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Mathematical Analysis I

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Mathematical Analysis I (2013-2014)

Basic Notions 1 - Sets and some logic

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Sets

A **set** is defined writing a list of its members (called **elements**), all different one from another; the elements are said to **belong to the set**. Examples: $A = \{1, 2, 3\}$, $B = \{a, b, c, d, e\}$. We use capital letters for sets, lowercase letters for elements.

If x is an element of the set E, we write $x \in E$; if x is not an element of the set E, we write $x \notin E$

A set with no elements is called **empty set**; the notation is $\emptyset$.

Subsets

Definition

Given a set B, the set A is a **subset** of B (we say also that A is included in B) when all the elements of A belong also to B. We write $A \subseteq B$.

The empty set $\emptyset$ is assumed to be a subset of any set A.

If there exists at least one element of B that does not belong to A, we say that A is a **proper subset** of B (the notation is $A \subset B$).

If $A \subseteq B$ and $B \subseteq A$ (i.e. all the elements of A belong also to B and all elements of B belong also to A), then we say that the two sets are **equal** and we write $A = B$.

The power set

Definition

The set of all subsets of A is called the **power set** of A. The symbol is $\mathcal{P}(A)$.

Example. If $A = \{a, b, c\}$ the power set of A is:

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.

Operating with sets - 2

Some remarks.

- $\mathcal{C}_X A = X \setminus A$
- $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$
- for other properties: see textbook (page 3)
- the **De Morgan laws** show the relation between complement, union and intersection:

$$\mathcal{C}(A \cup B) = \mathcal{C}A \cap (\mathcal{C}B)$$

$$\mathcal{C}(A \cap B) = \mathcal{C}A \cup (\mathcal{C}B)$$

Propositions and predicates

In Mathematics, we use only statements that can be true or false: for instance "$3 < 4$" is a (true) mathematical statement, while "$3 > 4$ is a (wrong) mathematical statement.

Frequently a mathematical statement depends upon one or more variables; in this case it is called a **predicate**. A predicate is neither true nor false, until we fix the value of the variable(s).

As an example, "x is a prime number"' is a predicate containing the variable x; setting $x = 19$ we get a true statement, setting $x = 10$ a false one.

For a predicate with one variable (called also **property**) we use the notation $p(x)$.

Predicates with two or more variables

A predicate with two or more variables is called also a **relation**. We consider here predicates with two variables: $p(x, y)$

Using two quantifiers we have eight possible cases:

$$\begin{array}{ll}
\exists x,\ \exists y\ : p(x,y) & \exists y,\ \exists x\ : p(x,y) \\
\exists x,\ \forall y\ : p(x,y) & \forall y,\ \exists x\ : p(x,y) \\
\forall x,\ \exists y\ : p(x,y) & \exists y,\ \exists x\ : p(x,y) \\
\forall x,\ \forall y\ : p(x,y) & \forall y,\ \forall x\ : p(x,y)
\end{array}$$

and eight statements with a completely different meaning (try to understand their meanings, using the relation "$p(x, y)$ = the person x can do the job y").

The negation of a multiply quantified predicate is obtained by changing the quantifiers and by negating the relation. For instance:

$$\neg(\forall x,\ \exists y\ : p(x,y)) \Longleftrightarrow \exists x,\ \forall y\ : \neg(p(x,y))$$

QUANTIFIERS

- The UNIVERSAL quantifier: $\forall$ "for all"
  $\forall x: p(x)$ = the property $p(x)$ is true for all x
- The EXISTENTIAL quantifier: $\exists$ "there exists a"
  $\exists x: p(x)$ = the property $p(x)$ is true for at least one x.

NEGATION OF A QUANTIFIED PREDICATE

$\forall x: p(x)$ negation → $\exists x \neg(p(x)) \Leftrightarrow \neg(\forall x: p(x))$

$\exists x: p(x)$ negation → $\forall x \neg(p(x)) \Leftrightarrow \neg(\exists x: p(x))$

RELATIONS = predicates with two or more variables
$p(x,y)$ = "the person x can do the job y"

- $\exists x, \forall y$ = "there exists at least one person who can do all the jobs"

  $\neg(\exists x, \forall y: p(x,y)) \Leftrightarrow \forall x \, \exists y: \neg(p(x,y))$

- $\forall y, \exists x: p(x,y)$ = "for all the jobs, there exists at least one person able to do them all" = "there's not any impossible job"

Algebraic operations in $\mathbb{Z}$

In the set $\mathbb{N}$ it is not possible to solve (with the exception of $x = 0$) the following problem: given $x \in \mathbb{N}$ find a number $y \in \mathbb{N}$ such that $x + y = 0$.

It is possible to solve this problem if we consider a larger set, the set $\mathbb{Z}$ of integer numbers. In this set we have

$$\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z} : x + y = 0$$

The number y is the **inverse** di x and it is written $-x$.

We can define the inverse operation of the sum, the **difference**, by setting

$$x - y = x + (-y).$$

Algebraic operations in $\mathbb{Q}$

In the set $\mathbb{Z}$ it is not possible to solve (with the exception of $x = 1$) the following problem: given $x \in \mathbb{Z} \setminus \{0\}$ find a number $y \in \mathbb{Z}$ such that $xy = 1$.

Again, it is possible to solve this problem if we consider a larger set, the set $\mathbb{Q}$ of rational numbers. In this set we have

$$\forall x \in \mathbb{Q} \setminus \{0\},\ \exists y \in \mathbb{Q} : xy = 1$$

The number y is the **inverse** of x and it is written x^{-1} (if $x = p/q$ with $p \neq 0$, then $x^{-1} = q/p$).

We can define the inverse operation of the product, the **quotient**, by setting

$$x/y = x \cdot y^{-1}.$$

Rational and non-rational numbers

The real numbers that are non rational (i.e. the elements of the set $\mathbb{R} \setminus \mathbb{Q}$) are called **irrational numbers.**

Remarks.

- A rational number has a finite decimal expansion (for instance, $1/25 = 0,04$) when the denominator contains only 2 and/or 5 as prime factors.
- In the other cases, it has an infinite periodic expansion (some examples are $1/3 = 0,\bar{3}$, $1/6 = 0,1\bar{6}$).
- A non-rational number has an infinite non-periodic decimal expansion (an example is the non rational number π; the first digits of its expansion are 3,1415926535).

Rational and non-rational numbers - 2

- Given two rational numbers q_1 and q_2 there are infinitely many rational numbers (and infinitely many irrational numbers) between them. The same holds if we consider two irrational numbers r_1 and r_2: there are infinitely many irrational numbers (and infinitely many rational numbers) between them.
- We can approximate an irrational number as well as we please with rational numbers (and viceversa).
- It is impossible to find a non empty interval of $\mathbb{R}$ containing only rational (or only irrational) numbers.

Unbounded intervals

The unbounded intervals are:

$(-\infty, a]$	the set of $x \in \mathbb{R}$ such that $x \leq a$,
$(-\infty, a)$	the set of $x \in \mathbb{R}$ such that $x < a$,
$[a, +\infty)$	the set of $x \in \mathbb{R}$ such that $x \geq a$,
$(a, +\infty)$	the set of $x \in \mathbb{R}$ such that $x > a$,
$\mathbb{R}$	the real line

$(-\infty, a]$	unbounded closed interval,
$(-\infty, a)$	unbounded open interval,
$[a, +\infty)$	unbounded closed interval,
$(a, +\infty)$	unbounded open interval.

General property of intervals

The word **interval** (without any specification) indicates a subset of $\mathbb{R}$ of one of the nine types introduced. All intervals (and only intervals) satisfy this property:

Property

A subset I of $\mathbb{R}$ in an interval if and only if

$$\forall x \in I, \forall y \in I, \forall z \in \mathbb{R} : \quad x < z < y \Longrightarrow z \in I .$$

The absolute value - 2

If $x_0 \in \mathbb{R}$, the quantity $|x - x_0|$ measures the **distance between the points** x **and** x_0.

The following property holds.

$$\forall x \in \mathbb{R},\ |x| \geq 0, \quad \text{and } |x| = 0 \Longleftrightarrow x = 0$$

If we fix $a > 0$, we can define the following sets:

- $\{x \in \mathbb{R} :\ |x - x_0| = a\} = \{x_0 - a, x_0 + a\}$
- $\{x \in \mathbb{R} :\ |x - x_0| < a\} = (x_0 - a, x_0 + a)$
- $\{x \in \mathbb{R} :\ |x - x_0| \leq a\} = [x_0 - a, x_0 + a]$
- $\{x \in \mathbb{R} :\ |x - x_0| > a\} = (-\infty, x_0 - a) \cup (x_0 + a, +\infty)$
- $\{x \in \mathbb{R} :\ |x - x_0| \geq a\} = (-\infty, x_0 - a] \cup [x_0 + a, +\infty)$

But rational numbers don't complete the line → some points don't correspond to rational number

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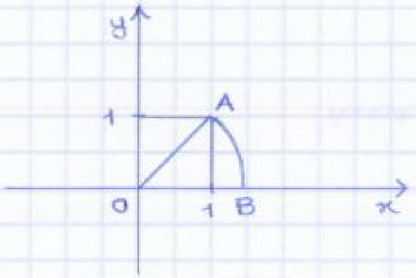


$\overline{OB}$ is not a rational number

$\overline{OA}^2 = \overline{OB}^2 = 1^2+1^2 = 2$

$x^2 = 2 \not\Rightarrow p/q \not\Rightarrow \frac{p^2}{q^2} = 2$

$\mathbb{R}$ = real numbers

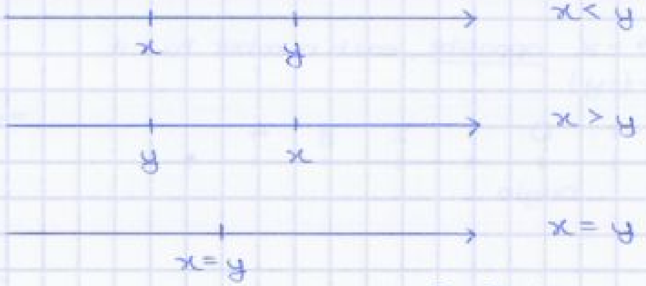
$$\mathbb{R} \begin{cases} \mathbb{Q} = \text{rational numbers} \\ \mathbb{R} \setminus \mathbb{Q} = \text{irrational numbers} \end{cases}$$

$x = \sqrt{2}$ $\quad \pi, e, \sqrt{3}, \sqrt{5}$

fixed O, unit
- $\forall x \in \mathbb{R}$ $\exists$ one and only one corresponding point on the line
- $\forall$ point on the line $\exists x \in \mathbb{R}$

$\mathbb{R}$ is COMPLETE → operations $+, \cdot, -, \%$

In the set it's possible to define an ORDER RELATION → $\le$ $\quad x, y \in \mathbb{R}$

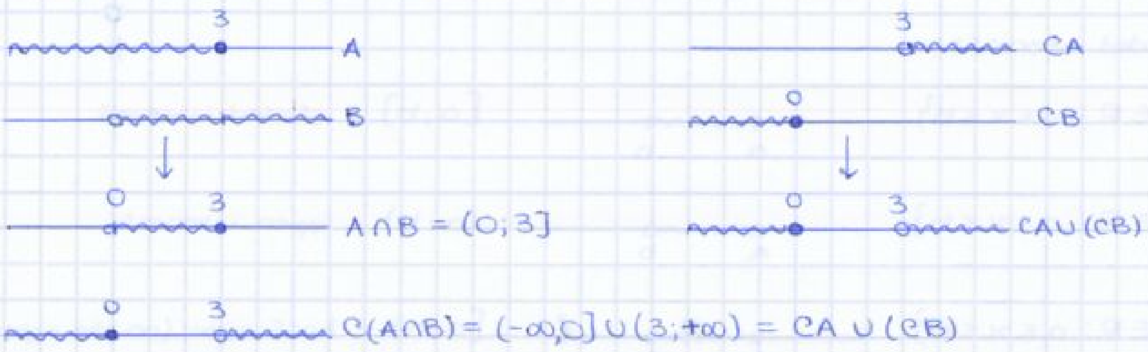


$$x \le y \begin{cases} x < y \\ \text{or} \\ x = y \end{cases} \qquad 3 \le 5 \begin{cases} 3 < 5 \ ✓ \\ \text{or} \\ 3 = 5 \ ✗ \end{cases} \qquad 3 \le 3 \begin{cases} 3 < 3 \ ✗ \\ \text{or} \\ 3 = 3 \ ✓ \end{cases}$$

- SUM
  $\forall x, y, z \in \mathbb{R}$
  $x \le y \Rightarrow x + z \le z + y$
- PRODUCT
  $\forall x, y, z \in \mathbb{R}, z > 0$
  $x \le y \Rightarrow x \cdot z \le y \cdot z$

De Morgan laws: $A = (-\infty; 3]$ $B = (0; +\infty)$

① $C(A \cap B) = CA \cup (CB)$

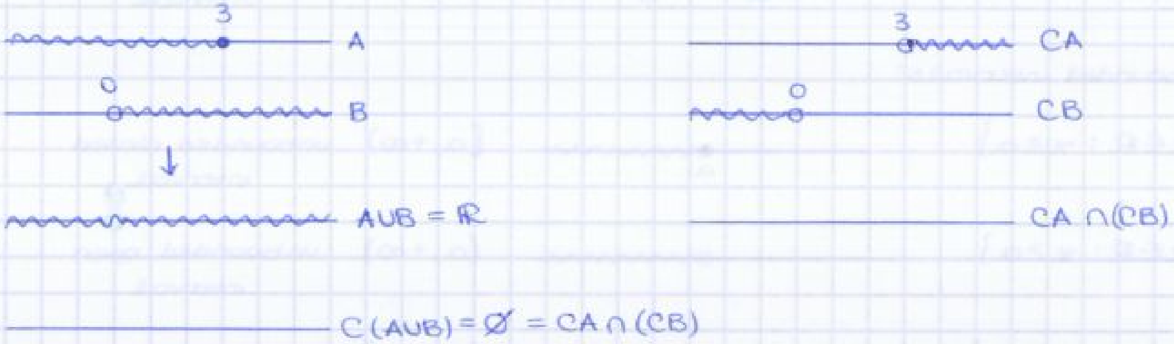


$A \cap B = (0; 3]$

$CA \cup (CB)$

$C(A \cap B) = (-\infty, 0] \cup (3; +\infty) = CA \cup (CB)$

② $C(A \cup B) = CA \cap (CB)$



$A \cup B = \mathbb{R}$

$CA \cap (CB)$

$C(A \cup B) = \emptyset = CA \cap (CB)$

Cartesian product - 2

- We know that the set $\mathbb{R}$, thanks to completeness, is a mathematical model of the line.
- The Cartesian product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is a mathematical model of the plane (every element of $\mathbb{R}^2$ corresponds to one and only one point of the plane and viceversa); x e y are the **Cartesian coordinates** of the point.
- The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ is a mathematical model of 3-dimensional space (every ordered triple of $\mathbb{R}^3$ corresponds to one and only one point of the space and viceversa).

Relations

- A property $p(x)$ (see the first lesson) defines a subset (possibly empty) of the real line: for instance,
    - the property $x^2 - 3x = 0$ (called **equation**) defines the set $\{0, 3\}$
    - the property $x(1 - x) > 0$ (called **inequality**) defines the interval $(0, 1)$
    - the property $x^2 + 1 \leq 0$ is never satisfied; it defines the empty set
- A relation $p(x, y)$ (defined in first lesson) relating the coordinates of a point of the plane defines a subset of $\mathbb{R}^2$. The corresponding points are the elements of the **graph of the relation**.
    - The relation $x^2 + y^2 - 1 = 0$ defines the **unit circle** in the plane
    - The relation $y > 2x$ defines the half-plane above the line $y = 2x$ (the line is excluded)
    - The relation $x = 5$ defines a vertical line
    - The relation $3x^2 + 5y^2 < 0$ is never satisfied; its graph is empty.

Some notations

Let X, Y be two sets and $f : \mathrm{dom} f \subseteq X \to Y$ be a function.

- If $X = \mathbb{R}$, f is a function of **one real variable**
- If $Y = \mathbb{R}$, f is a **real function** or **real-valued function**
- If $f : \mathrm{dom} f \subseteq \mathbb{R} \to \mathbb{R}$, the graph $\Gamma(f) \subseteq \mathbb{R}^2$.

Remark. The graph of a real-valued function of a real variable is always a subset of the plane. It is not true, in general, that a subset of the plane is the graph of a function.
This happens when the intersection of A with a vertical line contains to most one point.

Remark. If the graph of a function is known, we can find its domain with a **projection on the** x **axis** and its range with a **projection on the** y **axis**.

Some elementary functions

We describe here some basic functions.

Constant valued functions	$y = c, \quad c \in \mathbb{R}$
Powers	$y = x^n, \quad n \in \mathbb{N} \setminus \{0\}$
Exponential function	$y = a^x, \quad a \in \mathbb{R}, a > 0$
Trigonometric functions	$y = \sin x, y = \cos x$

Some other basic functions are obtained as **inverse functions** of those listed above:

Root functions	$y = \sqrt[n]{x}, \quad n \in \mathbb{N} \setminus \{0\}$
Logarithmic function	$y = \log_a x, \quad a \in \mathbb{R}, a > 0, a \neq 1$
Inverse trig. functions	$y = \arcsin x, y = \arccos x, \dots$

An **elementary function**, roughly speaking, is a function obtained from the functions shown above, using the following operations:

1. the algebraic operations
2. the **composition of functions**.

Examples

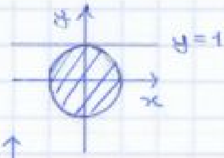
We consider the **linear function** or, better, **affine function**

$$f(x) = ax + b\,, \qquad a, b \in \mathbb{R}$$

- $a \neq 0$
  - f is surjective on $\mathbb{R}$, since $\operatorname{im} f = \mathbb{R}$
  - f is injective since se $x_1 \neq x_2$ then $f(x_1) = ax_1 + b \neq ax_2 + b = f(x_2)$

  then f is bijective between $\mathbb{R}$ and $\mathbb{R}$.
- $a = 0$
  - f is not bijective since $\operatorname{im} f = \{b\}$
  - f is not injective since when $x_1 \neq x_2$ we have that $f(x_1) = b = f(x_2)$

$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$



$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$

$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 0\} = \emptyset$

A RELATION $p(x,y)$ relating the coordinates of a point of the plane defines a subset of $\mathbb{R}^2$. The corresponding points are the elements of the graph of the relation.

FUNCTIONS

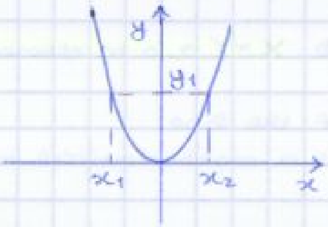
$x \rightarrow \boxed{f(x)} \rightarrow y$

$\overset{\text{dependent}}{y} = f(x)$

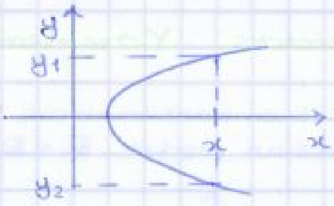
independent variable $x \in X$

For each $x \in X$, I have at most one $y \in Y$ (property of univocity)

$x \to x^2$
$1 \to 1$
$2 \to 4$
$3 \to 9$
$-5 \to 25$



This is a function



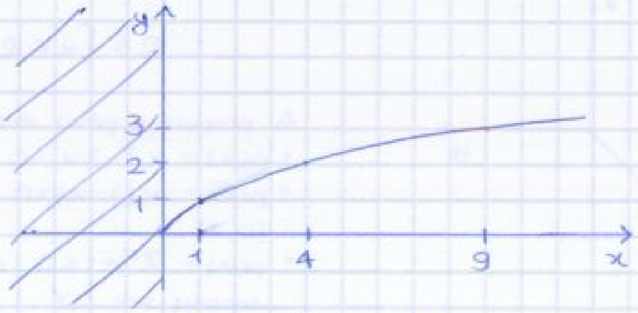
This is NOT a function

DOMAIN, RANGE AND GRAPH

- Domain of f = dom$f$ = $\{x \in X : \exists y \in Y$ such that $y = f(x)\}$ dom$f \subseteq X$
- Range of f = im$f$ = $\{y \in Y : \exists x \in X$ such that $y = f(x)\}$ im$f \subseteq Y$
- Graph of f = $\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in \text{dom}f\}$ $\Gamma(f) \subseteq X \times Y$

$f : \text{dom}f \subseteq X \to Y$

$x \to \sqrt{x}$ (y)	
$1 \to 1$	$1 \in$ dom$f$
$4 \to 2$	$4 \in$ dom$f$
$9 \to 3$	$9 \in$ dom$f$
$-1 \to //$	$-1 \notin$ dom$f$
$-10 \to //$	$-10 \notin$ dom$f$
$0 \to 0$	$0 \in$ dom$f$



dom$f$ = $[0; +\infty)$

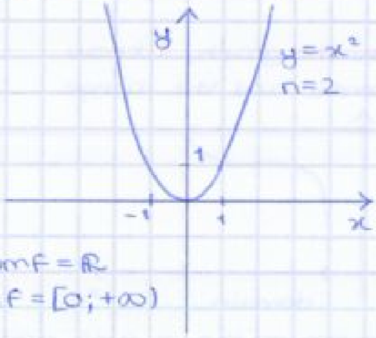
③ $f(x) = x^n$ $n=0 \rightarrow y=x^0 \rightarrow y=1$

$n=1 \rightarrow$ identity function

$n \in \mathbb{N} \rightarrow$ positive exponent
$n \in \mathbb{Z} \rightarrow$ negative exponent

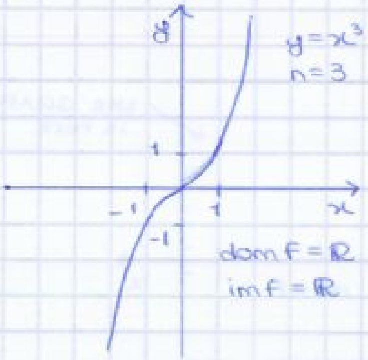
$n \in \mathbb{N}$, POSITIVE EXPONENT

$f(x) = x^n$ n = even positive



dom f = $\mathbb{R}$
im f = $[0; +\infty)$

n = odd positive

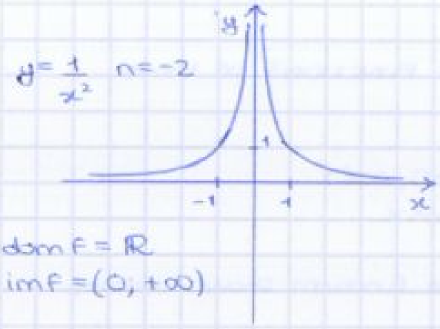


dom f = $\mathbb{R}$
im f = $\mathbb{R}$

$n \in \mathbb{Z}$, NEGATIVE EXPONENT

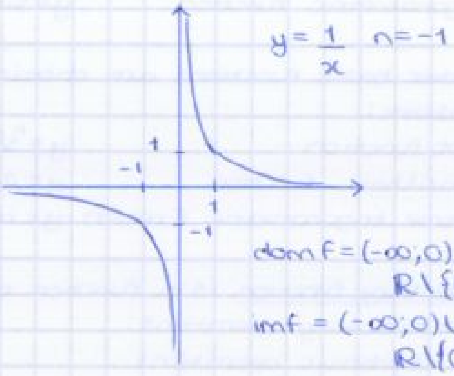
$f(x) = x^{-n} \rightarrow f(x) = \frac{1}{x^n}$

n = even negative



dom f = $\mathbb{R}$
im f = $(0, +\infty)$

n = odd negative



dom f = $(-\infty; 0) \cup (0; +\infty)$
$\mathbb{R} \setminus \{0\}$
im f = $(-\infty; 0) \cup (0; +\infty)$
$\mathbb{R} \setminus \{0\}$

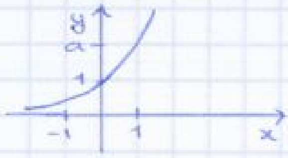
④ $f(x) = a^x$, $a \in \mathbb{R}$, $a > 0$

$(0;1)$ $\left(-1; \frac{1}{a}\right)$ $(1;a)$

- a=1

dom f = $\mathbb{R}$
im f = $\{1\}$

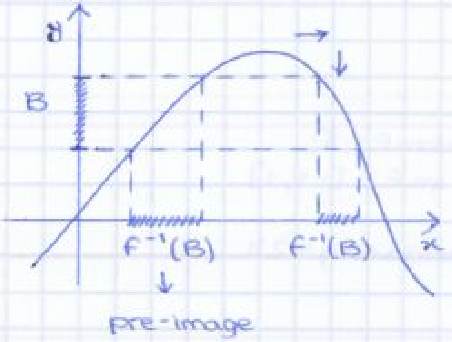
- a>1

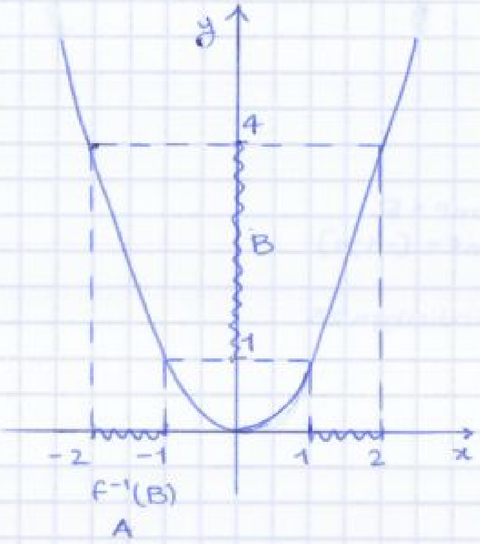


dom f = $\mathbb{R}$
im f = $(0; +\infty)$

- 0<a<1

dom f = $\mathbb{R}$
im f = $(0; +\infty)$





$f^{-1}(\underbrace{[+1;4]}_{B}) = [-2;-1] \cup [1;2]$

$f(\underbrace{[1;2]}_{A}) = [1;4]$

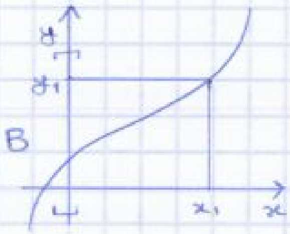
$f^{-1}(f(A)) \geq A$

↳ "is included in"

$f(f^{-1}(B)) = B$

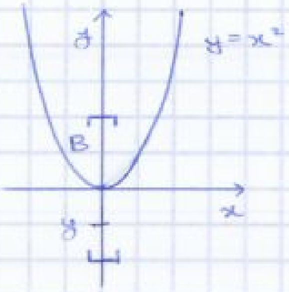
SURJECTIVE, INJECTIVE AND BIJECTIVE FUNCTIONS

① SURJECTIVE



$\exists x: f(x) = y$
SURJECTIVE on B
or f is ONTO B

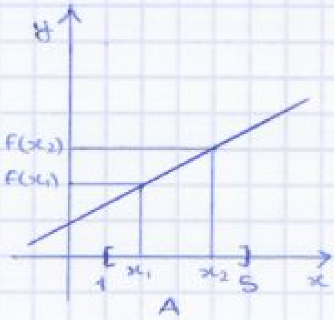
$\forall y \in B \ \exists x \in \text{dom} f : y = f(x)$



f is NOT onto B
because there are elements of B that don't correspond to any element x.

② INJECTIVE

- $\forall x_1, x_2 \in A \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- $\forall x_1, x_2 \in A \quad x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$

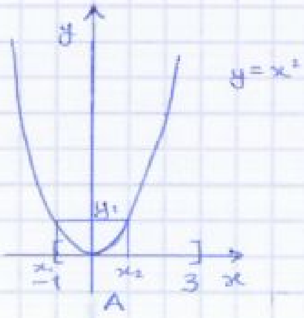


$f(x) = \frac{x}{2} + 1$

$x_1 \neq x_2 \quad f(x_1) \neq f(x_2)$

$\frac{x_1}{2} \neq \frac{x_2}{2} \qquad \frac{x_1}{2} + 1 \neq \frac{x_2}{2} + 1$

$f(x_1) = f(x_2) \quad \frac{x_1}{\cancel{2}} + \cancel{1} = \frac{x_2}{\cancel{2}} + \cancel{1} \Rightarrow x_1 = x_2$



$x_1 \neq x_2$ BUT $f(x_1) = f(x_2)$
f is NOT injective

③ BIJECTIVE

$$\begin{cases} f(A) = B \text{ (surjective)} \\ f \text{ is INJECTIVE in } A \end{cases} \longrightarrow f \text{ is BIJECTIVE or ONE-TO-ONE} \quad f: A \to B$$

The graph of the inverse function

The graph of f^{-1} coincides with the graph of the function f:

$$\begin{aligned} \Gamma(f^{-1}) &= \{(y, f^{-1}(y)) \in Y \times X \ : \ y \in \mathrm{dom} f^{-1}\} \\ &= \{(f(x), x) \in Y \times X \ : \ x \in \mathrm{dom} f\}. \end{aligned}$$

Here the independent variable is the the second coordinate and the dependent one is the first.
If we want to use the standard representation of graphs (independent variable as first coordinate) we must swap (i.e. interchange) the coordinates.
Geometrically this corresponds to a

symmetry with respect to the line $y = x$

.

Exponential and logarithm

We consider the exponential function $f(x) = a^x$ with $a > 0$.

- If $a \neq 1$
  $f(x) = a^x, \qquad f : \mathbb{R} \to (0, +\infty), \qquad$ is injective and surjective

  We define the inverse function, called the **logarithm**:

  $$f^{-1}(x) = \log_a x, \qquad f^{-1} : (0, +\infty) \to \mathbb{R}$$

  The relationship between exponential and logarithm is:

  $$\log_a(a^x) = x, \ \forall x \in \mathbb{R}, \qquad a^{\log_a x} = x, \ \forall x \in (0, +\infty)$$

- If $a = 1$
  $f(x) = 1^x = 1, \qquad f : \mathbb{R} \to \{1\}, \qquad$ is not injective
  then it is impossible to define the inverse function.

Even and odd functions

Let $f : \mathrm{dom}f \subseteq \mathbb{R} \to \mathbb{R}$ be a function with dom f symmetric with respect to the origin ($x \in \mathrm{dom}f \Rightarrow -x \in \mathrm{dom}f$)

- The function f is **even** if $f(x) = f(-x)$, $\quad \forall x \in \mathrm{dom}f$
- The function f is **odd** if $f(x) = -f(-x)$, $\quad \forall x \in \mathrm{dom}f$

Examples.

- The function $f(x) = x^n$, $n \in \mathbb{N} \setminus \{0\}$ is even when n is even and it is odd when n is odd.
- The exponential function a^x $(a > 0$ and $a \neq 1)$ is neither even nor odd.

Periodic functions

Fix a real and positive p; consider a function f such that

- if $x \in \mathrm{dom}f$ then $x \pm p \in \mathrm{dom}f$, $\forall x \in \mathrm{dom}f$
- $\forall x \in \mathrm{dom}f \quad f(x) = f(x + p)$

This function is said to be **periodic** of period p.

Tangent and inverse tangent

The **function tangent**

$$f : \mathbb{R} \setminus \{\tfrac{\pi}{2} + k\pi : k \in \mathbb{Z}\} \to \mathbb{R}\,, \quad f(x) = \tan x$$

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

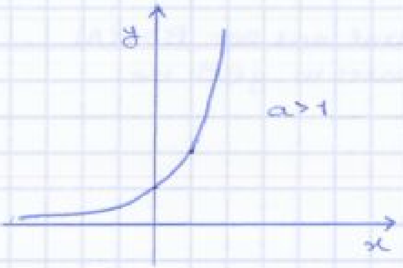
is odd and periodic of period $p = \pi$; then it is not injective in $\mathbb{R}$.

Considering the restriction $f_1(x) = \tan x\,, \quad f_1 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ injective (and surjective) we define the inverse function

$$f_1^{-1}(x) = \arctan x\,, \quad f_1^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

This function is called **inverse tangent** or **arctangent**; a different notation is $\tan^{-1}$.

EXPONENTIAL FUNCTION

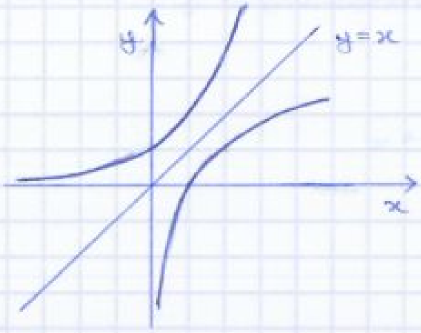


dom f   im f

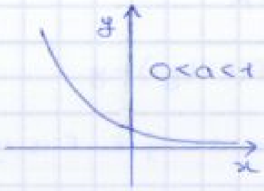
$$\begin{tikzcd} \mathbb{R} \arrow[r] & (0;+\infty) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

dom f^{-1} im f^{-1} inverse function

$(0,+\infty) \to \mathbb{R}$



$y=a^x$ $a>0$ $a\neq 1$

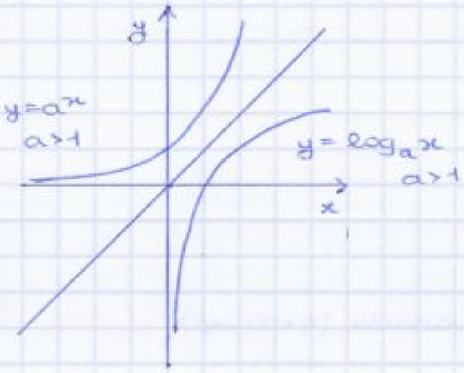


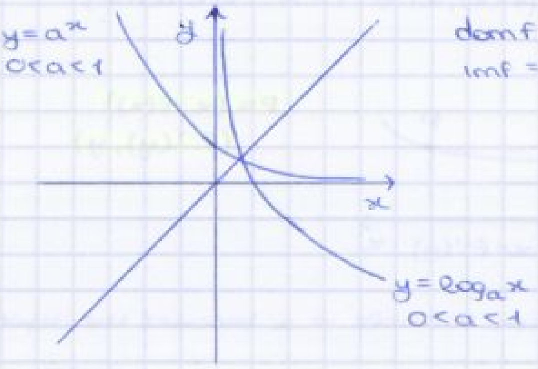
$(0,1)$ $(1;a)$ $(-1;1/a)$

$dom(f)=\mathbb{R}$ $im(f)=(0;+\infty)$

→ inverse function: $$\begin{tikzcd} x \arrow[r] & a^x=y \arrow[l, bend left, "\log_a y"] \end{tikzcd}$$

$(1;0)$ $(a;1)$ $(1/a;-1)$

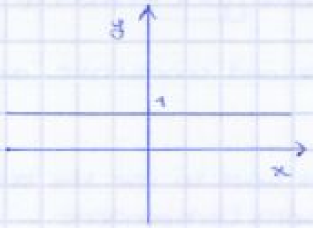




dom f = $(0;+\infty)$

im f = $\mathbb{R}$

$$\begin{cases} \log_a 1 = 0 \\ \log_a a = 1 \\ \log_a \frac{1}{a} = -1 \end{cases}$$

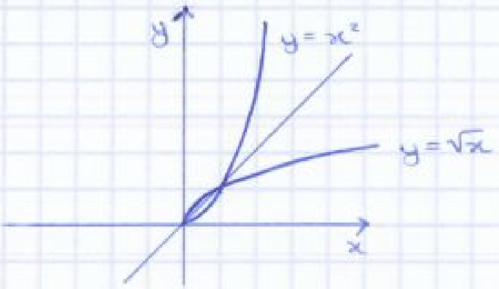


$y=a^x$ $a=1$

it's not possible to have the inverse function, because $y=a^x$, $a=1$ is NOT an injective function

F PARI (simm. asse y): $f(x)=f(-x)$

F DISPARI (simm. O). $f(-x)=-f(x)$, cioè $f(x)=-f(-x)$

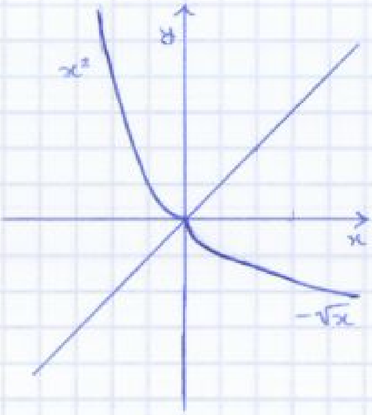


$\sqrt[2]{x^2} = x \quad \forall x \in [0;+\infty)$

$(\sqrt{x})^2 = x \quad \forall x \in [0;+\infty)$

If I consider the other part: $f: (-\infty;0] \to [0;+\infty)$

$-\sqrt{x}$



because x is negative

$-\sqrt{9} = -3 \quad x=-3$

$-\sqrt{x^2} = \boxed{x} \quad \forall x \in (-\infty;0]$

$(-\sqrt{x})^2 = x \quad \forall x \in [0;+\infty)$

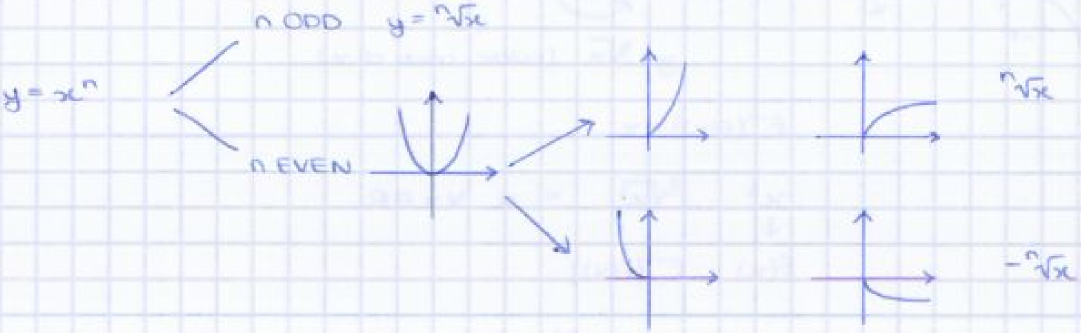
$$\begin{cases} \sqrt{x^2} = x & x \geq 0 \\ \sqrt{x^2} = -x & x<0 \end{cases} \qquad \sqrt{x^2} = |x|$$

- $\sqrt{k} \quad k>0$

  it does not mean the solution of the equation $x^2 = k$, but it means the positive solution of the equation $x^2 = k$

  $\sqrt{9} = 3$, not $\pm 3 \qquad -\sqrt{9} = -3$

$y = a^x \quad a>0 \quad a \neq 1$ → inverse function → $y = \log_a x$



$\arcsin(\sin x) = x \quad \forall x \in [-\pi/2; \pi/2]$

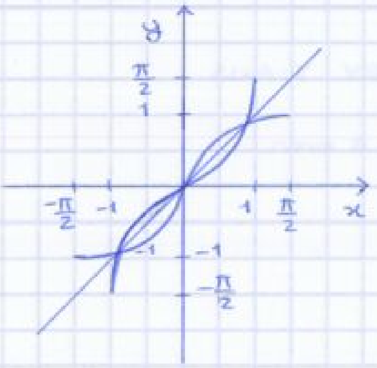
$\sin(\arcsin x) = x \quad \forall x \in [-1; 1]$

$f^{-1}(f(x)) = x$

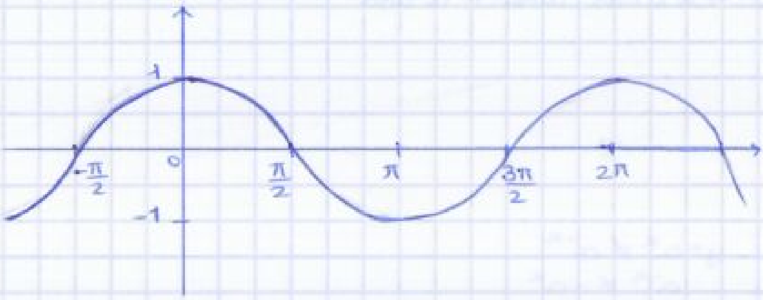
$f(f^{-1}(x)) = x$

$\arcsin x = \sin^{-1} x$

$\sin^{-1} x \neq \frac{1}{\sin x}$



$f(x) = \cos x$ (FUNCTION COSINE)

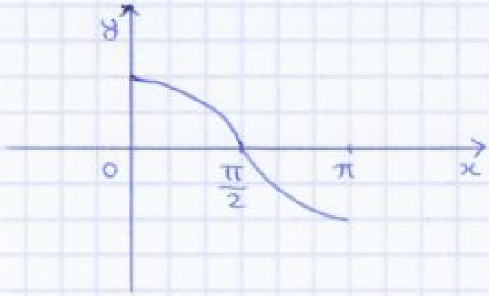


Period = 2π

$f: \mathbb{R} \rightarrow [-1; 1]$

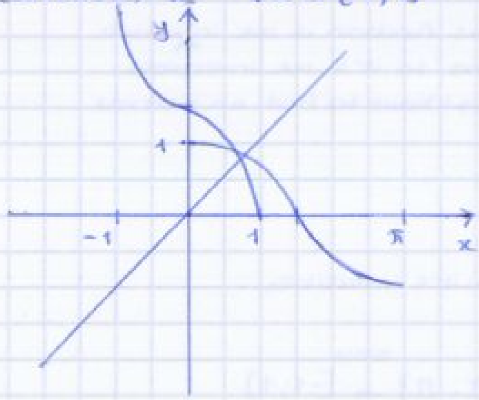
RESTRICTION: $f: [0; \pi] \xrightarrow{\cos x} [-1; 1]$, $\arccos x$

$\arccos x = \cos^{-1} x$



$\arccos(\cos x) = x \quad \forall x \in [0; \pi]$

$\cos(\arccos x) = x \quad \forall x \in [-1; 1]$



$f^{-1}: [-1; 1] \rightarrow [0; \pi]$

Mathematical Analysis I (2013-2014)

Basic Notions 5 - Some properties of functions

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Monotone functions

Definition

Consider a function $f : \operatorname{dom} f \subseteq \mathbb{R} \to \mathbb{R}$ *and an interval* $I \subseteq \operatorname{dom} f$ *(our definitions apply also to a generic subset* A *of* $\operatorname{dom} f$*, but they are usually referred to intervals).*

- f *is* **monotone increasing** *on* I *if*
$$\forall x_1, x_2 \in I, \quad x_1 < x_2 \quad \Longrightarrow \quad f(x_1) \leq f(x_2)$$
- f *is* **monotone strictly increasing** *on* I *if*
$$\forall x_1, x_2 \in I, \quad x_1 < x_2 \quad \Longrightarrow \quad f(x_1) < f(x_2)$$
- f *is* **monotone decreasing** *on* I *if*
$$\forall x_1, x_2 \in I, \quad x_1 < x_2 \quad \Longrightarrow \quad f(x_1) \geq f(x_2)$$
- f *is* **monotone strictly decreasing** *on* I *if*
$$\forall x_1, x_2 \in I, \quad x_1 < x_2 \quad \Longrightarrow \quad f(x_1) > f(x_2)$$
- f *is* **monotone** *on* I *if it is increasing or decreasing on* I*;* f *is* **strictly monotone** *on* I *if it is strictly increasing or strictly decreasing on* I*.*

Upper and lower bound - 2

Definition

*The function f is **bounded in** I if it is bounded from above and bounded by below in I.*

Some examples.

- The function $f(x) = x^2$ in $I = [-2, 5]$ is bounded from above by all real $c \geq 25$ and by below by all real $c \leq 0$.
- The functions $y = \sin x$, $y = \cos x$, $y = \arctan x$ are bounded in $\mathbb{R}$.
- The function $f(x) = \frac{1}{x}$ in $I = (0, 1]$ is bounded by below but it is not bounded from above.

Convex functions

Consider a function f defined on an interval I. Given two points x_1 and x_2 in I, with $x_1 < x_2$, we consider the segment $S(x_1, x_2)$ joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Definition

*The function f is **convex** in I if for all $x_1, x_2 \in I$ the segment $S(x_1, x_2)$ lies above (or coincide with) the graph of f in $[x_1, x_2]$.*

Definition

*The function f is **concave** in I if the function $-f(x)$ is convex in I.*

Piecewise functions - 2

- **The integer part function (or floor function)**

  $f(x) = [x] =$ the greatest integer $\leq x$
  $\operatorname{dom} f = \mathbb{R}; \qquad \operatorname{im} f = \mathbb{Z}$

- **The mantissa function**

  $f(x) = M(x) = x - [x]$
  $\operatorname{dom} f = \mathbb{R}; \qquad \operatorname{im} f = [0, 1)$

- **The "positive part" function**

  $$x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
  $\operatorname{dom} f = \mathbb{R}; \qquad \operatorname{im} f = [0, +\infty)$

Piecewise functions - 3

- **The "negative part" function**

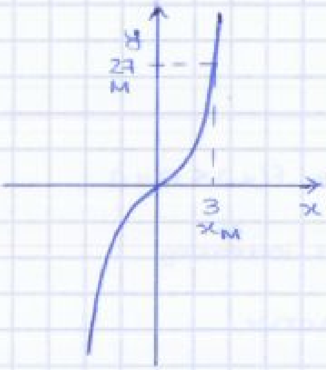
  $$x^- = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$
  $\operatorname{dom} f = \mathbb{R}; \qquad \operatorname{im} f = [0, +\infty)$

- **The maximum of two functions**

  Suppose that $f(x)$ and $g(x)$ are defined in an interval I; the maximum of the two functions is

  $$h(x) = \max\{f(x), g(x)\} = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

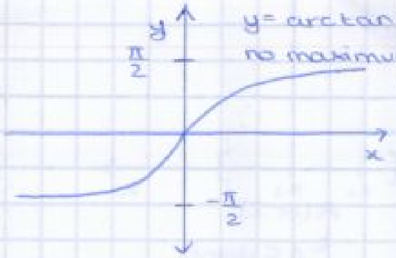
  With obvious changes we define the function $k(x) = \min\{f(x), g(x)\}$.



$y = x^3$ in $\mathbb{R}$ it has no maximum

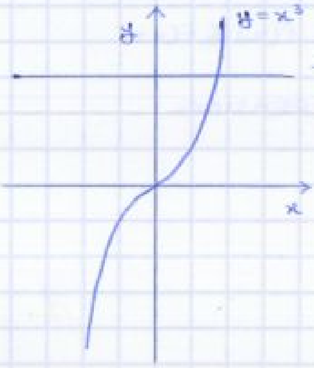
If we consider the interval $[0;3] \rightarrow \exists\, m, \exists\, M$
3 is the point of maximum,
while 27 is the maximum

$[0;3)$ $\exists\, m, \nexists\, M$
$(0;3)$ $\nexists\, m, \nexists\, M$
$(0;3]$ $\nexists\, m, \exists\, M$

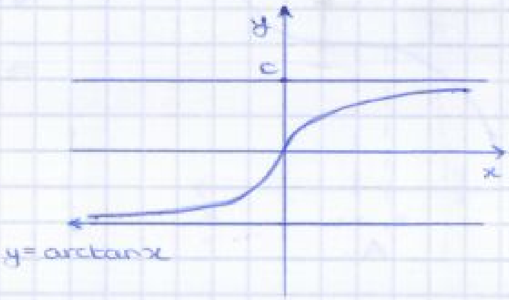


$y = \arctan x$ has no maximum in $\mathbb{R}$

UPPER AND LOWER BOUND



→ it's not possible to bound neither by below nor from above



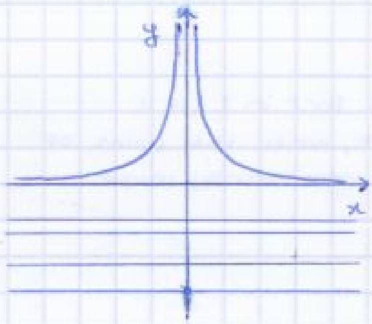
The number $c \in \mathbb{R}$ is a LOWER BOUND for f in I if $\forall x \in I : f(x) \geq c$
→ f is BOUNDED BY BELOW in I.

The number $c \in \mathbb{R}$ is an UPPER BOUND for f in I if $\forall x \in I : f(x) \leq c$
→ f is BOUNDED FROM ABOVE in I

$\arctan x \leq 3$ $c = 3$ upper bound
$\arctan x \leq 2$ $c = 2$ upper bound
$\arctan x \leq \frac{\pi}{2}$ $c = \frac{\pi}{2}$ (least) upper bound
↑ infinite upper bounds

- The function f is bounded in I if it is bounded from above and bounded by below in I → ex: $y = x^2$ in $I = [-2;5]$ is bounded from above by all real $c \geq 25$ and by below by all real $c \leq 0$

- BOUNDED FUNCTION IN $\mathbb{R}$ (from above and by below)
Ex: $y = \cos x$, $y = \operatorname{sen} x$, $y = \arctan x$



$y = \frac{1}{x^2}$ $\operatorname{dom} f = \mathbb{R} \setminus \{0\}$ it's bounded by below but not from above

$$f(x) = \begin{cases} 1-x & \text{if } x \le 1 \\ \log_2 x & \text{if } x > 1 \end{cases}$$

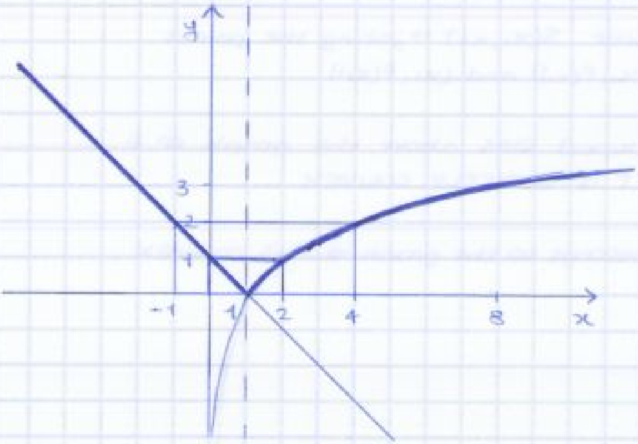


Image: $f([0;2]) = \{y \in \mathbb{R} : y = f(x) \quad x \in [0,2]\} = [0;1]$ from x to y $^{\leftarrow}\uparrow$

Pre-image: $f^{-1}([1;2]) = \{x \in \text{dom} f : f(x) \in [1;2]\} = [-1;0] \cup [2;4]$ from y to x $^{\rightarrow}\downarrow \; \downarrow^{\leftarrow}$

Exercises:

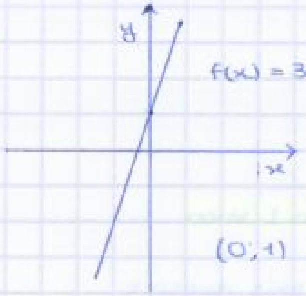
$f(x) = 3x+1$

1) is this function invertible? where?

2) graph

3) Can I find the formula of f^{-1}?

1)



$f(x) = 3x+1$

(0;1) (1;4)

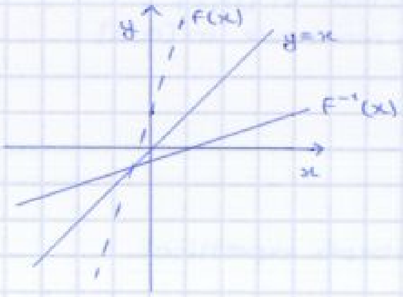
dom f = $\mathbb{R}$

im f = $\mathbb{R}$

This function is injective ⇒ INVERTIBLE

$f: \mathbb{R} \rightarrow \mathbb{R}$ $\underset{f^{-1}}{\curvearrowleft}$

2) Graph (1;0) (4;1)



3) Formula:

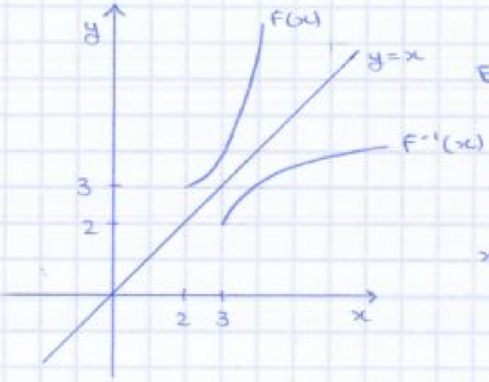
$f(x) = y$

$3x+1 = y$

$3x = y-1$

$x = \frac{y-1}{3}$ → exchange x and y ⇒ $y = \frac{x-1}{3}$

$f^{-1}(f(x)) = x \quad \forall x \in \mathbb{R}$ $\qquad f(f^{-1}(x)) = x \quad \forall x \in \mathbb{R}$



Equation of f^{-1}:

$y = x^2 - 4x + 7$

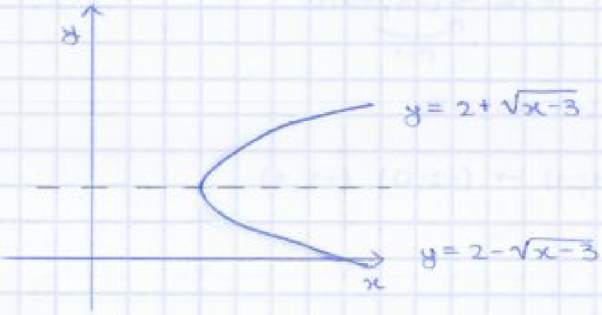
$x^2 \quad 4x + (7-y) = 0$

$x_{1,2} = 2 \pm \sqrt{4-(7-y)} = 2 \pm \sqrt{4-7+y} = 2 \pm \sqrt{y-3}$

$x_{1,2} = 2 \pm \sqrt{y-3}$ exchange x and y

$\Rightarrow f^{-1}(x) = 2 \pm \sqrt{x-3} = \begin{cases} 2+\sqrt{x-3} \\ 2-\sqrt{x-3} \end{cases}$

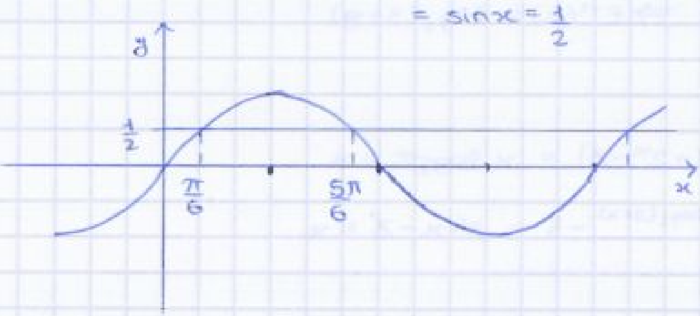
$x \geq 3 \quad f^{-1} = 2+\sqrt{x-3}$ becouse $\text{dom} f^{-1} = [3;+\infty)$



- $y = \sin x \qquad f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$ ← pre-image

$f^{-1}\left(\left\{\frac{1}{2}\right\}\right) = \left\{x \in \mathbb{R} : \sin x \in \left\{\frac{1}{2}\right\}\right\}$ → infinitely many solutions

$= \sin x = \frac{1}{2}$



$f^{-1}\left(\left\{\frac{1}{2}\right\}\right) = \left\{\frac{\pi}{6} + 2k\pi ; k \in \mathbb{Z}\right\} \cup \left\{\frac{5}{6}\pi + 2k\pi ; k \in \mathbb{Z}\right\}$

$\sin x = \frac{1}{2} \Leftrightarrow x = \arcsin \frac{1}{2}$

Neighbourhoods - 1

Definition

We consider a point $x_0 \in \mathbb{R}$ and a real number $r > 0$.
A **neighbourhood of x_0 of radius r** *is the open and bounded interval*

$$I_r(x_0) = (x_0 - r, x_0 + r) = \{x \in \mathbb{R} \ : \ |x - x_0| < r\}.$$

Remarks.

- The point x_0 is called the centre of the neighbourhood.
- The neighbourhood $I_r(x_0)$ is the set of points x having a distance from x_0 smaller than r.
- The American spelling of "neighbourhood" is "neighborhood".

Neighbourhoods - 2

Definition

We consider a point $x_0 \in \mathbb{R}$ and a real number $r > 0$.
A **right neighbourhood of x_0 of radius r** *is the half-open and bounded interval*

$$I_r^+(x_0) = [x_0, x_0 + r) = \{x \in \mathbb{R} \ : \ 0 \leq x - x_0 < r\}.$$

A **left neighbourhood of x_0 of radius r** *is the half-open and bounded interval*

$$I_r^-(x_0) = (x_0 - r, x_0] = \{x \in \mathbb{R} \ : \ 0 \leq x_0 - x < r\}.$$

From these definitions it follows that

$$I_r^+(x_0) \cup I_r^-(x_0) = I_r(x_0),$$
$$I_r^+(x_0) \cap I_r^-(x_0) = \{x_0\}.$$

Infinite limits at infinity - 2

Now we define limits for $x \to -\infty$.

- We consider a function $f : (-\infty, b] \to \mathbb{R}$. Then $\lim_{x \to -\infty} f(x) = +\infty$ if

$$\forall A > 0, \exists B \geq 0 : \ \forall x : x \in \mathrm{dom}\, f, x < -B \ \Rightarrow \ f(x) > A.$$

- In the same way, we say that $\lim_{x \to -\infty} f(x) = -\infty$ if

$$\forall A > 0, \exists B \geq 0 : \ \forall x : x \in \mathrm{dom}\, f, x < -B \ \Rightarrow \ f(x) < -A.$$

Examples

- We verify that $\lim_{x \to +\infty} \sqrt{x} = +\infty$.
  In order to do this, we fix $A > 0$; noticing that

$$\sqrt{x} > A \iff x > A^2$$

  we can define $B = A^2$ and we get the result.
- We verify that $\lim_{x \to -\infty} \left(\frac{1}{2}\right)^x = +\infty$.
  In order to do this, we fix $A > 0$; noticing that

$$\left(\frac{1}{2}\right)^x > A \iff \log_{1/2} \left(\frac{1}{2}\right)^x < \log_{1/2} A \iff x < \log_{1/2} A$$

  we can define $B = \log_{1/2} A = -\log_2 A$ and we get the result.

Examples - 2

- We verify that $\lim_{x \to -\infty} \dfrac{1}{\sqrt{1-x}} = 0$.

  Fix $\varepsilon > 0$; we have

$$\begin{aligned} \left|\frac{1}{\sqrt{1-x}}\right| = \frac{1}{\sqrt{1-x}} &< \varepsilon \\ \sqrt{1-x} &> \frac{1}{\varepsilon} \\ 1-x &> \frac{1}{\varepsilon^2} \\ x &< 1-\frac{1}{\varepsilon^2} \end{aligned}$$

  Then, setting $B = \max\left(0, \dfrac{1}{\varepsilon^2} - 1\right)$, we have

$$x < -B \quad \Rightarrow \quad \left|\frac{1}{\sqrt{1-x}}\right| < \varepsilon.$$

Infinite limit for $x \to x_0$

Definition

We consider a function f defined in $I(x_0)$, except possibly at x_0. If

$\forall A > 0,\ \exists \delta > 0 : \ \forall x \in \operatorname{dom} f,\ 0 < |x - x_0| < \delta \ \Rightarrow \ f(x) > A.$

we say that the function f **has limit** *$+\infty$ for x going to x_0 and we write*

$$\lim_{x \to x_0} f(x) = +\infty.$$

Remarks.

- We can also say that f **tends to** $+\infty$ or **diverges to** $+\infty$.
- We the obvious changes we define $\lim_{x \to x_0} f(x) = -\infty$.
- The function f may have a value in x_0; since we use the "neighbourhood without centre" $0 < |x - x_0| < \delta$ the value at x_0 (even if it exists) **is not** considered in our definition of limit.

  Limit is looking at the neighbourhood and not at the point!

Left limit of a function

Definition

The function f is defined in $I^-(x_0) \setminus \{x_0\}$; if

$$\forall A > 0\,, \exists \delta > 0 \text{ such that } \forall x :$$

$$x \in \mathrm{dom}\, f \text{ and } 0 < x_0 - x < \delta \Longrightarrow f(x) > A$$

we say that f **has left limit** $+\infty$ *and we write*

$$\lim_{x \to x_0-} f(x) = +\infty$$

Remarks.

- With the obvious changes we define
$$\lim_{x \to x_0+} f(x) = -\infty, \qquad \lim_{x \to x_0-} f(x) = -\infty\ .$$
- It is easy to verify that $\lim_{x \to 0+} \frac{1}{x} = +\infty$ and $\lim_{x \to 0-} \frac{1}{x} = -\infty$.

Limits of elementary functions - 1

$$\lim_{x \to +\infty} x^n = +\infty \quad (n \in N),$$

$$\lim_{x \to -\infty} x^n = +\infty \quad (n \text{ even}), \qquad \lim_{x \to -\infty} x^n = -\infty \quad (n \text{ odd})$$

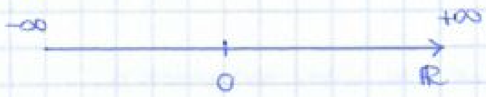
$$\lim_{x \to +\infty} a^x = +\infty\,, \qquad \lim_{x \to -\infty} a^x = 0 \qquad a > 1$$

$$\lim_{x \to +\infty} a^x = 0\,, \qquad \lim_{x \to -\infty} a^x = +\infty \qquad a < 1$$

$$\lim_{x \to +\infty} \log_a x = +\infty\,, \qquad \lim_{x \to 0^+} \log_a x = -\infty \qquad a > 1$$

$$\lim_{x \to +\infty} \log_a x = -\infty\,, \qquad \lim_{x \to 0^+} \log_a x = +\infty \qquad a < 1$$

LIMITS



For limits we need an extension of the real line

$\mathbb{R} \cup \{-\infty; +\infty\} = \mathbb{R}^*$

extension in the real line:
- operation ✗
- order ✓ $-\infty < +\infty$ $\forall x \in \mathbb{R}$ $-\infty < x < +\infty$

operation: we cannot extend operations in the extended real line keeping the properties of the real line

$(\cancel{+\infty}) + (-\infty) = \cancel{+\infty}$ → contradiction

$-\infty = 0$

$(+\infty) + (-\infty) = 0$ → contradiction

NEIGHBOURHOOD OF A POINT

x is close to / far from a point x_0

$|x - x_0| < r$

x_0 = midpoint = CENTER

$r > 0$

$I_r(x_0) = (x_0 - r; x_0 + r) = \{x \in \mathbb{R} : |x - x_0| < r\}$ = set of points x having a distance from x_0 smaller than radius r.

r = radius $(-r; r)$

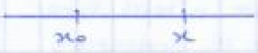
$|x|$ = distance of x from the origin

distance $\overline{ab} = b - a$

$|x| = a$, $a \geq 0$ points having distance a from the origin

$|x| < a$ $\{x \in \mathbb{R} : |x| < a\} = (-a; a)$

$|x - x_0|$



$x - x_0$ $x \geq x_0$

$x_0 - x$ $x < x_0$

$I_r(x_0) = (x_0 - r; x + r) = \{x \in \mathbb{R} : |x - x_0| < r\}$

$x \in I_r(x_0)$

RIGHT NEIGHBOURHOOD OF x_0 OF RADIUS r

$I^+_r(x_0) = [x_0, x_0 + r)$

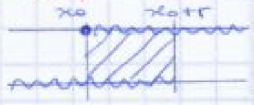
$|x \cancel{- x_0|} < r$

$0 \leq x - x_0 < r$

$= \{x \in \mathbb{R} : 0 \leq x - x_0 < r\}$

$x - x_0 \geq 0$ $x \geq x_0$

$x - x_0 \leq r$ $x < x_0 + r$



LEFT NEIGHBOURHOOD OF x_0 OF RADIUS r

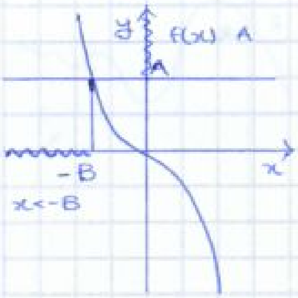
$I^-_r(x_0) = (x_0 - r; x_0] = \{x \in \mathbb{R} : 0 \leq x_0 - x < r\}$

$x_0 - x \geq 0$ $x \leq x_0$

$x_0 - x < r$ $x > x_0 - r$

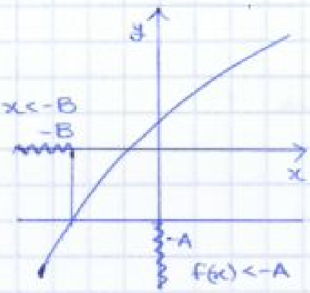
- $I^+_r(x_0) \cup I^-_r(x_0) = I_r(x_0)$
- $I^+_r(x_0) \cap I^-_r(x_0) = \{x_0\}$

2) x going at $-\infty$



$\lim_{x\to-\infty} f(x) = +\infty$

$\forall A>0\ \exists B\geq 0 : \forall x : \left.\begin{matrix} x\in \text{dom}f \\ x<-B \end{matrix}\right\} \Rightarrow f(x)>A$



$\lim_{x\to-\infty} f(x) = -\infty$

$\forall A>0\ \exists B\geq 0 : \forall x : \left.\begin{matrix} x\in \text{dom}f \\ x<-B \end{matrix}\right\} \Rightarrow f(x)<-A$

Examples:

$\lim_{x\to+\infty} x^2 = +\infty$ $\qquad \forall A>0\ \exists B\geq 0 : \forall x : \left.\begin{matrix} x\in \text{dom}f \\ x>B \end{matrix}\right\} \Rightarrow f(x)>A$

$x>B \rightarrow x^2>A \qquad x^2>A \rightarrow x>\sqrt{A}$ comparing $\sqrt{A}$ and B $\sqrt{A}=B$

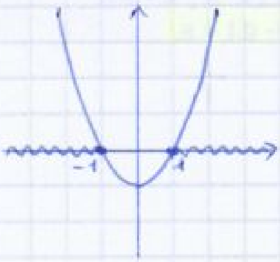
$x>B$

PRE-IMAGE

- $f(x)=x^2-1 \qquad f^{-1}(\underbrace{[0,+\infty)}_{A}) \qquad f^{-1}(\underbrace{(-1;1)}_{B})$

↳ pre-image $f^{-1}(B)=\{x\in \text{dom}f : f(x)\in B\}$

$f^{-1}([0,+\infty)) = \{x\in\mathbb{R} : x^2-1\in[0,+\infty)\}$
$= \{x\in\mathbb{R} : x^2-1\geq 0\}$



$(-\infty;-1]\cup[1;+\infty)$

$f(x)=x^2-1 \qquad B=(-1;1)$

$\{x\in\mathbb{R} : x^2-1\in(-1;1)\} = \{x\in\mathbb{R} : -1<x^2-1<1\}$

$= 0<x^2<2 = \begin{cases} x^2\cancel{-1}>\cancel{-1} & x^2>0 \quad \forall x\in\mathbb{R}\ x\neq 0 \\ x^2-1<1 & x^2<2 \quad -\sqrt{2}<x<\sqrt{2} \end{cases}$

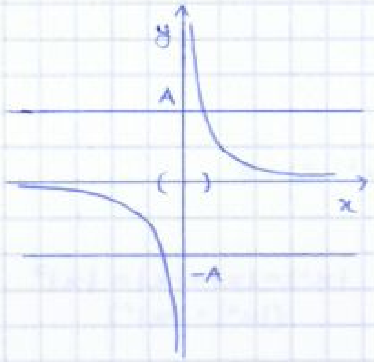
$= (-2;0)\cup(0;\sqrt{2})$

$\lim_{x\to x_0} f(x) = -\infty$

$$\forall A>0 \quad \exists \delta>0 \quad \left.\begin{array}{l}\forall x: x\in \text{dom}f \\ 0<|x-x_0|<\delta\end{array}\right\} \Rightarrow f(x)<-A$$

Ex: $f(x)=\frac{1}{x^2}$ $\quad \lim_{x\to x_0} f(x) = +\infty \quad \frac{1}{x^2}>A \quad x^2<\frac{1}{A} \rightarrow -\frac{1}{\sqrt{A}}<x<\frac{1}{\sqrt{A}}$

$|x|<\frac{1}{\sqrt{A}} \qquad \delta\leq\frac{1}{\sqrt{A}}$



$f(x)=\frac{1}{x}$

$\lim_{x\to+\infty}\frac{1}{x}=0$

$\lim_{x\to-\infty}\frac{1}{x}=0$

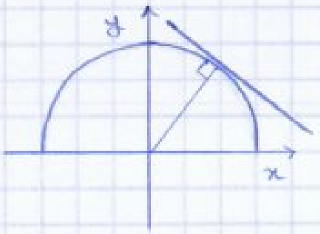
ONE - SIDED LIMITS

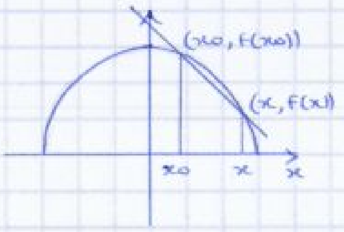
$\lim_{x\to x_0^+} f(x) = \dots$ RIGHT LIMIT $\quad 0<x-x_0<\delta$

$\lim_{x\to x_0^-} f(x) = \dots$ LEFT LIMIT $\quad 0<x_0-x<\delta$

$$\lim_{x\to x_0^+} f(x)=+\infty \quad \forall A>0 \quad \exists\delta>0 \quad \left.\begin{array}{l}\forall x: x\in\text{dom}f \\ 0<x-x_0<\delta\end{array}\right\} \Rightarrow f(x)>A$$

limit / point	$+\infty$	$-\infty$	ℓ
$+\infty$			
$-\infty$			
x_0			○ → $\lim_{x\to x_0} f(x)=\ell$





FUNCTION SLOPE $= \frac{f(x)-f(x_0)}{x-x_0}$

when these 2 points become the same $(x=x_0) \rightarrow$ tangent

Mathematical Analysis I (2013-2014)

Limits 2 - Limits at $x_0 \in \mathbb{R}$

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October 2013

Finite limit for $x \to x_0$

The definition of finite limit for $x \to x_0 \in \mathbb{R}$ is not unexpected, if we consider the meaning of "having a finite limit l" and of "x tends to $x_0 \in \mathbb{R}$".

Definition

Let f be a function defined in $I(x_0) \setminus \{x_0\}$. We say that f **has limit** *$l \in \mathbb{R}$ (o* **tends to** *l) and we write $\lim_{x \to x_0} f(x) = l$, if*

$$\forall \varepsilon > 0\,, \exists \delta > 0 : \forall x : x \in \operatorname{dom} f \text{ and } 0 < |x - x_0| < \delta \implies |f(x) - l| < \varepsilon$$

Remark.

Given a real number l and using this definition we can **decide whether l is the limit or not**; the definition does not help us in the task of computing l.

Examples

Example.

- The function $f(x) = x^3 + 1$ is continuous at $x_0 = 0$, since $\lim_{x \to 0} f(x) = 1$ and $f(0) = 1$.
- The constant function $l(x) = c$ is continuous at $x_0 \in \mathbb{R}$, since $\lim_{x \to x_0} l(x) = c = l(x_0)$.
- The affine function $m(x) = mx + q$ is continuous at $x_0 \in \mathbb{R}$, since $\lim_{x \to x_0} m(x) = mx_0 + q = m(x_0)$.

Remark.

- The question "Are the functions $f(x) = \dfrac{\sin x}{x}$ or $g(x) = \dfrac{1}{x}$ continuous at $x_0 = 0$?" has no meaning.
  In order to say whether a function is continuous or not at a point the function **must be defined at that point**.

Second case: removable discontinuity

Definition

Let f be a function defined in a neighbourhood $I(x_0)$. The function f has a **removable discontinuity** *or a* **removable singularity** *at x_0 if*

$$\lim_{x \to x_0} f(x) = l \neq f(x_0).$$

Example. The function $g(x) = \operatorname{sign}(x^2)$ is not continuous at $x_0 = 0$, since $\lim_{x \to 0} g(x) = 1$, while $g(0) = 0$. The point $x_0 = 0$ is a removable discontinuity for this function.

One-sided limits at x_0 - 2

Definition

Let f be a function defined in $I^-(x_0) \setminus \{x_0\}$. We say that f **has left limit** *$l \in \mathbb{R}$ (or* **tends to** *l* **from the left***) and we write* $\lim_{x \to x_0^-} f(x) = l$, *if*

$\forall \varepsilon > 0 \,, \exists \delta > 0 : \ \forall x : x \in \operatorname{dom} f$ *and* $0 < x_0 - x < \delta \implies |f(x) - l| < \varepsilon$

The function defined in $I_-(x_0)$ is **continuous on the left** *or* **left -continuous** *at x_0 if* $\lim_{x \to x_0^-} f(x) = f(x_0)$.

Examples.

- The mantissa function $M(x)$ is right-continuous at $x_0 = 1$, since $\lim_{x \to 1^+} M(x) = 0 = M(1)$, but it is not left-continuous since $\lim_{x \to 1^-} M(x) = 1 \neq M(1)$.
- The sign function $\operatorname{sign} x$ is neither right-continuous nor left-continuous at $x_0 = 0$, since $\lim_{x \to 0-} \operatorname{sign} x = -1 \neq \operatorname{sign} 0$ and $\lim_{x \to 0^+} \operatorname{sign} x = 1 \neq \operatorname{sign}(0)$.

Jump discontinuity

Definition

Let f be a function defined in $I(x_0) \setminus \{x_0\}$. We say that f has a **jump (discontinuity)** *at x_0 if*

- *the limits* $\lim_{x \to x_0^+} f(x)$ *and* $\lim_{x \to x_0^-} f(x)$ *exist and they are finite*
- $\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$

The quantity $\lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x)$ *is the* **jump** *of the function at x_0.*

Other limits of elementary functions

$$\lim_{x\to\pm1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm1)$$

$$\lim_{x\to+1} \arccos x = 0 = \arccos 1\,, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$$

$$\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$$

Limits and neighbourhoods

We have given several definitions of limit: nine for limits and six for one-sided limits.
All these definitions can be resumed in one definition, using a suitable notation for points and neighbourhoods:

- The Greek letter γ indicates the limit point ($x_0 \in \mathbb{R}$, $+\infty$, $-\infty$)
- The Greek letter λ indicates the limit ($l \in \mathbb{R}$, $+\infty$, $-\infty$)
- The symbol $I(\gamma)$ indicates the neighbourhood of the limit point γ; we have different cases:
  - Neighbourhood of $x_0 \in \mathbb{R}$: $I_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$
  - Neighbourhood of $+\infty$: $I_a(+\infty) = (a, +\infty)$
  - Neighbourhood of $-\infty$: $I_a(-\infty) = (-\infty, a)$
  - Right neighbourhood of $x_0 \in \mathbb{R}$: $I_\delta^+(x_0) = (x_0, x_0 + \delta)$
  - Left neighbourhood of $x_0 \in \mathbb{R}$: $I_\delta^-(x_0) = (x_0 - \delta, x_0)$

$x \to \gamma \begin{cases} +\infty \\ -\infty \\ x_0 \end{cases} \qquad \lambda \begin{cases} +\infty \\ -\infty \\ \ell \end{cases}$

$\lim_{x\to+\infty} f(x) = \ell$

$$\left[\forall \varepsilon > 0 \quad \exists B \geq 0 \quad \forall x : \left.\begin{matrix} x \in \text{dom} f \\ x > B \\ (I_B(+\infty)) \end{matrix}\right\} \Rightarrow \begin{matrix} |f(x) - \ell| < \varepsilon \\ (f(x) \in I_\varepsilon(\ell)) \end{matrix} \right.$$

$$= \left[\forall I_\varepsilon(\ell) \quad \exists I_B(+\infty) \quad \forall x : \left.\begin{matrix} x \in \text{dom} f \\ x \in I_B(+\infty) \end{matrix}\right\} \right] \Rightarrow f(x) \in I_\varepsilon(\ell)$$

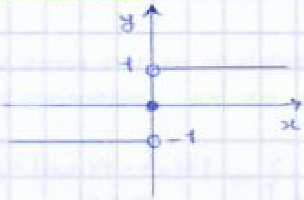
$$\forall I(\lambda) \quad \exists I(\gamma) \quad \forall x : x \in \text{dom} f \cap I(\gamma) \setminus \{\gamma\} \Rightarrow f(x) \in I(\lambda)$$

— — — — — —

$\lim_{x\to x_0} f(x) = \ell \qquad x_0 \in \mathbb{R} \quad \ell \in \mathbb{R}$

$f(x) = \text{sign}\, x = \text{sgn}\, x$ SIGN or SIGNUM

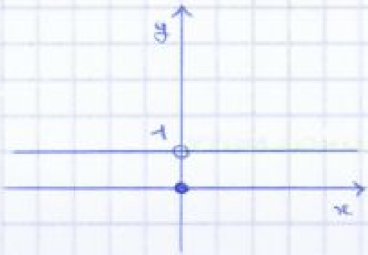
$$= \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$



$\text{dom} f = \mathbb{R}$

$\text{im} f = \{-1; 0; 1\}$

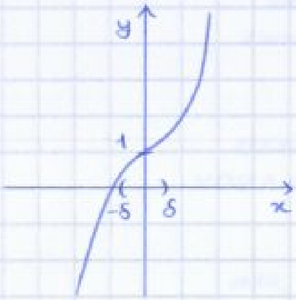
$g(x) = \text{sign}(x^2)$



$$\rightarrow \begin{cases} 1 & \text{if } x > 0 \vee x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\lim_{x\to x_0} f(x) = \ell$

1) $f(x) = x^3 + 1$



$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^3 + 1 = 1$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x : \left.\begin{matrix} x \in \text{dom} f \\ 0 < |x - x_0| < \delta \end{matrix}\right\} \Rightarrow |x^3 + \cancel{1} - \cancel{1}| < \varepsilon$$

$|x|^3 < \varepsilon \qquad |x| < \sqrt[3]{\varepsilon} \qquad \delta \leq \sqrt[3]{\varepsilon}$

$\boxed{f \text{ defined in } I(x_0)} \xrightarrow[\text{the point}]{\text{including}} \boxed{\lim_{x\to x_0} f(x)=\ell\in\mathbb{R}}$

① $\ell = f(x_0)$
f CONTINUOUS

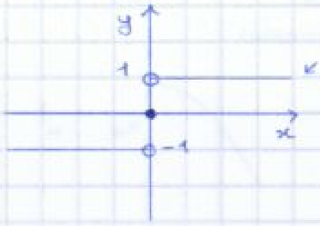
② $\ell \neq f(x_0)$
REMOVABLE
SINGULARITY

③ $\boxed{f \text{ is defined in } I(x_0)\setminus\{x_0\}}$

$\downarrow$

$\boxed{\lim_{x\to x_0} f(x)=\ell\in\mathbb{R}}$
- REMOVABLE SINGULARITY
- CONTINUOUS PROLONGATION

CONTINUITY AND DISCONTINUITY

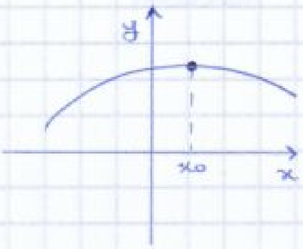


constant function

$\lim_{x\to x_0^+} f(x)=\ell$ $\qquad \lim_{x\to x_0^-} f(x)=m$

$\lim_{x\to 0^+} \operatorname{sign} x = 1$ $\qquad \lim_{x\to 0^-} \operatorname{sign} x = -1$

1)



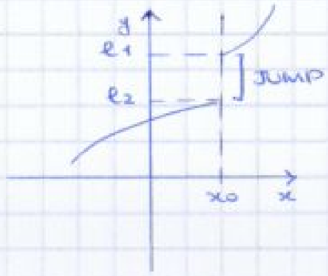
$\lim_{x\to x_0^+} f(x)=\ell_1$

$\lim_{x\to x_0^-} f(x)=\ell_2$

$\ell_1=\ell_2 \Rightarrow \lim_{x\to x_0} f(x)=\ell_1=\ell_2$

f is CONTINUOUS

2)



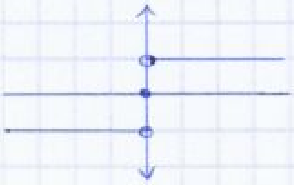
$\lim_{x\to x_0^+} f(x)=\ell_1$

$\lim_{x\to x_0^-} f(x)=\ell_2$

$\ell_1\neq\ell_2$ DISCONTINUITY OF THE FIRST KIND

Jump discontinuity $=|\ell_1-\ell_2|$

ex:



$\lim_{x\to 0^+} \operatorname{sign} x = 1$

$\lim_{x\to 0^-} \operatorname{sign} x = -1$

Jump $= 2$ at $x_0 = 0$

Mathematical Analysis I (2013-2014)

Limits 3 - Limits and algebraic operations

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October 2013

Algebra of limits - Finite limits

We study here the relations between limits and algebraic operations of functions. All the functions are supposed to be defined in $I(\gamma) \setminus \gamma$.

Theorem

If $\lim_{x\to\gamma} f(x) = l$ *and* $\lim_{x\to\gamma} g(x) = m$ *then*

$$\begin{aligned} \lim_{x\to\gamma} (f(x) \pm g(x)) &= l \pm m, \\ \lim_{x\to\gamma} f(x)\,g(x) &= l\,m, \\ \lim_{x\to\gamma} \frac{f(x)}{g(x)} &= \frac{l}{m} \text{ if } m \neq 0. \end{aligned}$$

Continuity of elementary inverse functions

In order to study the continuity of inverse functions we need the following result.

Theorem

If the function f is continuous and invertible on an interval I, then its inverse function f^{-1} is continuous on the interval $J = f(I)$.

Using this result, we can prove the following theorem.

Theorem

The results about continuity of elementary inverse functions are summarized in the following statement.

- *The root function $y = \sqrt[n]{x}$ is continuous in its domain for all $n \geq 2$.*
- *The logarithmic function is continuous on $(0, +\infty)$.*
- *The functions arcsine, arccosine and arctangent are continuous in their domains.*

Continuity of elementary inverse functions

A simple, but very useful result is given by the following theorem.

Theorem

Suppose that two functions $f(x)$ and $g(x)$ are equal in a neighbourhood $I(\gamma) \setminus \{\gamma\}$ and that $\lim_{x \to \gamma} f(x)$ exists. Then also $\lim_{x \to \gamma} g(x)$ exists and the two limits are equal.

Example. Consider the function

$$f(x) = \begin{cases} \cos x & \text{if } x \leq 0 \\ 1 + x^2 & \text{if } x > 0 \end{cases}$$

Using this result we can immediately say that it is continuous in $(-\infty, 0)$ and in $(0, +\infty)$; then we have to prove only the continuity at $x = 0$.

Algebra of limits - The reciprocal

$f(x)$	$1/g(x)$	Note
$m \neq 0$	$1/m$	
∞	0	
0		(1)

(1) If $f(x) > 0$ in $I(\gamma) \setminus \{\gamma\}$ the limit is $+\infty$; if $f(x) < 0$ in $I(\gamma) \setminus \{\gamma\}$ the limit is $-\infty$. If the function has no constant sign in $I(\gamma) \setminus \{\gamma\}$ sometimes it is possible to consider the right and the left neighbourhood and check the one-sided limits.

Algebra of limits - The quotient

We study the limit of the quotient of two functions using the previous result and writing the quotient as:

$$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$$

The possible indeterminate forms of the product written above are:
1) $0 \cdot \infty$ (this happens when $f(x) \to 0$ and $g(x) \to 0$),
2) $\infty \cdot 0$ (this happens when $f(x) \to \infty$ and $g(x) \to \infty$).
These indeterminate forms are denoted using the symbols:

$$\frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty}$$

Polynomial functions at infinity

We know the

$$\lim_{x\to+\infty} x^n = +\infty, \qquad \lim_{x\to-\infty} x^n = +\infty \ (n \text{ even}), \qquad \lim_{x\to-\infty} x^n = -\infty \ (n \text{ odd}).$$

If we consider the **polynomial function** of degree n

$$P(x) = a_n x^n + \cdots + a_1 x + a_0 \qquad (a_n \neq 0)$$

for $x \to \pm\infty$ it is possible to have an indeterminate form $\infty - \infty$.

In order to solve it, we factor out the monomial of highest degree $a_n x^n$:

$$P(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

The expression in the round brackets tends to 1 for $x \to \pm\infty$; then

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} a_n x^n = \infty.$$

The sign of the limit depends on the sign of a_n, on the value of n (odd or even) and on the limite point ($+\infty$ or $-\infty$).

Limits of rational functions - 1

Given two polynomial functions $P(x)$ and $Q(x)$ a **rational function** is a function of the form:

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} \qquad (a_n,\ b_m \neq 0,\ m > 0)$$

Acting on numerator and denominator as done in the polynomial case, we have that

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m} = \\ = \begin{cases} \infty & \text{if } n > m, \\ \dfrac{a_n}{b_m} & \text{if } n = m, \\ 0 & \text{if } n < m. \end{cases}$$

LIMITS AND ALGEBRAIC OPERATIONS

$f(x), \; g(x) \qquad f(x) \pm g(x)$

$\qquad\qquad f(x) \cdot g(x)$

$\qquad\qquad f(x) / g(x)$

1) finite limite
2) continuous function (defined also at x_0)
3) finite / infinite

- **THEOREM** f, g DEFINED in $I(\gamma) \setminus \{\gamma\}$ $\qquad \gamma \begin{cases} +\infty \\ -\infty \\ x_0 \end{cases}$

$\lim_{x \to \gamma} f(x) = l \in \mathbb{R}, \; \lim_{x \to \gamma} g(x) = m \in \mathbb{R}$

$\Rightarrow$ 1) $\lim_{x \to \gamma} f(x) \pm g(x) = l \pm m$

2) $\lim_{x \to \gamma} f(x) \cdot g(x) = l \cdot m$

3) $m \neq 0 \rightarrow \lim_{x \to \gamma} \frac{f(x)}{g(x)} = \frac{l}{m}$

- **THEOREM** f, g DEFINED in $I(x_0)$

f, g continuous at x_0 $\qquad \lim_{x \to x_0} f(x) = f(x_0)$

$\qquad \lim_{x \to x_0} g(x) = g(x_0)$

$\lim_{x \to x_0} f(x) + g(x) = f(x_0) + g(x_0)$

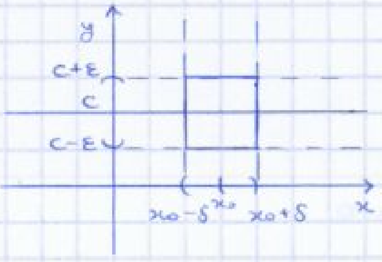
1) $f(x) \pm g(x)$ is continuous at x_0
2) $f(x) \cdot g(x)$ is continuous at x_0
3) $\frac{f(x)}{g(x)}$ is continuous at x_0 if $g(x) \neq 0$

$A \subseteq \mathbb{R}$ CONTINUOUS at a point x_0

f is continuous in $A \subseteq \mathbb{R} \Leftrightarrow f$ is continuous at $x_0 \; \forall x_0 \in A$

$f \in \underbrace{C^0(A)}_{\text{set of function continuous in } A}$

1) $f(x) = c$



f continuous

$\lim_{x \to x_0} f(x) = c$

$\left.\begin{matrix} \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x : x \in dom f \\ 0 < |x - x_0| < \delta \end{matrix}\right\} \Rightarrow |f(x) - c| < \varepsilon$

$|c - c| < \varepsilon$

$\underline{\underline{0 < \varepsilon}}$

$f(x) \in C^0(\mathbb{R})$

- THEOREM

  f continuous on an interval I , $f(I) = J$
  f is invertible in I
  $\Rightarrow f^{-1}$ is continuous in J

ex: $f(x) = \sin x \quad C^0(\mathbb{R})$ continuous in $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

f is invertible in $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$f^{-1} = \arcsin x \quad \in C^0([-1;1])$

5) ROOT FUNCTIONS

$y = \sqrt[n]{x}$ n-th root of x

$\log_a x$ continuous $\in C^0((0,+\infty))$
$a \neq 1 \quad a > 0$

$$\lim_{x\to 3} \frac{\log_2 x+1}{\log_2 x-1} = \frac{\log_2 3+1}{\log_2 3-1} = 2$$

SUM

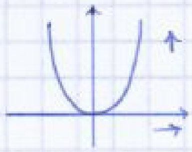
f(x)	g(x)	f(x) + g(x)
l	m	$l + m$
l	$+\infty$	$+\infty$
l	$-\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$
$-\infty$	$-\infty$	$-\infty$
$+\infty$	$-\infty$	$[+\infty - \infty]$

→ INDETERMINATE FORM

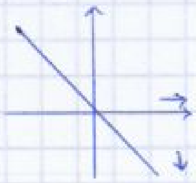
Example:

$f(x) = x^2 \qquad g(x) = -x$

$\lim_{x\to+\infty} x^2 = +\infty$



$\lim_{x\to+\infty} -x = -\infty$



$\lim_{x\to+\infty} x^2 - x = +\infty$

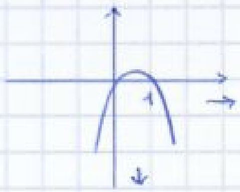
$f(x) = -x^2$

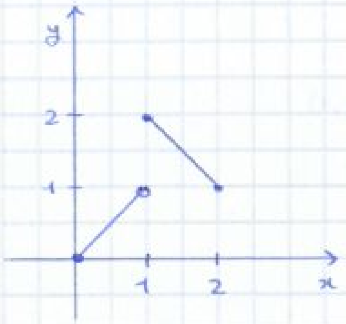
$\lim_{x\to+\infty} -x^2 = -\infty$

$g(x) = x$

$\lim_{x\to+\infty} x = +\infty$

$\lim_{x\to+\infty} x - x^2 = -\infty$



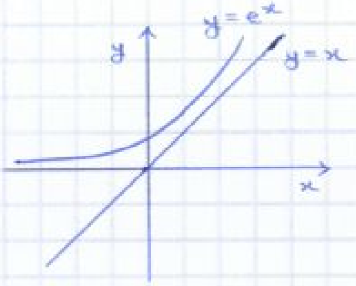


$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 3-x & \text{if } 1 \leq x \leq 2 \end{cases}$$

$f: [0;2] \to [0;2]$

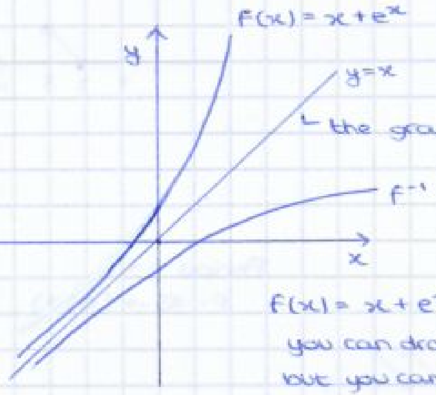
this function is injective, but it's not strictly monotone

$f(x) = x + e^x$



$f(x) = x$
$g(x) = e^x$

f, g are monotone strictly increasing
→ f + g → monotone strictly increasing



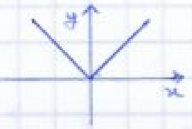
└ the graph is above this line

$f: \mathbb{R} \to \mathbb{R}$
strictly monotone → injective
⇒ invertible:
$f^{-1}: \mathbb{R} \to \mathbb{R}$

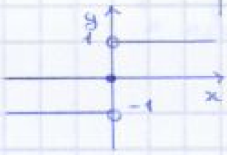
$f(x) = x + e^x$
you can draw the graph
but you cannot do the expression
$x + e^x = y$ solve for x → it's impossible

PIECEWISE DEFINED FUNCTIONS

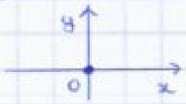
1) $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$



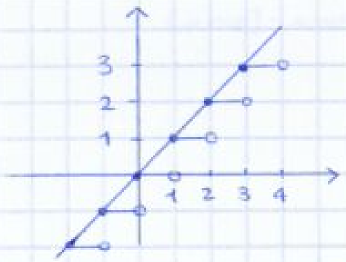
2) $\text{sign } x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$



3) $f(x) = [x]$ = greatest integer $\leq x$
integer part of x = floor(x)
$x = 0$

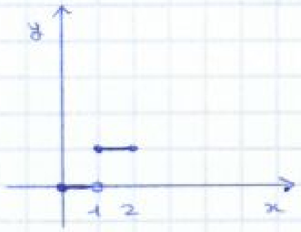


greatest integer ≤ 0
$= \{\ldots -4, -3, -2, -1, 0\}$ → the greatest is 0



$[x] \leq x$ f is always below the line $y = x$

If I take $x \in (0,1)$
$x - 1 < [x] \leq x$
$f: \mathbb{R} \to \mathbb{Z}$

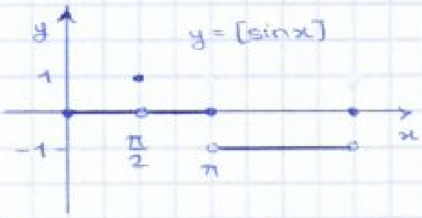


$\lim_{x\to 1^-} [x] = 0$ $\lim_{x\to 1^+} [x] = 1$

$[1] = 1$ $\underbrace{\qquad\qquad\qquad}$

Jump discontinuity $= 1$ (positive jump)

→ Right continuous function



$\left.\begin{array}{l}\lim_{x\to \frac{\pi}{2}} [\sin x] = 0 \\ \left[\sin \frac{\pi}{2}\right] = 1\end{array}\right)$ removable singularity

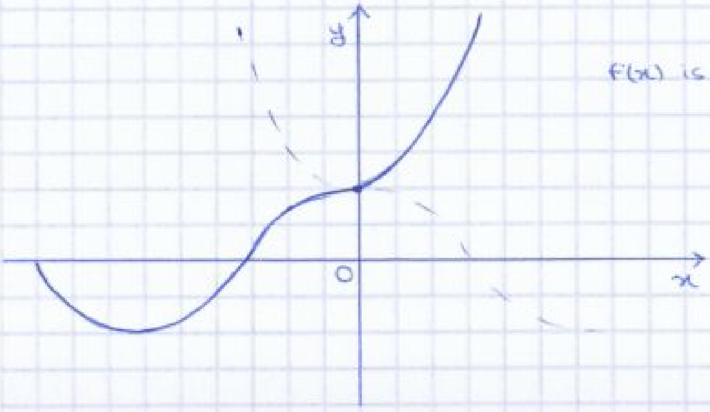
$\nearrow l = 0 = f(\pi)$

$\left.\begin{array}{l}\lim_{x\to \pi^-} [\sin x] = 0 \\ \lim_{x\to \pi^+} [\sin x] = -1\end{array}\right\}$ Jump discontinuity $J = -1$ $\left\{\begin{array}{l}\text{Removable singularity} \\ \text{and jump discontinuity}\end{array}\right.$

↳ left continuous function

$f(x) = \begin{cases} \cos x & x \leq 0 \\ 1 + x^2 & x > 0 \end{cases}$

- what happens at $x_0 = 0$?
- is this function continuous?



f(x) is continuous in $x = 0$

Mathematical Analysis I (2013-2014)

Limits 4 - Composition of functions

Paolo Boieri

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Composition of functions

Let X, Y and Z be three sets and $f : \operatorname{dom} f \subseteq X \to Y$ and $g : \operatorname{dom} g \subseteq Y \to Z$.

Definition. The composition of f and g is the function $h = g \circ f$ (read "g composed with f") defined as $h(x) = (g \circ f)(x) = g(f(x))$.

Remarks.

- It is possible to define the composition if and only if
  $$\operatorname{dom} g \cap \operatorname{im} f \neq \emptyset.$$
- We have that
  $$x \in \operatorname{dom}(g \circ f) \iff x \in \operatorname{dom} f \text{ and } f(x) \in \operatorname{dom} g.$$
- The composition (in general) is not commutative: $g \circ f \neq f \circ g$.

Limit of a composition - 1

We study the problem of the limit of a composition of two functions. We begin with some particular cases.

Theorem

Suppose that:

1. $\lim_{x\to\gamma} f(x) = l$;
2. *g is defined in $I(l)$ and continuous at l;*

then the composition $g(f(x))$ has a limit and we have that

$$\lim_{x\to\gamma} g(f(x)) = g(l) = g(\lim_{x\to\gamma} f(x)).$$

Limit of a composition - 2

Corollary

Let f be continuous at x_0. If $y_0 = f(x_0)$ and the function g is defined in a neighbourhood $I(y_0)$ and continuous at y_0. Then the composition $g(f(x))$ is continuous at x_0.

Remark. The identity

$$\lim_{x\to\gamma} g(f(x)) = g(\lim_{x\to\gamma} f(x)) = g(l).$$

can be put into words saying that a continuous function commutes (exchanges places) with the symbol of limit.

Examples - 3

- Compute $\lim_{x\to+\infty} \log\sin\frac{1}{x}$.

  Setting $h(x) = \sin\frac{1}{x}$, we observe that $h(x) > 0$ for $x \geq 1/\pi$; then the function is defined in a neighbourhood of $+\infty$. With the substitution $y = 1/x$ we have

$$\lim_{x\to+\infty} \sin\frac{1}{x} = \lim_{y\to0+} \sin y = 0$$

  With a second substitution $z = \sin\frac{1}{x}$ we have that

$$\lim_{x\to+\infty} \log\sin\frac{1}{x} = \lim_{z\to0+} \log z = -\infty\ .$$

Limit of a composition - 2

We consider now the case of infinite limits.

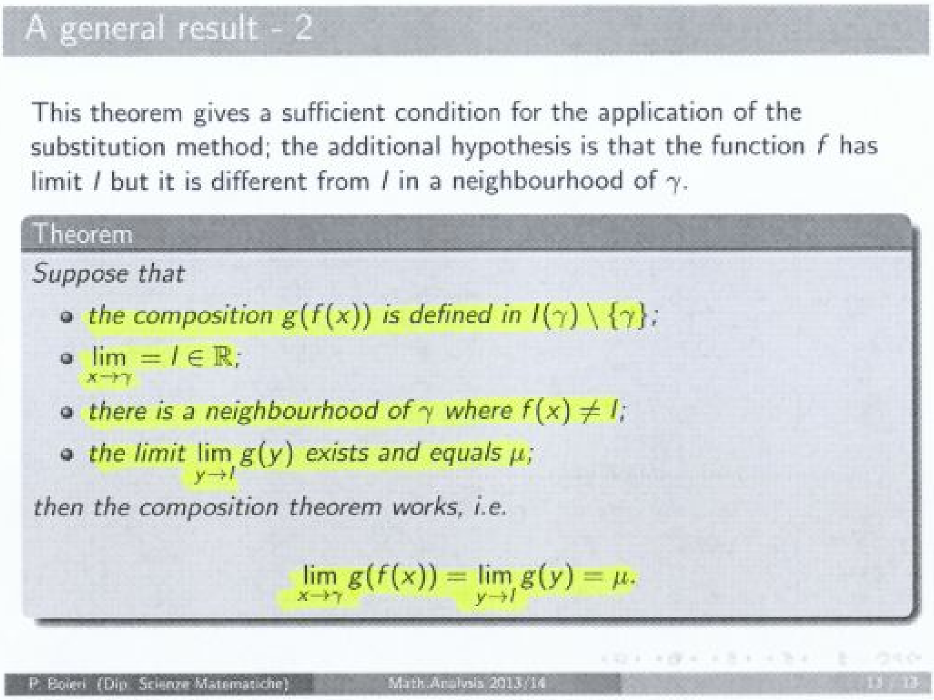
Theorem

Suppose that:

1. $\lim_{x\to\gamma} f(x) = +\infty$ *(respect.* $-\infty$*);*
2. *there exists* $\lim_{y\to+\infty} g(y)$ *(respect.* $\lim_{y\to-\infty} g(y)$*);*

then the composition $g(f(x))$ *has a limit and we have that*

$$\lim_{x\to\gamma} g(f(x)) = \lim_{y\to+\infty} g(y) \quad (\textit{respectively } \lim_{y\to-\infty} g(y))$$



A general result - 2

This theorem gives a sufficient condition for the application of the substitution method; the additional hypothesis is that the function f has limit l but it is different from l in a neighbourhood of γ.

Theorem

Suppose that

- *the composition* $g(f(x))$ *is defined in* $I(\gamma) \setminus \{\gamma\}$;
- $\lim_{x\to\gamma} = l \in \mathbb{R}$;
- *there is a neighbourhood of* γ *where* $f(x) \neq l$;
- *the limit* $\lim_{y\to l} g(y)$ *exists and equals* μ;

then the composition theorem works, i.e.

$$\lim_{x\to\gamma} g(f(x)) = \lim_{y\to l} g(y) = \mu.$$

SUBSTITUTION RULES FOR LIMITS

1) $\lim_{x\to\gamma} f(x) = \ell \in \mathbb{R}$ ‖ $\lim_{x\to x_0} f(x) = f(x_0)$ (particular case)

2) g is defined in $I(\ell)$ LIMIT OF A COMPOSITION
g is continuous at ℓ

$$\Rightarrow \underbrace{\lim_{x\to\gamma} g(f(x))}_{\text{limit of } g} = \underbrace{g(\lim_{x\to\gamma} f(x))}_{g \text{ of the limit}} = g(\ell)$$

it's possible to interchange the function and the limit (commute)

ex:

$\lim_{x\to 0} \arctan\left(\frac{\sin x}{x}\right)$ $f(x) = \frac{\sin x}{x}$ $g(x) = \arctan x$
$\rightarrow (g\circ f)(x) = \arctan\left(\frac{\sin x}{x}\right)$

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $g(x) = \arctan x$ → it's continuous at $\ell = 1$ and continuous at $I(\ell) = I(1)$
  ↳ continuous function in the domain ($\mathbb{R}$)

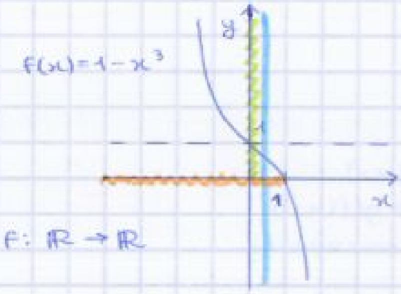
$$\lim_{x\to 0} \arctan\left(\frac{\sin x}{x}\right) = \arctan\left(\lim_{x\to 0} \frac{\sin x}{x}\right) = \arctan 1 = \frac{\pi}{4}$$

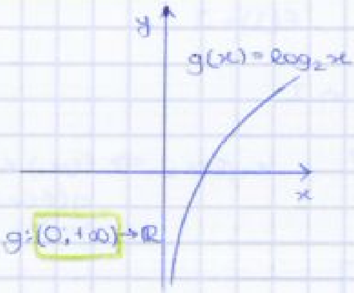
THEOREM:
1) f continuous at x_0 ($y_0 = f(x_0)$)
2) g defined in $I(y_0)$
g continuous at y_0

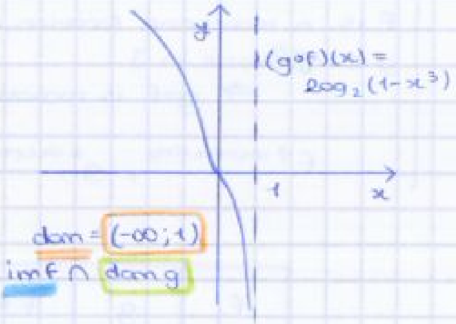
$\Rightarrow (g\circ f)(x)$ continuous at x_0

THE COMPOSITION OF A CONTINUOUS FUNCTION IS A CONTINUOUS FUNCTION

$h(x) = \log_2(1-x^3)$
↳ this function is a composition
$f(x) = 1-x^3$ $g(x) = \log_2 x$ $(g\circ f)(x) = \log_2(1-x^3)$



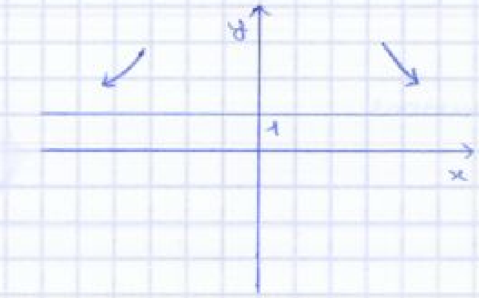




$\text{dom}\,(g\circ f)(x) = \text{im}f \cap \text{dom}g = (-\infty, 1)$

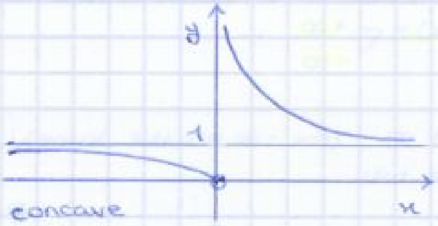
$\lim_{x\to -3} h(x) = h(-3)$ → because it's a continuous function
$= \log_2 28$

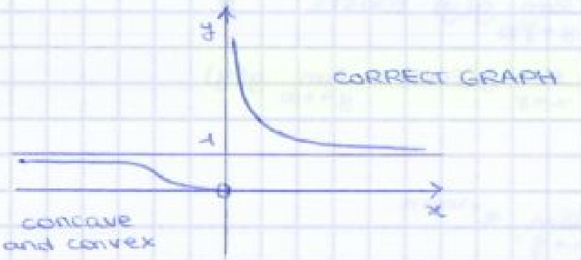
$\lim_{x\to\pm\infty} 2^{\frac{1}{x}}$ $y=\frac{1}{x} \xrightarrow{x\to+\infty} =0$ $\lim_{y\to 0} 2^y = 2^0 = 1$



What happens at 0?

- $\lim_{x\to 0^+} 2^{\frac{1}{x}} \overset{y=\frac{1}{x}}{=} \lim_{y\to+\infty} 2^y = +\infty$
- $\lim_{x\to 0^-} 2^{\frac{1}{x}} \overset{y=\frac{1}{x}}{=} \lim_{y\to-\infty} 2^y = 0$





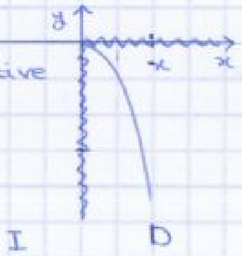
Following this method it's possible to find limits and monotonicity, but it's not possible to find concavity and convexity (→ derivatives)

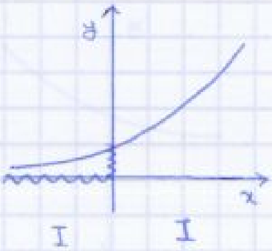
ex: $y=e^{-x^2}$ (very important function in the course of probability) GAUSS FUNCTION

$f(x)=-x^2 \in C^0(\mathbb{R})$
$g(x)=e^x \in C^0(\mathbb{R})$ $\longrightarrow$ $h(x)=e^{-x^2} \in C^0(\mathbb{R})$
$= (g\circ f)(x)$

- EVEN FUNCTION $h(-x)=h(x)$
  → I plot the graph only for $x\geq 0$ and I get the negative part by simmetry

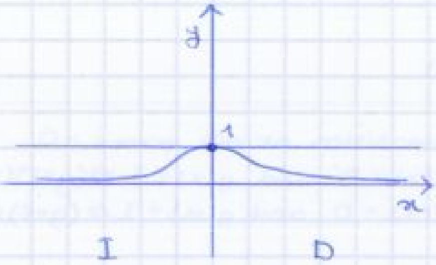
for each positive x, $f(x)<0$





for each negative x, $0<g(x)\leq 1$ $(0,1]$

⇒ D



$\lim_{x\to+\infty} e^{-x^2} \overset{y=-x^2}{=} \lim_{\to-\infty} e^y = 0$

Mathematical Analysis I (2011-2012)

Limits 5 - Theorems on limits

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October 2013

Uniqueness of the limit

We start studying some general results about limits: the first one ensures us that the limit process in well defined, i.e. we never end up with two or more limits.

Theorem (Uniqueness of the limit)

Suppose that f admits limit λ as $x \to \gamma$. Then f admits no other limit for $x \to \gamma$ (in other words: if limit exists, it is unique).

Proof: Textbook 4.1.1 page 90

The "Sign and limit" theorem

The two following results concern the relation between the limit and the function in a neighbourhood of the limit point.

Theorem ("Sign and limit" theorem)

Suppose that the function f has a limit λ for $x \to \gamma$. If $\lambda = l > 0$ or $\lambda = +\infty$, then there exists a neighbourhood $I(\gamma)$ of γ such that $f(x) > 0, \ \forall x \in I(\gamma) \setminus \{\gamma\}$.
In the same way, if $\lambda = l < 0$ or $\lambda = -\infty$, then there exists a neighbourhood $I(\gamma)$ of γ such that $f(x) < 0, \ \forall x \in I(\gamma) \setminus \{\gamma\}$.

Proof: Textbook 4.1.1 page 90
Remark. From the proof we see immediately that a stronger version of this theorem holds: if $\lambda = l > 0$ then there exists a neighbourhood where $f(x) > h, \ \ \forall 0 \leq h < l$. When $\lambda = +\infty$ we can find a neighbourhood where $f(x) > h, \ \forall h \geq 0$.

Corollary

Corollary

Suppose f admits a limit λ for $x \to \gamma$.
If there exists a neighbourhood $I(\gamma)$ such that $f(x) \geq 0$ in $I(\gamma) \setminus \{\gamma\}$, then $\lambda = l \geq 0$ or $\lambda = +\infty$.
A similar assertion holds when $f(x) \leq 0$ in $I(\gamma) \setminus \{\gamma\}$.

Proof: Textbook 4.1.1 page 91

Remark. If we suppose $f(x) > 0$ in $I(\gamma) \setminus \{\gamma\}$ we can not conclude that the limit is strictly positive. For instance, the function $f(x) = 1/x$ is strictly positive in all $I_a(+\infty)$, but $\displaystyle\lim_{x \to +\infty} \frac{1}{x} = 0$.

Second comparison theorem - the infinite case

Theorem

Suppose that f and g are defined in $I_1(\gamma) \setminus \{\gamma\}$ and there is a neighbourhood $I_2(\gamma)$ such that $f(x) \leq g(x), \forall x \in I_2(\gamma) \setminus \{\gamma\}$. Then:

- $\lim_{x\to\gamma} f(x) = +\infty \quad \Rightarrow \quad \lim_{x\to\gamma} g(x) = +\infty$
- $\lim_{x\to\gamma} g(x) = -\infty \quad \Rightarrow \quad \lim_{x\to\gamma} f(x) = -\infty$

Two corollaries

Corollary

$\lim_{x\to\gamma} f(x) = 0$ *if and only if* $\lim_{x\to\gamma} |f(x)| = 0$.

Corollary

Suppose that

- *the function $f(x)$ is* **bounded** *in $I(\gamma) \setminus \{\gamma\}$, i.e.*

$$\exists C > 0 : |f(x)| \leq C, \quad \forall x \in I(\gamma) \setminus \{\gamma\}.$$

- *the function $g(x)$ is* **infinitesimal** *for $x \to \gamma$, i.e.* $\lim_{x\to\gamma} g(x) = 0$.

Then the product function $f(x)g(x)$ is infinitesimal for $x \to \gamma$, i.e.

$$\lim_{x\to\gamma} f(x)g(x) = 0.$$

Other trigonometrical limits

Using the fundamental limit we can obtain other important limits involving the trig. and inverse trig. functions.

- $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$;
- $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$;
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$;
- $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$;
- $\lim_{x\to 0} \frac{\arctan x}{x} = 1$.

Other examples - 1

- We prove that $\lim_{x\to +\infty} \frac{\sin x}{x}$.
  - We recall that $-1 \leq \sin x \leq 1, \forall x \in \mathbb{R}$ (i.e. $\sin x$ is bounded in $\mathbb{R}$;
  - we have that $\lim_{x\to +\infty} \frac{1}{x} = 0$;
  - then applying the corollary of the second comparison theorem we get the result.
- We prove that $\lim_{x\to +\infty} (x + \sin x) = +\infty$.
  - We use again the fact that $-1 \leq \sin x \leq 1, \forall x \in \mathbb{R}$;
  - then we have that $x - 1 \leq x + \sin x, \forall x \in \mathbb{R}$;
  - applying the second comparison theorem (infinite case) we get the result.

THEOREMS ON LIMITS

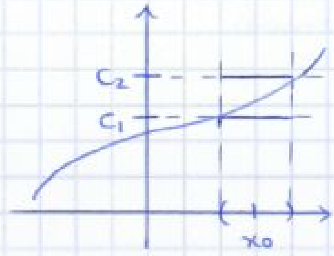
1) UNIQUENESS OF LIMIT (page 90)

Suppose that f admits a limit $(\ell, \pm\infty)$ λ as $x \to \gamma$
Then this limit is unique.

2) LOCAL BOUNDEDNESS

Suppose that $\lim\limits_{x \to \gamma} f(x) = \ell \in \mathbb{R}$

Then there exists a neighbourhood $I(\gamma)$ such that
$f(x)$ is bounded in $I(\gamma) \setminus \{\gamma\}$
$(\exists c_1, c_2 : c_1 \leq f(x) \leq c_2$



<u>PROOF</u>: $\lim\limits_{x \to \gamma} f(x) = \ell$

$\forall \varepsilon > 0 \quad \exists I(\gamma)$

$$\left.\begin{array}{l} \forall x : x \in \text{dom} f \\ x \in I(\gamma) \setminus \{\gamma\} \end{array}\right\} \Rightarrow |f(x) - \ell| < \varepsilon$$

This is true $\forall \varepsilon$, so I can choose any ε

$$\varepsilon = 1 \quad \exists I_1(\gamma) \quad \left.\begin{array}{l} \forall x : x \in \text{dom} f \\ x \in I_1(\gamma) \setminus \{\gamma\} \end{array}\right\} \Rightarrow |f(x) - \ell| < 1$$

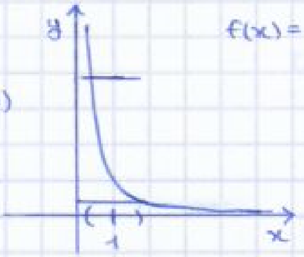
$-1 < f(x) - \ell < 1$

$\ell - 1 < f(x) < \ell + 1$

$f(x)$ is bounded

Example:

we consider only $(0; +\infty)$



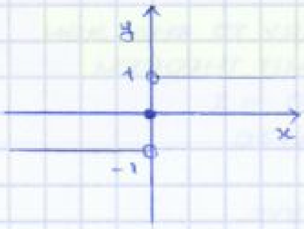
$f(x) = \frac{1}{x}$

$\lim\limits_{x \to 1} \frac{1}{x} = 1$

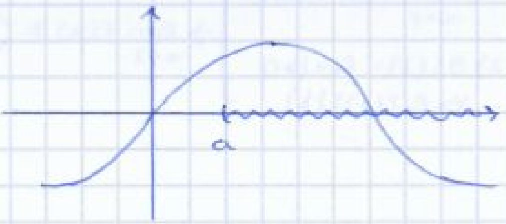
I have to take radius < 1, because otherwise the function is not bounded

$\lim\limits_{x \to \gamma} f(x) = \ell \in \mathbb{R} \Rightarrow f$ is bounded in $I(\gamma) \setminus \{\gamma\}$

$\Leftarrow$ is the opposite true? $\underline{\underline{\text{NO}}}$



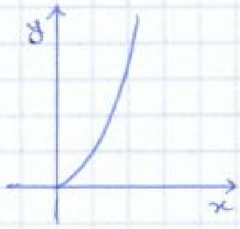
Bound $-1 \leq \text{sign}x \leq 1$
$R(x)$ is bounded in $I(0)$
BUT $\lim\limits_{x \to 0} \text{sign}x \nexists$ (they're different $x \to 0^+$ $x \to 0^-$)



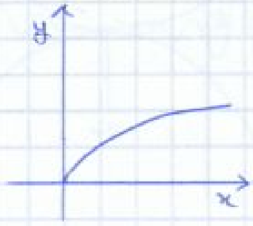
$f(x) = \sin x$ Bounded in $I_a(+\infty)$
$\lim\limits_{x \to +\infty} \sin x \nexists$

Examples:

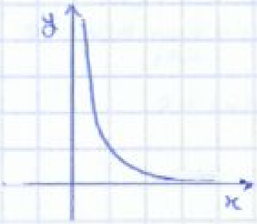
1) $f(x)=x^2$



2) $f(x)=\arctan x$



3) $f(x)=\frac{1}{x}$



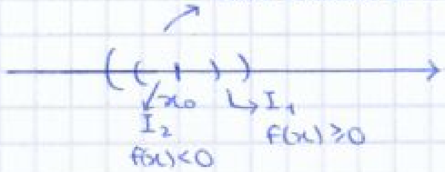
PROOF given by contradiction → I suppose the 2 points (conditions) and the negation of the result

$$\begin{cases} 1) \lim_{x\to\gamma} f(x) \text{ EXISTS} \\ 2) \exists I_1(\gamma): f(x)\geq 0 \quad \forall x\in I_1(\gamma)/\{\gamma\} \\ 3) \lim_{x\to\gamma} f(x) = \begin{cases} -\infty \\ l<0 \end{cases} \end{cases}$$

I apply the sign and limit theorem

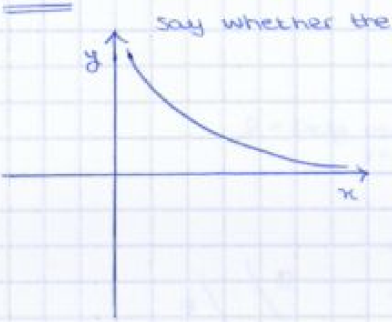
$3) \Rightarrow \exists I_2(\gamma): \quad f(x)<0 \quad \forall x\in I_2(\gamma)\setminus\{\gamma\}$

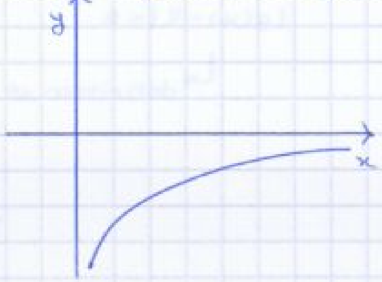
this should be : $f(x)\geq 0$ at the same time → extremely impossible
$f(x)<0$

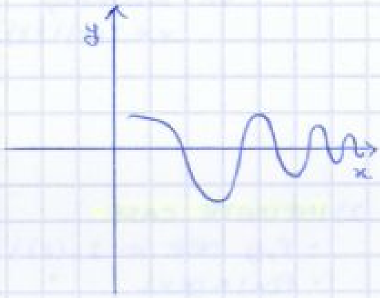


$I = I_1 \cap I_2$

l = 0 If I know that the limit is 0 it's impossible to say whether the function is positive or negative







I COMPARISON THEOREM

- f, g DEF in $I(\gamma)\setminus\{\gamma\}$
- $\lim_{x\to\gamma} f(x)=\lambda, \quad \lim_{x\to\gamma} g(x)=\mu$
- $\exists I(\gamma): \; f(x)\leq g(x)$

$\Rightarrow \quad \lambda \leq \mu$

$\lambda=-\infty \quad \lambda\leq\mu$ → there's nothing to prove
$\mu=+\infty \quad \lambda\leq\mu$

$f(x)-g(x)\leq 0$
$\Downarrow$
$\lambda-\mu\leq 0 \quad \lambda\leq\mu$

TH

- $f(x)$ BOUNDED in $I(\gamma)\setminus\{\gamma\}$
- $\lim_{x\to\gamma} g(x)=0$

$\Rightarrow \lim_{x\to\gamma} f(x)g(x)=0$

A bounded function does not have necessarily a limit, and even if it has not one I can say something about the limit of the product of this function and another one that has limit =0

A function whose limit is 0 is called an INFINITESIMAL FUNCTION

$\rightarrow \lim_{x\to\gamma} g(x)=0 \qquad g(x)=$ infinitesimal

BOUNDED × INFINITESIMAL = INFINITESIMAL

PROOF: $f(x)$ is bounded when $c_1\le f(x)\le c_2$ $\qquad |f(x)|\le c$

$-c\le f(x)\le c$

$|f(x)|\le c \qquad |f(x)\,g(x)|=|f(x)|\cdot|g(x)|$

$$\underset{\substack{\downarrow\\0}}{0}\le \underset{\substack{\downarrow\\0}}{|f(x)\,g(x)|}=|f(x)|\cdot|g(x)|\le \underset{\substack{\downarrow\\0}}{c\cdot|g(x)|}$$

Then $|f(x)\,g(x)|\to0$

$f(x)\,g(x)\to0$

Examples:

- $\lim_{x\to+\infty}\frac{\sin x}{x}$ I can consider $\left|\frac{\sin x}{x}\right|$ because if I prove that the abs v. is 0, then the function without the abs v. is 0.

$\left|\frac{\sin x}{x}\right|=\frac{|\sin x|}{\textcircled{x}}\rightarrow$ I can consider $x>0$ because $x\to+\infty$

$$|\sin x|\le1 \rightarrow \underset{\substack{x\to+\infty\downarrow\\0}}{0}\le \boxed{\underset{\substack{\downarrow\\0}}{\frac{|\sin x|}{x}}}\le \underset{\substack{\downarrow x\to+\infty\\ \frac{1}{\infty}=0}}{\frac{1}{x}}$$

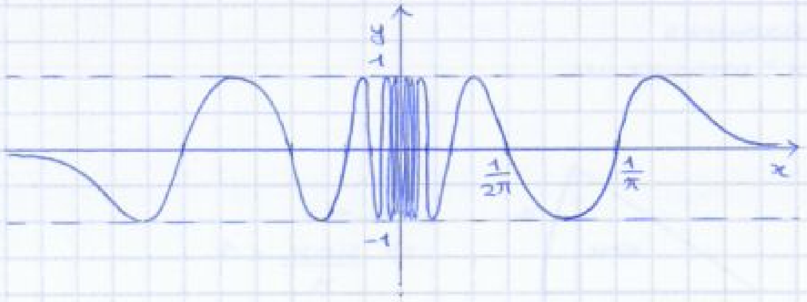
$\lim_{x\to+\infty}\left|\frac{\sin x}{x}\right|=0 \Rightarrow \lim_{x\to+\infty}\frac{\sin x}{x}=0$

- $f(x)=\sin\frac{1}{x}$ composition of $\sin x$ and $\frac{1}{x}$

$\text{dom} f=\mathbb{R}\setminus\{0\} \qquad -1\le\sin\frac{1}{x}\le1$ Bounded function

$\lim_{x\to+\infty}\sin\frac{1}{x}=\sin\frac{1}{\infty}=\sin0=0 \qquad \sin z=0 \quad z=k\pi$

$\sin\frac{1}{x}=0 \quad \frac{1}{x}=k\pi \quad x=\frac{1}{k\pi}$



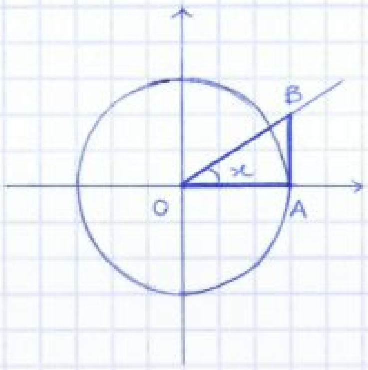
$f(-x)=\sin\frac{1}{-x}=-\sin\frac{1}{x}=-f(x)$

ODD function

I want to apply the comparison theorem

$$\begin{array}{ccc} \bigcirc & \dfrac{\sin x}{x} < & 1 \\ \downarrow & & \downarrow x\to 0 \\ & & 1 \end{array}$$

I need something that has limit 1 to prove that $\frac{\sin x}{x}$ goes to 1



Triangle $\hat{OAB}$

Area$_1$: $\frac{1\cdot \tan x}{2}$

Area$_2$: $\frac{1\cdot x}{2}$

Area$_2$ < Area$_1$ $\Rightarrow \frac{x}{\cancel{2}} < \frac{\tan x}{\cancel{2}} \Rightarrow \frac{\sin x}{\cos x} > x$

$\frac{\sin x}{x} > \cos x$

$$\Rightarrow \begin{array}{ccccc} \cos x & < & \boxed{\begin{array}{c}\frac{\sin x}{x}\\ \downarrow \\ 1\end{array}} & < & 1 \\ x\to 0^+\downarrow & & & & \downarrow x\to 0^+ \\ 1 & & & & 1 \end{array} \qquad x\in\left(0,\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{\sin x}{x} = 1$$

- $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} =$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} =$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{1+\cos x} = 1\cdot 1\cdot\frac{1}{2} = \frac{1}{2}$$

limit of a product
= product of the limit

$$\boxed{\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}}$$ Fundamental limit

- $\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{x}{x} = \lim_{x\to 0}\underset{\substack{\downarrow\\ \frac{1}{2}}}{\frac{1-\cos x}{x^2}}\cdot\underset{\substack{\downarrow\\ 0}}{x} = 0$

$$\boxed{\lim_{x\to 0}\frac{1-\cos x}{x} = 0}$$ Fundamental limit

Examples

1. $a_n = n^2, \quad n \geq 0$
   The first values of the sequence are:
   $a_0 = 0\,, \quad a_1 = 1\,, \quad a_2 = 4\,, \quad a_3 = 9$
   This sequence is the restriction of the function $f(x) = x^2$ to the set $\mathbb{N}$.
2. $a_n = \dfrac{n}{n^2+1}, \quad n \geq 1$
   The first values of the sequence are: $a_1 = \frac{1}{2}\,, \quad a_2 = \frac{2}{5}\,, \quad a_3 = \frac{3}{10}$
   This sequence is the restriction of the function $f(x) = \dfrac{x}{x^2+1}$ to the set $\mathbb{N} \setminus \{0\}$.
3. $a_n = (-1)^n = \begin{cases} +1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$
   The values of this sequence are $a_{2k} = 1$ and $a_{2k+1} = -1$.

Limit of a sequence - 1

Definition.

- We say that the sequence $a : n \mapsto a_n$ **tends to the limit** $l \in \mathbb{R}$ (or **converges to** l), and we write $\lim\limits_{n\to\infty} a_n = l$,
  if
  $$\forall \varepsilon > 0\,, \exists n_\varepsilon \; \forall n : n \geq n_0 \text{ and } n > n_\varepsilon \Longrightarrow |a_n - l| < \varepsilon .$$
- We say that the sequence $a : n \mapsto a_n$ **tends to** $+\infty$ (or **diverges to** $+\infty$), and we write $\lim\limits_{n\to\infty} a_n = +\infty$,
  if
  $$\forall A > 0\,, \exists n_A : \forall n : n \geq n_0 \text{ and } n > n_\varepsilon \Longrightarrow a_n > A \,.$$
- We say that the sequence $a : n \mapsto a_n$ **tends to** $-\infty$ (or **diverges to** $-\infty$), and we write $\lim\limits_{n\to\infty} a_n = -\infty$,
  if
  $$\forall A > 0\,, \exists n_A : \forall n \geq n_0,\, n > n_\varepsilon \Rightarrow a_n < -A \,.$$
- A sequence non convergent and non divergent is said to be **indeterminate**.

Properties of sequences

It is not always possible (or convenient) to consider a sequence as the restriction of a function. This happens when:

- we can define the sequence and not the function: for instance, the sequence $n \mapsto (-1)^n$ is defined for all $n \in \mathbb{N}$, while we can not define the function $f(x) = (-1)^x$;
- we do not know a function whose restriction to $\mathbb{N}$ is the given sequence: an example is the sequence $n \mapsto n!$;
- we know the sequence and the function, but it is easier to study the sequence.

For these reasons, it is necessary to have a theory of limit of sequences; this task is easier than expected, since it is possible to extend to sequences the majority of our definitions and results about functions, with the appropriate changes in terminology (Textbook pages 137-138).

1. uniqueness of the limit, the "sign and limit" theorem and its consequences;
2. the algebra of limits and the classification of indeterminate forms;
3. the comparison theorems.

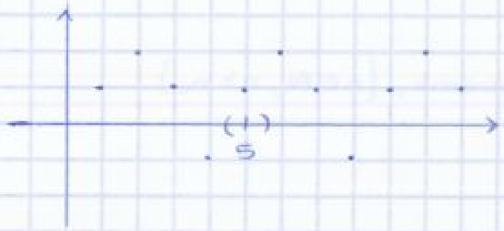
Some examples

- The sequence $a_n = n!$ diverges to $+\infty$.
  Since we have that $n! > n, \quad \forall n > 2$, we may apply the comparison theorem.
- The sequence $a_n = n + (-1)^n$ diverges to $+\infty$.
  Since $\forall n \in \mathbb{N}, \quad -1 \leq (-1)^n \leq 1$ we have that
  $$\forall n \in \mathbb{N}, \quad n - 1 \leq n + (-1)^n \leq n + 1.$$
  Applying the comparison theorem we reach the conclusion.
- Given $a \in \mathbb{R}$ the **geometric sequence** is the sequence $n \mapsto a^n$, $n \in \mathbb{N}$. We can prove that

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } -1 < a < 1 \\ \nexists & \text{if } a \leq -1 \end{cases}$$

Examples

- The function $y = \sin x$ does not have a limit for $x \to +\infty$.
  In fact, if we consider the sequences
  $$a_n = 2n\pi \quad \text{and} \quad b_n = \frac{\pi}{2} + 2n\pi, \qquad n \in \mathbb{N}$$
  we have that
  $$\lim_{n\to\infty} \sin a_n = \lim_{n\to\infty} 0 = 0 \quad \text{and} \quad \lim_{n\to\infty} \sin b_n = \lim_{n\to\infty} 1 = 1.$$
- The function $y = M(x)$ does not have a limit for $x \to 0$.
  In fact, if we consider the sequences
  $$a_n = -\frac{1}{n} \quad \text{and} \quad b_n = \frac{1}{n}, \qquad n \in \mathbb{N}$$
  we have that
  $$\lim_{n\to\infty} M(a_n) = \lim_{n\to\infty} 1 - \frac{1}{n} = 1 \quad \text{and} \quad \lim_{n\to\infty} M(b_n) = \lim_{n\to\infty} \frac{1}{n} = 0.$$



I can't define the limit of this sequence $\to S$, because there's no neighbourhood

But I can say when $\underline{\underline{n \to +\infty}}$

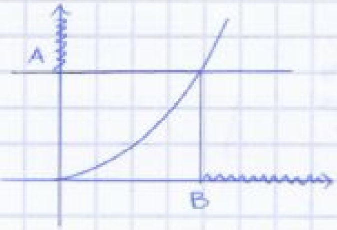
$\lim_{n \to +\infty} a_n$ or $\lim_{n \to \infty} a_n$

n is a natural number, and so positive, $n \to \infty$ means always $n \to +\infty$, because n is always positive

$$\lim_{x \to +\infty} f(x) = \begin{cases} +\infty \\ -\infty \\ l \end{cases}$$

$\lim_{n \to \infty} a_n = l$	the SEQUENCE is CONVERGENT
$\lim_{n \to \infty} a_n = \begin{cases} +\infty \\ -\infty \end{cases}$	the SEQUENCE is DIVERGENT (to $+\infty$ or to $-\infty$)
$\nexists \lim_{n \to \infty} a_n$	INDETERMINATE

$\lim_{x \to +\infty} f(x) = +\infty$ (First def of limits)

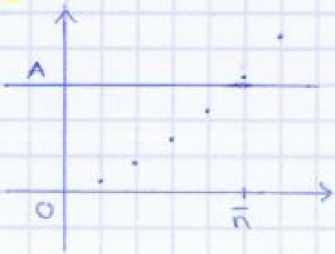


$\forall A > 0 \quad \exists B > 0$

$\forall x : \left. \begin{matrix} x \in \text{dom} f \\ x > B \end{matrix} \right\} \Rightarrow f(x) > A$

INFINITE LIMIT OF SEQUENCE

1) $\lim_{n \to \infty} a_n = +\infty$ (divergent at $+\infty$)



$\forall A > 0 \quad \exists \bar{n} \in \mathbb{N}$

$\forall n : n > \bar{n} \Rightarrow a_n > A$

2) $\lim_{n \to \infty} a_n = -\infty$ (divergent to $-\infty$)

$\forall A > 0 \quad \exists \bar{n} \in \mathbb{N}$

$\forall n : n > \bar{n} \Rightarrow a_n < -A$

- $\lim\limits_{n\to\infty} \dfrac{1}{\log(n^2+1)}$ $f(x) = \dfrac{1}{\log(x^2+1)}$ composition of 3 functions: $x \to x^2+1 \to \log(x^2+1) \to \dfrac{1}{\log(x^2+1)}$

1) $\lim\limits_{x\to+\infty} \underbrace{(x^2+1)}_{\hookrightarrow y} = +\infty$

2) $\lim\limits_{x\to+\infty} \log(x^2+1) \overset{y=x^2+1}{=} \lim\limits_{y\to+\infty} \log y = +\infty$

$\Rightarrow \lim\limits_{x\to+\infty} \dfrac{1}{\log(x^2+1)} = 0 \Rightarrow \lim\limits_{n\to\infty} \dfrac{1}{\log(n^2+1)}$

- $\lim\limits_{n\to\infty} \sqrt{n^2+1} - n$ $f(x) = \sqrt{x^2+1} - x$

$\lim\limits_{x\to+\infty} \sqrt{x^2+1} - x = [+\infty - \infty]$ (indeterminate form)

$\lim\limits_{x\to+\infty} \dfrac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} \to (a-b)(a+b) = a^2 - b^2$

$\lim\limits_{x\to+\infty} \dfrac{\cancel{x^2}+1-\cancel{x^2}}{\underset{\downarrow\,+\infty}{\sqrt{x^2+1}} + \underset{\downarrow\,+\infty}{x}} = \dfrac{1}{+\infty} = 0 \Rightarrow \lim\limits_{n\to\infty} \sqrt{n^2+1} - n = 0$

$\{a_n\} \Rightarrow f(x) = \underbrace{f(n) = a_n}$ sequences can be seen as restrictions of the functions

$\lim\limits_{x\to+\infty} f(x) = \lambda \Rightarrow \lim\limits_{n\to\infty} a_n = \lambda$ The sequence must be associated to the function

1) $\{a_n\} \longrightarrow f(x)$

$a_n = (-1)^n$ $\forall n \in \mathbb{N}$ $\longrightarrow$ $f(x) = (-1)^x$, $a>0$ this function does not exist, there's not an associated function

2) $n!$ FACTORIAL OF n

$n \geq 2$ $n! = 1 \cdot \ldots n$ $1! = 1$ $0! = 1$

$2! = 1 \cdot 2$

$3! = 1 \cdot 2 \cdot 3$

RECURVISE FUNCTION

$$\begin{cases} 0! = 1 \\ 1! = 1 \\ n! = n \cdot (n-1)! \end{cases}$$

ex: $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2! = 4 \cdot 3 \cdot 2 \cdot 1! = 4 \cdot 3 \cdot 2 \cdot 1$

$a_n = n! \to f(x)$ there is a corresponding function

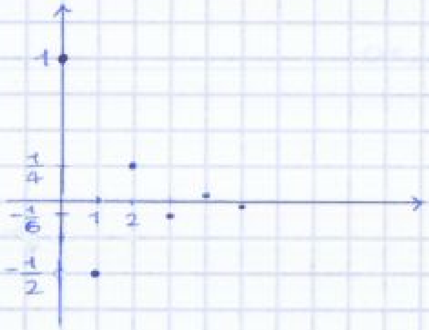
3) a_n $f(x)$ there is a sequence and a function corresponding to it, BUT the study of $f(x)$ is more difficult than studying the sequence

2) $a=0$ $n \geq 1$ → constant sequence $\equiv 0$

$$0^n \xrightarrow{n\to\infty} 0$$

3) $-1<a<0$

ex: $a=-\frac{1}{2}$ $n \to \left(-\frac{1}{2}\right)^n$



this sequence goes to 0

$$\underset{\overset{\downarrow}{0}}{-|a|^n} \leq \underset{\overset{\downarrow}{0}}{a^n} \leq \underset{\overset{\downarrow}{0}}{|a|^n}$$

$-1<a<0 \Rightarrow 0<|a|<1$

$a_n \to 0$

4) $a \leq -1$

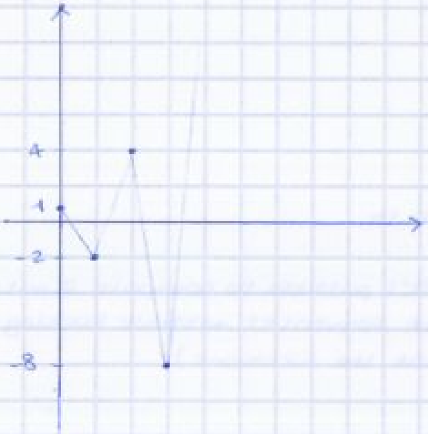
- $a=-1$ $a_n=(-1)^n$

$\lim\limits_{n\to\infty} a_n \nexists$ because it changes always behaviour: $-1, 1$

This is a BOUNDED sequence, but the limit does not exist

- $a<-1$

ex: $a=-2$ $a_n=(-2)^n$



$\lim\limits_{n\to\infty} a_n \nexists$ This sequence is NOT bounded

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty & a>1 \\ 1 & a=1 \\ 0 & -1<a<1 \\ \nexists & a \leq -1 \end{cases}$$

How can I prove that a limit does not exist?

$a_n \to 0$ then I take another $b_n \to 0$

$f(a_n) \to \ell_1$ sequence convergent to 0 $f(b_n) \to \ell$

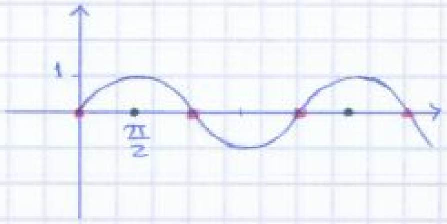
$$\ell_1 \neq \ell_2 \Rightarrow \lim f(x) \nexists$$

CONDITION FOR NON-EXISTENCE OF A LIMIT

if $\lim f(x)$ exists, ℓ_1 and ℓ_2 are the same

$f(x) = \sin x$ $\lim_{x \to +\infty} \sin x \nexists$ how can I prove this?

$$\begin{cases} a_n \to +\infty & \sin(a_n) \to \ell_1 \\ b_n \to +\infty & \sin(b_n) \to \ell_2 \end{cases} \neq$$



🟥 $a_n = n\pi$ $\lim_{n \to \infty} a_n = +\infty$ $\lim_{n \to \infty} \sin(n\pi) = 0$

🟩 $b_n = \frac{\pi}{2} + 2n\pi$ $\lim_{n \to \infty} b_n = +\infty$

$$\lim_{n \to \infty} \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$$

$$\Rightarrow \lim_{x \to +\infty} \sin x \nexists$$

- $\lim_{x \to \gamma} f(x)$ IS NOT A GIVEN VALUE
- $\lim_{x \to \gamma} f(x) \nexists$ proved with sequences

Maximum and minimum of a subset of $\mathbb{R}$ - 2

Examples.

- The bounded closed interval $X = [a, b]$ admits a maximum (b) and a minimum (a).
- The bounded open interval $Y = (a, b)$ does not admit neither a maximum nor a minimum.
- The set $\mathbb{N}$ admits a minimum (zero), but does not admit a maximum.

If the maximum (resp. the minimum) exists, then it is unique.
As shown in the examples, there are sets that do not admit a maximum nor a minimum.

Upper and lower bounds

Definition

- *A set* $X \subset \mathbb{R}$ *is* **bounded from above** *if*
$$\exists b \in \mathbb{R}, \quad \forall x \in X : \; x \leq b$$
The number b *is an* **upper bound** *of* X.
- *A set* $X \subset \mathbb{R}$ *is* **bounded by below** *if*
$$\exists a \in \mathbb{R}, \quad \forall x \in X : \; a \leq x$$
The number a *is a* **lower bound** *of* X.
- *A set*
$X \subset \mathbb{R}$ *is* **bounded** *if it is bounded from above and bounded by below:*
$$\exists a, b \in \mathbb{R}, \; \forall x \in X : a \leq x \leq b$$
or
$$\exists c \in \mathbb{R}, \quad \forall x \in X \; : \; |x| \leq c.$$

The supremum of a set

Definition. Let $X \subset \mathbb{R}$ be a set bounded from above. The **least upper bound** or **supremum** of X is the smallest upper bound of X:

$$S = \sup X = \text{supremum of } X$$

In other words

- $\forall x \in X\,,\ x \leq S$; (i.e. S is an upper bound for X);
- $\forall r < S\,,\ \exists x \in X : x > r$ (any number smaller than S is not an upper bound, i.e. S the smallest upper bound)

Definition. If X is not bounded from above, one defines

$$\sup X = +\infty$$

The infimum of a set

Definition. Let $X \subset \mathbb{R}$ be a set bounded by below. The **greatest lower bound** or **infimum** of X is the largest lower bound of X:

$$s = \inf X = \text{infimum of } X$$

In other words

- $\forall x \in X\,,\ x \geq s$; (i.e. s is a lower bound for X);
- $\forall r > s\,,\ \exists x \in X : x < r$ (any number larger than s is not a lower bound, i.e. s the greatest lower bound)

Definition. If X is not bounded by below, we say that

$$\inf X = -\infty.$$

Supremum of a function

Given a function f and a set $I \subseteq \mathrm{dom} f$, recall the definitions (Basic notions 4) of:

- max and min of f in I;
- lower and upper bound of f in I;
- the definitions of function bounded from above...

Definition

The minimum S of the set of the upper bounds of f in I is called the **supremum of f in I**: $S = \sup\{f(x) : x \in I\} = \sup_{x \in I} f(x)$.

If f is not bounded from above in I we set

$$\sup\{f(x) : x \in I\} = \sup_{x \in I} f(x) = +\infty$$

.

In the same way we define the **infimum of f in I**. *When the function is not bounded by below in I we set*

$$\inf\{f(x) : x \in I\} = \inf_{x \in I} f(x) = -\infty$$

.

Monotone sequences

We define monotonicity for sequences.

Definition

A sequence $\{a_n\}$ is

- **monotone increasing** *if* $\forall n \geq n_0, a_n \leq a_{n+1}$,
- **monotone decreasing** *if* $\forall n \geq n_0, a_n \geq a_{n+1}$,
- **monotone strictly increasing** *if* $\forall n \geq n_0, a_n < a_{n+1}$,
- **monotone strictly decreasing** *if* $\forall n \geq n_0, a_n > a_{n+1}$.

Monotonicity is a sufficient condition for the existence of the limit of a sequence. The following theorem is closely related to the similar result about functions.

Napier's number

The sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

is strictly increasing and bounded from above. Then there exists the limit for $n \to \infty$. This number is called **Napier's number** e

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}.$$

Napier's number e is a non-rational number; the first digits of its decimal expansion are

$$\mathrm{e} = 2.71828182845905\cdots$$

Definition. The exponential function $f(x) = \mathrm{e}^x$ is invertible; its inverse function is called **natural logarithm**: $\ln x$ or $\log x$.

Fundamental limits 1

Consider the real variable function

$$h(x) = \left(1 + \frac{1}{x}\right)^x;$$

its domain is $A = (-\infty, -1) \cup (0, +\infty)$ and, restricted to natural $n \leq 1$, it is the sequence defining Napier's number. We prove that this function also tends to the Napier' s number.

Fix $x > 0$; consider $[x] = n$; from the inequality $n \leq x < n + 1$ we have that

$$1 + \tfrac{1}{n} \geq 1 + \tfrac{1}{x} > 1 + \frac{1}{n+1}$$

$$\left(1 + \tfrac{1}{n}\right)^{n+1} > \left(1 + \tfrac{1}{x}\right)^x > \left(1 + \frac{1}{n+1}\right)^n$$

$$\left(1 + \tfrac{1}{n}\right)^n \left(1 + \tfrac{1}{n}\right) > \left(1 + \tfrac{1}{x}\right)^x > \frac{\left(1 + \dfrac{1}{n+1}\right)^{n+1}}{\left(1 + \dfrac{1}{n+1}\right)}$$

Fundamental limits 3

We verify that

$$\boxed{\lim_{x\to 0}(1+x)^{1/x} = \mathrm{e}.}$$

We set $y = \dfrac{1}{x}$ and we have that

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{y\to\pm\infty}\left(1+\frac{1}{y}\right)^{y} = \mathrm{e}.$$

Fundamental limits 4

We verify that

$$\boxed{\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \frac{1}{\log a}, \qquad \forall a > 0}$$

We have that:

$$\begin{aligned}\lim_{x\to 0}\frac{\log_a(1+x)}{x} &= \lim_{x\to 0}\log_a(1+x)^{1/x}\\ &= \log_a \lim_{x\to 0}(1+x)^{1/x}\\ &= \log_a \mathrm{e} = \frac{1}{\log a}.\end{aligned}$$

In particular, when $a = \mathrm{e}$ we have

$$\boxed{\lim_{x\to 0}\frac{\log(1+x)}{x} = 1.}$$

A table of fundamental limits (exp and log)

- $\lim\limits_{x\to\pm\infty}\left(1+\dfrac{a}{x}\right)^x = \mathrm{e}^a \quad (a \in \mathbb{R})$
- $\lim\limits_{x\to 0}\,(1+x)^{1/x} = \mathrm{e}$
- $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x} = \dfrac{1}{\log a}$ $(a > 0)$; in particular $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x} = 1$
- $\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \log a \quad (a > 0)$; in particular $\lim\limits_{x\to 0}\dfrac{\mathrm{e}^x-1}{x} = 1$
- $\lim\limits_{x\to 0}\dfrac{(1+x)^\alpha-1}{x} = \alpha \qquad (\alpha \in \mathbb{R}).$

IF a maximum (resp. the minimum) exists, then it is unique.

- $X \subset \mathbb{R}$ BOUNDED FROM ABOVE

$\exists b \in \mathbb{R}: x \leq b \quad \forall x \in X$ b is an UPPER BOUND for X

the maximum belongs to the set
the upper bound can be an element
that don't belongs to the set X.

ex: $X = [0,3)$ 4 is an upper bound
$\forall x \in [0,3) \quad x \leq 4$

There are INFINITELY MANY upper bounds

geometrically, 4 is an upper bound, because
it lies at the right of the set X

$X = \mathbb{N}$ $\nexists \max X$
$\nexists$ upper bound for X

$\rightarrow \mathbb{N}$ UNBOUNDED FROM ABOVE
$\nexists b \in \mathbb{R}: x \leq b \quad \forall x \in X$

- $X \subset \mathbb{R}$ BOUNDED BY BELOW

$\exists b \in \mathbb{R}: x \geq b \quad \forall x \in X$ b is a LOWER BOUND for X

ex: $X = [0,3)$ -1 is a lower bound for X
$X = (-\infty; 1]$ } these two sets are
$X = \mathbb{Z}$ } UNBOUNDED BY BELOW $\nexists b \in \mathbb{R}: x \geq b \quad \forall x \in X$

$$\left\{ x \in \mathbb{R}: x = \frac{n}{n+1}, n \in \mathbb{N} \right\} = \left\{ 0; \frac{1}{2}; \frac{2}{3}; \frac{3}{4}; \frac{4}{5}; \frac{5}{6}; \ldots \right\}$$

when n grows, the values become closer
and closer to 1 but they never reach
the point 1

lower bounds → $x \leq 0$ $(-\infty; 0]$
upper bounds → $x \geq 1$ $[1; +\infty)$

THE SUPREMUM OF A SET
sup X
1) if X is not bounded from above
$\sup X = +\infty$
2) if X is bounded from above
sup X = MINIMUM of the set of upper bounds
↳ LEAST UPPER BOUND
$X \subset \mathbb{R}$ $s = \sup X$
- $\forall x \in X: x \leq s$ s is an upper bound
- $\forall \varepsilon > 0 \ \exists x \in X: x > s - \varepsilon$

s-ε cannot be an
upper bound

MONOTONE SEQUENCES

- monotone INCREASING $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$
- monotone STRICTLY increasing $a_{n+1} > a_n$
- monotone DECREASING $a_{n+1} \leq a_n$
- monotone STRICTLY decreasing $a_{n+1} < a_n$

THEOREM "LIMIT OF MONOTONE SEQUENCES"

① Let $\{a_n\}$ be an INCREASING SEQUENCE (this is a sufficient condition to have the existence of the limit)

$\Rightarrow \lim_{n\to\infty} a_n$ EXISTS

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} \{a_n\}$$

1) if $\{a_n\}$ is bounded from above ($\exists k \in \mathbb{R} : a_n \leq k$)
$\lim_{n\to\infty} a_n = \sup\{a_n\} = \ell \in \mathbb{R}$

2) if $\{a_n\}$ is not bounded from above
$\lim_{n\to\infty} a_n = \sup\{a_n\} = +\infty$

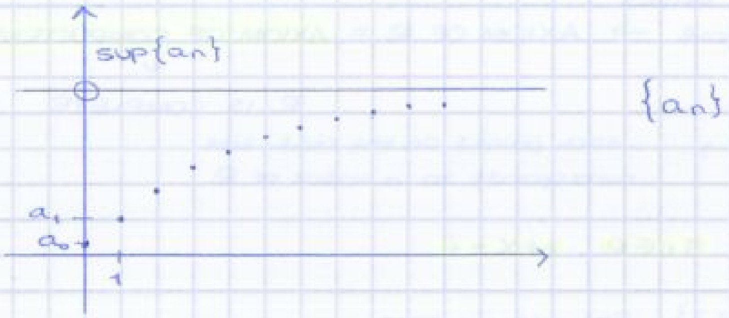
② Let $\{a_n\}$ be a DECREASING SEQUENCE

$\Rightarrow \lim_{n\to\infty} a_n$ EXISTS

$$\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} \{a_n\}$$

1) if $\{a_n\}$ is bounded by below ($\exists k \in \mathbb{R} : a_n \geq k$)
$\lim_{n\to\infty} a_n = \inf\{a_n\} = \ell \in \mathbb{R}$

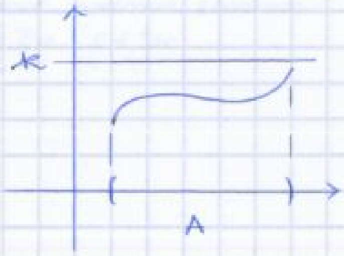
2) if $\{a_n\}$ is not bounded by below
$\lim_{n\to\infty} a_n = \inf\{a_n\} = -\infty$



MONOTONICITY OF FUNCTIONS

- monotone in $A \subseteq \text{dom} f$
- $f(x)$ UPPER BOUND in A
  $\forall x \in A \quad f(x) \leq k$

$\sup f(x) = x \in A$ → minimum value of k that satisfies this condition ($f(x) \leq k$)
$\sup f(x) = \sup(f(A))$

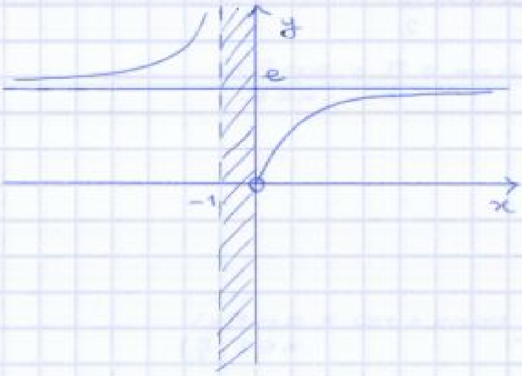


$a_n = \left(1+\frac{1}{n}\right)^n \rightarrow f(x) = \left(1+\frac{1}{x}\right)^x$ **FUNDAMENTAL LIMITS**

exponential function → base > 0

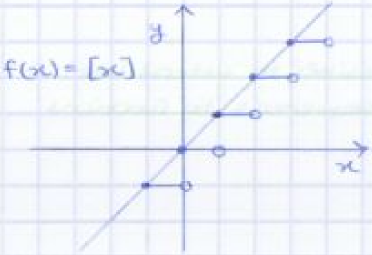
dom f = ? $1+\frac{1}{x} > 0$ $\frac{x+1}{x} > 0$

$(-\infty; -1) \cup (0; +\infty)$



comparison theorem: I consider the limit at $+\infty \rightarrow x>0$

$x>0$ $[x]=n$



$[x] \leq x \leq [x]-1$

$\frac{1}{n} \geqslant \frac{1}{x} \geqslant \frac{1}{n+1}$

$1+\frac{1}{n} \geqslant 1+\frac{1}{x} \geqslant 1+\frac{1}{n+1}$

$\left(1+\frac{1}{n}\right)^{n+1} \geq \left(1+\frac{1}{x}\right)^x \geq \left(1+\frac{1}{n+1}\right)^n$

$$\underbrace{\underset{\underset{e}{\downarrow}}{\left(1+\frac{1}{n}\right)} \cdot \underset{\underset{1}{\downarrow}}{\left(1+\frac{1}{n}\right)^n}}_{\underset{e}{\downarrow}} \geq \boxed{\underset{\underset{e}{\downarrow}}{\left(1+\frac{1}{x}\right)^x}} \geq \underset{\underset{e}{\downarrow}}{\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}}}$$

① $\boxed{\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e}$

- $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$ write $x = -|x|$

  $y = |x|-1$

- $\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to+\infty}\left(\left(1+\frac{a}{x}\right)^{\frac{x}{a}}\right)^a$ $a>0$ $y=\frac{x}{a}$

  $\lim_{y\to+\infty}\left(\left(1+\frac{1}{y}\right)^y\right)^a = e^a$

  $a<0 \ldots = e^a$

② $\boxed{\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = e^a}$

Exercises - limits of SEQUENCES

1) $\lim\limits_{n\to\infty} \left[\log(n^2) - (\log n)^2\right] = [+\infty - \infty]$ indeterminate form

$$a_n = \log(n^2) - (\log n)^2 = 2\log n - \log^2 n = \log n(2 - \log n)$$

$$\lim_{n\to\infty} \underset{+\infty}{\underset{\downarrow}{\log n}}(\underset{-\infty}{\underset{\downarrow}{2 - \log n}}) = -\infty$$

2) $\lim\limits_{n\to\infty} \frac{2^n + 3^n}{5^n} = \left[\frac{\infty}{\infty}\right]$ indeterminate form

$$\lim_{n\to\infty}\left(\frac{2^n}{5^n} + \frac{3^n}{5^n}\right) = \lim_{n\to\infty}\left[\underset{0}{\underset{\downarrow}{\left(\frac{2}{5}\right)^n}} + \underset{0}{\underset{\downarrow}{\left(\frac{3}{5}\right)^n}}\right] = 0$$

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty & a > 1 \\ 1 & a = 1 \\ 0 & -1 < a < 1 \\ \nexists & a \leq -1 \end{cases}$$

3) $\lim\limits_{n\to\infty} \frac{3^n + 4^n}{4^n + 5^n} = \left[\frac{\infty}{\infty}\right]$ indeterminate form $\qquad \lim\limits_{x\to\infty} \frac{P(x)}{Q(x)}$

$$\lim_{n\to\infty} \frac{4^n\left(1 + \frac{3^n}{4^n}\right)}{5^n\left(1 + \frac{4^n}{5^n}\right)} = \lim_{n\to\infty}\left(\frac{4}{5}\right)^n \cdot \frac{1 + \left(\frac{3}{4}\right)^n}{1 + \left(\frac{4}{5}\right)^n} = 0 \cdot 1 = 0$$

Existence of zeroes - 1

Definition. A **zero** of a real-valued function f is a point $x_0 \in \operatorname{dom} f$ at which the function vanishes.

Theorem (Existence of zeroes)

Let f be a continuous function on a closed bounded interval $[a, b]$. If $f(a)f(b) < 0$ (i.e., if the images of the endpoints under f have different signs) then f admits a zero within the open interval (a, b).
If moreover f is strictly monotone on $[a, b]$, the zero is unique.

Existence of zeroes - 2

Corollary

Let f be continuous on the interval I and suppose it admits non-zero limits (finite or infinite) that are different in sign for x tending to the end-points of I. Then f has a zero in I, which is unique if f is strictly monotone on I.

Corollary

Let f and g be continuous functions on a closed bounded interval $[a, b]$; if

$$f(a) < g(a) \quad \text{and} \quad f(b) > g(b) \qquad \text{(or vice versa)}$$

then there exists at least a point x_0 in (a, b) such that $f(x_0) = g(x_0)$.

The Weierstrass theorem

Theorem (Weierstrass theorem)

A continuous function f on a closed and bounded interval $[a, b]$ is bounded and admits minimum $m = \min f(x)$ and maximum $M = \max f(x)$.

Theorem (Intermediate value theorem - II version)

If a function f is continuous on a closed and bounded interval $[a, b]$, it assumes all values between m and M: i.e. $f([a, b]) = [m, M]$.

Monotonicity, continuity and invertibility

We know that if f is strictly monotone on an interval I then f is injective in I and that the opposite implication is not true.
But injectivity is a necessary and sufficient condition for strict monotonicity, if we consider continuous functions.

Theorem

Let f be a continuous function on the interval I; then f is injective on I if and only if f is strictly monotone on I.

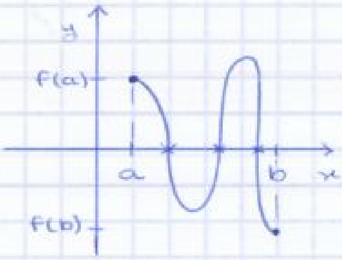
We have already mentioned the following theorem, that has been used to state the continuity of the inverse functions $y = \log_a x$, $y = \arcsin x$, $y = \arccos x$ and $y = \arctan x$.

Theorem

Let the function f be continuous and invertible on the interval I; then the inverse function f^{-1} is continuous on the interval $J = f(I)$.

THEOREM OF THE EXISTENCE OF ZEROES

$$\left.\begin{array}{l}\bullet\ f \text{ CONT in } [a,b] \\ \bullet\ f(a)\cdot f(b)<0\end{array}\right\} \Rightarrow \begin{array}{l}\exists x_0 \in (a,b) \\ f(x_0)=0\end{array}$$



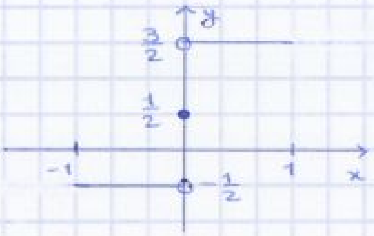
x_0 is not unique
→ there exists at least x_0

$$\left.\begin{array}{l}\bullet\ f \text{ CONT in } [a,b] \\ \bullet\ f(a)\cdot f(b)<0 \\ \bullet\ f \text{ is STRICTLY MONOTONE on } [a,b]\end{array}\right\} \Rightarrow \begin{array}{l}\exists!\ x_0 \in (a,b) \\ f(x_0)=0\end{array} \qquad x_0 \text{ is UNIQUE}$$

PROOF based on the BISECTION METHOD
↳ divide the interval in 2 parts
and look at the properties of f in each part

What happens if I remove an hypothesis?
ex: $f(a)\cdot f(b)<0$
BUT: f is defined at all points of an interval, but not continuous
Is it true that: $\exists x_0 \in (a,b) : f(x_0)=0$

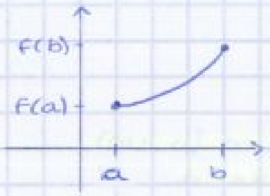
1) $[-1;1]$ $f(x)=\text{sign}\,x+\frac{1}{2}$



- f is defined on a closed interval
- $f(-1)\cdot f(1)<0$

BUT $\nexists x_0 \in [-1;1]$
$f(x_0)=0$ because f is NOT CONTINUOUS

2) • $[a,b]$ CONT.
• $f(a)\cdot f(b)>0$ (they're both positive or both negative



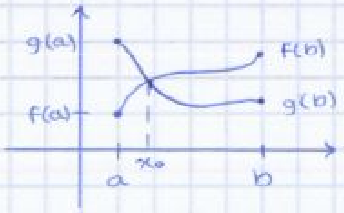
$\nexists x_0 \in [a,b]$
$f(x_0)=0$ because $f(a)\cdot f(b)>0$
and not <0

TH. APPLICATION 1

$$\left.\begin{array}{l}\bullet\ f, g \text{ CONTINUOUS on } [a,b] \\ \bullet\ \begin{array}{l}f(a)<g(a) \\ f(b)>g(b)\end{array} \text{ or } \begin{array}{l}f(a)>g(a) \\ f(b)<g(b)\end{array}\end{array}\right\} \Rightarrow \begin{array}{l}\exists x_0 \in (a,b) \\ f(x_0)=g(x_0)\end{array}$$

the two functions must intersect at a point

⇒



x_0 is not unique
→ if f, g are strictly monotone
then x_0 is UNIQUE

TH. APPLICATION 3

A Polynomial function of ODD degree must have at least one real zero

$$\lim_{x\to\pm\infty} (a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0)$$

$$a_n x^n \left(1 + \frac{a_{n-1}}{a_n \cdot x} + \dots \underset{\hookrightarrow 1}{\frac{1}{x^2}} + \dots \frac{a_0}{a_n} \cdot \frac{1}{x^n}\right)$$

$$= \lim_{x\to\pm\infty} a_n x^n$$

n ODD : $n \to +\infty \begin{cases} a_n > 0 \to +\infty \\ a_n < 0 \to -\infty \end{cases}$

$n \to -\infty \begin{cases} a_n > 0 \to -\infty \\ a_n < 0 \to +\infty \end{cases}$

- $f \in C^0(\mathbb{R})$
- the limits have different signs

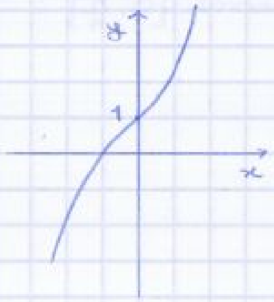
$\Big\} \Rightarrow \exists x_0 \in \mathbb{R}: f(x_0) = 0$

ex. $f(x) = x^5 + 3x^3 + 1$ degree $= 5 \to$ odd $\Rightarrow$ f(x) must have at least one x_0
$f(x_0) = 0$

Is x_0 unique?
- x^5 strictly increasing
- x^3 strictly increasing

$\Rightarrow$ f(x) is STRICTLY INCREASING
x_0 is UNIQUE

Is x_0 positive or negative?



$x \to +\infty$ $f(x) \to +\infty$
$x \to -\infty$ $f(x) \to -\infty$
$f(0) = 1$
x_0 is negative
$x_0 < 0$

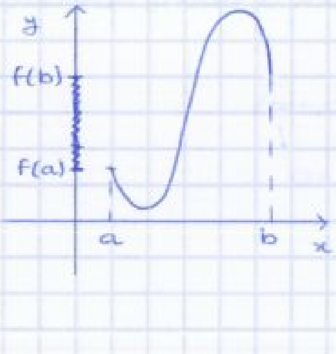
ex: $f(x) = \underset{\nearrow}{\smile x^5} + \underset{\searrow\nearrow}{\smile x^4} + 1$ degree $= 5 =$ odd there is at least one x_0
Is this x_0 unique? we don't know

INTERMEDIATE VALUE THEOREM I VERSION (WEAK VERSION)

- f CONTINUOUS on [a,b]

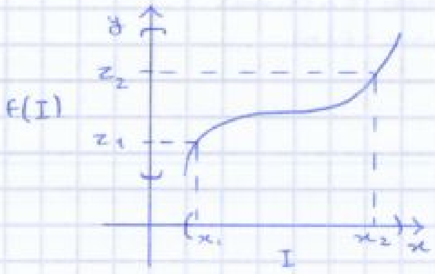
$\Rightarrow$ f assumes all the values between f(a) and f(b)

Can I say something about the range of f?
$f([a,b])$



- f continuous on an interval I
  ⇒ f(I) is an interval

PROOF:



$z_1 \in f(I)$ $z_2 \in f(I)$
$z_1 < z_2$
I choose $z \in (z_1, z_2)$
⇒ I have to prove that
$z \in f(I)$

$\left.\begin{array}{ll} \exists x_1 \in I & f(x_1)=z_1 \\ \exists x_2 \in I & f(x_2)=z_2 \end{array}\right]$ consider $[x_1,x_2]$ (or $[x_2;x_1]$)

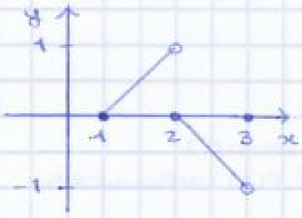
I appey the intermediate value theorem to f in $[x_1,x_2]$
⇒ f assumes all the values in (z_1,z_2),
in particular also the value z

⇒ $z \in f(I)$ this proves that f(I) is an interval

WEIERSTRASS THEOREM (FIRST PART)

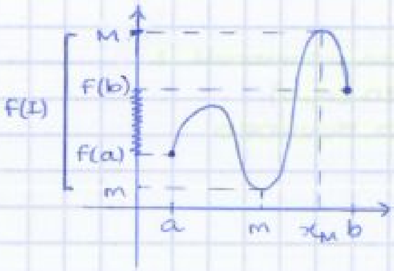
$\left.\begin{array}{l} f \text{ continuous on an interval } I \\ I \text{ closed bounded} \end{array}\right\} \Rightarrow f(I)$ closed bounded

Both conditions must be satisfied, otherwise the theorem is not true.
ex:



[1;3] → (-1;1) because f is not continuous
This function has not an absolute minimum and absolute maximum

(SECOND PART)
f CONTINUOUS on [a,b] (closed and bounded
⇒ • f is BOUNDED on [a,b]
• f admits the ABSOLUTE MAXIMUM M and the ABSOLUTE MINIMUM m in [a,b]
↳ $\exists x_M : f(x_M)=M$
$\exists x_m : f(x_m)=m$



Mathematical Analysis I (2013-2014)

Differential Calculus 1 - The derivative

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Difference quotient - 1

If $x_0 \in \mathrm{dom}\, f$ and f is defined in a neighbourhood $I_r(x_0)$, then for all $I_r(x_0)$ we define the following quantities:

- the **increment of the independent variable** between x_0 and x is the difference $h = \Delta x = x - x_0$
- the **increment of the dependent variable** between x_0 and x is the difference $\Delta f = f(x) - f(x_0)$

From the definitions we have that:

$$x = x_0 + h, \qquad f(x) = f(x_0) + \Delta f.$$

Other definitions

Definition. If f is differentiable at x_0, the line

$f'(x_0) = m$

$$\boxed{y = t(x) = f(x_0) + f'(x_0)(x - x_0), \qquad x \in \mathbb{R}.}$$

is the **tangent line** to the graph of f at $(x_0, f(x_0))$. The derivative $f'(x_0)$ is the slope of the tangent line.

Definition. If $I \subseteq \operatorname{dom} f$ and f is differentiable $\forall x \in I$ we say that f is **differentiable on** I. The function

$$f' : \operatorname{dom} f' \subseteq \mathbb{R} \to \mathbb{R}, \quad f' : x \mapsto f'(x)$$

is called **(first) derivative of** f.

Continuity and differentiability

Differentiability is "stronger" than continuity; in fact the following theorem holds.

Theorem

Let f be a functions defined in $I(x_0)$. If f is differentiable at x_0, then f is continuous at x_0.

Proof: Textbook 4.1 page 169

Remarks.

- The opposite implication is **NOT** true; we will see later examples of functions continuous at a point, but non differentiable.
- The contrapositive of this statement is

  if f is not continuous at x_0 then f is not differentiable at x_0.

Algebra of the derivatives

If the functions f and g are differentiable at x_0 and $\alpha, \beta \in \mathbb{R}$, then the functions

$$f(x) \pm g(x),\ \alpha f(x),\ \alpha f(x) + \beta g(x),\ f(x)g(x),\ \frac{f(x)}{g(x)} \text{ if } g(x_0) \neq 0$$

are differentiable at x_0 and we have that

$$\begin{aligned}
(f \pm g)'(x_0) &= f'(x_0) \pm g'(x_0) \\
(\alpha f)'(x_0) &= \alpha f'(x_0) \\
(\alpha f + \beta g)'(x_0) &= \alpha f'(x_0) + \beta g'(x_0) \\
(f\,g)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0)
\end{aligned}$$

Algebra of the derivatives (continued)

$$\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2} \qquad (g(x_0) \neq 0)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2} \qquad (g(x_0) \neq 0)$$

Theorem

If the function $f(x)$ is differentiable at $x_0 \in \mathbb{R}$ and the function $g(y)$ is differentiable at $y_0 = f(x_0)$, then the composition $(g \circ f)(x) = g(f(x))$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0).$$

Examples

- Compute the derivative of the function $g(x) = \arctan x$.
  - Setting $y = f(x) = \tan x$, we have that $f'(x) = 1 + \tan^2 x$;
  - since $x = f^{-1}(y) = \arctan y$, applying the theorem we have that

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

  We conclude that

$$\boxed{(\arctan x)' = \frac{1}{1+x^2}}$$

- In the same way we can prove that

$$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{(\log_a x)' = \frac{1}{x \ln a}, \qquad (\ln x)' = \frac{1}{x}}$$

Table of elementary derivatives

$D\, x^\alpha = \alpha x^{\alpha-1} \qquad \forall \alpha \in \mathbb{R}$

$D \sin x = \cos x \qquad D \cos x = -\sin x$

$D \tan x = 1 + \tan^2 x = \dfrac{1}{\cos^2 x}$

$D \arcsin x = \dfrac{1}{\sqrt{1-x^2}} \qquad D \arccos x = -\dfrac{1}{\sqrt{1-x^2}}$

$D \arctan x = \dfrac{1}{1+x^2}$

$D\, a^x = a^x \log a$ in particular, $D\, e^x = e^x$

$D \log_a |x| = \dfrac{1}{x \log a}$ in particular, $D \log |x| = \dfrac{1}{x}$

Derivative and one-sided derivatives

Recalling the relations between limit and one-sided limits, we have that:

Theorem

The function f is differentiable at x_0 if and only if f is differentiable on the right and on the left at x_0 and the one-sided derivatives coincide:

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

Examples - 1

- Consider the function

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ \cos x & \text{if } x > 0 \end{cases}$$

It is continuous at the origin and

$$f'_+(x_0) = \lim_{x\to 0^+} \frac{\cos x - 1}{x} = 0 \text{ and } f'_-(x_0) = \lim_{x\to 0^-} \frac{x^2}{x} = 0.$$

Then it is differentiable at x_0 and $f'(0) = 0$.

- Consider the function $f(x) = |\sin x|$ e $x_0 = 0$. Since

$$f'_+(0) = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \text{ and } f'_-(0) = \lim_{x\to 0^-} \frac{-\sin x}{x} = -1$$

we have that

$$f'_+(0) = 1 \qquad \text{e} \qquad f'_-(0) = -1;$$

the function is differentiable on the right and on the left at 0, but the one-sided derivatives do not coincide; then the function is not differentiable at 0.

Examples

- The point $x_0 = 0$ is a point with vertical tangent for the function $f(x) = \sqrt[3]{x}$.
- The point $x_0 = 0$ is a cusp point for the function $f(x) = \sqrt{|x|}$.
- The point $x_0 = 1$ is a corner point for the function
$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \leq 1 \\ \ln x & \text{if } x > 1 \end{cases}$$
- The point $x_0 = 2$ is a corner point for the function
$$f(x) = \begin{cases} \sqrt{2-x} & \text{if } x \leq 2 \\ x-2 & \text{if } x > 2 \end{cases}$$

Higher order derivatives - 1

If a function f is differentiable in $I(x_0)$, we can define its first derivative $f'(x)$ in $I(x_0)$.

Definition. If the limit of the difference quotient of the first derivative

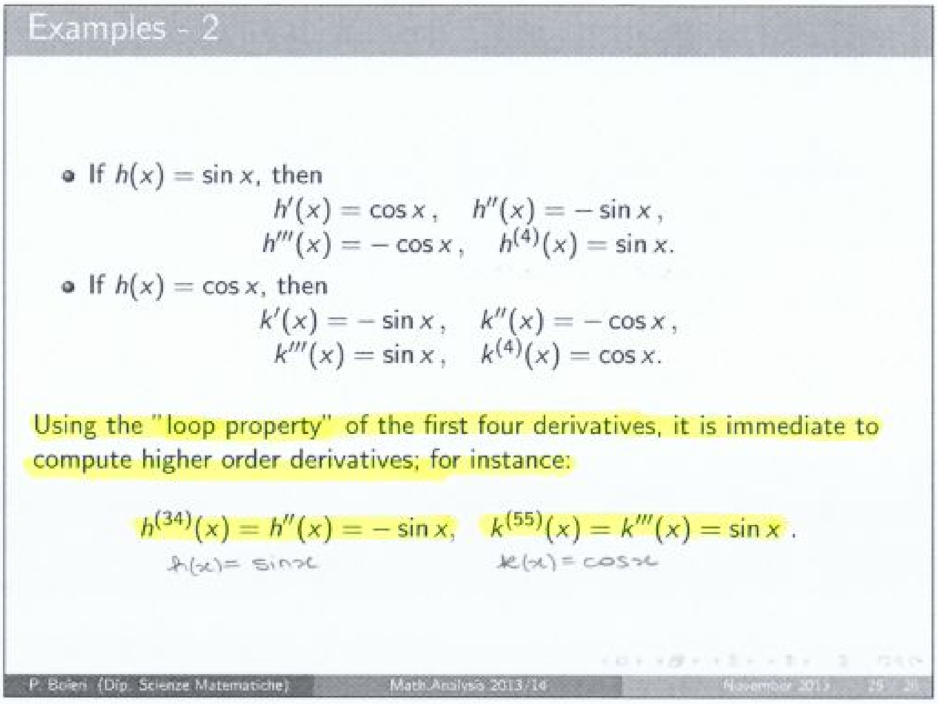
$$\lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

exists and is finite, we say that f is **twice differentiable at x_0** and the number

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

is called the **second derivative of f at x_0**.

Other symbols for the second derivative are $y''(x_0)$, $\frac{d^2f}{dx^2}(x_0)$, $D^2f(x_0)$.



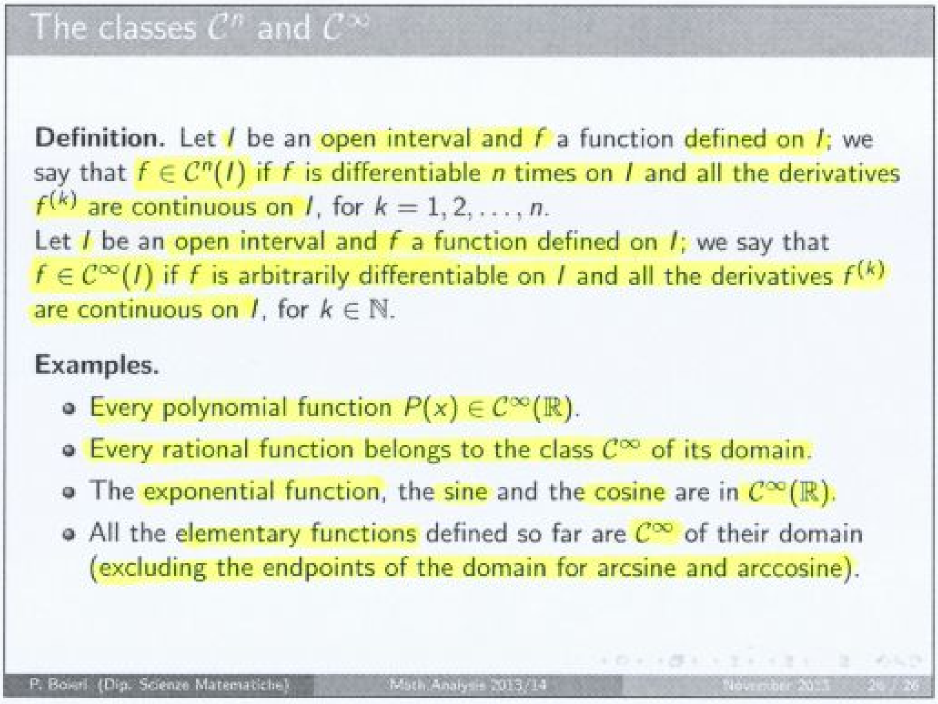
Examples - 2

- If $h(x) = \sin x$, then
$$h'(x) = \cos x\,, \quad h''(x) = -\sin x\,,$$
$$h'''(x) = -\cos x\,, \quad h^{(4)}(x) = \sin x.$$
- If $h(x) = \cos x$, then
$$k'(x) = -\sin x\,, \quad k''(x) = -\cos x\,,$$
$$k'''(x) = \sin x\,, \quad k^{(4)}(x) = \cos x.$$

Using the "loop property" of the first four derivatives, it is immediate to compute higher order derivatives; for instance:

$$h^{(34)}(x) = h''(x) = -\sin x, \quad k^{(55)}(x) = k'''(x) = \sin x\,.$$

$h(x) = \sin x$ $\qquad$ $k(x) = \cos x$

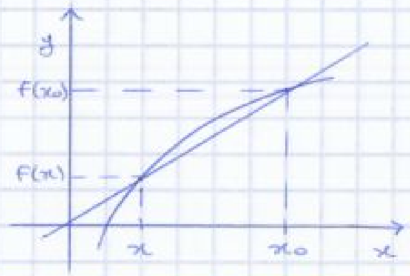


The classes C^n and C^∞

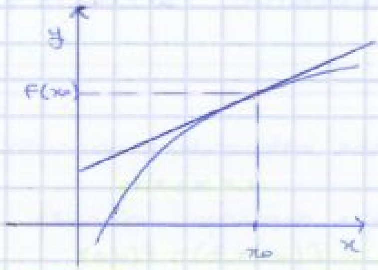
Definition. Let I be an open interval and f a function defined on I; we say that $f \in C^n(I)$ if f is differentiable n times on I and all the derivatives $f^{(k)}$ are continuous on I, for $k = 1, 2, \ldots, n$.
Let I be an open interval and f a function defined on I; we say that $f \in C^\infty(I)$ if f is arbitrarily differentiable on I and all the derivatives $f^{(k)}$ are continuous on I, for $k \in \mathbb{N}$.

Examples.

- Every polynomial function $P(x) \in C^\infty(\mathbb{R})$.
- Every rational function belongs to the class C^∞ of its domain.
- The exponential function, the sine and the cosine are in $C^\infty(\mathbb{R})$.
- All the elementary functions defined so far are C^∞ of their domain (excluding the endpoints of the domain for arcsine and arccosine).



secant slope: $\frac{f(x)-f(x_0)}{x-x_0}$



tangent: $y=f(x_0)+f'(x_0)(x-x_0)$
$(x_0, f(x_0))$
slope $= f'(x_0)$
= limit of the difference quotient

f CONTINUOUS at x_0 || f DIFFERENTIABLE at x_0

THEOREM: RELATION BETWEEN DERIVATIVE AND CONTINUITY

- f defined in $I(x_0)$
- f DIFFERENTIABLE at x_0 ⇒ f CONTINUOUS at x_0

PROOF

I know that f is differentiable, so $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)\in\mathbb{R}$

I have to prove that f is continuous at x_0:

$$\lim_{x\to x_0} f(x)=f(x_0) \iff \lim_{x\to x_0}(f(x)-f(x_0))=0$$

$$\Rightarrow \lim_{x\to x_0}(f(x)-f(x_0)) = \lim_{x\to x_0} \underset{\substack{\downarrow \\ f'(x_0)}}{\frac{f(x)-f(x_0)}{x-x_0}} \cdot \underset{\substack{\downarrow \\ 0}}{(x-x_0)} =$$

$$=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot\lim_{x\to x_0}(x-x_0) = f'(x_0)\cdot 0 = 0$$

$\boxed{\text{f DIFF at } x_0 \Rightarrow \text{f CONT. at } x_0}$

$P \Rightarrow Q$

CONTRAPOSITIVE FORM: $\neg Q \Rightarrow \neg P$

$\boxed{\text{f NOT CONT. at } x_0 \Rightarrow \text{f NOT DIFF at } x_0}$

f CONT at x_0 $\not\Rightarrow$ f DIFF at x_0 NO
A function can be continuous, but not differentiable

Ex: IF Paolo is Italian (P)
THEN Paolo is European (Q)
$P\Rightarrow Q$
- $Q \Rightarrow P$ NOT TRUE
- $\neg P \Rightarrow \neg Q$ NOT TRUE
- $\neg Q \Rightarrow \neg P$ TRUE

IF Paolo is not European
THEN Paolo is not Italian
CONTRAPOSITIVE FORM

4) $f(x) = \sin x$

- $x_0 = 0$

$$\lim_{x\to0} \frac{\sin x - \sin 0}{x-0} = \frac{\sin x}{x} = 1$$

- $x_0 \in \mathbb{R}$

$$\lim_{h\to0} \frac{\sin(x_0+h) - \sin x_0}{h} = \frac{\sin x_0 \cos h + \cos x_0 \sin x - \sin x}{h}$$

$$\lim_{h\to0}\left(\cos x_0 \frac{\sin h}{h} + \sin x_0 \frac{\cos h - 1}{h}\right) = \cos x_0 \underset{\overset{\downarrow}{1}}{\lim_{h\to0} \frac{\sin h}{h}} + \sin x_0 \underset{\overset{\downarrow}{0}}{\lim_{h\to0} \frac{\cos h - 1}{h}} =$$

$$= \cos x_0$$

$f(x) = \sin x$ $\quad$ $f'(x) = \cos x$

$f(x) = \cos x$ $\quad$ $f'(x) = -\sin x$

5) $f(x) = a^x$

$$\lim_{h\to0} \frac{a^{x_0+h} - a^{x_0}}{h} = \lim_{h\to0} \frac{a^{x_0}\cdot a^h - a^{x_0}}{h} = a^{x_0} \lim_{h\to0} \frac{a^h - 1}{h} = a^{x_0}\cdot k(a)$$

$D(a^x) = a^x \cdot k(a)$

└ constant that depends only on the base

$$a^{x_0} \lim_{h\to0} \frac{a^h - 1}{h} = a^{x_0} \cdot \underline{\ln a}$$

is there an a

ST. $\ln a = 1 \rightarrow a = e$

$D(e^x) = e^x$

$y'(x) = y(x)$

Differential equation

the solution is the exponential function in base e and also the constant function 0.

RULES FOR DIFFERENTIATION

- f, g DIFF at x_0, $f'(x_0)$, $g'(x_0)$, $\alpha, \beta \in \mathbb{R}$

1) $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

2) $(\alpha f)'(x_0) = \alpha f'(x_0)$

3) $(1,2) \rightarrow \underline{(\alpha f \pm \beta g)}'(x_0) = \alpha f'(x_0) \pm \beta g'(x_0)$

$=$ LINEAR COMBINATION

The derivative of linear combination is the linear combination of derivative

The DERIVATIVE is a LINEAR OPERATOR

4) $(f\, g)'(x_0) = f(x_0)\cdot g'(x_0) + f'(x_0)\cdot g(x_0)$

5) $\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2}$ $\quad$ $g(x_0) \neq 0$

6) $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\cdot g(x_0) - f(x_0)\cdot g'(x_0)}{(g(x_0))^2}$ $\quad$ $g(x_0) \neq 0$

ex: $y = f(x) = \tan x = \frac{\sin x}{\cos x}$

$$f'(x) = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \begin{cases} 1 + \tan^2 x \\ \frac{1}{\cos^2 x} \end{cases}$$

$y = f(x) = \tan x = y_0$

$x = f^{-1}(y) = \arctan x$

$$(D\arctan)(y_0) = \frac{1}{1+\tan^2 x} = \frac{1}{1+y_0^2}$$

$\Rightarrow h(x) = \arctan x \qquad h'(x) = \frac{1}{1+x^2}$

ex: $f(x) = a^x$

$x = f^{-1}(y) = \log_a y$

$$(D\log_a)(y_0) = \frac{1}{f'(x_0)} = \frac{1}{a^{x_0} \cdot \log a} = \frac{1}{y_0 \cdot \log a}$$

$a^{x_0} = y_0$

$\Rightarrow h(x) = \log_a x \qquad h'(x) = \frac{1}{x \ln a}$

$h(x) = \log x \qquad h'(x) = \frac{1}{x}$

Direct computation of the derivative of the log, without considering it as the inverse function of a^x

$$\lim_{h\to0} \frac{\log_a(x_0+h) - \log_a x_0}{\cancel{x}\ h} = \lim_{h\to0} \frac{\log_a\left(x_0\left(1+\frac{h}{x_0}\right)\right) - \log_a x_0}{h} =$$

$$\lim_{h\to0} \frac{\cancel{\log_a x_0} + \log_a\left(1+\frac{h}{x_0}\right) - \cancel{\log_a x_0}}{h} = \lim_{h\to0} \frac{\log_a\left(1+\frac{h}{x_0}\right)}{\frac{h}{x_0} \enclose{circle}{x_0}} \stackrel{y=\frac{h}{x_0}}{=}$$

it's a constant

$h\to0 \quad y\to0$

change variable

$$\frac{1}{x_0} \lim_{y\to0} \frac{\log_a(1+y)}{y} = \frac{1}{x_0 \ln a}$$

ex. $x^n \to nx^{n-1} \quad n\geq2 \quad n=1 \quad x^0 \equiv 1$

$x^3 \quad x^{3/2} \quad x^{\sqrt{2}}$

$D\sqrt[n]{x}$

$y = f(x) = x^n$

$x = f^{-1}(y) = y^{\frac{1}{n}} = \sqrt[n]{y}$

$$D\left(y^{\frac{1}{n}}\right) = \frac{1}{Dx^n} = \frac{1}{nx^{n-1}} = \frac{1}{n\left(y^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{ny^{1-\frac{1}{n}}} = \frac{1}{n} \cdot y^{\frac{1}{n}-1}$$

$$\begin{cases} h(x) = x^{\enclose{circle}{\frac{1}{n}}} \to h'(x) = \enclose{circle}{\frac{1}{n}} x^{\enclose{circle}{\frac{1}{n}}-1} \\ h(x) = x^{\enclose{circle}{n}} \to h'(x) = \enclose{circle}{n} x^{\enclose{circle}{n}-1} \end{cases}$$

$x \neq 0$ in any case

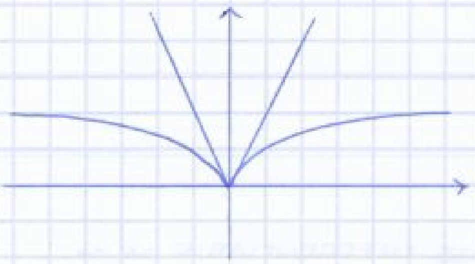
$x > 0$ when n is even

③ $f(x) = \sqrt{|x|}$

$$f(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ \sqrt{-x} & x < 0 \end{cases}$$

Even function:

$f(-x) = \sqrt{|-x|} = \sqrt{|x|} = f(x)$



$\lim_{x\to0^+} \frac{\sqrt{x}}{x}$ x is positive ⇒ $x = \sqrt{x^2}$

$= \lim_{x\to0^+} \sqrt{\frac{1}{x}} = +\infty$ RIGHT derivative

$\lim_{x\to0^-} \sqrt{\frac{1}{-x}} = -\infty$

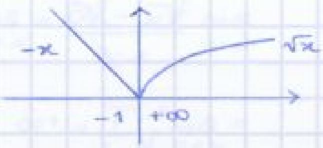
CUSP POINT

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \nexists \text{, BUT } \begin{cases} \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = +\infty & \text{or} \quad -\infty \\ \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = -\infty & \quad\;\; +\infty \end{cases}$$

INFINITE, but DIFFERENT

1) VERTICAL TANGENT POINT
2) CORNER POINT
3) CUSP POINT

another case, not included in these ones:



HIGHER ORDER DERIVATIVES

- f DIFFERENTIABLE in $I(x_0)$

⇓

f' DEFINED in $I(x_0)$ at all points

⇒ $\frac{f'(x)-f'(x_0)}{x-x_0}$ DIFFERENCE QUOTIENT for f'

$\lim_{x\to x_0} \frac{f'(x)-f'(x_0)}{x-x_0}$ EXISTS and is FINITE ⇒ $f''(x_0)$

f(x) is TWICE DIFFERENTIABLE at x_0

$f''(x_0)$ $y''(x_0)$ $D^2(f(x_0))$ $\frac{d^2 f}{dx^2}(x_0)$ → LEIBNIZ notation

IF the second derivative exists in a neighbourhood, one can consider the difference quotient of it and the limit of this will define the THIRD DERIVATIVE → $f'''(x_0)$

$$\boxed{f^{(k)}(x_0) = (f^{(k-1)}(x_0))'}$$

$f^{(k)}(x)$, $y^{(k)}(x)$, $D^k f(x)$, $\frac{d^k f}{dx^k}$

$f^{(4)}(x) = f^{(IV)}(x)$

PROBLEMS ON DERIVATIVES

1) $f(x) = e^{\cos x^3}$ $x \to x^3$

$y \to \cos y$

$z \to e^z$

I have to use the chain rule for the composition of 3 functions

$f'(x) = e^{\cos x^3} \cdot (-\operatorname{sen} x^3) \cdot 3x^2$

2) $f(x) = \log(\log 5x)$ $x \to 5x$

$y \to \log 5x$

$z \to \log z$

$f'(x)\ \frac{1}{\log 5x} \cdot \frac{1}{\cancel{5}x} \cdot \cancel{5} = \frac{1}{x \log 5x}$

- f def in $(-a;a) = I$ $(\mathbb{R})$
- f even
- f differentiable in I

} can I say something about f'? $\Rightarrow$ f' is ODD

Even function:

$f(x) = f(-x)\ \forall x \in \operatorname{dom} f$

$\to f'(x) = f'(-x) \cdot (-1) \Rightarrow -f'(x) = f'(-x)$ f' is ODD

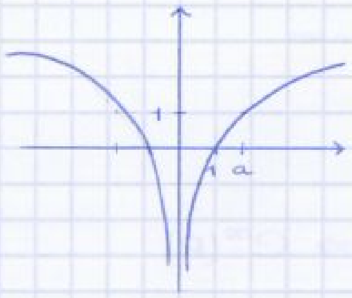
$\left(x \to -x \quad y \to f(y) \right.$

The derivative of an even function is an odd function and viceversa

- F EVEN ⇒ F' ODD $f(x) = f(-x) \Rightarrow -f'(x) = f'(-x)$
- F ODD ⇒ F' EVEN $-f(x) = f(-x) \Rightarrow f'(x) = f'(-x)$

3) $f(x) = \log_a |x|$ domain? $|x| > 0 \Leftrightarrow x \neq 0$ $\operatorname{dom} f = \mathbb{R} \setminus \{0\}$

$f(-x) = \log_a |-x| = \log_a |x| = f(x) \Rightarrow f$ even



$x > 0$ $f(x) = \log_a x \to f' = \frac{1}{x \log a}$

$x < 0$ $f(x) = \log_a(-x) \to f' = \frac{1}{\cancel{-}x \log a} \cdot (\cancel{-}1) =$

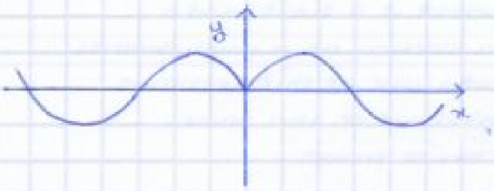
$= \frac{1}{x \log a}$

$f(x) = \log_a |x| \to f'(x) = \frac{1}{x \log a}$ $x \neq 0$

If I put the base e:

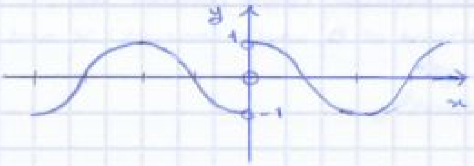
$f(x) = \log|x| \to f'(x) = \frac{1}{x}$ $x \neq 0$

$h(x) = \sin|x|$



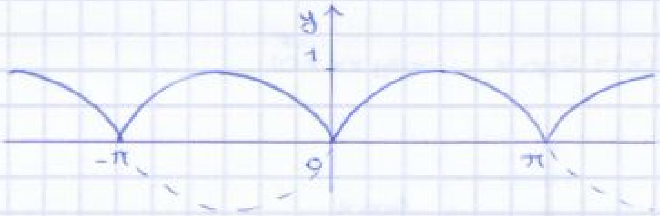
$$\sin|x| = \begin{cases} \sin x & x \geq 0 \\ -\sin x & x < 0 \end{cases}$$

$h'(x) = (\sin|x|)'$



$$(\sin|x|)' = \begin{cases} \cos x & x > 0 \\ -\cos x & x < 0 \end{cases}$$

② $h(x) = |\sin x|$ $|f(x)|$



$\forall x \neq k\pi,\ k \in \mathbb{Z}$

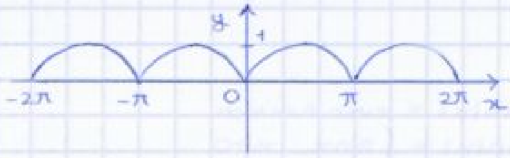
$D|\sin x| = \text{sign}(\sin x) \cdot \cos x$

$\sin x \neq 0 \rightarrow \forall x \neq k\pi,\ k \in \mathbb{Z}$

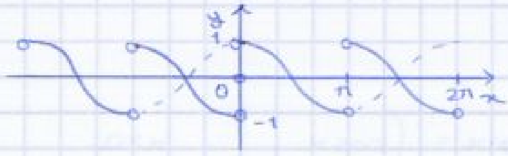
$$|\sin x| = \begin{cases} \sin x & \sin x \geq 0 & 0 \leq x \leq \pi + k\pi \\ -\sin x & \sin x < 0 & \pi < x < 2\pi + k\pi \end{cases}$$

$$(|\sin x|)' = \begin{cases} \cos x & 0 < x < \pi + k\pi \\ -\cos x & \pi < x < 2\pi + k\pi \end{cases}$$

$h(x) = |\sin x|$



$h'(x) = (|\sin x|)'$

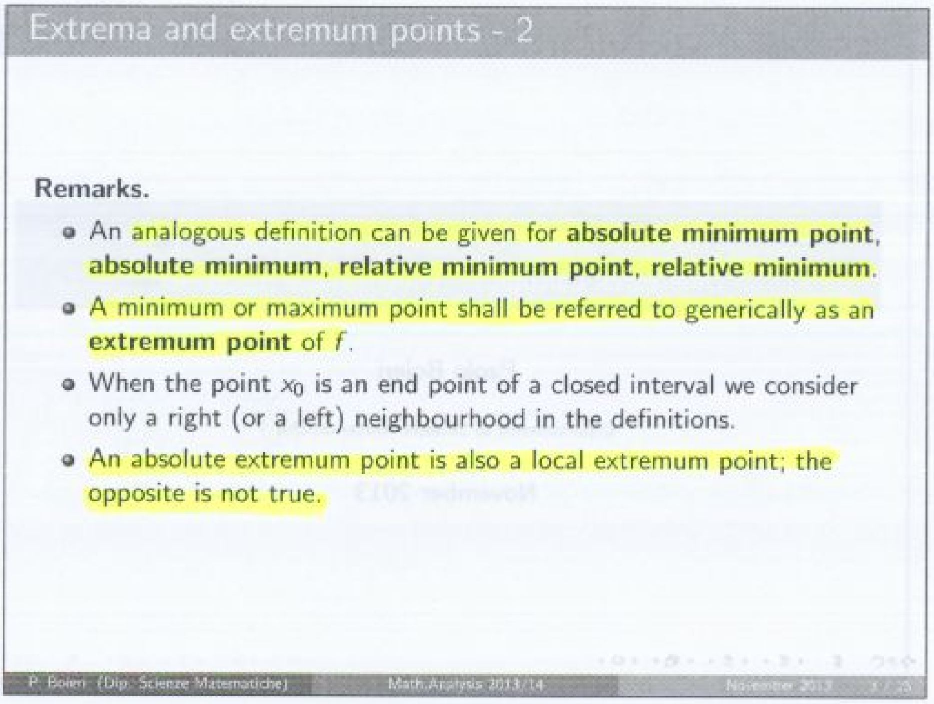


$h(x) = x^{\sin x} = (f(x))^{g(x)} = (e^{\ln f(x)})^{g(x)} = e^{g(x) \cdot \ln f(x)}$

$f(x) > 0$

↓

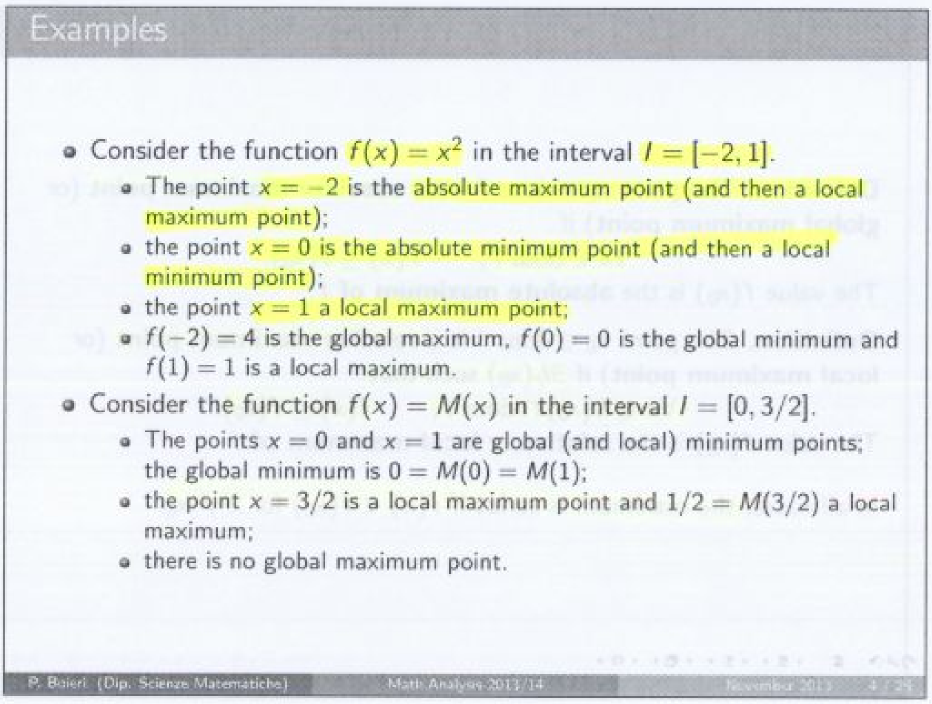
$h(x) = e^{\sin x \cdot \ln x}$ $h'(x) = e^{\sin x \cdot \ln x} \cdot \left(\frac{\sin x}{x} + \cos x \cdot \ln x \right)$



Extrema and extremum points - 2

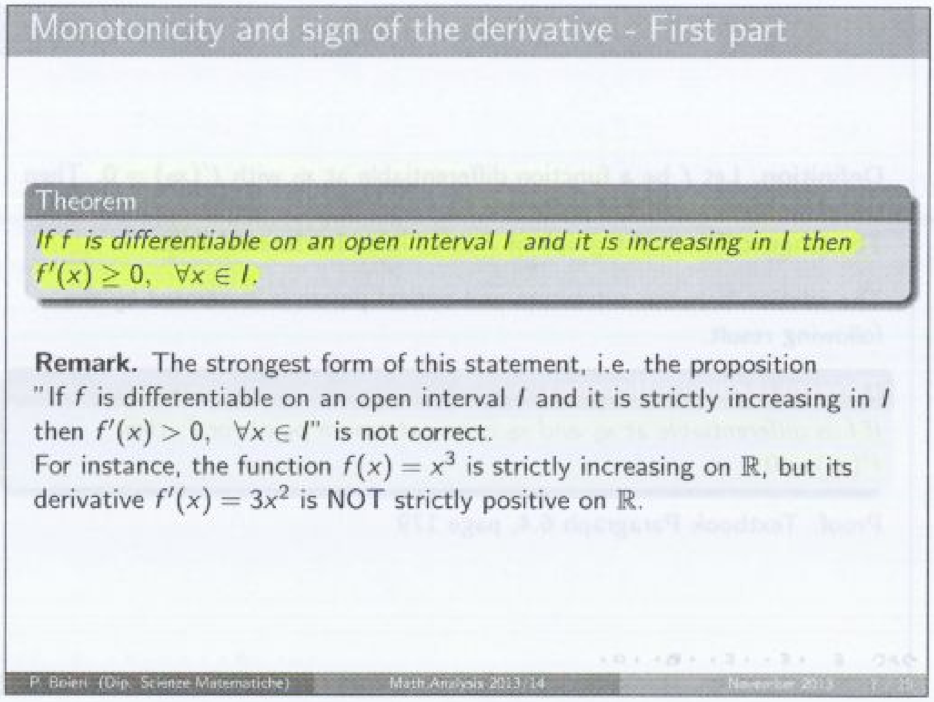
Remarks.

- An analogous definition can be given for **absolute minimum point**, **absolute minimum**, **relative minimum point**, **relative minimum**.
- A minimum or maximum point shall be referred to generically as an **extremum point** of f.
- When the point x_0 is an end point of a closed interval we consider only a right (or a left) neighbourhood in the definitions.
- An absolute extremum point is also a local extremum point; the opposite is not true.



Examples

- Consider the function $f(x) = x^2$ in the interval $I = [-2, 1]$.
  - The point $x = -2$ is the absolute maximum point (and then a local maximum point);
  - the point $x = 0$ is the absolute minimum point (and then a local minimum point);
  - the point $x = 1$ a local maximum point;
  - $f(-2) = 4$ is the global maximum, $f(0) = 0$ is the global minimum and $f(1) = 1$ is a local maximum.
- Consider the function $f(x) = M(x)$ in the interval $I = [0, 3/2]$.
  - The points $x = 0$ and $x = 1$ are global (and local) minimum points; the global minimum is $0 = M(0) = M(1)$;
  - the point $x = 3/2$ is a local maximum point and $1/2 = M(3/2)$ a local maximum;
  - there is no global maximum point.

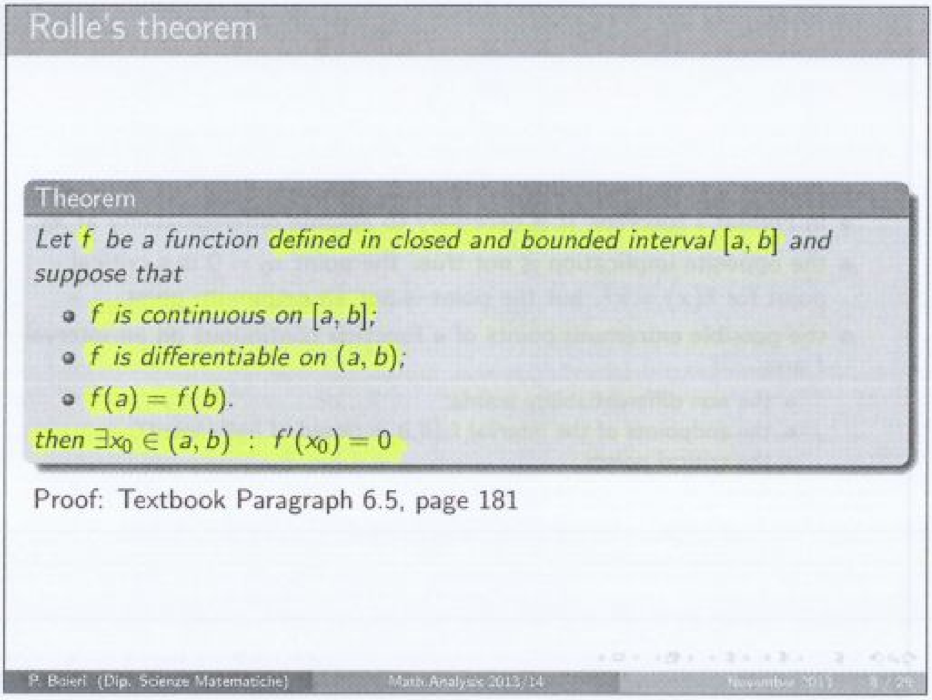


Monotonicity and sign of the derivative - First part

Theorem

If f is differentiable on an open interval I and it is increasing in I then $f'(x) \geq 0, \quad \forall x \in I$.

Remark. The strongest form of this statement, i.e. the proposition "If f is differentiable on an open interval I and it is strictly increasing in I then $f'(x) > 0, \quad \forall x \in I$" is not correct.
For instance, the function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, but its derivative $f'(x) = 3x^2$ is NOT strictly positive on $\mathbb{R}$.



Rolle's theorem

Theorem

Let f be a function defined in closed and bounded interval $[a, b]$ *and suppose that*

- f *is continuous on* $[a, b]$;
- f *is differentiable on* (a, b);
- $f(a) = f(b)$.

then $\exists x_0 \in (a, b) \; : \; f'(x_0) = 0$

Proof: Textbook Paragraph 6.5, page 181

Monotonicity and sign of the derivative - Second part

Theorem

Let f be a differentiable function on the open interval I; then

$$f \text{ is constant } \iff f'(x) = 0, \quad \forall x \in I.$$

Proof: Textbook Paragraph 6.6 page 185

Theorem

Let f be a differentiable function on the open interval I; then

$$f'(x) \geq 0, \quad \forall x \in I \Longrightarrow f \text{ is increasing in } I.$$

In the same hypotheses,

$$f'(x) > 0, \quad \forall x \in I \Longrightarrow f \text{ is strictly increasing in } I.$$

Analogous results hold for $f'(x) \leq 0$ and $f'(x) < 0$.

Proof: Textbook Paragraph 6.7 page 185

Classification of critical points

Theorem

Let f be a differentiable function on the open interval I and let $x_0 \in I$ a critical point for f.

- *If $f'(x) \geq 0$ in $I_r^-(x_0)$ and $f'(x) \leq 0$ in $I_r^+(x_0)$*

  $$\Longrightarrow \quad x_0 \text{ is a local maximum point for } f$$

- *If $f'(x) \leq 0$ in $I_r^-(x_0)$ and $f'(x) \geq 0$ in $I_r^+(x_0)$*

  $$\Longrightarrow \quad x_0 \text{ is a local minimum point for } f$$

Remark

Remark.
If $\nexists \lim\limits_{x\to\gamma} \dfrac{f'(x)}{g'(x)}$ it is NOT true that $\nexists \lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$.

Example.
We compute the limit

$$\lim_{x\to+\infty} \frac{2x+\sin x}{3x-\cos x} = \lim_{x\to+\infty} \frac{x\left(2+\frac{\sin x}{x}\right)}{x\left(3-\frac{\cos x}{x}\right)} = \frac{2}{3}$$

If we compute the derivatives and the limit of their quotient, we have

$$\lim_{x\to+\infty} \frac{2+\cos x}{3+\sin x} \qquad \text{that does not exist.}$$

Some important limits - 1

$$\boxed{\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty\,, \qquad \forall\alpha\in\mathbb{R}}$$

- If $\alpha \le 0$ it is obvious (the limit is not an indeterminate form).
- If $\alpha = 1$

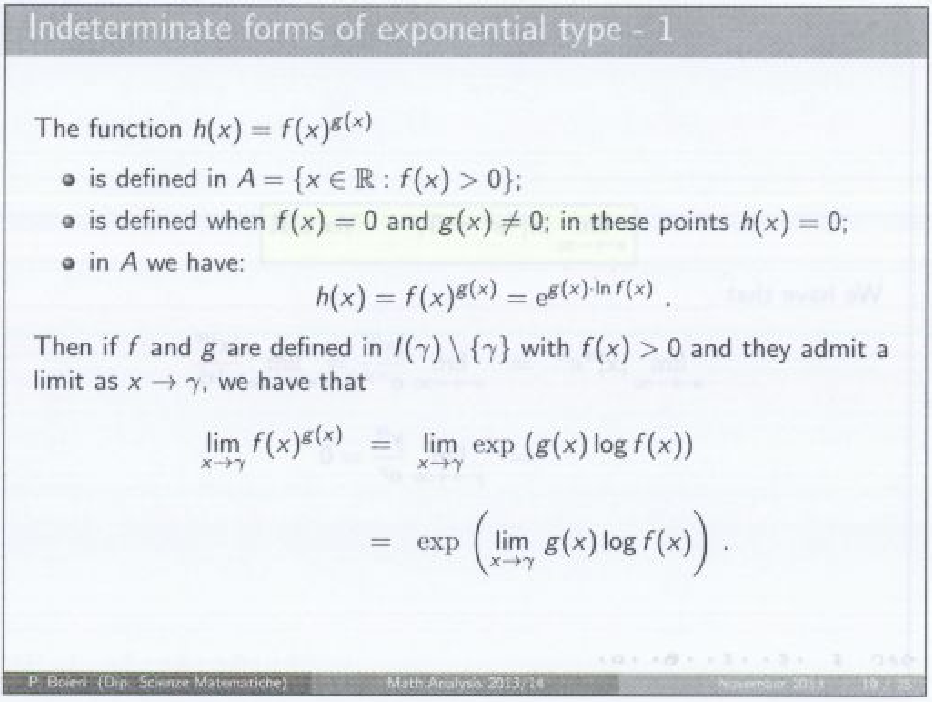
$$\lim_{x\to+\infty} \frac{e^x}{x} \overset{\text{Hôp}}{=} \lim_{x\to+\infty} \frac{e^x}{1} = +\infty$$

- If $\alpha > 0$

$$\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = \lim_{x\to+\infty} \left(\frac{e^{x/\alpha}}{\alpha\cdot x/\alpha}\right)^\alpha = \frac{1}{\alpha^\alpha}\left(\lim_{y\to+\infty} \frac{e^y}{y}\right)^\alpha = +\infty$$

Remark. The same result holds for

$$\lim_{x\to+\infty} \frac{a^x}{x^\alpha} = +\infty, \qquad \forall a>1,\ \forall\alpha\in\mathbb{R}$$



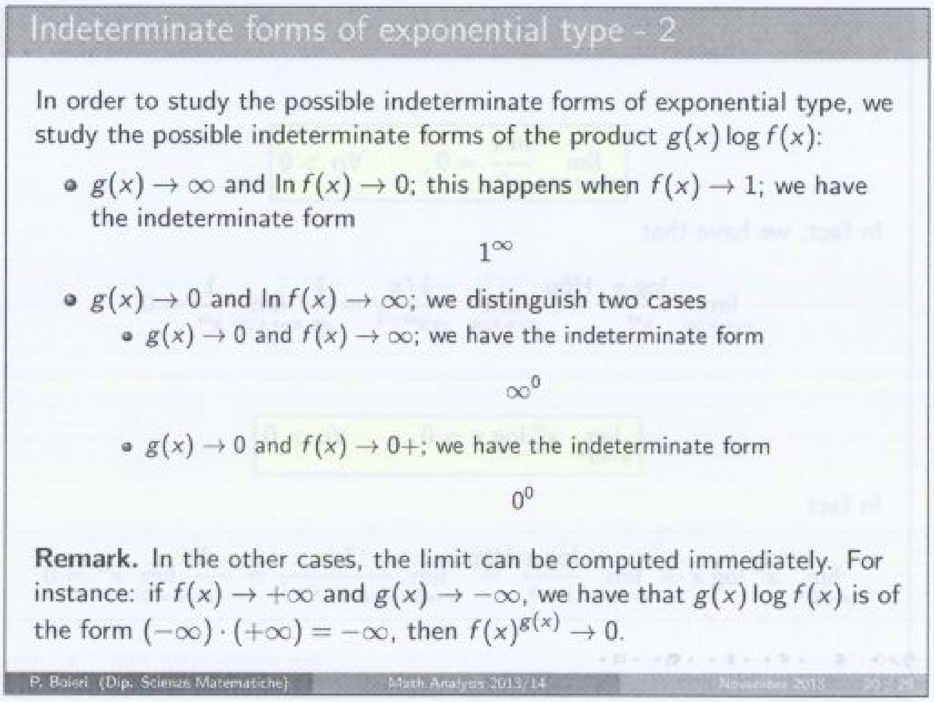
Indeterminate forms of exponential type - 1

The function $h(x) = f(x)^{g(x)}$

- is defined in $A = \{x \in \mathbb{R} : f(x) > 0\}$;
- is defined when $f(x) = 0$ and $g(x) \neq 0$; in these points $h(x) = 0$;
- in A we have:
$$h(x) = f(x)^{g(x)} = e^{g(x) \cdot \ln f(x)} .$$

Then if f and g are defined in $I(\gamma) \setminus \{\gamma\}$ with $f(x) > 0$ and they admit a limit as $x \to \gamma$, we have that

$$\begin{aligned} \lim_{x \to \gamma} f(x)^{g(x)} &= \lim_{x \to \gamma} \exp\left(g(x) \log f(x)\right) \\ &= \exp\left(\lim_{x \to \gamma} g(x) \log f(x)\right) . \end{aligned}$$



Indeterminate forms of exponential type - 2

In order to study the possible indeterminate forms of exponential type, we study the possible indeterminate forms of the product $g(x) \log f(x)$:

- $g(x) \to \infty$ and $\ln f(x) \to 0$; this happens when $f(x) \to 1$; we have the indeterminate form
$$1^{\infty}$$
- $g(x) \to 0$ and $\ln f(x) \to \infty$; we distinguish two cases
  - $g(x) \to 0$ and $f(x) \to \infty$; we have the indeterminate form
$$\infty^{0}$$
  - $g(x) \to 0$ and $f(x) \to 0+$; we have the indeterminate form
$$0^{0}$$

Remark. In the other cases, the limit can be computed immediately. For instance: if $f(x) \to +\infty$ and $g(x) \to -\infty$, we have that $g(x) \log f(x)$ is of the form $(-\infty) \cdot (+\infty) = -\infty$, then $f(x)^{g(x)} \to 0$.

Remarks

- The hypothesis "f continuous in $I(x_0)$" is necessary! Consider the following example

$$f(x) = \begin{cases} x-1 & \text{if} x < 0 \\ 0 & \text{if} x < 0 \\ x+1 & \text{if} x < 0 \end{cases}$$

is NOT differentiable at $x = 0$, since it is not continuous.
The derivative is $f'(x) = 1, \ \ \forall x \neq 0$ and then $\lim_{x \to x_0} f'(x) = 1$; if we apply the theorem (and we forget checking the continuity) we get the wrong conclusion that $f'(0) = 1$.

Example

We consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if} x \neq 0 \\ 0 & \text{if} x \neq 0 \end{cases}$$

- The function is continuous in $\mathbb{R}$.
- It is differentiable if $x \neq 0$ (as composition of differentiable functions) and its derivative is

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad \text{if } x \neq 0$$

- It is differentiable at zero, since

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

- The derivative at zero is NOT the limit of the derivative (the limit of the derivative does not exist).

DIFFERENTIABLE FUNCTIONS

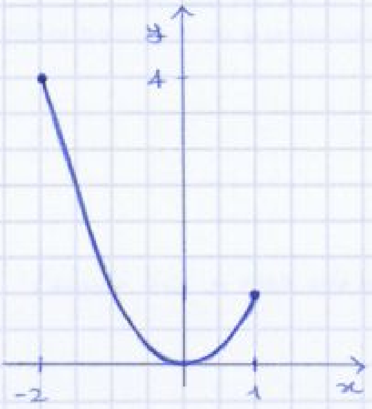
$x_0 \in \text{dom}f$ ABSOLUTE MAXIMUM POINT
(or GLOBAL)
$\forall x \in \text{dom}f$ $f(x) \leq f(x_0)$

$x_0 \in \text{dom}f$ ABSOLUTE MINIMUM POINT
(or GLOBAL)
$\forall x \in \text{dom}f$ $f(x) \geq f(x_0)$

$x_0 \in \text{dom}f$ RELATIVE MAXIMUM POINT (or LOCAL) for MINIMUM → $f(x) \geq f(x_0)$
$\exists I_r(x_0)$: $f(x) \leq f(x_0)$ $\forall x \in I_r(x_0) \cap \text{dom}f$

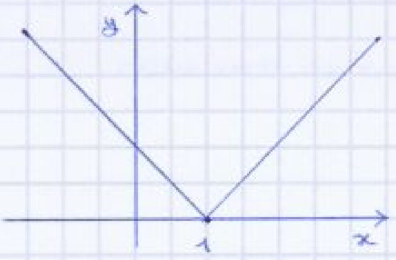
x_0 is an ABS. MAX. POINT $\Rightarrow$ x_0 is a REL. MAX. POINT
$\not\Leftarrow$

x_0 is a $\left.\begin{array}{l}\text{MAXIMUM point}\\ \text{MINIMUM point}\end{array}\right\} \Rightarrow x_0$ EXTREMUM

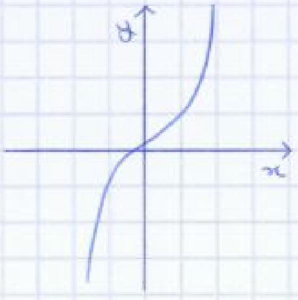


$f(x) = x^2$ $[-2;1]$
f is continuous, I is closed and bounded, according to Weierstrass theorem f has an absolute maximum and an absolute minimum.
How many GLOBAL EXTREMUM POINTS?
$x_1 = -2$ $f(x) \leq f(-2)$ $\forall x \in [-2;1]$ MAX.
$x_2 = 0$ $f(x) \geq f(0)$ $\forall x \in [-2;1]$ MIN
How many LOCAL EXTREMUM POINTS?
$x_1 = -2$ (also global)
$x_2 = 0$ (also global)
$x_3 = 1$ $\exists I_r(1)$: $f(x) \leq f(1)$ $\forall x \in I_r(1) \cap \text{dom}f$

$f'(x)$ at 0 is 0 → horizontal tangent line
$x_0 = 0$ is also a point of minimum



$f(x) = |x-1|$
$x_0 = 1$ is a minimum point
but the derivative at this point is not zero, because it does not exist

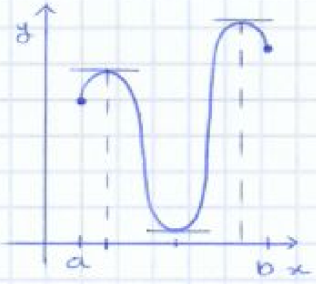


$f(x) = x^3$
$f'(0) = 0$ but x_0 is not a maximum and not a minimum

$\left.\begin{array}{l}f \text{ DIFFERENTIABLE at } x_0\\ f'(x_0) = 0\end{array}\right\} \Rightarrow x_0$ CRITICAL or STATIONARY POINT for f

ROLLE'S THEOREM

$$\left.\begin{array}{l}\bullet\ f \text{ CONTINUOUS on } [a,b]\\ \bullet\ f \text{ DIFFERENTIABLE on } (a,b)\\ \bullet\ f(a)=f(b)\end{array}\right\} \Rightarrow \begin{array}{l}\exists x_0 \in (a,b)\\ f'(x_0)=0\end{array}$$



x_0 is not unique

PROOF: f is continuous on $[a,b]$ ⇒ I can apply Weierstrass theorem:

$\exists x_M : f(x_M)=M$

$\exists x_m : f(x_m)=m$

where are these two points?

1) x_m and x_M are the end points

$\{x_m; x_M\} = \{a,b\}$

$f(a)=f(b) \Rightarrow m= f(x_m) = f(x_M) = M$ this function is a constant, because $m=M$

$f(x)=m \Rightarrow f'(x)=0$

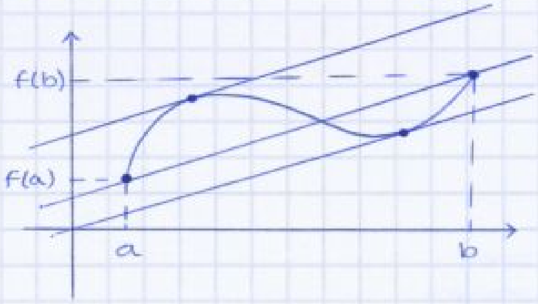
2) At least one of x_m and x_M is not an end point ⇒ it belongs to (a,b)

$$\left.\begin{array}{r} x_m \in (a,b) \Rightarrow f \text{ diff at } x_m\\ x_m \text{ local minimum point}\end{array}\right\} \underset{\text{Fermat's theorem}}{\Rightarrow} f'(x_m)=0$$

LAGRANGE'S THEOREM

$$\left.\begin{array}{l}\bullet\ f \text{ CONTINUOUS on } [a,b]\\ \bullet\ f \text{ DIFFERENTIABLE on } (a,b)\end{array}\right\} \Rightarrow \begin{array}{l}\exists x_0 \in (a,b)\\ f'(x_0)=\dfrac{f(b)-f(a)}{b-a}\end{array}$$

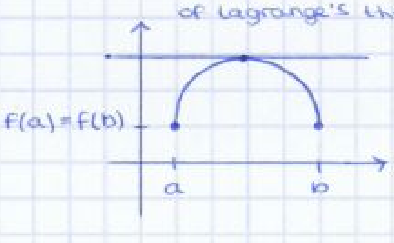
x_0 = LAGRANGE POINT for f in $[a,b]$



slope = $\dfrac{f(b)-f(a)}{b-a}$

$\exists x_0$ such that the tangent line of the graph passing through x_0 is parallel to the line joining A and B (same slope)

$f(a)=f(b)$ particular case of Lagrange's theorem ⇒ Rolle's theorem ⇒ $f'(x_0)=0$



slope = 0

→ horizontal line

THEOREM APPLICATION 1 - PROPERTIES OF FUNCTIONS WITH ZERO DERIVATIVE

- f DIFFERENTIABLE on an open interval I
- f CONSTANT in I $\Leftrightarrow$ $f'(x)$ is ZERO $\forall x \in I$

PROOF:

$\Rightarrow$ the derivative of a constant is zero $f(x) = c \to f'(x) = 0$

$\Leftarrow$ choose $x_1, x_2 \in I$ $x_1 < x_2$

$\Delta f = f'(t)\Delta x$ $f(x_2) - f(x_1) = f'(t)(x_2 - x_1)$ $t \in (x_1, x_2)$

$f(x_2) - f(x_1) = 0$ $f(x_2) = f(x_1)$

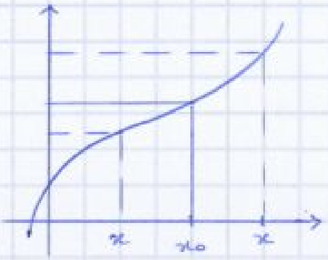
x_2 fixed, vary $x_1 \in I$

$\Rightarrow$ the function is a CONSTANT

TH. APPLICATION 2

- f DIFFERENTIABLE on an open interval I
- f INCREASING in I $\Rightarrow$ $f'(x) \geq 0$ $\forall x \in I$

PROOF: (similar the the proof of Fermat's theorem)



$x > x_0$ $\dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$

$x < x_0$ $\dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$

f increasing $\Rightarrow$ in $I(x_0) \setminus \{x_0\}$ $\dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$ $\xrightarrow{x \to x_0}$ $f'(x_0) \geq 0$

corollary of sign and limit

If I consider a strictly increasing function, can I say that the derivative is strictly positive?

$\begin{cases} f \text{ strictly increasing} \Rightarrow f'(x) > 0 \\ f \text{ differentiable} \end{cases}$ This is not true, $\Rightarrow f'(x) \geq 0$

ex: $f(x) = x^3$

strictly increasing on $\mathbb{R}$

$f'(x) = 3x^2 \geq 0$ not strictly positive

$\Rightarrow$ f STRICTLY INCREASING or INCREASING implies always $f'(x) \geq 0$

(f strictly decreasing or decreasing implies $f'(x) \leq 0$)

All these results are not true if I consider sets that are not intervals

Ex: $f(x) = \arctan x + \arctan \frac{1}{x}$ F def in $\mathbb{R} \setminus \{0\}$ → it's not an interval

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} + \frac{1}{\frac{x^2+1}{\cancel{x^2}}} \cdot \left(-\frac{1}{\cancel{x^2}}\right) =$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

This function has derivative zero: according to the previous theorem application result, if $f'(x)=0 \Rightarrow f(x)$ is a constant

$f(1) = \frac{\pi}{2}$

$f(-1) = -\frac{\pi}{2}$

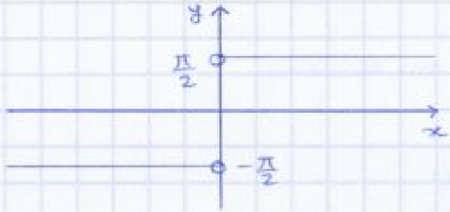
⇒ f is not a constant function (even if $f'(x)=0$) because it has different values

$\mathbb{R} \setminus \{0\}$ is not an interval → the theorem cannot be applied

1) I consider $(0;+\infty)$ → it's an interval and I can apply the theorem $f(x)=c$ in this interval

2) I consider $(-\infty;0)$ → it's an interval $f(x)=c$ but I don't know if the constant is the same as the previous, the two constants may be different

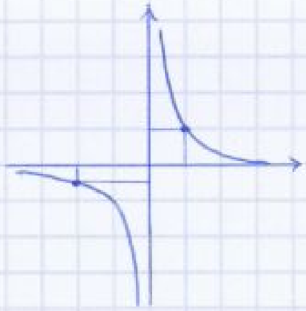
$f(1) = \frac{\pi}{2}$ $f(-1) = -\frac{\pi}{2}$



f(x) is not constant in its domain, which is not an interval but it's constant in the two intervals

Ex: $f(x) = \frac{1}{x}$ $\mathbb{R} \setminus \{0\}$

$f'(x) = -\frac{1}{x^2} < 0 \Leftrightarrow$ strictly decreasing



this function is not decreasing in its domain $\mathbb{R} \setminus \{0\}$ └ not an interval ⇒ f(x) is strictly decreasing in two intervals:

1) $(-\infty;0)$

2) $(0;+\infty)$

Ex: $\lim_{x\to+\infty} \frac{e^x}{x^\alpha} \quad \alpha>0$

$\lim_{x\to+\infty} \frac{e^x}{x} = \frac{\infty}{\infty}$ I can appply De l'Hôpital

$\lim_{x\to+\infty} \frac{e^x}{1} = +\infty \Rightarrow \lim_{x\to+\infty} \frac{e^x}{x} = +\infty$

$$\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = \lim_{x\to+\infty} \frac{\left(e^{\frac{x}{\alpha}}\right)^\alpha}{\left(\alpha\frac{x}{\alpha}\right)^\alpha} \overset{y=\frac{x}{\alpha}}{=} \lim_{y\to+\infty} \frac{(e^y)^\alpha}{\alpha^\alpha y^\alpha} = \frac{1}{\alpha^\alpha} \lim_{y\to+\infty} \underbrace{\left(\frac{e^y}{y}\right)^\alpha}_{+\infty} = +\infty$$

$f(x) = \frac{e^x}{x^{1000}} \xrightarrow{x\to+\infty} +\infty$ $\quad \lim_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty$

Ex: $\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = \frac{\infty}{\infty} \quad \alpha>0$

f, g are diff, $g'(x) \neq 0$ at $x\to+\infty$
I can appply De l'Hôpital

$$\lim_{x\to+\infty} \frac{1}{x} \cdot \frac{1}{\alpha x^{\alpha-1}} = \lim_{x\to+\infty} \frac{1}{\alpha x^\alpha} = \frac{1}{+\infty} = 0$$

Ex: $\lim_{x\to0^+} \underset{\substack{\downarrow \\ 0}}{x^\alpha} \cdot \underset{\substack{\downarrow \\ -\infty}}{\log x} = 0 \cdot \infty \quad \alpha>0$

∟ not $\frac{0}{0}$ or $\frac{\infty}{\infty}$

I have to try to rewrite

$0 \cdot \infty \rightarrow \frac{0}{0}, \frac{\infty}{\infty}$

$\lim_{x\to0^+} \frac{\log x}{\frac{1}{x}} = \left[\frac{\infty}{\infty}\right]$ or $\lim_{x\to0^+} \frac{x}{\frac{1}{\log x}} = \left[\frac{0}{0}\right]$

$\frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x \to 0$ $\quad \frac{1}{-\frac{1}{\log^2 x} \cdot \frac{1}{x}} = \frac{x}{-\frac{1}{\log^2 x}}$ → it's not possible to solve it
I have the same type of limit

$\lim_{x\to0^+} x^\alpha \log x = 0 \quad \alpha>0$

$$\frac{\log x}{x^{-\alpha}} \quad \frac{\frac{1}{x}}{(-\alpha)x^{-\alpha-1}} = \frac{1}{-\alpha} \cdot \frac{1}{x^{-\alpha}} = -\frac{1}{\alpha} x^\alpha \to 0$$

$\forall \alpha>0$

Ex: $\lim_{x\to-\infty} \underset{\substack{\downarrow \\ +\infty}}{|x|^\alpha} \underset{\substack{\downarrow \\ 0}}{e^x} = \infty \cdot 0 \quad \alpha>0$

$$\lim_{x\to-\infty} |x|^\alpha e^x = \lim_{x\to-\infty} \frac{|x|^\alpha}{\frac{1}{e^x}} = \lim_{x\to-\infty} \frac{|x|^\alpha}{e^{-x}} = \lim_{x\to-\infty} \frac{|x|^\alpha}{e^{|x|}} =$$

$-x$ for $x\to-\infty$ $= |x|$

$y=|x|$

$= \lim_{y\to+\infty} \frac{y^\alpha}{e^y} = 0$ $\quad \lim_{x\to+\infty} \frac{x^\alpha}{e^x} = 0$

Mathematical Analysis I (2013-2014)

Differential Calculus 3 - Convexity

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Convexity and concavity at a point

Let f be differentiable at x_0 and let $y = t(x) = f(x_0) + f'(x_0)(x - x_0)$ be the equation of the tangent line to the graph of f at x_0.

Definition. The function f differentiable at x_0 is **convex at** x_0 if $\exists I_r(x_0) \subseteq \mathrm{dom}\, f$ such that

$$\forall x \in I_r(x_0), \quad f(x) \geq t(x)$$

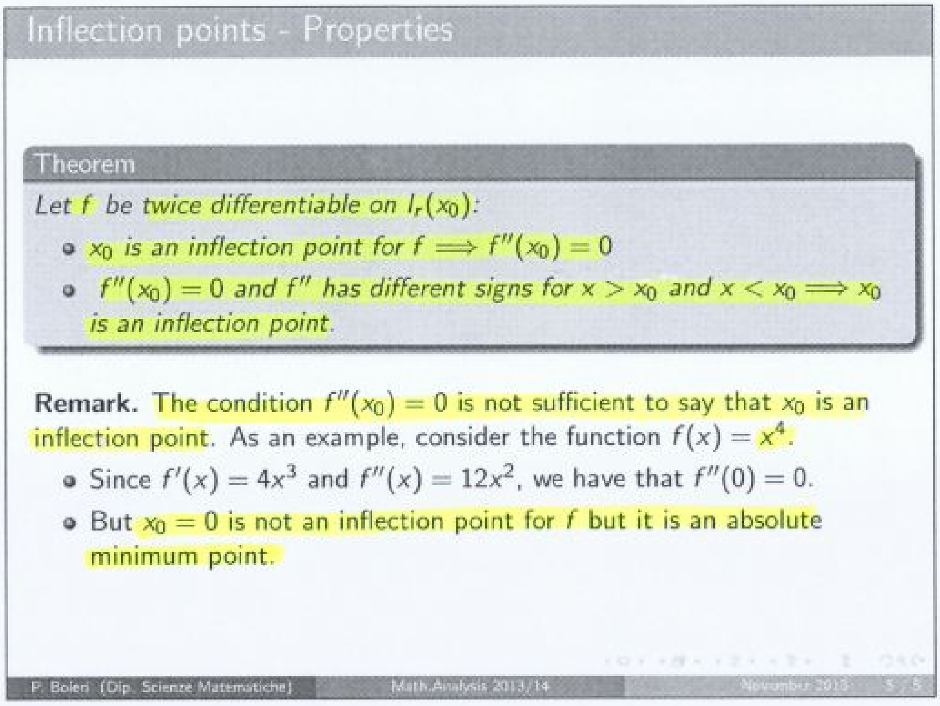
We say that f is **strictly convex** if $f(x) > t(x)$ for $x \neq x_0$ (in $I_r(x_0)$).

Definition. The function f differentiable at x_0 is **concave at** x_0 if $\exists I_r(x_0) \subseteq \mathrm{dom}\, f$ such that

$$\forall x \in I_r(x_0), \quad f(x) \leq t(x)$$

We say that f is **strictly concave** if $f(x) < t(x)$ for $x \neq x_0$ (in $I_r(x_0)$).

Definition. If f is differentiable on I (open interval in $\mathbb{R}$) we say that f is convex on I if it is convex for all $x \in I$.
Analogous definition for f is convex on I .



Inflection points - Properties

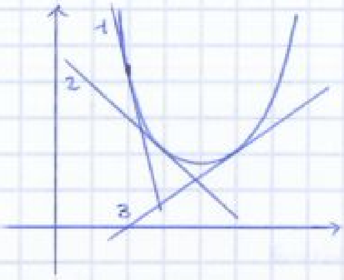
Theorem

Let f be twice differentiable on $I_r(x_0)$:

- x_0 *is an inflection point for* $f \Longrightarrow f''(x_0) = 0$
- $f''(x_0) = 0$ *and* f'' *has different signs for* $x > x_0$ *and* $x < x_0 \Longrightarrow x_0$ *is an inflection point.*

Remark. The condition $f''(x_0) = 0$ is not sufficient to say that x_0 is an inflection point. As an example, consider the function $f(x) = x^4$.

- Since $f'(x) = 4x^3$ and $f''(x) = 12x^2$, we have that $f''(0) = 0$.
- But $x_0 = 0$ is not an inflection point for f but it is an absolute minimum point.



tangent lines:
1. decreasing faster than 2
2. decreasing, but less than 1
3. increasing

this function is convex because the slope of the tangent line is an increasing function

THEOREM

- f DIFFERENTIABLE on an interval I
- f CONVEX in I $\Leftrightarrow$ f' INCREASING in I

THEOREM

- f TWICE DIFFERENTIABLE on an interval I

f CONVEX in I $\Leftrightarrow$ $f''(x) \geq 0$ $\forall x \in I$

because the derivative of an increasing function is a positive function

CLASSIFICATION OF INFLECTION POINTS

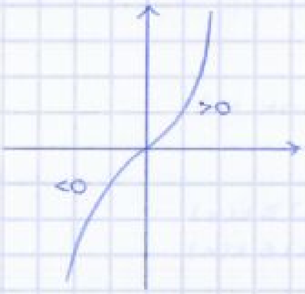
f TWICE DIFFERENTIABLE at x_0

- x_0 is an inflection point $\Rightarrow$ $f''(x_0) = 0$ (analogous of Fermat's theorem)

  the opposite is not true:

  $f''(x_0) = 0 \not\Rightarrow x_0$ is an inflection point
- $f''(x_0) = 0$ and f'' has different signs for $x > x_0$ and $x < x_0$ $\Rightarrow$ x_0 is an inflection point

ex: $f(x) = x^3$ $f'(x) = 3x^2$ $f''(x) = 6x$

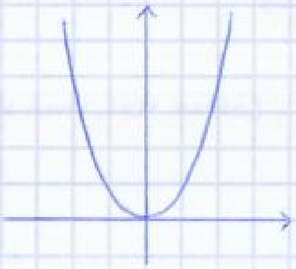


$x_0 = 0$ $f''(0) = 0$

└ INFLECTION POINT

$f''(x)$ has different signs for $x > x_0$ and $x < x_0$

ex: $f(x) = x^4$ $f'(x) = 4x^3$ $f''(x) = 12x^2$



$x_0 = 0$

$f''(0) = 0$ but $x_0 = 0$ is not an inflection point because $f(x)$ is a convex function at all points

Landau's symbols - 2

Remarks.

- It is possible to define another Landau's symbol (not used in this course).

Definition

If $\lambda = l \in \mathbb{R}$, we say that f **is controlled by** *g for $x \to \gamma$;*

$$f = O(g), \qquad x \to \gamma$$

read as "f is big-o of g".

- We did not give a definition for λ infinite; since when $f/g \to \infty$ we have that $g/f \to 0$, we say that

Definition

If $\lambda = \infty$, we say that g **is negligible with respect to** *f for $x \to \gamma$, i.e.*

$$g = o(f), \qquad x \to \gamma \; .$$

Landau's symbols - 3

- From the definitions it is immediate to see that $f \sim g$ is a particular case of $f \asymp g$, i.e. $f \sim g \Longrightarrow f \asymp g$.
- The opposite is not true; but since

$$f \asymp g \Longleftrightarrow \lim_{x\to\gamma} \frac{f(x)}{g(x)} = l \neq 0 \Rightarrow \lim_{x\to c} \frac{f(x)}{lg(x)} = 1 \Rightarrow f \sim lg$$

we have that $\quad f \asymp g \Longrightarrow \exists l \neq 0 \;$ such that $\; f \sim lg$.

- We have that

$$f \text{ is infinitesimal} \Longleftrightarrow f = o(1).$$

In fact

$$f = o(1)\,,\; x \to \gamma \quad \Longleftrightarrow \quad \lim_{x\to\gamma} \frac{f(x)}{1} = \lim_{x\to\gamma} f(x) = 0.$$

Comparison of infinite functions

Definition

A function f is said to be **infinite at** γ if $\lim_{x\to\gamma} f(x) = \infty$.

Definition

Let f and g be two infinite functions at γ.

- If $f \asymp g$ for $x \to \gamma$, then f and g are said to be **infinite of the same order**;
- If $f = o(g)$ for $x \to \gamma$, then f is called **infinite of smaller order than** g; ex: $x \to \infty$ $x = o(x^3)$
- If $g = o(f)$ for $x \to \gamma$, then f is called **infinite of bigger order than** g; ex: $x \to \infty$ $x^5 = o(x^7)$
- If none of the above are satisfied, the infinite functions f and g are said **non-comparable**.

Infinite functions at $+\infty$

The functions $f(x) = x^\alpha$ $(\alpha > 0)$ are infinite at $+\infty$.

If $\alpha_1 > \alpha_2 > 0$, we have that $\lim_{x+\infty} \dfrac{x^{\alpha_1}}{x^{\alpha_2}} = +\infty$.

Then a power function with a lower exponent is negligible with respect to a power function with a higher exponent.

It is natural to classify the behaviour at $+\infty$ of these functions according to the exponent: the higher is the exponent, the higher is the order.

For other infinite functions at $+\infty$ we introduce the following definition:

Definition

An infinite function at $+\infty$ is said to be an **infinite of (real) order** α if there exists a real number $l \neq 0$ such that

$$f \sim lx^\alpha \quad (\text{for } x \to +\infty).$$

$f \asymp g$ not ($f = o(g)$
or $g = o(f)$)

Order and principal part

In order to classify the infinite functions at $+\infty$ we introduce a test function ($\varphi(x) = x$) and we compare the other functions with the powers of this test function. This can be done for infinite functions (and infinitesimal functions) at all points.
Then we can give the general definition.

Definition

Let φ be the infinitesimal (resp. infinite) test function at γ.
If $\exists \alpha > 0,\ \exists l \neq 0$ such that $f \sim l\varphi^\alpha$ for $x \to \gamma$ then:

- the positive number α is called the **order of** f at γ with respect to the infinitesimal (resp. infinite) test function φ.
- the function $\varphi^\alpha(x)$ is called **principal part** of the infinitesimal (resp. infinite) with respect to the infinitesimal (resp. infinite) test function φ.

Infinite test function

The choice of a test function is arbitrary; usually the following functions are used as test functions:

- $\gamma = +\infty \qquad \varphi(x) = x,$
- $\gamma = -\infty \qquad \varphi(x) = |x|,$
- $\gamma = 0+ \qquad \varphi(x) = \frac{1}{x},$ $\frac{1}{0} = \infty$
- $\gamma = 0- \qquad \varphi(x) = -\frac{1}{x} = \frac{1}{|x|},$
- $\gamma = x_0+ \qquad \varphi(x) = \frac{1}{x - x_0},$ $\frac{1}{0} = \infty$
- $\gamma = x_0- \qquad \varphi(x) = \frac{1}{x_0 - x} = \frac{1}{|x - x_0|}.$

In particular, when we are interested in integer powers only we can use

- $\gamma = \pm\infty \qquad \varphi(x) = x$
- $\gamma = 0 \qquad \varphi(x) = \frac{1}{x}$
- $\gamma = x_0 \qquad \varphi(x) = \frac{1}{x - x_0}$

Some properties of Landau's symbols - 1

Theorem

If $\lambda \in \mathbb{R} \setminus \{0\}$, for $x \to \gamma$, we have that

$$a) \quad o(\lambda f) = \lambda o(f) = o(f)$$

$$b) \quad f \sim g \iff f = g + o(g)$$

Proof.

- a) Suppose that $g = o(\lambda f)$, $x \to \gamma$. Then

$$g = o(\lambda f) \iff \lim_{x \to \gamma} \frac{g(x)}{\underset{\text{costante}}{\lambda} f(x)} = 0$$

$$\iff \lim_{x \to \gamma} \frac{g(x)}{f(x)} = 0$$

$$\iff g = o(f),\ x \to \gamma .$$

Some properties of Landau's symbols - 2

- b) We have that

$$f \sim g \iff \lim_{x \to \gamma} \frac{f(x)}{g(x)} = 1 \iff \lim_{x \to \gamma} \frac{f(x)}{g(x)} - 1 = 0$$

$$\iff \lim_{x \to \gamma} \frac{f(x) - g(x)}{g(x)} = 0 \iff f - g = o(g)$$

$$\iff f = g + o(g) .$$

Example

We compute $L = \lim\limits_{x\to 0} \dfrac{\sin^2 2x}{1-\cos 3x}$. Since

- $\sin^2 2x \sim (2x)^2\,, \quad x \to 0 \quad \text{i.e.} \quad \sin^2 2x \sim 4x^2\,, \quad x\to 0$
- $1-\cos 3x \sim \dfrac{1}{2}(3x)^2\,, \quad x\to 0 \quad \text{i.e.} \quad 1-\cos 3x \sim \dfrac{9}{2}x^2\,, \quad x \to 0$

we have that

$$L = \lim_{x\to 0} \frac{4x^2}{\frac{9}{2}x^2} = \frac{8}{9}$$

Elimination of negligible summands

The second property allows us to ignore negligible summands with respect to others within one factor.

Theorem

If $f_1 = o(f)$ and $g_1 = o(g)$ for $x \to c$ then

$$\lim_{x\to\gamma} \frac{f(x)+f_1(x)}{g(x)+g_1(x)} = \lim_{x\to\gamma} \frac{f(x)}{g(x)}\,,$$

$$\lim_{x\to\gamma} \big(f(x)+f_1(x)\big)\big(g(x)+g_1(x)\big) = \lim_{x\to\gamma} f(x)\,g(x)\,.$$

Algebra of little-o's

The Landau's symbols allow us to simplify formulas when studying limits. We consider here the most important case: the limit for $x \to 0$. All the following properties, which define a special "algebra of little o's" can be extended to $x \to x_0$, substituting $x - x_0$ to x.

$$o(x^n) \pm o(x^n) = o(x^n)\,;$$

$$o(x^n) \pm o(x^m) = o(x^p)\,; \quad \text{where } p = \min(n, m)\,;$$

$$o(\lambda x^n) = \lambda o(x^n) = o(x^n), \qquad \forall \lambda \in \mathbb{R} \setminus \{0\}\,;$$

$$\varphi(x) o(x^n) = o(x^n) \qquad \text{if } \varphi \text{ is bounded in } I_r(0)\,;$$

$$x^m o(x^n) = o(x^{m+n})\,;$$

$$o(x^m) o(x^n) = o(x^{m+n})\,;$$

$$[o(x^n)]^k = o(x^{kn})\,.$$

4) $\lambda = \infty$ $\quad \lim_{x \to \gamma} \frac{f(x)}{g(x)} = \infty \Rightarrow \lim_{x \to \gamma} \frac{g(x)}{f(x)} = 0$

$g = o(f)$

5) $\lim_{x \to \gamma} \frac{f(x)}{g(x)} \nexists$ $\quad$ f and g are not comparable

$\lambda = \ell \in \mathbb{R}$ $\quad f = O(g)$ big o

Examples:

1) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$ $\quad \log(1+x) \sim x \quad (x \to 0)$

they're equivalent in a small neighborhood

2) $\lim_{x \to 0} \frac{\log\left(1+\frac{x}{2}\right)}{x} = \lim_{x \to 0} \frac{\log\left(1+\frac{x}{2}\right)}{\frac{x}{2} \cdot 2} = \frac{1}{2}$ $\quad \ell \in \mathbb{R}$

$\log\left(1+\frac{x}{2}\right) \asymp x \quad (x \to 0)$ same order (same ratio, proportion between them)

3) $\lim_{x \to 0} \frac{\log(1+x^2)}{x} = \lim_{x \to 0} \frac{\log(1+x^2)}{x^2} \cdot x = 0$

$\log(1+x^2) = o(x) \quad (x \to 0)$ negligible (it disappears with respect to the other)

Remark: • $\underset{1 \neq 0}{f \sim g} \Rightarrow \underset{\ell \neq 0}{f \asymp g}$

- $f \asymp g$ $\quad \lim \frac{f}{g} = \ell \neq 0$ $\quad \lim \frac{f}{\ell g} = \frac{1}{\ell}$ $\quad \lim \frac{f}{g} - \frac{1}{\ell} \cdot \ell = 1$

  $\Rightarrow f \sim \ell g$

  $$\left.\begin{array}{l} f \asymp g \\ \lim \frac{f}{g} = \ell \end{array}\right\} \Rightarrow f \sim \ell g$$

- $o(\lambda f) = o(f)$ $\quad \lambda \neq 0$

  if $g = o(\lambda f)$ then $g = o(f)$

  if $g = o(f)$ then $g = o(\lambda f)$

  $g = o(f) \Leftrightarrow g = o(\lambda f)$

  $g = o(f) \Leftrightarrow \lim \frac{g}{f} = 0$ hypothesis

  $g = o(\lambda f) \Leftrightarrow \lim \frac{g}{\lambda f} = 0$

  $\lim \frac{g}{\lambda f} = \frac{1}{\lambda} \lim \frac{g}{f} = \frac{1}{\lambda} \cdot 0 = 0 \Rightarrow g = o(\lambda f)$

  $o(\lambda f) = \lambda o(f) = o(f)$

$\boxed{o}$ 1) $f \neq o(f)$ because $f \sim f$

2) $f = o(g) \Rightarrow g = o(f)$ NO

$\frac{f}{g} \to 0 \qquad \frac{g}{f} \to \infty$

3) $f = o(g),\ g = o(h) \Rightarrow f = o(h)$

$\frac{f}{g} \to 0 \qquad \frac{g}{h} \to 0$

$\frac{f}{h} = \frac{f}{g} \cdot \frac{g}{h} \to 0 \Rightarrow f = o(h)$

examples:

- $f \asymp g,\ g = o(h) \Rightarrow f \,?\, h$

  $\frac{f}{g} \to \ell \neq 0 \qquad \frac{g}{h} \to 0$

  $\frac{f}{h} = \frac{f}{g} \cdot \frac{g}{h} \to 0 \Rightarrow f = o(h)$

- $f = o(g),\ g \sim h \Rightarrow f \,?\, h$

  $\frac{f}{g} \to 0 \qquad \frac{g}{h} \to 1$

  $\frac{f}{h} = \frac{f}{g} \cdot \frac{g}{h} \to 0 \Rightarrow f = o(h)$

$f, g \quad x \to \gamma \quad \left\langle \begin{array}{l} f \sim g \\ f \asymp g \\ f = o(h) \end{array} \right.$

$\lim_{x \to \gamma} \frac{f(x)}{g(x)}$ I cannot give immediately an answer when I get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (critical cases)

$x \to 0 \quad f(x) = \cos x \quad g(x) = e^x + 5 \qquad \lim_{x \to 0} \frac{\cos x}{e^x + 5} = \frac{1}{6} \quad \ell \neq 0 \quad \cos x \asymp e^x + 5$

classification depends on γ

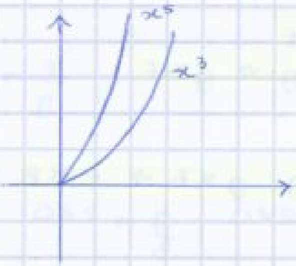
$\boxed{x \to +\infty}$ $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = \infty$

$\lim_{x \to \gamma} f(x) = \infty$ INFINITE FUNCTION for $x \to \gamma$

ex: $f(x) = x^3$ $g(x) = x^5$ for $x \to +\infty$ f and $g \to +\infty$ but how they tend to ∞?

	f	g
1	1	1
2	8	32
3	27	243

g(x) grows much faster than f(x)



$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{x^3}{x^5} = \frac{1}{x^2} = 0 \Rightarrow x^3 = o(x^5)$

x^3 tends to $+\infty$ slower than x^5

- $f(x) = x^{\alpha_1}$
- $g(x) = x^{\alpha_2}$

$\Rightarrow 0 < \alpha_1 < \alpha_2 \Rightarrow x^{\alpha_1} = o(x^{\alpha_2})$ for $x \to +\infty$

the lowest exponent corresponds to the negligible function ex:

$\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty \quad \forall\alpha>0 \qquad x^\alpha = o(e^x)$

Exponential functions are INFINITE of HIGHER ORDER than the all power functions

$\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = 0 \quad \forall\alpha>0 \qquad \log x = o(x^\alpha)$

$\xrightarrow{\log x} \qquad \xrightarrow{x^\alpha} \qquad \xrightarrow{e^x}$

these functions tend to ∞ slower than x^α
$(\log x)^\gamma$ $\gamma>0$ classification according to the order γ

these functions tend to ∞ faster than x^α
$e^{\beta x}$ $\beta>0$ classification according to the order β

$x^{10} = o(e^{0,3x})$
$(\log x)^{20} = o(x^{0,001})$

it's not possible to classify all functions. e.g: e^{x^2} higher order than all possible exponentials
$\log(\log x)$ infinite of lower order than $\log x$

$f(x) = x\log x$ higher than $x^1 \to x^{1+\varepsilon}$

$\lim_{x\to+\infty} \frac{x\log x}{x} = +\infty \qquad \frac{x}{x\log x} \to 0 \qquad x = o(x\log x)$ $x\log x$ tends to ∞ faster than x

$\lim_{x\to+\infty} \frac{x\log x}{x^{1+\varepsilon}} = \lim_{x\to+\infty} \frac{x\log x}{x\cdot x^\varepsilon} = 0 \qquad x\log x = o(x^{1+\varepsilon}) \quad \forall\varepsilon>0$

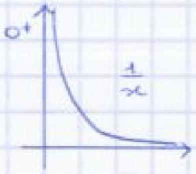
$x\log x \qquad \frac{x}{\log x}$ I cannot classify these functions in the scale

$x\log^\delta x \to$ classify according to δ
ex: $x\ln^3 x$ is an infinite of higher order than $x\ln^2 x$

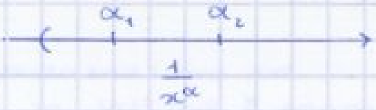
At $-\infty$ it's necessary to put an absolute value : $|x|^\alpha$

INFINITE FUNCTIONS at $x_0 = 0$

$\boxed{x\to0^+}$ ex: $f(x) = \frac{1}{x} \qquad \lim_{x\to0^+} \frac{1}{x} = +\infty$



$\frac{1}{x^2} \to x^2 < x \qquad \frac{1}{x^2} > \frac{1}{x}$



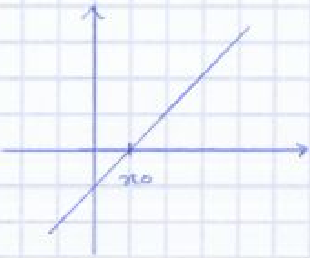
$\frac{l}{x^\alpha} \sim f(x)$

ex: $f(x) = \frac{1}{\sin 3x} \qquad \lim_{x\to0^+} \frac{1}{\sin 3x} = +\infty$

$\lim_{x\to0^+} \frac{\frac{1}{\sin 3x}}{\frac{1}{3x}} = \lim_{x\to0^+} \frac{3x}{\sin 3x} = 1 \qquad \frac{1}{\sin 3x} \sim \frac{1}{3x} \quad x\to0^+$

$x \to 0^-$ $f(x) = x^n$
$f(x) = |x|^\alpha$

$x \to x_0^+$ $f(x) = (x - x_0)^\alpha$



$x \to x_0^-$ $f(x) = |x - x_0|^\alpha$ or $f(x) = (x_0 - x)^\alpha$

TEST FUNCTIONS

- $x \to +\infty$
  - INFINITE $f(x) = x^\alpha$ $\varphi(x) = x$
  - INFINITESIMAL $f(x) = \frac{1}{x^\alpha}$ $\varphi(x) = \frac{1}{x}$
- $x \to 0^+$
  - INFINITE $f(x) = \frac{1}{x^\alpha}$ $\varphi(x) = \frac{1}{x}$
  - INFINITESIMAL $f(x) = x^\alpha$ $\varphi(x) = x$
- $x \to x_0^+$
  - INFINITE $f(x) = \frac{1}{(x - x_0)^\alpha}$ $\varphi(x) = \frac{1}{x - x_0}$
  - INFINITESIMAL $f(x) = (x - x_0)^\alpha$ $\varphi(x) = x - x_0$

(for $-\infty$, 0^- and x_0^- put the absolute value)

$g(x) \sim \ell \cdot (\varphi(x))^\alpha$ $\ell \in \mathbb{R}$ $\alpha > 0$

- $g(x)$ is an $\begin{cases}\text{INFINITE} \\ \text{INFINITESIMAL}\end{cases}$ of order α
- $\ell\,(\varphi(x))^\alpha$ is the PRINCIPAL PART of $g(x)$ for $x \to \gamma$

Examples:

$g(x) = e^{x^2} - 1$ $x_0 = 0$

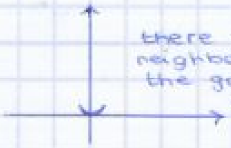
$\lim_{x \to 0} e^{x^2} - 1 = 1 - 1 = 0$ $g(x)$ is an infinitesimal at $x_0 = 0$

$e^{x^2} - 1 \sim \ell x^\alpha$ the test function for an infinitesimal function at $x_0 = 0$ is $\varphi(x) = x$

$\lim_{x \to 0} \frac{e^{x^2} - 1}{\ell x^\alpha} = 1$ $\lim_{x \to 0} \frac{e^z - 1}{z} = 1$ $\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} = 1$

$\Rightarrow e^{x^2} - 1 \sim x^2$ $(x \to 0)$

- $g(x)$ is an infinitesimal of order 2
- p.p is x^2



there is a small neighborhood where the graph of $g(x)$ is like x^2

THEOREM 1 - Substitution of equivalent functions

$f_1 \sim f \quad g_1 \sim g \quad x \to \gamma$

a) $\lim_{x\to\gamma} \frac{f(x)}{g(x)} = \lim_{x\to\gamma} \frac{f_1(x)}{g_1(x)}$ QUOTIENT

b) $\lim_{x\to\gamma} f(x)\cdot g(x) = \lim_{x\to\gamma} f_1(x)\cdot g_1(x)$ PRODUCT

PROOF: $\lim_{x\to\gamma} \frac{f}{g} = \lim_{x\to\gamma} \frac{f}{f_1}\cdot f_1 \cdot \frac{g_1}{g}\cdot\frac{1}{g_1} = \lim_{x\to\gamma} \underset{\substack{\downarrow\\1}}{\frac{f}{f_1}} \cdot \underset{\substack{\downarrow\\1}}{\frac{g_1}{g}} \cdot \underset{\substack{\downarrow\\1}}{\frac{f_1}{g_1}} = \lim_{x\to\gamma}\frac{f_1}{g_1}$

For quotients and products it's possible to substitute equivalent functions

ex: $\lim_{x\to 0} \frac{1-\cos x^2}{(\sin^3 x)x}$ $\quad 1-\cos x \sim \frac{1}{2}x^2 \to 1-\cos x^2 \sim \frac{1}{2}x^4$

$\sin x \sim x \to \sin x^3 \sim x^3$

$\Rightarrow \lim_{x\to 0} \frac{\frac{1}{2}x^4}{x^3\cdot x} = \lim_{x\to 0}\frac{\frac{1}{2}\cancel{x^4}}{\cancel{x^4}} = \frac{1}{2}$

THEOREM 2 · Elimination of negligible functions

$f_1 = o(f) \quad g_1 = o(g)$

a) $\lim_{x\to\gamma} \frac{f(x)+f_1(x)}{g(x)+g_1(x)} = \lim_{x\to\gamma}\frac{f(x)}{g(x)}$

b) $\lim_{x\to\gamma} (f(x)+f_1(x))(g(x)+g_1(x)) = \lim_{x\to\gamma} f(x)\,g(x)$

ELIMINATION OF NEGLIGIBLE FUNCTIONS (little-o's)

PROOF: $\frac{f+f_1}{g+g_1} = \frac{f}{g}\underbrace{\frac{\left(1+\frac{f_1}{f}\right)}{\left(1+\frac{g_1}{g}\right)}}_{\substack{\downarrow\\1}}$

ex: $\lim_{x\to 0}\frac{2\sin^2 x + \log(1+6x^3)}{x^2+2\sin x}$ $\quad \frac{f+f_1}{g+g_1}$

$\begin{cases} 2\sin^2 x \sim 2x^2 \\ \log(1+6x^3) \sim 6x^3 \end{cases}$ $\quad 6x^3 = o(2x^2) \quad (x\to 0)$ → I can cancel $\log(1+6x^3)$

$2\sin x \sim 2x \to x^2 = o(2x) \to$ I can cancel x^2

$\lim_{x\to 0}\frac{2\sin^2 x + \cancel{\log(1+6x^3)}}{\cancel{x^2}+2\sin x} = \lim_{x\to 0}\frac{2\sin^2 x}{2\sin x} = \lim_{x\to 0}\frac{\cancel{2}x^{\cancel{2}}}{\cancel{2}\cancel{x}} = \lim_{x\to 0} x = 0$

- $\lim_{x\to+\infty}\left(\cos\frac{1}{x}\right)^{\frac{1}{x^2}} = 1^0$ this is NOT an indeterminate form

$f(x) = e^{\frac{1}{x^2}\log(\cos\frac{1}{x})}$ $\quad\lim_{x\to+\infty}\underset{\substack{\downarrow\\0}}{\frac{1}{x^2}}\underset{\substack{\downarrow\\0}}{\log(\cos\frac{1}{x})} = 0 \to e^0 = 1$

- $\lim_{x\to0^+}(\cos x)^{\frac{1}{x^2}} = [1^\infty]$

$f(x) = e^{\frac{1}{x^2}\log(\cos x)}$ $\quad\lim_{x\to0^+}\frac{1}{x^2}\log(\cos x) = \lim_{x\to0^+}\frac{\log(\cos x)}{x^2} = \frac{0}{0}$

$\overset{H}{\Rightarrow}\lim_{x\to0^+}\frac{\frac{-\sin x}{\cos x}}{2x} = \lim_{x\to0^+} -\frac{1}{2}\cdot\underset{\substack{\downarrow\\1}}{\frac{1}{\cos x}}\cdot\underset{\substack{\downarrow\\1}}{\frac{\sin x}{x}} = -\frac{1}{2}$ $\quad\lim_{x\to0^+} e^{-\frac{1}{2}} = e^{-\frac{1}{2}}$

- $\lim_{n\to\infty}(2+n)^{\frac{1}{\log n}} = \infty^0$

$f(x) = e^{\frac{1}{\log x}\cdot\log(2+x)}$ $\quad\lim_{x\to+\infty}\frac{\log(x+2)}{\log x} = \left[\frac{\infty}{\infty}\right]$

$\Rightarrow \lim_{x\to+\infty}\frac{\log(x(1+\frac{2}{x}))}{\log x} = \lim_{x\to+\infty}\frac{\log x + \log(1+\frac{2}{x})}{\log x} =$

$= 1 + \lim_{x\to+\infty}\frac{\ln(1+\frac{2}{x})}{\ln x} = 1 \quad\Rightarrow e^1 = e$

- $f(x) = x^x$ $\quad$ dom $f = (0;+\infty)$ $\quad$ is $f(x)$ continuous and differentiable in the domain?

$f(x)\, e^{x\log x}$

$f(x) \in C^0(\text{dom} f)$ and differentiable in dom f because it's a composition of continuous functions

$\lim_{x\to+\infty} e^{x\log x} = e^{+\infty} = +\infty$

$\lim_{x\to0^+} e^{x\log x} = e^0 = 1$

$f'(x) = e^{x\log x}\cdot\left(\cancel{x}\cdot\frac{1}{\cancel{x}} + 1\cdot\log x\right) = e^{x\log x}\cdot(1+\log x)$

$f'(x) = 0 \quad e^{x\log x}\cdot(1+\log x) = 0 \quad e^{x\log x}$ is always >0 $\quad \ln x = -1$

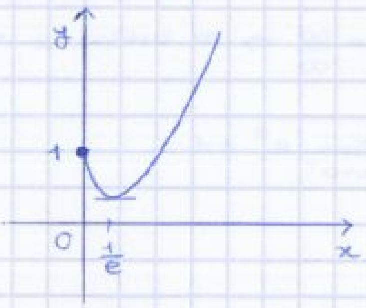
$x = e^{-1} = \frac{1}{e}$

$f(x)$ is differentiable in the domain
and there is a point $\frac{1}{e}$ where $f'(x) = 0$ → critical or stationary point

$f'(x) > 0 \Leftrightarrow 1+\ln x > 0 \Leftrightarrow \ln x > -1 \Leftrightarrow x > e^{-1} \quad x > \frac{1}{e}$

$f'(x) < 0 \Leftrightarrow 1+\ln x < 0 \Leftrightarrow x < \frac{1}{e}$

CONTINUOUS PROLONGATION
at $x=0$ setting $f(0)=1$
→ RIGHT CONTINUOUS at $x=0$



Theorem 1 and theorem 2 are true for quotients and products, not for sums or differences

ex: $\lim f(x) \pm g(x)$ $\quad g_1 \sim g$, $f_1 \sim f$ $\quad$ in this case I cannot substitute equivalent functions

$f(x) = \sqrt{x^2+2x}$

$g(x) = \sqrt{x^2-1}$

$$\lim_{x\to+\infty} (f(x) - g(x)) = \lim_{x\to+\infty} \sqrt{x^2+2x} - \sqrt{x^2-1} = [+\infty - \infty]$$

$$\lim_{x\to+\infty} \frac{(\sqrt{x^2+2x} - \sqrt{x^2-1})(\sqrt{x^2+2x} + \sqrt{x^2-1})}{\sqrt{x^2+2x} + \sqrt{x^2-1}} =$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^2}+2x-\cancel{x^2}+1}{\sqrt{x^2+2x}+\sqrt{x^2-1}} = \lim_{x\to+\infty} \frac{2x+1}{x\Big(\underbrace{\sqrt{x+\frac{2}{x}}}_{\downarrow\,1} + \underbrace{\sqrt{1-\frac{1}{x^2}}}_{\downarrow\,1}\Big)} = \lim_{x\to+\infty} \frac{2x+1}{2x} = 1$$

i try to substitute an equivalent function to f(x)

$\lim_{x\to+\infty} \frac{\sqrt{x^2+2x}}{x} = 1$ $\quad$ x is the principal part of $f(x)$

$$\lim_{x\to+\infty} (\sqrt{x^2+2x} - \sqrt{x^2-1}) \overset{\text{substitution}}{=} \lim_{x\to+\infty} \frac{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}{x+\sqrt{x^2-1}} =$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^2}-\cancel{x^2}+1}{x+\sqrt{x^2-1}} = \frac{1}{\infty} = 0$$

making a substitution of equivalent functions I get a wrong result

RULES FOR LITTLE-O'S (for $x\to0$)

(this is true also for $x \to x_0$ substituting $x - x_0$ to x)

1) $o(x^n) \pm o(x^n) = o(x^n)$

2) $o(x^n) \pm o(x^m) = o(x^p)$ $\quad$ p= minimum of n and m)

3) $o(\lambda x^n) = \lambda o(x^n) = o(x^n)$

4) $\varphi(x) \cdot o(x^n) = o(x^n)$ $\quad$ $\varphi(x)$ bounded function in $I_r(0)$

5) $x^m \cdot o(x^n) = o(x^{n+m})$

6) $o(x^n) \cdot o(x^m) = o(x^{m+n})$

7) $(o(x^n))^k = o(x^{nk})$

⑥ $f = o(x^m)$ $\quad g = o(x^n)$

$\frac{f}{x^m} \to 0$ $\quad \frac{g}{x^n} \to 0$ $\quad \frac{f\cdot g}{x^m \cdot x^n} \to 0$ $\quad \frac{f\cdot g}{x^{m+n}} \to 0$

④ $f = o(x^n)$ $\quad \frac{f}{x^n} \to 0$

$f \cdot \varphi(x)$

$$\frac{\varphi(x)\cdot f(x)}{x^n} = \underbrace{\varphi(x) \cdot \frac{f(x)}{x^n}}_{\text{bounded} \times 0} = 0$$

- $h(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{3}} \to 0$

$$\lim_{x \to 3} \frac{\frac{-\sqrt{x}-\sqrt{3}}{\sqrt{3}\cdot\sqrt{x}}}{\ell(x-3)^{\alpha}} = 1 \qquad \sqrt{x}-\sqrt{3} \sim \frac{1}{2\sqrt{3}}(x-3) \quad (x \to 3)$$

substitution

$$\lim_{x\to 3} \frac{-1}{\sqrt{3}\cdot\sqrt{x}} \cdot (\sqrt{x}-\sqrt{3}) \cdot \frac{1}{\ell(x-\alpha)^{\alpha}} = -\frac{1}{2\sqrt{3}} \lim_{x\to 3} \frac{\frac{1}{2\sqrt{3}}(x-3)}{(x-3)} =$$

$$= -\frac{1}{12} \lim_{x\to 3} \frac{(x-3)}{-\frac{1}{12}(x-3)} = 1 \qquad \Rightarrow \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{3}} \text{ infinitesimal of order } 1$$

principal part $= -\frac{1}{12}(x-3)$

ex: $\lim_{x\to 0} \frac{\sin^2(x^3)}{1-\cos 3x} = \frac{0}{0}$

$\sin z \sim z$ $\qquad$ $\sin x^3 \sim x^3 \to \sin^2 x^3 \sim x^6$

$1-\cos z \sim \frac{1}{2} z^2$ $\qquad$ $1-\cos 3x \sim \frac{1}{2} 9x^2$

$$f(x) \underset{x\to 0}{\sim} \frac{x^{\cancel{6}\,4}}{\frac{9}{2}\cancel{x^2}} = \frac{2}{9}x^4$$

ex: $\lim_{n\to\infty} \frac{n^2+1}{2^n+(-1)^n}$ $\qquad$ $\frac{f+f_1}{g+g_1}$

$1 = o(n^2)$ $\qquad$ $\frac{1}{n^2} \to 0 \quad n\to\infty$

$$\Rightarrow \lim_{n\to\infty} \frac{n^2+\cancel{1}}{2^n+\cancel{(-1)^n}}$$

$(-1)^n = o(2^n)$ $\qquad$ $\frac{(-1)^n}{2^n} \to 0$

$\lim_{n\to\infty} \frac{n^2}{2^n} = 0$ $\qquad$ all powers are negligible with respect to exponentials $\qquad$ $x^a = o(a^x)$ $\quad a>1$

$n^2 = o(2^n)$

Taylor formula with Peano's remainder: the general case

In general, we have the following fundamental result.

Theorem (Taylor formula with Peano's remainder)

If f in n times differentiable at x_0, then there is one and only one polynomial $Tf_{n,x_0}(x)$ such that

$$f(x) = Tf_{n,x_0}(x) + o((x - x_0)^n), \quad x \to x_0$$

with

$$\begin{aligned} Tf_{n,x_0}(x) &= f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \end{aligned}$$

- $Tf_{n,x_0}(x)$ is the **Taylor polynomial of f at x_0 of degree n**
- if $x_0 = 0$, $Tf_{n,0}(x)$ is the **McLaurin polynomial of f of degree n**

Uniqueness of the Taylor polynomial

As stated in the previous theorem, **the Taylor polynomial is unique.** In more precise terms:

Theorem

If

- *$f : (a, b) \to \mathbb{R}$ is differentiable n times at $x_0 \in (a, b)$*
- *there exists a polynomial $P_n(x)$ of degree $\leq n$ such that*

$$f(x) = P_n(x) + o((x - x_0)^n), \quad x \to x_0$$

then $P_n(x) = Tf_{n,x_0}(x)$.

Examples - 2

We compute the Taylor polynomial of $f(x) = 5 - 2x + 3x^2 + x^4$ at $x_0 = 1$, using the definition.

Since

$$f'(x) = -2 + 6x + 4x^3\,, \qquad f''(x) = 6 + 12x^2\,,$$

$$f'''(x) = 24x\,, \qquad f^{(4)}(x) = 24\,, \qquad f^{(n)}(x) = 0\,,\ \forall n > 4$$

and

$$f(1) = 7\,, \qquad f'(1) = 8\,, \qquad f''(1) = 18\,,$$

$$f'''(1) = 24\,, \qquad f^{(4)}(1) = 24\,, \qquad f^{(n)}(1) = 0\,,\ \forall n > 4$$

we have that

$$Tf_{0,x_0}(x) = 7$$

$$Tf_{1,x_0}(x) = 7 + 8(x-1)$$

$$Tf_{2,x_0}(x) = 7 + 8(x-1) + 9(x-1)^2$$

$$Tf_{3,x_0}(x) = 7 + 8(x-1) + 9(x-1)^2 + 4(x-1)^3$$

Examples - 2 (continued)

The Taylor polynomial of degree 4 is

$$Tf_{4,x_0}(x) = 7 + 8(x-1) + 9(x-1)^2 + 4(x-1)^3 + (x-1)^4.$$

It is easy to check that this polynomial equals $f(x)$ for all $x \in \mathbb{R}$; simply it is written using powers of $(x-1)$ instead of powers of x.

Also in this case we have that

$$Tf_{n,x_0}(x) = Tf_{4,x_0}(x)\,,\ \forall n > 4,\ \forall x \in \mathbb{R}$$

The logarithmic function - 2

We compute the Maclaurin polynomial of $f(x) = \ln(1+x)$ of order n. Since

$$\log y = (y-1) - \frac{(y-1)^2}{2} + \cdots + (-1)^{n-1}\frac{(y-1)^n}{n} + o\big((y-1)^n\big)\,,\ y \to 1$$

with the substitution $y = 1+x$, we get

$$\boxed{\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n) \\ &= \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k} + o(x^n)\,, \qquad x \to 0 \end{aligned}}$$

Trigonometric functions - 1

We compute the Maclaurin polynomial of $f(x) = \sin x$ of order n. Since the function is odd, the expansion contains odd powers only. Since

$$f'(x) = \cos x\,, \qquad f'''(x) = -\cos x$$

and, in general, for all integer k,

$$f^{(2k+1)}(x) = (-1)^k \cos x\,, \qquad f^{(2k+1)}(0) = (-1)^k$$

Then, if $n = 2m+2$,

$$\boxed{\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2}) \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2m+2})\,, \qquad x \to 0 \end{aligned}}$$

Power functions - 2

We compute the McLaurin polynomial of $f(x) = (1+x)^\alpha$ of order n (with $\alpha \in \mathbb{R}$). Since

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1}, \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, \\ f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \\ &\dots \\ f^{(k)}(x) &= \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k} \\ \frac{f^{(k)}(0)}{k!} &= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} = \binom{\alpha}{k}. \end{aligned}$$

Power functions - 3

Then we have the general formula:

$$\boxed{\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + o(x^n) \\ &= \sum_{k=0}^{n}\binom{\alpha}{k}x^k + o(x^n), \qquad x \to 0 \end{aligned}}$$

Remark. This formula in useful for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$; when $\alpha \in \mathbb{Z}$ it is not necessary to use it, since

- if $\alpha \in \mathbb{N}$ the function $f(x) = (1+x)^\alpha$ is a polynomial.
- If $\alpha = -1$ we obtained the expansion with a different method.
- If α is integer and < -1, we will get the expansion using the possibility of differentiating Taylor polynomial

Some important McLaurin expansions - 2

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)\,.$$

First and second finite increment formula - 1

- If $f(x)$ is differentiable at x_0 we can write the Taylor formula (with Peano's remainder) of order one, that is also called **first formula of the finite increment**:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0).$$

$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x - x_0)$ $\Delta f = f'(x_0)\Delta x - o(x - x_0)$

- We already know that, if f is differentiable in an interval I, the **second formula of the finite increment** holds: for all $x \in I$ there exists a point t such that

$$f(x) = f(x_0) + f'(t)(x - x_0)$$

$f(x) - f(x_0) = f'(t)(x - x_0)$

$\Delta f = f'(t)\Delta x$

.

These formulas state that:

the increment of the function $\Delta f = f(x) - f(x_0)$ is proportional to the increment of the variable $\Delta x = x - x_0$.

LOCAL COMPARISON 2

$f \sim g \Rightarrow f = g + o(g)$ $\quad x \to r$

$\frac{f}{g} \to 1 \quad \frac{f}{g} - 1 \to 0 \Rightarrow \frac{f-g}{g} \to 0 \quad f - g = o(g)$

- $\sin x \sim x \quad (x \to 0)$
  $\sin x = x + o(x)$

- $e^x - 1 \sim x \quad (x \to 0)$
  $e^x - 1 = x + o(x)$

$\sin x = x + o(x) \quad pp = -2x$ $\qquad$ I cancel it because it's a negligible function

- $$\lim_{x\to 0^+} \frac{\sin x - 3x}{x^2} \overset{\vee}{=} \lim_{x \to 0^+} \frac{\boxed{x} + o(x) - \boxed{3x}}{x^2} = \lim_{x\to 0^+} \frac{-2x + \cancel{o(x)}}{x^2} =$$

$$= \lim_{x \to 0^+} -\frac{2}{x} = -\infty$$

- $$\lim_{x\to 0^+} \frac{\sin x - x}{x^2} = \lim_{x\to 0^+} \frac{\cancel{x} + o(x) - \cancel{x}}{x^2} = \lim_{x \to 0^+} \frac{o(x)}{x^2}$$

we cannot say anything about this limit because $o(x)$ is a function negligible with respect to x

$o(x) \Rightarrow \frac{g(x)}{x} \to 0 \qquad pp \equiv 0$

$g(x) = x^{3/2} = o(x) \to \infty$

$g(x) = x^3 = o(x) \to 0 \qquad \underline{x^\alpha \quad \alpha > 1}$ $\quad$ all these functions satisfy $o(x)$

$g(x) = x^2 = o(x) \to 1$

In the 1st case $pp \neq 0$, so $o(..)$ can be cancelled, but in the 2nd case $pp = 0$ and it's very important what it's left, the little-o, it cannot be cancelled the information given by $o(x)$ are not enough to say something about the limit

↓

CANCELLATION OF PRINCIPAL PARTS

$\sin x = x + o(x) \quad (x \to 0)$

1 $\boxed{\sin x - x = o(x)}$ $\qquad \lim_{x \to 0} \frac{\sin x - x}{x} = 0$

these functions tend to 0 faster than $x \to \alpha > 1$

Fundamental limit ≠ 0

- $$\lim_{x\to 0} \frac{\sin x - x}{x^2} = \frac{0}{0} \overset{H}{\Rightarrow} \lim_{x\to 0} \frac{\cos x - 1}{2x} = 0$$

↳ If the limit is a number ≠ 0 it means that $\sin x - x \sim \ell x^2$

2 $\boxed{\sin x - x = o(x^2)}$

$\sin x - x$ goes to 0 faster than x^2

I know something more

$\to \alpha > 2$

I try again (3rd order)

Fundamental limit $\to -\frac{1}{2}$

- $$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \frac{0}{0} \overset{H}{\Rightarrow} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = -\frac{1}{6}$$

$$\frac{\sin x - x}{x^3} \to -\frac{1}{6} \qquad \frac{\sin x - x}{-\frac{1}{6}x^3} \to 1 \quad \Rightarrow \quad \sin x - x \sim -\frac{1}{6}x^3$$

3 $\boxed{\sin x - x = -\frac{1}{6}x^3 + o(x^3)}$

$$\sin x - \left[x - \frac{1}{6}x^3\right] = o(x^3)$$

$$\frac{\sin x - \left[x - \frac{1}{6}x^3\right]}{x^5} \Rightarrow \frac{1}{120} \quad \Rightarrow \quad \sin x - \left[x - \frac{1}{6}x^3\right] \sim \frac{1}{120}x^5$$

4 $\boxed{\sin x = x - \frac{x^3}{\ } + \frac{x^5}{\ } + o(x^5)}$

0) f CONTINUOUS at x_0
1) f DIFFERENTIABLE at x_0
2) f TWICE DIFFERENTIABLE at x_0
n) f has n DERIVATIVES at x_0

$f(x)$ defined in $I(x_0) \setminus \{x_0\}$

$\lim_{x \to x_0} f(x) = \ell$ can I say that $f(x)$ is ℓ in the neighbourhood?
No, because if the function is constant, the limit is ℓ
but if the limit is ℓ, it doesn't mean that $f(x)$ is constant

the limit is ℓ, but in the neighbourhood the function is not ℓ

$\lim_{x \to x_0} f(x) = \ell \Rightarrow \lim_{x \to x_0} f(x) - \ell = 0$ $f(x) - \ell$ is an infinitesimal function at x_0

$f(x) - \ell = o(1)$
$f(x) = \ell + o(1)$

- f DEFINED in $I(x_0)$
- f CONTINUOUS at $x_0 \Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$ $\lim_{x \to x_0} f(x) - f(x_0) = 0$ $f(x) - f(x_0) = o(1)$

$$\Rightarrow f(x) = f(x_0) + \underbrace{o(1)}_{\text{infinitesimal function}} \quad (x \to x_0)$$ ← Taylor formula of order 0

- f DIFFERENTIABLE at x_0

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \qquad \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + o(1)$$

$$f(x) - f(x_0) = (f'(x_0) + o(1))(x - x_0)$$
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(1)(x - x_0) \quad (x \to x_0)$$
$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{tangent line to the graph at } (x_0; f(x_0)), \text{ with } m = f'(x_0)} + o(x - x_0)$$

a differentiable function in a neighborhood can be written as its tangent line with an error that goes to 0 faster than $x - x_0$ $(x \to x_0)$

TAYLOR FORMULA OF ORDER 1

$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$ (1st derivative)

- f TWICE DIFFERENTIABLE (2nd derivative)

$f(x) - [f(x_0) + f'(x_0)(x - x_0)] = o(x - x_0)$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)^2} = \frac{0}{0} \overset{H}{\Rightarrow} \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{2(x - x_0)} = \frac{f''(x_0)}{2}$$

difference quotient of the first derivative

$$\frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)^2} = \frac{f''(x_0)}{2} + o(1)$$

$$\to f(x) - f(x_0) - f'(x_0)(x - x_0) = \frac{f''(x_0)}{2}(x - x_0)^2 + o(1)(x - x_0)^2$$ TAYLOR FORMULA OF ORDER 2

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2}_{\text{Parabola (2nd degree polynomial)}} + o((x - x_0)^2)$$

2nd degree term
now the approximation is better because the error term is $o((x - x_0)^2)$ and this goes to 0 faster than $o(x - x_0)$

$f(x) = Tf_{n,x_0}(x) + o((x-x_0)^n) \qquad x \to x_0$

$Tf_{n,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \dots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$

↪ n FACTORIAL

ex:
$n=5 \qquad f(x) = 3x^2 + 4x^3 - x^5 + o(x^5) \qquad x \to 0$
$x_0 = 0 \qquad f(0) = 0$
f is differentiable 5 times at $x_0=0$, so it's continuous
f is an infinitesimal function
what is the principal part? ex^α: $\lim_{x\to 0} \frac{f(x)}{ex^\alpha} = 1$

$$\lim_{x\to 0} \frac{3x^2+4x^3-x^5+o(x^5)}{3x^2}$$

$$1 + \frac{4}{3}x - \frac{1}{3}x^2 + \frac{o(x^5)}{o(x^2)} = 1 + \underset{\substack{\downarrow\\0}}{\frac{4}{3}x} - \underset{\substack{\downarrow\\0}}{\frac{1}{3}x^2} + \underset{\substack{\downarrow\\0}}{o(x^3)} = 1$$

$3x^2$ is the principal part
If you have an infinitesimal function and you know the Taylor expansion until a certain point, the principal part is the term of lowest degree.

TAYLOR EXPANSION AT $x_0 = 0$ ⇒ MAC LAURIN EXPANSION

$f(x) = e^x \in C^\infty(\mathbb{R})$ it's differentiable of any order in $\mathbb{R}$
$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

Mac Laurin expansion:

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + o(x^n)$ Mac Laurin polynomial of order n

$$e^x = \underset{f(0)}{1} + \underset{f'(0)}{1} \cdot (x-0) + \underset{\frac{f''(0)}{2}}{\frac{1}{2}}(x-0)^2 + \underset{\frac{f'''(0)}{3!}}{\frac{1}{3!}}(x-0)^3 + \dots + \frac{1}{n!}(x-0)^n + o(x^n)$$

↓

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$$ Remark: $k=0 \quad \frac{x^0}{0!} = 1$

$$= 1 + \sum_{k=1}^{n} \frac{x^k}{k!}$$

$$f^{(n)}(x_0) = e^{x_0} \Rightarrow \underset{x\to x_0}{e^x} = e^{x_0} + e^{x_0}(x-x_0) + \frac{e^{x_0}}{2}(x-x_0)^2 + \dots + \frac{e^{x_0}}{n!}(x-x_0)^n + o(x-x_0)^n$$

$f(x) = \sin x$ at 0: 0
$f'(x) = \cos x$ 1
$f''(x) = -\sin x$ 0
$f'''(x) = -\cos x$ -1
$f^{IV}(x) = \sin x$ 0

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ only ODD POWERS

there are alternating signs → $(-1)^n$
$n=1 \to -$
$n=2 \to 0$

$f(x) = (1+x)^\alpha \quad \alpha \in \mathbb{R} \quad x > -1 \quad f(x) \in C^\infty((-1; +\infty))$

$x_0 = 0$ Mac Laurin

$f(x) = (1+x)^\alpha \qquad f(0) = 1$

$f'(x) = \alpha(1+x)^{\alpha-1} \qquad f'(0) = \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \qquad f''(0) = \alpha(\alpha-1)$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \qquad f'''(0) = \alpha(\alpha-1)(\alpha-2)$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots + \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-(n-1))}{n!}x^n + o(x$$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-(n-1))}{n!} \qquad n > 1$$

$$\binom{\alpha}{1} = \alpha \qquad \binom{\alpha}{0} = 1$$

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + o(x^n)$$

$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$ Maclaurin expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} + o(x^3)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 + o(x^3)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

EXERCISES:

a) $\lim\limits_{x\to 0} \dfrac{(1-\cos x)^2 + \sin^2 x}{x^3 - x^2}$

b) $\lim\limits_{x\to +\infty} \dfrac{\sqrt[3]{1+x^3} + x^2}{3x^2 + \sin x}$

cancellation of negligible terms

Equivalent functions at $x_0 = 0$

$$\begin{cases} \sin z \sim z \\ 1-\cos z \sim \dfrac{z^2}{2} \\ e^z - 1 \sim z \\ (1+z)^a \sim 1 + az \\ \ln(1+z) \sim z \end{cases}$$

a) $\lim\limits_{x\to 0} \dfrac{(1-\cos x)^2 + \sin^2 x}{x^3 - x^2}$

- $\sin x \sim x \qquad \sin^2 x \sim x^2$

  $1 - \cos x \sim \dfrac{x^2}{2} \qquad (1-\cos x)^2 \sim \dfrac{x^4}{4}$

  $\dfrac{x^4}{4} = o(x^2) \quad (x \to 0) \qquad (1-\cos x)^2 = o(\sin^2 x)$

  $x^3 = o(x^2)$

  $$\lim_{x\to 0} \frac{\cancel{(1-\cos x)^2} + \sin^2 x}{\cancel{x^3} - x^2} = \lim_{x\to 0} \frac{\sin^2 x}{-x^2} = \lim_{x\to 0} \frac{\sin x}{-x} \cdot \frac{\sin x}{x} = -1 \cdot 1 = -1$$

  $$\text{or} = \lim_{x\to 0} \frac{x^2}{-x^2} = -1$$

b) $\lim\limits_{x\to +\infty} \dfrac{\sqrt[3]{1+x^3} + x^2}{3x^2 + \sin x}$

$\sin x \sim x \qquad x = o(3x^2) \quad (x \to +\infty) \qquad \sin x = o(3x^2)$

$\sqrt[3]{1+x^3} \sim x \qquad x = o(x^2) \quad (x \to +\infty) \qquad \sqrt[3]{1+x^3} = o(x^2)$

$$\lim_{x\to +\infty} \frac{\cancel{\sqrt[3]{1+x^3}} + x^2}{3x^2 + \cancel{\sin x}} = \lim_{x\to +\infty} \frac{\cancel{x^2}}{3\cancel{x^2}} = \frac{1}{3}$$

Mathematical Analysis I (2013-2014)

Local comparison 3 - Applications of Taylor's formula

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Taylor expansion and algebraic operations

We suppose $x_0 = 0$; substituting $x - x_0$ to x, analogous results are obtained for Taylor's polynomial centered at x_0.

Consider the McLaurin expansions of the functions f and g:

$$f(x) = a_0 + a_1x + ... + a_nx^n + o(x^n) = p_n(x) + o(x^n),$$
$$g(x) = b_0 + b_1x + ... + b_nx^n + o(x^n) = q_n(x) + o(x^n)$$

We study how to write the Taylor expansion of the sum, difference, product and quotient of the two functions.

Quotient - 1

- **First method.**
  Suppose that
  $f(x) = p_n(x) + o(x^n)$,
  $g(x) = q_n(x) + o(x^n)$, with $g(0) \neq 0$,
  and set
  $$h(x) = \frac{f(x)}{g(x)}.$$
  This expansion of the quotient can be computed as a division with the increasing powers method (see the next example).

Example

We compute the third order McLaurin polynomial of $h(x) = \tan x$.
Since $\sin x = x - \frac{x^3}{6} + o(x^3)$ e $\cos x = 1 - \frac{x^2}{2} + o(x^3)$; dividing

$$\begin{array}{r|l}
x - \frac{x^3}{6} + o(x^3) & 1 - \frac{x^2}{2} + o(x^3) \\
\cline{2-2}
x - \frac{x^3}{2} + o(x^3) & x + \frac{x^3}{3} + o(x^3) \\
\cline{1-1}
\frac{x^3}{3} + o(x^3) & \\
\frac{x^3}{3} + o(x^3) & \\
\cline{1-1}
o(x^3) &
\end{array}$$

Then $\tan x = x + \frac{x^3}{3} + o(x^3)$.

Composition of functions - 1

We study how to find the Taylor polynomial of the composition $g(f(x))$; we suppose that

$$\begin{aligned} f(x) &= a_1x + ... + a_nx^n + o(x^n), & x \to 0, \\ g(x) &= b_0 + b_1y + ... + b_ny^n + o(y^n), & y \to 0, \\ g(x) &= b_0 + b_1y + ... + b_ny^n + y^n o(1), & y \to 0. \end{aligned}$$

Substituting $y = f(x)$ in the expansion of g, we have

$h(x) = g(f(x)) =$
$= b_0 + b_1f(x) + b_2(f(x))^2 + ... + b_n(f(x))^n + o\,(f(x))^n) =$
$= b_0 + b_1f(x) + b_2(f(x))^2 + ... + b_n(f(x))^n + o\,(f(x)^n)\,o(1) =$
$= b_0 + b_1f(x) + b_2(f(x))^2 + ... + b_n(f(x))^n + o\,(f(x)^n)\,o(1)$

Composition of functions - 2

- If $a_1 \neq 0$, then $(f(x))^n = (a_1x)^n + o(x^n)$ and then $(f(x))^n o(1) = o(x^n) \quad x \to 0$.
  Developing $(f(x))^k$ $(1 \leq k \leq n)$ with respect to x up to the order n, we obtain the expansion of $g(f(x))$.
- If $a_1 = a_2 = \ldots = a_{m-1} = 0$ and $a_m \neq 0$, then $(f(x))^n = a_m^n x^{mn} + o(x^{mn})$ and $(f(x))^n o(1) = o(x^{mn}) \quad x \to 0$.
  Developing $(f(x))^k$ $(1 \leq k \leq n)$ with respect to x up to the order mn, we obtain the expansion of $g(f(x))$.

Order and principal part

Suppose that the function f is differentiable n times at x_0 and that its Taylor expansion of order n at x_0 is

$$f(x) = a_0 + a_1(x - x_0) + ... + a_n(x - x_0)^n + o((x - x_0)^n)$$

If there exists an integer m $(1 \leq m \leq n)$ such that

$$a_0 = a_1 = ... = a_{m-1} = 0, \qquad \text{and} \quad a_m \neq 0$$

then

$$f(x) = a_m(x - x_0)^m + o((x - x_0)^m).$$

Dividing by $a_m(x - x_0)^m$ and computing the limit, we have that:

- the function $p(x) = a_m(x - x_0)^m$ is the **principal part** of f with respect to the infinitesimal test function $\varphi(x) = x - x_0$
- $f(x)$ is an infinitesimal of order m with respect to the infinitesimal test function $\varphi(x) = x - x_0$.

Local behaviour

Suppose that the function f is differentiable n times at x_0 and that its Taylor expansion of order n at x_0 is

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + o((x - x_0)^n), \qquad x \to x_0,$$

then

$$f(x_0) = a_0\,, \ f'(x_0) = a_1\,, \ f''(x_0) = 2a_2\,, \ldots, f^{(n)}(x_0) = n!a_n.$$

If $f,\ f', f''$ are continuous in a neighbourhood of x_0 and $a_0,\ a_1,\ a_2$ do not vanish, then (sign and limit theorem) the signs of a_0, a_1 and a_2 coincide with the signs of $f(x)$, $f'(x)$ and $f''(x)$ in a neighbourhood of x_0.

Then, analyzing the signs of $a_0,\ a_1,\ a_2$ we can determine the sign, the local monotonicity and the convexity of $f(x)$ in a neighbourhood of x_0.

Classification of points where f'' vanishes

Theorem

If

- f *is differentiable* n *times* $(n \geq 3)$ *at* x_0
- *there exists* $m \in \mathbb{N}$ *such that* $3 \leq m \leq n$ *and*

$$f''(x_0) = \cdots = f^{(m-1)}(x_0) = 0, \qquad f^{(m)}(x_0) \neq 0$$

then

i) *If* m *is odd, then* x_0 *is an inflection point;*

ii) *if* m *is even, then* x_0 *is not an inflection point.*

THEOREM

- f DIFFERENTIABLE n times at x_0
  $f(x) = Tf_{n,x_0}(x) + o((x-x_0)^n)$

  $\Rightarrow$ f' is DIFFERENTIABLE n-1 times at x_0
  $f'(x) = (Tf_{n,x_0}(x))' + o((x-x_0)^{n-1})$

If a function is EVEN, the McLaurin expansion contains only EVEN powers
ODD ODD

f EVEN → f' ODD → f'' EVEN → f''' ODD ...

odd: $-g(x) = g(-x)$
$-g(0) = g(0)$
$g(0) = 0$

$f'(0) = 0$ $f'''(0) = 0$
because f' and f''' are odd
so in the McLaurin expansion there are not odd powers because the coefficient is always 0
only even powers for even functions

f is DIFFERENTIABLE n times at x_0 (i can write a Taylor polynomial of order n)

$f(x_0) = 0$ f is an INFINITESIMAL function

$\hookrightarrow a_0 = 0$ constant term in the Taylor polynomial is the value of the function at the point

$a_1 = a_2 \dots a_{k-1} = 0$

$a_k \neq 0$

$$f(x) = a_k(x-x_0)^k + a_{k+1}(x-x_0)^{k+1} + \dots a_n(x-x_0)^n + o((x-x_0)^n)$$

$$\lim_{x \to x_0} \frac{f(x)}{a_k(x-x_0)^k} = 1 \qquad f(x) \sim a_k(x-x_0)^k$$

$\Rightarrow$ f is an infinitesimal function at x_0 of order k, the principal part is $a_k(x-x_0)^k$ (first non-zero term of Taylor expansion)

ex: $f(x) = 1 - \cos(x^3)$ infinitesimal for $x \to 0$

$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} + o(z^4)$ or $\cos z = 1 - \frac{z^2}{2} + o(z^2)$ when $x \to 0$, $z \to 0$, I can substitute

$\cos x^3 = 1 - \frac{x^6}{2} + o(x^6)$

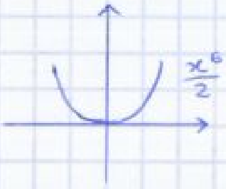
$1 - \cos x^3 = \frac{x^6}{2} + o(x^6)$ for $x \to 0$ f is an infinitesimal of order 6
principal part: $\frac{x^6}{2}$

$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!} + \dots \frac{f^{(6)}(0)}{6!}x^6 + o(x^6)$

the derivatives up to order 5 are 0 $\frac{f^{(6)}(0)}{6!} = \frac{1}{2}$ $f^{(6)}(0) = \frac{1}{2} \cdot 6!$

I don't know the behavior of this function in general,
but I know that in a very small neighborhood of the origin
$f(x) = 1 - \cos(x^3)$ looks like $\frac{x^6}{2}$, its equivalent function for $x \to 0$

and I can say that the origin is a local minimum point



HYPERBOLIC FUNCTIONS

- Hyperbolic cosine:

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

- Hyperbolic sine:

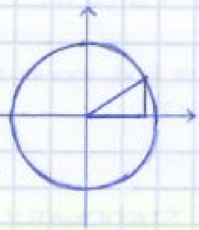
$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\cosh^2 x - \sinh^2 x =$$

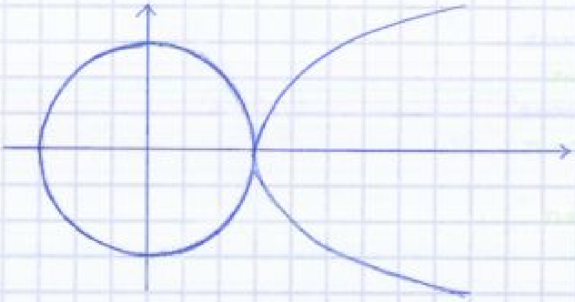
$$\frac{1}{4}(\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - \cancel{e^{2x}}$$

$$- \cancel{e^{-2x}} + 2) = \frac{1}{\cancel{4}} \cdot \cancel{4} = 1$$



$$\cosh^2 t - \sinh^2 t = 1$$

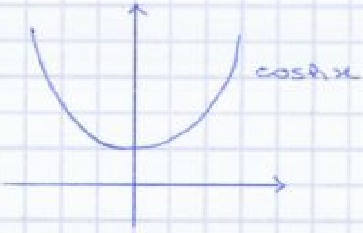
$$x^2 - y^2 = 1$$



$$\cosh x = \frac{e^{x}+e^{-x}}{2} \qquad \cosh(-x) = \frac{e^{-x}+e^{x}}{2} \qquad f(x) = f(-x) \rightarrow \text{even function}$$

$$\cosh x = \frac{e^{x}}{2} + \frac{e^{-x}}{2} \qquad \text{at } +\infty \quad \frac{e^{-x}}{2} \to 0 \qquad = \frac{e^{x}}{2} + o(1) \qquad \text{ASYMPTOTIC}$$

the hyperbolic cosine is asymptotic with respect to $\frac{e^{x}}{2}$

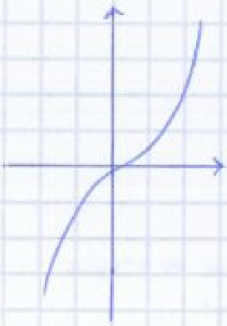


$\cosh x$ is never 0

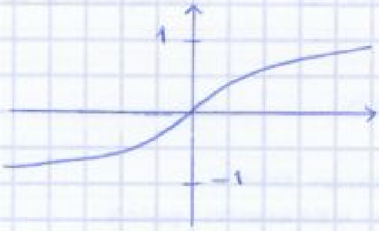
$\neq 0$

$$\sinh x = \frac{e^{x}-e^{-x}}{2} \qquad \sinh(-x) = \frac{e^{-x}-e^{x}}{2} = -\frac{e^{x}-e^{-x}}{2} \qquad \text{ODD}$$

$$\sinh x = \frac{e^{x}}{2} + o(1)$$



$$\tanh x = \frac{\sinh x}{\cosh x}$$



$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + o(x^4)$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$

Product

$1 + x^2 + \frac{x^4}{2} + o(x^4)$

$-\frac{x^2}{2} - \frac{x^4}{2}$

$+\frac{x^4}{24}$

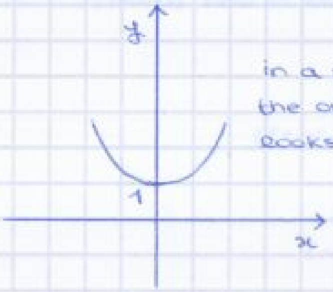
If I have a term like x^6 I don't need it because $x^6 = o(x^4)$, so it's included in it it vanishes. All the terms >4 are $o(x^4)$

$1 + \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$ → McLaurin polynomial of $e^{x^2} \cdot \cos x$

$h(x) = e^{x^2} \cos x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$

$h^{(4)}(0) = ?$

$\frac{h^{(4)}(0)}{4!} x^4 = \frac{x^4}{24} \to h^{(4)}(0) = 1$



in a small neighborhood of the origin the graph of h(x) looks like the graph of $1 + \frac{x^2}{2}$

$\Rightarrow x_0 = 0$ is a local minimum point

4) QUOTIENT we need to work on the composition before

5) COMPOSITION

$h(x) = g(f(x))$ we suppose that $f(0) = 0$ f is an infinitesimal function $x \to 0$

ex: $h(x) = e^{\sin x}$ McLaurin polynomial of order 3 (n=3)

$x \xrightarrow{f} \sin x$

$y \xrightarrow{g} e^{\sin x}$

$x_0 = 0 \Rightarrow y_0 = \sin x_0 = 0$

$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + o(y^3)$ $\Big(o(y^n) \to o((\sin x)^n) = o(x^n)$

$\sin x \sim x \ (x \to 0) \Rightarrow o(\sin x) = o(x)$

this is true also for $f \underset{x}{\sim} g$

$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3!} + o(x^3)$

This is not a polynomial I have to substitute

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3!} + o(x^3)$$

$$= 1 + x - \frac{x^3}{6} + o(x^3) + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + o(x^3)\right)^3 + o(x^3) =$$

$$= 1 + x - \cancel{\frac{x^3}{6}} + o(x^3) + \frac{1}{2}(x^2) + \frac{1}{\cancel{6}}(\cancel{x^3}) + o(x^3) =$$

$$= 1 + x + \frac{x^2}{2} + o(x^3)$$

Mathematical Analysis I (2013-2014)

Local comparison 4 - Asymptotes

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Asymptotic functions - 1

Definition. The function f is called **asymptotic** to a function g for $x \to +\infty$ if

$$\lim_{x \to +\infty} f(x) - g(x) = 0 \ .$$

A similar definition for $x \to -\infty$. Using Landau's symbols, g and g are asymptotic for $x \to +\infty$ if

$$f - g = o(1), \quad \text{i.e.} \quad f = g + o(1), \ (x \to +\infty) \ .$$

Examples.

- The function $f(x) = x^2 + \frac{1}{x}$ and $g(x) = x^2 + \frac{1}{x^3}$ are asymptotic for $x \to +\infty$.
- The function $f(x) = \mathrm{e}^x + \mathrm{e}^{-x}$ and $g(x) = \mathrm{e}^{-x}$ are asymptotic for $x \to -\infty$.

The coefficients m and q

Theorem

If $y = mx + q$ is a right asymptote for f, then

$$m = \lim_{x\to+\infty} \frac{f(x)}{x} \qquad \text{and} \qquad q = \lim_{x\to+\infty} (f(x) - mx)$$

In fact

$$\begin{aligned} 0 &= \lim_{x\to+\infty} \frac{f(x) - mx - q}{x} = \lim_{x\to+\infty} \frac{f(x)}{x} - \lim_{x\to+\infty} \frac{mx}{x} - \lim_{x\to+\infty} \frac{q}{x} \\ &= \lim_{x\to+\infty} \frac{f(x)}{x} - m \implies m = \lim_{x\to+\infty} \frac{f(x)}{x} \end{aligned}$$

and

$$\lim_{x\to+\infty} (f(x) - mx - q) = 0 \implies q = \lim_{x\to+\infty} (f(x) - mx)$$

Examples

- The function $f(x) = \sqrt{1 + x^2}$ has the right oblique asymptote $y = x$ and the left oblique asymptote $y = -x$.
- The function $f(x) = x + \sqrt{x}$ has no right asymptotes.
- The function $f(x) = 2x - \arctan x^3$ the right oblique asymptote $y = 2x - \frac{\pi}{2}$ and the left oblique asymptote $y = 2x + \frac{\pi}{2}$.