NUMERO: 1500A - ANNO: 2015

A P P U N T I

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MATERIA: Physics I (GB) Prof.Iazzi

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PHYSICS I

Theory

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PHYSICS I

FELICE IAZZI

Textbook: M. Alonso - E. Finn: Physics

Exam: - Written (2 exercises, 16 pt (GET 14 pt), 16 pt)
- Oral

Laboratory: 2 experiences (3 hrs. each) [Monday afternoon, Wedner. morning]

Imp. Quantities:

- LENGTH L
- MASS M
- TIME T
- TEMPERATURE Θ
- ~~CURRENT~~
- CHARGE (ELECTRIC) Q

Direct measurement: using a tool
Indirect measurement: using other quantities

Volume = $[V] = L^3$
Surface = $[S] = L^2$
Speed = $[v] = L \cdot T^{-1}$
Acceleration = $[a] = L \cdot T^{-2}$
Force = $[F] = M \cdot L \cdot T^{-2}$

Angle = $[\alpha] = \dfrac{\text{length of arch}}{\text{radius}} = \dfrac{L}{L}$ → dimensionless

$[\sin\alpha] = \dfrac{L}{L}$ → dimensionless (like all trigonometric functions)

DER. $\left[\dfrac{dy}{dx}\right] = \left[\lim_{\Delta x \to 0} \dfrac{y(x+\Delta x) - y(x)}{\Delta x}\right] = \dfrac{[y]}{[x]}$ → ~~or dimensions~~

INT. $\left[\int f(x)\,dx\right] = \left[\lim \sum f(x)\Delta x\right] = \left[f(x)\cdot\Delta x\right] = [f(x)]\cdot[\cancel{\Delta} x]$

2nd way to measure errors

$$* \quad (\Delta x)^2 = \sum_{i=1}^{M} (x_i - \bar{x})^2 / M = \sigma^2$$

$$\Delta x = \sqrt{\sigma^2} = \sigma$$

Δx is, in this case, called statistical error.

RELATIVE ERROR

$$\left|\frac{\Delta x}{\bar{x}}\right| = \varepsilon_r < \begin{matrix} 0,2 \\ 20\% \end{matrix}$$

$$f(x,y,z) = x^{\alpha} \cdot y^{\beta} \cdot z^{\gamma}$$

$$\Delta f = \left|\left(\frac{\partial f}{\partial x}\right)\right| \cdot |\Delta x| + \left|\left(\frac{\partial f}{\partial y}\right)\right| \cdot |\Delta y| + \left|\left(\frac{\partial f}{\partial z}\right)\right| \cdot |\Delta z|$$

$$\frac{\partial}{\partial x}(x^{\alpha} \cdot y^{\beta} \cdot z^{\gamma}) = (\alpha \bar{x}^{\alpha-1})(\bar{y}^{\beta} \cdot \bar{z}^{\gamma})$$

$$\frac{\partial}{\partial y}(x^{\alpha} \cdot y^{\beta} \cdot z^{\gamma}) = (\beta \cdot \bar{y}^{\beta-1})(\bar{x}^{\alpha} \cdot \bar{z}^{\gamma})$$

$$\frac{\partial}{\partial z}(\ldots) = (\gamma \cdot \bar{z}^{\gamma-1})(\bar{x}^{\alpha} \bar{y}^{\beta})$$

$$(\alpha \cdot |\bar{x}^{\alpha-1} \cdot \bar{y}^{\beta} \cdot \bar{z}^{\gamma}| \cdot \Delta x) + (\beta \cdot |\bar{y}^{\beta-1} \cdot \bar{x}^{\alpha} \cdot \bar{z}^{\gamma}| \cdot \Delta y) + \ldots$$

$$\text{Relative error of } f = \frac{\Delta f}{f(\bar{x},\bar{y},\bar{z})} = \frac{\Delta f}{|\bar{x}^{\alpha} \cdot \bar{y}^{\beta} \cdot \bar{z}^{\gamma}|}$$

symplifying

$$\square \quad \frac{\Delta f}{|f(\bar{x},\ldots)|} = \alpha \frac{\Delta x}{|\bar{x}|} + \beta \frac{\Delta y}{|\bar{y}|} + \gamma \frac{\Delta z}{|\bar{z}|}$$

~~number of measurements~~

~~$N \to$ very big~~ ~~x_i - age of a person~~

~~$N \approx 6 \cdot 10^9$~~

~~$x_1 = 0$~~ ~~$x_{MAX} = 121$ years~~

~~$x_{MAX} = 64 \cdot 10^6$ minutes~~

~~1: $x_1 - x_0 = x_1 - x_{MIN}$~~ ~~$f_1$~~

~~2: $x_2 - x_1$~~ ~~f_2~~

~~3: $x_3 - x_2$~~ ~~f_3~~

~~k: $x_k - x_{k-1}$~~ ~~f_k~~

N measurements

$x_1, x_2, x_3, x_4, \ldots x_N$

18 18 20 30 25

$$\bar{x} = \frac{18+18+20+30+\ldots+25}{N}$$

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{2 \cdot 18 + 1 \cdot 30 + \ldots}{N}$$

I make a number of groups of equal values

Number of groups = H

$H \leq N$

f_k = FREQUENCY OF x_k

f_1 = number of values equal to x_1

$f_2 = \ldots x_2$ / $f_k = \ldots x_k$

$1 \leq k \leq H$

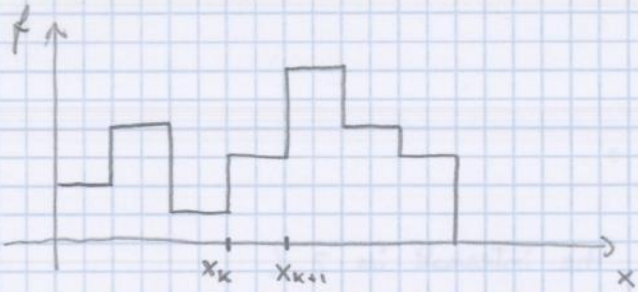
So:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\square \quad \bar{x} = \frac{1}{N}\sum_{k=1}^{H} f_k x_k$$

Recall

$$\bar{x} = \frac{1}{N} \sum_{k=1}^{M} f_k x_k$$



↳ Inside this interval there are some values of x (as many as "f" indicates)

$$\Delta x_k = x_{k+1} - x_k$$

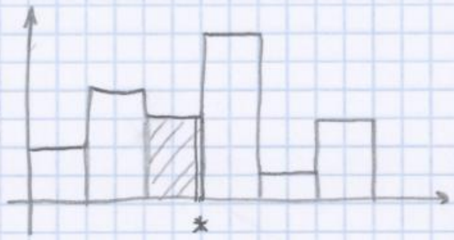
In a histogram, "f" is related to the number of different values inside the interval.

I consider $\sum x_m$, $m \in \Delta x_k$

$$\sum_{m \in \Delta x_k} x_m = f_k \cdot \left(\frac{\sum_{m \in \Delta x_k} x_m}{f_k} \right) = f_k \cdot \bar{x}_k$$

↳ sum of all the values divided by the n° of values

average



As a definition, $\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i = \frac{1}{N} \cdot \sum_{k=1}^{M} \left(\sum_{m \in \Delta x_k} x_m \right)$

$$= \frac{1}{N} \sum_{k=1}^{M} f_k \cdot \bar{x}_k$$

1) If $N \to \infty$ and Δx_k small

2) $\Rightarrow \Delta x_k \to$ infinitesimal

1) $\bar{x} \approx \frac{1}{N} \sum_{k=1}^{M} f_k \cdot x_k$ because $\bar{x}_k$ becomes very close to x_k

(the difference between one of the boundaries and the average gets smaller and smaller) *

2) $\bar{x} = \frac{1}{N} \sum_{k=1}^{M} f_k \cdot x_k$

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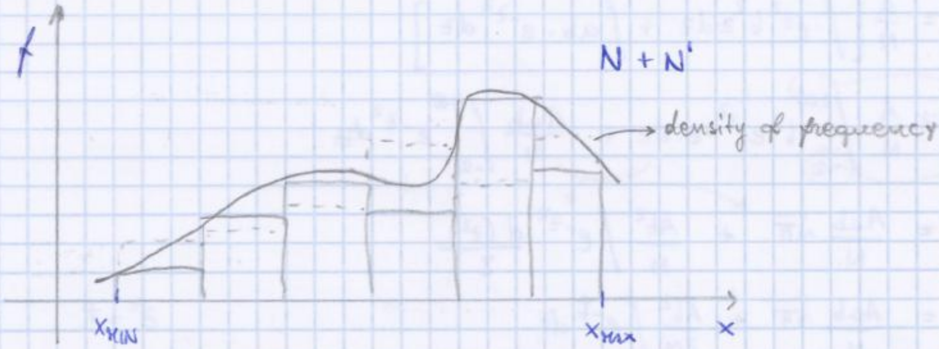
$$\bar{x} = \sum_{k=1}^{M} \frac{\Delta N_k}{N \cdot \Delta x} x_k \Delta x \xrightarrow[\Delta x \to dx]{} \int_{x_{MIN}}^{x_{MAX}} \frac{dN(x)}{N \cdot dx} \cdot x \cdot dx$$

and $\frac{dN}{dx}$ is called "density of frequency" or "distribution of the variable x"

Good to represent situations where N is very big and the x intervals are very small!

$$\bar{x} = \frac{1}{N} \sum_{k=1}^{M} \frac{f_k}{\Delta x} \cdot \Delta x \cdot x_k$$

$$\bar{x} = \frac{1}{N} \int_{x_{MIN}}^{x_{MAX}} \left(\frac{dN}{dx}\right) \cdot x \cdot dx$$



* If we add a new series of measurements (N') (- - -) the istogram is deformed, changes. Therefore, also the density of frequency changes.

* As you increase the number of columns, the curve starts to represent the function $\frac{dN}{dx}$ pretty accurately.

$$\sum_{k=1}^{N} f_k = N \quad \rightarrow \quad \int_{-\infty}^{+\infty} \frac{dN}{dx}\, dx = N$$

$$N = \int_{-\infty}^{+\infty} A \cdot e^{-\frac{(x-a)^2}{b^2}}\, dx =$$

$$\frac{x-a}{b} = z$$

$$x = bz + a$$

$$dx = (b)(dz)$$

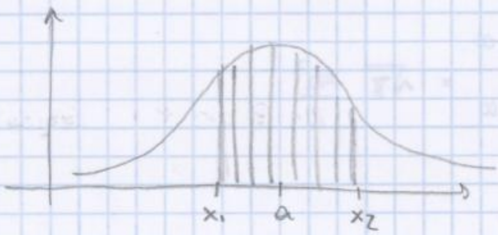
$$= \int A e^{-z^2}(bz+a) b \cdot dz$$

$$= \int A e^{-z^2} bz \cdot dz + \int A e^{-z^2} ab\, dz = A a \sqrt{\pi}$$

$$\begin{cases} N = A a \sqrt{\pi} & \rightarrow A = \dfrac{N}{a\sqrt{\pi}} \\ \bar{x} = \dfrac{Aab}{N} \sqrt{\pi} \end{cases}$$

Recall

$\hookrightarrow$ $$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2$$



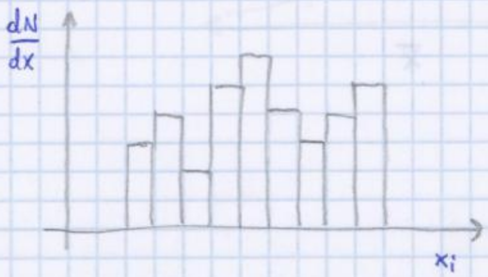
How many measurements between x_1 and x_2 ?

$$N(x_1, x_2) = \int_{x_1}^{x_2} \frac{dN}{dx}\, dx$$

$\hookrightarrow$ number of values between x_1 and x_2

$$\bar{x} = \frac{Aab\sqrt{\pi}}{N} = \frac{aN}{N} = a$$

We want to find the density of frequency of $(x_i - \bar{x})^2$.



The density of frequency of a function is the same as the density of frequency of the indep. variable!

$f(x_i)$

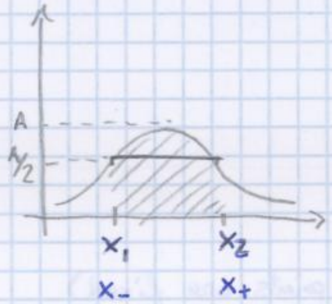
$$\Downarrow$$

$$\frac{dN}{dx} = N \cdot \boxed{\frac{e^{-\frac{(x-\bar{x})^2}{6^2}}}{\sqrt{\pi}\cdot 6}}$$

The n° of measurements affects the height of the function

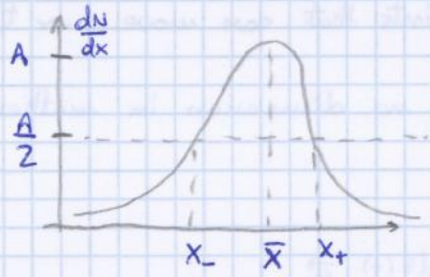
This is always the same

The Gaussian function is actually the density of probability



$$P(x_1, x_2) = \int_{x_1}^{x_2} \frac{dN}{dx} \cdot dx$$

Probability



$$\frac{A}{2} = A \cdot e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

$$-\ln 2 = -\frac{(x-\bar{x})^2}{2\sigma^2}$$

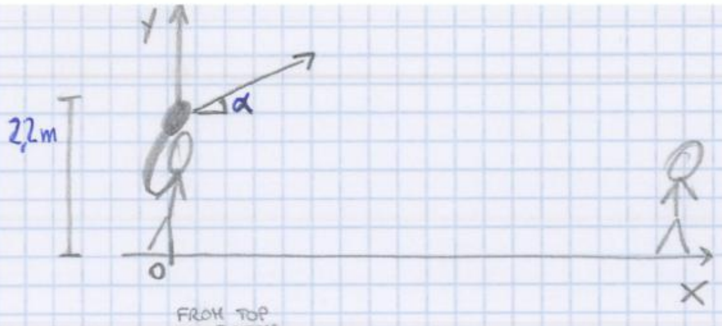
$$\sigma^2 \cdot 2\ln 2 = \frac{(x-\bar{x})^2}{\cancel{\sigma^2}} \qquad \text{So} \Rightarrow x - \bar{x} = \pm\sqrt{2\ln 2} \cdot \sigma$$

$$x_+ = \bar{x} + \sqrt{2\ln 2} \cdot \sigma$$

$$x_- = \bar{x} - \sqrt{2\ln 2} \cdot \sigma$$

FWHM (Full width @ half maximum) $= (x_+) - (x_-) = 2\sqrt{2\ln 2} \cdot \sigma$

From the variance, we get σ. Then, w/ this formula, we know that most of our measurements fall in this area. (just for big N)



$\alpha = 45°$

$v_0 = 10 \frac{m}{s}$

$\boxed{\downarrow \; g = 9{,}8\ N}$

FROM TOP TO BOTTOM

$\vec{a} = -g \cdot \vec{j}$ → UNIT VECTOR

(1)
$$\begin{cases} v_x = \int \overset{=0}{a_x}\, dt + c_1 = c_1 \\ v_y = \int a_y\, dt + c_2 = \int -g\, dt + c_2 = -gt + c_2 \\ v_z = \int a_z\, dt + c_3 = c_3 \end{cases}$$

→ 3 components of velocity (2 constant, 1 varies)

(2) Now, I want to determine c_1, c_2, c_3

$v_x(0) = v_0 \cos\alpha$

$v_y(0) = v_0 \sin\alpha$

$v_z(0) = 0$

(1+2)

$v_x = c_1 = v_0 \cos\alpha$

$v_y = -gt + c_2 = v_0 \sin\alpha = c_2$ (because $t=0$ @ the beginning)

$v_z = c_3 = 0$

$$\begin{cases} \frac{dx}{dt} = c_1 \\ \frac{dy}{dt} = -gt + c_2 \\ \frac{dz}{dt} = c_3 \end{cases} \quad \text{integrating} \quad \begin{cases} x(t) = c_1 t + c_4 \to = 0 \ * \\ y(t) = -\frac{1}{2} g t^2 + c_2 t + c_5 \to = 2{,}2\,m \ * \\ z(t) = c_3 t + c_6 \to = 0 \ * \end{cases}$$

At $t=0$ $x(0) = 0$

$y(0) = 2{,}2$

$z(0) = 0$

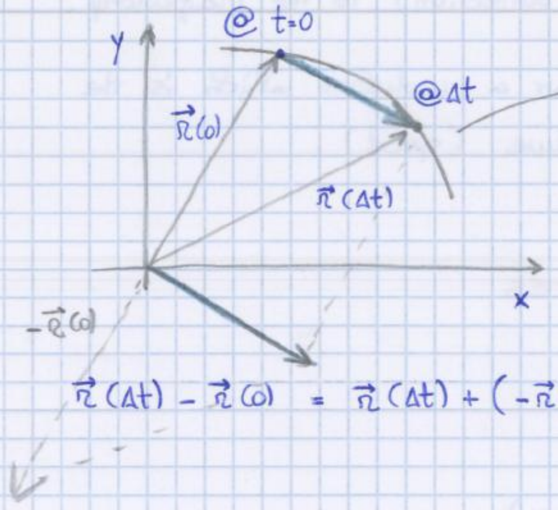
and then we need the initial speed:

$\frac{dy}{dt}(t=0) = v_0 \sin\alpha$

$\frac{dx}{dt}(t=0) = v_0 \cos\alpha$

$\frac{dz}{dt}(t=0) = 0$

(the x-y plane has to contain the direction of the velocity)



Either a tridimensional trajectory or a planar trajectory (infinitesimal time interval)

$$\vec{r}(\Delta t) - \vec{r}(0) = \vec{r}(\Delta t) + (-\vec{r}(0))$$

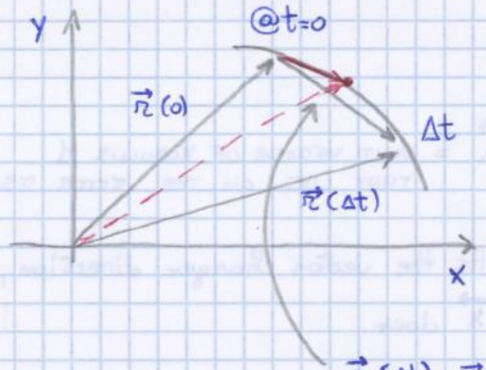
are equal and parallel

When Δt goes to 0 $(\Delta t \to 0)$, the vector becomes tangent to the curve.

I consider

$$\lim_{\Delta t \to 0} \frac{\vec{r}(\Delta t) - \vec{r}(0)}{\Delta t} =$$

$$= \frac{d\vec{r}(t)}{dt} = \vec{v}(t)$$ $\Rightarrow$ So, the velocity is always tangent ~~parallel~~ to the trajectory!



Δs = length of the cord

$\vec{r}(\Delta t) - \vec{r}(0)$ whose length is $|\vec{r}(\Delta t) - \vec{r}(0)|$

If I reduce the length of the arch (to the point •), the length of the cord is now closer to the length of the arch.

So, the smaller the arch, the closer the cord to the arch.

$$\lim |\vec{r}(\Delta t) \cdot \vec{r}(0)| = \lim \Delta s \Rightarrow \lim_{\Delta t \to 0} \frac{|\vec{r}(\Delta t) \cdot \vec{r}(0)|}{|\Delta t|} = \underbrace{\frac{|\Delta s|}{|\Delta t|}}_{\lim_{\Delta t \to 0}} \Rightarrow$$

$$\lim_{\Delta t \to 0} \left| \frac{- - -}{- - -} \right| = \lim_{\Delta t \to 0} \left| \frac{- -}{- -} \right| \Rightarrow$$

$$\Rightarrow \left| \lim_{\Delta t \to 0} \frac{- - -}{- - -} \right| = \lim_{\Delta t \to 0} \left| \frac{- -}{- -} \right| \Rightarrow |\vec{v}(t)| = \lim_{\Delta t \to 0} \left| \frac{\Delta s}{\Delta t} \right|$$ □

□ So speed is actually the modulus of velocity! (Velocity can have positive or negative direction; it is a vector)

Vector displacement :

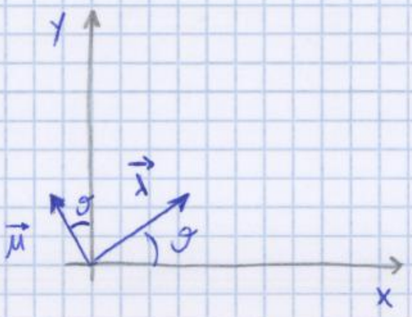
$\vec{r}(t) = r(t) \cdot \vec{\lambda}(t)$ (You use only 2 dimensions)

$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \overset{(DER)}{=} \dot{r}(t) \cdot \vec{\lambda} + r(t) \cdot \dot{\vec{\lambda}}$

transform this into s.thing better, an actual unit vector

NOTATION

$\boxed{\frac{df}{dt} = \dot{f}}$



$\vec{\lambda} = \cos\vartheta \, \vec{i} + \sin\vartheta \, \vec{j}$

$\vec{\mu} = -\sin\vartheta \, \vec{i} + \cos\vartheta \, \vec{j}$

$\dot{\vec{\lambda}} = -\dot{\vartheta}\sin\vartheta \, \vec{i} + \dot{\vartheta}\cos\vartheta \, \vec{j} = \dot{\vartheta}(\vec{\mu})$

$\dot{\vec{\mu}} = -\dot{\vartheta}\cos\vartheta \, \vec{i} - \dot{\vartheta}\sin\vartheta \, \vec{j} = -\dot{\vartheta}(\vec{\lambda})$

$\vec{v}(t) = \dot{r}\,\vec{\lambda} + r \cdot \dot{\vartheta} \cdot \vec{\mu}$

der. of length → velocity

LENGTH

$\frac{1}{\text{TIME}}$

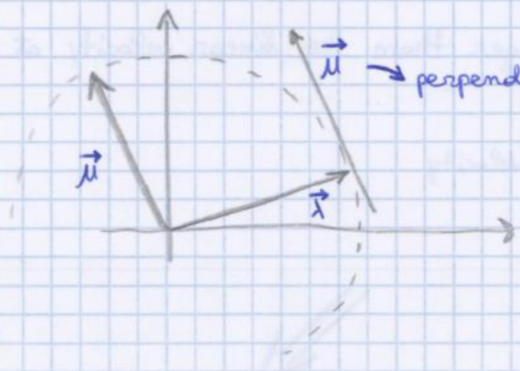
[$\vec{\lambda}$, $\vec{\mu}$, or directions → angles] are dimensionless

If the trajectory is a circumference, the vector displacement is the radius. The radius is constant so

distance of the

$r(t) = \text{const} \Rightarrow \dot{r} = 0$

So $\vec{v}(t) = \cancel{\dot{r}\vec{\lambda}} + r \cdot \dot{\vartheta} \cdot \vec{\mu}$ (velocity only along the direction of $\vec{\mu}$)



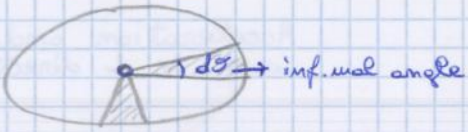
$\vec{\mu}$ → perpendicular to $\vec{\lambda}$, perpendicular to the radius

Proof :

$$\vec{a} = (\ddot{r} - r\dot{\vartheta}^2)\vec{\lambda} + (2\dot{r}\dot{\vartheta} + r\ddot{\vartheta})\vec{\mu}$$

Suppose $\vec{a} \parallel \vec{\lambda}$ (parallel)

So $2\dot{r}\dot{\vartheta} + r\ddot{\vartheta} = 0$



$dA = \frac{1}{2} r \cdot r\, d\vartheta$ (assuming that the ellypse behaves as a circle)

$$\frac{dA}{dt} = \frac{1}{2} r \cdot r \cdot \frac{d\vartheta}{dt} = \text{AREAL SPEED}$$

$$\frac{d\dot{A}}{dt} = \frac{1}{2} \cdot 2r \cdot \dot{r} \cdot \dot{\vartheta} + \frac{1}{2} r^2 \ddot{\vartheta} = \frac{1}{2} r \left(2\dot{r}\dot{\vartheta} + r\ddot{\vartheta}\right)$$

$$= \frac{1}{2} r \cdot a_\mu$$

$\to$ COMPONENT μ OF ACCELERATION

If $a_\mu = 0$

then $\frac{d\dot{A}}{dt} = 0$

so $\dot{A} = \text{const}$ $\Rightarrow$ Areal speed is constant

---

Using cylindrical coordinates, we obtained

$$\begin{cases} \vec{v}(t) = \dot{r}(t)\vec{\lambda} + r\dot{\vartheta}(t)\vec{\mu} \\ \vec{a}(t) = (\ddot{r} - r\dot{\vartheta}^2)\vec{\lambda} + (2\dot{r}\dot{\vartheta} + r\ddot{\vartheta})\vec{\mu} \end{cases}$$

PLANAR MOTION

SPACIAL

FOR TRIDIMENSIONAL MOTION,

WE SIMPLY ADD $z(t) \cdot \vec{k}$

Problem using cylindrical coordinates

$$\vec{r}(t) = r(t) \cdot \vec{\lambda} + z(t)\vec{k}$$

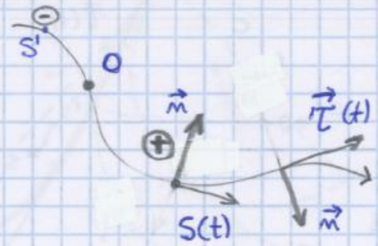
$$\vec{v}(t) = \dot{r}(t)\vec{\lambda} + r\dot{\vartheta}(t)\vec{\mu} + \dot{z}(t) \cdot \vec{k}$$

$$\vec{a}(t) = (\ddot{r} - r\dot{\vartheta}^2)\vec{\lambda} + (2\dot{r}\dot{\vartheta} + r\ddot{\vartheta})\vec{\mu} + \ddot{z}(t) \cdot \vec{k}$$

INTRINSIC COORDINATES

This reference system is used when you know the trajectory of a body.

(e.g. you know the trajectory of a train)



(PLANAR MOTION)

We define
- the origin
- the direction

S is the distance travelled (can be ⊕ or ⊖)

Knowing the trajectory, there is only 1 point at, say 3 meters from O.

If the body moves, S depends on t $\rightarrow$ S(t)

Unit vector : TAU (τ)

" " : $\vec{n}$: ⊥ TO TAU (always directed toward concavity)

$$\vec{v} = \dot{s} \cdot \vec{\tau}$$

→ ONLY THIS COMPONENT BECAUSE

velocity is tangent to the trajectory (so $\vec{n}$ won't appear in the formula)

$$\left(\text{Consider } \lim_{\Delta t \to 0} \left| \frac{S(t+\Delta t) - S(t)}{\Delta t} \right| = |\dot{s}(t)| \right)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{s}\vec{\tau} + \dot{s}\dot{\vec{\tau}}$$

→ BUT, THIS IS NOT A UNIT VECTOR!

$$\vec{\tau}^2 = 1$$

$$\frac{d(\vec{\tau}^2)}{dt} = 0$$

DERIVATIVE OF A PRODUCT

$$\frac{d(\vec{\tau}\cdot\vec{\tau})}{dt} = \frac{d\vec{\tau}}{dt}\cdot\vec{\tau} + \vec{\tau}\cdot\frac{d\vec{\tau}}{dt} = 2\vec{\tau}\cdot\frac{d\vec{\tau}}{dt}$$

So, $0 = \vec{\tau}\cdot\dot{\vec{\tau}} = |\vec{\tau}|\cdot|\dot{\vec{\tau}}|\cdot\cos\alpha \rightarrow$ For $\cos\alpha = 0$, $\alpha = 90°$

so $\vec{\tau} \perp \dot{\vec{\tau}}$

How can this be 0? $\vec{\tau} = 1$ ALWAYS

So, $\dot{\vec{\tau}} = a\vec{n}$

$$\Rightarrow \quad |\dot{\vec{\tau}}| = \frac{1}{S(t)} \cdot |\dot{s}(t)|$$

Recall: $\dot{\vec{\tau}} = a \cdot \vec{m}$ So, $|\dot{\vec{\tau}}| = |a|$ (because $|\vec{m}| = 1$)

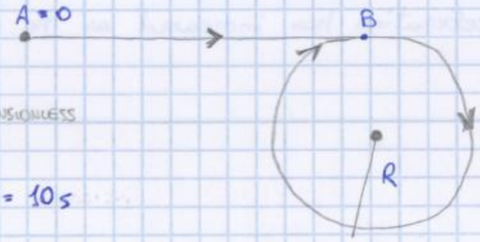
$$\Rightarrow \quad |a| = \frac{1}{\rho} \cdot |\dot{s}(t)|$$

just a number

$$\vec{a} = \ddot{s}\vec{\tau} + \dot{s} \cdot \frac{\dot{s}}{\rho} \cdot \vec{m} = \ddot{s}\vec{\tau} + \frac{\dot{s}^2}{\rho} \cdot \vec{m}$$

Taken the trajectory, the acceleration is always perpendicular to the tangent to the curve.

(ex) $\vec{v}(t) = \dot{s}\vec{\tau}$; $\vec{a} = \dot{s}(t)\,\vec{\tau}(t) + \frac{\dot{s}^2(t)}{\rho} \cdot \vec{m}$



$R = 150\ m$

DECREASE IN ACCELERATION AS YOU RELEASE THE PEDAL ← $a(t) = a_0 \cdot e^{-t/\tau}$ → DIMENSIONLESS

$a_0 = 40\ m/s^2$ $\quad \tau = 10\,s$

@ $t = 10\,s$ → CAR IS IN B

$\overline{AB} = ?$ $\quad |\vec{a}|$ one infinitesimal before B ?
" " after B ?

$$\vec{a}(t) = \ddot{s}(t) = a_0 \cdot e^{-t/\tau}$$

NO $\vec{\tau}$ BECAUSE IT'S A LINEAR ACCELERATION ?

Integrating

INITIAL CONDITION
$$\begin{cases} s(0) = 0 \\ \dot{s}(0) = 0 \end{cases}$$

$$\dot{s}(t) = -a_0 \tau e^{-t/\tau} + c_1$$

Integrating $\quad s(t) = +a_0\tau^2 e^{-t/\tau} + c_1 t + c_2$

$$\dot{s}(0) = 0 = -a_0\tau e^{-0/\tau} + c_1 = -a_0\tau + c_1$$

$$\Rightarrow \quad \dot{s}(t) = -a_0\tau e^{-t/\tau} + a_0\tau$$

$$s(t) = a_0\tau^2 e^{-t/\tau} + a_0\tau t + c_2$$

$$s(0) = 0 = a_0\tau^2 e^{-t/\tau} + a_0\tau t + c_2 = a_0\tau^2 + c_2 \quad \Rightarrow \quad c_2 = -a_0\tau^2$$

$$\Rightarrow \quad s(t) = a_0\tau^2 e^{-t/\tau} + a_0\tau t - a_0\tau^2$$

21/03/13

Recall:

CYL

$\vec{r} = r\vec{\lambda}$

$\vec{v} = \dot{r}\vec{\lambda} + r\dot{\vartheta}\vec{\mu}$

$\vec{a} = (\ddot{r} - r\dot{\vartheta}^2)\vec{\lambda} + (2\dot{r}\dot{\vartheta} + r\ddot{\vartheta})\vec{\mu}$

INT

NO VECTOR DISPLACEMENT BECAUSE TRAJECTORY IS KNOWN

$\vec{v} = \dot{s}\vec{\tau}$

$\vec{a} = \ddot{s}\vec{\tau} + \dfrac{\dot{s}^2}{\rho}\vec{m}$

Circular trajectory (radius = R)

Cyl:

$\vec{r} = R\vec{\lambda}$

$\vec{v} = R\dot{\vartheta}\vec{\mu} \rightarrow \perp$ to R

$\vec{a} = -R\dot{\vartheta}^2\vec{\lambda}^{*} + R\ddot{\vartheta}\vec{\mu}$

Int:

$\vec{v} = \dot{s}\vec{\tau} \rightarrow$ TANGENT TO TRAJ.

$\vec{a} = \ddot{s}\vec{\tau} + \underbrace{\dfrac{\dot{s}^2}{R}}_{\text{OSCULANT CIRC. IN CASE OF CIRCULAR TRAJ.}}\vec{m}^{*}$

* $\vec{\lambda}$ and $\vec{m}$ are parallel but have opposite direction $(-\vec{\lambda} = \vec{m})$

s = arch

$s = R\cdot\vartheta$

$s = R\cdot\vartheta$

$\Delta s = R\cdot\Delta\vartheta$

$\dfrac{\Delta s}{\Delta\vartheta} = R$

$\dfrac{\Delta s}{\Delta t} = R\cdot\dfrac{\Delta\vartheta}{\Delta t}$

$(\vec{v})$

$\dfrac{ds}{dt} = R\cdot\dot{\vartheta}$

$\dot{s} = R\cdot\dot{\vartheta}$ (components of $\vec{\mu}$ and $\vec{\tau}$)

$(\vec{a})$

$\ddot{s} = R\ddot{\vartheta}$ (components of $\vec{\mu}$ and $\vec{\tau}$)

$\dot{s}^2 = R^2\cdot\dot{\vartheta}^2$

$\dfrac{\dot{s}^2}{R} = R\cdot\dot{\vartheta}^2$ (components of $\vec{\lambda}$ and $\vec{m}$)

There's a perfect correspondance between cyl. and int. coordinates. When you're dealing w/ circular trajectories, you can use both systems.

$$\vec{a} = \dot{\vec{v}} = \dot{\vec{v}}_{O'} + \dot{\vec{v}}' + \frac{d}{dt}(\text{-----})$$

$$\ddot{x}'\vec{i}' + \dot{x}'\dot{\vec{i}}' + \ddot{y}'\vec{j}' + \dot{y}'\dot{\vec{j}}' + \ddot{z}'\vec{k}' + \dot{z}'\dot{\vec{k}}'$$

$$\vec{a} = \vec{a}_{O'} + \vec{a}' + \text{- - - - - - - - - -}$$ COMPLICATED TERMS NOT REQUIRED

acceleration seen from ref. sys. at rest | acceleration seen from moving ref. sys.

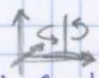
The origin of mov. ref. sys. can move anywhere, but z axis move and rotate w.r. to a fixed one.
So we'll reduce our discussion to rotation of z axis w.r.to a fixed one, which will be the z axis.

ANGULAR VELOCITY $= \dot{\vartheta}$

$$\vec{r} = \vec{r}_{O'} + \vec{r}'$$
$$\vec{v} = \vec{v}_{O'} + \vec{v}' + \vec{\omega} \times \vec{r}'$$
VECTOR PRODUCT
$$\vec{a} = \vec{a}_{O'} + \vec{a}' + 2\vec{\omega}\times\vec{v}' + \dot{\vec{\omega}}\times\vec{r}' + \underbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r}')}_{*}$$

THESE COME FROM DEMONSTRATION

$\vec{\omega}$ → has direction ⊥ to the rotation on a plane.
So, if x and y are changing, $\vec{\omega}$ has the direction of $\vec{k}$ (or the axis of rotation)



<u>DEF</u> $2\vec{\omega}\times\vec{v}'$ = CORIOLIS ACCELERATION

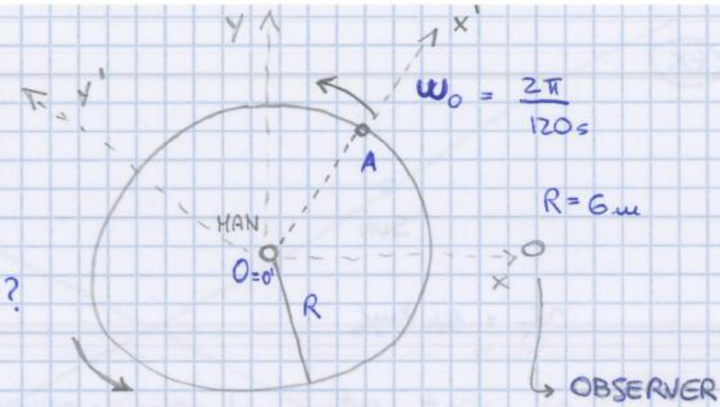
<u>Remark</u> * You cannot change the order of this cross products

ex EX. 2.6.14

Pace of man = 1 m/s = v_M

Man is going from O to A

Acceleration of man = a_M = ?



$\vec{v}\,' = v_M \cdot \vec{i}\,'$ → DIRECTION OF MAN WALKING

Here, x' and y' are rotating around z' (= z)

$\vec{\omega} = \omega_0 \cdot \vec{k}$

ORIGIN DOES NOT MOVE; MAN WALKS AT CONST. VELOCITY

$\vec{a} = \cancel{\vec{a}_{O'}} + \cancel{\vec{a}_r} + 2\,\vec{\omega} \times \vec{v}\,' + \cancel{\dot{\vec{\omega}} \times \vec{r}\,'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$?

$\vec{r}\,' = \int \vec{v}\,' dt = (v_M t + \cancel{c})\,\vec{i}\,' = v_M t\, \vec{i}\,'$

$\vec{\omega} \times \vec{v}\,' = \omega_0 \vec{k}\,' \times (v_M \vec{i}\,') = \omega_0 v_M \cdot \vec{j}\,'$

$\vec{\omega} \times \vec{r}\,' = \omega_0 \vec{k}\,' \times v_M t\, \vec{i}\,' = \omega_0 v_M t\, \vec{j}\,'$

$\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') = \omega_0 \vec{k}\,' \times \omega_0 v_M t\, \vec{j}\,' = -\omega_0^2 v_M t \cdot \vec{i}\,'$

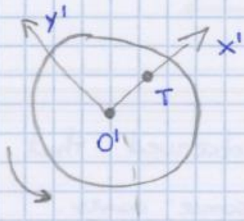
$\vec{a} = 2\omega_0 v_M \vec{j}\,' - \omega_0^2 v_M t\, \vec{i}\,'$

Acceleration along the radius, towards the center (x')

Acceleration from right to left, along y' (because of the direction of the rotation)

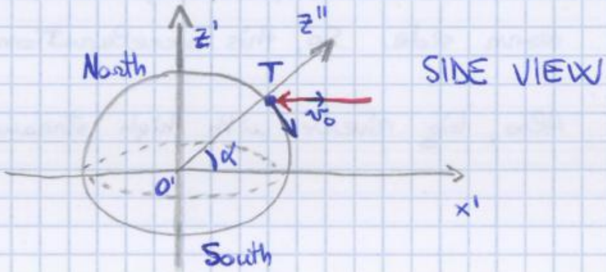
But, Torino is moving, rotating about the z axis.

I introduce a new ref. sys., w/ z' ∥ to z,
x' always including Torino.
Now, T is at rest, while the train is not.



For an observer in the station, the vertical will be a new z'' axis, from O' to the station.

$\alpha \to$ LATITUDE (45° for Torino)



$$\vec{v}' = v_0 (\sin\alpha \, \vec{i}' - \cos\alpha \, \vec{k}')$$

↳ Velocity seen from x', y', z' system. This system rotates w.r.t. X, Y, Z

$$\vec{a} = \underbrace{\cancel{\vec{a}_{O'}}}_{} + \overset{\text{NO ACCEL. ON A SHORT DISTANCE}}{\cancel{\vec{a}'}} + 2\vec{\omega} \times \vec{v}' + \underbrace{\cancel{\dot{\vec{\omega}} \times \vec{r}'}}_{\text{DERIVATIVE OF A CONSTANT}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}')$$

$\vec{\omega} = \omega \cdot \vec{k}'$ $\qquad \omega = \frac{2\pi}{24 \cdot 3600} = \frac{rad}{s}$

$\vec{r}' = R (\cos\alpha \, \vec{i}' + \sin\alpha \, \vec{k}')$

(Assume Earth → sphere)

$$\vec{\omega} \times \vec{v}' = \omega \cdot \vec{k}' \times v_0 (\sin\alpha \, \vec{i}' - \cos\alpha \, \vec{k}') = \omega v_0 \sin\alpha \, \vec{j}' - \cancel{\omega v_0 \cos\alpha \cdot 0}$$

$$\vec{\omega} \times \vec{r}' = \omega \vec{k}' \times R (\cos\alpha \, \vec{i}' + \sin\alpha \, \vec{k}') = \omega R \cos\alpha \, \vec{j}'$$

$\boxed{k' \times k' = 0}$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}') = \omega \vec{k}' \times \omega R \cos\alpha \, \vec{j}' = -\omega^2 R \cos\alpha \, \vec{i}$$

$$\vec{a} = \underline{2\omega v_0 \sin\alpha \, \vec{j}'} \; \underline{- \omega^2 R \cos\alpha \, \vec{i}'}$$

When a train goes on a straight line (v constant) on the surface of the Earth, from this ref. sys., it has no acceleration.
But, from the fixed stars, it is accelerated.

↳ ALONG $\vec{j}'$ → THIS ACCEL. IS NOT PRESENT WHEN $v = 0$ (i.e. WHEN THE TRAIN IS NOT MOVING)

↳ In the equator, $\alpha = 0$, $\sin\alpha = 0$ ⇒ no Coriolis acceleration
If it's in the southern emisphere, Cor. $\vec{a}$ is negative

← doesn't change sign from northern to southern emisphere

3) WEAK FORCE

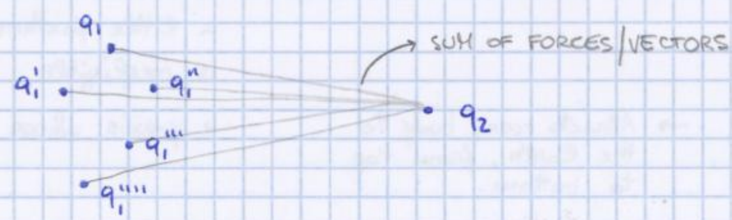
They are present when particles decay.
They're part of electro-magnetic forces;

(2) + 3) = ELECTROWEAK FORCES)

4) NUCLEAR or STRONG FORCE

This force maintains all protons (⊕) near each other, even though electro-magnetic forces tend to pull them away. This force, however, disappears out of the nucleus.

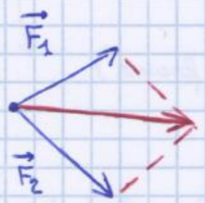
3) 4) NOT CLASSICAL PHYSICS



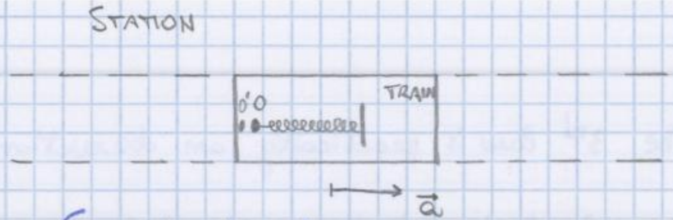
When we have two objects interacting, there are 10^{23} / 10^{25} charges per object, so it is very complex.

But, we can combine all these infinite forces to define new ones, produced by all of these invisible ones.

Force is a vector: it has an intensity and a direction, and it follows the rule of the parallelogram.



I can use a dynamometer to prove that $\vec{F_1} + \vec{F_2}$ give another vector (sum)



I measure $\vec{F} = 0$ $\begin{cases} \text{Train at rest} \\ \text{Body at rest} \\ \text{Dynamometer at rest} \end{cases}$

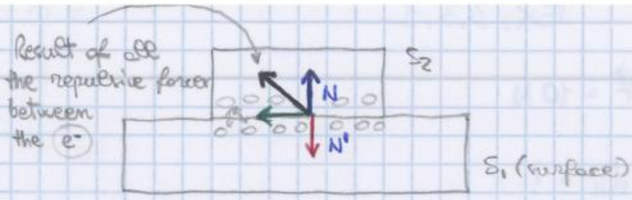
But, if the train leaves w/ some $\vec{a}$ the body remains a little on the left and the spring stretches.
When equilibrium is reached, the dym. keeps saying, let's say 2 N because it remains stretched.
The body is now at rest, no acceleration but there is a force (!)

If we consider o' as starting point, the spring, at the beginning, was compressed. DILEMMA

I LAW: In a ref. sys. at rest w.r.t. the fixed stars sys. or in all ref. sys's which have a constant velocity w.r.t. the fixed stars system, a point at rest has a dynamometer which indicates 0.

So, we should measure using the fixed stars ref. sys., but practically calibrations of dym. were done in a lab.
They were, however, very short in time and the system x', y', z' used for short time could be considered inertial. (good approximation of fixed stars system)

Two surfaces in contact



$\vec{N}$ = normal force

- always $\perp$ to the surface
- always repulsing

CAN BE VERY LARGE ← FRICTION FORCE $\vec{F}_a$

Modulus = ? $0 \leq N \leq N_{MAX}$

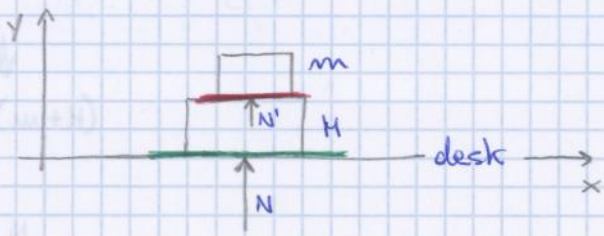
- it disappears when contact is lost

↓ According to the III law there is an equal and opposite force going from S_2 to S_1 = $\vec{N}'$

---

ex

1^{st} LAW • Ref. sys. at rest



2^{ND} LAW
$$\begin{cases} m\ddot{x} = F_{ax} \approx 0 \quad \text{(FRICTION)} \\ m\ddot{y} = N' - mg \\ m\ddot{z} = F_{az} \approx 0 \end{cases}$$

$N = ?$

$N' = ?$

$$\begin{cases} M\ddot{X} = F'_{ax} \approx 0 \\ M\ddot{Y} = N - N' - Mg \\ M\ddot{Z} = F'_{az} \approx 0 \end{cases}$$

$$\Rightarrow \begin{cases} m\ddot{y} = N' - mg \\ M\ddot{Y} = N - N' - M g \\ \ddot{y} = 0 \\ \ddot{Y} = 0 \end{cases}$$

→ THERE IS NO ACCELERATION

$$\Rightarrow \begin{cases} N' = mg \\ N = (m+M)g \end{cases}$$

$m = 1\,kg$
$M = 2\,kg$
$\Rightarrow$ Find N, N'

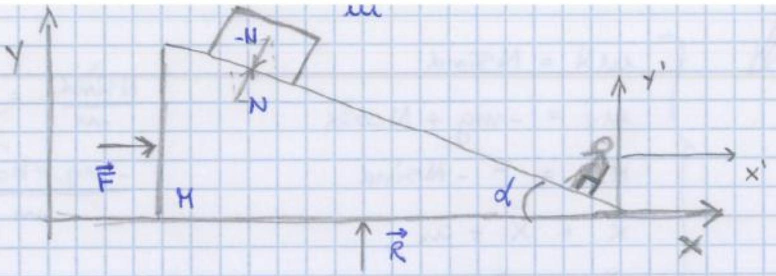
ex

$M = 2\,kg$

$m = 1\,kg$

$\alpha = 30°$

$F = 10\,N$



normal force applied by the surface on the triangle

[No friction]

$$m\ddot{x} = N\sin\alpha$$
$$m\ddot{y} = -mg + N\cos\alpha$$
$$M\ddot{X} = F - N\sin\alpha$$
$$M\ddot{Y} = -Mg - N\cos\alpha + R = 0$$

(No motion of M along y-axis)

$$\begin{cases} m\ddot{x} = N\sin\alpha \\ m\ddot{y} = -mg + N\cos\alpha \\ M\ddot{X} = F - N\sin\alpha \\ R = Mg + N\cos\alpha \end{cases}$$

→ THIS EQ. CANNOT CONTRIBUTE, IT'S GOING TO BE THE LAST ONE

I need a 4^{th} equation

I can introduce a new ref. sys. x', y'

velocity of m ← $\vec{v} = v_x'\,\vec{i}' + v_y'\,\vec{j}'$

$$\frac{v_y'}{v_x'} = -tg\,\alpha$$

$v_x' = v'\cos\alpha \qquad v_y' = -v'\sin\alpha$

$$\frac{v_y'}{v_x'} = \frac{-v'\sin\alpha}{v'\cos\alpha} = -tg\,\alpha$$

[N does not depend on the weight (mg) but on the forces between the electrons on the surfaces.]

$$v_y' = \underbrace{-tg\,\alpha}_{\text{CONST.}} \cdot v_x' \quad \text{DER.}$$

$$a_y' = -tg\,\alpha \cdot a_x'$$

Recall $\vec{a} = \vec{a}_0 + \vec{a}' +$ terms containing ω ($\omega = 0$ in this case → no rotation)

$$\ddot{x} = \ddot{X}_{O'} + a_x'$$

Acceleration of the origin of the moving ref. sys.; It is equal to the acceleration of the inclined plane $= \ddot{X}$

$$\ddot{y} = \underbrace{\ddot{Y}_{O'}}_{\to 0} + \underbrace{a_y'}_{-tg\alpha \cdot a_x'}$$

Now I have 5 unknowns and 5 equations

⇓

FRICTION

Two surfaces at rest w.r.to each other : STATIC FRICTION $\vec{F}_s$

In case of motion between two surfaces : DYNAMIC FRICTION $\vec{F}_d$
(KINETIC FRICTION)

$$\vec{F}_d = \mu_d \cdot N \left(- \frac{\vec{v}}{v} \right)$$

μ_d → DYNAMICAL COEFF. OF FRICTION

→ FRICTION HAS DIRECTION OPPOSITE TO THE VELOCITY

$0 \leq |F_s| \leq F_{s\,MAX}$ The direction is the one needed to maintain both surfaces at rest w.r.to each other.

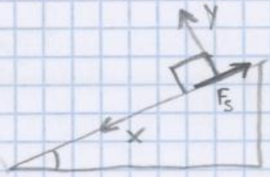
$F_{s\,MAX} = \mu_s \cdot N$

[μ_d, μ_s depend on the type of surfaces in contact]

A ball has a very small surface of contact, hence small friction!

In general, $\mu_s > \mu_d$

Will the body remain at rest or go back down?



$$\begin{cases} m\ddot{x} = mg\sin\alpha + F_s \\ m\ddot{y} = N - mg\cos\alpha = 0 \end{cases}$$

→ WE DON'T KNOW MODULUS OR DIRECTION

NO MOVEMENT, NO ACCELERATION IF STATIC FRICTION IS WORKING

$$0 = \begin{cases} m\ddot{x} = mg\sin\alpha + F_s \\ N = mg\cos\alpha \end{cases} \qquad \begin{cases} 0 = mg\sin\alpha + F_s \\ N = mg\cos\alpha \end{cases}$$

@ t=0

$$\begin{cases} F_s = -mg\sin\alpha \\ N = mg\cos\alpha \end{cases}$$

→ Now we know the direction of F_s (and its value = -5)

$$\boxed{\text{Initial: } \dot{x}(0) = 0, \quad x(0) = 0}$$

□ $|F_s| \leq F_{S_{MAX}}$ in order for the body to remain at rest.

$$F_{S_{MAX}} = \mu_S \cdot N = (0{,}5)(8{,}66) = 4{,}33 \qquad (N = 8{,}66)$$

OR

$$\cancel{mg}\sin\alpha \leq \mu_S \cancel{mg}\cos\alpha$$

$$tg\alpha \leq \mu_S$$

$$0{,}577 \leq 0{,}5 \quad \boxed{\text{NO}}$$

So the body will not remain at rest

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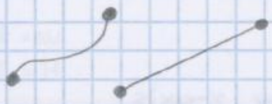
TENSION FORCES

cords, strings, ropes ...



↓

an object that has a maximum length.

When @ max, it is said to be "extended".

When the cord is not extended $\vec{T} = 0$ at both ends.

When the cord is extended there are two forces (T_1, T_2) going from the ends to the center.

$\vec{T} = \pm T$ (towards center) where $0 \leq T \leq T_{MAX}$

↓ depends on the cord

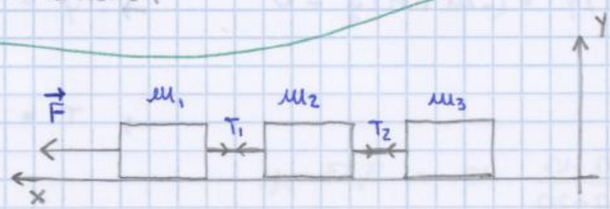
$|\vec{T}_1| = |\vec{T}_2|$ - equal in modulus
- opposite direction

EX. 3.3.3

ex

$m_1 = m_2 = m_3 = 10$ kg

$F = 100$ N



$\vec{F}$ is applied.

I want to know the acceleration and the tension forces.

NO FRICTION

T_1 = 2 tension forces equal in modulus

$$\begin{cases} m_1 \ddot{x}_1 = F - T_1 \\ m_2 \ddot{x}_2 = T_1 - T_2 \\ m_3 \ddot{x}_3 = T_2 \end{cases}$$

The cord cannot be extended further, so $\ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = \ddot{X}$

$$(m_1 + m_2 + m_3)\ddot{X} = F \cancel{- T_1} \cancel{+ T_1} \cancel{- T_2} \cancel{+ T_2}$$

$$\ddot{X} = \frac{F}{m_1 + m_2 + m_3}$$

THIS SEEMS OBVIOUS ($F = m \cdot a$)

BUT NOW I CAN ALSO CALCULATE T_1, T_2

$T_1 = F - m_1 \ddot{X}$

$T_2 = m_3 \ddot{X}$

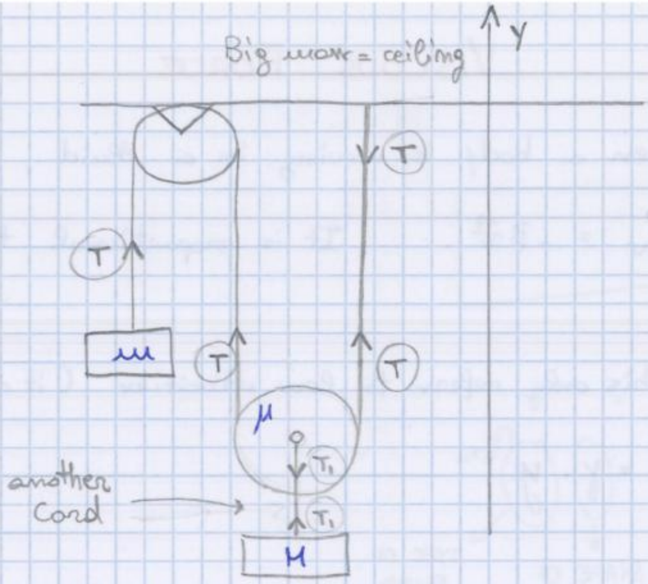
$\ddot{X} = \frac{100}{30} = 3{,}33$ m/s²

$T_1 = 100 - 33{,}3 = 66{,}7$ N

$T_2 = 10 \cdot 3{,}33 = 33{,}3$ N

We notice that $T_1 \gg T_2$; the first masses undergo a higher tension force.

ex $M = 2kg \quad m = 1{,}5\,kg$



$$\begin{cases} m\ddot{y} = T - mg \\ M\ddot{Y} = T_1 - Mg \end{cases}$$

How to connect T_1 and T?

Imagine that the lower pulley has a mass $= \mu$

$$\mu \ddot{Y} = 2T - T_1 - \mu g$$

As the pulley "becomes" massless, $\lim_{\mu \to 0} \quad 0 = 2T - T_1$

So $\Rightarrow \boxed{T_1 = 2T}$

As the mass M goes up by 1 unit, the mass m goes down by 2 unit!

So $\Rightarrow \boxed{\ddot{y} = -2\ddot{Y}}$

Now I have 4 equations and 4 unknowns.

$$\begin{cases} 2m\ddot{y} = 2T - 2mg \\ -\frac{M}{2}\ddot{y} = 2T - Mg \end{cases} \Rightarrow \left(2m + \frac{M}{2}\right)\ddot{y} = -2mg + Mg$$

$$\ddot{y} = \frac{(M - 2m)g}{2m + \frac{M}{2}}$$

$\ddot{y} = -2{,}5\ m/s^2$

So m goes down even though it's lighter!

IMPORTANT RESULT

$$z(t) = u \cdot v = e^{-\int f dt} \cdot \int g(t) \cdot e^{\int f dt} dt$$

In our course, we will have $f(t) = a$, $g(t) = b$ CONSTANTS

So, $\int f(t)\, dt = at$

$$\int g(t) \cdot e^{\int f(t) dt} dt = \int b \cdot e^{at} = \frac{b}{a} \cdot e^{at} + c$$

$$\Longrightarrow \boxed{e^{-at}\left(\frac{b}{a} e^{at} + c\right) = z(t)} \longrightarrow \text{GENERAL FORMULA}$$

Going back to the problem,

$$\cancel{m}\ddot{x} = \cancel{m}g\sin\alpha - \mu_d(\cancel{m}g\cos\alpha) - \frac{B}{m}\dot{x}$$

$$\ddot{x} + \frac{B}{m}\dot{x} = \underbrace{g\sin\alpha - \mu_d g\cos\alpha}_{=b}$$

$$\dot{x}(t) = z(t) \qquad \ddot{x}(t) = \dot{z}(t)$$

CALCULUS I

$$\boxed{\begin{aligned} &y' + a(x)\,y = b(x) \\ &y(x) = e^{-A(x)}\int b(x) \cdot e^{A(x)} dx \end{aligned}}$$

$$\dot{z} + \frac{B}{m} z = b$$

How to find c?

VELOCITY OF m ←

$$z(t) = e^{-\frac{B}{m}t}\left[\left(\frac{g\sin\alpha - \mu_d g\cos\alpha}{B/m}\right) e^{\left(\frac{B}{m}t\right)} + c\right]$$

USE INITIAL CONDITIONS

$$@\, t=0 \quad \begin{cases} x(t) = 0 \\ \dot{x}(t) = 10 \end{cases}$$

$$\Longrightarrow 10 = 1\left[\frac{(10)(\frac{1}{2}) - (0{,}5)(10)(\frac{\sqrt{3}}{2})}{1 \cdot 10^{-5}} \cdot 1 \cdot 1 + c\right]$$

$$c = 10 - (6{,}7 \cdot 10^{9})$$

To find $x(t)$ where the box stops, integrate $\dot{x}(t)$

$a\ddot{y} + b\dot{y} + cy = 0$

$\frac{b^2 - 4ac}{4a^2} = \Delta^2 > 0 \rightarrow y(t) = e^{-\frac{b}{2a}t} \cdot (A \cdot e^{-\Delta \cdot t} + Be^{+\Delta \cdot t})$

$-\frac{(b^2 - 4ac)}{4a^2} = \omega^2 > 0 \rightarrow y(t) = e^{-\frac{b}{2a}t} \left(A\sin(\omega t) + B\cos(\omega t) \right)$

$\rightarrow e^{-\frac{b}{2a}t}(c \cdot \cos(\omega t + \varphi))$

$=$ OR

$e^{-\frac{b}{2a}t}(d \cdot \sin(\omega t + \varphi'))$

If $\Delta^2 = 0$ $\quad y(t) = e^{-\frac{b}{2a}t}(A + Bt)$

From a physical point of view, it's nearly impossible to have $\Delta^2 = 0$ that is $b^2 - 4ac = 0 \rightarrow \beta^2 - 4m(mg\sin\alpha) = 0$

Δ^2 will always be either positive or negative.

$x_p(t)$: $\boxed{x = y + x_p} \rightarrow$ GENERAL + PARTICULAR SOLUTION

X_p : PARTIAL/PARTICULAR SOL.

$a(\ddot{y} + \ddot{x}_p) + b(\dot{y} + \dot{x}_p) + c(y + x_p) = f(t)$

$\underbrace{\cancel{a\ddot{y} + b\dot{y} + cy}}_{=0 \text{ (HOM. EQ.)}} + a\ddot{x}_p + b\dot{x}_p + cx_p = f$

$m\ddot{x} + \beta\dot{x} + Kx = \underline{mg\sin\alpha}$ CONST. *

$a = m$
$b = \beta$
$c = K$

$\frac{\beta^2 - 4mK}{4m^2} = \frac{\beta^2}{4m^2} - \frac{K}{m} < 0$

$\omega^2 = \frac{K}{m} - \frac{\beta^2}{4m^2} \quad \rightarrow x = x_p + c \cdot \cos(\omega t + \varphi) \cdot e^{-\frac{\beta}{2m}t}$ HOW TO FIND x_p ?

Let's suppose $x_p = D$ (const) $\quad \dot{x}_p = 0 \quad \ddot{x}_p = 0$

*

Now substitute $x_p, \dot{x}_p, \ddot{x}_p$ in our equation:

$m \cdot 0 + \beta \cdot 0 + K \cdot D = mg\sin\alpha \implies D = \frac{mg\sin\alpha}{K}$ $\quad$ D is a partial sol.

So, $x(t) = \frac{mg\sin\alpha}{K} + c \cdot \cos(\omega t + \varphi) \cdot e^{-\frac{\beta}{2m}t}$

$\dot{x}(t) = +c\left[-\sin(\omega t + \varphi)\,\omega \cdot e^{\cdots} + \cos(\omega t + \varphi)e^{\cdots} \cdot \left(-\frac{\beta}{2m}\right)\right]$

$= -c\left[+\omega\sin(\omega t + \varphi) + \cos(\omega t + \varphi)\left(+\frac{\beta}{2m}\right)\right] e^{-\frac{\beta}{2m}t}$

$$\begin{cases} m\ddot{x} = qB\dot{y} \\ m\ddot{y} = -qB\dot{x} \\ m\ddot{z} = 0 \end{cases} \Rightarrow 2 \text{ II ORDER DIFF. EQ. W/ CONST. COEFF'S}$$

Along z, there is no acceleration produced by the Lorentz force.

$$\begin{cases} \dot{x}\ddot{x} = \frac{qB}{m}\dot{y}\dot{x} \\ \dot{y}\ddot{y} = \frac{-qB}{m}\dot{y}\dot{x} \end{cases} +$$

So the movement along z has no acceleration, is at constant velocity.

$$\underline{\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0}$$

$$\vec{a} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath}$$
$$\vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath}$$

only considering components x, y

$$v^2 = \dot{x}^2 + \dot{y}^2 \rightarrow \frac{d(v^2)}{dt} = \underline{2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 0}$$

So, modulus of velocity in x, y plane : constant (not in direction)

$\vec{a} \cdot \vec{v} = \dot{x}\ddot{x} + \dot{y}\ddot{y}$ (but, this is equal to zero) $= 0$

$\downarrow$

Either $\vec{a}$ or $\vec{v}$ is $= 0$, or they are perpendicular.

NOT ZERO BECAUSE IT IS PROPORTIONAL TO THE FORCE

NOT ZERO BECAUSE IT IS CONSTANT $\neq 0$ (DOESN'T CHANGE THE MODULUS FROM THE BEGINNING)

$\Rightarrow \vec{a} \perp \vec{v}$ in the plane x, y

MODULUS SQUARED $\{$ $|\vec{F}_L|^2 = q^2B^2(\dot{y}^2 + (-\dot{x})^2) = q^2B^2 v^2$ this is constant

$\Rightarrow |\vec{F}_L|^2$ constant

In intrinsic coordinates (let's suppose trajectory is known)

$\vec{v} = \dot{s}\vec{\tau}$ $\quad (\dot{s} = v)$ CONSTANT $\quad ; \quad \vec{a} = \cancel{\ddot{s}\vec{\tau}} + \frac{\dot{s}^2}{\rho}\cdot\vec{n}$

because $\vec{a} \perp \vec{v}$

$\Rightarrow |\vec{a}| = \frac{\dot{s}^2}{\rho}$ const. $\quad$ but $|\vec{a}| = \frac{|\vec{F}_L|}{m}$ const. $\rightarrow \frac{\dot{s}^2}{\rho} = \frac{|\vec{F}_L|}{m}$ $\quad$ So $\underline{\rho = \text{constant}}$

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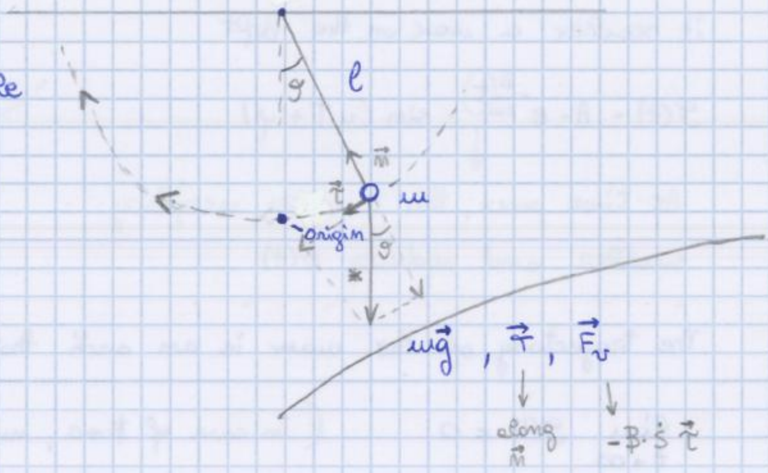
PENDULUM



Cartesian ref. sys. is not suitable

We'll use intrinsic coordinates.

$$\vec{\tau})\quad m\ddot{s} = -\beta\dot{s} - mg\sin\vartheta$$

$$\vec{n})\quad m\frac{\dot{s}^2}{\underbrace{\rho}_{=\ell}} = T - mg\cos\vartheta$$

The component of gravity along $\vec{\tau}$ can be either ⊕ or ⊖.

* $-mg$ w/ components along $\vec{\tau}$ and $\vec{n}$

Since this is a circ. we can write $s = \ell\cdot\vartheta$

Assuming $-90° \leq \vartheta \leq +90°$ and

ϑ positive on the left and negative on the right,

we can write $-mg\sin\alpha$ (along $\vec{\tau}$)

$s = \ell\vartheta \qquad \dot{s} = \ell\dot{\vartheta} \qquad \ddot{s} = \ell\ddot{\vartheta}$

$$\vec{\tau})\quad \ell\ddot{\vartheta} = -\frac{\beta}{m}\ell\dot{\vartheta} - g\sin\vartheta$$

$$\hookrightarrow \ell\ddot{\vartheta} + \frac{\beta}{m}\ell\dot{\vartheta} + g\sin\vartheta = 0$$

II ORDER DIFF. EQ. W/ CONST. COEFF'S (NOT LINEAR, ϑ APPEARS AS $\sin\vartheta$)

IMPOSSIBLE TO SOLVE ANALITICALLY

But, if we consider $|\vartheta| \leq 5{,}6°$ $\Rightarrow$ $\sin\vartheta \approx \vartheta$ (in radians)

$$\Longrightarrow \quad \ell\ddot{\vartheta} + \frac{\beta}{m}\ell\dot{\vartheta} + g\vartheta = 0$$

IT IS HOMOGENEOUS

$$\ddot{\vartheta} + \frac{\beta}{m}\dot{\vartheta} + \frac{g}{\ell}\vartheta = 0$$

$$\Delta^2 = \frac{\beta^2}{4m^2} - \frac{g}{\ell} < 0 \qquad \text{So, } \omega^2 = \frac{g}{\ell} - \frac{\beta^2}{4m^2} > 0$$

$$\vartheta(t) = e^{-\frac{\beta}{2m}t} \cdot A \cdot \sin(\omega t + \varphi)$$

Using init. cond. I can find A and φ

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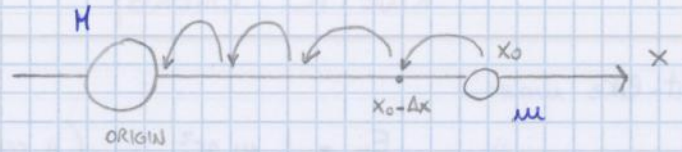
ex



M = earth

m = asteroid

Gravity : m goes towards M

$$m\ddot{x} = -\gamma \frac{mM}{x^2 \rightarrow d^2}$$

$$m\ddot{x} + \gamma m M x^{-2} = 0$$ DIFF. EQ. → IMPOSSIBLE TO SOLVE ANALITICALLY

How to solve this ?
- Approximation (but you lose terms that are useful to study and predict the results)
- look for other physical quantities

Numerical approximation :

$t = 0 \quad x(0) = X_0$

$m \xrightarrow{\text{goes}} X_0 \rightarrow X_0 - \Delta x$

In the interval $(x_0 ; x_0 - \Delta x)$ the force can be approximated as $-\gamma \frac{mM}{x_0^2}$ (STARTING POINT) or $-\gamma \frac{mM}{\left[\frac{x_0 + (x_0 - \Delta x)}{2}\right]^2}$ (MEAN VALUE)

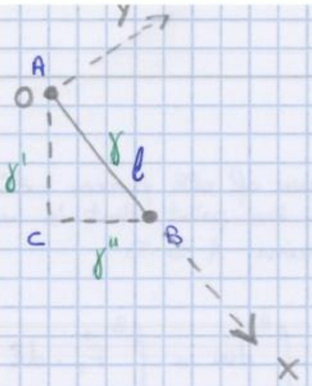
Then you keep considering intervals.

(the smaller the interval the smaller the final error)

[NOT TOO HARD WITH COMPUTERS]

ex

Mass → dynamical coeff. μ_d



1) going from A to B
→ evaluate work of friction

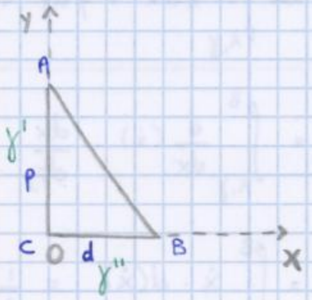
2) going from A to C to B
→ evaluate work of friction

1) $dw = -\mu_d \, mg \, \vec{i} \cdot dx \, \vec{i}$

N

what is left from $d\vec{r}$ (no displacement along y, z)

$$W_{AB,\gamma} = \int_{A,\gamma}^{B} dw = \int_{x=0}^{x=\ell} -\mu_d \, mg \, dx = -\mu_d \, mg \, x \Big|_{x=0}^{x=\ell} = \boxed{-\mu_d mg \ell}$$



2) Change ref. system

$$W_{ACB,\gamma',\gamma''} = \int_{A,\gamma'}^{C} dw + \int_{C,\gamma''}^{B} dw$$

$$= \int_{A,\gamma'}^{C} -\mu_d \, mg \, \vec{j} \cdot (-) \cdot dy \, \vec{j} \, (-) + \int_{C,\gamma''}^{B} -\mu_d \, mg \, \vec{i} \cdot dx \, \vec{i}$$

mass going from A to C

dy from A to C

$$= -\mu_d \, mg \, y \Big|_{A=p}^{C=0} + (-\mu_d \, mg \, x)\Big|_{C=0}^{B=d} = \boxed{+\mu_d mg p - \mu_d mg d}$$

Taking the absolute value, $\mu_d mg (p+d) > \mu_d mg \ell$

Work made by dynamical friction along $\gamma', \gamma'' > \gamma$

POTENTIAL ENERGY

Suppose you have a force :

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

We want to find a function of space $u(x,y,z)$ such that

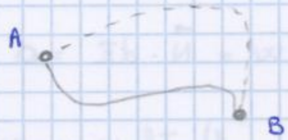
$$\frac{\partial u}{\partial x} = -F_x \qquad \frac{\partial u}{y} = -F_y \qquad \frac{\partial u}{\partial z} = -F_z$$

$\Rightarrow$ u is the potential energy of $\vec{F}$

Some forces have a potential energy, different forces have different pot. energies. Others don't.

Suppose you have a $\vec{F}$ with E_P ; we were able to find u

Work made by this force from A to B ?



$$\boxed{W_{AB,\gamma}} = \int_{A,\gamma}^{B} (F_x dx + F_y dy + F_z dz) =$$

$$= -\int_{A,\gamma}^{B} + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz =$$

- A function $\frac{\partial u}{\partial x} \cdot dx$ (DIFFERENTIAL) gives a differential

$$= -\int_{A,\gamma}^{B} du =$$

- Sum of differentials $\rightarrow$ differential

$$\boxed{= u(A) - u(B)}$$

☐ In case of potential energy, the work is the same independently on the trajectory

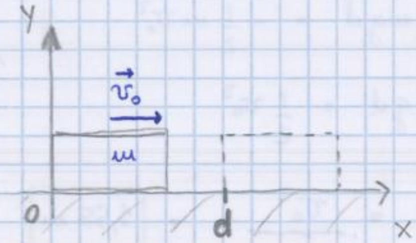
☐ Forces having potential energy are called conservative forces.

(ex)

$m = 2\ kg$

$\mu_d = 0{,}45$

$\vec{v}_0 = v_0 \vec{\imath} = 3{,}7\ m/s$



$$\begin{cases} m\ddot{x} = -\mu_d N \\ 0 = m\ddot{y} = -mg + N \end{cases} \Rightarrow \begin{cases} m\ddot{x} = -\mu_d mg \\ N = mg \end{cases} \Rightarrow \left\{ \ddot{x} = -\mu_d g \right.$$

$N = 19{,}6\ N$

$$\dot{x} = \int \ddot{x}\, dt = -\mu_d g t + c_1$$

$$x = \int \dot{x}\, dt = -\mu_d g \frac{t^2}{2} + c_1 t + c_2$$

INIT. COND. :

$$v_0 = -\mu_d g(0) + c_1 \rightarrow c_1 = v_0$$

$$0 = -\mu_d g \frac{(0)^2}{2} + c_1(0) + c_2 \rightarrow c_2 = 0$$

$$\begin{cases} \dot{x} = -\mu_d g t + v_0 \\ x = -\mu_d g \frac{t^2}{2} + v_0 t \end{cases}$$

The box stops at $\dot{x} = 0$ so, $0 = -\mu_d g t^* + 3{,}7$

$$\boxed{t^* = 0{,}83\ s}$$

$$\boxed{x(t^*)} = -\mu_d g \frac{(t^*)^2}{2} + v_0 t^* = \boxed{1{,}55\ m}$$

OTHER TECHNIQUES TO SOLVE THIS PROBLEM QUICKLY ?

We can look at the work made by the dynamical friction.

$$W^{F_d}_{(0,d)} = \int_0^d dW = \int_0^d \vec{F}_d \cdot \underset{\substack{\text{NO MOTION} \\ \text{ALONG } y,z \\ \text{so} \rightarrow dx}}{d\vec{r}} = \int_0^d -\mu_d mg \cdot dx = -\mu_d mg \cdot d$$

HOW DO WE FIND d?

POTENTIAL ENERGY ?

Now we have to check if the forces we have studied have a potential energy.

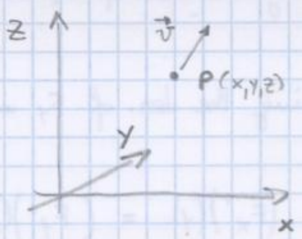
Recall   If we're able to find:

$$\vec{F} = -\frac{\partial u}{\partial x}\vec{i} - \frac{\partial u}{\partial y}\vec{j} - \frac{\partial u}{\partial z}\vec{k}$$

$\Rightarrow$ u is the potential energy of $\vec{F}$

☐ Let's start w/ F_v (viscousity)

$$\vec{F}_v = -\beta\vec{v}$$



$\vec{F}_v$ depends on the velocity

$\frac{\partial u}{\partial x}(x,y,z) \xrightarrow{\text{DER}} G(x,y,z)$ OR $G(x,y)$ OR $G(x,z)$

POT. ENERGY AS A FUNCTION OF x, y, z

This should be the x component of the velocity because $-\beta\vec{v} = -\frac{\partial u}{\partial x}\vec{i} - \frac{\partial u}{\partial y}\vec{j} - \frac{\partial u}{\partial z}\vec{k}$

BUT my $\frac{\partial u}{\partial x}(x,y,z)$ only gives me dependence on x, y, z, not on the velocity.

So, the viscous force is not a conservative force.

☐ Dynamical friction ?

$$\vec{F}_d = -\mu_d N \left(\frac{\vec{v}}{v}\right)$$

$\vec{F}_d$ depends on the velocity, for the direction.

I repeat the same discussion

$\Rightarrow$ the dynamical friction is not a conservative force.

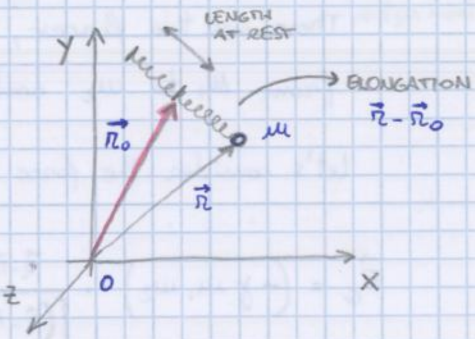
☐ Elastic force ?

$\boxed{d = \text{elongation}}$

$$\vec{F}_e = -k \cdot \text{elongation} \cdot \vec{m}$$

↳ SUITABLE UNIT VECTOR

$$\vec{F}_e = -k \cdot \cancel{|\vec{r} - \vec{r}_0|} \cdot \frac{\vec{r} - \vec{r}_0}{\cancel{|\vec{r} - \vec{r}_0|}} \quad \text{DIRECTION}$$



$$\vec{F}_e = -k \cdot (\vec{r} - \vec{r}_0)$$

$$\vec{F}_e = -k \cdot \left[(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} \right]$$

Let's try : $\mathcal{U}_e(x,y,z) = \frac{1}{2}k \cdot \left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \right] \boxed{= \frac{1}{2}k \cdot d^2}$

$$-\frac{\partial \mathcal{U}_e}{\partial x} = -\frac{1}{2}k \cdot \left[2(x - x_0) + 0 + 0 \right] = -k(x - x_0)$$

$$-\frac{\partial \mathcal{U}_e}{\partial y} = -k(y - y_0)$$

$$-\frac{\partial \mathcal{U}_e}{\partial z} = -k(z - z_0).$$

✓

THEOREM OF THE CONSERVATION OF THE MECHANICAL ENERGY

$W_{\gamma(A,B)} = E_{(B)} - E_{(A)}$ (work done by all forces)

$W_{\gamma(A,B)} = W_{\gamma\,Cons} + W_{\gamma\,NonCons} = \mathcal{U}_{(A)} - \mathcal{U}_{(B)} + W_{\gamma\,NON\,CONS}$ work done by forces w/ no potential energy

I set the 2 definitions equal to each other,

$$\left(E_{(B)} + \mathcal{U}_{(B)}\right) - \left(E_{(A)} + \mathcal{U}_{(A)}\right) = W_{\gamma\,NON\,CONS.} \overset{*}{=} 0$$

~~KINETIC + POTENTIAL ENERGY ... TOTAL MECHANICAL ENERGY~~

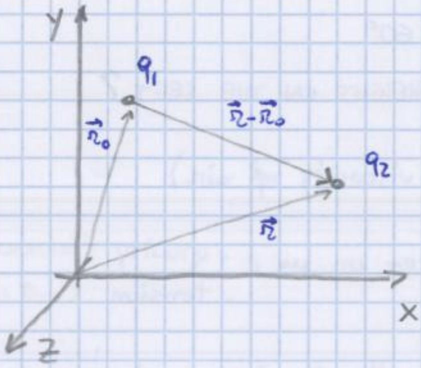
DEF Kinetic + Potential energy in a point : TOTAL MECHANICAL ENERGY (in a point)

In presence of zero non-conservative forces* (such as viscosity, dyn. friction...), the change from A to B of the total mechanical energy is 0.

Total mech. energy stays the same ⇒ it is CONSTANT

☐ Coulomb force ? (Two point-like charges → two forces, either attractive or repulsive)

$$\vec{F}_c = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{(\vec{r}-\vec{r}_0)^2} \cdot \underbrace{\frac{\vec{r}-\vec{r}_0}{((\vec{r}-\vec{r}_0)^2)^{1/2}}}_{\text{DIRECTION}}$$



Consider force produced by q_1 onto q_2 (if they have same sign)

If $\neq$ sign, the product $q_1 q_2$ changes the sign.

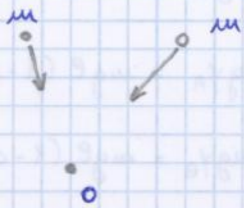
I try : $$\mathcal{U}_c(x,y,z) = \frac{q_1 q_2}{4\pi\varepsilon_0} \cdot \frac{1}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{1/2}}$$

I can take the derivatives (w.r.to x, y, z) of this and see if they match with the (x,y,z) components of $\vec{F}_c$

$$\Rightarrow \boxed{\mathcal{U}_c(x,y,z) = \frac{q_1 q_2}{4\pi\varepsilon_0} \cdot \frac{1}{d}}$$

d = distance between q_1 and q_2

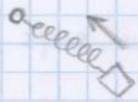
CENTRAL FORCE



$|\vec{F}| = F(\underset{\text{DISTANCE FROM } m \text{ TO } O}{r})$

Force always from m to O

Other central forces are
- coulomb force
- gravitational force (general)
- elastic force
- ~~tension force~~ DOESN'T DEPEND ON THE DISTANCE
- ~~gravity~~ IF WE ASSUME THAT GRAVITY FORCES ARE PARALLEL, THEY DON'T POINT TOWARDS A COMMON CENTER → NOT A CENTRAL FORCE



modulus depends on d

↳ F must depend on the distance (d)

ADDITIVE PROPERTY OF POTENTIAL ENERGY

$\vec{F}_1$ has U_1 : $-\frac{\partial U_1}{\partial x} = F_{x_1}$; $-\frac{\partial U_1}{\partial y} = F_{y_1}$ ...

$\vec{F}_2$ has U_2 : $-\frac{\partial U_2}{\partial x} = F_{x_2}$; $-\frac{\partial U_2}{\partial y} = F_{y_2}$ ...

$$\boxed{\vec{F} = \vec{F}_1 + \vec{F}_2}$$

$$\boxed{U = U_1 + U_2} \qquad -\frac{\partial U}{\partial x} = -\frac{\partial (U_1 + U_2)}{\partial x} = -\frac{\partial U_1}{\partial x} - \frac{\partial U_2}{\partial x} = F_{x_1} + F_{x_2} = F_x$$

$$\checkmark \qquad -\frac{\partial U}{\partial y} = -\frac{\partial (U_1 + U_2)}{\partial y} = -\frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial y} = F_{y_1} + F_{y_2} = F_y$$

Suppose $\vec{F} \rightarrow U(x,y,z)$

$$-\frac{\partial U}{\partial x} = F_x$$

define $\rightarrow U' = U + \underbrace{c}_{\text{constant}}$

$$-\frac{\partial U'}{\partial x} = -\frac{\partial U}{\partial x} + 0 = F_x$$

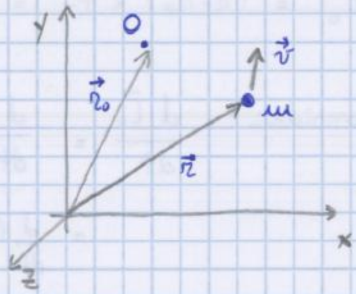
$\Rightarrow$ Potential energy is not unique.
By adding $\neq$ constants, you get $\neq$ potential energies. (in a point)

$$\begin{cases} U_A - U_B = \Delta \\ U'_A - U'_B = U_A + c - (U_B + c) = \Delta \end{cases}$$

When I calculate the difference in potential energy, however, the constant c does not affect my calculations.

ANGULAR MOMENTUM

mass m with velocity $\vec{v}$ $\quad \vec{L}_o$



We also have to define a point O (a pole) wherever we may like.

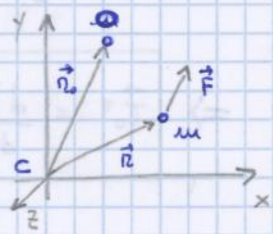
$$\vec{L}_o = (\vec{r} - \vec{r}_o) \times \vec{p} = $$
$$= (\vec{r} - \vec{r}_o) \times m\vec{v}$$

$\vec{L}_o$ will be $\perp$ to the plane containing - velocity - $\vec{r} - \vec{r}_o$

Angular momentum is a vector.

TORQUE (momentum of force)

$\vec{F}$



We call torque of the force $\vec{F}$ w.r.to the point O on the mass m:

$$\vec{\tau}_o = (\vec{r} - \vec{r}_o) \times \vec{F}$$

$\vec{\tau}_o$ will be $\perp$ to the plane containing - $\vec{F}$ - $\vec{r} - \vec{r}_o$

ADDITIVE PROPERTY:

$\vec{F}_1 \quad \vec{F}_2 \quad \vec{\tau}_o^{\,1} \quad \vec{\tau}_o^{\,2} \qquad \vec{F} = \vec{F}_1 + \vec{F}_2$

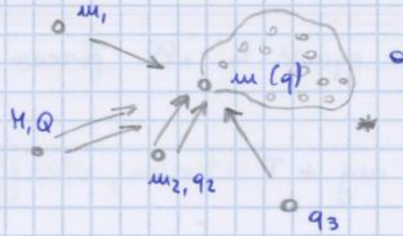
$$\vec{\tau}_o = (\vec{r} - \vec{r}_o) \times \vec{F} = (\vec{r} - \vec{r}_o) \times (\vec{F}_1 + \vec{F}_2) = (\vec{r} - \vec{r}_o) \times \vec{F}_1 + (\vec{r} - \vec{r}_o) \times \vec{F}_2 =$$
$$= \vec{\tau}_o^{\,1} + \vec{\tau}_o^{\,2}$$

Coming next ...

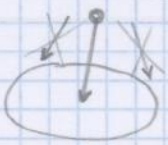
So far, we have dealt with single masses and forces applied on or made by single "point-like" masses.

What if we have more points applying a force on a mass?

What happens?



Why do we consider the distance up to the center of the Earth?



* What happens if I have a body with a mass and infinitely many points and connections inside?

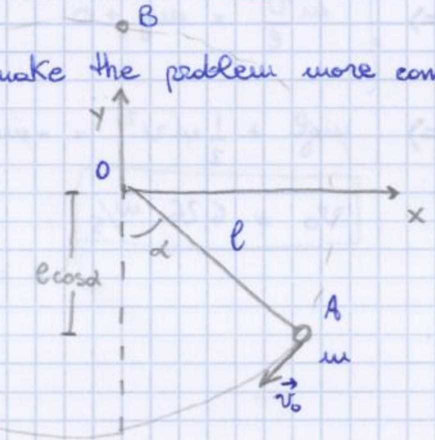
(ex) We go back to the pendulum and make the problem more complex.

$m = 1\,\text{Kg}$ (no viscous force)

$\ell = 1\,\text{m}$

$\alpha = 60°$

$\vec{v}_0 = 10\,\text{m/s}$



What will happen to the mass?

We cannot apply the 2nd law (non-linear diff. equation)

What is the min. $\vec{v}_0$ to make the mass make a circle?

Work made by tension = 0

Work made by gravity (conservative force) → TOTAL MECHANICAL ENERGY is maintained

$$E_B^k + U_B - (E_A^k + U_A) = 0 \quad \text{(in our case)}$$

(any point) $U + E^k = U_A + E_A^k$

$$mg\ell + \frac{1}{2}mv^2 = -mg\ell\cos\alpha + \frac{1}{2}mv_0^2$$

$\Downarrow$

$E_A^k = \frac{1}{2}mv_0^2$

$U_A = -mg(\ell\cos\alpha)$

$U_B = mg\ell$

$E_B^k = \frac{1}{2}mv^2$

FICTITIOUS FORCES

In an inertial reference system:

$$\overbrace{\vec{F}_T = m\vec{a}} = \underbrace{m \cdot (\vec{a}_{o'} + \vec{a}' + 2\vec{\omega} \times \vec{v}' + \dot{\vec{\omega}} \times \vec{r}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}'))}_{\text{for an observer on a moving ref. system which has some } \vec{a}}$$

$\vec{F}_T = m\vec{a}$ only if $\vec{a}$ is measured in an inertial ref. system.

$$\underbrace{\vec{a}_T}_{\text{TRANSLATION ACCELERATION}} + \underbrace{\vec{a}_R}_{\text{ROTATIONAL ACCELERATION}} \equiv \vec{a}_{o'} + \left(2\vec{\omega} \times \vec{v}' + \dot{\vec{\omega}} \times \vec{r}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}')\right)$$

$$\vec{F}_T = m\vec{a} = m\vec{a}' + m(\vec{a}_T + \vec{a}_R)$$

$$\underbrace{\vec{F}_T - m(\vec{a}_T + \vec{a}_R)} = m\vec{a}'$$

→ If you are in a moving ref. sys. this becomes the 2^{nd} law

- FICTITIOUS FORCE
- APPARENT

- The fictitious force is not a real force;
- It's just a term to maintain the structure of the 2^{nd} law.

If: $\vec{\omega} = 0$ (no rotation) $\Rightarrow \vec{a}_R = 0$

$\vec{a}_{o'} = 0$ (moving ref. sys. has constant velocity) $\Rightarrow \vec{a}_T = 0$

$\Big\}$ $\vec{F}_T = m\vec{a}' = m\vec{a}$

↳ Moving ref. systems having constant velocity w.r.to each other and to the fixed stars, are inertial.

That's why Earth = fixed ref. sys. (inertial)

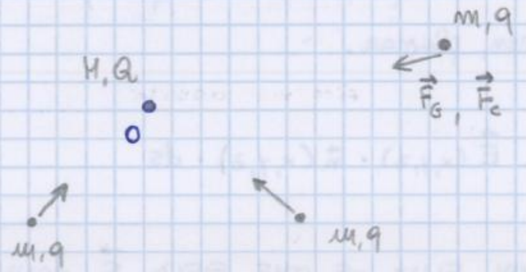
For short measurements
1) $\vec{v}$ constant $(\vec{a}_T = 0)$
2) motion is rectilinear $(\vec{a}_R = 0)$

FIELDS

$$\vec{F}_G = -\gamma \cdot \frac{m \cdot M}{r^2} \cdot \left(\frac{\vec{r}}{r}\right) \qquad \text{GRAVITY}$$

$$\vec{F}_c = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q \cdot Q}{r^2} \cdot \left(\frac{\vec{r}}{r}\right) \qquad \text{COULOMB FORCE}$$

(point-like - mass - charge)



With m,q moving,
in every point there is
a force (gravitational, electrostatic ...)

$$\vec{E}_G = \frac{\vec{F}_G}{m} \qquad \vec{E}_c = \frac{\vec{F}_c}{q}$$

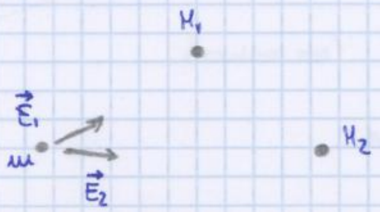
these new vectors are present in the space, even if there is no mass or charge in every point.
(you only need one mass or charge, M, Q)

Gravitational field / Electric field → they are present wherever there is a mass or a charge in the space

$$\vec{F}_G = m \cdot \vec{E}_G \quad // \quad \vec{F}_c = q \cdot \vec{E}_c$$

NOW...
Instead of one point-like charge/mass, there's a lot of them.

$$\vec{F}_{TOT} = \vec{F}_1 + \vec{F}_2 = m\vec{E}_1 + m\vec{E}_2 = m\underbrace{(\vec{E}_1 + \vec{E}_2)}_{\text{VECTOR PRODUCED BY } \vec{F}_{TOT}}$$

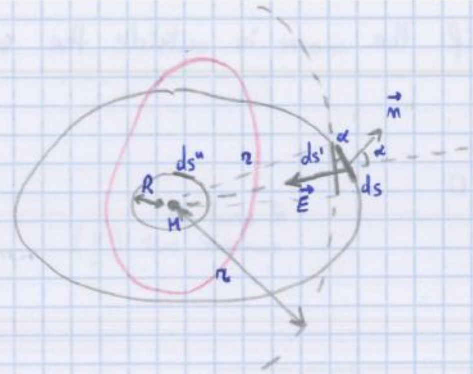


ADDITIVE PROPERTY OF THE FIELDS

The field produced by several point-like masses is equal to the sum of each field produced by each single mass.

$\rightarrow$ EUCLIDIAN GEOMETRY

$$\frac{ds''}{R^2} = \frac{ds'}{r^2}$$



$$\phi_s(\vec{E}) = \int_s \vec{E} \cdot \vec{m} \cdot ds =$$

$$= \int_s + \frac{\gamma M}{r^2} \cdot \underbrace{\frac{\vec{r}}{r} \cdot \vec{m}}_{} \cdot ds =$$

$\rightarrow$ 2 UNIT VECTORS. CONSIDER ANGLE BETWEEN THEM

$$= +\gamma M \int_s \frac{1}{r^2} (-\cos\alpha)\, ds =$$

$$= -\gamma M \int_s \frac{1}{r^2} \cancel{\cos\alpha} \frac{ds'}{\cancel{\cos\alpha}} =$$

$$= -\gamma M \int_s \underbrace{\frac{ds''}{R^2}}_{1} \cdot \cancel{\frac{r^2}{r^2}} = -\gamma M \cdot 4\pi \underbrace{R^2}_{1}$$

$\rightarrow$ surface of the sphere as a result of the integral of ds''

$$= -\gamma M 4\pi$$

So the flux related to the surface s depends on a constant $(-\gamma 4\pi)$ and on M (mass).

Even if I'd taken another surface ◯, the flux would have been the same!

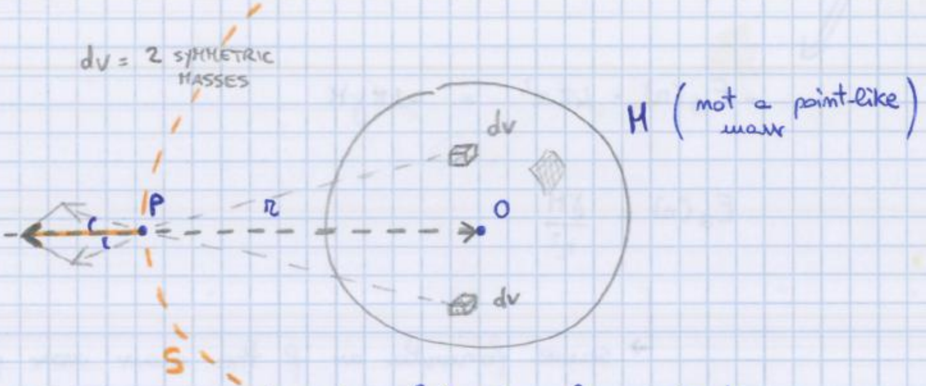
Also, the mass M can be anywhere inside the surface, and the flux will be the same.

$\rightarrow$ The flux on a closed surface, of a gravitational field produced by some pointlike masses, is $= -4\pi\gamma$ (M_{TOT} inside the surface).

all the masses

SAME FOR COULOMB FORCE, CHARGES

$$\vec{E}_G = \frac{\gamma}{d^2} \cdot \left(\text{point-like mass}\right) \cdot \frac{\vec{d}}{d}$$



* spherical approximation
* density = $\rho(x,y,z) = \rho_0$ constant

single mass = $\rho\, dv$

$\vec{E} = d\vec{E} + d\vec{E}' + \dots$ → sum of an infinite number of infinitesimal quantities
↳ integral

dv and dv produce two infinitesimal gravitational fields (dE_G) whose sum is along r (←)

Considering all symmetrical masses, the sum of the fields produced will be along r.

If the sphere is homogeneous, we have a radial field.

$$\vec{E}_G = E_G \cdot \frac{\vec{r}}{r}$$

(SAME FOR CHARGES)

I choose a surface as I want, passing through the point P and having center in O.

$$\phi = \int_S \vec{E} \cdot \vec{n} \cdot ds = \int E_G(r) \cdot \frac{\vec{r}}{r} \cdot \vec{n} \cdot ds =$$

↳ GRAVITATIONAL FIELD AS A FUNCTION OF r

Since r is the same along the surface S, the field E_G will be a constant here

$$= E_G(r) \int_S \left(\frac{\vec{r}}{r} \cdot \vec{n}\right) ds =$$

MODULUS = 1; DOT PRODUCT → $1 \cdot \cos\alpha$

($\alpha = \pi$ because $\vec{r}$ and $\vec{n}$ have opposite orientation)

$$= -E_G(r) \cdot \int_S ds =$$

$$= -E_G(r) \cdot 4\pi r^2$$

But, recalling the Gauss Theorem, we found that $\phi_S(\vec{E}_G) = -4\pi\gamma M$

⇓

NEW

Planar distribution of charges along an infinite plane

We cannot speak of volume, but ...

$dA \rightarrow$ infinitesimal area

$dq \rightarrow$ infinitesimal charge

$\sigma \equiv \frac{dq}{dA}$ $\rightarrow$ areal density

$$Q = \int dq = \int \frac{dq}{dA} \cdot dA = \int \sigma \cdot dA$$

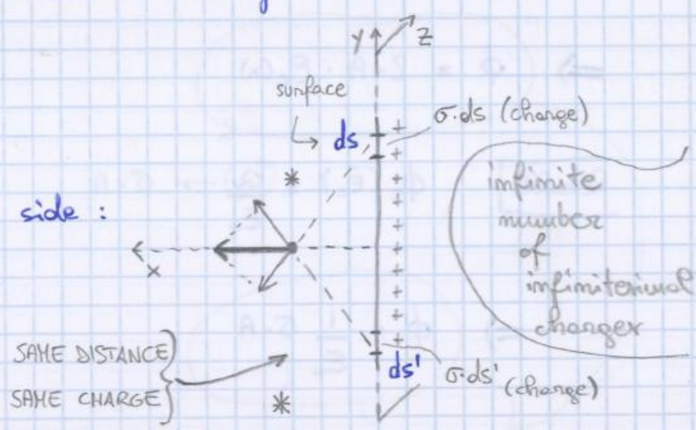
We cannot speak of an area because this is an infinite plane; but we can assume that the density is always σ and in every point of the plane there is a lot of area in every direction.

σ = uniform and constant

I look at the plane from one side:

$$\vec{E}_c = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\left(\text{point-like charge}\right)}{d^2} \cdot \vec{m}$$

$\vec{m}$ → unit vector for direction



We obtain two electric fields that are equal in modulus and are symmetrical. Their sum will be $\perp$ to the plane.

$$\vec{E}_c(x,y,z) = E_c(x,y,z) \cdot \vec{i}$$

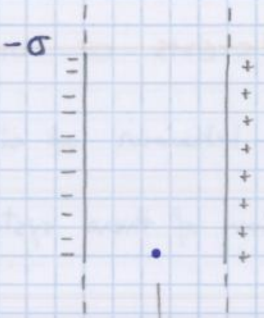
Since the plane is infinite, we can repeat the proof in other points ✱ and also in the z-axis direction.

$\vec{E}_c(x, \cancel{y}, \cancel{z})$ because the direction will always be $\perp$ to the plane

The charge felt by a outer point will be the same independently on y, z position.

CAPACITOR

If you have two equally charged surfaces, w/ opposite charges, the E_c outside will be $= 0$

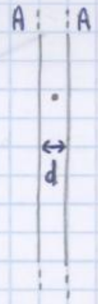


$$\left(\frac{\sigma}{2\varepsilon_0} + \frac{-\sigma}{2\varepsilon_0}\right) = 0$$

In between, there is an $\vec{E}_c$

$$\underset{\text{(left-surface)}}{-\frac{\sigma}{2\varepsilon_0}} + \underset{\text{(right-surface)}}{\left(-\frac{\sigma}{2\varepsilon_0}\right)} = \frac{\sigma}{\varepsilon_0}$$

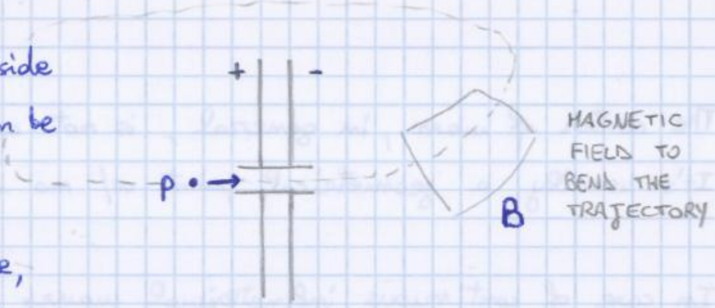
The distance between the plates must be very small.

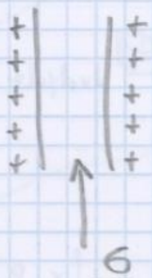


$d \ll \sqrt{A}$

With such a small hole inside the capacitor, a proton can be accelerated.

W/ a magnetic field outside, the proton can be accelerated as many times as we want.





$$\frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

$$\frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0$$

$$\frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0$$

$$\frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

In the exam → no calculating of center of mass!

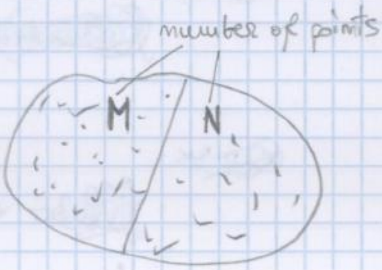
The center of mass can be at rest or moving.

If a rigid body has uniform density, the center of mass = center of symmetry
(ρ doesn't depend on x, y, z)

ADDITIVE PROPERTY OF THE CENTER OF MASS

Suppose to have a system of points.

Then I divide it in 2 sub-systems.



$$\vec{r}_{CM} = \frac{\sum_{i=1}^{N+M} m_i \vec{r}_i}{\sum_{i=1}^{N+M} m_i}$$

$$= \frac{\sum_{i=1}^{M} m_i \vec{r}_i + \sum_{j=1}^{N} m_j \vec{r}_j}{\underbrace{\sum_{i=1}^{M} m_i}_{m_M} + \underbrace{\sum_{j=1}^{N} m_j}_{m_N}}$$

$$\Longrightarrow (m_M + m_N)\,\vec{r}_{CM} = m_M \cdot \underbrace{\left(\frac{\sum_{i=1}^{M} m_i \vec{r}_i}{m_M}\right)}_{\text{CENTER OF MASS OF } M} + m_N \cdot \underbrace{\left(\frac{\sum_{j=1}^{N} m_j \vec{r}_j}{m_N}\right)}_{\text{CENTER OF MASS OF } N}$$

$$\Longrightarrow \boxed{\vec{r}_{CM} = \frac{m_M \cdot \vec{r}_{CM}^{(M)} + m_N \cdot \vec{r}_{CM}^{(N)}}{m_M + m_N}}$$

If you know the center of mass of each system you can use this formula.
It's like the mass of each sub-system were concentrated in its center of mass.

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$$\vec{r}_{CM}(t) \equiv \frac{\sum_{i=1}^{N} m_i \vec{r}_i(t)}{m_{TOT}}$$ function of time

VELOCITY OF THE CENTER OF MASS

$$\vec{v}_{CM} = \dot{\vec{r}}_{CM} = \frac{1}{m_{TOT}} \cdot \sum m_i \dot{\vec{r}}_i(t) = \frac{1}{m_{TOT}} \sum m_i \vec{v}_i$$

ACCELERATION " " "

$$\vec{a}_{CM} = \dot{\vec{v}}_{CM} = \frac{1}{m_{TOT}} \sum m_i \vec{a}_i$$

Also, $\vec{v}_{CM}$ and $\vec{a}_{CM}$ can be zero even if the particles are moving. For instance, fixing a wheel and making it rotate in air, this happens.

LINEAR MOMENTUM OF A SYSTEM

$$\boxed{\vec{P}_T \equiv \sum_{i=1}^{N} m_i \vec{v}_i = m_{TOT} \left(\frac{\sum_{i=1}^{N} m_i \vec{v}_i}{m_{TOT}} \right) = m_{TOT} \cdot \vec{v}_{CM}}$$

Total linear momentum = (Total mass)(velocity of the center of mass)

$$\boxed{\dot{\vec{P}}_T = \frac{d}{dt} \left(\sum m_i \vec{v}_i \right) = \sum m_i \vec{a}_i = m_{TOT} \cdot \left(\frac{\sum m_i \vec{a}_i}{m_{TOT}} \right) = m_{TOT} \cdot \vec{a}_{CM}}$$

Derivative of total linear momentum = (Total mass)(acceleration of center of mass)

TOTAL ANGULAR MOMENTUM OF A SYSTEM

$$\vec{L}_O^T \equiv \sum_{i=1}^{N} \vec{L}_O^i = \sum_{i=1}^{N} (\vec{r}_i - \vec{r}_O) \times m_i \vec{v}_i$$

2 reference frames:

$$(C, x, y, z) \qquad (C', x', y', z')$$

NO ROTATION OF THE MOVING REFERENCE FRAME

$$\vec{L}_O^T = \sum (\vec{r}_i - \vec{r}_O) \times m_i \vec{v}_i = \sum \left[(\vec{r}_i{}' + \vec{r}_{C'}) - (\vec{r}_O{}' + \vec{r}_{C'}) \right] \times m_i \vec{v}_i =$$

$$= \sum \left[\vec{r}_i{}' - \vec{r}_O{}' \right] \times m_i \vec{v}_i = \sum \left[\vec{r}_i{}' - \vec{r}_O{}' \right] \times m_i (\vec{v}_i{}' + \vec{v}_{C'}) =$$

$$= \underbrace{\sum (\vec{r}_i{}' - \vec{r}_O{}') \times m_i \vec{v}_i{}'}_{\text{TOT. ANGULAR MOMENTUM MEASURED IN MOVING REF. SYSTEM}} + \left[\sum (\vec{r}_i{}' - \vec{r}_O{}') \times m_i \right] \vec{v}_{C'} =$$

↳ THIS DOESN'T DEPEND ON i

$$= \vec{L}_O^{T'} + \left[\sum m_i \vec{r}_i{}' - \underbrace{\sum m_i \vec{r}_O{}'}_{\text{INDEPENDENT ON } i} \right] \times \vec{v}_{C'} =$$

$$= \vec{L}_O^{T'} + \left[M_T \cdot \vec{r}_{CM}{}' - M_T \cdot \vec{r}_O{}' \right] \times \vec{v}_{C'} =$$

$$\left(M_T (\vec{r}_{CM}{}' - \vec{r}_O{}') \right)$$

Recall:

$$\vec{v} = \vec{v}\,' + \vec{v}_{C'}$$

$$\boxed{\vec{L}_O^T = \vec{L}_O^{T'} + (\vec{r}_{CM}{}' - \vec{r}_O{}') \times M_T \vec{v}_{C'}}$$