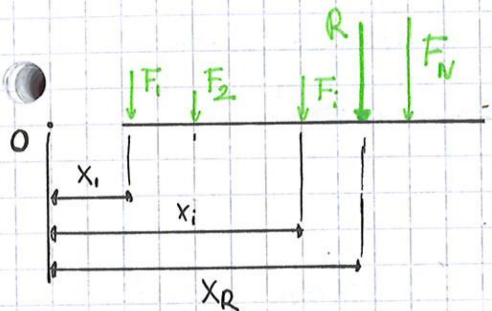
⑦ trave cantilever (a sbalzo)

$$\begin{cases} R_O = F\cos\theta \\ R_V = F\sin\theta \\ M_V = F\sin\theta\, L \end{cases}$$

⑧

$$\begin{cases} R_V = qL \\ R_O = 0 \\ M_V = q\frac{L^2}{2} \end{cases}$$

$$\begin{cases} R_{AO} = -\dfrac{FL}{h} \\ R_{AV} = F - \dfrac{FL}{l} = F\left(\dfrac{l-L}{l}\right) \\ R_{BO} = \dfrac{FL}{h} \\ R_{BV} = \dfrac{FL}{l} \\ R_{CO} = \dfrac{FL}{h} \\ R_{CV} = \dfrac{FL}{l} \end{cases}$$



la Risultante è pari alla somma di tutte le forze.

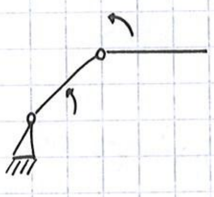
Per poter trovare il punto di applicazione di R occorre eguagliare il momento di R rispetto ad O con la somma dei momenti di ogni singola forza rispetto ad O. Il punto O è scelto a piacere

$$R = \sum_{i=1}^{N} F_i$$

$$x_R R = \sum_{i=1}^{N} x_i F_i \qquad x_R = \frac{\sum x_i F_i}{\sum F_i}$$

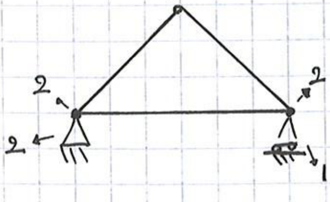
Gradi di vincolo di una cerniera interna

$m = (n° \text{ travi collegate} - 1)\cdot 2$



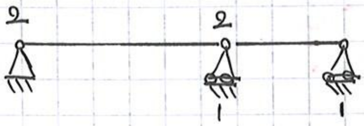
$W = 2\cdot 3 = 6$

$m = 2+2 = 4 \quad l = 2$



$W = 3\cdot 3 = 9$

$m = 2+2+2+2+1 = 9 \quad l = 0$ isostatico

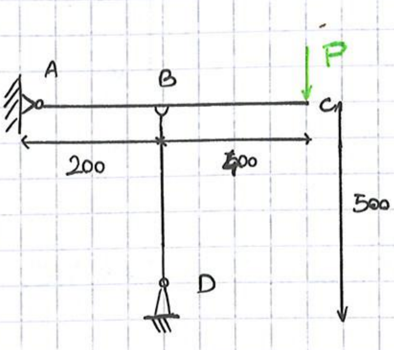


$W = 2\cdot 3 = 6$

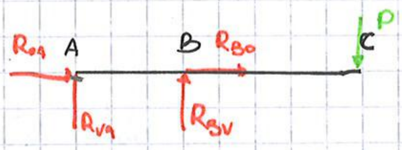
$m = 2 + (1+2) + 1 = 6$

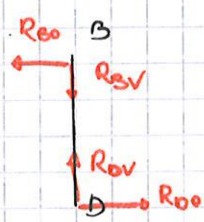
② $\begin{cases} R_{VC} = F \\ R_{OC} = 0 \\ M_C = Fh \end{cases}$ ① $\begin{cases} -R_{OD} - R_{OC} = 0 \\ R_{VD} - R_{VC} - qa = 0 \\ qa\left(c+b+\frac{a}{2}\right) + R_{VC}\,c - M_C + M_D = 0 \end{cases}$ $\begin{cases} R_{OD} = 0 \\ R_{VD} = F + qa \\ M_D = \cancel{Fh} - qa\left(c+b+\frac{a}{2}\right) - F \end{cases}$

④



SR ↰ $w = 6$
$m = 2+2+2 = 6$ isostatico

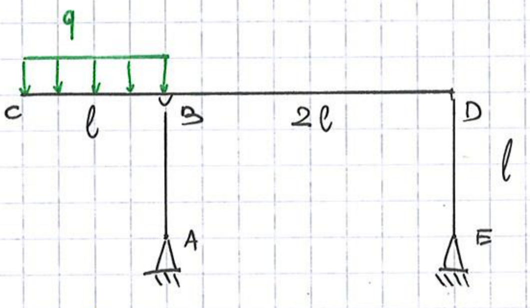




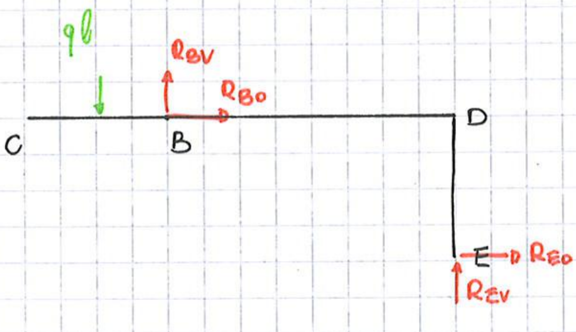
$\begin{cases} R_{AO} + R_{BO} = 0 \\ R_{AV} + R_{BV} - P = 0 \\ -P(600) + R_{BV}(200) \end{cases}$ $\begin{cases} R_{BV} = R_{DV} \\ R_{DO} = R_{BO} \\ R_{BO}(500) = 0 \end{cases}$

$R_{BO} = 0$; $R_{DO} = 0$; $R_{AO} = 0$; $R_{BV} = 3P$; $R_{AV} = -2P$; $R_{DV} = 3P$

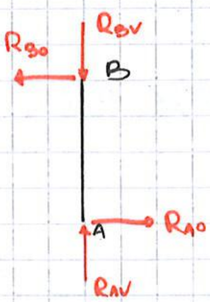
⑤



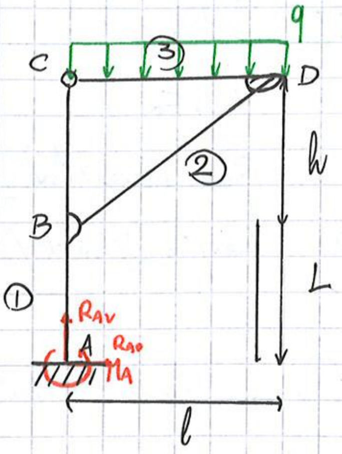
SR ↰ $w = 6$ $m = 2 \cdot 3 = 6$ isostatico



$\begin{cases} R_{EV} + R_{BV} - ql = 0 \\ R_{BO} + R_{EO} = 0 \\ R_{BV}\,2l - ql\left(2l + \frac{l}{2}\right) = 0 \\ R_{AV} = R_{BV} \\ R_{AO} = R_{BO} \\ R_{BO}\,l = 0 \end{cases}$



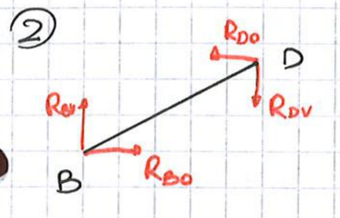
Esercizio



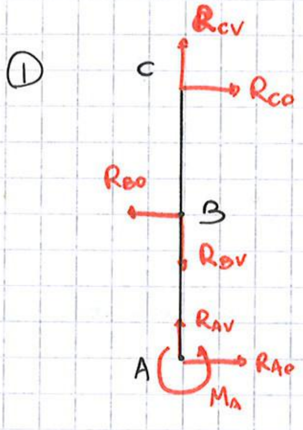
$w = 3+3+3 = 9 \quad m = 2\times3+3 = 9$ isostatico

staccando vincoli:

$$A \cdot \begin{cases} R_{AO} = 0 \\ R_{AV} - q\,l = 0 \\ M_A = q\dfrac{l^2}{2} \end{cases}$$

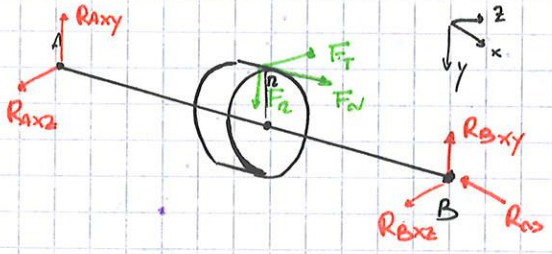


$$② \begin{cases} R_{BV} = R_{DV} \\ R_{BO} = R_{DO} \\ R_{DO}h - R_{DV}l = 0 \end{cases}$$



$$① \begin{cases} R_{CO} = R_{BO} \\ R_{CV} - R_{BV} + R_{AV} = 0 \\ -R_{BO}h + \cancel{R_{AO}}L + M_A = 0 \end{cases}$$

Reazioni vincolari di sistemi spaziali: albero con ruota dentata



$w = 6$

l'albero ruota attorno a x

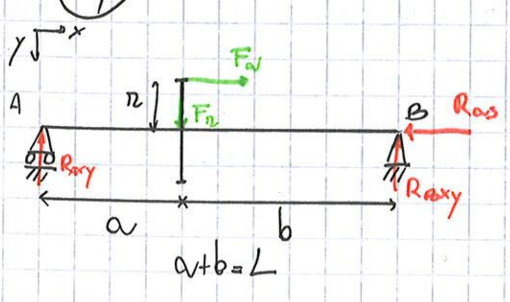
⇓

sist. labile $\quad m = 5 \quad$ equilibri x reazioni

$l = 1 \quad$ equazioni del moto

$\hookrightarrow C_M = F_T r$

piano (xy)



i vincoli sono a "salta"

F_r è sempre diretta verso il centro altrimenti la ruota non ingrana.

F_a c'è solo per ruote elicoidali.

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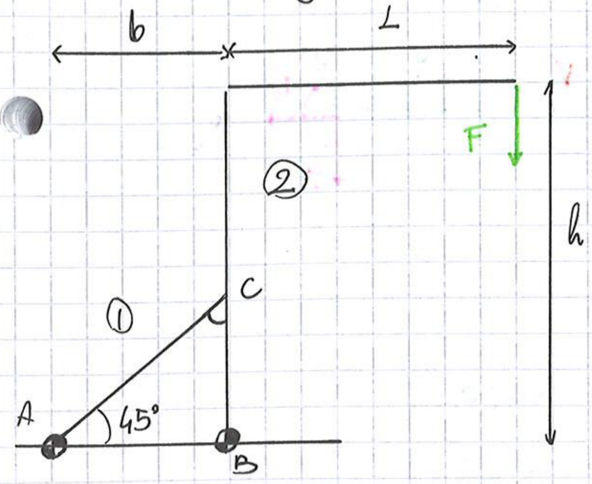
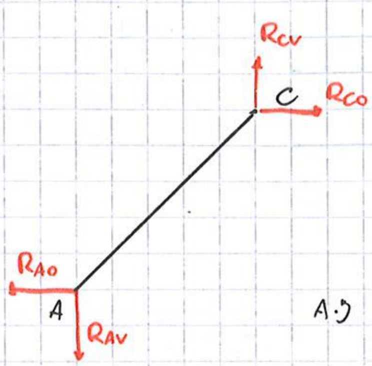
Schema di una gru

$w=6 \quad m=2\times3=6$ isostatico



(1)



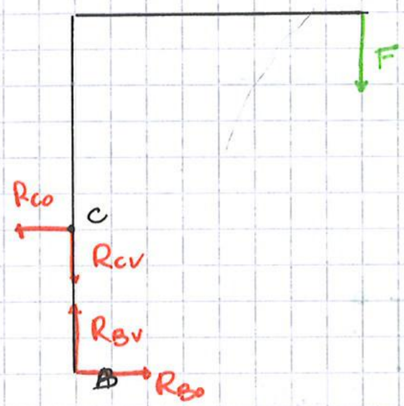
$R_{Co}=R_{Ao}$

$R_{Cv}=R_{Av}$

$A{:}\ R_{Cv}b-R_{Co}b=0$

$\hookrightarrow R_{Cv}=R_{Co}$

(2)



$R_{Bo}=R_{Co}$

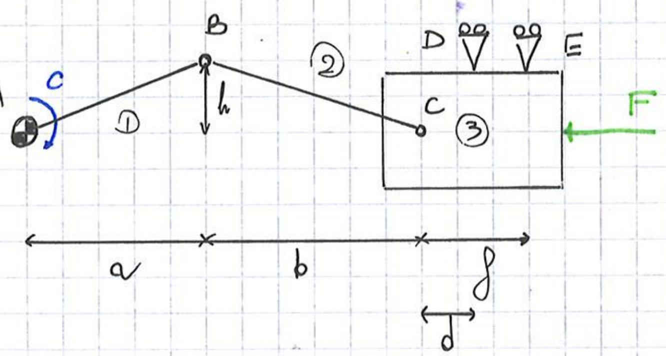
$R_{Bv}=R_{Cv}+F$

$B{:}\ R_{Co}b-FL=0$

$R_{Bv}=\left(\frac{L}{b}+1\right)F$

$R_{Ao}=R_{Av}=R_{Co}=R_{Cv}=R_{Bo}=F\frac{L}{b}$

Schema biella-manovella

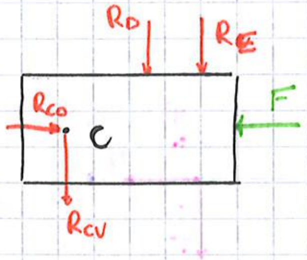


$w=9$

$m=2+2+2+1+1=8$

$l=1$

(3)



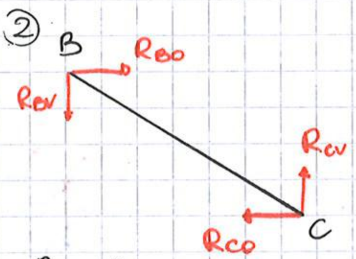
$R_{Co}=F$

$R_{Cv}+R_D+R_E=0$

$C{:}\ -R_D d-R_E l=0$

$R_E=\frac{Fhd}{b(d-l)} \quad R_D=\frac{Fhl}{b(l-d)}$

(2)



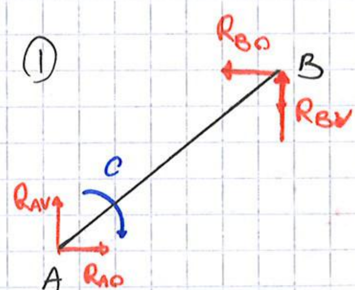
$R_{Bo}=R_{Co}=F$

$R_{Bv}=R_{Cv}$

$B{:}\ R_{Cv}b-R_{Co}h=0$

$R_{Bv}=R_{Cv}=R_{Co}\frac{h}{b}=F\frac{h}{b}$

(1)



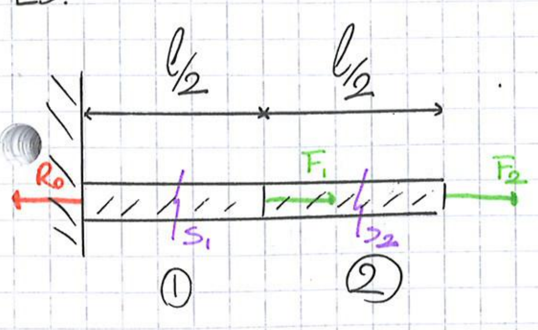
$R_{Ao}=R_{Bo}=F$

$-R_{Bv}=R_{Av}=-\frac{Fh}{b}$

$A{:}\ -C+R_{Bo}h+R_{Bv}a=0$

$C=Fh\left(1+\frac{a}{b}\right)$ eq. moto

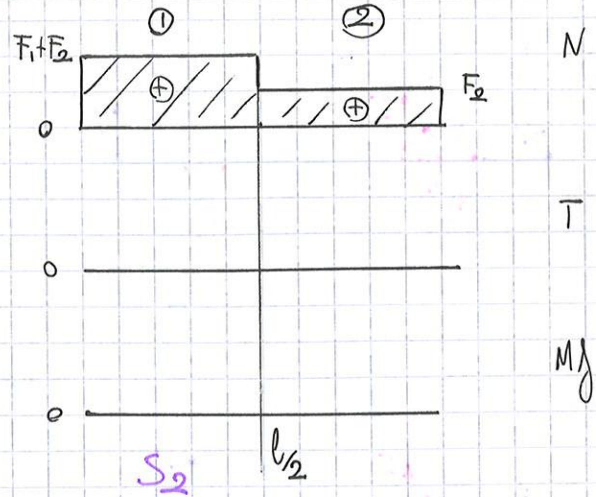
ES:



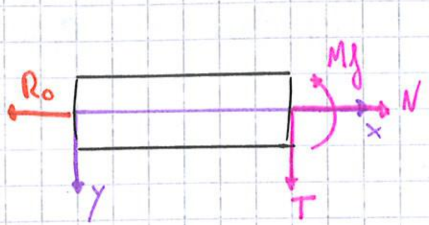
$R_0 = F_1 + F_2$

$R_V = 0$

$M_V = 0$



S_1

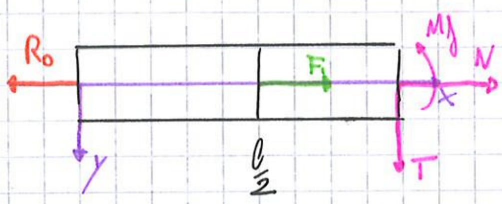


$N - R_0 = 0 \quad N = F_1 + F_2$

$T = 0$

$M_f = 0$

S_2



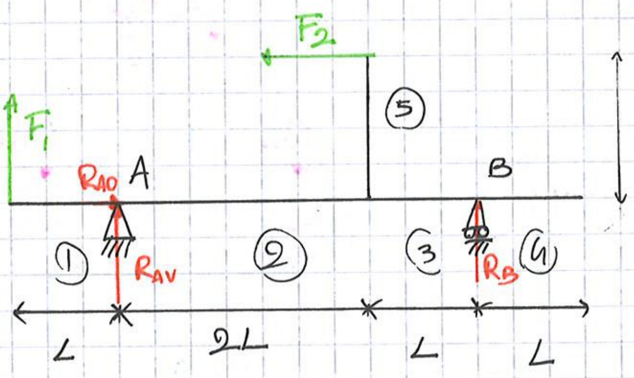
$N - R_0 + F_1 = 0 \quad N = R_0 - F_1 = F_2$

$T = 0$

$M_f = 0$

esercizi:

①

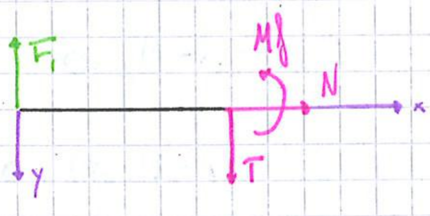


$R_{AV} = F_2 \frac{h}{3L} - \frac{4}{3}F_1 = -91.67\ N$

$R_{AO} = F_2 = 120\ N$

$R_B = \frac{F_1 L}{3L} - \frac{F_2 h}{3L} = 11.67\ N$

①



$N = 0$

$T = F_1 = 80N$

$M_f = F_1 x \quad x \in [0, L]$ → 32Nm

②

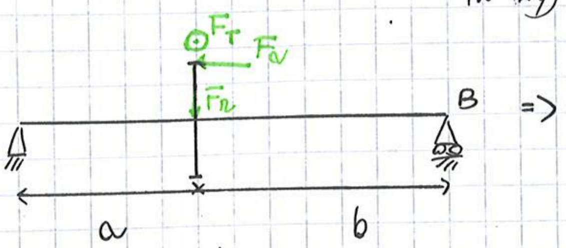
$N = -R_{AO} = -120N$

$T = R_{AV} + F_1 = 11.67N$

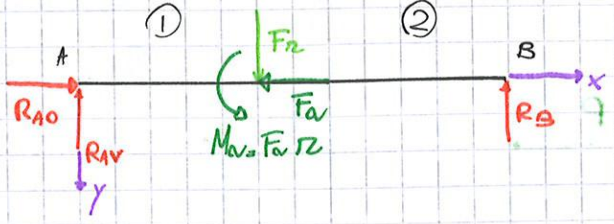
$M_f = R_{AV}(x-L) + F_1 x \quad x \in [L, 3L]$ → 32Nm, 22.67Nm

② albero con ruota dentata: sforzi

in (xy)



=>



$R_{AO} = F_a$; $R_{AV} = \dfrac{F_r b}{L} + \dfrac{M_a}{L}$; $R_B = \dfrac{F_r a}{L} - \dfrac{M_a}{L}$

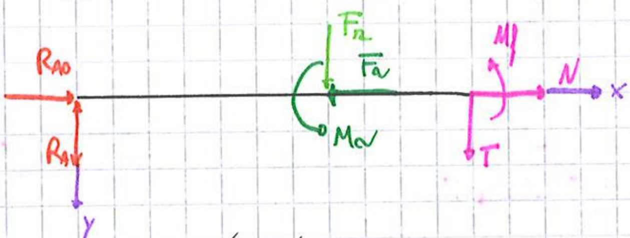
①

$N = -R_{AO}$

$T = R_{AV}$

$M_f = R_{AV}\, x \qquad x \in [0, a]$

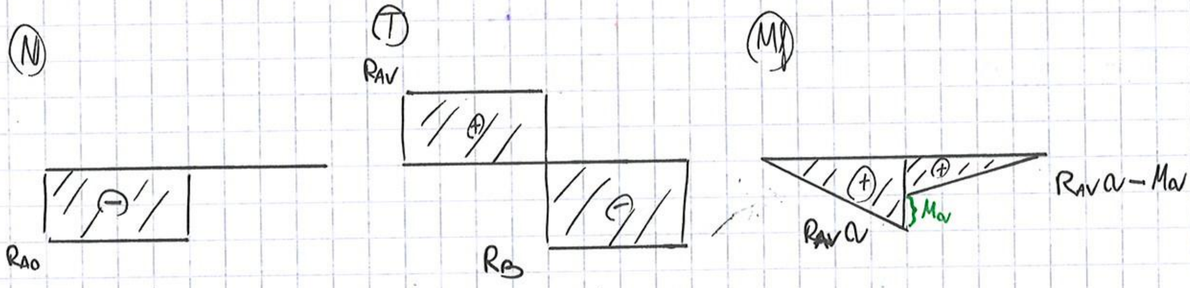
②



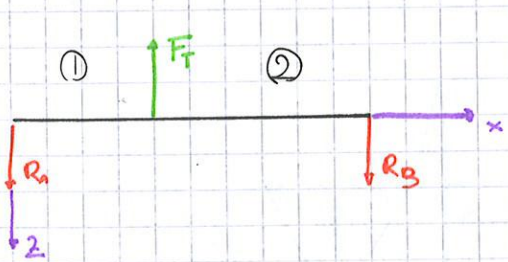
$N = \cancel{-R_{AO}} + \cancel{F_a} = 0$

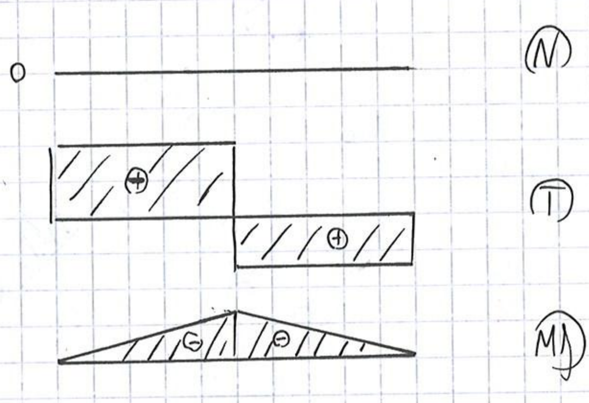
$T = R_{AV} - F_r$

$M_f = R_{AV}\, x - F_r (x - a) - M_a \qquad x \in [a, L]$

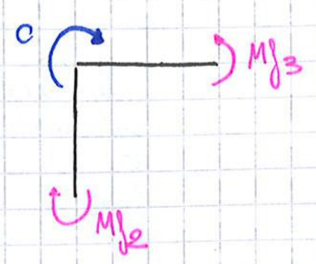


in (xz)





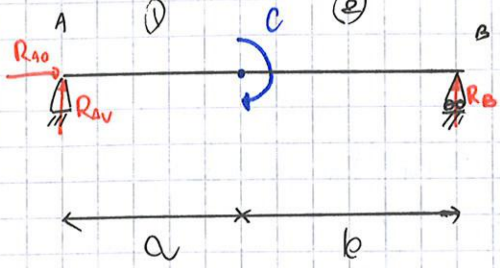
equilibrio al nodo



$$C + Mf_2 - Mf_3 = 0$$

$$600\ Nm - 10\ KNm + 9400\ Nm = 0 \checkmark$$

(4)

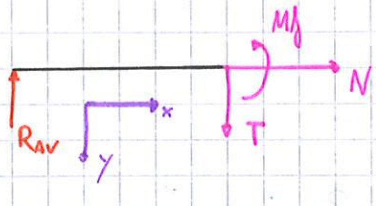


$$R_B = \frac{C}{L}$$

$$R_{AV} = -\frac{C}{L}$$

$$R_{AO} = 0$$

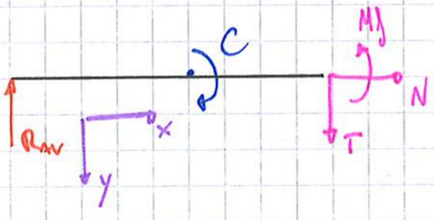
(1)



$N = 0$; $T = R_{AV} = -\frac{C}{L}$

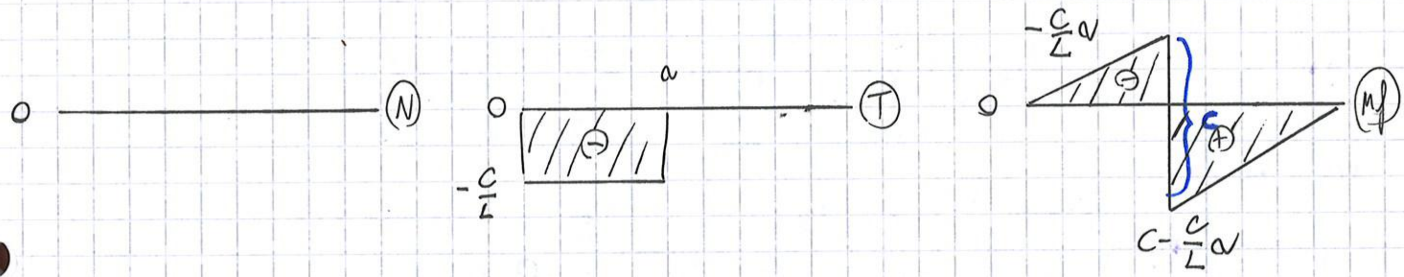
$Mf = + R_{AV} x = -\frac{C}{L} x \qquad x \in [0, a]$

(2)

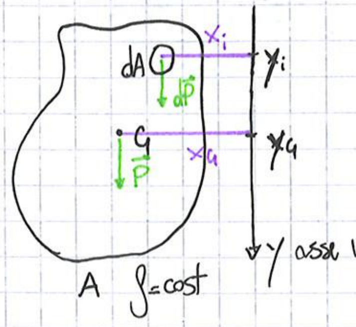


$N = 0$; $T = R_{AV}$

$Mf = R_{AV} x + C = -\frac{C}{L} x + C \qquad x \in [a, L]$



Geometria delle aree: Baricentri e momenti statici.



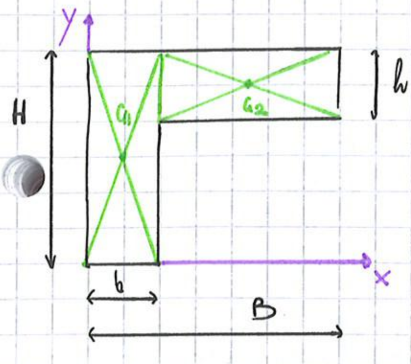
$$|P| = \int_A \rho \vec{g}\, dA = \rho \vec{g} A$$

eq. momenti: $\int_A \cancel{\rho \vec{g}}\, x\, dA = \cancel{\rho \vec{g}} A x_G \qquad \int_A x\, dA = x_G A$

S_y momento statico rispetto ad y

$$S_y = \int_A x\, dA = x_G A \qquad x_G = \frac{S_y}{A} = \frac{\int_A x\, dA}{A}$$

$$S_x = \int_A y\, dA = y_G A \qquad y_G = \frac{S_x}{A} = \frac{\int_A y\, dA}{A}$$



metodo di somma

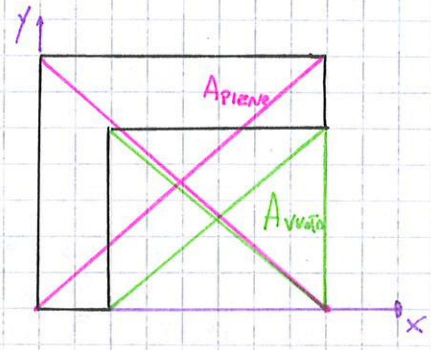
asse ∀

$$Y_G = \frac{S_{\boxtimes}}{A_{TOT}} = \frac{S_{x_1}+S_{x_2}}{A_1+A_2}$$

$$S_x = S_{x_1}+S_{x_2} = A_1\cdot\underbrace{\frac{H}{2}}_{Y_{G_1}} + A_2\cdot\underbrace{\frac{H+h}{2}}_{Y_{G_2}}$$

$$S_x = bH\cdot\frac{H}{2} + (B-b)(H-h)\frac{H+h}{2} = \frac{bH^2}{2} + (B-b)(H-h)(H+h)\frac{1}{2}$$

metodo di sottrazione (usabile solo se c'è un asse comune)

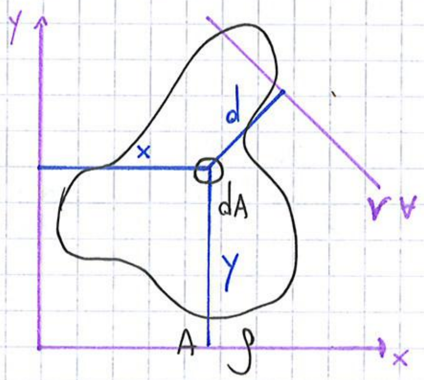


$A_V = (B-b)h$ $\qquad A_{TOT} = A_p - A_V$

$A_p = HB$ $\qquad S_x = S_{xp} - S_{xV} = A_p\frac{H}{2} - A_V\frac{h}{2} =$

$$= \frac{BH^2}{2} - \frac{(B-b)h^2}{2}$$

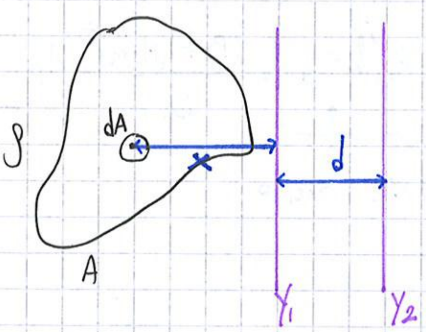
Geometria delle aree: momento d'inerzia.



$I_x = \int_A y^2 dA \qquad I_y = \int_A x^2 dA$

$I_r = \int d^2 dA$

Teorema di trasposizione



$$I_{y_1} = \int_A x^2 dA$$

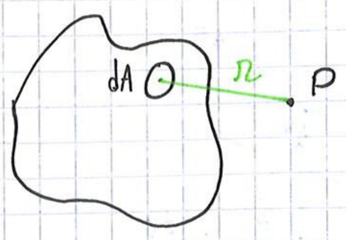
$$I_{y_2} = \int_A (x+d)^2 dA = \int_A x^2 dA + \int_A d^2 dA + \int_A 2dx\, dA$$

$$I_{y_2} = I_{y_1} + d^2 A + 2d S_{y_1}$$

$$\Downarrow$$

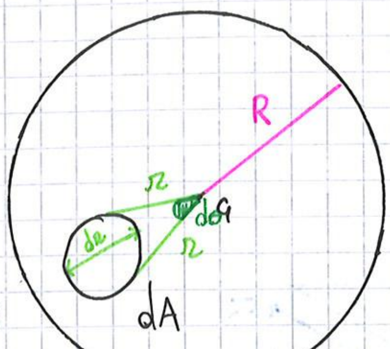
$I_y = I_{y_G} + Y_G^2 A$ il momento statico si annulla se y_1 è scelto baricentrico

Momento d'inerzia polare



$$I_p = \int_A r^2 \, dA$$

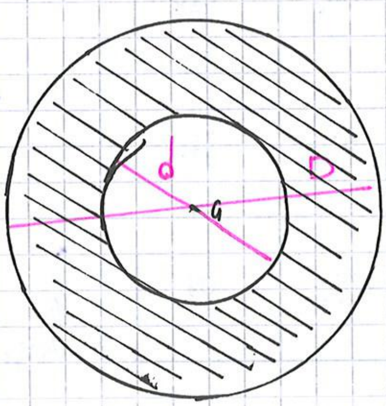
Momento d'inerzia polare sezione circolare e sezione circolare cava

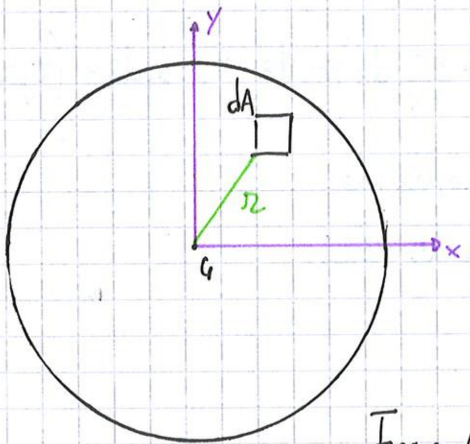


$$2R = D$$

$$I_p = \int_0^{2\pi}\int_0^{R} r^2 \, r \, d\theta \, dr = 2\pi \frac{R^4}{4} = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

$$I_p = \pi \frac{D^4}{32} - \pi \frac{d^4}{32} = \frac{\pi}{32}(D^4 - d^4)$$





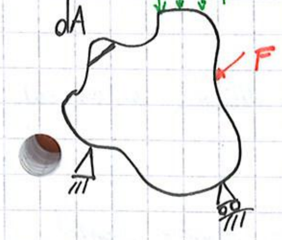
$$I_x = \int_A y^2 \, dA \qquad I_p = \int_A r^2 \, dA = \int_A x^2 + y^2 \, dA$$

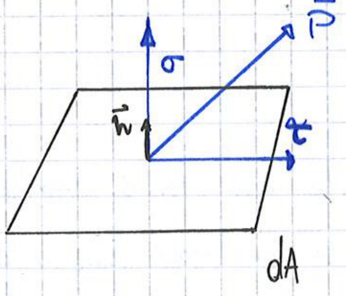
$$I_p = \int_A (x^2+y^2) \, dA = 2\int_A x^2 \, dA = 2\int_A y^2 \, dA$$

$$I_{asse\ diametrale} = \frac{I_p}{2} = \rightarrow \frac{\pi D^4}{64}$$

$$\frac{\pi (D^4 - d^4)}{64}$$

Tensioni (stress)





p tensione generica $[N/mm^2] = [MPa]$

$$p = \lim_{dA \to 0} \frac{dF_i}{dA} \leftarrow F_{interna}$$

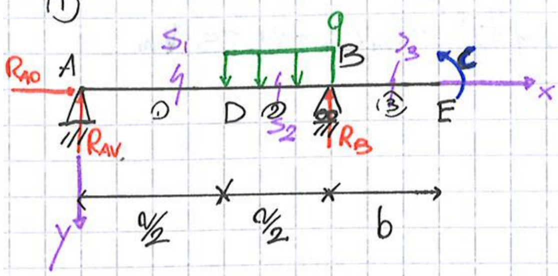
σ tensione normale (normal stress)

τ tensione tangente (shear stress)

$$\tau_{yz}\, dz dx \cdot dy - \tau_{zy}\, dx dy \cdot dz = 0 \implies \tau_{yz} = \tau_{zy}$$

esercizi:

①



$a = 0.8\ m$

$b = 0.3\ m$

$q = 80\ N/m$

$C = 100\ N \cdot m$

$n = 3$

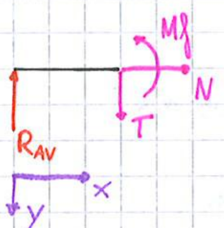
$m = 2+1 = 3$ isostatico

$R_{A0} = 0$

$R_{AV} + R_{VB} - q\frac{a}{2} = 0$ $\qquad R_B = q\frac{a}{2} - R_{AV} = -101\ N$

$C + q\frac{a}{2}\frac{a}{4} - R_{VA}\, a = 0$ $\qquad R_{AV} = \left(C + q\frac{a^2}{8}\right)\frac{1}{a} = 133\ N$

S_1 $x \in [0, \frac{a}{2}]$

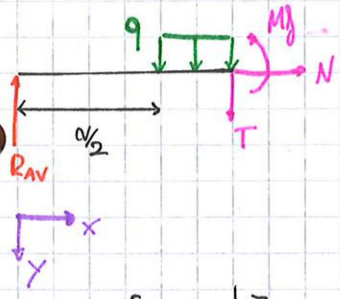


$N = 0$

$R_{AV} - T = 0$ $\qquad T = R_{AV} = 133\ N$

$Mf - R_{AV}\, x = 0$ $\qquad Mf = R_{AV}\, x = 133\, x$ $\begin{cases} x = 0 & 0 \\ x = \frac{a}{2} & 53{,}2\ N \cdot m \end{cases}$

S_2 $x \in [\frac{a}{2}, a]$

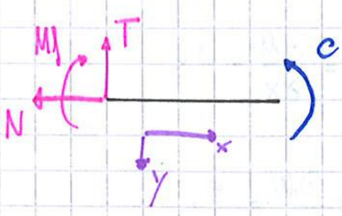


$N = 0$

$R_{AV} - q\left(x - \frac{a}{2}\right) = T$ $\begin{cases} x = \frac{a}{2} & 133\ N \\ x = a & 101\ N \end{cases}$

$Mf = R_{AV}\, x - \frac{1}{2} q \left(x - \frac{a}{2}\right)^2$ $\begin{cases} x = \frac{a}{2} & 53.2\ N \cdot m \\ x = a & 100\ N \cdot m \end{cases}$

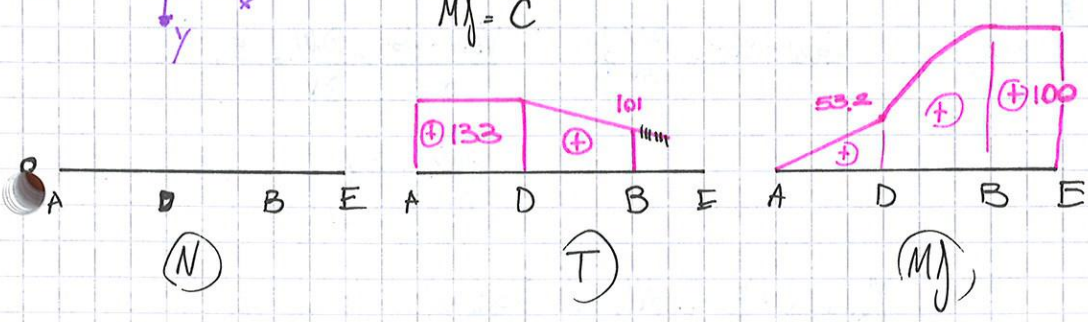
S_3 $x \in [a, a+b]$



$N = 0$

$T = 0$

$Mf = C$



$\operatorname{tg}\beta \simeq \beta$ per piccole deformazioni

$dy \operatorname{tg}\beta = u_{BB'} - u_{OO'}$ $\quad \operatorname{tg}\beta = \frac{u_{BB'} - u_{OO'}}{dy}$ $\quad \beta = \frac{\partial u}{\partial y}$

$dx \operatorname{tg}\alpha = v_{AA'} - v_{OO'}$ $\quad \alpha = \frac{\partial v}{\partial x}$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

Legame Tensione - deformazione

$\sigma_i \rightsquigarrow \varepsilon_i$

$\tau_{ij} \rightsquigarrow \gamma_{ij}$

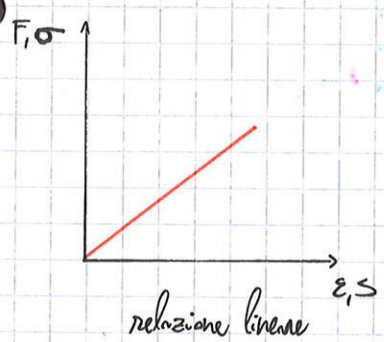
comportamenti della materia:

A. Elastico

B. Plastico

C. Viscoplastico

(A) scaricando la struttura ritorna nello stato iniziale

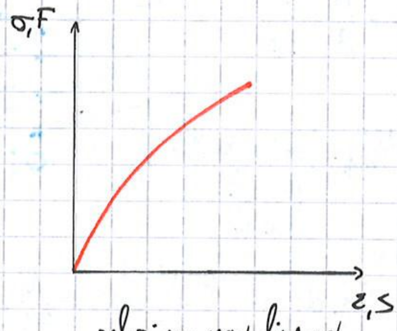


relazione lineare

legge di Hooke

$\sigma = E\varepsilon$

E: dipende dal materiale



relazione non lineare

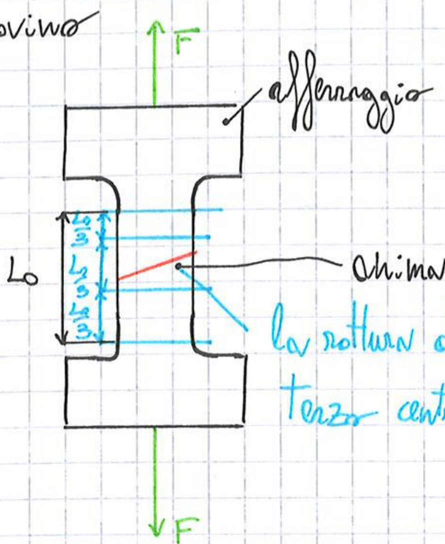
Teoria di Hertz

(B)

Caricando la struttura si provocano deformazioni permanenti

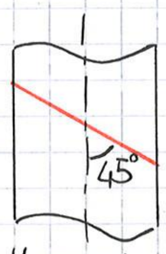
nel caso di sforzo normale: $\varepsilon_y = \nu \varepsilon_x$
con ν (o μ o m) coefficiente di Poisson (0.3 per gli acciai)

Provino

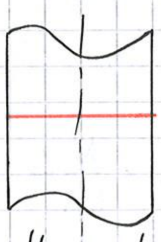


anima o tratto calibrato: la sezione minore assicura la rottura nell'anima.

la rottura deve avvenire nel terzo centrale per essere valida

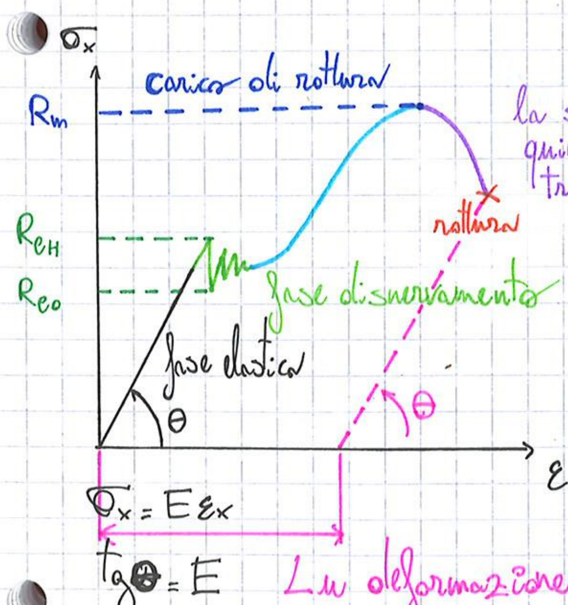


rottura dei materiali duttili



rottura dei materiali fragili

Acciaio duttile con evidente snervamento



la sezione si restringe a causa dell'allungamento, le σ_x sono quindi applicate su una superficie minore di quella considerata e quindi trascurate per la prova.

R_{eH} carico di snervamento superiore

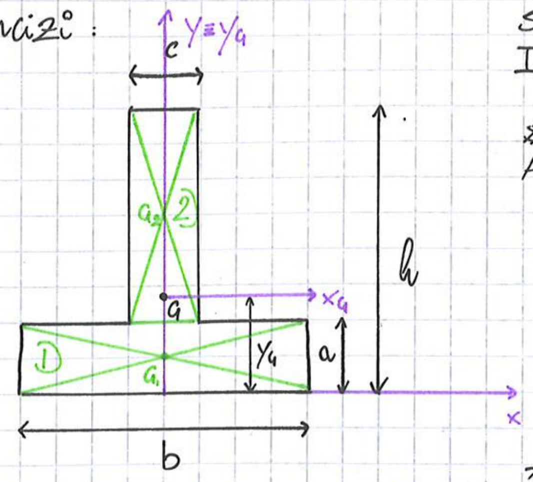
R_{eo} carico di snervamento inferiore

$$A\% = \frac{L_u - L_0}{L_0}$$

se $A\% > 5$ è duttile
se $A\% < 5$ è fragile

esercizi:

1



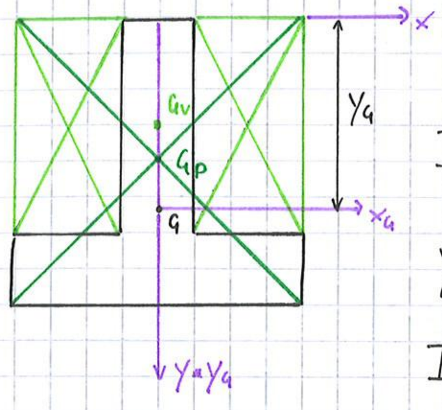
somma

$$I_{y_G} = I_{y_{G1}} + I_{y_{G2}} = \frac{ab^3}{12} + \frac{(h-a)c^3}{12}$$

$$A = A_1 + A_2$$

$$y_G = \frac{S_x}{A} = \frac{S_{x1}+S_{x2}}{A} = \left(A_1\frac{a}{2} + A_2\left(a+\frac{h-a}{2}\right)\right)\frac{1}{A}$$

$$I_{xG} = \left[\frac{ba^3}{12} + A_1\left(y_G-\frac{a}{2}\right)^2\right] + \left[\frac{c(h-a)^3}{12} + A\left(\frac{a+h}{2}-y_G\right)^2\right]$$



sottrazione

$$A = A_p - A_v = hb - (b-c)(h-a)$$

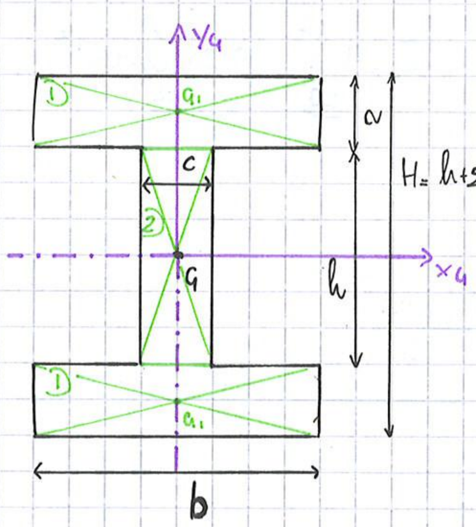
$$I_{y_G} = I_{y_{Gp}} - I_{y_{Gv}} = \frac{hb^3}{12} - 2\left[\frac{(h-a)}{12}\left(\frac{b-c}{2}\right)^3 + A_v\left(\frac{c}{2}+\frac{1}{2}\left(\frac{b}{2}-\frac{c}{2}\right)\right)^2\right]$$

$$y_G = \frac{S_x}{A} = \frac{S_{xp}-S_{xv}}{A} = \frac{1}{A}\left(A_p\frac{h}{2} - A_v\left(h-\frac{a}{2}\right)\right)$$

$$I_x = I_{xp} - I_{xv} = \frac{bh^3}{3} - \frac{(b-c)(h-a)^3}{3}$$

$I_x = I_{xG} + A y_G^2$ $\qquad$ $I_{xG} = I_x - A y_G^2$

2



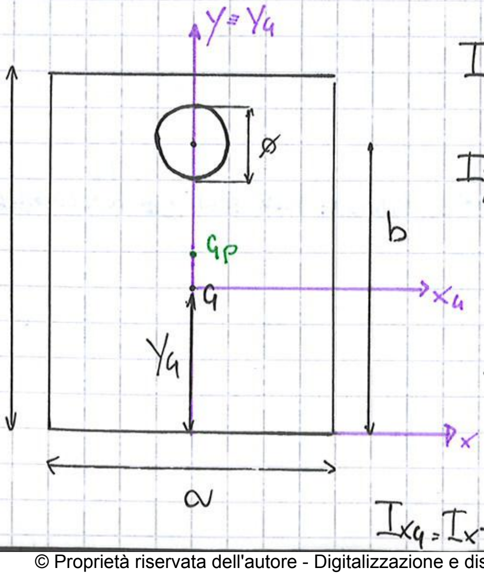
$$A = 2A_1 + A_2 = 2ba + hc$$

$$L = A_p - A_v = Hb - h(bc)$$

$$I_{xG} = I_{xGp} - I_{xGv} = \frac{bH^3}{12} - \frac{(b-c)h^3}{12}$$

$$I_{y_G} = 2I_{y_{G1}} + I_{y_{G2}} = 2\frac{ab^3}{12} + \frac{hc^3}{12}$$

3



$$I_d = \frac{\pi\phi^4}{64}$$

$$I_{y_G} = \frac{ha^3}{12} - \frac{\pi\phi^4}{64}$$

$$y_G = \frac{S_x}{A} = \left(A_{rett}\frac{h}{2} - A_{foro}\, b\right)\frac{1}{A_{rett}-A_{foro}}$$

$$I_{xG} = \left[\frac{ah^3}{12} + A_{rett}\left(\frac{h}{2}-y_G\right)^2\right] - \left[\frac{\phi^4\pi}{64} + A_{foro}(b-y_G)^2\right]$$

oppure

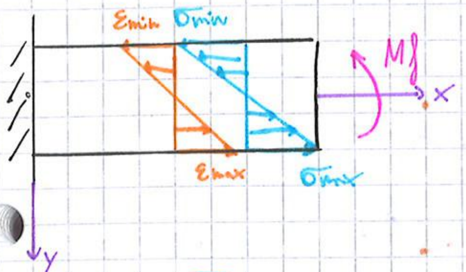
$$I_{xG} = I_x - A y_G^2 \rightarrow I_x = \frac{ah^3}{3} - \left[\frac{\pi\phi^4}{64} + A_{foro}\, b^2\right]$$

$$\varepsilon_x = \frac{dl'-dl}{dl} = \frac{1}{(r+y)dl}\left((r'+y)dl' - (r+y)dl\right) = \frac{1}{(r+y)dl}\left(\cancel{r'dl'} + y\,dl' - \cancel{r\,dl} - y\,dl\right) =$$

$$= \frac{y}{r+y}\,\frac{dl'-dl}{dl} = \frac{y}{r+y}\left(\frac{dl'}{dl}-1\right) = \frac{y}{r+y}\left(\frac{r}{r'}-1\right) = \frac{yr}{r+y}\left(\frac{1}{r'}-\frac{1}{r}\right)$$

se $r \to \infty$ trave rettilinea : $r' = R$

$$\varepsilon_x = \frac{y}{1+\underbrace{\frac{y}{r}}_{\to 0}}\,\frac{1}{R} = \frac{y}{R}$$ relazione lineare

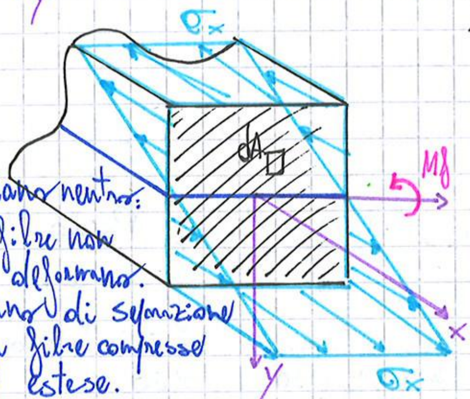


$$\varepsilon_x = \frac{y}{R}$$

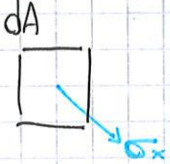
$$\sigma_x = E\varepsilon_x = \frac{E}{R}y$$

$$\int_A \sigma_x\, dA = 0$$

$$\int_A \frac{E}{R} y\, dA = \frac{E}{R}\int_A y\, dA = 0$$ $\nearrow S_z$ asse z passante per G



piano neutro: le fibre non si deformano. Piano di separazione tra fibre compresse ed estese.

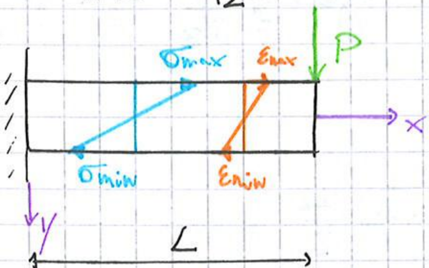


$$M_f = \int_A \sigma_x\, dA\, \underset{\frac{\sigma_x}{y}}{y} = \frac{E}{R}\int_A y^2\, dA = \frac{E}{R} I_z$$

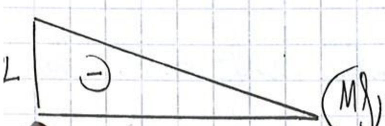
$$\sigma_x = \frac{M_f\, y}{I_z}$$

$$\sigma_{x\,max}\Big|_{y=\frac{h}{2}} = \frac{M_f}{\frac{bh^3}{12}}\,\frac{h}{2} = \frac{M_f}{\frac{bh^2}{6}}$$ $\rightarrow W_f$ modulo di resistenza a flessione per sezione rettangolare

$$\sigma_{x\,min}\Big|_{y=-\frac{h}{2}} = \frac{M_f}{\frac{bh^3}{12}}\left(-\frac{h}{2}\right) = -\frac{M_f}{W_f}$$



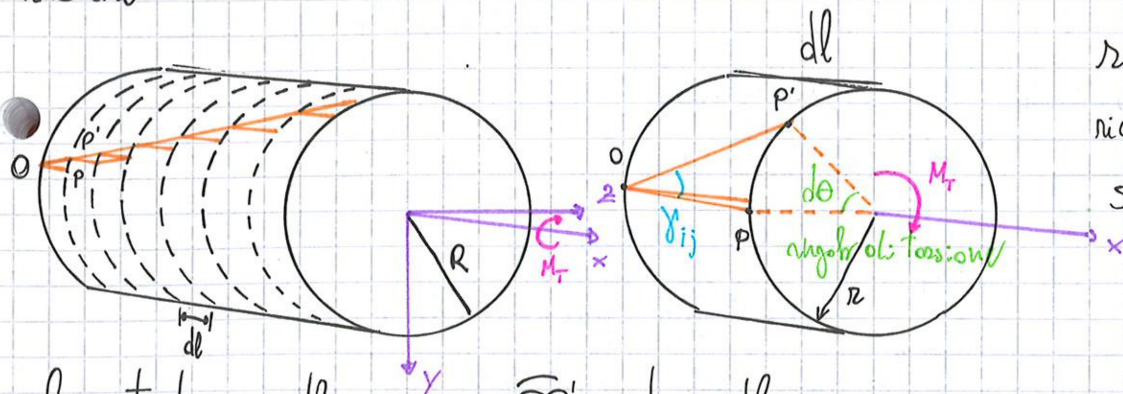
$$\sigma_x = \frac{M_f\, y}{I_z} = \frac{-PL}{\frac{bh^3}{12}}\cdot\left(\frac{h}{2}\right)$$



Per sezioni circolari

$$\sigma_{x\,max} = \frac{M_f}{I_z}\,\frac{D}{2} = \frac{M_f}{\frac{\pi D^4}{64}}\,\frac{D}{2} = \frac{M_f}{\frac{\pi D^3}{32}}$$ $\rightarrow W_0$ modulo resistenza a flessione

Torsione



r è un raggio generico, anche minore della sezione, tanto l'effetto della torsione è identico

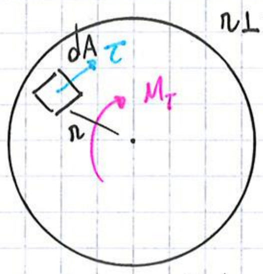
lo studio su dl è estendibile su l

$$\overline{PP'} = r\,d\theta = \gamma\,dl$$

$$\frac{d\theta}{dl} = \frac{\gamma}{r}$$

$\gamma = \frac{\tau}{G}$ → G modulo di elasticità tangenziale $G = \frac{E}{2(1+\nu)}$ → ν coeff. di Poisson

$r \perp \tau$



$$\frac{d\theta}{dl} = \frac{\tau}{Gr}$$

$$M_T = \int_A \tau\, dA\, r = \int_A \frac{d\theta}{dl} G r \cdot r \cdot 2\pi r\, dr$$

forza; tensione; braccio; π; $\int_0^{2\pi} r\,d\theta\,dr$ area anello infinitesimo

M_T è la risultante della distribuzione delle τ

$$M_T = \frac{d\theta}{dl} G 2\pi \int_0^R r^3 dr = \frac{d\theta}{dl} G 2\pi \frac{R^4}{4}$$

$$M_T = \frac{d\theta}{dl} G \frac{\pi R^4}{2} = \frac{d\theta}{dl} G I_p$$

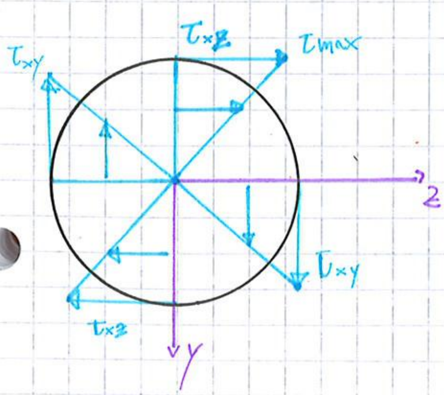
$I_{polare} = \frac{\pi D^4}{32}$

$$\frac{d\theta}{dl} = \frac{\tau}{Gr} \qquad \frac{d\theta}{dl} G = \frac{M_T}{I_p}$$

$$M_T = \frac{\tau}{r} I_p \Rightarrow \tau = \frac{M_T}{I_p} r$$

$$\gamma = \frac{\tau}{G} = \frac{M_T}{G I_p} r$$

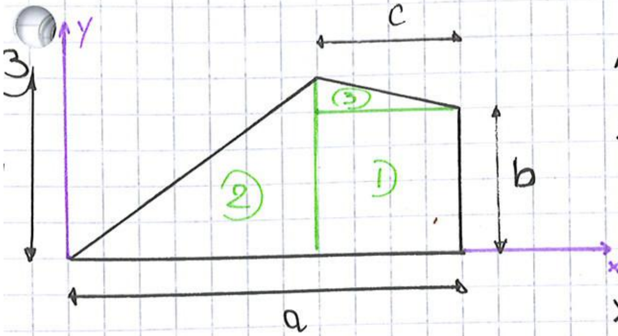
$$\frac{d\theta}{dl} = \frac{M_T}{I_p}\frac{r}{Gr} = \frac{M_T}{I_p G} \Rightarrow \Delta\theta = \frac{M_T}{I_p G} l$$



$$\tau_{max} = \frac{M_T}{\frac{\pi D^4}{32}} \cdot \frac{D}{2} = \frac{M_T}{\pi D^3} 16$$

$\frac{\pi D^3}{16} = W_T$ modulo di resistenza a torsione.

$$I_{xG} = \sum_{k=1}^{3} I_{xGk} + A_k (y_{Gk} - y_G)^2$$
$$I_{yG} = \sum_{k=1}^{3} I_{yGk} + A_k (x_{Gk} - x_G)^2$$



$A_1 = bc$

$A_2 = (a-c)\, h/2$

$A_3 = c(h-b)\, 1/2$

$\frac{2}{3}(a-c)$?

$x_{G1} = a - \frac{c}{2}$ $\quad x_{G2} = \frac{a-c}{3}$ $\quad x_{G3} = \frac{a-c}{} + \frac{c}{3}$

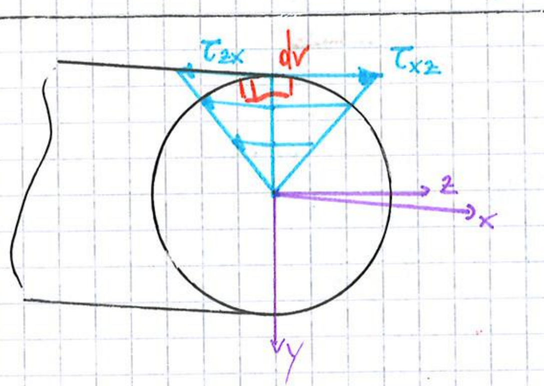
$y_{G1} = \frac{b}{2}$ $\quad y_{G2} = h/3$ $\quad y_{G3} = \frac{h-b}{3}$

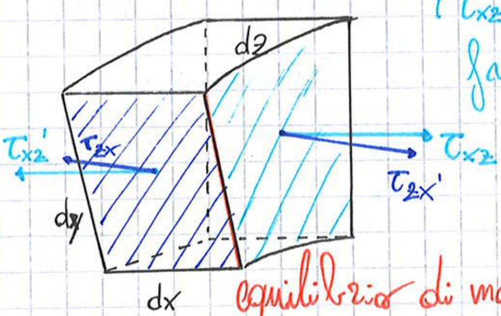
$I_{xG1} = \frac{b^3 c}{12}$ $\quad I_{yG1} = \frac{bc^3}{12}$

$I_{xG2} = \frac{(a-c)h^3}{36}$ $\quad I_{yG2} = \frac{h(a-c)^3}{36}$

$I_{xG3} = \frac{c(h-b)^3}{36}$ $\quad I_{yG3} = \frac{c^3(h-b)}{36}$

$$I_{xG} = \sum_{k=1}^{3} \left(I_{xGk} + A_k (y_{Gk} - y_G)^2\right)$$
$$I_{yG} = \sum_{k=1}^{3} \left(I_{yGk} + A_k (x_{Gk} - x_G)^2\right)$$





i τ_{xz} sono applicati sulle facce frontale e posteriore

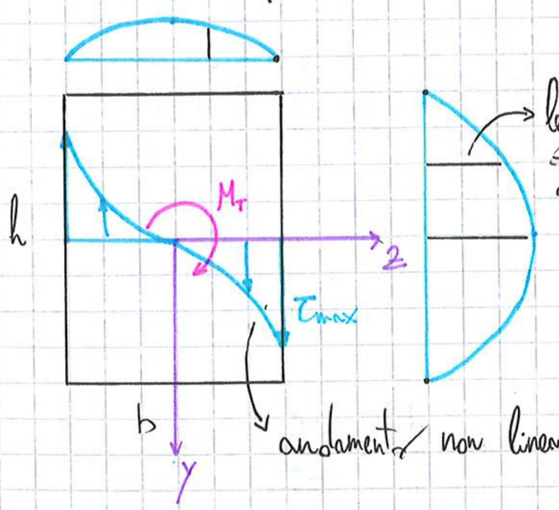
i τ_{zx} sono applicati sulle facce laterali

equilibrio di momento rispetto allo spigolo:

$\tau'_{xz}\, \cancel{dz}\, dy\, \cancel{dx} - \tau'_{zx}\, dx\, y\, dz = 0$

$\tau'_{xz} = \tau'_{zx}$ ← dim 2

Momento torcente per sezioni non circolari



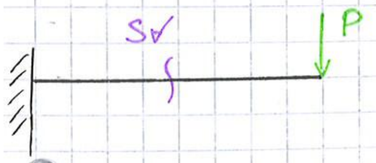
le altezze sono proporzionali al modulo di τ lungo il bordo.

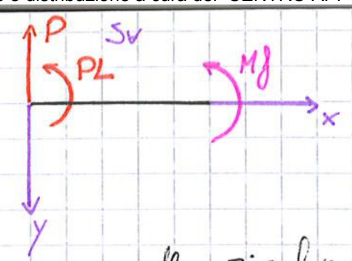
Negli spigoli $\tau = 0$ ed è max a metà lato.

andamento non lineare: $\tau_{max} = \frac{M_T}{\alpha b^2 h} \sim W_t$

α tabulato in base a h/b:

$\cdot\ h/b = 1 \quad \alpha = 0{,}208$; $\ h/b = 2 \quad \alpha = 0{,}246$





$Mf = Px - PL = P(x-L)$

$$\frac{d^2y}{dx^2} = \frac{P(L-x)}{EI_z}$$

cost integrazione

$$\frac{dy}{dx} = -\frac{P}{EI_z}\frac{(L-x)^2}{2} + A \quad (1)$$

$$y = \frac{P}{EI_z}\frac{(L-x)^3}{6} + Ax + B \quad (2)$$

per risolvere servono le condizioni al contorno determinate dal dominio (vincoli, continuità tra campate)

$$\begin{cases} \text{per } x=0 & \frac{dy}{dx}=0 \\ \text{per } x=0 & y=0 \end{cases}$$

all'incastro non c'è pendenza nè deformazione

(1) $0 = -\frac{P}{2EI_z}L^2 + A \qquad A = \frac{PL^2}{2EI_z}$

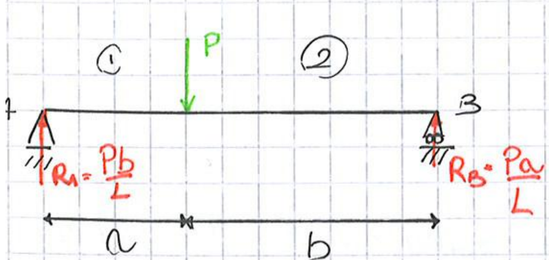
(2) $0 = \frac{P}{6EI_z}L^3 + B \qquad B = -\frac{PL^3}{6EI_z}$

eq. variazione pendenza: $\frac{dy}{dx} = -\frac{P}{2EI_z}(L-x)^2 + \frac{PL^2}{2EI_z}$

eq. deformazione: $y = \frac{P(L-x)^3}{6EI_z} + \frac{PL^2}{2EI_z}x - \frac{PL^3}{6EI_z}$

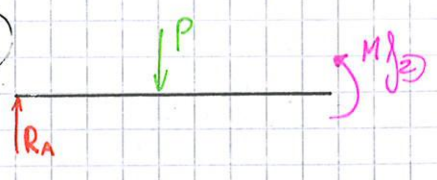
per $x=L \rightarrow y_{max} = \text{freccia} = \frac{PL^3}{2EI_z} - \frac{PL^3}{6EI_z} = \frac{PL^3}{3EI_z}$

per $x=0 \rightarrow y=0$



(1) $Mf_1 = \frac{Pb}{L}x$

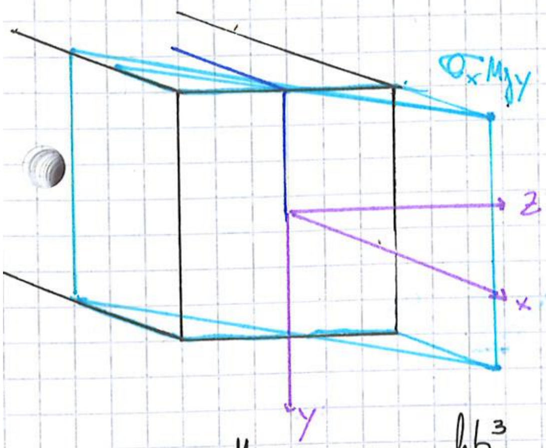
(2)



$Mf_2 = \frac{Pb}{L}x - P(x-a)$

(1) $\frac{d^2y}{dx^2} = -\frac{Pbx}{EI_zL} \qquad y = -\frac{Pbx^3}{6EI_zL} + A_1x + B_1$

$\frac{dy}{dx} = -\frac{Pbx^2}{2EI_zL} + A_1$



$$\sigma_{xA} = \frac{N}{bh} - \frac{M_z}{I_z}\frac{h}{2} + \frac{M_y}{I_y}\frac{b}{2}$$

$$\sigma_{xB} = \frac{N}{bh} - \frac{M_z}{I_z}\frac{h}{2} - \frac{M_y}{I_y}\frac{b}{2}$$

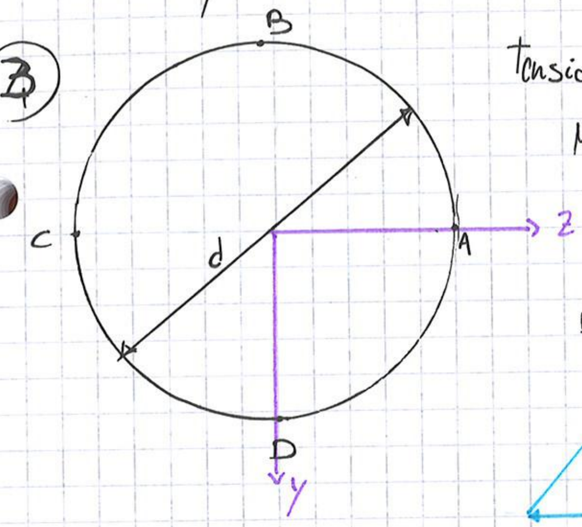
$$\sigma_{xC} = \frac{N}{bh} + \frac{M_z}{I_z}\frac{h}{2} - \frac{M_y}{I_y}\frac{b}{2}$$

$$\sigma_{xD} = \frac{N}{bh} + \frac{M_z}{I_z}\frac{h}{2} + \frac{M_y}{I_y}\frac{b}{2}$$

$$\sigma_x M_{fy} = \frac{M_y}{I_y} z \qquad I_y = \frac{hb^3}{12}$$

D spigolo più sollecitato

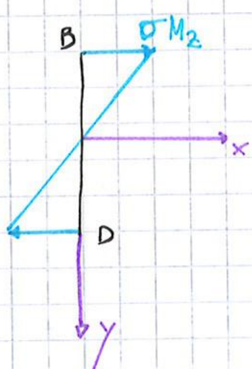
3)



tensione max e min

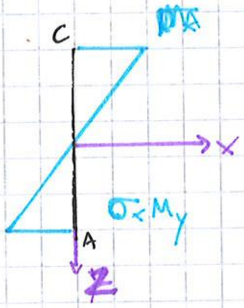
M_z ⊖, M_y ⊖

$$I_d = \frac{\pi d^4}{64}$$



$$\sigma_{xB} = \frac{(-M_z)(-\frac{d}{2})}{I_d} = \frac{M_z \frac{d}{2}}{I_d}$$

$$\sigma_{xD} = \frac{(-M_z)(\frac{d}{2})}{I_d}$$



$$\sigma_{xA} = \frac{(-M_y)(\frac{d}{2})}{I_d}$$

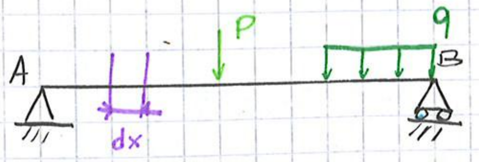
$$\sigma_{xC} = \frac{(-M_y)(-\frac{d}{2})}{I_d} = \frac{M_y \frac{d}{2}}{I_d}$$

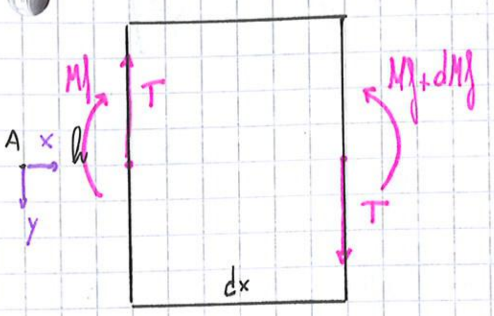
$$M_{f\,risultante} = \sqrt{(-M_y)^2 + (-M_z)^2} \qquad \sigma_{max} = \frac{M_f}{W_f}$$

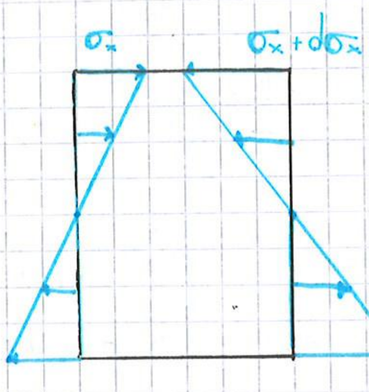
4) Momento torcente su sezione circolare (ϕ: D) e sezione circolare cava (ϕ_E: D, ϕ_i: d)

- $I_p = \frac{\pi D^4}{32}$ $\qquad \tau_{ij\,max} = \frac{M_T \frac{D}{2}}{\frac{\pi D^4}{32}} = \frac{16 M_T}{\pi D^3}$

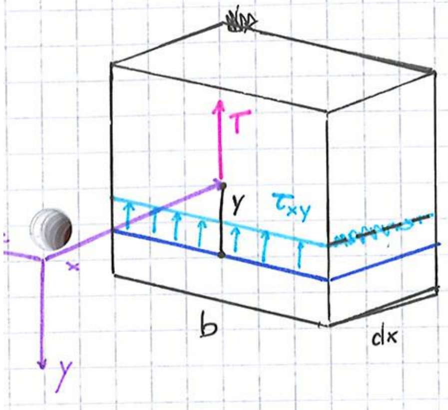
- $I_p = \frac{\pi (D^4 - d^4)}{32}$ $\qquad \tau_{ij\,max} = \frac{M_T \frac{D}{2}}{\frac{\pi}{32}(D^4 - d^4)} = \frac{16 M_T D}{\pi (D^4 - d^4)}$

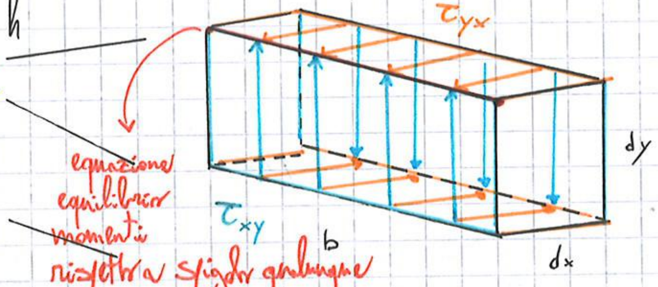




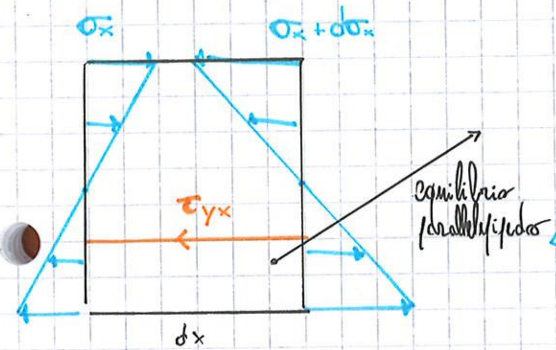


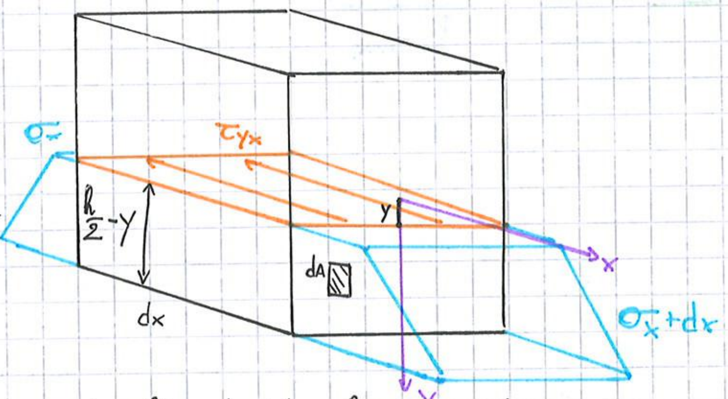
τ è costante lungo la corda, è parallela a T e ne ha la stessa convenzione





$\tau_{xy} \cdot b\,dy \cdot dx = \tau_{yx}\, b\,dx \cdot dy \qquad \tau_{xy} = \tau_{yx}$ cvd dim 3





$$-\tau_{yx}\, b\,dx + \int_A (\sigma_x + d\sigma_x)\,dA - \int \sigma_x\, dA = 0$$

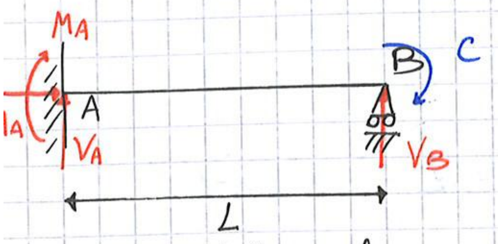
$$\tau_{yx}\, b\,dx = \int_A d\sigma_x\, dA$$

$$\int_A \underbrace{\frac{dM_J}{I_z}\, y}_{d\sigma_x}\, dA = \tau_{yx}\, dx\, b \qquad \tau_{yx} = \tau_{xy} = \frac{dM_J}{dx}\,\frac{1}{b I_z}\int_A y\, dA = \frac{T\, S_y^*}{I_z\, \text{corda}} \quad \text{cvd}$$

Taglio — corda — momento statico variabile (A è variabile)

Iperstatiche

Il modello di corpo deformabile usato per risolverle è valido solo in regime elastico, dove valgono la legge di Hooke e la sovrapposizione degli effetti



$n = 3$

$m = 4$

$l = -1$ iperstatico

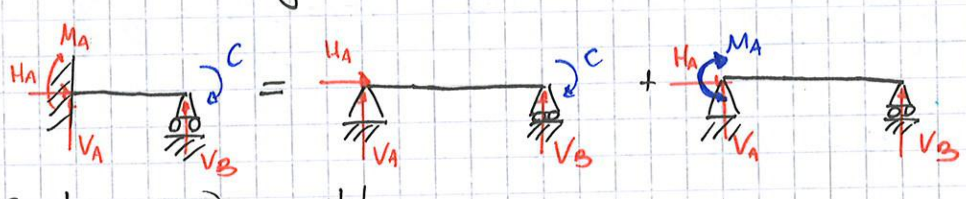
è scomponibile nel modo seguente:

Si elimina una reazione vincolare e la si suppone applicata dall'esterno

Il vincolo eliminato impediva il movimento lungo y quindi:

$y_1(L) + y_2(L) = 0$ equazione di congruenza

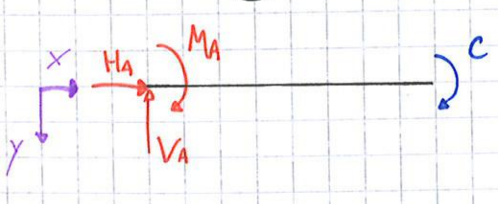
N.B: se si fosse scelto M_A invece di V_B si sarebbe ottenuto:



eq. congruenza alla rotazione:

$$\left.\frac{dy_I}{dx}\right|_{x=0} + \left.\frac{dy_{II}}{dx}\right|_{x=0} = 0$$

Sistema (I): isostatico



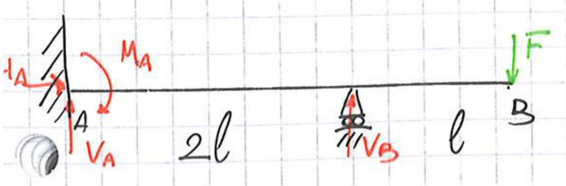
$H_A = 0$ $\qquad N = 0$

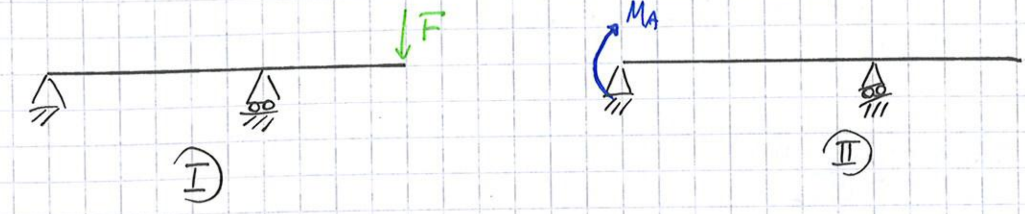
$V_A = 0$ $\qquad T = 0$

$M_A = -C$ $\qquad M_f = -C$

$$\frac{d^2 y_I}{dx^2} = -\frac{M_f}{EI_2} = \frac{C}{EI_2} \qquad \frac{dy_I}{dx} = \frac{C}{EI_2}x + cost_1 \qquad y = \frac{C}{EI_2}\frac{x^2}{2} + cost_1 x + cost_2$$

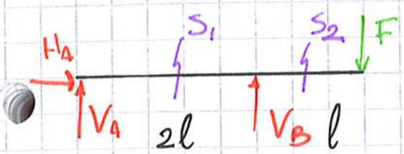
iperstatica





Condizione congruenza: $\frac{dy}{dx}(x=0)=0 \rightarrow \frac{dy_I}{dx}(0)+\frac{dy_{II}}{dx}(0)=0$

Sistema I



$H_A=0$

$V_A=-\frac{F}{2}$

$V_B=\frac{3}{2}F$

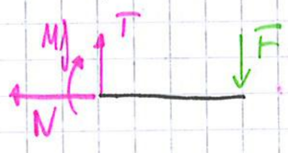
$S_1 \quad x\in[0,2l]$

$N=0$

$T=-\frac{F}{2}$

$M_f=-\frac{F}{2}x$

$S_2 \quad x\in[2l,3l]$



$N=0$

$T=F$

$M_f=-F(3l-x)$

$S_1 \quad x\in[0,2l]$

$$\frac{d^2y}{dx^2}=-\frac{M_f}{EI_2}=\frac{F}{2EI_2}x$$

$$\frac{dy}{dx}=\frac{F}{2EI_2}\frac{x^2}{2}+cost_1$$

$$y=\frac{F}{2EI_2}\frac{x^3}{6}+cost_1x+cost_2$$

$S_2 \quad x\in[2l,3l]$

$$\frac{d^2y}{dx^2}=\frac{F(3l-x)}{EI_2}$$

$$\frac{dy}{dx}=\frac{F}{EI_2}\left(3lx-\frac{x^2}{2}\right)+cost_3$$

$$y=\frac{F}{EI_2}\left(\frac{3}{2}lx^2-\frac{x^3}{6}\right)+cost_3x+cost_4$$

Condizioni di contorno: $y_{S_1}(x=0)=0 \rightarrow cost_2=0$

$y(x=2l)=0$

S_1: $0=\frac{F}{2EI_2}\frac{8}{6}l^3+cost_1\,2l \rightarrow cost_1=-\frac{l^2F}{3EI_2}$

S_2: $0=\frac{F}{2EI_2}\left(\frac{3}{2}4l^3-\frac{8}{6}l^3\right)+cost_3\,2l+cost_4$

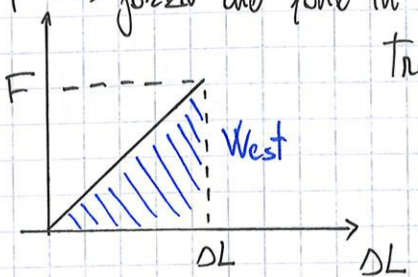
Energia elastica immagazzinata (valida per campo lineare ed elastico)

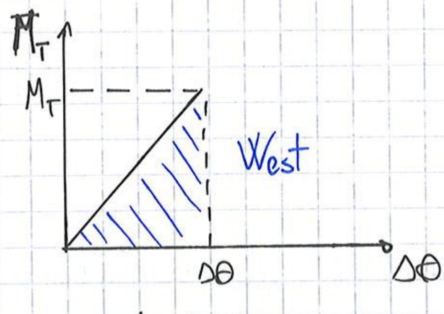
- Energia elastica: $\phi = \int_V \sigma \varepsilon \, dV + \int_V \tau \gamma \, dV$

$\phi = W_{interno} = W_{esterno}$

Teorema di Clapeyron: W_{est} = area sottesa

F → forza che pone in trazione una trave



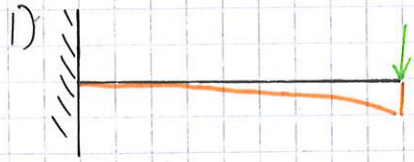


$W_{est} = \frac{1}{2} F \Delta L$

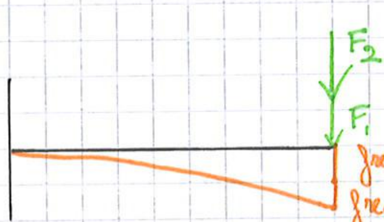
$W_{est} = \frac{1}{2} M_T \Delta\theta$

Gli effetti si sovrappongono solo se i sistemi sono disaccoppiati.

Teorema di Betti

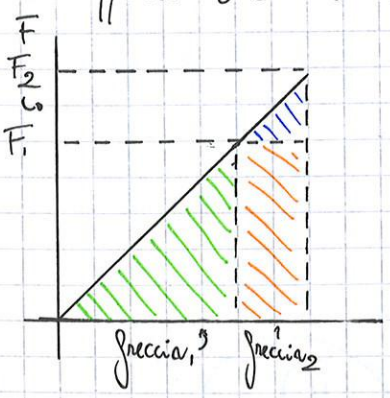
1) 

→

2) 

alla trave si applica il carico F_1

alla trave si aggiunge il carico F_2 ($<F_1$)

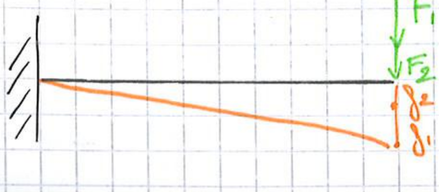


$W_{e_1} = \boxed{\frac{1}{2} f_1 F_1} + \boxed{\frac{1}{2} f_2 F_2} + \boxed{f_2 F_1}$

→ lavoro mutuo

Se c'è lavoro mutuo allora non vale la sovrapposizione degli effetti

- Se si inverte l'ordine di applicazione di F_1 ed F_2 si ottiene:



Coefficiente di sicurezza statico

C.S. alla rottura $= \dfrac{R_m}{\sigma_{eq}}$ → carico di rottura

σ_{eq} → tensione equivalente

C.S alla snervamento $= \dfrac{R_{p02}}{\sigma_{eq}}$ → carico snervamento / discostamento dalla proporzionalità

1) Stato di tensione in un punto 3D → tensore xyz

2) tensore xyz (cubetto elementare) → altro sistema di riferimento (tetraedro - Cauchy)

3) tensore tensioni 123 : $\sigma_1, \sigma_2, \sigma_3$. $\sigma_1, \sigma_2, \sigma_3$ tensioni generiche che definiscono lo stato di tensione in qualunque SR.

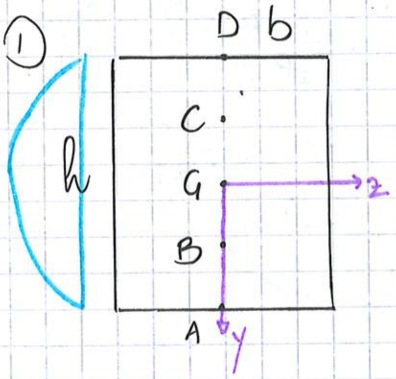
123 → direzioni principali

$\sigma_1 > \sigma_2 > \sigma_3$; le τ sono tutte nulle

4) Ipotesi di rottura e C.S.

esercitazione

① Calcolare le tensioni σ_x, τ_{xy} in A, B, C, D, G.



T, $M_z \oplus$, $M_2 \ominus$

$$\tau_{xy} = \tau_{yx} = \frac{T\, S_z^*}{I_z\ \text{corda}} = \frac{12\, T\, b\int_y^{h/2} y\,dy}{b^2 h^3} =$$

$S_z^* \to b\int_y^{h/2} y\,dy$; $I_z \to \dfrac{bh^3}{12}$; corda $\to b$

$$= \frac{12T}{b^2h^3}\,\frac{b}{2}\left[\frac{h^2}{4} - y^2\right] = \frac{6T}{bh^3}\left[\frac{h^2}{4} - y^2\right]$$

$$\tau_{xy_A} = \tau_{xy_D} = 0 = \tau_{xy_{y=\frac{h}{2}}} \qquad \tau_{xy_B} = \tau_{xy_C} = \tau_{xy_{y=\frac{h}{4}}} = \frac{6T}{bh^3}\, h^2\left[\frac{1}{4} - \frac{1}{16}\right]$$

$$\tau_{xy_G} = \tau_{xy\,max} = \tau_{xy_{y=0}} = \frac{6T}{bh^3}\,\frac{h^2}{4} = \frac{3}{2}\,\frac{T}{bh}$$

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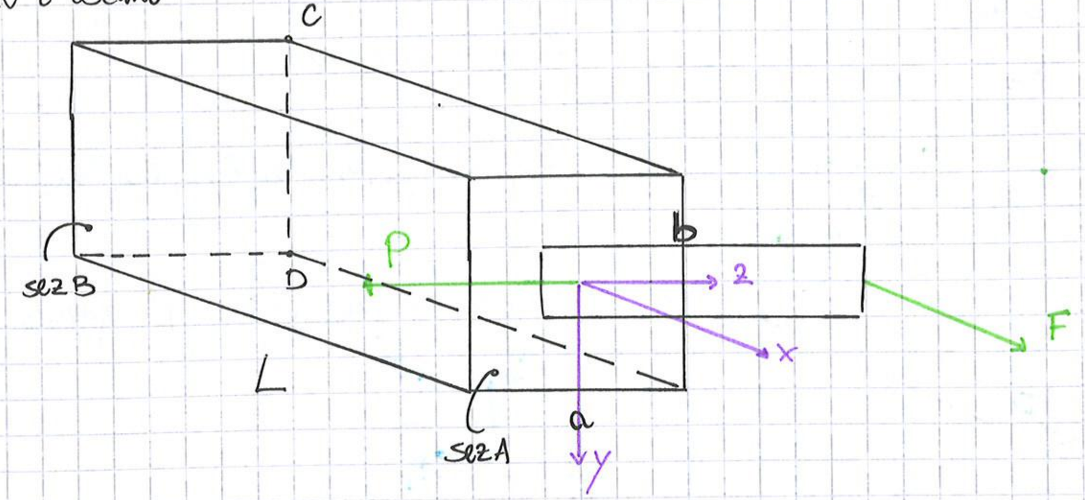
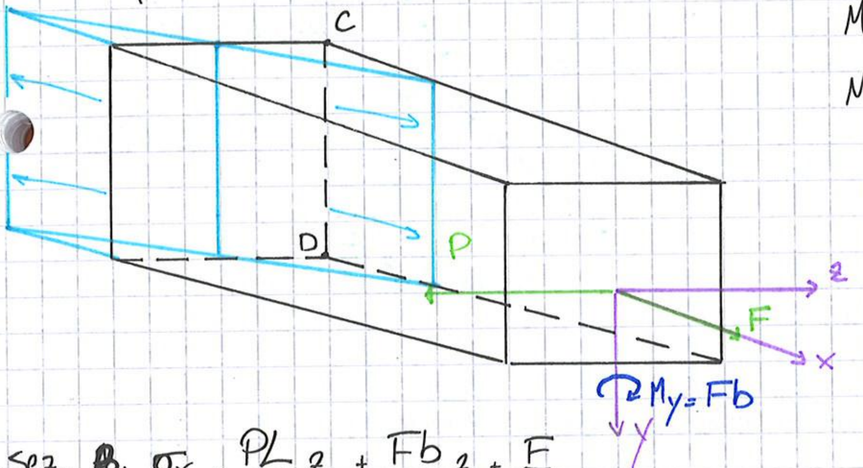
tema d'esame



sist. equivalente



$T_z = -P$ cost

$M_y = +Px' = P(L-x)$

$M_{y\,max}\big|_{x=0} = PL$

$\tau_{xz\,max} = \tau_{zx\,max} = \frac{3}{2}\frac{T}{a^2}$

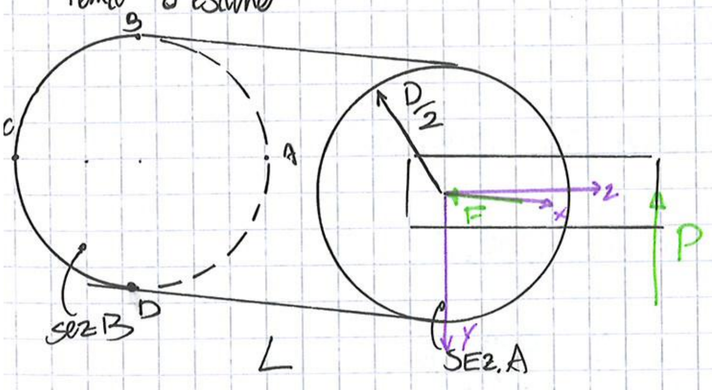
$Mf_{Fb} = \text{cost} = Fb$

sez. B: $\sigma_x = \frac{PL}{I_y} z + \frac{Fb}{I_y} z + \frac{F}{A}$

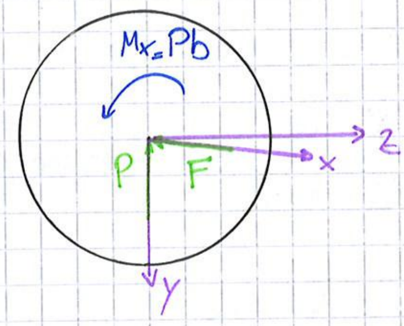
$I_y \to \frac{a^4}{12}$ $\quad A \to a^2$

lato CD: $\sigma_{x\,max} = \frac{PL}{I_y}\frac{a}{2} + \frac{Fb}{I_y}\frac{a}{2} + \frac{F}{a}$ area più sollecitata

tema d'esame



sist. equivalente



$\sigma_{xN} = -\frac{F}{A}$ cost

$T_y = -P$

$M_z = P(L-x)$

sez B: $\sigma_{xD} = -\frac{F}{A} + \frac{PL}{I_d}\frac{D}{2}$

$\tau_{xz_D} = \frac{M_x}{I_p}\frac{D}{2}$

le relazioni precedenti diventano: (Diviso tutto per ABC)

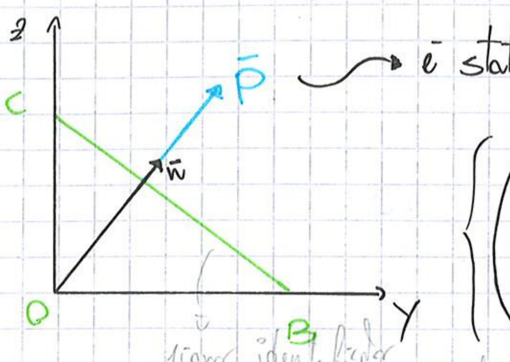
$\bar{x}: \sigma_x \lambda + \tau_{yx} \mu + \tau_{zx} \nu = p_x$

$\bar{y}: \tau_{xy} \lambda + \sigma_y \mu + \tau_{zy} \nu = p_y$

$\bar{z}: \tau_{xz} \lambda + \tau_{yz} \mu + \sigma_z \nu = p_z$

~~Dopo~~ rotazione: $p_x = \sigma \lambda$, $p_y = \sigma \mu$, $p_z = \sigma \nu$

se il piano è generatore di direzioni principali (no τ, solo σ)



piano ident. fuori dalle direzioni principali

è stato ruotato: $\bar{p} // n \rightarrow \bar{p} \equiv \bar{\sigma}$

$$\left\{ \begin{pmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} - \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} \right\} \cdot \begin{Bmatrix} \lambda \\ \mu \\ \nu \end{Bmatrix} = 0$$

$$\left\{ [T] - \underbrace{[I]}_{\text{identità}}\sigma \right\} \cdot \begin{Bmatrix} \lambda \\ \mu \\ \nu \end{Bmatrix} = 0$$

Tensioni principali 3D

Trovare autovalori:
(a meno di una costante arbitraria)

$$\left\{ \begin{array}{l} \begin{vmatrix} \sigma_x - \sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} \begin{vmatrix} \lambda \\ \mu \\ \nu \end{vmatrix} = 0 \\ \lambda^2 + \mu^2 + \nu^2 = 1 \end{array} \right.$$

λ, μ, ν autovettori

condizione ortogonalità

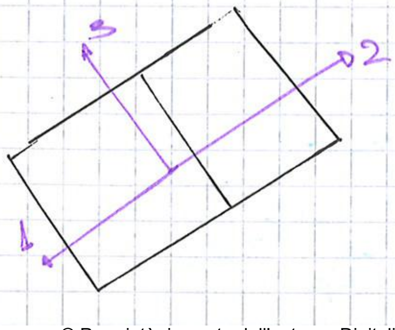
Tensioni principali: $\sigma_1\ \sigma_2\ \sigma_3$ autovalore direzione 1, 2, 3 = autovettori

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{zy} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{zx} \\ \tau_{xz} & \sigma_z \end{vmatrix}$$

Tensore delle tensioni in 1, 2, 3



È un tensore diagonale

$p_1 = \sigma \lambda$
$p_2 = \sigma \mu$
$p_3 = \sigma \nu$

$$\lambda^2 + \mu^2 + \nu^2 = 1 \Rightarrow \left(\frac{p_1}{\sigma}\right)^2 + \left(\frac{p_2}{\sigma}\right)^2 + \left(\frac{p_3}{\sigma}\right)^2 = 1$$

eq. ellissoide

calcolo $\sigma_1\ \sigma_2\ \sigma_3 \Rightarrow \det\begin{pmatrix} \sigma_x-\sigma & \tau_{yx} & 0 \\ \tau_{xy} & \sigma_y-\sigma & 0 \\ 0 & 0 & -\sigma \end{pmatrix} = 0$

- l'asse 2 non è caricato da τ $\Rightarrow$ è una direzione principale.

Risultato: $\frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2} > 0 > \frac{\sigma_x}{2} - \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}$

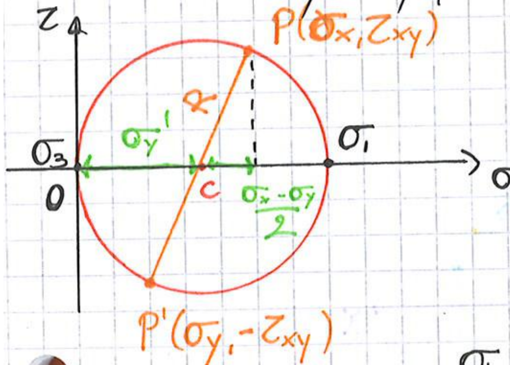
Formula generica se $\sigma_y = 0$: $\sigma_{b,c} = \frac{\sigma_x}{2} \pm \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}}$

le tensioni vanno numerate dopo il calcolo $(\sigma_1 > \sigma_2 > \sigma_3)$

Cerchi di Mohr

- elementi noti: $\sigma_x, \sigma_y, \tau_{xy}$ o $\sigma_{1,2,3}$ 1,2,3 $\longrightarrow$ Si ottengono con un procedimento grafico: $\sigma, \tau\ \forall$ direzione

Dato $\sigma_x, \sigma_y, \tau_{xy} \neq 0$ 1) si calcolano $\sigma_1\ \sigma_2\ \sigma_3$ ottenute graficamente



C centro del cerchio di Mohr

$\overline{PP'}$ diametro

$\sigma_3 \le \sigma_2 \le \sigma_1 \Rightarrow \sigma_3, \sigma_1$ estremi del cerchio su σ

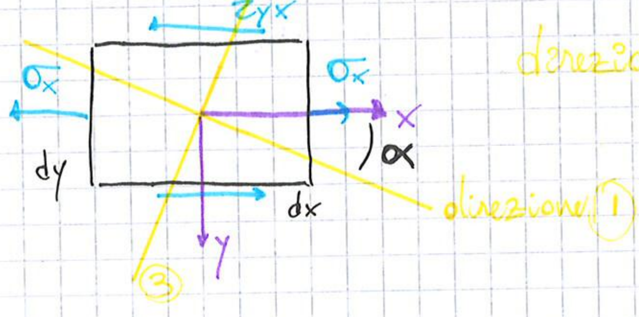
$$\sigma_{b,c} = \underbrace{\frac{\sigma_x+\sigma_y}{2}}_{\sigma_y'} \pm \underbrace{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}}_{R}$$

$\Rightarrow$ Soluzione grafica e analitica coincidono.

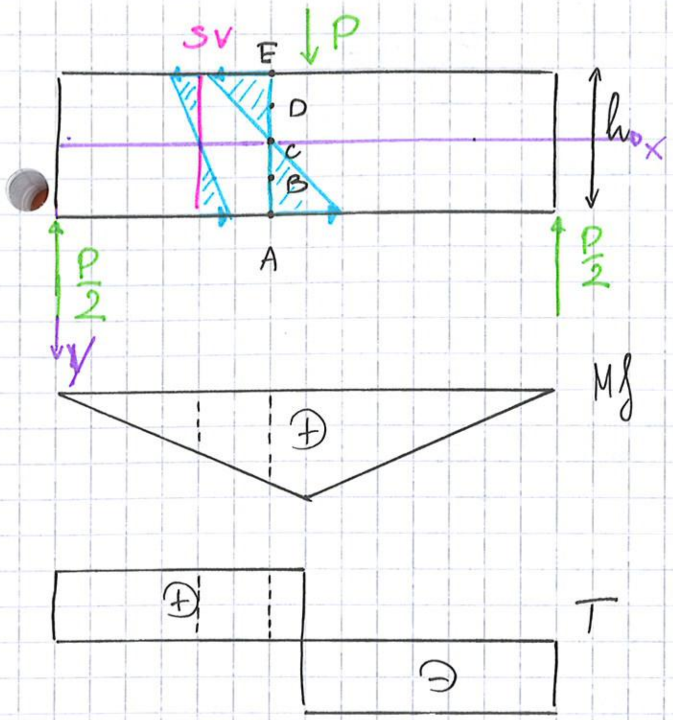
2) Si calcolano le direzioni principali 1,2,3

$\sigma_x \neq 0$

$\tau_{xy} \neq 0$



direzione 2 ≡ asse 2



$$\sigma_x = \frac{M_f}{I_z} y \qquad \tau_{xy} = \tau_{yx} = \frac{T}{corda} \frac{S_z^*}{I_z}$$

A) $\sigma_{x\,max}$, $\tau_{xy} = 0$ $\quad \frac{h}{2} = y$

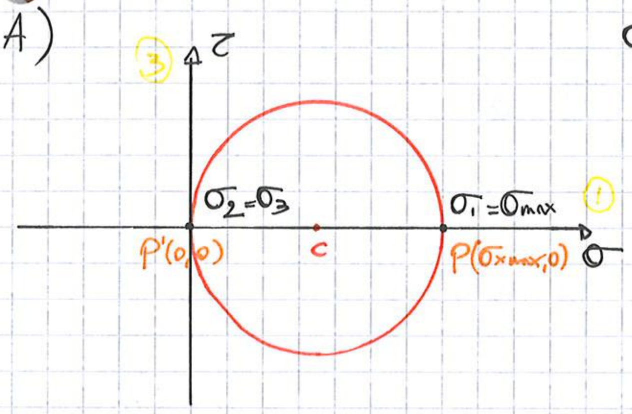
B) σ_{xB}, τ_{xyB} $\quad y = \frac{h}{4}$

C) $\sigma_x = 0$, $\tau_{xy\,max}$ $\quad y = 0$

D) $\sigma_{xD} < 0$, τ_{xyD} $\quad y = -\frac{h}{4}$

E) $\sigma_{x\,min} < 0$, $\tau_{xy} = 0$ $\quad y = -\frac{h}{2}$

A)

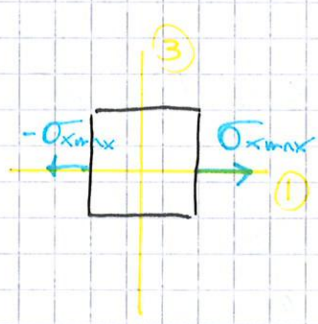


$$\sigma_{1,3} = \frac{\sigma_{x\,max}}{2} \pm \frac{\sigma_{max}}{2}$$

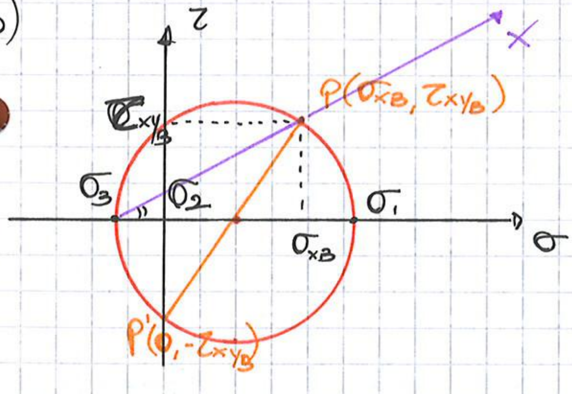
$\sigma_1 = \sigma_{max}$

$\sigma_2 = 0$

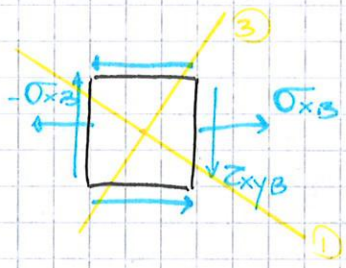
$\sigma_3 = 0$



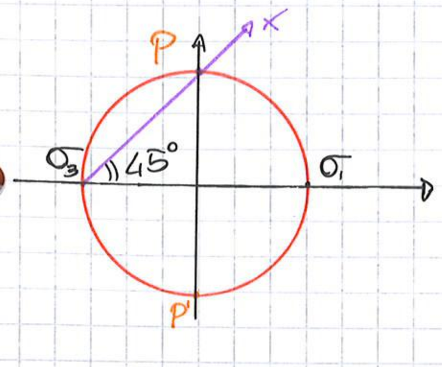
B)

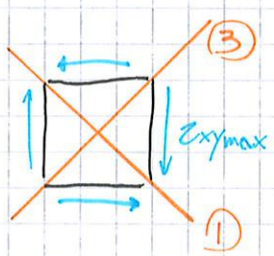


$$\sigma_{1,2} = \frac{\sigma_{xB}}{2} \pm \sqrt{\left(\frac{\sigma_{xB}}{2}\right)^2 + \tau_{xyB}^2}$$

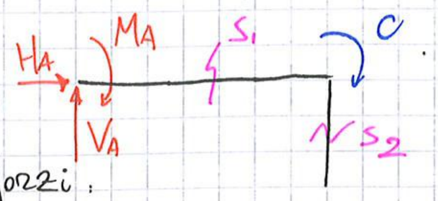


C) coincide con il piano neutro → comportamento come gesso in torsione.





(I) Reazioni Vincolari:

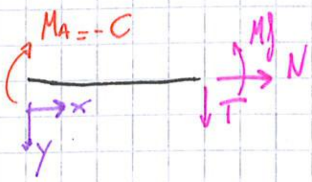


$V_A = 0$ $\quad M_A = -C$

$H_A = 0$

Sforzi:

$S_1 \; x \in [0, l]$

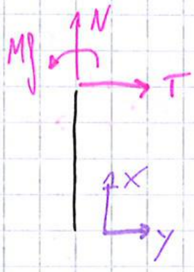


$N = 0$

$T = 0$

$Mf = -C$

$S_2 \; x \in [0, h]$



$N = 0$

$Mf = 0$

$T = 0$

linea elastica:

$x \in [0, l]$

$$\frac{d^2 y_{I_1}}{dx^2} = \frac{C}{EI_2} \qquad \frac{dy_{I_1}}{dx} = \frac{C}{EI_2}x + cost_1 \qquad y = \frac{C}{2EI_2}x^2 + cost_1 x + cost_2$$

$x \in [0, h]$

$$\frac{d^2 y_{I_2}}{dx^2} = 0 \qquad \frac{dy_{I_2}}{dx} = cost_3 \qquad y_{I_2} = cost_3 x + cost_4$$

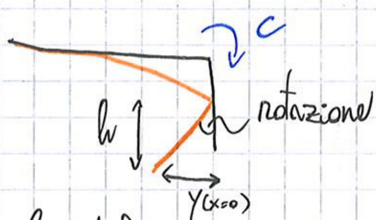
Condizioni al contorno:

incastro: $\cdot \; y_{I_1}(x=0) = 0$

$\cdot \; \frac{dy_{I_1}}{dx}(x=0) = 0$

spigolo rigido: $\cdot \; \frac{dy_{I_1}}{dx}(x=l) = \frac{dy_{I_2}}{dx}(x=h)$

rotazione: $\cdot \; y_{I_2}(x=0) = \frac{dy_{I_2}}{dx}(x=h) \cdot h$



x piccole deformazioni

1) $\rightarrow cost_2 = 0$ $\qquad$ 4) $cost_4 = \frac{C}{EI_2} l h$

2) $\rightarrow cost_1 = 0$

3) $\frac{C}{EI_2} l = cost_3$

SVBSI $= y_{I_1}(l)$ $\quad$ x spigolo rigido $\qquad y_{I_1}(x) = \frac{C}{2EI_2}x^2$

$$\sigma_x = \frac{-V_B}{A}$$

$$\varepsilon_x = \frac{\sigma_x}{E} = -\frac{V_B}{EA} \qquad \Delta l = \int_0^h \varepsilon_x \, dx = \frac{-V_B}{EA} h$$

$$SVBS\,II = y_{II}(x=l) - \frac{V_B h}{AE}$$

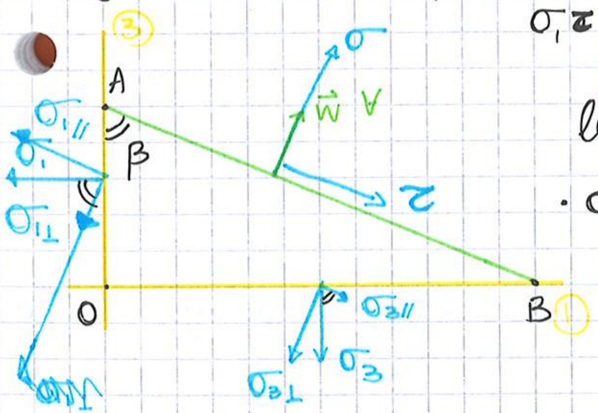
Condizione congruenza:

$$\frac{C}{2EI_2} l^2 + \frac{V_B}{EI_2}\left(\frac{l^3}{6} - \frac{l^3}{2}\right) - \frac{V_B h}{AE} = 0$$

$$V_B = \frac{\dfrac{C\, l^2}{2EI_2}}{\dfrac{h}{EA} - \dfrac{1}{EI_2} \cdot \dfrac{l^3}{3}}$$

Corrispondenza equilibri di tensione e cerchi di Mohr

elemento infinitesimo (in coordinate principali) tagliato in una direzione generica anche non cartesiana (piano di rottura).



σ, τ generici ed incogniti: hanno provocato la frattura

equilibrio lungo $\vec{w}$:

- $\sigma AB = \sigma_1 \cos\beta \, AO + \sigma_3 \sin\beta \, OB$

equilibrio lungo $\perp \vec{w}$:

- $\tau AB = \sigma_1 \sin\beta \, AO + \sigma_3 \cos\beta \, OB$

$$AO = AB \cos\beta \; ; \; OB = AB \sin\beta$$

$$\sigma AB = \sigma_1 \, AB \cos^2\beta + \sigma_3 \, AB \underbrace{\sin^2\beta}_{1-\cos^2\beta}$$

$$\tau AB = \sigma_1 \, AB \sin\beta\cos\beta + \sigma_3 \, AB \sin\beta \cos\beta$$

$$\sigma = \sigma_3 + (\sigma_1 - \sigma_3)\cos^2\beta \qquad \tau = (\sigma_1 + \sigma_3)\sin\beta\cos\beta$$

la tensione tangenziale massima ($\sigma_{eq} = \sigma_1 - \sigma_3$). Usata per gli acciai fragili.

3) Von Mises: In un punto si arriva a rottura quando si arriva alla massima energia di distorsione.

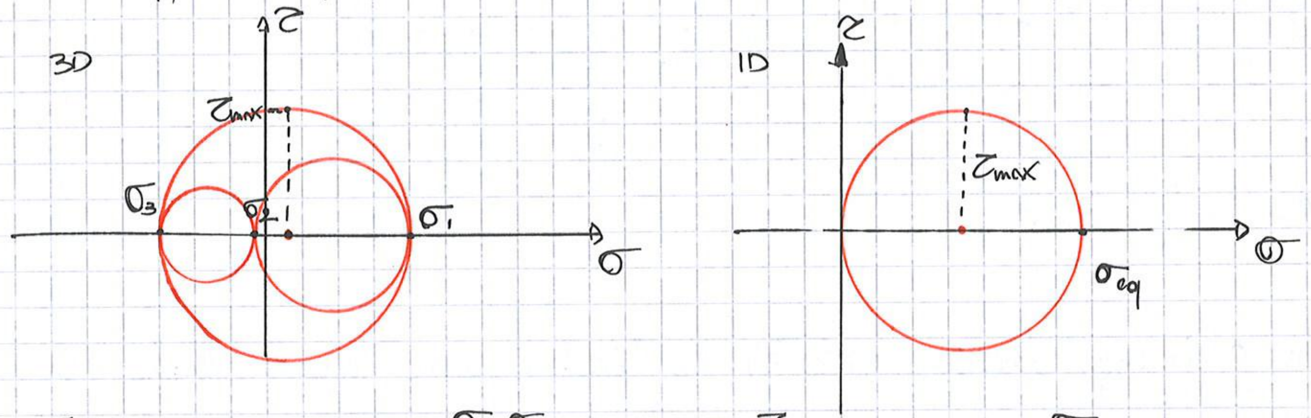
($\sigma_{eq} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2}$)

Vale per gli acciai duttili

Dim ipotesi 2:

3D $\sigma_1, \sigma_2, \sigma_3$

1D $\sigma_{eq} \neq 0$ $\sigma_{eq} = \sigma_1$ $\sigma_2, \sigma_3 = 0$



τ_{max} = raggio maggiore = $\frac{\sigma_1 - \sigma_3}{2}$

τ_{max} = raggio = $\frac{\sigma_{eq}}{2}$

Si eguagliano i due stati $\rightarrow$ $\frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{eq}}{2} \Rightarrow \sigma_{eq} = \sigma_1 - \sigma_3$

Dim ipotesi 3:

energia elastica immagazzinata $\phi = \int_V \sigma \varepsilon \, dV$

$\phi = \phi' + \phi''$

componente deviatorica: deforma la struttura.

componente idrostatica: fa variare solo il volume senza deformazioni.

Se si raggiunge ϕ''_{max} si arriva a rottura.

$\phi'' = \phi - \phi'$

$\phi = \frac{1}{2}(\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_3 \varepsilon_3)$

$$\phi''_{3D} = \frac{1}{2E}\left[A\left(1-\frac{1}{3}+\frac{2\nu}{3}\right)+B\left(-2\nu-\frac{2}{3}+\frac{4\nu}{3}\right)\right]$$

$$\phi''_{3D} = \frac{1}{2E}\left[A\frac{2}{3}(1+\nu)-B\frac{2}{3}(1+\nu)\right] = \frac{1}{2E}\frac{2}{3}(1+\nu)(A-B)$$

$$\phi''_{max_{3D}} = \frac{1}{3E}(1+\nu)\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)\right]$$

$$\phi''_{max_{1D}} = \frac{1}{3E}(1+\nu)\sigma_{eq}^2$$

$\downarrow$

$\sigma_1 = \sigma_{eq}\ ;\ \sigma_2, \sigma_3 = 0$

$$\phi''_{max_{3D}} = \phi''_{max_{1D}} \Rightarrow \sigma_{eq}^2 = \sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)$$

$\Downarrow$

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}$$

ipotesi	3D $\sigma_{1,2,3} \neq 0$	2D $\sigma_i \neq 0$ $\sigma_2 = 0$ $\tau_{ij} \neq 0$	trazione $\sigma_i \neq 0$ $\tau_{ij} = 0$	torsione $\sigma_i = 0$ $\tau_{ij} \neq 0$
1	$\sigma_{eq} = \sigma_1$	$\sigma_{eq} = \frac{\sigma}{2}+\sqrt{\left(\frac{\sigma}{2}\right)^2+\tau^2}$	$\sigma_{eq} = \sigma$	$\sigma_{eq} = \tau$
2	$\sigma_{eq} = \sigma_1 - \sigma_3$	$\sigma_{eq} = \sqrt{\sigma^2+4\tau^2}$	$\sigma_{eq} = \sigma$	$\sigma_{eq} = 2\tau$
3	$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}$	$\sigma_{eq} = \sqrt{\sigma^2+3\tau^2}$	$\sigma_{eq} = \sigma$	$\sigma_{eq} = \sqrt{3}\tau$

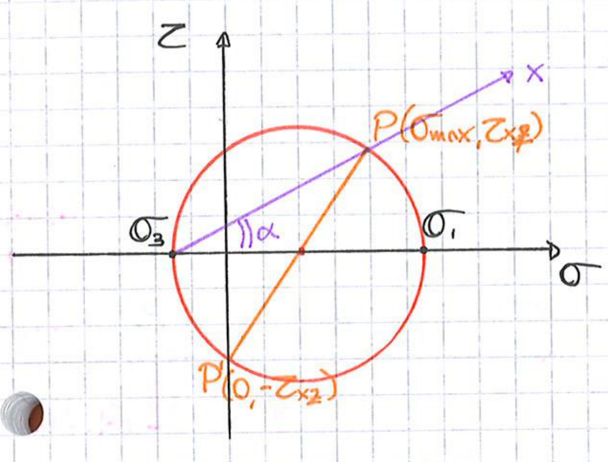
il caso 2D si ottiene sostituendo $\sigma_{1,3} = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2+\tau^2}$ e $\sigma_2 = 0$ al caso 3D.

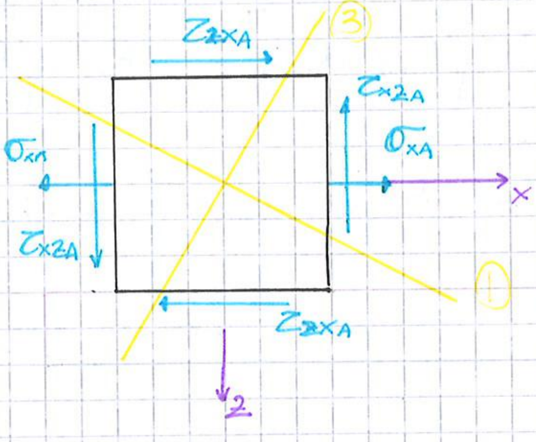
Confronto grafico delle ipotesi di cedimento

Il confronto è poco utile siccome slega le ipotesi dal tipo di materiale su cui sono basate.

$$\left.\begin{array}{l}\sigma_{x\,max} = \sigma_{xN} + \sigma_{xMf\,max} \\ \tau_{max} = \tau_{xz_A}\end{array}\right\} \Rightarrow \text{stato di tensione sul piano } xz$$

$$\sigma_{1,3} = \frac{\sigma_{x\,max}}{2} \pm \sqrt{\left(\frac{\sigma_{x\,max}}{2}\right)^2 + \tau_{xz_A}} \qquad \sigma_2 = 0$$





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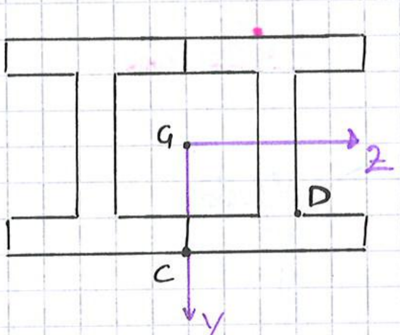
(3)



$C: \sigma_{xC\,max} \; ; \; \tau_{xyC} = 0 \rightarrow \sigma_1 = \sigma_{xC\,max} = \sigma_{eq}$

$D: \sigma_{xD} \; ; \; \tau_{xyD}$

$$\sigma_{1,3D} = \frac{\sigma_{xD}}{2} \pm \sqrt{\left(\frac{\sigma_{xD}}{2}\right)^2 + \tau_{xyD}^2} \qquad \sigma_2 = 0$$

$$\sigma_{eq} = \sigma_{1D} - \sigma_{3D}$$

M_z +

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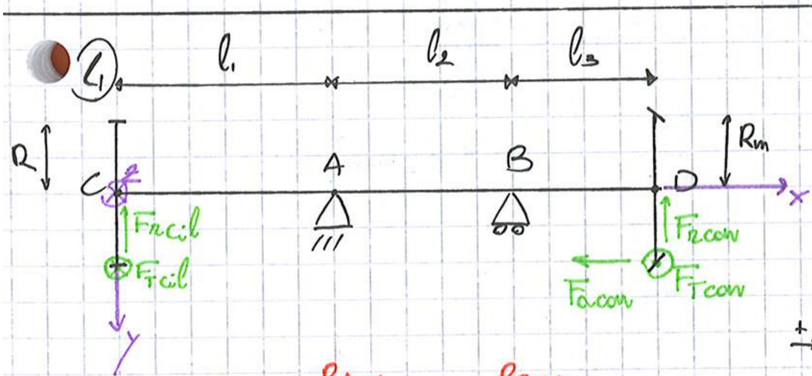
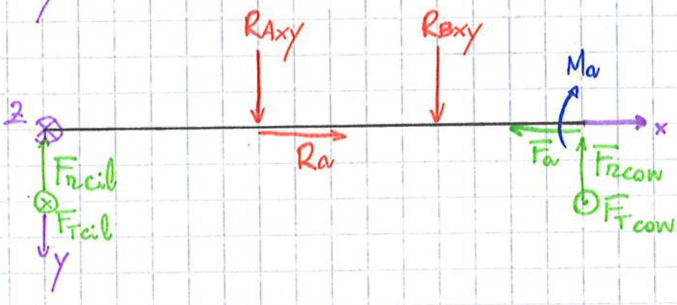
(1)





$\xrightarrow{+} \quad R_a = F_{acon}$

$+\uparrow \quad +F_{Rcil} - R_{Axy} - R_{Bxy} + F_{Rcon} = 0$

$A) \quad -F_{Rcil}\, l_1 - R_{Bxy}\, l_2 - M_a + F_{Rcon}(l_2 + l_3) = 0$

$\downarrow$

$$R_{Bxy} = \frac{M_a + F_{Rcil}\, l_1 - F_{Rcon}(l_2 + l_3)}{l_2}$$

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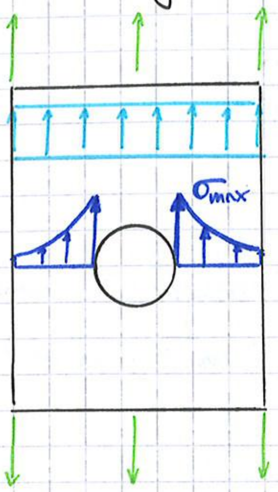
Effetto d'intaglio

piastra forata in trazione

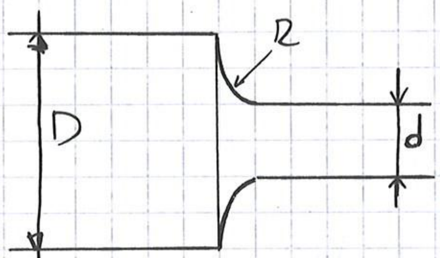


$\sigma_N = \frac{F}{Area}$ costante su tutta la sezione

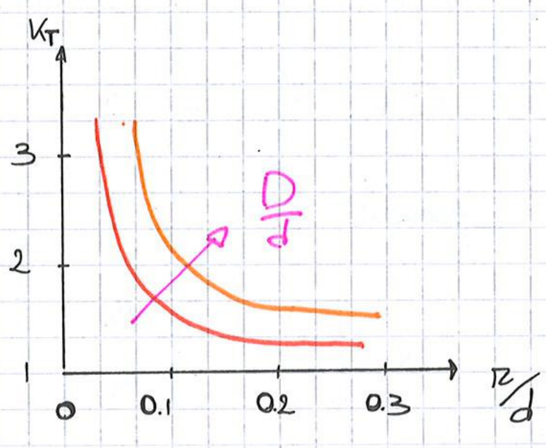
cresce all'avvicinarsi al foro

$K_T = \frac{\sigma_{max}}{\sigma_{nom}}$ → fattore di forma

$\sigma_{nom} = \frac{F}{\text{Area netta}}$ → area sezione privata dell'intaglio



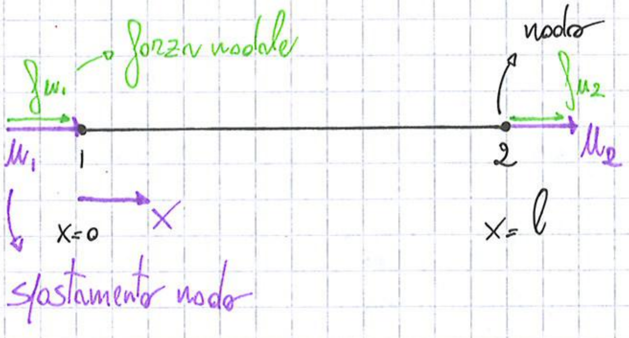
albero sottoposto a M_T



Calcolo strutturale

- Formulazione rigidezza degli elementi
- Forze equivalenti applicate ai nodi degli elementi
- Combinazione degli elementi
- Modellazione di una struttura globale

Elemento asta



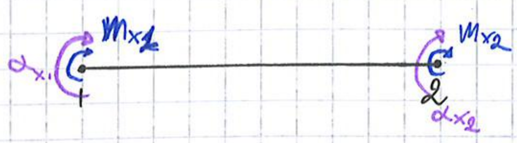
Ogni elemento ha i nodi

$n_i \Rightarrow$ n° gradi di libertà del nodo o n° di forze per nodo

$n = i \cdot n_i \Rightarrow$ DOF'S elemento

ci sono $2n$ variabili in n equazioni

Elemento barra di torsione



$$\frac{G I_p}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{pmatrix} \alpha_{x1} \\ \alpha_{x2} \end{pmatrix} = \begin{pmatrix} m_{x1} \\ m_{x2} \end{pmatrix}$$

Trave di Eulero - Bernulli (trave sottile dove lo sforzo di taglio è trascurabile)

$i = 2$

$n_i = 2$

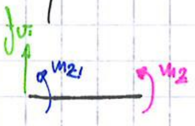
$n = 4$

4 equazioni:

equilibrio statico: $f_{v_1} + f_{v_2} = 0$

$2)\ m_{z1} + m_{z2} - f_{v_1} l = 0$

equilibrio spostamento:



$m_z = -m_{z1} + f_{v_1} x$

$$\frac{d^2 y}{dx^2} = \frac{m_{z1} - f_{v_1} x}{E I_z}$$

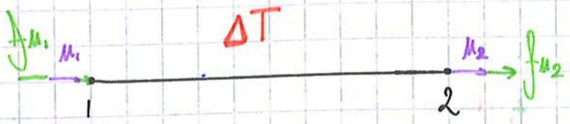
$$\alpha_z = \frac{dy}{dx} = \int \frac{m_{z1}}{E I_z} - \frac{f_{v_1} x}{E I_z} = \frac{m_{z1}}{E I_z} x - \frac{f_{v_1} x^2}{2 E I_z} \Rightarrow \alpha_{z2} - \alpha_{z1} = \frac{m_{z1} l}{E I_z} - \frac{f_{v_1} l^2}{2 E I_z}$$

$$v = y = \frac{m_{z1} x^2}{2 E I_z} - \frac{f_{v_1} x^3}{6 E I_z} \Rightarrow v_2 - v_1 - \underset{\substack{\downarrow \\ \sin\theta \sim \theta}}{\alpha_{1z} l} = \frac{m_{z1} l^2}{2 E I_z} - \frac{f_{v_1} l^3}{6 E I_z}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & -l & 1 & 0 \end{bmatrix} \cdot \begin{pmatrix} v_1 \\ \alpha_{z1} \\ v_2 \\ \alpha_{z2} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -l & 1 & 0 & 1 \\ -\frac{l^2}{2 E I_z} & \frac{l}{E I_z} & 0 & 0 \\ -\frac{l^3}{6 E I_z} & \frac{l^2}{2 E I_z} & 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} f_{v_1} \\ m_{z1} \\ f_{v_2} \\ m_{z2} \end{pmatrix}$$

Variazione temperatura

$\varepsilon_x = \alpha \Delta T$ → coeff. dilatazione termica



$\Delta l = \alpha \Delta T l = u_2 - u_1$

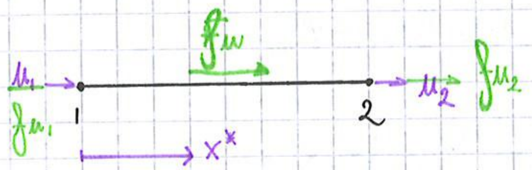
(x gli equilibri si tralasciano le componenti che compaiono sia in reale che nell'equivalente)

$f_{e_1} + f_{e_2} = 0$

$\alpha \Delta T l = \frac{f_{e_2} l}{EA} \qquad f_{e_2} = \alpha \Delta T EA = - f_{e_1}$

$\vec{f}_e = \alpha \Delta T EA \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Carico concentrato



$f_u = f_{e_1} + f_{e_2}$

$$\Delta u = \Delta l = \frac{N}{EA} l = \frac{1}{EA} \int_0^{(l-x^*)} f_u \, dx + \frac{1}{EA} \int_0^{l} f_{u_2} \, dx$$

$$\Delta u = \underbrace{\frac{1}{EA} f_u (l - x^*)} + \frac{1}{EA} f_{u_2} l$$

$\frac{1}{EA} f_u (l - x^*) = \frac{1}{EA} f_{e_2} l$

$f_{e_2} = f_u \cdot \frac{l - x^*}{l}$ e $f_{e_2} = \frac{x^*}{l} f_u$

$\vec{f}_e = \frac{f_u}{l} \begin{pmatrix} x^* \\ l - x^* \end{pmatrix}$

Attraverso la sovrapposizione degli effetti si ottiene:

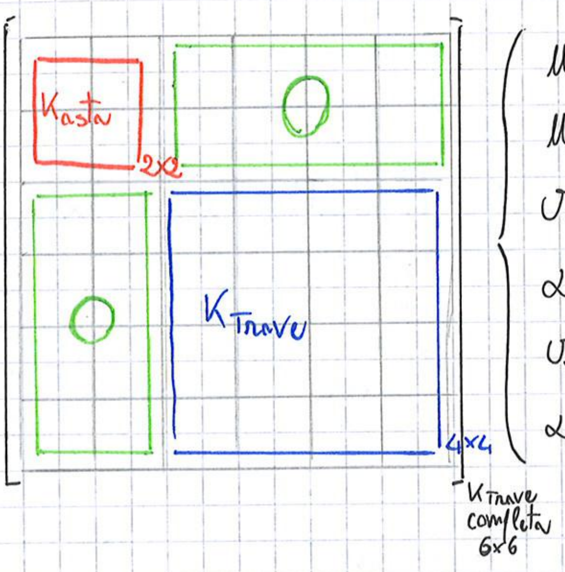
$$[K] \cdot \vec{S} = \vec{f} + \frac{ql}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha \Delta T EA \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{f_u}{l} \begin{pmatrix} x^* \\ l - x^* \end{pmatrix}$$

$$(4) \rightarrow m_{e1} = -\frac{ql^2}{2} + \frac{ql}{2}\cdot\frac{l}{3} = \frac{ql^2}{12} \rightarrow (2) \Rightarrow m_{e2} = -q\frac{l^2}{12}$$

$$\vec{f}_e = \left(q\frac{l}{2},\ q\frac{l}{2},\ q\frac{l^2}{12},\ -q\frac{l^2}{12}\right)$$

$$[K]\cdot\vec{s} = \vec{f} + \vec{f}_e$$

Trave completa

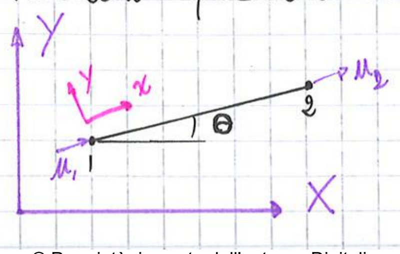


$$\begin{Bmatrix} u_1 \\ u_2 \\ v_1 \\ \alpha_{z1} \\ v_2 \\ \alpha_{z2} \end{Bmatrix} = \begin{Bmatrix} f_{u1} \\ f_{u2} \\ f_{v1} \\ m_{z1} \\ f_{v2} \\ m_{z2} \end{Bmatrix}$$

Riordinando gli spostamenti si ottiene:

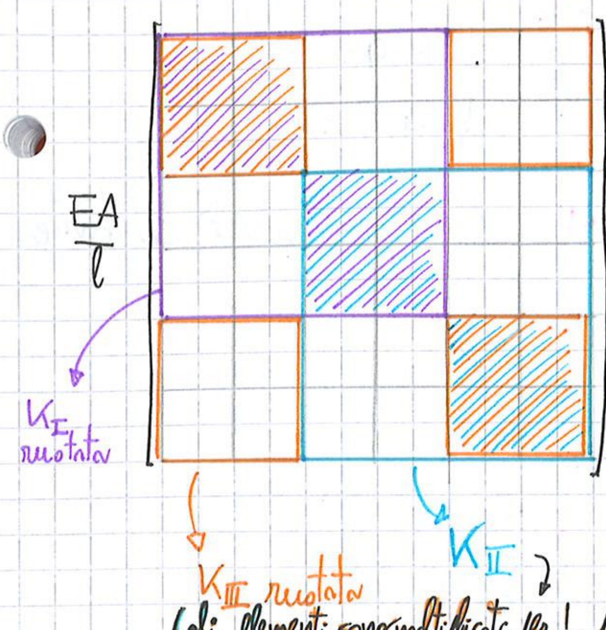
$$\begin{bmatrix} 1,1 & 0 & 0 & 1,2 & 0 & 0 \\ 0 & 1,1 & 1,2 & 0 & 1,3 & 1,4 \\ 0 & 2,1 & 2,2 & 0 & 2,3 & 2,4 \\ 2,1 & 0 & 0 & 2,2 & 0 & 0 \\ 0 & 3,1 & 3,2 & 0 & 3,3 & 3,4 \\ 0 & 4,1 & 4,2 & 0 & 4,3 & 4,4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \alpha_1 \\ u_2 \\ v_2 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} f_{u1} \\ f_{v1} \\ m_{z1} \\ f_{u2} \\ f_{v2} \\ m_{z2} \end{Bmatrix}$$

Rotazioni: passare da coordinate locali xyz a coordinate globali XYZ



$u_{x1}\cos\theta = u_{X1}$ $\qquad$ $u_{x2}\cos\theta = u_{X2}$

$u_{x1}\sin\theta = u_{Y1}$ $\qquad$ $u_{x2}\sin\theta = u_{Y2}$



= 0 direzioni vincolate

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \qquad F_3 \to 0,\quad F_4 \to -F,\quad F_6 \to 0,\quad u_5 \to 0$$

Alle direzioni vincolate corrispondono le forze incognite (F_1 F_2 F_5)

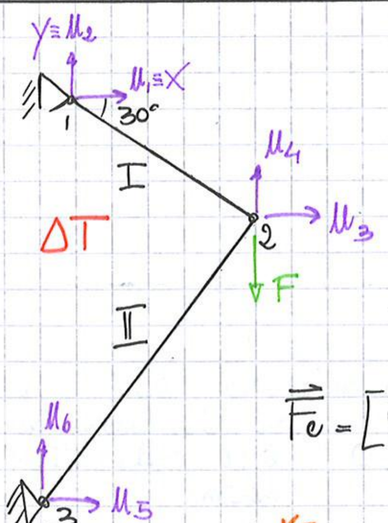
Agli spostamenti non vincolati corrispondono le forze esterne note (F_3, F_4, F_6)

(gli elementi sono moltiplicati per $\frac{1}{\sqrt{2}}$ perché è lunga $\sqrt{2}\,l$)

nelle intersezioni si sommano gli elementi delle matrici

dalle righe 3,4,6 si trovano u_3, u_4, u_6.

dalle righe 1,2,5 noti u_3, u_4, u_6 si trovano F_1, F_2, F_5



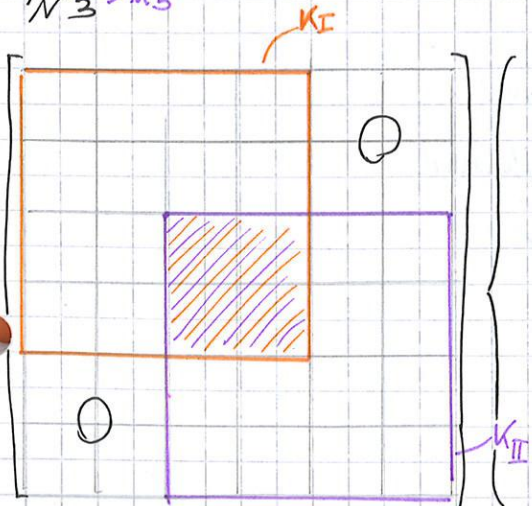
I	1,2,3,4	−30°
II	3,4,5,6	−120°

$C = \cos\theta$
$S = \sin\theta$

ΔT (asta 1 — 2)

$$\begin{Bmatrix} f_{e1} \\ f_{e2} \end{Bmatrix} = \alpha \Delta T E A \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad \{f_e\}_{xyz} = [R]\{\vec{F}_e\}_{xyz} \qquad [R] = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

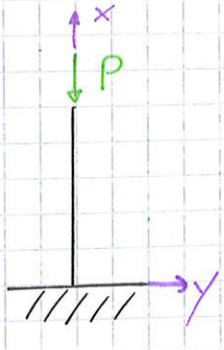
$$\vec{F}_e = [R]^{-1} \vec{f}_e = \alpha \Delta T E A \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \alpha \Delta T E A \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}$$



$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} + \begin{Bmatrix} F_{e1} \\ F_{e2} \\ F_{e3} \\ F_{e4} \\ F_{e5} \\ F_{e6} \end{Bmatrix}$$

$u_1, u_2 \to 0$; $u_5, u_6 \to 0$; $F_3 \to 0$; $F_4 \to -F$

$F_{e1} \to -C_I$; $F_{e2} \to -S_I$; $F_{e3} \to C_I - C_{II}$; $F_{e4} \to S_I - S_{II}$; $F_{e5} \to C_{II}$; $F_{e6} \to S_{II}$



il carico non è mai concentrato sull'asse, si ipotizza quindi di flettere la trave

$$M_f + P(\delta - y) = 0$$

$$M_f = -P(\delta - y)$$

$$\frac{d^2y}{dx^2} = -\frac{M_f}{EI_z}$$

$$\frac{d^2y}{dx^2} = +\frac{P}{EI_z}(\delta - y) \qquad \frac{d^2y}{dx^2} + \frac{P}{EI_z}y = \frac{P}{EI_z}\delta$$

l'equazione differenziale ha soluzione:

$$y = A\cos\sqrt{\frac{P}{EI_z}}\,x + B\sin\sqrt{\frac{P}{EI_z}}\,x + \delta$$

condizioni al contorno:

per $x = 0$ $\quad y = 0 \quad \rightarrow \quad A + \delta = 0 \Rightarrow A = -\delta$

per $x = 0$ $\quad \frac{dy}{dx} = 0 \quad \rightarrow \quad B = 0$

$$y = -\delta\cos\sqrt{\frac{P}{EI_z}}\,x + \delta \qquad y = \delta\left(1 - \cos\sqrt{\frac{P}{EI_z}}\right)$$

condizione trave flessa:

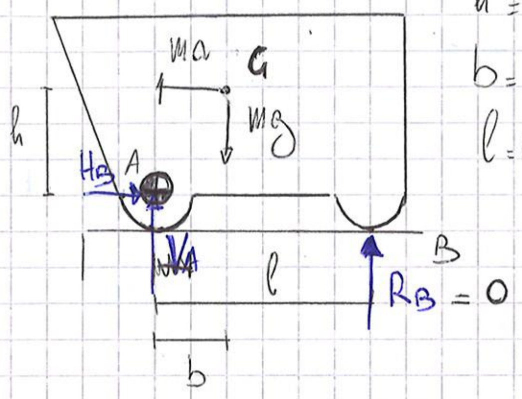
per $x = l$ $\quad y = \delta \qquad \delta = -\delta\cos\sqrt{\frac{P}{EI_z}}\,l + \delta$

$$\cos\sqrt{\frac{P}{EI_z}}\,l = 0 \implies \sqrt{\frac{P}{EI_z}}\,l = \frac{\pi}{2} + n\pi$$ per n fondamentale $(=0)$:

$$\sqrt{\frac{P_{CR}}{EI_z}}\,l = \frac{\pi}{2} \qquad \frac{P_{CR}}{EI_z}l^2 = \frac{\pi^2}{4} \qquad P_{CR} = \frac{\pi^2 EI_z}{4l^2}$$

$$(2l)^2 = l_0^2$$

• ESERCIZIO 4: Cassone ribaltabile su veicolo: individuare l'accelerazione del veicolo per cui si annulla la reazione di B



$h = 0.9\,m$

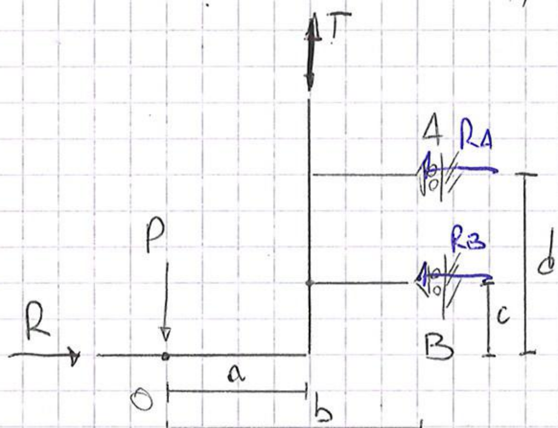
$b = 0.2\,m$

$l = 0.6\,m$

$$\begin{cases} V_A + \underbrace{R_B}_{=0} = mg \\ H_B = ma \\ mah + \underbrace{R_B l}_{=0} = mgb \end{cases} \quad \begin{cases} \cancel{m}ah = \cancel{m}gb \end{cases}$$

$$a = g\frac{b}{h} = 2.18\ m/s^2$$

• ESERCIZIO 5: Calcolare R_A, R_B, T della struttura.



$a = 400\ mm \qquad b = a + 100\ mm$

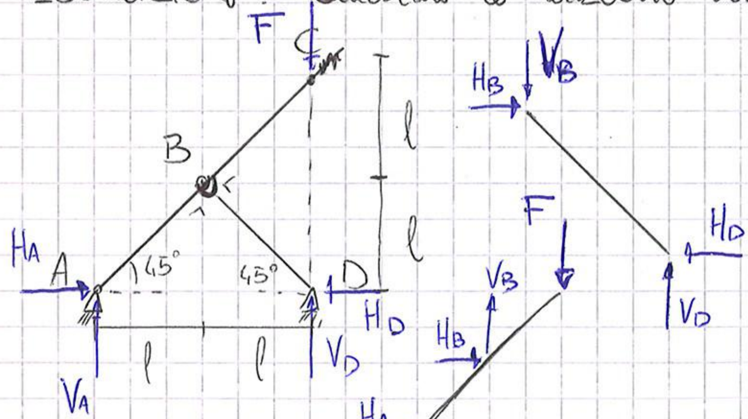
$c = 160\ mm \qquad d = 600\ mm$

$P = 10\ kN \qquad R = 1200\ N$

$$\begin{cases} R_A + R_B = R \\ P = T \\ Ta + R_A d + R_B c = 0 \end{cases} \quad \begin{cases} R_A = R - R_B \\ T = P \\ Pa + Rd - R_B d + R_B c = 0 \end{cases}$$

$$\begin{cases} R_B = \dfrac{Pa + Rd}{d - c} = \cancel{17.2\ kN}\ \ 10727\ N \checkmark \\ R_A = R - R_B = \cancel{-46\ kN}\ \ 9527\ N \checkmark \\ T = P = 10\ kN \checkmark \end{cases}$$

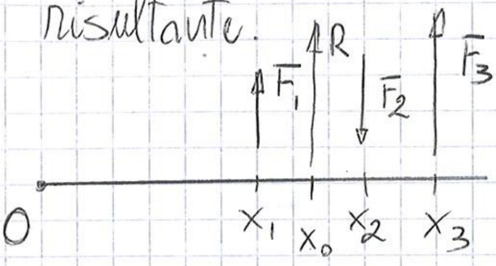
• ESERCIZIO 6: Calcolare le reazioni vincolari della struttura



$$\begin{cases} H_A = H_D \\ V_A + V_D = F \\ V_D 2\cancel{l} = F 2\cancel{l} \\ V_B = V_D \\ H_B = H_D \\ \cancel{H_B = H_B} \end{cases} \quad \begin{cases} V_D = F = 2\ kN \checkmark \\ V_A = F - V_D = 0\ N \\ V_B = V_D = F = 2\ kN \\ H_B = \dfrac{V_B l - 2Fl}{l} = -2\ kN \\ H_D = H_B = -2\ kN \checkmark \\ H_A = H_D = -2\ kN \checkmark \end{cases}$$

$2Fl = V_B l - H_B l$

• ESERCIZIO 9: Sostituire il sistema di forze con una sola risultante.



$F_1 = 500\,N \quad F_2 = 800\,N \quad F_3 = 1\,kN$

$x_1 = 5\,m \quad x_2 = 7\,m \quad x_3 = 8\,m$

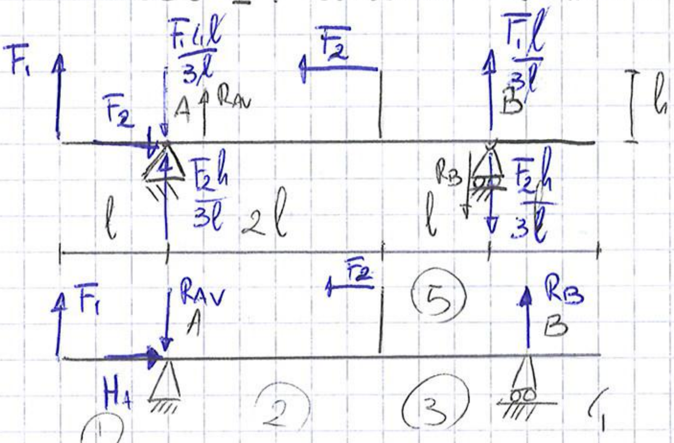
$R = F_1 + F_3 - F_2 = 700\,N$ ✓

$R x_0 = F_2 x_2 - F_1 x_1 - F_3 x_3$

$x_0 = \dfrac{-F_2 x_2 + F_1 x_1 + F_3 x_3}{R} = 7\,m$ ✓

ESERCITAZIONE 3 DIAGRAMMI DEGLI SFORZI

• ESERCIZIO 1: Calcolare le reazioni vincolari e gli sforzi della struttura



$l = 0.4\,m \quad h = 0.15\,m \quad F_1 = 80\,N$

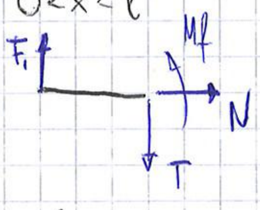
$F_2 = 120\,N$

$R_B = \dfrac{F_2 h}{3l} - \dfrac{F_1}{3} = -11.67\,N$ ✓

$H_A = F_2 = 120\,N$ ✓

$V_A = \dfrac{h F_2}{3l} - \dfrac{4F_1}{3} = -91.67\,N$ ✓

1) $0 < x < l$

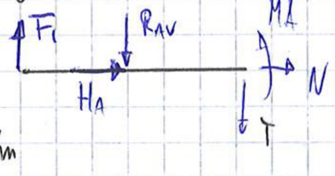


$N = 0$

$T = F_1 = 80\,N$

$Mf = +F_1 x \left\langle \begin{array}{l} 0 \\ +F_1 l = 32\,Nm \end{array} \right.$

(2) $l < x < 3l$

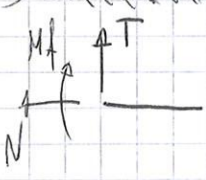


$N = -H_A = -F_2 = 120$

$T = F_1 - V_A = -11.67$

$Mf = F_1 x - V_A (x - l) \left\langle \begin{array}{l} 32 \\ 22.67 \end{array} \right.$

4) $0 < x' < l$



$N = 0$

$T = 0$

$Mf = 0$

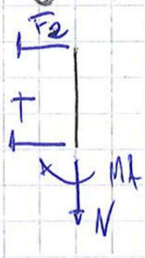
(3) $l < x' < 2l$

$N = 0$

$T = -R_B = -11.67$

$Mf = R_B (x' - l) \left\langle \begin{array}{l} 0 \\ 4.67\,Nm \end{array} \right.$

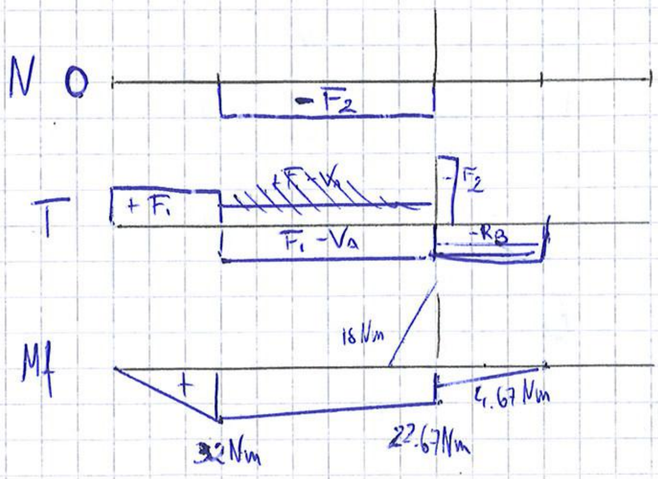
(5) $0 < x < h$



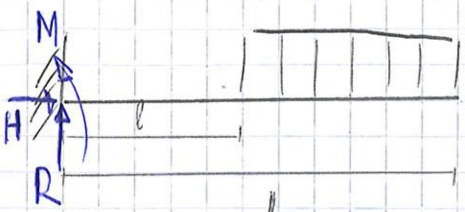
$N = 0$

$T = -F_2$

$Mf = F_2 x'$



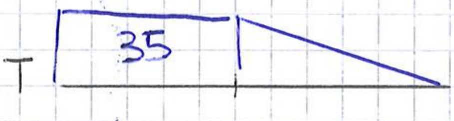
• ESERCIZIO 4: Calcolare reazioni e sforzi. $l = 0.5\,m$ $L = 1.2\,m$ $q = 50\,N/m$



$H = 0$ $\quad R = q(L-l) = 35\,N$ $\quad M = q(L-l)\left(\frac{L-l}{2}\right) = 29.75\,Nm$

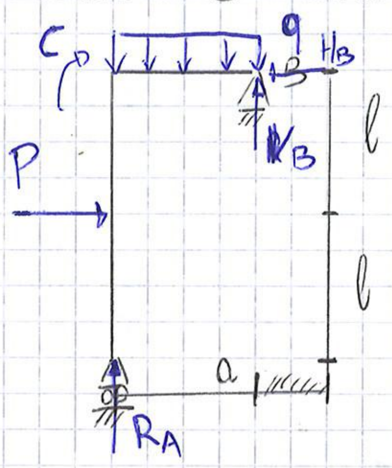
$[0,l]$ $\quad N = 0$ $\quad T = R = 35\,N$ $\quad Mf = Rx - M = \begin{cases} -29.75\,Nm \\ -12.25\,Nm \end{cases}$

$[l,L]$ $\quad N = 0$ $\quad T = R - q(x-l) = \begin{cases} 35\,N \\ 0\,N \end{cases}$ $\quad Mf = Rx - M - q\frac{(x-l)^2}{2} = \begin{cases} -12.25 \\ 0 \end{cases}$



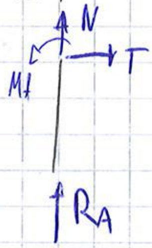
$N = 0$

• ESERCIZIO 5: Calcolare reazioni vincolari e sforzi. $l = 10\,m$ $\quad a = 7\,m$ $\quad q = 8\,\frac{kN}{m}$ $\quad C = 600\,Nm$ $\quad P = 1\,kN$



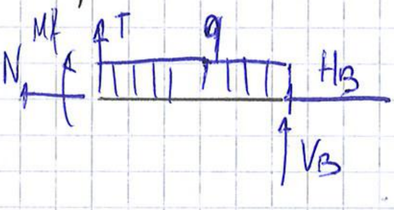
$$\begin{cases} H_B = P = 1\,kN \\ R_A + V_B = qa \rightarrow V_B = 26657 \\ R_A = \dfrac{q\frac{a^2}{2} - C + Pl}{a} = 29343\,N \end{cases}$$

$[0,l]$ $\quad N = -R_A = -29343$ $\quad T = 0$ $\quad Mf = 0$



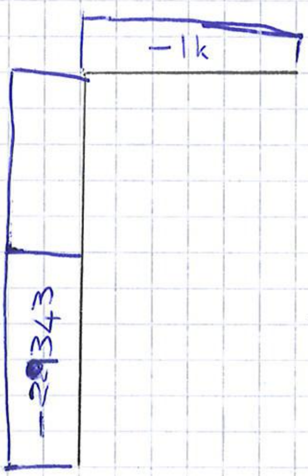
$[l,2l]$ $\quad N = -R_A$ $\quad T = -P = -1\,kN$ $\quad Mf = P(x-l) = \begin{cases} 0 \\ 10 \end{cases}$

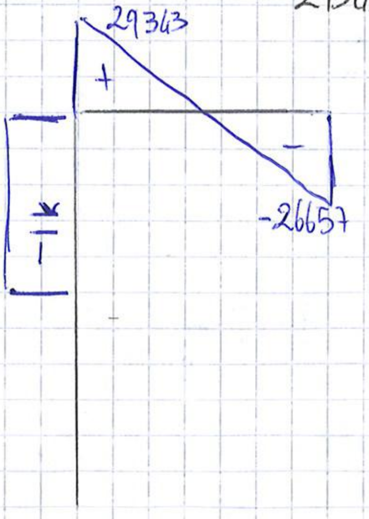
$(0,a]$

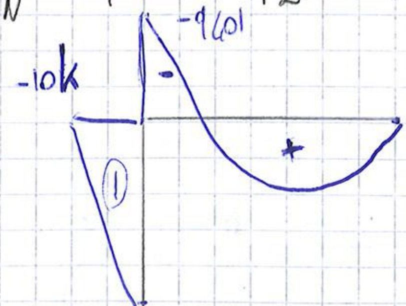


$N = -H_B = -1\,kN$

$T = qx - V_B = \begin{cases} -26657\,N \\ 29343\,N \end{cases}$ $\quad Mf = V_B x - q\frac{x^2}{2} = \begin{cases} 0 \\ -9401 \end{cases}$







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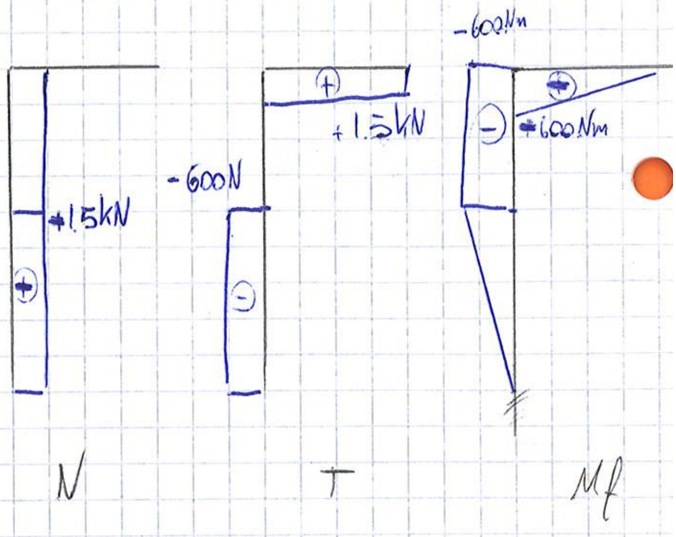
$1\,[h, s]$

$N = -V_A = -1.5\ kN$

$T = R_B - H_A = 0$

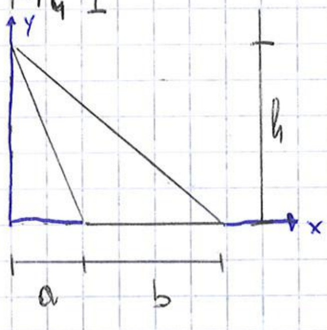
$M_f = -H_A x + R_B (x-h) = \begin{cases} -600\ Nm \\ -6\ Nm \end{cases}$



ESERCITAZIONE 4. GEOMETRIA DELLE AREE

ESERCIZIO 1: Calcolare la posizione di G e i momenti d'inerzia I_G delle figure 1-~~10~~18.

FIG 1



$a = 100\ mm$

$b = 600\ mm$

$h = 500\ mm$

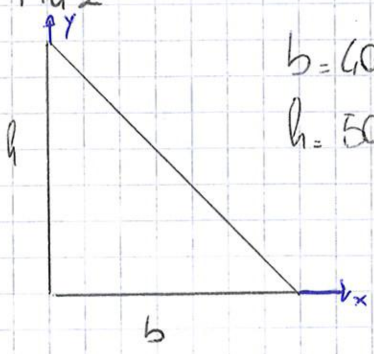
$Y_G = \frac{h}{3} = 166.67\ mm$

$$X_G = \frac{\left(\frac{a+b}{3}\right)\frac{1}{2}(a+b)h - \frac{a}{3}\left(\frac{1}{2}ah\right)}{\frac{bh}{2}} = 200\ mm$$

$$I_{xG} = \frac{(a+b)h^3}{36} = 138888\ cm^4$$

$$I_{yG} = \frac{(a+b)^3 h}{36} - \frac{a^3 h}{36} + \frac{(a+b)h}{2}\left(\frac{a+b}{3} - X_G\right)^2 - \frac{ah}{2}\left(\frac{a}{3} - X_G\right)^2 = 11666\ cm^4$$

FIG 2



$b = 400\ mm$

$h = 500\ mm$

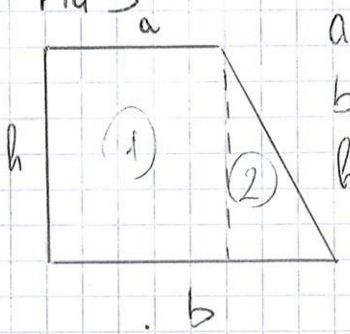
$X_G = \frac{b}{3} = 133.34\ mm$

$Y_G = \frac{h}{3} = 166.67\ mm$

$I_{xG} = \frac{bh^3}{36} = 138888\ cm^4$

$I_{yG} = \frac{hb^3}{36} = 88888\ cm^4$

FIG 3



$a = 360\ mm$

$b = 600\ mm$

$h = 400\ mm$

$A_1 = ah = 152 \cdot 10^3\ mm^2$

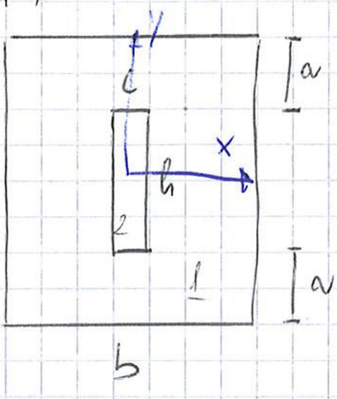
$A_2 = (b-a)h = 48 \cdot 10^3\ mm^2$

$A_T = A_1 + A_2 =$

$$X_G = \frac{\frac{a}{2}A_1 + \left(b - \frac{2}{3}a\right)A_2}{A_T} = 215\ mm$$

$$Y_G = \frac{\frac{h}{2}A_1 + \frac{h}{3}A_2}{A_T} = 186\ mm$$

FIG 7



a = 100 mm
b = 600 mm
c = 100 mm
h = 600 mm

$X_G = 0$ $X_{G1} = 0$ $\cancel{X}_{G2} = 0$

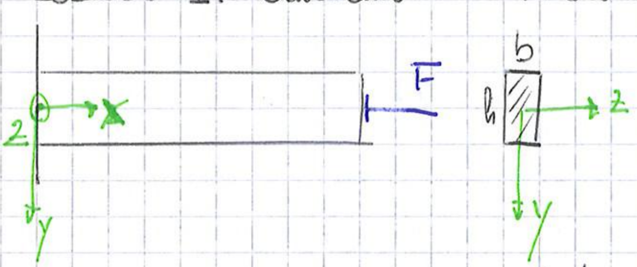
$Y_G = 0$ $\cancel{Y}_{G1} = 0$ $Y_{G2} = 0$

$$I_{xG} = \frac{b(h+2a)^3}{12} - \frac{ch^3}{12} = 666667 \text{ cm}^4$$

$$I_{yG} = \frac{b^3(h+2a)}{12} - \frac{c^3h}{12} = 316667 \text{ cm}^4$$

ESERCITAZIONE 5 - CALCOLO DELLE TENSIONI

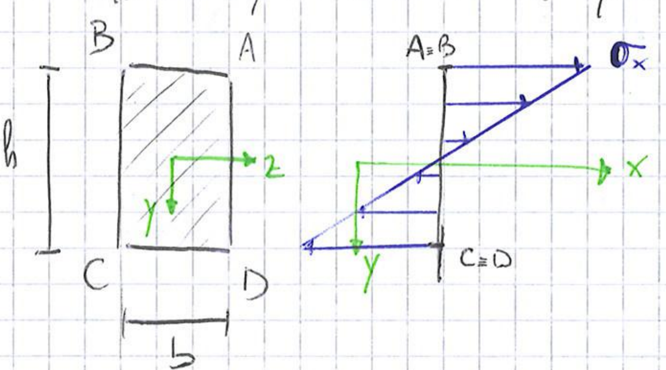
• ESERCIZIO 1: Calcolare la tensione normale della trave F = 60 kN b = 60 mm h = 100 mm



F = -N
N = - F

$$\sigma_x = \frac{N}{A} = -\frac{F}{bh} = -10 \text{ MPa}$$

• ESERCIZIO 2: La trave precedente è soggetta ad un Mf = -12 kNm. Tracciare l'andamento delle tensioni lungo l'asse, calcolare le tensioni massima, minima e nel punto z = -20 mm y_1 = -30 mm



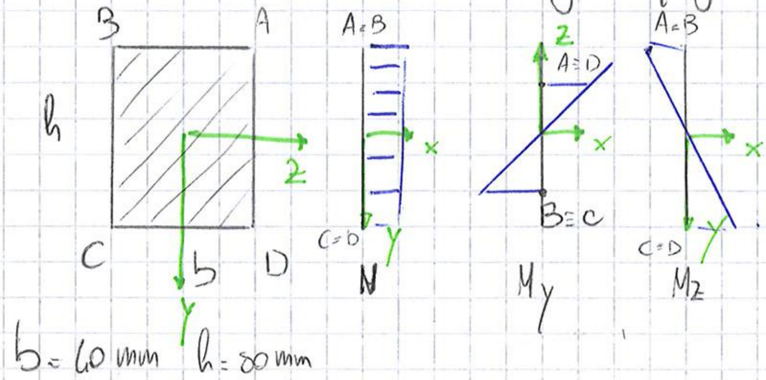
$$\sigma_x = \frac{M_z y}{I_z} \qquad I_z = \frac{bh^3}{12} = 5 \cdot 10^{-6} m^4$$

$$\sigma_{max} = \frac{M_z y_A}{I_z} = 120 \text{ MPa}$$

$$\sigma_{min} = \frac{M_z y_C}{I_z} = -120 \text{ MPa}$$

$$\sigma_1 = \frac{M_z y_1}{I_z} = 72 \text{ MPa}$$

• ESERCIZIO 3: La trave è soggetta a M_z = 10 kNm, M_y = 6 kNm, N = 64 kN. Calcolare la tensione negli spigoli



b = 60 mm h = 80 mm

$$\sigma_A = \frac{N}{bh} + \frac{M_y \frac{b}{2}}{\frac{b^3h}{12}} + \frac{M_z(-\frac{h}{2})}{\frac{bh^3}{12}} = -26.88 \text{ MPa}$$

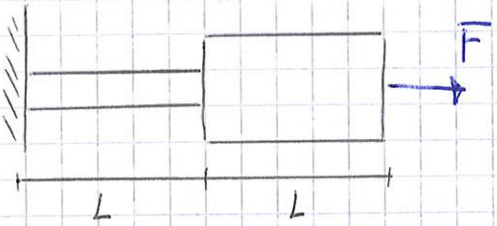
$$\sigma_B = \frac{N}{bh} + \frac{M_y(-\frac{b}{2})}{\frac{b^3h}{12}} + \frac{M_z(-\frac{h}{2})}{\frac{bh^3}{12}} = -602 \text{ MPa}$$

$$\sigma_C = \frac{N}{bh} + \frac{M_y(-\frac{b}{2})}{\frac{b^3h}{12}} + \frac{M_z(\frac{h}{2})}{\frac{bh^3}{12}} = 68 \text{ MPa}$$

$$\sigma_D = \frac{N}{bh} + \frac{M_y(\frac{b}{2})}{I} + \frac{M_z(\frac{h}{2})}{\frac{bh^3}{12}} = 662 \text{ MPa}$$

• ESERCIZIO 8: Trovare la legge di variazione degli sforzi normali, delle tensioni σ e degli spostamenti u.

$F = 50\ kN$ $A_1 = 375\ mm^2$ $L = 300\ mm$ $A_2 = 2A_1$



$N = F = 50\ kN$

$\sigma_1 = \frac{N}{A_1} = 133\ MPa$

$\sigma_2 = \frac{N}{A_2} = \frac{N}{2A_1} = \frac{\sigma_1}{2} = 66.67\ MPa$

$u_1 = \varepsilon_1 L = \frac{N}{EA_1} L = \sigma \frac{L}{E} = 0.19\ mm$

$u_{TOT} = u_1 + u_2 = 0.285\ mm$

$u_2 = \varepsilon_2 L = \sigma_2 \frac{L}{E} = 0.095\ mm$

• ESERCIZIO 9: Un binario ferroviario di acciaio ($\alpha = 11.7 \cdot 10^{-6}\ °C^{-1}$) è stato posato ad una temperatura di 6°C. Determinare la tensione dei binari supponendo una lunghezza iniziale di 10m e una temperatura finale di 48°C.

$l_f = l_0(1 + \alpha\Delta T) = l_0 + l_0\alpha\Delta T$ $l_f - l_0 = \Delta l = l_0 \alpha \Delta T$

$\Delta l = \frac{\sigma}{E} l_0 = l_0 \alpha \Delta T$ $\sigma = E\alpha\Delta T = 103\ MPa$

• ESERCIZIO 10: Calcolare la deformazione angolare nel caso di sezione piena e di sezione cava.

$M_x = 260\ Nm$ $G = 77\ GPa$ $\phi = 30\ mm$ $D = 32\ mm$ $d = 22\ mm$

$\gamma_1 = \frac{M_x}{GI_p}\frac{\phi}{2} = \frac{16 M_x \cancel{\phi}}{G\pi \cancel{\phi^4}^3} = 0.00063$

$\gamma_2 = \frac{M_x}{GI_p}\frac{D}{2} = \frac{16 M_x D}{G\pi(D^4 - d^4)} = 0.00068$

• ESERCIZIO 6: Calcolare le tensioni principali.

$$\sigma = \begin{bmatrix} 120 & -150 & 0 \\ -150 & 240 & 0 \\ 0 & 0 & 108 \end{bmatrix} \text{MPa} \quad \det \begin{vmatrix} 120-\sigma & -150 & 0 \\ -150 & 240-\sigma & 0 \\ 0 & 0 & 108-\sigma \end{vmatrix} = (108-\sigma)\left[(120-\sigma)(240-\sigma) - 150^2\right] = 0$$

$$(108-\sigma)\left[\sigma^2 - 360\sigma + 6300\right] = \begin{cases} 108 \text{ MPa} \rightsquigarrow \sigma_2 \\ 342 \text{ MPa} \rightsquigarrow \sigma_1 \\ 18.4 \text{ MPa} \rightsquigarrow \sigma_3 \end{cases}$$

• ESERCIZIO 7: Barra a sezione circolare ($\phi = 40$ mm) sollecitata da $N = 3 \cdot 10^4$ N, $M_f = 500$ Nm, $M_T = 850$ Nm. Determinare le tensioni principali del punto più sollecitato

$$\sigma_x = \frac{N}{A} + M_f \frac{r}{I_z} = \frac{N}{\pi\left(\frac{D}{2}\right)^2} + \frac{M_f \cancel{D}\, 64}{2\pi D^{4}_{3}} = 103.43 \text{ MPa}$$

$$\tau_{xy} = \frac{M_T}{I_p} r = \frac{32 M_T \cancel{D}}{\pi D^{4}_{3}\, 2} = 67.64 \text{ MPa}$$

$$\sigma_1, \sigma_3 = \frac{\sigma_x}{2} \pm \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2} = \begin{cases} 136.8 \text{ MPa} \\ -33 \text{ MPa} \end{cases}$$

$$\sigma_2 = 0$$

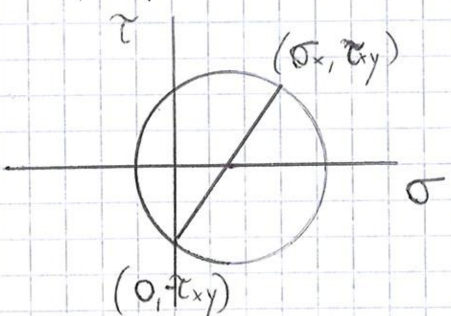
• ESERCIZIO 8: Barra a sezione circolare ($\phi = 40$ mm) sollecitata da $N = 5 \cdot 10^4$ N e $M_T = 850$ Nm. Determinare le tensioni principali del punto più sollecitato e tracciare i cerchi di Mohr.

$$\sigma_x = \frac{N}{A} = \frac{N}{\pi\left(\frac{D}{2}\right)^2} = 39.79 \text{ MPa}$$

$$\tau_{xy} = \frac{M_T}{I_p} \frac{D}{2} = \frac{16 M_T}{\pi D^3} = 67.64 \text{ MPa}$$

$$\sigma_1, \sigma_3 = \frac{\sigma_x}{2} \pm \sqrt{\frac{\sigma_x^2}{2} + \tau_{xy}^2} = \begin{cases} 125 \text{ MPa} \\ -33 \text{ MPa} \end{cases}$$

$$\sigma_2 = 0$$



• ESERCIZIO 9: Barra a sezione quadrata cava di lato $a = 40$ mm e spessore $s = 4$ mm sollecitata da $N = 6 \cdot 10^4$ N e $M_{f\,max} = 6 \cdot 10^5$ Nmm. Tracciare i cerchi di Mohr e determinare le tensioni principali sul punto più sollecitato.