NUMERO: 1355 ANNO: 2014

A P P U N T I

STUDENTE: Preatto

MATERIA: Analisi Matematica I Eserc., Prof.Tilli

ESERCIZI SUI DOMINI DELLE FUNZIONI (PAG 15)

a) $a(x) = \frac{x^2-6x+5}{x-7}$

$x-7 \neq 0 \quad x \neq 7 \quad \text{dom} f = \mathbb{R} \setminus \{7\} = (-\infty, 7) \cup (7, +\infty)$

b) $b(x) = x - \frac{x^2}{x^2+3} = \frac{x^3-x^2+3x}{x^2+3}$

$x^2+3 \neq 0 \quad x^2 \neq -3 \quad \forall x \in \mathbb{R} \Rightarrow \text{dom} f = \mathbb{R}$

c) $c(x) = \frac{x-9}{x^2-5x+6}$

$x^2-5x+6 \neq 0 \quad (x-3)(x-2) \neq 0 \quad x \neq 3 \lor x \neq 2$

$\text{dom} f = \mathbb{R} - \{2, 3\} = (-\infty, 2) \cup (2, 3) \cup (3, +\infty)$

d) $d(x) = \frac{x}{(x^2-5)^2}$

$x^2-5 \neq 0 \quad (x-\sqrt{5})(x+\sqrt{5}) \neq 0 \quad x \neq \pm\sqrt{5}$

$\text{dom} f = \mathbb{R} \setminus \{\pm\sqrt{5}\} = (-\infty; -\sqrt{5}) \cup (-\sqrt{5}, \sqrt{5}) \cup (\sqrt{5}; +\infty)$

e) $e(x) = \sqrt{x} - \sqrt{1-x}$

$\begin{cases} x \geq 0 \\ 1-x \geq 0 \end{cases} \quad \begin{cases} x \geq 0 \\ -x \geq -1 \to x \leq 1 \end{cases}$

	0		1	
−		+		+
+		+		−
−		+		−

$\text{dom} f = [0, 1]$

f) $f(x) = \sqrt{x^2-9x+8}$

$x^2-9x+8 \geq 0 \quad (x-8)(x-1) \geq 0 \to x \leq 1 \cup x \geq 8$

$\text{dom} f = (-\infty, 1] \cup [8, +\infty)$

g) $g(x) = \underbrace{\sqrt[3]{x^2-7}}_{\text{def: } \mathbb{R}} - \sqrt{12-x}$

$12-x \geq 0 \to -x \geq -12 \to x \leq 12$

$\text{dom} f = (-\infty; -12]$

h) $h(x) = \frac{x-6}{x+1-\sqrt{2x+3}}$

$\begin{cases} 2x+3 \geq 0 & x \geq -\frac{3}{2} \\ x+1-\sqrt{2x+3} \neq 0 & \to x+1 \neq \sqrt{2x+3} \end{cases}$

$(x+1)^2 \neq 2x+3$

$x^2+\cancel{2x}-\cancel{2x}+1-3 \neq 0$

$x^2-2 \neq 0 \quad x \neq \pm\sqrt{2}$

$\text{dom} f = \mathbb{R} \setminus \{\pm\sqrt{2}\} \cap [-\frac{3}{2}, +\infty) = [-\frac{3}{2}, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, +\infty)$

n) $n(x) = (1+x)^{3x^2+12x} \rightarrow (1+x) > 0$

$1+x > 0 \qquad x > -1 \rightarrow \text{dom} f = (-1; +\infty)$

p) $p(x) = \dfrac{5\,\text{sen}2x - 3\cos x}{4\,\text{sen}x}$

$4\,\text{sen}x \neq 0 \qquad \text{sen}x \neq 0 \qquad x \neq k\pi, +\pi + k\pi$

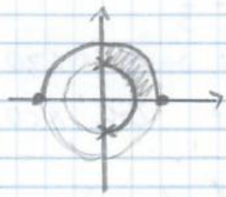
$\text{dom} f = \mathbb{R} \setminus \{0 + k\pi, \pi + k\pi\}$

q) $q(x) = \sqrt{\text{tg}x - \text{sen}x}$

$\text{tg}\,x - \text{sen}x \geq 0 \qquad \dfrac{\text{sen}x}{\cos x} - \text{sen}x \geq 0 \qquad \dfrac{\text{sen}x - \text{sen}x\cos x}{\cos x} \geq 0$

$$\begin{cases} \dfrac{\text{sen}x\,(1-\cos x)}{\cos x} \geq 0 \\ \cos x \neq 0 \end{cases}$$

$$\begin{cases} \text{sen}x \geq 0 & 0 + k\pi \leq x \leq \pi + k\pi \\ -\cos x \geq -1 \rightarrow \cos x \leq 1 & \forall x \\ \cos x > 0 & -\dfrac{\pi}{2} + k\pi < x < \dfrac{\pi}{2} + k\pi \end{cases}$$



$\text{dom} f = 0 + k\pi \leq x < \dfrac{\pi}{2} + k\pi$

r) $r(x) = \text{arcsen}\left(\dfrac{9x}{x^2-1}\right)$

$$\begin{cases} x^2 - 1 \neq 0 & \rightarrow x \neq \pm 1 \\ -1 \leq \dfrac{9x}{x^2-1} \leq 1 & \rightarrow \bullet\ \dfrac{9x}{x^2-1} \geq -1 \qquad \dfrac{x^2+9x-1}{x^2-1} \geq 0 \end{cases}$$

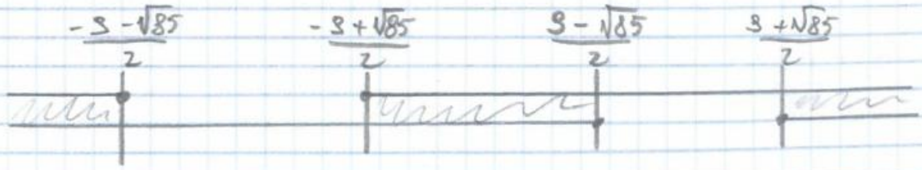
$x_{1,2} = \dfrac{-9 \pm \sqrt{81+4}}{2} = \dfrac{-9 \pm \sqrt{85}}{2}$

$x \leq \dfrac{-9-\sqrt{85}}{2} \cup x \geq \dfrac{-9+\sqrt{85}}{2}$

$\bullet\ \dfrac{9x}{x^2-1} \leq 1 \qquad \dfrac{-x^2+9x+1}{x^2-1} \leq 0$

$\dfrac{x^2-9x-1}{x^2-1} \geq 0$

$x_{1,2} = \dfrac{9 \pm \sqrt{85}}{2} \rightarrow x \leq \dfrac{9-\sqrt{85}}{2} \cup x \geq \dfrac{9+\sqrt{85}}{2}$

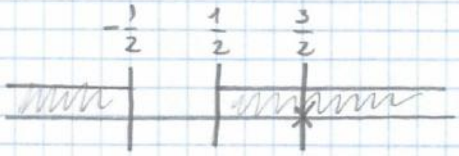


$\text{dom} f = \left(-\infty, \dfrac{-9-\sqrt{85}}{2}\right] \cup \left[\dfrac{-9+\sqrt{85}}{2}, \dfrac{9-\sqrt{85}}{2}\right] \cup \left[\dfrac{9+\sqrt{85}}{2}; +\infty\right)$

v) $v(x) = \dfrac{\sqrt{2|x|-1}}{|2x-3|}$

$$\begin{cases} 2|x| - 1 \geq 0 \\ |2x-3| \neq 0 \end{cases} \rightarrow \begin{cases} 2x - 1 \geq 0 & x \geq 0 \\ -2x - 1 \geq 0 & x < 0 \end{cases} \rightarrow \begin{cases} x \geq \frac{1}{2} & \text{con } x \geq 0 \\ x < -\frac{1}{2} & \text{con } x < 0 \end{cases}$$

$$\begin{cases} 2x - 3 \neq 0 & x \geq \frac{3}{2} \\ -2x + 3 \neq 0 & x < \frac{3}{2} \end{cases} \rightarrow x \neq \frac{3}{2}$$



$\text{dom}f = \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{3}{2}\right) \cup \left(\frac{3}{2}; +\infty\right)$

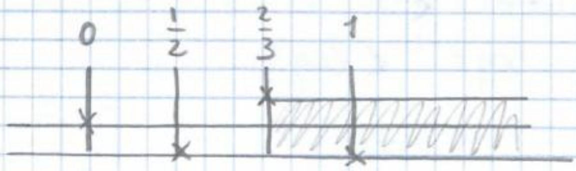
w) $w(x) = \dfrac{e^{2x} - 1}{\log(3x-2) - \log|x|}$

$$\begin{cases} 3x - 2 > 0 \\ |x| > 0 \\ \log(3x-2) - \log|x| \neq 0 \end{cases} \quad \rightarrow x > 2/3 \longrightarrow \begin{cases} x > 0 & \text{con } x \geq 0 \\ x < 0 & \text{con } x < 0 \end{cases} \Leftrightarrow \forall x \neq 0$$

$$\begin{cases} \log(3x-2) - \log x \neq 0 & x \geq 0 \\ \log(3x-2) - \log(-x) \neq 0 & x < 0 \end{cases}$$

$$\begin{cases} \log 3x - \log 2 - \log x \neq 0 \rightarrow \log(3x - 2 - x) \neq 0 & x \geq 0 \\ \log(3x - 2 + x) \neq 0 & x < 0 \end{cases}$$

$$\begin{cases} 3x - 2 - x \neq 0 \\ 3x - 2 + x \neq 0 \end{cases} \quad \begin{cases} 2x \neq 2 \\ 4x \neq 2 \end{cases} \quad \begin{cases} x \neq 1 \\ x \neq \frac{1}{2} \end{cases}$$



$\text{dom}f. = \left(\frac{2}{3}, 1\right) \cup (1, +\infty)$

ESERCIZI SUI LIMITI DI FUNZIONE

1) FORME INDETERMINATE DI TIPO RAZIONALE

a) $$\lim_{x\to\pm\infty} \frac{x^3-3x^2+4x}{x^5-x} = \lim_{x\to\pm\infty} \frac{\cancel{x^5}\left(\frac{x^3}{x^5}-\frac{3x^2}{x^5}+\frac{4x}{x^5}\right)}{\cancel{x^5}\left(1-\frac{x}{x^5}\right)} =$$

$$= \lim_{x\to\pm\infty} \frac{\underbrace{\frac{1}{x^2}}_{\to 0}-\underbrace{\frac{3}{x^3}}_{\to 0}+\underbrace{\frac{4}{x^4}}_{\to 0}}{1-\underbrace{\frac{1}{x^4}}_{\to 0}} = \frac{0}{1} = 0$$

b) $$\lim_{x\to\pm\infty} \frac{x^3-3x^2-5}{2x^3-x^2+1} = \lim_{x\to\pm\infty} \frac{\cancel{x^3}\left(1-\underbrace{\frac{3x^2}{x^3}}_{\to 0}-\underbrace{\frac{5}{x^3}}_{\to 0}\right)}{\cancel{x^3}\left(2-\underbrace{\frac{x^2}{x^3}}_{0}+\underbrace{\frac{1}{x^3}}_{0}\right)} = \frac{1}{2}$$

c) $$\lim_{x\to\pm\infty} \frac{6x^4-x^2-1}{1+12x-x^3} = \lim_{x\to\pm\infty} \frac{x^4\left(6-\frac{x^2}{x^4}-\frac{1}{x^4}\right)}{\cancel{x^3}\left(-1+\frac{12x}{x^4}+\frac{1}{x^4}\right)} = \lim_{x\to\pm\infty} -6x = \mp\infty$$

d) $$\lim_{x\to -1} \frac{x^5+1}{x^9+1} = \frac{0}{0}$$

$x^5+1 \to P(-1) = -1+1 = 0$

$$\begin{array}{c|ccccc|c} & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & // \end{array}$$

$$(x^5+1) = (x+1)(x^4-x^3+x^2-x+1)$$

$x^9+1 \quad P(-1) = -1+1 = 0$

$$\begin{array}{c|ccccccccc|c} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & // \end{array}$$

$$x^9+1 = (x+1)(x^8-x^7+x^6-x^5+x^4-x^3+x^2-x+1)$$

$$\lim_{x\to -1} \frac{\cancel{(x+1)}(x^4-x^3+x^2-x+1)}{\cancel{(x+1)}(x^8-x^7+x^6-x^5+x^4-x^3+x^2-x+1)} = \frac{1+1+1+1+1}{1+1+1+1+1+1+1+1+1} = \frac{5}{9}$$

e) $$\lim_{x\to 1} \frac{x^2-3x+2}{x^2-5x+4} = \lim_{x\to 1} \frac{(x-2)\cancel{(x-1)}}{(x-4)\cancel{(x-1)}} = \lim_{x\to 1} \frac{x-2}{x-4} = \frac{1-2}{1-4} = +\frac{1}{3}$$

f) $$\lim_{x\to 2} \frac{x^4-16}{x^2-3x+2} = \lim_{x\to 2} \frac{(x^2-4)(x^2+4)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)(x^2+4)}{\cancel{(x-2)}(x-1)} = \frac{4\cdot(8)}{1} = 32$$

g) $$\lim_{x\to 0} \frac{x^3-x^4+x^2}{x(x-1)} = \lim_{x\to 0} \frac{x^{\cancel{2}}(x-x^2+1)}{\cancel{x}(x-1)} = \lim_{x\to 0} \frac{x(x-x^2+1)}{x-1} = \frac{0}{1} = 0$$

h) $$\lim_{x\to 1} \frac{x^2-3x+2}{(x^3-2x^2+x)^2} = \lim_{x\to 1} \frac{(x-1)(x-2)}{[x(x^2-2x+1)]^2} = \lim_{x\to 1} \frac{(x-1)(x-2)}{x^2[(x-1)^2]^2} =$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x-2)}{x^2(x-1)^{\cancel{4}\,3}} = \lim_{x\to 1} \frac{x-2}{x^2(x-1)^3} \Longrightarrow \nexists$$

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2) LIMITI CON I TEOREMI DEL CONFRONTO

a) $\lim_{x \to \pm\infty} (x + \cos x) = \pm\infty$

$$-1 \le \cos x \le 1$$

$$\underset{\pm\infty}{\underset{\downarrow}{x-1}} \le \underset{\pm\infty}{\underset{\Downarrow}{\cos x + x}} \le \underset{\pm\infty}{\underset{\downarrow}{x+1}}$$

b) $\lim_{x \to 0} x^2 \operatorname{sen} \frac{1}{x} = 0$

$$-1 \le \operatorname{sen} \frac{1}{x} \le 1$$

$$\underset{0}{\underset{\downarrow}{-x^2}} \le \underset{0}{\underset{\Downarrow}{x^2 \operatorname{sen} \frac{1}{x}}} \le \underset{0}{\underset{\downarrow}{x^2}} \qquad \forall x \neq 0$$

c) $\lim_{x \to \pm\infty} \frac{\cos x}{\log x}$

$$-1 \le \cos x \le 1$$

$$\forall x > 1 \quad \Rightarrow \quad \underset{0}{\underset{\downarrow}{-\frac{1}{\log x}}} \le \underset{0}{\underset{\Downarrow}{\frac{\cos x}{\log x}}} \le \underset{0}{\underset{\downarrow}{\frac{1}{\log x}}}$$

d) $\lim_{x \to 1} (x-1)[x] = 0$

$$(x-1) \le [x] \le x$$

$$(x-1)(x-1) \le [x] \le x(x-1)$$

$$\underset{0}{\underset{\downarrow}{(x-1)^2}} \le \underset{0}{\underset{\downarrow}{(x-1)[x]}} \le \underset{0}{\underset{\downarrow}{x(x-1)}}$$

e) $\lim_{x \to \pm\infty} e^x \left(1 - \frac{1}{2}\cos x\right)$

$$-1 \le \cos x \le 1$$

$$+\frac{1}{2} \le -\frac{1}{2}\cos x \le -\frac{1}{2}$$

$$-\frac{1}{2} \le \frac{1}{2}\cos x \le \frac{1}{2}$$

$$1 - \frac{1}{2} \le 1 - \frac{1}{2}\cos x \le \frac{1}{2} + 1$$

$$e^x\left(1 - \frac{1}{2}\right) \le \left(1 - \frac{1}{2}\cos x\right)e^x \le e^x\left(1 - \frac{1}{2}\right)$$

$+\infty \downarrow +\infty \qquad -\infty \downarrow 0 \qquad\qquad +\infty \downarrow +\infty \qquad -\infty \downarrow 0$

$\lim_{x \to -\infty} e^x = 0$

perché

$\lim_{x \to +\infty} e^x \left(1 - \frac{1}{2}\cos x\right) = +\infty \qquad \lim_{x \to -\infty} e^x \left(1 - \frac{1}{2}\cos x\right) = 0$

3) FORME INDETERMINATE DI TIPO TRIGONOMETRICO

a) $\lim_{x\to 0} \frac{\operatorname{sen}x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \lim_{x\to 0} \frac{\operatorname{sen}x \cdot \sqrt{x}}{x} = \lim_{x\to 0} \frac{\operatorname{sen}x}{x} \cdot \sqrt{x} = 1 \cdot 0 = 0$

b) $\lim_{x\to 0} \frac{x+\operatorname{sen}3x}{x-\operatorname{sen}2x} = \lim_{x\to 0} \frac{\cancel{x}\left(1+\frac{\operatorname{sen}3x}{x}\right)}{\cancel{x}\left(1-\frac{\operatorname{sen}2x}{x}\right)} = \lim_{x\to 0} \frac{1+\frac{\operatorname{sen}3x}{x}\cdot\frac{3}{3}}{1-\frac{\operatorname{sen}2x}{x}\cdot\frac{2}{2}} =$

$= \lim_{x\to 0} \frac{1+\frac{3\operatorname{sen}3x}{3x}}{1-\frac{2\operatorname{sen}2x}{2x}} = \frac{1+3}{1-2} = -4$

?c) $\lim_{x\to \alpha} \frac{\operatorname{sen}x - \operatorname{sen}\alpha}{x-\alpha} = \lim_{x\to \alpha} \frac{2\operatorname{sen}\frac{x-\alpha}{2}\cos\frac{x+\alpha}{2}}{x-\alpha} = \cos\alpha$

d) $\lim_{x\to 0^+} \frac{\operatorname{sen}x + x^2}{x+\sqrt{x}} = \lim_{x\to 0} \frac{\cancel{x}\left(\frac{\operatorname{sen}x}{x}+x\right)}{\cancel{\sqrt{x}}\left(1+\sqrt{x}\right)} = \lim_{x\to 0} \frac{\sqrt{x}\left(\frac{\operatorname{sen}x}{x}+x\right)}{1+\sqrt{x}} = \frac{0(1)}{1} = 0$

e) $\lim_{x\to 0} \frac{\operatorname{tg}3x - 3x}{x} = \lim_{x\to 0} \frac{\frac{\operatorname{sen}3x}{\cos 3x} - 3x}{x\cos 3x} = \lim_{x\to 0} \frac{\operatorname{sen}3x - 3x\cos 3x}{x\cos 3x} \cdot \frac{3}{3}$

$= \lim_{x\to 0} \frac{3}{\cos 3x}\left(\frac{\operatorname{sen}3x}{3x} - \frac{\cancel{3}x\cos 3x}{\cancel{3x}}\right) = \lim_{x\to 0} 3\left(\frac{\operatorname{sen}3x}{3x} - \cos 3x\right)\cdot\frac{1}{\cos 3x} = 3\cdot 1(1-1) = 0$

f) $\lim_{x\to 0} \frac{\operatorname{sen}(x^2-x)}{x} \cdot \frac{(x^2-x)}{(x^2-x)} = \lim_{x\to 0} \frac{\operatorname{sen}(x^2-x)}{(x^2-x)} \cdot \frac{(x^2-x)}{x} =$

$\lim_{x\to 0} \frac{\operatorname{sen}(x^2-x)}{(x^2-x)} \cdot \frac{\cancel{x}(x-1)}{\cancel{x}} = 1\cdot(-1) = -1$

g) $\lim_{x\to 0} \frac{\operatorname{arctg}x}{x} = \lim_{x\to 0} \frac{\frac{\cos x}{\operatorname{sen}x}}{x} = \lim_{x\to 0} \frac{\cos x}{x\operatorname{sen}x} = \lim_{x\to 0} \frac{\cos x}{x^2\left(\frac{\operatorname{sen}x}{x}\right)} =$

$\lim_{x\to 0} \frac{\cos x}{x^2} \cdot \frac{1}{\left(\frac{\operatorname{sen}x}{x}\right)} = \infty \cdot 1 = +\infty$

h) $\lim_{x\to \pi} \frac{\operatorname{sen}x}{\pi - x} =$ $\quad t = \pi - x \quad x = -\pi - t \quad x\to\pi \quad t\to(\pi-\pi) = 0$

$= \lim_{t\to 0} \frac{\operatorname{sen}(t-\pi)}{t} = \lim_{t\to 0} \frac{\operatorname{sen}t - \operatorname{sen}\pi}{t} = \lim_{t\to 0} \frac{\operatorname{sen}t}{t} = 1$

g) $\lim_{x\to\pm\infty} \frac{\sqrt[3]{x^4+x-1}}{x+\sqrt[3]{x}} = \lim_{x\to\pm\infty} \frac{x\sqrt[3]{x+\frac{x}{x^3}-\frac{1}{x^3}}}{x\left(1+\frac{\sqrt[3]{x}}{x}\right)} = \pm\infty$

h) $\lim_{x\to\pm\infty} \frac{\sqrt[5]{x^4+1}}{x-\sqrt[3]{x}} = \lim_{x\to\pm\infty} \frac{\sqrt[5]{x^4}\cdot\sqrt[5]{1+\frac{1}{x^4}}}{x\left(1+\frac{\sqrt[3]{x}}{x}\right)} = \lim_{x\to\pm\infty} \frac{\sqrt[5]{1+\frac{1}{x^4}}}{\sqrt[5]{x}\left(1+\frac{\sqrt[3]{x}}{x}\right)} = 0$

$|3| = 3$ perchè $x\to+\infty$

i) $\lim_{x\to+\infty} \log \frac{\sqrt{9x^2-1}-x}{x} = \lim_{x\to+\infty} \log \frac{x\left(\frac{\sqrt{9x^2-1}}{x}-1\right)}{x} = \lim_{x\to+\infty} \log\left(\sqrt{9-\frac{1}{x}}-1\right) = \log(3-1) = \log 2$

l) $\lim_{x\to 0} \frac{\sqrt{x^2+x+1}-1}{x} = \lim_{x\to 0} \frac{x^2+x+\cancel{1}-\cancel{1}}{x\left(\sqrt{x^2+x+1}+1\right)} = \lim_{x\to 0} \frac{\cancel{x}(x+1)}{\cancel{x}\left(\sqrt{x^2+x+1}+1\right)} = \frac{1}{2}$

m) $\lim_{x\to\frac{\pi}{2}} \frac{1-\sqrt{\operatorname{sen}x}}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}} \frac{1-\operatorname{sen}x}{\left(x-\frac{\pi}{2}\right)\left(1+\sqrt{\operatorname{sen}x}\right)}$ pongo $t = x-\frac{\pi}{2}$, $x = t+\frac{\pi}{2}$, $x\to\frac{\pi}{2}$, $t\to 0$

$\lim_{t\to 0} \frac{1-\operatorname{sen}\left(t+\frac{\pi}{2}\right)}{t\left(1+\sqrt{\operatorname{sen}\left(t+\frac{\pi}{2}\right)}\right)} = \lim_{t\to 0} \frac{1-\cos t}{t\left(1+\sqrt{\cos t}\right)} = \lim_{t\to 0} \frac{\left(1-\sqrt{\cos t}\right)\cancel{\left(1+\sqrt{\cos t}\right)}}{t\,\cancel{\left(1+\sqrt{\cos t}\right)}} =$

$= \lim_{t\to 0} \frac{1-\sqrt{\cos t}}{t}\cdot\frac{t}{t} = \lim_{t\to 0} \frac{1-\sqrt{\cos t}}{t^2}\cdot t = \frac{1}{2}\cdot 0 = 0$

6. INFINITI E INFINITESIMI

ESERCIZIO 1

a) $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{(x-1)^3} =$ $\quad t = x-1 \quad x = t+1 \quad x\to 1 \; t\to 0$

$\lim_{t\to 0} \frac{\sqrt[3]{t+1}-1}{t^3} =$

$\lim_{t\to 0} \frac{(t+1)^{\frac{1}{3}}-1}{t^3} = \lim_{t\to 0} \frac{1/3\, t + o(t)}{t^3+o(t^3)} = \lim_{t\to 0} \frac{1}{3}\frac{1+o(t)}{t^2+o(t)} = \frac{1}{3t^2} + o(\frac{1}{t^2})$

ordine 2 $\quad$ pp: $\frac{1}{3t^2} = \frac{1}{3(x-1)^2} \quad x\to 1$

$\lim_{x\to\pm\infty} \frac{\sqrt[3]{x}-1}{(x-1)^3} = \lim_{x\to\pm\infty} \frac{x^{\frac{1}{3}}+o(x^{\frac{1}{3}})}{x^3+o(x^3)} = \lim_{x\to\pm\infty} \frac{1}{x^{8/3}} + o(x^{\frac{8}{3}})$

ordine $\frac{8}{3}$ $\quad$ pp: $\frac{1}{x^{8/3}} \quad x\to\pm\infty$

b) $f(x) = \sqrt{3+x^2} - \sqrt{3+2x^2}$

$\lim_{x\to 0} \sqrt{3+x^2} - \sqrt{3+2x^2} = \lim_{x\to 0} \sqrt{3}\left(\sqrt{1+\frac{x^2}{3}} - \sqrt{1+\frac{2x^2}{3}}\right) =$

$= \lim_{x\to 0} \sqrt{3}\left(\frac{1}{2}\frac{x^2}{3} + 1 + o(x^2) - \left(\frac{1}{2}\cdot\frac{2x^2}{3} + 1 + o(x^2)\right)\right) = \lim_{x\to 0} \sqrt{3}\left(\frac{x^2}{6} + \cancel{1} - \frac{2x^2}{6} - \cancel{1}\right) =$

$= \lim_{x\to 0} -\frac{\sqrt{3}}{6}x^2 + o(x^2) \quad x\to 0$

ordine: 2 $\quad$ pp: $-\frac{\sqrt{3}}{6}x^2 \quad x\to 0$

$\lim_{x\to\pm\infty} \sqrt{3+x^2} - \sqrt{3+2x^2} = \lim_{x\to\pm\infty} \frac{\cancel{3}+x^2-\cancel{3}-2x^2}{\sqrt{3+x^2}+\sqrt{3+2x^2}} = \lim_{x\to\pm\infty} \frac{-x^2}{\sqrt{3+x^2}+\sqrt{3+2x^2}}$

$= \lim_{x\to\pm\infty} \frac{-x^2}{|x|\left(\sqrt{\frac{3}{x^2}+1} + \sqrt{\frac{3}{x^2}+2}\right)}$

$\lim_{x\to+\infty} \frac{-x^2}{x\left(\sqrt{\frac{3}{x^2}+1} + \sqrt{\frac{3}{x^2}+2}\right)} = \lim_{x\to+\infty} \frac{-x^2}{x(1+\sqrt{2})} = \lim_{x\to+\infty} -\frac{x}{1+\sqrt{2}}$

$\lim_{x\to-\infty} \frac{-x^2}{-x\left(\sqrt{\frac{3}{x^2}+1} + \sqrt{\frac{3}{x^2}+2}\right)} = \lim_{x\to+\infty} \frac{x}{1+\sqrt{2}}$

$x\to+\infty$ $\quad$ ordine: 1 $\quad$ pp. $-\frac{x}{1+\sqrt{2}}$

$x\to-\infty$ $\quad$ ordine: 1 $\quad$ pp: $\frac{x}{1+\sqrt{2}}$

7. ESERCIZI DI RIEPILOGO

1) $$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{\frac{1}{2}x+o(x)+\frac{1}{2}x+o(x)}{x} = \lim_{x\to 0} \frac{x+o(x)}{x} = 1$$

2) $$\lim_{x\to 0^+} (1-x^2)^{\frac{1}{x}} = \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} (1-x)^{\frac{1}{x}} = e \cdot \frac{1}{e} = 1$$

3) $$\lim_{x\to 0^-} (1+3x)^{\frac{1}{x}} = \lim_{x\to 0^-} e^3+o(x) = e^3$$

4) $$\lim_{x\to 0} \frac{\log(1+2x)}{3x\cos x - \operatorname{tg} x} = \lim_{x\to 0} \frac{2x+o(x)}{3x\left(1-\frac{x^2}{2}+o(x^2)\right)-x+o(x)} = \lim_{x\to 0} \frac{2x+o(x)}{2x-\frac{x^3}{2}+o(x^3)+o(x)}$$

$$= \lim_{x\to 0} \frac{2x+o(x)}{2x+o(x)} = 1$$

5) $$\lim_{x\to 0} \frac{2x\cos x - 2x}{x \operatorname{sen}^2 x} = \lim_{x\to 0} \frac{2x(\cos x-1)}{x(\operatorname{sen}^2 x)} = \lim_{x\to 0} \frac{2x\left(-\frac{x^2}{2}+o(x^2)\right)}{x(x+o(x))^2} =$$

$$= \lim_{x\to 0} \frac{-x^3+o(x^3)}{x(x^2+2x\,o(x)+o(x^2))} = \lim_{x\to 0} \frac{-x^3+o(x^3)}{x^3+o(x^3)} = -1$$

6) $$\lim_{x\to \pm\infty} \left(1+\frac{1}{5x-1}\right)^{5x} = \lim_{x\to\pm\infty} \left(1+\frac{1}{5x-1}\right)^{5x+1-1} = \lim_{x\to\pm\infty} \left(1+\frac{1}{5x-1}\right)^{5x-1} \cdot \left(\frac{1}{5x-1}+1\right)^{1} =$$

$$\boxed{\lim_{x\to\pm\infty} \left(1+\frac{a}{x}\right)^x = e^a}$$

$$= \lim_{x\to\pm\infty} \left(1+\frac{1}{5x-1}\right)^{5x-1} \Big(\underbrace{\frac{1}{5x-1}}_{0}+1\Big) = e \cdot 1 = e$$

7) $$\lim_{x\to\pm\infty} \left(1+\frac{1}{5x-1}\right)^{4x} = \lim_{x\to\pm\infty} \left(1+\frac{1}{5x-1}\right)^{4x\cdot\frac{5x-1}{5x-1}} =$$

$$= \lim_{x\to\pm\infty} \left(\left(1+\frac{1}{5x-1}\right)^{5x-1}\right)^{\frac{4x}{5x-1}} = (e)^{\frac{4}{5}} = e^{\frac{4}{5}}$$

8) $$\lim_{x\to -2} \frac{\operatorname{tg} \pi x}{x+2}$$

$x+2=t$
$x=t-2$
$x\to -2$
$t\to 0$

$$\lim_{t\to 0} \frac{\operatorname{tg} \pi(t-2)}{t} =$$

$$= \lim_{t\to 0} \frac{\operatorname{tg}(\pi t-2\pi)}{t} = \lim_{x\to 0} \frac{\operatorname{tg}\pi t - \operatorname{tg} 2\pi}{t} = \lim_{x\to 0} \frac{\operatorname{tg}\pi t}{t} = \pi$$

9) $$\lim_{x\to 1} \frac{x^3-1}{\operatorname{sen}(x-1)} =$$

$t=x-1$
$x=t+1$
$x\to 1 \quad t\to 0$

$$\lim_{t\to 0} \frac{(t+1)^3-1}{\operatorname{sen} t} = \lim_{t\to 0} \frac{t^3+3t^2+3t+\cancel{1}-\cancel{1}}{t+o(t)} = \lim_{t\to 0} \frac{3t+o(t)}{t+o(t)} = 3$$

20) $\lim_{x\to\frac{1}{2}} \frac{3^{2x-1}-1}{\operatorname{sen}(2x-1)}$ $\quad t = x-\frac{1}{2} \quad x\to\frac{1}{2} \quad t\to 0$

$x = t+\frac{1}{2}$

$\lim_{t\to 0} \frac{3^{2(t+\frac{1}{2})-1}-1}{\operatorname{sen}(2(t+\frac{1}{2})-1))} = \lim_{t\to 0} \frac{3^{2t+\cancel{1}-\cancel{1}}-1}{\operatorname{sen}(2t+\cancel{1}-\cancel{1})} = \lim_{t\to 0} \frac{3^{2t}-1}{\operatorname{sen}(2t)} =$

$= \lim_{t\to 0} \frac{2t\log 3)^{2t}+o(t)}{2t+o(t)} = \lim_{t\to 0} \frac{\cancel{2t}\log 3+o(t)}{\cancel{2t}+o(t)} = \log 3$

21) $\lim_{x\to-\infty} (x+\log(2e^{-x}+1)) = \lim_{x\to-\infty} (x+\log e^{-x}(2+\frac{1}{e^{-x}})) =$

$\lim_{x\to-\infty} x+\log e^{-x}+\log(2+\frac{1}{e^{-x}}) = \lim_{x\to-\infty} \cancel{x}-\cancel{x}+\log(2+e^{x}) =$

$= \lim_{x\to-\infty} \log 2(1+\frac{e^x}{2}) = \log 2 \lim_{x\to-\infty} (1+\frac{e^x}{2}) = \log 2$

22) $\lim_{x\to 0} \frac{\operatorname{tg}x-\operatorname{sen}x}{x^3} = \lim_{x\to 0} \cancel{\frac{x+o(x)-x+o(x)}{x^3}} =$

$\lim_{x\to 0} \frac{\frac{\operatorname{sen}x}{\cos x}-\operatorname{sen}x}{x^3} = \lim_{x\to 0} \frac{\operatorname{sen}x\,(\frac{1}{\cos x}-1)}{x^3} = \lim_{x\to 0} \frac{\operatorname{sen}x\,(1-\cos x)}{x^3} =$

$= \lim_{x\to 0} \frac{(x+o(x))(\frac{x^2}{2}+o(x^2))}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{2}+o(x^3)}{x^3} = \frac{1}{2}$

23) $\lim_{x\to(-1)^-} \left(\frac{1}{(x+1)}+\frac{3}{x^2-x-2}\right) = \lim_{x\to(-1)^-} \left(\frac{1}{x+1}+\frac{3}{(x+1)(x-2)}\right) =$

$= \lim_{x\to(-1)^-} \frac{(x-2)+3}{(x+1)(x-2)} = \lim_{x\to(-1)^-} \frac{x-2+3}{(x+1)(x-2)} = \lim_{x\to(-1)^-} \frac{\cancel{(x+1)}}{\cancel{(x+1)}(x-2)}$

$= \lim_{x\to(-1)^-} \frac{1}{x-2} = -\frac{1}{3}$

24) $\lim_{x\to 0} \frac{\log(1+x^2)}{x^2\, e^{x+x^2}} = \lim_{x\to 0} \frac{\log(1+x^2)}{x^2} \cdot \frac{1}{e^{x+x^2}} = 1\cdot\frac{1}{e^0} = 1$

25) $\lim_{x\to 0} \frac{\operatorname{sen}x+x^3}{4\operatorname{sen}x-x} = \lim_{x\to 0} \frac{x+o(x)+x^3}{4x+o(x)-x} = \lim_{x\to 0} \frac{x+o(x)}{3x+o(x)} = \frac{1}{3}$

26) $\lim_{x\to 0} \frac{\operatorname{sen}x}{x+\sqrt{3x}} = \lim_{x\to 0} \frac{x+o(x)}{\sqrt{3x}+o(\sqrt{x})} = \lim_{x\to 0} \frac{\sqrt{x}}{\sqrt{3}} + o(\sqrt{x}) = 0$

27) $\lim_{x\to+\infty} \frac{1-\cos\frac{1}{x}}{2(\operatorname{sen}^2\frac{1}{x}-\frac{1}{x^2})}$ $\quad t = \frac{1}{x} \quad x\to+\infty \quad t\to 0$

$\lim_{t\to 0} \frac{1-\cos t}{2(\operatorname{sen}^2 t-t^2)} = \lim_{t\to 0} \frac{\frac{t^2}{2}+o(t^2)}{2(\cancel{t^2}+o(t^2)-\cancel{t^2})} = \lim_{t\to 0} \frac{\frac{t^2}{2}+o(t^2)}{0} = -\infty$

37) $\lim_{x\to-\infty}\left(1-\frac{x}{\sqrt{4+x^2}}\right) = \lim_{x\to-\infty}\left(1-\frac{x}{-x\sqrt{\frac{4}{x^2}+1}}\right) = 1+1=2$

38) $\lim_{x\to-\infty}\frac{\sqrt{|x^2-x|}\,e^{\frac{1}{x}}}{x} = \lim_{x\to-\infty}\frac{-x\sqrt{|1-\frac{1}{x}|}\,e^{\frac{1}{x}}}{x} = -1$

39) $\lim_{x\to 0^-} 3^{\frac{1}{x}}\,\text{sen}\,\frac{1}{x} = 0$

3

$-1\le \text{sen}\,\frac{1}{x}\le 1 \iff \underset{\downarrow\,0}{-3^{\frac{1}{x}}}\le 3^{\frac{1}{x}}\text{sen}\,\frac{1}{x}\le \underset{\downarrow\,0}{3^{\frac{1}{x}}} \quad \forall x\neq 0$

40) $\lim_{x\to\pi^+} e^{\frac{1}{x-\pi}}\,\text{sen}\,x$

$t=x-\pi$
$x\to\pi \quad t\to 0$
$x=t+\pi$

$=\lim_{t\to 0^+} e^{\frac{1}{t}}\cdot\text{sen}(t+\pi) = \lim_{t\to 0^+} e^{t^{-1}}\cdot\text{sen}\,t = \lim_{t\to 0} e^{t^{-1}}\cdot(t+o(t)) =$

$= e^0\cdot -\infty = 1(-\infty) = -\infty$

41) $\lim_{x\to 0}(1-\text{sen}\,x)^{\frac{3}{\text{sen}\,x}}$

$t=\text{sen}\,x$
$x\to 0 \quad t\to 0$

$\lim_{t\to 0}(1-t)^{\frac{3}{t}} = e^{-3}$

$\frac{1}{2}$

42) $\lim_{x\to 2^-}\frac{\cos\frac{\pi}{4}x}{\sqrt{4-x^2}}$

$t=2-x \quad x=2-t$
$x\to 2 \quad t\to 0$

$\lim_{t\to 0}\frac{\cos\frac{\pi}{4}(2-t)}{\sqrt{4-(2-t)^2}} = \lim_{t\to 0}\frac{\cos\left(\frac{\pi}{2}-\frac{\pi}{4}t\right)}{\sqrt{\cancel{4}-\cancel{4}+4t-t^2}} =$

$\lim_{t\to 0}\frac{\text{sen}\left(\frac{\pi}{4}t\right)}{\sqrt{4t-t^2}} = \lim_{t\to 0}\frac{\frac{\pi}{4}t+o(t)}{\sqrt{t}\sqrt{\frac{4t}{t^2}-\frac{t^2}{t^2}}} = 0$

43) $\lim_{x\to\pi}\frac{\sqrt{1+\text{sen}\,x}-\sqrt{1-\text{sen}\,x}}{e^{\pi-x}-1} =$

$t=\pi-x$
$x=\pi-t$
$x\to\pi \quad t\to 0$

$\lim_{t\to 0}\frac{\sqrt{1+\text{sen}(\pi-t)}-\sqrt{1-\text{sen}(\pi-t)}}{e^t-1} =$

$=\lim_{t\to 0}\frac{\sqrt{1+\text{sen}\,t}-\sqrt{1-\text{sen}\,t}}{e^t-1} = \lim_{t\to 0}\frac{\sqrt{1+t+o(t)}-\sqrt{1-t+o(t)}}{t+o(t)} =$

$=\lim_{t\to 0}\frac{\frac{1}{2}t+o(t)+\frac{1}{2}t+o(t)}{t+o(t)} = \lim_{t\to 0}\frac{t+o(t)}{t+o(t)} = 1$

c) $f(x) = \begin{cases} x^\alpha \log x & x>0 \\ \dfrac{e^{x^2}-\alpha}{\operatorname{sen}^2 x} & x<0 \end{cases}$

$\lim_{x\to 0^-} x^\alpha \log x = 0$

$\lim_{x\to 0^+} \dfrac{e^{x^2}-\alpha}{\operatorname{sen}^2 x} =$ $\qquad \forall \alpha \in \mathbb{R}$

d) $f(x) = \begin{cases} x+1 & x \leq 1 \\ 3-\alpha x^2 & x>1 \end{cases}$ $\qquad$ dom $f = \mathbb{R}$

$\lim_{x\to 1^-} x+1 = 2 \qquad f(1) = 2$

$\lim_{x\to 1^+} 3-\alpha x^2 = 3-\alpha \qquad 3-\alpha = 2 \Rightarrow \alpha = 1$

e) $f(x) = \begin{cases} x^2+\alpha & x<0 \\ (x-\alpha)^2 & x \geq 0 \end{cases}$ $\qquad$ dom $f = \mathbb{R}$

$\lim_{x\to 0^-} x^2+\alpha = \alpha$

$\lim_{x\to 0^+} (x-\alpha)^2 = \alpha^2 \qquad f(0) = \alpha^2$

$\alpha = \alpha^2 \qquad \alpha^2 - \alpha = 0 \qquad \alpha(\alpha-1) = 0 \qquad \alpha = 1 \vee \alpha = 0$

f) $f(x) = \begin{cases} e^{-\frac{1}{x}} & x<0 \\ \alpha & x=0 \\ \frac{1}{x} & x>0 \end{cases}$

$\lim_{x\to 0^-} e^{-\frac{1}{x}} = e^{-\infty} = 0$

$\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$\alpha = 0 = +\infty \rightarrow$ NO $\Rightarrow \nexists \alpha \in \mathbb{R}$

g) $f(x) = \begin{cases} \alpha x - 2x^2 + 1 & x \leq -1 \\ \log(2+x) - \alpha e^x & x > -1 \end{cases}$

$\lim_{x\to -1^-} \alpha x - 2x^2 + 1 = -\alpha - 2 + 1 = -\alpha - 1 \qquad f(-1) = -\alpha - 1$

$\lim_{x\to -1^+} \log(2+x) - \alpha e^x = \log(2-1) - \alpha e^{-1} = 0 - \frac{\alpha}{e} = -\frac{\alpha}{e}$

$-\alpha - 1 = -\frac{\alpha}{e} \qquad \alpha + 1 = \frac{\alpha}{e} \qquad \frac{\alpha}{\alpha+1} = e \qquad \alpha = \frac{e}{1-e}$

DERIVATE DI FUNZIONI

ESERCIZIO 1

a) $a(x) = 4x^3 - \frac{7}{2}x^2 + 4x + 5$

$a'(x) = 12x^2 - 7x + 4 \qquad \forall x \in \mathbb{R}$

b) $b(x) = \frac{x^2 - 3x + 1}{x + 1}$

$b'(x) = \frac{(2x-3)(x+1) - (x^2 - 3x + 1)}{(x+1)^2} = \frac{2x^2 + 2x \cancel{-3x} - 3 - x^2 \cancel{+3x} - 1}{(x+1)^2} = \frac{x^2 + 2x - 4}{(x+1)^2} \qquad \forall x \neq -1$

c) $c(x) = \sqrt[3]{1+x} \qquad c'(x) = \frac{1}{3\sqrt[3]{(1+x)^2}} \qquad \forall x \neq -1$

d) $d(x) = \operatorname{sen} x - \cos x \qquad d'(x) = \cos x + \operatorname{sen} x \qquad \forall x \in \mathbb{R}$

e) $e(x) = x \operatorname{sen} x \qquad e'(x) = \operatorname{sen} x + x \cos x \qquad \forall x \in \mathbb{R}$

f) $f(x) = \log x^2 \qquad f'(x) = \frac{1}{x^2} \cdot 2x = \frac{2}{x} \qquad \forall x \neq 0$

g) $g(x) = \operatorname{tg} 2x \qquad g'(x) = (1 + \operatorname{tg}^2 x)2 = \frac{2}{\cos^2 2x} \qquad \forall x \neq \frac{\pi}{4} + k\frac{\pi}{2}$

h) $h(x) = x^{\frac{3}{2}} - 6e^{5x^4} = \sqrt[2]{x^3} - 6e^{5x^4}$

$h'(x) = \frac{3x^2}{2\sqrt{x^3}} - 6e^{5x^4} \cdot 20x^3 = \frac{3}{2\sqrt{x}} - 120x^3 \cdot e^{5x^4} \qquad \forall x > 0$

i) $i(x) = \sqrt{1+x^2}$

$i'(x) = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}} \qquad \forall x \in \mathbb{R}$

j) $j(x) = e^{\operatorname{tg} x^3}$

$j'(x) = e^{\operatorname{tg} x^3} \cdot 3x^2 (\operatorname{tg}^2 x^3 + 1)$

k) $k(x) = x \operatorname{arctg} x$

$k'(x) = \operatorname{arctg} x + \frac{x}{1+x^2} \qquad \forall x \in \mathbb{R}$

l) $l(x) = \log(\log x)$

$l'(x) = \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \log x} \qquad \forall x > 1$

m) $m(x) = 2^{\operatorname{sen} x} = e^{\operatorname{sen} x \log 2}$

$m'(x) = e^{\operatorname{sen} x \log 2} \cdot (\cos x \log 2) \qquad \forall x \in \mathbb{R}$

LA FORMULA DI TAYLOR

ESERCIZIO 1

a) $f(x) = (x - \operatorname{sen} x)\log(1+x)$ $\quad x_0 = 0 \quad n = 6$

$$f(x) = \left(\cancel{x} - \cancel{x} + \frac{x^3}{3!} - \frac{x^5}{5!} + o(x^5)\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right) = \frac{x^4}{6} - \frac{x^5}{12} + \frac{17x^6}{360} + o(x^6) \quad (x \to 0)$$

b) $f(x) = \operatorname{sen} x^2 - \operatorname{sen}^2 x$ $\quad x_0 = 0 \quad n = 6$

$$f(x) = x^2 - \frac{x^6}{3!} + o(x^6) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^3)\right)^2 = \cancel{x^2} - \frac{x^6}{6} - \cancel{x^2} - \frac{x^6}{36} - \frac{x^6}{60} + \frac{x^4}{3} + o(x^6) \quad (x \to 0)$$

$$= \frac{x^4}{3} - \frac{19}{90}x^6 + o(x^6) \quad (x \to 0)$$

c) $f(x) = e^{\operatorname{sen} x}$ $\quad n = 4 \quad x_0 = 0$

$$f(x) = e^{\operatorname{sen} x} = 1 + \operatorname{sen} x + \frac{\operatorname{sen}^2 x}{2} + \frac{\operatorname{sen}^3 x}{3!} + \frac{\operatorname{sen}^4 x}{4!} + o(x^4) = \quad x \to 0$$

$$= 1 + \left(x + \frac{x^3}{3!} + o(x^4)\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + o(x^4)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{3!} + o(x^4)\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{3!} + o(x^4)\right)^4 =$$

$$= 1 + x - \cancel{\frac{x^3}{6}} + \frac{1}{2}x^2 - \frac{x^4}{6} + \cancel{\frac{x^3}{6}} + \frac{x^4}{24} + o(x^4) = 1 + x + \frac{1}{2}x^2 - \frac{x^4}{8} + o(x^4) \quad (x \to 0)$$

d) $f(x) = \log(\cos x)$ $\quad n = 4 \quad x_0 = 0$

$$f(x) = \log\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) =$$

$$\overset{x\to 0}{=} -\frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + \frac{1}{3}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^3 + o(x^4) =$$

$$\overset{x\to 0}{=} -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + o(x^4) = -\frac{x^2}{2} - \frac{2}{24}x^4 + o(x^4) =$$

$$\overset{x\to 0}{=} -\frac{x^2}{2} - \frac{1}{12}x^4 + o(x^4) \quad (x \to 0)$$

e) $f(x) = \dfrac{1}{2+x} = \dfrac{1}{2\left(1+\frac{x}{2}\right)} = \dfrac{1}{2}\left(1+\dfrac{x}{2}\right)^{-1}$ $\quad n = 3 \quad x_0 = 0$

$$f(x) \overset{x\to 0}{=} \frac{1}{2}\cdot\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + o(x^3)\right) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + o(x^3) \quad (x \to 0)$$

g) $f(x) = \operatorname{tg} x$ $\quad x = 0 \quad n = 5$

$$f(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$$

h) $f(x) = \dfrac{1}{\sqrt{8}\operatorname{sen} x - 2\cos x}$ $\quad n = 3 \quad x_0 = 0$

$$f(x) = \frac{1}{\sqrt{8}\left(x - \frac{x^3}{3!} + o(x^3)\right) - 2\left(1 - \frac{x^2}{2} + o(x^3)\right)} = \frac{1}{\sqrt{8}x - \frac{\sqrt{8}}{6_3}x^3 - 2 + x^2 + o(x^3)} = \left(-\frac{\sqrt{2}}{3}x^3 + x^2 + \sqrt{8}x - 2 + o\right.$$

$$\overset{x\to 0}{=} 1 - \left(-\frac{\sqrt{2}}{3}x^3 + x^2 + \sqrt{8}x - 2\right) + \left(-\frac{\sqrt{2}}{3}x^3 + x^2 + \sqrt{8}x - 2\right)^2 + \left(-\frac{\sqrt{2}}{3}x^3 + x^2 + \sqrt{8}x - 2\right)^3 + o(x^3) = -\frac{1}{2} + \frac{\sqrt{2}}{2}x - \frac{5}{4}x^2 + \frac{17\sqrt{2}}{12}x^3$$

ESERCIZIO 2

a) $$\lim_{x\to 0} \frac{x^2 - \operatorname{sen}^2 x}{x^3(e^x - \cos x)} = \lim_{x\to 0} \frac{x^2 - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right)^2}{x^3\left(\cancel{1} + x + \frac{x^2}{2} - \cancel{1} + \frac{x^2}{2} + o(x^2)\right)} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^2} - \cancel{x^2} - \frac{x^6}{36} + \frac{x^4}{3} + o(x^4)}{x^4 + o(x^4)} = \lim_{x\to 0} \frac{x^4 + o(x^4)}{3x^4 + o(x^4)} = \frac{1}{3}$$

b) $$\lim_{x\to 0} \frac{x^5\left[(e^{x^3})^2 - 1\right]}{\sqrt{1+x^8} - \sqrt[3]{1+x^8}} = \lim_{x\to 0} \frac{x^5\left((1+x^3)^2 - 1\right)}{\cancel{1} + \frac{1}{2}x^8 - \frac{1}{8}x^{16} - \cancel{1} - \frac{1}{3}x^8 + \frac{1}{9}x^{16} + o(x^{16})} =$$

$$\frac{1}{2}\frac{1}{3}\left(\frac{1}{3} - 1\right) = \frac{1}{6} \cdot \frac{-2}{3} = -\frac{1}{9}$$

$$= \lim_{x\to 0} \frac{x^5\left(\cancel{1} + 2x^3 + x^6 - \cancel{1}\right) + o(x^8)}{\frac{1}{6}x^8 + o(x^8)} = \lim_{x\to 0} \frac{2x^8 + o(x^8)}{\frac{x^8}{6} + o(x^8)} = 12$$

c) $$\lim_{x\to 0^+} \frac{\sqrt{1-x} - \cos\sqrt{x}}{\log(\log(e+x^2))} = \lim_{x\to 0^+} \frac{\cancel{1} - \left(\frac{1}{2}x\right) + \left(\frac{1}{8}(x)^2 - \left(\frac{1}{16}(x)\right)^3 + \ldots - \cancel{1} + \frac{(\sqrt{x})^2}{2} - \frac{(\sqrt{x})^4}{4!} + o}{\log\left(\log e\left(1 + \frac{x^2}{e}\right)\right)} \to \log\left(1 + \log\left(1 + \frac{x^2}{e}\right)\right)$$

$$= \lim_{x\to 0^+} \frac{-\frac{4}{24}x^2 + o(x^2)}{\log\left(1 + \frac{x^2}{e}\right)} = \lim_{x\to 0^+} \frac{-\frac{1}{6}x^2 + o(x^2)}{\frac{x^2}{e} + o(x^2)} = -\frac{e}{6}$$

d) $$\lim_{x\to 0} \frac{\operatorname{sen}^4 x\,(\operatorname{sen} x^2 - \operatorname{sen}^2 x)}{1 - \cos x^4} = \lim_{x\to 0} \frac{(x^4 + o(x^4))\left(x^2 - \frac{x^6}{3!} + o(x^5) - \left(x - \frac{x^3}{3!}\right)^2\right)}{1 - \left(1 - \frac{x^8}{2} + o(x^8)\right)} =$$

$$= \lim_{x\to 0} \frac{(x^4 + o(x^4))\left(\cancel{x^2} - \frac{x^6}{6} - \cancel{x^2} + \frac{x^4}{3} + o(x^4)\right)}{\frac{x^8}{2} + o(x^8)} = \lim_{x\to 0} \frac{\frac{1}{3}x^8 + o(x^8)}{\frac{x^8}{2} + o(x^8)} = \frac{2}{3}$$

e) $$\lim_{x\to 0} \frac{x \operatorname{arcsen} x - x^2}{\sqrt{1+x^4} - \cos x^2} = \lim_{x\to 0} \frac{x\left(x + \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right) - x^2}{1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 - 1 + \frac{1}{2}x^4 + o(x^4)} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^2} + \frac{x^4}{6} + o(x^4) - \cancel{x^2}}{x^4 + o(x^4)} = \frac{1}{6}$$

f) $$\lim_{x\to 0^+} \frac{\log(1 - \cos 2x)}{\log(\operatorname{tg} 2x)}$$

$$\log(1 - \cos 2x) = \log\left(\cancel{1} - \cancel{1} + \frac{4x^2}{2} - \frac{16x^4}{4!} + o(x^4)\right) =$$

$$= \log(2x^2 + o(x^2)) = \log x^2\left(2 + \frac{o(x^2)}{x^2}\right) = \log x^2 + \log\left(2 + \frac{1}{x^2}(o(x^2))\right)$$

$$= 2\log x + o(\log x) \qquad x\to 0$$

$$\log(\operatorname{tg} 2x) = \log(2x + o(x)) = \log x\left(2 + \frac{o(x)}{x}\right) = \log x + \log\left(2 + \frac{o(x)}{x}\right) =$$

$$= \log x + o(\log x) \qquad (x\to 0)$$

$$\lim_{x\to 0} \frac{2\log x + o(\log x)}{\log x + o(\log x)} = 2$$

j) $\lim_{x\to 0} \dfrac{(\sin^2 x - \log(\cos x))\log(1+\sin x)}{x \sin x \sin 2x}$

$(\sin^2 x - \log(\cos x))\log(1+\sin x)$

$\sin^2 x = \left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)^2 = x^2 - \frac{x^4}{3} + \frac{x^6}{36} + \frac{x^6}{60} + o(x^6) \qquad x\to 0$

$\log(\cos x) = \log\left(1 - \frac{x^2}{2}\right) = -\frac{x^2}{2} - \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + o(x^2) = -\frac{x^2}{2} - \frac{1}{8}x^4 + o(x^4)$

$\log(1+\sin x) = \log(1 + x + o(x)) = x - \frac{x^2}{2} + o(x^2)$

$\lim_{x\to 0} \dfrac{\left(x^2 - \frac{x^4}{3} + \frac{x^2}{2} + \frac{1}{8}x^4\right)\left(x - \frac{x^2}{2} + o(x^2)\right)}{x\,(x + o(x))(2x + o(x))} =$

$= \lim_{x\to 0} \dfrac{\frac{3}{2}x^3 + o(x^3)}{2x^3 + o(x^3)} = \frac{3}{4}$

k) $\lim_{x\to 0} \dfrac{\cos\sqrt[3]{x} - \sqrt[3]{\cos x}}{x^2} =$

$\cos\sqrt[3]{x} = 1 - \frac{(\sqrt[3]{x})^2}{2!} + \frac{(\sqrt[3]{x})^4}{4}$

$\sqrt[3]{\cos x} = \left(1 - \frac{x^2}{2}\right)^{\frac{1}{3}} = 1 + \frac{1}{3}\left(-\frac{x^2}{2}\right) + o(x^2)$

$\lim_{x\to 0} \dfrac{\cancel{1} - \frac{\sqrt[3]{x^2}}{2} - \cancel{1} + \frac{1}{6}x^2}{x^2} = \lim_{x\to 0} \dfrac{-\sqrt[3]{x^2}/2 + o(\sqrt[3]{x^2})}{x^2} = -\infty$

l) $\lim_{x\to 0} \dfrac{\cos\sqrt[3]{x} - \sqrt[3]{\cos x}}{\sqrt[3]{x^2}} = \lim_{x\to 0} \dfrac{-\frac{\sqrt[3]{x^2}}{2} + o(\sqrt[3]{x^2})}{\sqrt[3]{x^2}} = -\frac{1}{2}$

f) $f(x) = \frac{1 + \frac{1}{2}x^2}{\cos x} - \sqrt{1+2x^2}$ $\qquad x \to 0$

$f(x) = \frac{1 + \frac{1}{2}x^2}{\left(1 - \frac{1}{2}x^2 + o(x^4) + \frac{x^4}{24}\right)} - \left(1 + \frac{1}{2}\cdot 2x^2 - \frac{1}{8}(2x^2)^2 + o(x^4)\right) =$ $\qquad x \to 0$

$= \left(1 + \frac{1}{2}x^2\right)\left(1 - \left(-\frac{1}{2}x^2 + \frac{x^4}{24}\right)\right) - \left(1 + x^2 - \frac{1}{\cancel{8}2}\cancel{4}x^4\right) =$ $\qquad x \to 0$

$= \left(1 + \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x^2 - \frac{x^4}{24} + \frac{x^4}{4}\right) - 1 - x^2 + \frac{1}{2}x^4 + o(x^4) =$ $\qquad x \to 0$

$= \cancel{1} + \frac{1}{\cancel{2}}x^2 - \frac{x^4}{24} + \frac{1}{\cancel{2}}x^2 + \frac{1}{4}x^4 + \frac{1}{4}x^4 - \cancel{1} - \cancel{x^2} + \frac{1}{2}x^4 + o(x^4) = \frac{23}{24}x^4 + o(x^4)$ $\qquad x \to 0$

$PP = \frac{23}{24}x^4$ PER $x \to 0$

q) $\int \frac{x}{1-x^2}\,dx = \int -\frac{x}{-x^2+1}\,dx = -\frac{1}{2}\int \frac{-2x}{1-x^2} = -\frac{1}{2}\log|1-x^2| + c$

r) $\int \frac{3x+2}{x^2+1}\,dx = \int\left(\frac{3x}{x^2+1} + \frac{2}{x^2+1}\right)dx = \frac{3}{2}\int \frac{2x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx =$

$= \frac{3}{2}\log(x^2+1) + 2\,\mathrm{arctg}\,x + c$

s) $\int \frac{x+1}{x(x^2+1)}\,dx = \int \frac{\cancel{x}}{\cancel{x}(x^2+1)} + \frac{1}{x(x^2+1)}\,dx = \int \frac{1}{x^2+1}\,dx + \int \frac{1}{x(x^2+1)}\,dx =$

$= \mathrm{arctg}\,x + c + \int \frac{1+x^2-x^2}{x(x^2+1)}\,dx = \mathrm{arctg}\,x + \int\left(\frac{\cancel{x^2+1}}{x(\cancel{x^2+1})} + \frac{-x^{\cancel{2}}}{\cancel{x}(x^2+1)}\right)dx =$

$= \mathrm{arctg}\,x + \log|x| - \frac{1}{2}\log(x^2+1) + c = \mathrm{arctg}\,x + \log\frac{|x|}{\sqrt{x^2+1}} + c$

t) $\int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx = \int \frac{\sqrt{1-x}}{\sqrt{1+x}}\cdot\frac{\sqrt{1-x}}{\sqrt{1-x}} = \int \frac{1-x}{\sqrt{(1+x)(1-x)}}\,dx = \int \frac{1-x}{\sqrt{1-x^2}}\,dx =$

$= \int \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\,dx = \mathrm{arcsen}\,x - \int x(\sqrt{1-x^2})^{-1}dx = \mathrm{arcsen}\,x - \int x(1-x^2)$

$= \mathrm{arcsen}\,x + \sqrt{1-x^2} + c$

u) $\int \mathrm{sen}\,x\cos x\,dx = \frac{1}{2}\int 2\,\mathrm{sen}\,x\cos x\,dx = \frac{1}{2}\mathrm{sen}^2 x + c$

v) $\int \mathrm{sen}^2 x\cos x\,dx = \frac{1}{3}\int 3\,\mathrm{sen}^2 x\cos x\,dx = \frac{1}{3}\mathrm{sen}^3 x + c$

w) $\int \mathrm{sen}\,x\cos^3 x\,dx = -\frac{1}{4}\int 4\cos^3 x\,\mathrm{sen}\,x\,dx = -\frac{1}{4}\cos^4 x + c$

z) $\int x(x^2-4)^5\,dx = \frac{1}{2}\int 2x(x^2-4)^5\,dx = \frac{1}{2}\cdot\frac{1}{6}\int 2x\cdot 6(x^2-4)^5\,dx = \frac{1}{12}(x^2-4)^6 + c$

f) $\int \frac{\log x}{x^3}dx = \int \underset{g'}{x^{-3}} \underset{f}{\log x}\, dx = \log x \cdot \frac{1}{-2}x^{-2} - \int \underbrace{\frac{1}{x} \cdot \frac{1}{2x^2}}_{x^{-3}} dx =$

$= -\frac{1}{2x^2}\log x + \frac{1}{4}x^{-2} + c$

g) $\int \underset{f}{\sqrt{1-x^2}} \cdot \underset{g'}{1}\, dx = x\sqrt{1-x^2} + \int \frac{\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} \cdot x\, dx = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}dx =$

$= x\sqrt{1-x^2} + \int \frac{x^2-1+1}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}} =$

$= x\sqrt{1-x^2} + \operatorname{arcsen} x - \int \sqrt{1-x^2}\,dx$

$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \operatorname{arcsen} x - \int \sqrt{1-x^2}\,dx$

$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \operatorname{arcsen} x\right) + c$

h) $\int \underset{g'}{x^2} \underset{f}{\log x}\, dx = \log x \cdot \frac{1}{3}x^3 - \int \frac{1}{\cancel{x}} \cdot \frac{1}{3}x^{\cancel{3}\,2} dx = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + c$

i) $\int \underset{f}{x^2} \underset{g'}{\cos x}\, dx = x^2 \operatorname{sen} x - \int \underset{f}{2x} \cdot \underset{g'}{\operatorname{sen} x}\, dx = x^2 \operatorname{sen} x + 2x\cos x - \int 2 \cdot (\cos x)dx$

$= x^2 \operatorname{sen} x + 2x\cos x - 2\operatorname{sen} x$

l) $\int \underset{g'}{x^2} \underset{f}{\log^3 x}\, dx = \log^3 x \cdot \frac{1}{3}x^3 - \int \left(\frac{3(\log^2 x)}{\cancel{x}}\right) \cdot \frac{1}{\cancel{3}}x^{\cancel{3}\,2}dx = \frac{1}{3}x^3\log^3 x - \int \underset{g'}{x^2} \underset{f}{\log^2 x}$

$= \frac{1}{3}x^3\log^3 x - \log^2 x \cdot \frac{1}{3}x^3 + \int \frac{2\log x}{\cancel{x}} \cdot \frac{1}{3}x^{\cancel{3}\,2}dx =$

$= \frac{1}{3}x^3\log^3 x - \frac{1}{3}x^3\log^2 x + \frac{2}{3}\int \underset{g'}{x^2}\underset{f}{\log x}\, dx =$

$= \frac{1}{3}x^3\log^3 x - \frac{1}{3}x^3\log^2 x + \frac{2}{3}\left(\log x \cdot \frac{1}{3}x^3 - \frac{1}{9}x^3\right) =$

$= \frac{1}{3}x^3\left(\log^3 x - \log^2 x + \frac{2}{3}\log x - \frac{2}{9}\right)$

3
$+c$

e) $\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx = \int \frac{u^2}{\cancel{\sqrt{1-x^2}}} \cdot \cancel{\sqrt{1-x^2}}\,du = \frac{1}{3}u^3 + c \Rightarrow \frac{1}{3}\arcsin^3 x + c$

$u = \arcsin x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

$dx = \sqrt{1-x^2}\,du$

f) $\int e^{\sqrt{x}}\,dx = \int 2ue^u\,du = 2\int \underset{f}{u}\,\underset{g'}{e^u}\,du = 2ue^u - 2\int e^u = (2u-2)e^u + c$

$\Downarrow$

$(2\sqrt{x}-2)e^{\sqrt{x}} + c = 2e^{\sqrt{x}}(\sqrt{x}-1) + c$

$u = \sqrt{x} \qquad x = u^2$

$du = \frac{1}{2\sqrt{x}}\,dx \qquad 2\sqrt{x}\,du = dx$

$2u\,du = dx$

g) $\int \frac{\log(\log x)}{x}\,dx = \int \frac{\log(u)}{\cancel{x}} \cdot \cancel{x}\,du = u\log u - u + c$

$\Downarrow$

$\log x(\log(\log x)) - \log x + c$

$\log x\,(\log(\log x) - 1) + c$

$\log x = u$

$\frac{1}{x}dx = du \rightarrow dx = x\,du$

h) $\int \frac{1}{x\log x}\,dx = \int \frac{1}{x} \cdot \frac{1}{\log x}\,dx = \int \frac{1}{\cancel{x}} \cdot \frac{1}{u} \cdot \cancel{x}\,du = \log|u| + c$

$\Downarrow$

$\log|\log x| + c$

$u = \log x$

$du = \frac{1}{x}\,dx \qquad dx = x\,du$

i) $\int \frac{1}{7x^2+1}\,dx = \int \frac{1}{(\sqrt{7}x)^2+1}\,dx = \int \frac{1}{u^2+1} \cdot \frac{du}{\sqrt{7}} = \frac{1}{\sqrt{7}}\int \frac{1}{u^2+1}\,du = \frac{1}{\sqrt{7}}\operatorname{arctg} u + c$

$\Downarrow$

$\frac{1}{\sqrt{7}}\operatorname{arctg}\sqrt{7}x + c$

$u = \sqrt{7}x^2 \rightarrow$

$du = \sqrt{7}\,dx$

$\frac{du}{\sqrt{7}} = dx$

$$\begin{cases} A = -B-C \\ 3C = 5B + 5C - 4B \\ 6A + 3B + 2C = 1 \end{cases} \quad \begin{cases} A = -B-C \\ -2C = B \\ 6(-B-C) + 3B + 2C = 1 \end{cases} \quad \begin{cases} A = -B-C \\ B = -2C \\ 6(2C-C) + 3(-2C) + 2C = 1 \end{cases}$$

$$\begin{cases} \cancel{6C} - \cancel{6C} + 2C = 1 \\ B = -2C \\ A = -B-C \end{cases} \quad \begin{cases} C = \frac{1}{2} \\ B = -1 \\ A = \frac{1}{2} \end{cases}$$

$$\Rightarrow \int \frac{1}{2}\frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx + \frac{1}{2}\int \frac{1}{x+3}\,dx =$$

$$= \frac{1}{2}\log|x+1| - \log|x+2| + \frac{1}{2}\log|x+3| + c$$

$$= \log\sqrt{|x+1|} + \log\sqrt{|x+3|} - \log|x+2| + c = \log \frac{\sqrt{|x+1||x+3|}}{|x+2|} + c$$

e) $$\int \frac{x}{x^3-1}\,dx = \int \frac{x}{(x-1)(x^2+x+1)}\,dx$$

$$\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\frac{A(x^2+x+1) + (Bx+C)(x-1)}{(x-1)(x^2+x+1)}$$

$$Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$(A+B)x^2 + (A-B+C)x + (A-C)$$

$$\begin{cases} A + B = 0 \\ A - B + C = 1 \\ A - C = 0 \end{cases} \quad \begin{cases} A = -B \\ A + A + A = 1 \\ A = C \end{cases} \quad \begin{cases} B = -\frac{1}{3} \\ A = \frac{1}{3} = C \end{cases}$$

$$\int \frac{1}{3}\frac{1}{x-1}\,dx + \int \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1}\,dx = \frac{1}{3}\log|x-1| - \frac{1}{3}\int \frac{x-1}{x^2+x+1}\,dx =$$

$$= \frac{1}{3}\log|x-1| - \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx =$$

$$= \frac{1}{3}\log|x-1| - \frac{1}{6}\log|x^2+x+1) + \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + c =$$

$$= \log\sqrt[3]{|x-1|} - \log\sqrt[6]{|x^2+x+1|} + \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + c =$$

$$= \log\sqrt[6]{\frac{(x-1)^2}{x^2+x+1}} + \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + c$$

2.5. INTEGRAZIONE DI FUNZIONI IRRAZIONALI

a) $\int \frac{1+\sqrt{x}}{1+x+\sqrt{x}}\,dx$

$t=\sqrt{x} \qquad x=t^2$

$dx=2t\,dt$

$$\int \frac{1+t}{1+t+t^2}\cdot 2t\,dt = 2\int \frac{t+t^2}{1+t+t^2}\,dt = 2\int \frac{t^2+t+1-1}{t^2+t+1}\,dt = 2\int 1-\frac{1}{t^2+t+1}\,dt =$$

$$= 2t - 2\int \frac{1}{t^2+t+\frac{1}{4}+\frac{3}{4}}\,dt = 2t - 2\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,dt = 2t - \ldots \frac{4}{3}\int \frac{1}{\left(\frac{2t+1}{\sqrt{3}}\right)^2+1}\,dt$$

$$= 2t - \frac{4}{3}\sqrt{3}\,\mathrm{arctg}\,\frac{2t+1}{\sqrt{3}} + c \Rightarrow 2\sqrt{x} - \frac{4}{3}\sqrt{3}\,\mathrm{arctg}\,\frac{2\sqrt{x}+1}{\sqrt{3}} + c$$

b) $\int \frac{1+\sqrt{x}}{x(1+\sqrt[3]{x})}\,dx \qquad t=\sqrt[6]{x} \qquad \sqrt{x}=t^3 \qquad \sqrt[3]{x}=t^2$

$dx=6t^5\,dt$

$$\int \frac{(1+t^3)6t^5}{t^6(1+t^2)}\,dt = 6\int \frac{1+t^3}{t(1+t^2)}\,dt = 6\int \frac{t^3+1}{t^3+t}\,dt = 6\int \frac{t^3+t+1-t}{t^3+t}\,dt =$$

1^3+

$$= 6\int 1+\frac{1-t}{t^3+t}\,dt = 6t + 6\int \frac{1-t}{t(t^2+1)}\,dt$$

$$\frac{A}{t}+\frac{Bt+C}{t^2+1} = \frac{At^2+A+Bt^2+Ct}{t(t^2+1)} = (A+B)t^2+Ct+A$$

$$\begin{cases} A+B=0 \\ C=-1 \\ A=1 \end{cases} \qquad \begin{cases} B=-1 \\ C=-1 \\ A=1 \end{cases}$$

$$6t + 6\int \frac{1}{t} - \frac{t+1}{t^2+1}\,dt = 6t + 6\log|t| - 3\int \frac{2t}{t^2+1} + \frac{2}{t^2+1}\,dt =$$

$$= 6t + 6\log|t| - 3\log|t^2+1) + 6\,\mathrm{arctg}\,t + c$$

$$\Downarrow$$

$$6\sqrt[6]{x} + 6\log\sqrt[6]{x} - 3\log|\sqrt[3]{x}+1| + 6\,\mathrm{arctg}\sqrt[6]{x} + c =$$

$$= 6\sqrt[6]{x} + 2\log\frac{(\sqrt[6]{x})^6}{(\sqrt[3]{x}+1)^3} + 6\,\mathrm{arctg}\sqrt[6]{x} + c$$

2.6. INTEGRAZIONE DI FUNZIONI TRIGONOMETRICHE

a) $\int \frac{\operatorname{sen} x \cos x}{\operatorname{sen}^2 x - 3\operatorname{sen} x + 2}\, dx = \int \frac{t \cdot \cancel{\cos x}}{t^2 - 3t + 2} \cdot \frac{dt}{\cancel{\cos x}} =$

$t = \operatorname{sen} x$

$dt = \cos x\, dx$

$= \int \frac{t}{(t-2)(t-1)}\, dt = \int \frac{A}{t-2} + \frac{B}{t-1}\, dt$

$\frac{A(t-1) + B(t-2)}{(t-2)(t-1)} = At - A + Bt - 2B$

$(A+B)t + (-A-2B)$

$\begin{cases} A+B = 1 \\ -A-2B = 0 \end{cases} \quad \begin{cases} A = 1-B \\ B = -1 \end{cases} \rightarrow A = 2$

$= \int \frac{2}{t-2} + \frac{-1}{t-1}\, dt = 2\log|t-2| - \log|t-1| + c = 2\log(\operatorname{sen} x - 2) - \log(\operatorname{sen} x - 1) + c$

b) $\int \frac{\operatorname{sen} x}{\cos^2 x - 1}\, dx = \int \frac{\operatorname{sen} x}{(\cos x - 1)(\cos x + 1)} = \int \frac{\operatorname{sen} x}{\cos x - 1}\, dx + \int \frac{1}{\cos x + 1}\, dx =$

$t = \cos x$

$dt = \operatorname{sen} x\, dx$

$\int \frac{\cancel{\operatorname{sen} x}}{t^2 - 1} \cdot \frac{dt}{\cancel{\operatorname{sen} x}} = \int \frac{1}{(t-1)(t+1)}\, dt = \frac{A}{t-1} + \frac{B}{t+1}$

$A(t+1) + B(t-1)$

$At + A + Bt + B = 0 \quad (A+B)t + A - B = 0$

$\begin{cases} A+B = 0 \\ A-B = 1 \end{cases} \quad \begin{cases} A = -B \\ B = -\frac{1}{2} \end{cases} \quad A = \frac{1}{2}$

$\int \frac{1}{2}\frac{1}{t-1} - \frac{1}{2}\frac{1}{t+1}\, dt = \frac{1}{2}\log|t-1| - \frac{1}{2}\log|t+1| + c$

$\Downarrow$

$\frac{1}{2}\log(\cos x - 1) - \frac{1}{2}\log(\cos x + 1) + c$

c) $\int \frac{1}{\cos x}\, dx = \int \frac{1}{\frac{1-t^2}{\cancel{1+t^2}}} \cdot \frac{2}{\cancel{1+t^2}}\, dt = 2\int \frac{1}{(1-t)(1+t)}\, dt =$

$t = \operatorname{tg}\frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2}{1+t^2}\, dt$

$= 2\int \frac{A}{1-t} + \frac{B}{1+t}$

$A(1+t) + B(1-t)$

$A + At + B - Bt = (A-B)t + A + B$

$\begin{cases} A-B = 0 \\ A+B = 1 \end{cases} \quad \begin{cases} A = B \\ B = \frac{1}{2} \end{cases}$

$2\int \frac{1}{2}\frac{1}{1-t} + \frac{1}{2}\frac{1}{1+t}\, dt = \cancel{2}\frac{1}{\cancel{2}}\int \frac{1}{1-t} + \frac{1}{1+t}\, dt =$

$= \log|1-t| + \log|1+t| + c = \log\left|1 - \operatorname{tg}\frac{x}{2}\right| + \log\left|1 + \operatorname{tg}\frac{x}{2}\right| + c$

PRIMITIVE DI FUNZIONI DEFINITE A TRATTI

a) $f(x) = \begin{cases} xe^x & x \leq 0 \\ \operatorname{sen} x & x > 0 \end{cases}$

$$\int xe^x\,dx = xe^x - \int e^x = xe^x - e^x = e^x(x-1) + c_1$$

$$\int \operatorname{sen} x\,dx = -\cos x + c_2$$

$$F(x) = \begin{cases} e^x(x-1) + c_1 & x \leq 0 \\ -\cos x + c_2 & x > 0 \end{cases}$$

$$\lim_{x \to 0^+} -\cos x + c_2 = \lim_{x \to 0^-} e^x(x-1) + c_1 = F(0)$$

$$-\cancel{1} + c_2 = -\cancel{1} + c_1$$

$$\Downarrow$$

$$F(x) = \begin{cases} e^x(x-1) + c & x \leq 0 \\ c - \cos x & x > 0 \end{cases}$$

$$e^x(x-1) = 0 \rightarrow \begin{matrix} x = 0 \\ x = 1 \end{matrix}$$

$$\lim_{x \to 0^-} e^x(x-1) + c = -1 + c$$

$$\lim_{x \to 0^+} c - \cos x = c - 1$$

$$F(0) = 0$$

$$\Downarrow$$

$$-1 + c = 0 \rightarrow c = 1$$

$$F(x) = \begin{cases} e^x(x-1) + 1 & x \leq 0 \\ 1 - \cos x & x > 0 \end{cases}$$

$$G(0) = 1$$

$$G(x) = \begin{cases} e^x(x-1) & x \leq 0 \\ -\cos x & x > 0 \end{cases}$$

INTEGRALI IMPROPRI

ESERCIZIO 1. DISCUTERE CONVERGENZA POI CALCOLARLA

a) $\int_0^1 \frac{x-1}{(x+2)\sqrt{x}}\,dx$ dom f: -2, 0, $+\infty$

$\int_0^1 \frac{O(1)}{x^{\frac{1}{2}}+o(x^{\frac{1}{2}})} \to \frac{1}{2} < 1 \Rightarrow$ integrale convergente

$\int_0^1 \frac{x-1}{(x+2)\sqrt{x}}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{x-1}{(x+2)\sqrt{x}}\,dx$

$\sqrt{x} = t$

$\frac{dx}{2\sqrt{x}} = dt$ $\quad \lim_{c\to 0^+} \int_c^1 \frac{t^2-1}{(t^2+1)\sqrt{x}} \cdot 2\sqrt{x}\,dt = \lim_{c\to 0^+} 2\int \frac{t^2-1}{t^2+1}$

$= \lim_{c\to 0^+} 2\int \frac{t^2+1}{t^2+1} - \frac{2}{t^2+1}\,dt =$

$= \lim_{c\to 0^+} 2t - 4\operatorname{arctg} t \Big|_c^1 = 2 - 4\operatorname{arctg} 2$

b) $\int_0^{+\infty} (x^2-x)e^{-x}\,dx = \int_0^{+\infty} \frac{x^2-x}{e^x}\,dx$ $\quad \mathbb{R} \to +\infty$

$\frac{x^2-x}{e^x} = o\left(\frac{1}{x^2}\right)$ $\quad 2>1 \Rightarrow$ convergente

$\int_0^{+\infty} \frac{x^2-x}{e^x} = \lim_{c\to+\infty} \int_0^c \left[\frac{x^2-x}{e^x}\right] = \lim_{c\to+\infty} \int_0^c \underset{f}{(x^2-x)}\,\underset{g'}{e^{-x}} =$

$= \lim_{c\to+\infty} \left[-(x^2-x)e^{-x}\right]_0^c + \int_0^{+\infty} e^{-x}(2x-1)\,dx =$

$= \lim_{c\to+\infty} \underset{0}{(c-c^2)e^{-c}} + \int_0^{+\infty} \underset{g'}{e^{-x}}\,\underset{f}{(2x-1)}\,dx =$

$= \lim_{c\to+\infty} \left[e^{-x}(1-2x)\right]_0^c + 2\int_0^{+\infty} e^{-x} =$

$= \lim_{c\to+\infty} e^{-c}(1-2c) - 1 + 2e^{-c} + 2 =$

$= -1+2 = 1$

g) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt[3]{x}\log x}$ $x \neq 0$ $x > 0$
$x \neq 1$

$x \to 0^+$ $\frac{1}{\sqrt[3]{x}\log x} = o\left(\frac{1}{\sqrt[3]{x}}\right)$ $\frac{1}{3} < 1 \Rightarrow$ converge

h) $\int_0^1 \frac{1}{\log(1+x)}\,dx$ dom f $x \neq -1$
$x \neq 0$

$\int_0^1 \frac{1}{\log(1+x)}\,dx$ $x \to 0^+$ $\frac{1}{x} = o\left(\frac{1}{\log(1+x)}\right)$ $\to$ $1 = 1$ diverge

i) $\int_1^{+\infty} \frac{1}{x\log^2 x}\,dx$ f continua $(1; +\infty)$

$= \int_1^2 \frac{1}{x\log^2 x}\,dx + \int_2^{+\infty} \frac{1}{x\log^2 x}\,dx$ $t = \log x \to \int_1^{+\infty} \frac{1}{t^2}\,dx$ $\frac{1}{t^{\alpha}}$ >1

$x \to +\infty$ $\frac{1}{x^2} = o\left(\frac{1}{x\log^2 x}\right)$ $2 > 1 \Rightarrow$ converge

j) $\int_0^1 \log x\,dx$ continua in $(0, 1]$

$x \to 0^+$ $\log x = o\left(\frac{1}{x^{\alpha}}\right)$ $\forall \alpha > 0$ diverge

per $\alpha < 1$ converge

ESERCIZIO 2 FORMA < ALGEBRICA / ESPONENZIALE

a) $z = 4\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right)$

$z = 4e^{i\frac{\pi}{4}}$

$x = |z|\cos\theta = 4 \cdot \cos\frac{\pi}{4} = \cancel{4}^{2} \cdot \frac{\sqrt{2}}{\cancel{2}} = 2\sqrt{2}$

$y = |z|\operatorname{sen}\theta = \cancel{4}^{2} \cdot \frac{\sqrt{2}}{\cancel{2}} = 2\sqrt{2}$

$z = 2\sqrt{2} + i\,2\sqrt{2} = 2\sqrt{2}(1+i)$

b) $z = 6\left(\cos\frac{\pi}{6} + i \operatorname{sen}\frac{\pi}{6}\right)$

$z = 6e^{i\frac{\pi}{6}}$

$x = |z|\cos\theta = 6\cos\frac{\pi}{6} = \cancel{6}^{3} \cdot \frac{\sqrt{3}}{\cancel{2}}$

$y = |z|\operatorname{sen}\theta = 6\operatorname{sen}\frac{\pi}{6} = \cancel{6}^{3} \cdot \frac{1}{\cancel{2}}$

$z = 3\sqrt{3} + i3 = 3(\sqrt{3} + i)$

c) $z = 2\left(\cos\frac{\pi}{3} + i \operatorname{sen}\frac{\pi}{3}\right)$

$z = 2e^{i\frac{\pi}{3}}$

$x = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$

$y = 2\operatorname{sen}\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

$z = 1 + i\sqrt{3}$

d) $z = 7i \operatorname{sen}\frac{\pi}{2}$

$z = 7e^{i\frac{\pi}{2}}$

$x = 7\cos\frac{\pi}{2} = 7 \cdot 0 = 0$

$y = 7\operatorname{sen}\frac{\pi}{2} = 7 \cdot 1 = 7$

$z = 7i$

e) $z = 2\cos\pi$

$z = 2e^{i\pi}$

$x = 2\cos\pi = 2\cdot(-1) = -2$

$y = 0$

$z = -2$

f) $z = 8\left(\cos\frac{3}{4}\pi + i \operatorname{sen}\frac{3}{4}\pi\right)$ $z = 8e^{i\frac{3}{4}\pi}$

$x = 8\cos\frac{3}{4}\pi = 8 \cdot \left(-\frac{\sqrt{2}}{2}\right)$ $y = 8\operatorname{sen}\frac{3}{4}\pi = 8\,\frac{\sqrt{2}}{2}$

$z = \frac{8\sqrt{2}}{2}(-1+i)$

ESERCIZIO 4 RISOLVERE EQUAZIONI

a) $z^2 - 6z + 5 = 0$

$(z-5)(z-1) = 0 \qquad z = +1 \ \vee \ z = 5$

b) $z^2 + 7iz + 4 = 0$

$z_{1/2} = \dfrac{-7i \pm \sqrt{49i^2 - 16}}{2} = \dfrac{-7i \pm \sqrt{-49-16}}{2} = \dfrac{-7i \pm \sqrt{-67}}{2} =$

$= \dfrac{-7i \pm i\sqrt{67}}{2} = \dfrac{(-7 \pm \sqrt{67})i}{2}$

c) $z^3(z-i) - 1 - 2z^2 - z^4 = 0$

$z^3(z-i) - (z^2+1)^2 = 0$

$z^3(z-i) - (z+i)^2(z-i)^2 = 0$

$(z-i)[z^3 - (z-i)(z+i)^2] = 0 \qquad (z-i)[z^3 - (z-i)(z^2-2iz+i^2$

$(z-i)[\cancel{z^3} - \cancel{z^3} - 2iz^2 + i^2 z \ \overset{+iz^2}{} + 2i^2 z - i^3] = 0$

$(z-i)[-2iz^2 - z - i] = 0$

$-(z-i)(iz^2 + z + i) = 0$

$z = i$

$z_{2,3} = \dfrac{-1 \pm \sqrt{1-4(-1)}}{2i} = -\dfrac{1 \pm \sqrt{5}}{2}\, i$

d) $iz^6 - 3z^3 + 4i = 0$

$z^3 = \omega$

$i\omega^2 - 3\omega + 4i = 0$

$\omega_{1,2} = \dfrac{+3 \pm \sqrt{9 - 4(4)(-1)}}{2i} = -\dfrac{3 \pm \sqrt{25}}{2}\, i = -\dfrac{3 \pm 5}{2}\, i \quad \begin{cases} \frac{8}{2} = 4i \\ -1 \end{cases}$

$z = \sqrt[3]{\omega} \qquad z_1 = \sqrt[3]{-4i}\, e^{i0} = \sqrt[3]{-4i} = -\sqrt[3]{4i} \qquad k = 0,1,2$

$z_{1,2,3} = \sqrt[3]{4i} \quad z_1 = \sqrt[3]{-4i}\, e^{i\frac{\pi}{6}} = -\sqrt[3]{4i}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$z_2 = \sqrt[3]{-4i}\, e^{-i\frac{5}{6}\pi} = -\sqrt[3]{4i}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$z_{4,5,6}$:

$z_4 = -i$

$z_5 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$z_6 = +\frac{\sqrt{3}}{2} - i\frac{1}{2}$

c) $z = \rho e^{i\vartheta}$ $\quad \rho \geq 0$ e $\vartheta \in \mathbb{R}$ risulta $Re(z) - (Re z)^2 \geq 0$

$z = \rho(\cancel{sen}\,\vartheta + i\,sen\,\vartheta)$

$\rho^2 \cos\vartheta - \rho^2 \cos^2\vartheta \geq 0$

$\rho^2(\cos\vartheta - \cos^2\vartheta) \geq 0$

$\rho^2 \cos\vartheta\,(1-\cos\vartheta) \geq 0$

$\rho = 0$

$\cos\vartheta \leq 1 \rightarrow \vartheta = 0 + k\pi$

e) $\dfrac{z^2 - |\bar{z}|^2}{z - \bar{z}} = 1$ $\quad$ pongo $z = a + ib$

$$\frac{(a+ib)^2 - \left(\sqrt{a^2+b^2}\right)^2}{\cancel{a} + ib - \cancel{a} + ib} = 1$$

$$\frac{\cancel{a^2} + 2aib - b^2 - \cancel{a^2} - b^2}{2ib} = 1$$

$$\frac{\cancel{b}(2ai - 2b)}{2i\cancel{b}} = 1$$

$$\frac{\cancel{2}(ai - b)}{\cancel{2}i} = 1$$

$a + ib = 1$ $\quad z = 1$ $\quad$ MA $\quad z = \bar{z} \Rightarrow$ NON È POSSIBILE

$\nexists z$

f) per quali z $\quad (Re z + Im z)^2 = (Re \bar{z} + Im \bar{z})^2$

Pongo $z = a + ib$

$\bar{z} = a - ib$

$(a + b)^2 = (a - b)^2$

$\cancel{a^2} + 2ab + \cancel{b^2} = \cancel{a^2} - 2ab + \cancel{b^2}$

$4ab = 0 \rightarrow a = 0$

$b = 0$

c) $y' = \frac{2xy}{x^2-1}$ $\qquad y=0$ integrale particolare

$\int \frac{1}{y}\,dy = \int \frac{2x}{x^2-1}\,dx$

$\cancel{\log}|y| = \cancel{\log}|x^2-1| + c$

$y = c\,(x^2-1) \qquad c \in \mathbb{R}$

d) $(x-3)^2 y' = x\,(y-1)$ $\qquad y=1$ integrale particolare

$\frac{y'}{y-1} = \frac{x}{(x-3)^2}$

$\int \frac{1}{y-1}\,dy = \int \frac{x}{(x-3)^2}\,dx$

$\log|y-1| = \int \frac{x}{x^2-6x+9}\,dx$

$\int \frac{x}{x^2-6x+9}\,dx = \int \frac{x-3+3}{(x-3)^2}\,dx = \int \frac{1}{(x-3)} + \frac{3}{(x-3)^2}\,dx = \log|x-3| + 3\int \frac{1}{(x-3)^2}$

$= \log|x-3| - \frac{3}{(x-3)} + c$

$\log|y-1| = \log|x-3| - \frac{3}{x-3} + c$

$|y-1| = |x-3| \cdot c\,e^{-\frac{3}{x-3}}$

$y-1 = c(x-3)e^{-\frac{3}{x-3}}$

$y = c\,(x-3)e^{-\frac{3}{x-3}} + 1$

e) $y' = y\,(1+y)$ $\qquad y=0$ e $y=-1$ sol particolari

$\int \frac{1}{y(1+y)}\,dy = \int dx$

$\int \frac{A}{y} + \frac{B}{1+y}\,dy \qquad A(1+y)+By = A + Ay + By$

$= (A+B)y + A$

$\begin{cases} A+B=0 \\ A=1 \end{cases} \rightarrow B = -A = -1$

$\int \frac{1}{y} - \frac{1}{1+y}\,dy =$

$= \log|y| - \log|1+y| = \log\left|\frac{y}{1+y}\right|$

$$y = cx\left(\sqrt{1+y^2}\right)$$

$$y = \sqrt{c^2x^2 + c^2x^2y^2}$$

$$y^2 = c^2x^2 + c^2x^2y^2$$

$$y^2 - c^2x^2y^2 = c^2x^2$$

$$y^2(1 - c^2x^2) = c^2x^2$$

$$y^2 = \frac{c^2x^2}{1-c^2x^2}$$

$$y = \sqrt{\frac{c^2x^2}{1-c^2x^2}} = \frac{cx}{\sqrt{1-c^2x^2}}$$

k) $y' \, tg x = y$

$$\int \frac{dy}{y} = \int \frac{1}{tg x} dx$$

$$\log|y| = \int \frac{1}{tg x} dx$$

$$t = tg x$$

$$dt = 1+t^2 dx$$

$$\frac{dt}{1+t^2} = dx$$

$$\int \frac{1}{tg x} dx = \int \frac{1}{t(1+t^2)} dt =$$

$$= \log\left|\frac{t}{(1+t^2)^{\frac{1}{2}}}\right|$$

$$\log|y| = \log\left|\frac{tg x}{\sqrt{1+tg^2 x}}\right| + c$$

$$y =$$

$$\log|y| = \int \frac{\cos x}{\operatorname{sen} x} dx$$

$$\log|y| = \log|\operatorname{sen} x| + c$$

$$y = c \cdot (\operatorname{sen} x)$$

n) $y' = (1+y^2)(1+t^2)$

$$\int \frac{dy}{1+y^2} = \int 1+t^2 \, dt$$

$$\operatorname{arctg} y = t + \frac{t^3}{3} + c$$

$$y = \operatorname{tg}\left(t + \frac{t^3}{3} + c\right)$$

o) $y' = (1+e^y)\,t$

$$\int \frac{dy}{1+e^y} = \int t \, dt$$

$e^y = u$

$y = \log u$

$dy = \frac{1}{u} du$

$$\int \frac{1}{u(1+u)} du = \int t \, dt$$

$$\int \frac{1+u-u}{u(1+u)} du = \int \frac{\cancel{(1+u)}}{u\cancel{(1+u)}} - \frac{\cancel{u}}{\cancel{u}(1+u)} du$$

$$= \log|u| - \log|1+u| = \log\left|\frac{u}{1+u}\right| = \log\left|\frac{e^y}{1+e^y}\right|$$

$$\log\left|\frac{e^y}{1+e^y}\right| = \frac{t^2}{2} + c$$

$$\frac{e^y}{1+e^y} = ce^{\frac{t^2}{2}+c}$$

$$e^y = (1+e^y)\left(c \cdot e^{\frac{t^2}{2}}\right)$$

$$e^y = ce^{\frac{t^2}{2}} + ce^{\frac{t^2}{2}}e^y$$

$$e^y - ce^{\frac{t^2}{2}}e^y = ce^{\frac{t^2}{2}} \Rightarrow e^y\left(1 - ce^{\frac{t^2}{2}}\right) = ce^{\frac{t^2}{2}}$$

$$e^y = \frac{ce^{\frac{t^2}{2}}}{1-ce^{\frac{t^2}{2}}} \Rightarrow y = \log\left(\frac{ce^{\frac{t^2}{2}}}{1-ce^{\frac{t^2}{2}}}\right)$$

p) $y' = e^t y \log y$ $\quad y = 0$ integrale particolare

$$\int \frac{1}{y \log y} dy = \int e^t dt$$

$u = \log y$

$du = \frac{1}{y} dy$

$y\,du = dy$

$$\int \frac{1}{y \cdot u} \cdot y \, du = \int e^t dt$$

$$\log|u| = e^t + c$$

$$\log(\log y) = e^t + c \qquad \log|y| = ce^{e^t}$$

ce^{e^t}

s) $x' = \dfrac{(t-1)}{\log x}\, x$ $\qquad x=0$ ~~sol~~ integrale particolare

$$\int \frac{\log x\, dx}{x} = \int t-1\, dt$$

$u = \log x$

$du = \frac{1}{x}dx$

$$\int \frac{u}{\cancel{x}}\, \cancel{x}\, du = \int (t-1)\, dt$$

$$\frac{u^2}{2} = \frac{t^2}{2} - t + c$$

$$\frac{\log^2 x}{2} = \frac{t^2}{2} - t + c \rightarrow \log^2 x = t^2 - 2t + c$$

$$\log x = \pm\sqrt{t^2-2t+c}$$

$$x = e^{\pm\sqrt{t^2-2t+c}}$$

t) $x' = 2t\sqrt{1-x^2}$ $\qquad$ ~~sol~~ integrale particolare $x = \pm 1$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int 2t\, dt \qquad \arcsin x = t^2 + c \rightarrow x = \sin(t^2+c)$$

u) $x' = -2tx^2$ $\qquad x = 0$ int particolare

$$\int \frac{dx}{x^2} = \int -2t\, dt$$

$$-\frac{1}{x} = -t^2 + c \rightarrow x = +\frac{1}{t^2+c}$$

v) $x' = \dfrac{3x-2}{t^2+1}$ $\qquad x = \frac{2}{3}$ int part

$$\int \frac{dx}{3x-2} = \int \frac{1}{t^2+1}\, dt$$

$$\int \frac{1}{3\left(x-\frac{2}{3}\right)} = \operatorname{arctg} t + c$$

$$\frac{1}{3}\log\left|x-\frac{2}{3}\right| = \operatorname{arctg} t + c$$

$$\log\left|x-\frac{2}{3}\right| = 3\operatorname{arctg} t + c$$

$$x - \frac{2}{3} = e^{3\operatorname{arctg} t + c}$$

$$x = e^{3\operatorname{arctg} t + c} + \frac{2}{3}$$

2.2. EQUAZIONI LINEARI DEL 1° ORDINE

a) $y' = xy + x^3$ $\qquad a(x) = x \quad b(x) = x^3$

$A(x) = \frac{x^2}{2}$

$$y = e^{\frac{x^2}{2}}\left(\int e^{-\frac{x^2}{2}} x^3\,dx + c\right)$$

$$y = e^{\frac{x^2}{2}}\left(-x^2 e^{-\frac{x^2}{2}} + 2\int x e^{-\frac{x^2}{2}}\,dx\right) =$$

$$= e^{\frac{x^2}{2}}\left(-x^2 e^{-\frac{x^2}{2}} - 2e^{-\frac{x^2}{2}} + c\right) = c e^{\frac{x^2}{2}} - x^2 - 2$$

b) $y' = -xy + x$ $\qquad a(x) = -x \quad b(x) = x$

$A(x) = -\frac{x^2}{2}$

$$y = e^{-\frac{x^2}{2}}\left(\int e^{\frac{x^2}{2}} x\,dx + c\right)$$

$$y = e^{-\frac{x^2}{2}}\left(e^{\frac{x^2}{2}} + c\right)$$

$$y = c \cdot e^{-\frac{x^2}{2}} + 1$$

2.3. EQUAZIONI LINEARI DEL 2° ORDINE A COEFFICIENTI COSTANTI

b) $y'' - 4y = 0$ $\qquad c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = y(x)$

$\lambda^2 - 4 = 0 \qquad \lambda = \pm 2$

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$

a) $y'' + y = 0$

$\lambda^2 + 1 = 0 \qquad \Delta = -4$

$$\lambda_{1,2} = \frac{\pm\sqrt{-4}}{2} = \frac{\pm 2i}{2} = \pm i \qquad \pm i$$

$$\lambda_1 = +i \qquad \lambda_2 = -i$$

$$y(x) = (c_1 \sin x + c_2 \cos x)$$