NUMERO : 92

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A P P U N T I

STUDENTE : Pignato

MATERIA : Analisi 2
Prof. Codegone

RIPASSO

Studio di funzione:

- $w = x^2 - 4z\log x + xy^2 z$

1) $\mathrm{dom}(w) = \{x > 0,\ y, z \in \mathbb{R}\}$ DOMINIO

2) $\nabla w = \left(\underbrace{2x - \frac{4z}{x} + y^2 z}_{\frac{\partial w}{\partial x}},\ \underbrace{2xyz}_{\frac{\partial w}{\partial y}},\ \underbrace{-4\log x + xy^2}_{\frac{\partial w}{\partial z}}\right)$ GRADIENTE

3) MATRICE HESSIANA

$$H = \begin{bmatrix} 2 + \frac{4z}{x^2} & 2yz & -\frac{4}{x} + y^2 \\ 2yz & 2xz & 2xy \\ -\frac{4}{x} + y^2 & 2xy & 0 \end{bmatrix}$$

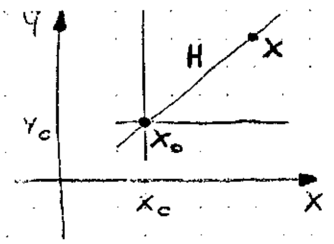
DERIVATA DIREZIONALE

$f(x, y, z)$ $\quad \bar{v} = (v_1, v_2, v_3)$ $\quad \|v\| = 1$ versore

dati $x_0 = (x_0, y_0, z_0)$

$x = (x_0 + hv_1, y_0 + hv_2, z_0 + hv_3)$

$$\frac{\delta f}{\delta v} = \lim_{h \to 0} \frac{1}{h} \left\{ f(x_0 + hv_1, y_0 + hv_2, z_0 + hv_3) - f(x_0, y_0, z_0) \right\}$$



$f(x,y)$

$$\lim_{\substack{\bar{H} \to (0,0,0) \\ \|H\| \to 0}} \frac{w(\bar{x}_0 + \bar{H}) - w(\bar{x}_0)}{\|\bar{H}\|}$$

- w è DIFFERENZIABILE in x_0 se:

$$\lim_{\|H\| \to 0} \frac{w(x_0 + H) - w(x_0) - \nabla w(x_0) \cdot H}{\|H\|} = 0$$

cioè se $w(x_0 + H) = w(x_0) + \nabla w(x_0) \cdot H + o\|H\|$

es. di funzione vettoriale

CURVA GEOMETRICA

$$\gamma(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad \mathbb{R} \to \mathbb{R}^3 \qquad \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 3t \end{cases} \quad t \in (0, 2\pi)$$

$$\sigma(u,v) = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \qquad \mathbb{R}^2 \to \mathbb{R}^3$$

- **COMPOSIZIONE di FUNZIONE**

$$G(Y) = \begin{cases} z_1 = g_1(y_1, \dots, y_m) \\ \vdots \\ z_q = g_q(y_1, \dots, y_m) \end{cases} \qquad \begin{matrix} y_1 = f_1 \\ \vdots \\ y_m = f_m \end{matrix}$$

$\downarrow$

$$(G \circ F)(X) \qquad \mathbb{R}^n \to \mathbb{R}^q$$

per esempio

$$f(x,y,z) = f(x(t), y(t), z(t)) \qquad (f \circ \gamma)(t) \qquad \underset{t}{\mathbb{R}} \to \underset{f}{\mathbb{R}}$$

$\downarrow$

$$J_G(Y) = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots \\ \frac{\partial g_q}{\partial y_1} & \frac{\partial g_q}{\partial y_2} & \cdots & \frac{\partial g_q}{\partial y_m} \end{bmatrix}$$

$\downarrow$

$$J_{G \circ F}(X) = J_G(X) \times J_F(X) \qquad \text{prodotto righe per colonne}$$

REGOLA di DERIVAZIONE per un CAMPO SCALARE

$$J_f(X) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \qquad J_\gamma(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$

$$J_{(f \circ \gamma)}(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y} + \frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{d(f \circ \gamma)}{dt}$$

es. MASSA $= \int_a^b \partial(x(t), y(t), z(t)) \cdot \|\gamma'(t)\|\, dt$

$\partial(P)$
DENSITÀ

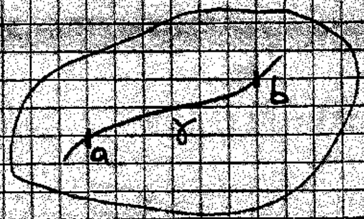
BARICENTRO $x_G = \dfrac{\int_a^b x(t)\, \partial(t) \cdot \|\gamma'(t)\|}{\text{massa}} = \dfrac{\int_a^b x(t)\, \partial(t)\, \|\gamma'(t)\|\, dt}{\int_a^b \partial(P) \cdot \|\gamma'(t)\|\, dt}$

$$G = \frac{\int_a^b P\, \partial(P) \cdot \|\gamma'(t)\|\, dt}{\int_a^b \partial(P) \cdot \|\gamma'(t)\|\, dt} \xrightarrow{\text{se il filo è omogeneo}} = \frac{\int_a^b P\, \|\gamma'(t)\|\, dt}{\int_a^b \|\gamma'(t)\|\, dt}$$

- **INTEGRALE di LINEA**

$\gamma(t) = (x(t), y(t), z(t)) \qquad t \in [a, b]$

$F(P) = f_1(P)\, \hat{i} + f_2(P)\, \hat{j} + f_3(P)\, \hat{k}$



$\int F(P)\, dP \qquad dP = (dx, dy, dz)$

$\downarrow$

$$\int_{\gamma(a,b)} F(P)\, dP = \int_{\gamma(a,b)} \left[f_1(P)\, dx + f_2(P)\, dy + f_3(P)\, dz \right] =$$

$$= \int_{\gamma(a,b)} \left[f_1(x(t), y(t), z(t)) \frac{dx}{dt} dt + f_2\, P(t) \frac{dy}{dt} dt + f_3\, P(t) \frac{dz}{dt} dt \right] = \int F(P(t)) \cdot \gamma'(t)\, dt$$

l'integrale di linea è orientato, a seconda si determinano gli estremi di integrazione

es. 1 $F(P) = \dfrac{x}{x^2+y^2}\, \hat{i} + \dfrac{y}{x^2+y^2}\, \hat{j} + \hat{k}$

$\gamma(t) = (\cos t, \operatorname{sen} t, t) = \cos t\, \hat{i} + \operatorname{sen} t\, \hat{j} + t\, \hat{k} \qquad t \in (0, 3\pi)$

ELICA $\longrightarrow$

$F(P) = \frac{\partial g(P)}{\partial x}\hat{i} + \frac{\partial g(P)}{\partial y}\hat{j} + \frac{\partial g(P)}{\partial z}\hat{k} = f_1(P)\hat{i} + f_2(P)\hat{j} + f_3(P)\hat{k}$

$g(P)$ = FUNZIONE POTENZIALE DEL CAMPO $F(P)$

$$\int_{\gamma_1} F(P)\,dP = \int_{\gamma_2} F(P)\,dP \qquad \forall\, \gamma_{(A,B)}$$

Il LAVORO in un campo conservativo non dipende dal percorso ma solo dagli estremi del percorso

Dim. Hp: $\exists\, g(P) \,/\, F(P) = \text{grad}\, g(P)$

D = dom $F(P)$ è aperto connesso

$F(P)$ è continua

Th: $\int_\gamma F(P)\,dP = g(B) - g(A)$

$$\int_{\gamma(A,B)} F(P)\,dP \qquad \gamma(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t \in [t_A, t_B]$$

$$= \int \left[f_1(x(t),y(t),z(t))\frac{dx}{dt}dt + f_2(x(t),y(t),z(t))\frac{dy}{dt}dt + f_3(x(t),y(t),z(t))\frac{dz}{dt}dt \right] =$$

$$= \int_{t_A}^{t_B} \left[\frac{\partial g(x(t),y(t),z(t))}{\partial x}\frac{dx}{dt} + \frac{\partial g(P(t))}{\partial y}\frac{dy}{dt} + \frac{\partial g(P(t))}{\partial z}\frac{dz}{dt} \right] dt =$$

$$= \int_{t_A}^{t_B} \frac{dg(P \circ \gamma(t))}{dt}\,dt = \left[g(P \circ \gamma(t)) \right]_{t_A}^{t_B} = g(B) - g(A)$$

es. $L_{(0,B)} = \int_0^B F \cdot dP = [xyz]_0^B = \frac{3\sqrt{3}}{27} - 0 = \frac{\sqrt{3}}{9}$

$F(P) = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k} \qquad B = \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

$g(P) = xyz + C$ quali altri POTENZIALI sarebbero andati bene

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Dim.
2) $\Longrightarrow$ 1)

Hp: $\displaystyle\int_{\gamma_1} F(P)\,dP = \int_{\gamma_2} F(P)\,dP$

Th: $\exists\, g(P) \,/\, F(P) = \text{grad } g(P)$

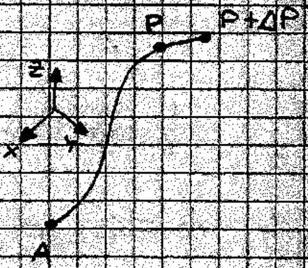
$\downarrow$

$$g(P) = \int_{A(a,b,c)}^{P(x,y,z)} F(P)\,dP \longrightarrow \frac{\partial g(P)}{\partial x} = f_1(x,y,z), \quad \frac{\partial g(P)}{\partial x} = f_2(x,y,z), \quad \frac{\partial g(P)}{\partial z} = f_3(x,y,z)$$

ne dimostro solo uno:

$$\lim_{h\to 0} \frac{1}{h}\Big(g(x+h,y,z) - g(x,y,z)\Big) = \lim_{h\to 0}\frac{1}{h}\left[\int_A^{(P+\Delta P)(x+h,y,z)} F(P)\,dP - \int_A^{P(x,y,z)} F(P)\,dP\right] =$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\cancel{\int_{\gamma(A,P)} F(P)\,dP} + \int_P^{P+\Delta P} F(P)\,dP - \cancel{\int_{\gamma(A,P)} F(P)\,dP}\right] =$$



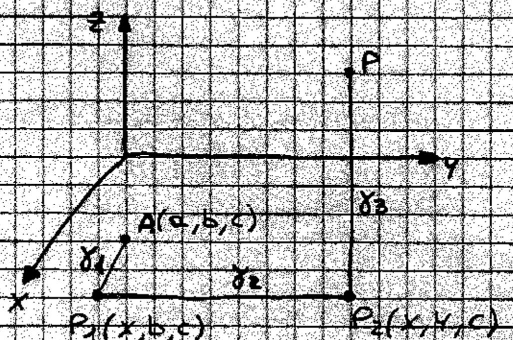
$$(P, P+\Delta P): \begin{cases} x = x+t \\ y = y \\ z = z \end{cases} \qquad \downarrow \qquad = \lim_{h\to 0}\frac{1}{h}\int_0^h f_1(x+t,y,z)\,dt =$$

$$= \lim_{h\to 0}\frac{1}{\cancel{h}}\,\cancel{h}\, f_1(x+\theta \cancel{h}, y, z) \qquad 0 \le \theta \le 1 \longrightarrow = f_1(x,y,z)$$

per il TEOREMA DELLA MEDIA

la curva che unisce $A-P$ DEVE $\in D$

$$g(P) = \int_{A(a,b,c)}^{P(x,y,z)} F(P)\,dP$$



$\gamma_1(t) = (t, b, c) \qquad \gamma_1' = (1,0,0)$

$\gamma_2(t) = (x, t, c) \qquad \gamma_2' = (0,1,0)$

$\gamma_3(t) = (x, y, t) \qquad \gamma_3' = (0,0,1)$

$$g(P) = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = \int_a^x f_1(t,b,c)\,dt + \int_b^y f_2(x,t,c)\,dt + \int_c^z f_3(x,y,t)\,dt$$

$\longrightarrow$ CALCOLO del POTENZIALE

Dim.

$$\mathrm{rot}(F(P)) = \left(\frac{\partial}{\partial y}\left(\frac{\partial g(P)}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial g(P)}{\partial y}\right)\right)\hat{\imath} + \left(\frac{\partial^2 g(P)}{\partial z \partial x} - \frac{\partial^2 g(P)}{\partial x \partial z}\right)\hat{\jmath} + \left(\frac{\partial^2 g(P)}{\partial x \partial y} - \frac{\partial^2 g(P)}{\partial y \partial x}\right)\hat{k}$$

per il TEOREMA di SCHWARZ (inversione delle derivate):

$$\frac{\partial^2 g(P)}{\partial x \partial y} = \frac{\partial^2 g(P)}{\partial y \partial x} \longrightarrow \mathrm{rot}(F(P)) = (0,0,0) = \vec{0}$$

adesso devo verificare che, se $\mathrm{rot} = \vec{0} \Longleftrightarrow F(P) = \mathrm{grad}\, g(P)$

$$\mathrm{rot}(F(P)) = \mathrm{rot}\left\{f_1(x,y)\hat{\imath} + f_2(x,y)\hat{\jmath} + 0\hat{k}\right\} =$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]\hat{k} = \left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]\hat{k} = \frac{0}{(x^2+y^2)^2}\hat{k} = \vec{0}$$

$\downarrow$

$$F(P) = \mathrm{grad}\, g(P) \Longleftrightarrow \begin{cases} \mathrm{rot}(F(P)) = \vec{0} \\ \mathrm{dom}\, F(P) \text{ connesso in } \mathbb{R}^3 \end{cases}$$

semplicemente connesso (senza buchi) in $\mathbb{R}^2$

- **FUNZIONI IMPLICITE/ESPLICITE**

$g(x_1, x_2, \dots, x_n) = 0$ implicita

$x_i = \varphi(x_1, x_2, \dots, x_n)$ esplicita

- **TEOREMA del DINI**

$g(x_1, x_2, \dots, x_n) = 0 \qquad \mathbb{R}^n \to \mathbb{R}$

Hp: $g(x) = g(x_1, x_2, \dots, x_n) = 0$ Th: $x_i = f_i(x_1, x_2, \dots, x_n)$

$$x_0 = (x_1^0, x_2^0, \dots, x_n^0) \longrightarrow \begin{cases} g(x_0) = 0 \\ \dfrac{\partial g(x_0)}{\partial x_i} \neq 0 \end{cases}$$

es 3) CAMPI e INTEGRALI di LINEA

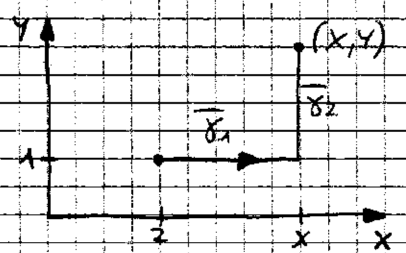
$$\bar{f}(x,y) = \begin{pmatrix} xy \\ x^2+y^2 \end{pmatrix} \qquad \int_\gamma \bar{f}(x,y)\cdot\underset{\text{VERSORE TANGENTE}}{\bar{\tau}} = \int_a^b f(\bar{\gamma}(t))\cdot\gamma'(t)\,dt$$

$$\bar{\tau} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

Calcolo del POTENZIALE:

$$\bar{f}(x,y) = \begin{pmatrix} 2xy \\ x^2+y^2 \end{pmatrix} \qquad \frac{\partial f_2}{\partial x} = 2x \qquad \frac{\partial f_1}{\partial y} = 2x$$

$$\text{rot}\,\bar{f} = 0$$



$$\bar{\gamma}_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} \quad 2 \le t \le x \qquad \bar{\gamma}_1'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\bar{\gamma}_2(t) = \begin{pmatrix} x \\ t \end{pmatrix} \quad 1 \le t \le y \qquad \bar{\gamma}_2'(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\int_2^x \begin{pmatrix} 2\cdot t\cdot 1 \\ t^2+1^2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \int_1^y \begin{pmatrix} 2xt \\ x^2+t^2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \int_2^x 2t\,dt + \int_1^y (x^2+t^2)\,dt =$$

$$= \left[t^2\right]_2^x + \left[x^2 t + \frac{t^3}{3}\right]_1^y = \cancel{x^2} - 4 + x^2 y + \frac{y^3}{3} - \cancel{x^2} - \frac{1}{3}$$

$$\underset{\text{POTENZIALE}}{g(x)} = x^2 y + \frac{y^3}{3} + C$$

- TEOREMA del DINI

$g(x_1, x_2, \dots, x_n) = 0 \qquad \mathbb{R}^n \to \mathbb{R}$

Hp: $x_0 = (x_1^0, x_2^0, \dots, x_n^0)$

$g(x_0) = 0$

$\dfrac{\partial g(x_0)}{\partial x_i} \neq 0$

Th: $x_i = f_i(x_1, x_2, \dots, x_n) \qquad \mathbb{R}^{n-1} \to \mathbb{R}$

COROLLARI

1) c.s. in x_0, sia $x_j = f_j(x_1, x_2, \dots, x_n)$

$\downarrow$

$\nabla g(x_0) \neq (0,0,0)$

2) $\dfrac{\partial f_i}{\partial x_j} = - \dfrac{\dfrac{\partial g(x)}{\partial x_j}}{\dfrac{\partial g(x)}{\partial x_i}} \qquad j \neq i$

es. $y^2 + xz + z^2 - e^z - 4 = 0$

$\dfrac{\partial g}{\partial z} = x + 2z - e^z \qquad x_0 \begin{cases} x = 0 \\ y = \sqrt{5} \\ z = 0 \end{cases}$

non so ricavare $z = f(x,y)$

però so ricavare $\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$

$\downarrow$

$\nabla g(x) = (z, 2y, x + 2z - e^z)$

$\dfrac{\partial f(x,y)}{\partial x} = - \dfrac{z}{x + 2z - e^z}$

$\dfrac{\partial f(x,y)}{\partial y} = - \dfrac{2y}{x + 2z - e^z}$

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date le premesse di prima posso esplicitare:

$$\begin{cases} x_1 = \varphi(x_{m+1}, \dots, x_n) \\ x_2 = \varphi(x_{m+1}, \dots, x_n) \\ \vdots \\ x_m = \varphi_m(x_{m+1}, \dots, x_n) \end{cases}$$

derivo e pongo $= 0$
cerco max e min liberi di $(w \circ \phi)$

$$\phi(x_{m+1}, \dots, x_n)$$

es. 1 $w = x^2 + y^2 + z^2$ $\quad \mathbb{R}^3 \to \mathbb{R}$

$x + y + z + 1 = 0 \longrightarrow z = -1 - x - y$

$$(w \circ z)(x,y) = x^2 + y^2 + (-1 - x - y)^2$$

$$\frac{\partial (w \circ z)}{\partial x} = 2x - 2(-1 - x - y) = 4x + 2y + 2 = 0$$

$$\frac{\partial (w \circ z)}{\partial y} = 2x + 4y + 2 = 0$$

pongo a sistema

$y = x = -\frac{1}{3} \qquad P\left(-\frac{1}{3}, -\frac{1}{3}\right)$

$$H_{(w \circ z)} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \longrightarrow \det(H - \lambda I) = 0$$

$\lambda_1 = 6$
$\lambda_2 = 2$ $\longrightarrow$ P è MIN

- MAX e MIN per funzioni a più variabili

guardo gli autovalori e posso avere 3 casi:

1) tutti POSITIVI $\longrightarrow$ MINIMO
2) tutti NEGATIVI $\longrightarrow$ MASSIMO
3) + e − $\longrightarrow$ FLESSO

$$\begin{cases} \frac{\partial w}{\partial x}\frac{\partial g}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial g}{\partial x} \\ \frac{\partial w}{\partial y}\frac{\partial g}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial g}{\partial y} \end{cases} \longrightarrow \begin{cases} \frac{\partial w}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial w}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ \frac{\partial w}{\partial z} = \lambda \frac{\partial g}{\partial z} \end{cases} \qquad \nabla f = \lambda \nabla g$$

$$g(x,y,z) = 0$$

continuazione

es. 1

$$(x,y,z,\lambda) \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases}$$

$$x + y + z + 1 = 0$$

es. 2

1) moltiplicatore di Lagrange

$$\begin{cases} \underbrace{z = x^2 - xy + y^2}_{w(x,y)} \\ \underbrace{x^2 + y^2 - 4 = 0}_{g(x,y)} \end{cases} \qquad n_{\text{INCOGNITE}} = 2 \qquad m_{\text{VINCOLI}} = 1$$

$$\nabla w = (2x - y, -x + 2y)$$

$$\nabla g = (2x, 2y)$$

$$\downarrow$$

$$\begin{cases} \nabla w = \lambda \nabla g \\ g(x,y) = 0 \end{cases} \longrightarrow \begin{cases} 2x - y = 2\lambda x \\ -x + 2y = 2\lambda y \\ x^2 + y^2 - 4 = 0 \end{cases} \longrightarrow \lambda_1 = \frac{1}{2} \qquad \lambda_2 = \frac{3}{2}$$

$$\downarrow$$

$$P_1(\sqrt{2}, \sqrt{2}) \quad P_2(-\sqrt{2}, -\sqrt{2}) \quad P_3(\sqrt{2}, -\sqrt{2}) \quad P_4(-\sqrt{2}, \sqrt{2})$$

$$\downarrow$$

$w(P_1) = w(P_2) = 2 \qquad 2 = \min w(g)$

$w(P_3) = w(P_4) = 6 \qquad 6 = \max w(g)$

$g(x,y) = 0$

2) altra strada sostituzione

esplicito y da $g(x,y)$ $\longrightarrow$ $y = \pm\sqrt{4-x^2}$ e sostituisco in $w(x,y)$ $\longrightarrow$ $z = x^2 \pm \sqrt{4-x^2} + (4-x^2)$

derivo e ottengo i punti stazionari

cambiamento di coordinate (inversione):

$\mathbb{R}^n \to \mathbb{R}^n$

$Y = F(X) \longrightarrow$ $X = (x_1, x_2, \dots, x_n)$
$Y = (f_1, f_2, \dots, f_n)$

es. 1 $\mathbb{R}^2 \to \mathbb{R}^2$

$\begin{cases} x = \rho\cos\theta \\ y = \rho \,\mathrm{sen}\,\theta \end{cases}$ $(\rho, \theta) \to (x, y)$ $\quad \phi(x,y) = \begin{cases} \rho = \varphi_1(x,y) \\ \theta = \varphi_2(x,y) \end{cases}$

• forma ESPLICITA / IMPLICITA:

$F(X) - Y = \bar{0} \longrightarrow \begin{cases} f_1(x_1, x_2, \dots, x_n) - y_1 = 0 \\ \vdots \end{cases}$

$\mathbb{R}^{2n}$

$X = \phi(Y)$ $\quad J(x,y) = (J_F(X) - I)$ $\quad \phi(Y) = F^{-1}(X)\big|_{Y=F(X)} = (F \circ \phi)(Y) = F(\phi(Y))$

$\det J_F(x) \neq 0$

$J_y = I$ $\quad J_{F\circ\phi} = J_F \, J_\phi = I$ $\quad J_\phi = J_F^{-1} = J_{F^{-1}}$

(CON)TINUAZIONE

es. 1

$J(\rho, \theta) = \begin{bmatrix} \cos\theta & -\rho\,\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \rho\cos\theta \end{bmatrix}$ $\quad \det J(\rho,\theta) = \rho\cos^2\theta + \rho\,\mathrm{sen}^2\theta = \rho$

es. 2 "COMPOSIZIONE" di FUNZIONE CON VINCOLI

$\begin{cases} w = f(x,y,z) \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases}$ $\quad n = 3 \quad m = 2 \quad \begin{cases} y = \varphi_1(x) \\ z = \varphi_2(x) \end{cases}$ $\phi(x)$

$\det J = \begin{bmatrix} \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{bmatrix} \neq 0 \quad \frac{\partial(g_1, g_2)}{\partial(y, z)}$

$\begin{cases} w = x^2 + y^2 + z^2 \\ z = 3x^2 - 4y^2 \\ x^2 + y^2 = 4 \end{cases}$ $\quad \begin{cases} 3x^2 - 4y^2 - z = 0 \\ x^2 + y^2 - 4 = 0 \end{cases}$

$\longrightarrow$ n variabili, m vincoli
$$\begin{cases} \nabla w = \lambda_1 \nabla g_1 + \dots + \lambda_m \nabla g_m \\ G(\ ) = 0 \end{cases}$$

CONTINUAZIONE
<u>es. 2</u>

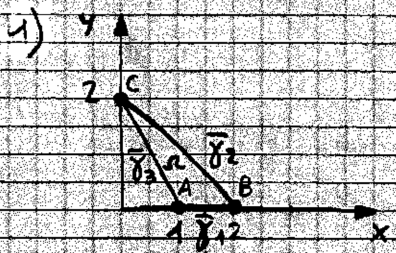
$\nabla w = (2x, 2y, 2z) \qquad \nabla g_1 = (6x, -8y, -1) \qquad \nabla g_2 = (2x, 2y, 0)$

$\downarrow$

$$\begin{cases} 2x = 6\lambda x + 2\mu x \\ 2y = -8\lambda y + 2\mu y \quad \dots \\ 2z = -\lambda_1 \\ 3x^2 - 4y^2 - z = 0 \\ x^2 + y^2 - 4 = 0 \end{cases}$$

<u>es. POME</u>

1)



$f(x,y) = 4x^2 + 3y^2$

A(1,0)
B(2,0) $\quad \Omega$
C(0,2)

Cercare max e min di f(x,y) nel triangolo ABC:

$\bar{\nabla} f = \bar{0} \longrightarrow O(0,0) \notin \Omega$

$\downarrow$

bordi di $\Omega = \bar{\gamma}_1 \cup \bar{\gamma}_2 \cup \bar{\gamma}_3$

li parametrizzo:

- $\bar{\gamma}_1(t) = \begin{bmatrix} t \\ 0 \end{bmatrix} \quad 1 < t < 2 \longrightarrow f(\bar{\gamma}_1(t)) = 4t^2 + 3 \cdot 0^2 = 4t^2 \qquad \dfrac{d f(\bar{\gamma}_1(t))}{dt} = 8t = 0 \quad t \notin \bar{\gamma}_1$

- $\bar{\gamma}_2(t) = \begin{bmatrix} t \\ -(t-2) \end{bmatrix} \longrightarrow f(\gamma_2(t)) = 4t + 3(-(t-2))^2 \qquad \dfrac{d f(\bar{\gamma}_2(t))}{dt} = 14t - 12 \quad t = \dfrac{6}{7}$

$0 < t < 2$

$P_1\left(\dfrac{6}{7}, \dfrac{8}{7}\right) \in \bar{\gamma}_2$

b) SUL BORDO con il moltiplicatore di Lagrange
nei punti di max e min per $f(x,y)$ vincolati su $g(x,y)=1$

$$\begin{cases} \overline{\nabla} f = \lambda \overline{\nabla} g \\ g(x,y)=1 \end{cases} \longrightarrow \begin{cases} \begin{bmatrix} 4x \\ 3+2y \end{bmatrix} = \lambda \begin{bmatrix} 4x+1 \\ 2y+\frac{5}{6} \end{bmatrix} \\ 2x^2+y^2+x+\frac{5}{6}y = 1 \end{cases}$$

osservazione:

$$\begin{matrix} 4x+1 \neq 0 \\ 2y+\frac{5}{6} \neq 0 \end{matrix} \longrightarrow \frac{4x}{4x+1} = \lambda = \frac{3+2y}{2y+\frac{5}{6}}$$

$\downarrow$

$$4x\left(2y+\frac{5}{6}\right) = (4x+1)(3+2y) \longrightarrow y = -\frac{13}{3}x - \frac{3}{2}$$

$\downarrow$ sostituisco nel vincolo

$$2x^2 + \left(-\frac{13}{3}x - \frac{3}{2}\right)^2 + x + \frac{5}{6}\left(-\frac{13}{3}x - \frac{3}{2}\right) = 1$$

$\downarrow$

$x_1 = 0 \qquad x_2 = -\frac{1}{2} \qquad P_1\left(0, -\frac{3}{2}\right) \qquad P_2\left(-\frac{1}{2}, \frac{2}{3}\right)$

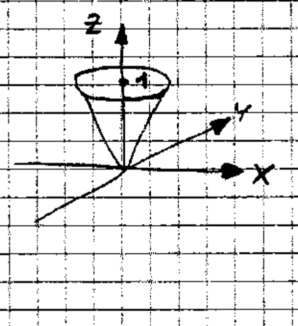
$y = -\frac{3}{2} \qquad y_2 = \frac{2}{3} \qquad f(P_1) = -\frac{9}{4} \qquad f(P_2) = \frac{53}{18}$

min MAX

es. 3 cercare MAX e min

$$f(x,y,z) = 2x + 2\sqrt{3}y + z^2$$

vincolo $\begin{cases} g(x,y,z) = x^2+y^2-z^2 \leq 0 \\ 0 \leq z \leq 1 \end{cases}$ CONO



a) $\overline{\nabla} f = \overline{0} \quad \begin{bmatrix} 2 \\ 2\sqrt{3} \\ 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ MAI

b) • COPERCHIO $\begin{cases} x^2+y^2<1 \\ z=1 \end{cases} \longrightarrow f(x,y,1) = 2x + 2\sqrt{3}y + 1$

$$\overline{\nabla} f = \overline{0} \longrightarrow \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ MAI}$$

$$\rightarrow \begin{cases} 2 = \lambda 2x + \mu 0 \\ 2\sqrt{3} = \lambda 2y + \mu 0 \\ 2z = \lambda \cdot 0 + \mu \\ x^2 + y^2 - 1 = 0 \\ z - 1 = 0 \end{cases} \rightarrow \quad y = \sqrt{3}x \quad x^2 + 3x^2 = 1 \quad x = \pm\frac{1}{2} \quad y = \pm\frac{\sqrt{3}}{2} \quad z = 1$$

$P_2\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\right)$

$P_3\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1\right)$

$O(0,0,0)$

$f(P_2) = 5$ MAX

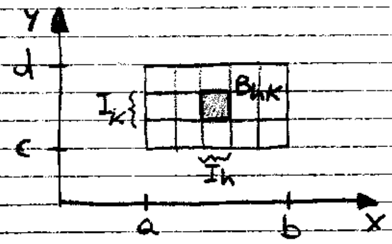
$f(P_3) = -3$ min

$f(O) = 0$

• INTEGRALI di FUNZIONI a PIÙ VARIABILI

~~INTEGRALE DOPPIO~~

INTEGRALE DOPPIO su RETTANGOLI



$[a,b] \times [c,d]$

$B_{hk} = I_h \times I_k$

$z = f(x,y) \quad \mathbb{R}^2 \rightarrow \mathbb{R}$ LIMITATA

$0 \leq z \leq f(x,y) \quad \vee \quad f(x,y) \leq z \leq 0$

$\mathbb{D}$ insieme di tutte le suddivisioni B_{hk}

$m_{hk} = \inf_{B_{hk}} f \qquad M_{hk} = \sup_{B_{hk}} f$

somme superiori e inferiori:

$s = s(\mathbb{D}, f) = \Sigma_k \Sigma_h \, m_{hk} (x_h - x_{h-1})(y_k - y_{k-1})$

$S = S(\mathbb{D}, f) = \Sigma_k \Sigma_h \, M_{hk} (x_h - x_{h-1})(y_k - y_{k-1})$

es. 1 $B = [0, \frac{\pi}{4}] \times [0, \frac{\pi}{2}]$

$$\iint_B \cos(x+y)\,dx\,dy = \int_0^{\frac{\pi}{2}} \left(\int_0^{\pi/4} \cos(x+y)\,dx \right) dy = \int_0^{\pi/2} \Big[+\operatorname{sen}(x+y) \Big]_{x=0}^{x=\frac{\pi}{4}} dy =$$

$$= \int_0^{\pi/2} \left(\operatorname{sen}\left(\frac{\pi}{4}+y\right) - \operatorname{sen} y \right) dy = \left[-\cos\left(\frac{\pi}{4}+y\right) + \cos y \right]_0^{\pi/2} =$$

$$= -\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + \cos\frac{\pi}{4} + \cos\frac{\pi}{2} - \cos 0 = -\cos\frac{3}{4}\pi + \frac{\sqrt{2}}{2} - 1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

si può risolvere anche integrando prima su y e poi su x

per verticali

$B = [1,2] \times [0,\pi]$

es. 2 $$\iint_B x \cos xy\,dx\,dy = \int_1^2 \left(\int_0^{\pi} x \cos xy\,dy \right) dx = \int_1^2 \Big[\operatorname{sen} xy \Big]_{y=0}^{y=\pi} dx =$$

$$= \int_1^2 (\operatorname{sen}\pi x)\,dx = \left[-\frac{1}{\pi}\cos\pi x \right]_1^2 = -\frac{1}{\pi} - \left(+\frac{1}{\pi} \right) = -\frac{2}{\pi}$$

INSIEMI MISURABILI

$$\chi_{\Omega}(\bar{x}) = \begin{cases} 1 & \text{se } x \in \Omega \\ 0 & \text{se } x \notin \Omega \end{cases}$$ funzione caratteristica

$\Omega \subseteq B$ B rettangolo

$$\iint_B \chi_{\Omega}(\bar{x})\,dx\,dy = \text{Area di } \Omega = |\Omega| = \iint_{\Omega} \chi_{\Omega}(x,y)\,dx\,dy = \iint_{\Omega} dx\,dy$$

INSIEMI di MISURA NULLA

$A \subseteq \mathbb{R}^2$ $|A| = \iint_A dx\,dy = 0$

$\Omega \subseteq \mathbb{R}^2$ è MISURABILE $\Longleftrightarrow$ $\underbrace{\partial\Omega}_{\text{FRONTIERA DI }\Omega}$ ha misura nulla

f definita su Ω (insieme misurabile)

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{se } x \in \Omega \\ 0 & \text{se } x \notin \Omega \end{cases}$$

b) $\Omega \subseteq \mathbb{R}^2$ sia ORIZZONTALMENTE CONVESSO, $\Omega \to \mathbb{R}$ CONTINUA allora

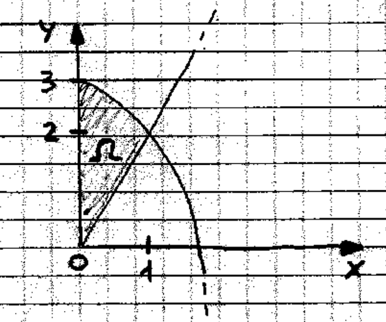
$$\iint_\Omega f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

es.

$$\iint_\Omega (x+2y)\,dx\,dy$$

Ω è la regione del 1° quadrante $(x \geq 0, y \geq 0)$, limitata dalle curve di equazione

$y = 2x$, $y = 3 - x^2$, $x = 0$



$$\begin{cases} y \leq 3 - x^2 \\ y \geq 2x \\ x \geq 0 \\ y \geq 0 \end{cases}$$

descriviamo Ω per VERTICALI:

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \;/\; 0 \leq x \leq 1 \quad \vee \quad 2x \leq y \leq 3 - x^2 \right\}$$

se volessi descrivere per ORIZZONTALI:

$$\begin{cases} 0 \leq y \leq 2 & 0 \leq x \leq \frac{1}{2} y \\ \\ 2 \leq y \leq 3 & 0 \leq x \leq \sqrt{3-y} \end{cases}$$

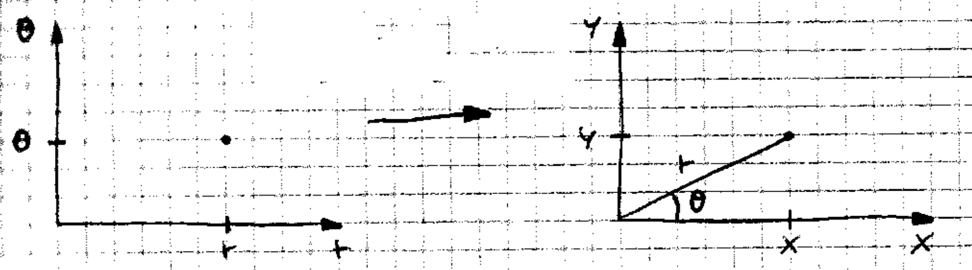
↓ svolgo per verticali

$$\iint_\Omega (x+2y)\,dx\,dy = \int_0^1 \left(\int_{2x}^{3-x^2} (x+2y)\,dy \right) dx = \int_0^1 \left[xy + y^2 \right]_{y=2x}^{y=3-x^2} dx =$$

$$= \int_0^1 \left[x(3-x^2) + (3-x^2)^2 - 2x^2 - 4x^2 \right] dx = \int_0^1 (3x - x^3 + 9 - 6x^2 + x^4 - 6x^2)\,dx =$$

$$= \int_0^1 (x^4 - x^3 - 12x^2 + 3x + 9)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4} - \frac{12}{3}x^3 + \frac{3}{2}x^2 + 9x \right]_0^1 = \frac{1}{5} - \frac{1}{4} - \frac{12}{3} + \frac{3}{2} + 9$$ [illegible]

CAMBIO di VARIABILI con COORDINATE POLARI nel PIANO (caso importante)



$$\begin{pmatrix} x \\ y \end{pmatrix} = \overline{\Phi}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

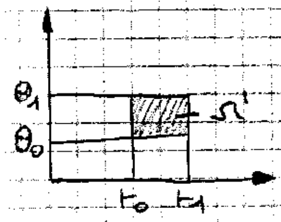
$$\det J\phi = \left|\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}\right| = r\cos^2\theta + r\sin^2\theta = r$$

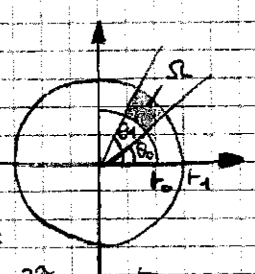
$\downarrow$

$$\int_\Omega dx\,dy = \int_{\Omega'} r\,dr\,d\theta$$

$$\int_\Omega f(x,y)\,dx\,dy = \int_{\Omega'} f(\phi(u,v))\,|\det J\phi(u,v)|\,du\,dv$$

es. 1 $\iint_\Omega dx\,dy \qquad \Omega : \{x^2+y^2 \le r^2\}$





se voglio l'area del cerchio:

$$\iint_\Omega dx\,dy = \iint_{\Omega'} r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{r_0} r\,dr = [\theta]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{r_0} = 2\pi\,\frac{r_0^2}{2} = \pi r_0^2$$

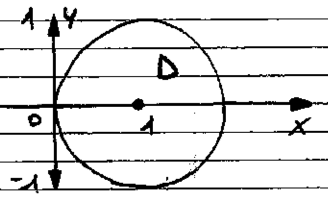
se voglio calcolare un settore:

$$\iint_\Omega dx\,dy = \iint_{\Omega'} r\,dr\,d\theta = \int_{\theta_0}^{\theta_1} d\theta \int_{r_0}^{r_1} r\,dr = [\theta]_{\theta_0}^{\theta_1}\left[\frac{r^2}{2}\right]_{r_0}^{r_1} = (\theta_1-\theta_0)\frac{(r_1^2-r_0^2)}{2}$$

es. 3 $\iint_D \sqrt{x^2+y^2}\, dx\, dy$ $\qquad D = \{(x,y) \in \mathbb{R}^2 / x^2+y^2-2x \leq 0\}$

ricostruisco il quadrato:

$(x-1)^2-1+y^2 \leq 0 \longrightarrow (x-1)^2+y^2 \leq 1$



coordinate polari

$$\phi: \begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases} \qquad \det|J\phi| = r$$

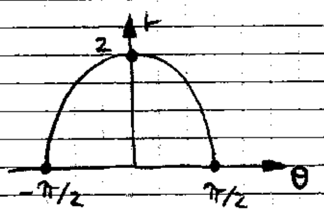
cosa è D': $r^2\cos^2\theta + r^2\text{sen}^2\theta - 2r\cos\theta \leq 0$

$r^2 - 2r\cos\theta \leq 0 \qquad$ per $r \geq 0$

$r(r-2\cos\theta) \leq 0$

$\downarrow$

$r = 2\cos\theta$



$$\iint_D \sqrt{x^2+y^2}\, dx\, dy = \iint_{\Omega'} \sqrt{r^2} \cdot r\, dr\, d\theta \overset{\text{per verticali}}{=}$$

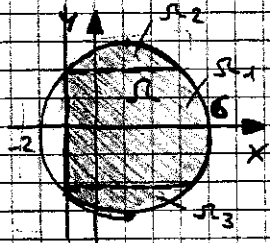
$$= \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\cos\theta} r^2\, dr \right) d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \right]_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} \frac{8\cos^3\theta}{3}\, d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \frac{8}{3}(1-\text{sen}^2\theta)\cos\theta\, d\theta \longrightarrow \begin{matrix} t = \text{sen}\,\theta \\ dt = \cos\theta\, d\theta \end{matrix} \longrightarrow \int_{-1}^{1} \frac{8}{3}(1-t^2)\, dt =$$

$$= \left[\frac{8}{3}t - \frac{8}{9}t^3 \right]_{-1}^{1} = \frac{16}{3} - \frac{16}{9} = \frac{32}{9}$$

es. 2

$$\iint_{\Omega} (2x+y-4)\,dx\,dy \qquad \Omega : \begin{cases} x^2+y^2-4x \le 12 & (x-2)^2+y^2=16 \\ x \ge 2-2\sqrt{2} & x=2-2\sqrt{2} \end{cases}$$



① per verticali:

$$\Omega = \{(x,y) / -\sqrt{12-x^2+4x} \le y \le \sqrt{12-x^2+4x}$$
$$2-2\sqrt{2} \le x \le 6\}$$

$$\downarrow$$

$$\int_{2-2\sqrt{2}}^{6}\left(\int_{-\sqrt{12-x^2+4x}}^{\sqrt{12-x^2+4x}} (2x+y-4)\,dy\right)dx \qquad + \text{ MEGLIO}$$

② per orizzontali:

$$\Omega = \Omega_1+\Omega_2+\Omega_3$$

$$\Omega_1 = \{(x,y) / 2-2\sqrt{2} \le x \le 2+\sqrt{16-y^2} \,,\, -2\sqrt{2} \le y \le 2\sqrt{2}\}$$

$$\Omega_2 = \{(x,y) / 2-\sqrt{16-y^2} \le x \le 2+\sqrt{16-y^2} \,,\, 2\sqrt{2} \le y \le 4\}$$

$$\Omega_3 = \{(x,y) / 2-\sqrt{16-y^2} \le x \le 2+\sqrt{16-y^2} \,,\, -4 \le y \le 2\sqrt{2}\}$$

$$\downarrow$$

$$\iint_{\Omega_1} (2x+y-4)\,dx\,dy + \iint_{\Omega_2} (2x+y-4)\,dx\,dy + \iint_{\Omega_3} (2x+y-4)\,dx\,dy =$$

$$= \int_{-2\sqrt{2}}^{\sqrt{2}}\left(\int_{2-2\sqrt{2}}^{2+\sqrt{16-y^2}} (2x+y-4)\,dx\right)dy + \int_{2\sqrt{2}}^{4}\left(\int_{2-\sqrt{16-y^2}}^{2+\sqrt{16-y^2}} (2x+y-4)\,dx\right)dy +$$

$$+ \int_{-4}^{2\sqrt{2}}\left(\int_{2-\sqrt{16-y^2}}^{2+\sqrt{16-y^2}} (2x+y-4)\,dx\right)dy = \dots \qquad \text{PEGGIO}$$

considerazioni:

- in $\mathbb{R}$

$m(x)$ è la densità di massa del filo

$\int_a^b m(x)$ è la massa del filo

- in $\mathbb{R}^2$

$m(x,y)$ è la densità di massa di Ω

$\iint_\Omega m(x,y)\,dx\,dy$ è la massa di Ω

$\iint_\Omega dx\,dy$ è l'area di Ω

se $m(x,y) = z \longrightarrow \iint_\Omega m(x,y)\,dx\,dy$ è un volume

- in $\mathbb{R}^3$

$f(x,y,z)$ è la densità di massa $\iiint_V dx\,dy\,dz$ è un volume

$\iiint_V f(x,y,z)\,dx\,dy\,dz$ è la massa di V

- INTEGRALE TRIPLO

$$\overline{\iiint_V} f\,dx\,dy\,dz = \underline{\iiint_V} f\,dx\,dy\,dz$$

somme superiori = somme inferiori

FORMULE di RIDUZIONE per FILI PARALLELI

a) supponiamo che Ω sia convesso nella direzione z (semplice o normale rispetto a z)

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 \,/\, (x,y) \in D \subseteq \mathbb{R}^2,\ g_1(x,y) \le z \le g_2(x,y)\}$$

il piano che cerco è normale a $\bar{N}$ e passante per $(1,0,0)$:

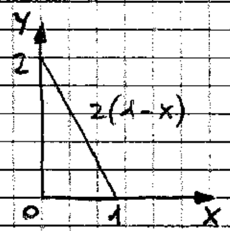
$$\bar{N} \cdot \begin{pmatrix} x-1 \\ y \\ z \end{pmatrix} = 0 \longrightarrow 6(x-1) + 3y + 2z = 0 \longrightarrow z = \frac{1}{2}\left(-6(x-1) - 3y\right)$$

$$z = -3x - \frac{3}{2}y + 3$$

$$\downarrow$$

$$\Omega = \left\{ (x,y) \in D \,/\, 0 \le z \le -3x - \frac{3}{2}y + 3 \right\}$$

$$\iiint_\Omega dx\,dy\,dz = \iint_D \left(\int_0^{-3x-\frac{3}{2}y+3} dz \right) dx\,dy = \iint_D \left(-3x - \frac{3}{2}y + 3\right) dx\,dy =$$



per VERTICALI

$$= \int_0^1 \left(\int_0^{2(1-x)} \left(-3x - \frac{3}{2}y + 3\right) dy \right) dx =$$

$$= \int_0^1 \left[-3xy - \frac{3}{2}\frac{y^2}{2} + 3y \right]_{y=0}^{y=2(1-x)} dx = \int_0^1 \left(-\cancel{6x} + 6x^2 - 3 + \cancel{6x} - 3x^2 + 6 - 6x \right) dx =$$

$$= \left[x^3 - 3x^2 + 3x \right]_0^1 = 1$$

FORMULE di RIDUZIONE per STRATI

d) Ω sia incontrato da tutti i piani paralleli al piano x,y con $a \le z \le b$

indichiamo con C_z l'intersezione di Ω con il piano parallelo al piano x,y e a quota z con $a \le z \le b$

allora:

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\iint_{C_z} f(x,y,z)\,dx\,dy \right) dz$$

per STRATI PARALLELI a x,y

lo stesso vale per l'integrazione per STRATI PARALLELI ai piani y,z e x,z

$$\bar{\varphi}(r,\theta,\varphi) = \begin{pmatrix} r\,\text{sen}\theta\cos\varphi \\ r\,\text{sen}\theta\,\text{sen}\varphi \\ r\cos\theta \end{pmatrix}$$

$$|\det J\varphi| = \left|\det\begin{pmatrix} \text{sen}\theta\cos\varphi & r\cos\theta\cos\varphi & -r\,\text{sen}\theta\,\text{sen}\varphi \\ \text{sen}\theta\,\text{sen}\varphi & r\cos\theta\,\text{sen}\varphi & r\,\text{sen}\theta\cos\varphi \\ \cos\theta & -r\,\text{sen}\theta & 0 \end{pmatrix}\right| =$$

$$= \left|\cos\left(r^2\text{sen}\theta\cos\theta\cdot 1\right) + r\,\text{sen}\theta\left(r\,\text{sen}^2\theta\cdot 1\right)\right| = r^2\,\text{sen}\theta \geq 0$$

VOLUME SFERA

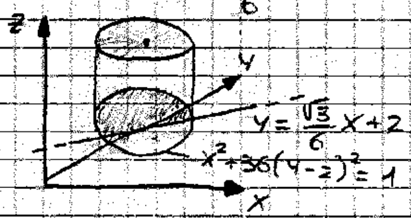
$$\Omega = x^2+y^2+z^2 \leq r^2$$

$$\iiint_{\Omega} dx\,dy\,dz = \iiint_{\Omega'} r^2\,\text{sen}\theta\,dr\,d\theta\,d\varphi = \int_0^{r_0} r^2\,dr \int_0^{\pi}\text{sen}\theta\,d\theta \int_0^{2\pi} d\varphi =$$

$$= \frac{r_0^3}{3}\cdot 2\cdot 2\pi = \frac{4}{3}\pi r_0^3$$

es. FOME

1) $\Omega : \begin{cases} 0 \leq z \leq x^2+36(y-2)^2 \\ (x,y)\in D \end{cases}$ $\qquad D : \begin{cases} x^2+36(y-2)^2 \leq 1 \rightarrow \frac{x^2}{1}+\frac{(y-2)^2}{\left(\frac{1}{6}\right)^2}=1 \\ y \geq \frac{\sqrt{3}}{6}x+2 \end{cases}$



$$\iiint_{\Omega} dx\,dy\,dz = \text{per FILI // } z$$

$\downarrow$

$$\iint_D\left(\int_0^{x^2+36(y-2)^2} dz\right)dx\,dy = \iint_D [z]_0^{x^2+36(y-2)^2}\,dx\,dy = \iint_D \left(x^2+36(y-2)^2\right)dx\,dy =$$

cambio variabile $\begin{cases} u=x \\ v=y-2 \\ v\geq\frac{\sqrt{3}}{6}u \end{cases} \longrightarrow u^2+36v^2=1$ $\qquad \begin{cases} t=r\cos\theta \\ s=\frac{1}{6}r\,\text{sen}\theta \end{cases}$ $\qquad \frac{y}{x}=\frac{\sqrt{3}}{6}=\text{tg}\,\alpha$

$\alpha = \text{arctg}\frac{\sqrt{3}}{6}$

$\downarrow$

cambio coordinate 2 volte, prima con una parabola e poi una circonferenza $\begin{cases} u=t \\ v=\frac{1}{6}s \end{cases}$ $\quad t^2+s^2=1$

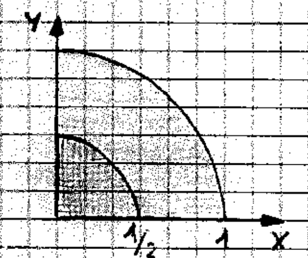
$\alpha \leq \theta \leq \alpha+\pi \longrightarrow$

$J = \frac{1}{6}r$

$z_0 = \text{cost}$ taglio la superficie con un piano z_0

$$\sqrt{x^2+y^2} = \frac{1}{z_0} \longrightarrow x^2+y^2 = \left(\frac{1}{z_0}\right)^2$$

ogni piano che taglia individua una circonferenza di raggio $= \frac{1}{z_0}$



Facciamo l'integrale per strati // al piano x,y

$1 \le z_0 \le 2$

$$C_z = \left\{(x,y) \in \mathbb{R}^2 \,/\, x^2+y^2 \le \frac{1}{z^2},\ 1 \le z \le 2\right\}$$

$$I = \int_1^2 \left(\iint_{C_z} xy\,dx\,dy\right) dz \longrightarrow \begin{cases} x = r\cos\theta & 0 \le \theta \le \frac{\pi}{2} \\ y = r\operatorname{sen}\theta & 0 \le r \le \frac{1}{z} \end{cases}$$

$$\int_1^2 \left(\iint_{C'_z} \underbrace{r\cos\theta}_{x} \cdot \underbrace{r\operatorname{sen}\theta}_{y} \cdot \underbrace{r}_{J}\,dr\,d\theta\right) dz = \int_1^2 \left(\iint_{C'_z} r^3 \cos\theta \operatorname{sen}\theta\,dr\,d\theta\right) dz =$$

$$= \int_1^2 \left(\int_0^{1/z} r^3\,dr \int_0^{\pi/2} \cos\theta\operatorname{sen}\theta\,d\theta\right) dz = \int_1^2 \left[\frac{r^4}{4}\right]_0^{1/z} \cdot \left[\frac{\operatorname{sen}^2\theta}{2}\right]_0^{\pi/2} dz =$$

$$= \int_1^2 \frac{1}{4z^4} \cdot \frac{1}{2}\,dz = \left[\frac{1}{8}\,\frac{1}{(-3)z^3}\right]_1^2 = -\frac{1}{192} + \frac{1}{24} = \frac{7}{192}$$

es. 4 Calcolare il volume del solido di rotazione dato dalle seguenti eq. (intorno all'asse z)

$$\begin{cases} y^2 - z^2 \ge 0 & (y+z)(y-z) \ge 0 \\ y^2 + z^2 - 4 \le 0 & z = \sqrt{y^2-4} \\ 2 \le y \le 3 \\ z \ge 0 \end{cases}$$

per Guldino

$$V = 2\pi y_G \iint_A dz\,dy = 2\pi \frac{\iint_A y\,dz\,dy}{\cancel{\iint_A dz\,dy}} \cancel{\iint_A dz\,dy} = 2\pi \iint_A y\,dz\,dy = \dots = \frac{2}{3}\pi\left(19 - 5\sqrt{5}\right)$$

se avessi una funzione all'interno dell'integrale:

$$\int_\Sigma f(x,y,z)\, d\sigma = \iint_D f(\sigma_1(u,v), \sigma_2(u,v), \sigma_3(u,v)) \cdot \left\| \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\| du\, dv$$

OSSERVAZIONE

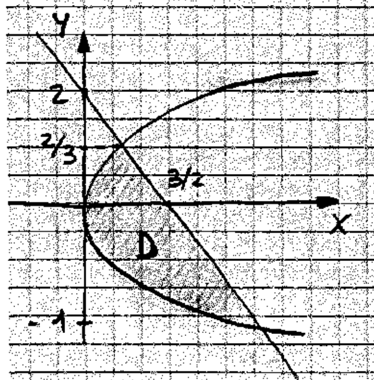
Se la superficie è descritta in modo esplicito

$z = g(x,y)$ — es. $z = \sqrt{1-x^2-y^2}$

$$\bar\sigma(x,y) = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix} \qquad \bar\sigma(x,y) = \begin{pmatrix} x \\ y \\ \sqrt{1-x^2-y^2} \end{pmatrix}$$

$$\frac{\partial \bar\sigma}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial g}{\partial x} \end{pmatrix} \quad \frac{\partial \bar\sigma}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial g}{\partial y} \end{pmatrix} \longrightarrow \|\bar N\| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$$

es. $\int_\Sigma yz\, d\sigma$ dove Σ è la superficie di eq. $z = \sqrt{x-y^2}$ che si proietta nel piano x,y nella regione delimitata dalla parabola $x = y^2$ e dalla retta $3x + y - 2 = 0$



$$\bar\sigma(x,y) = \begin{pmatrix} x \\ y \\ \sqrt{x-y^2} \end{pmatrix} \quad (x,y) \in D$$

$$\|\bar N\| = \sqrt{1 + \left(\frac{1}{2\sqrt{x-y^2}}\right)^2 + \left(\frac{-2y}{2\sqrt{x-y^2}}\right)^2} = \sqrt{\frac{4x - \cancel{4y^2} + 1 + \cancel{4y^2}}{4(x-y^2)}} = \sqrt{\frac{1+4x}{4(x-y^2)}}$$

$$\int_\Sigma yz\, d\sigma = \iint_D y\sqrt{x-y^2} \cdot \frac{1}{2}\sqrt{\frac{1+4x}{(x-y^2)}}\, dx\,dy = \iint \frac{1}{2} y \sqrt{1+4x}\, dx\, dy \overset{\text{per orizzontali}}{=}$$

$$= \int_{-1}^{2/3} \left(\int_{y^2}^{\frac{2-y}{3}} \frac{y\sqrt{1+4x}}{2}\, dx \right) dy = \dots$$

$$\rightarrow \int_{\Sigma} \bar{f}\cdot\bar{N}\,d\sigma = \int_{\Sigma_1} \bar{f}\cdot\bar{N}\,d\sigma + \int_{\Sigma_2} \bar{f}\cdot\bar{N}\,d\sigma + \int_{\Sigma_3} \bar{f}\cdot\bar{N}\,d\sigma + \int_{\Sigma_4} \bar{f}\cdot\bar{N}\,d\sigma$$

$\downarrow$

$$I_1 = \int_{\Sigma_1} \bar{f}\,\bar{N}\,d\sigma = \iint_{D_1} \begin{pmatrix} x-y^2 \\ x^2+\frac{1}{3}y^3 \\ 4 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\,dy = \iint_{D_1} -4\,dx\,dy = -4\cdot\frac{1}{2}\pi(\sqrt{2})^2 = -4\pi$$

$$I_2 = \int_{\Sigma_2} \bar{f}\cdot\bar{N}\,d\sigma = \iint_{D_2} \begin{pmatrix} z^3 \\ x^2 \\ x^2 z+\frac{1}{3}z^3+2 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dx\,dz = \iint_{D_2} -x^2\,dx\,dz = \dots$$

$$I_3 = \int_{\Sigma_3} \bar{f}\cdot\bar{N}\,d\sigma = \iint_{D_3} \begin{pmatrix} xy^2+\left(\sqrt{2-x^2-y^2}\right)^3 \\ x^2+\frac{1}{3}y^3 \\ x^2\sqrt{2-x^2-y^2}+\frac{1}{3}\left(\sqrt{2-x^2-y^2}\right)^3+2 \end{pmatrix}\cdot\bar{N}\,dx\,dy = \dots$$

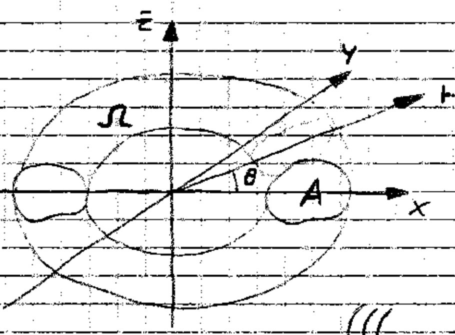
$$\bar{N} = \begin{pmatrix} -\frac{-x}{\sqrt{2-x^2-y^2}} \\ -\frac{-y}{\sqrt{2-x^2-y^2}} \\ 1 \end{pmatrix}$$

$$I_4 = \int_{\Sigma_4} \bar{f}\,\bar{N}\,d\sigma = \iint_{D_4} \begin{pmatrix} xy^2+\left(\sqrt{x^2+y^2}\right)^3 \\ x^2+\frac{1}{3}y^3 \\ x^2\sqrt{x^2+y^2}+\frac{1}{3}\left(\sqrt{x^2+y^2}\right)^3+2 \end{pmatrix}\cdot\begin{pmatrix} -\frac{x}{\sqrt{x^2+y^2}} \\ -\frac{y}{\sqrt{x^2+y^2}} \\ 1 \end{pmatrix} dx\,dy = \dots$$

• TEOREMI di GULDINO

SOLIDI di ROTAZIONE

1°)



Se voglio il volume di Ω

$\downarrow$

$$\iiint_{\Omega} dx\,dy\,dz \qquad \begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \\ z = z \end{cases}$$

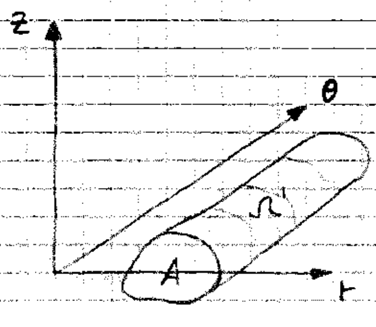
$\downarrow$

fili // θ

$$\iiint_{\Omega'} r\,dr\,d\theta\,dz = \qquad |\det J| = r = \sqrt{x^2+y^2}$$

$0 \le \theta \le 2\pi$

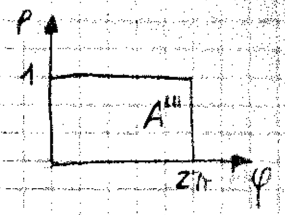
$(r,z) \in A$



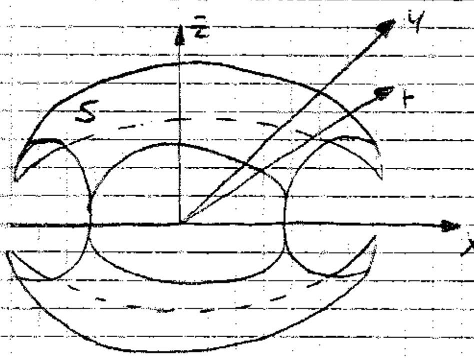
$$= \iint_A \left(\int_0^{2\pi} r\,d\theta\right) dr\,dz =$$

$\longrightarrow$

$$= 0 + 2\pi r_0 ab \left[\frac{\rho^2}{2}\right]_0^1 2\pi = 2\pi^2 r_0 ab = 2\pi r_0 \cdot \pi ab$$

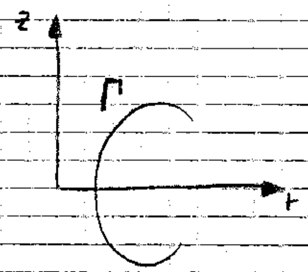


2°) SUPERFICI di ROTAZIONE



Si fa ruotare nello spazio intorno all'asse z una curva γ

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \\ z = z \end{cases} \qquad |\det J| = r$$

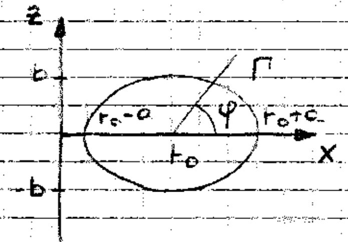


$$\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} \qquad \begin{cases} r = \gamma_1(t) \\ z = \gamma_2(t) \end{cases}$$

$$I = \int_S dS = 2\pi \int_\Gamma x\, d\gamma = 2\pi \underbrace{\frac{\int_\Gamma x\, d\gamma}{\int_\Gamma d\gamma}}_{x_G} \int_\Gamma d\gamma =$$

$$I = 2\pi x_G \cdot \int_\Gamma \|\gamma'(\varphi)\|\, d\varphi$$ 2° TEOREMA di GULDINO

<u>es.</u> superficie di rotazione del TORO



$$I = \int_S dS = 2\pi \int_\Gamma x\, d\varphi \longrightarrow$$

$$\frac{(x-r_0)^2}{a^2} + \frac{z^2}{b^2} = 1$$

$$\gamma(\varphi): \begin{cases} x - r_0 = a\cos\varphi \\ z = b\,\mathrm{sen}\,\varphi \end{cases} \longrightarrow \gamma(\varphi) = \begin{pmatrix} a\cos\varphi + r_0 \\ b\,\mathrm{sen}\,\varphi \end{pmatrix} \longrightarrow \gamma'(\varphi) = \begin{pmatrix} -a\,\mathrm{sen}\,\varphi \\ b\cos\varphi \end{pmatrix}$$

$$\|\gamma'(\varphi)\| = \sqrt{a^2\,\mathrm{sen}^2\varphi + b^2\cos^2\varphi}$$

$$\overline{\gamma}(x) = \begin{pmatrix} x \\ \alpha(x) \end{pmatrix} \longrightarrow \overline{\gamma}'(x) = \begin{pmatrix} 1 \\ \alpha'(x) \end{pmatrix} \qquad a \leq x \leq b$$

$$\int_{\gamma_1} f(x,y)\,dx = \int_a^b f(x,\alpha(x))\,dx$$

$$-\int_{\gamma_4} f\,dx = -\int_a^b f(x,\beta(x))\,dx$$

$$\int_{\partial D} f\,dx = \int_a^b f(x,\alpha(x))\,dx - \int_a^b f(x,\beta(x))\,dx$$

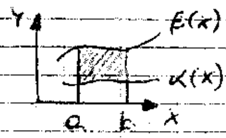
allo stesso modo si può dimostrare che

$$\iint_D \frac{\partial f}{\partial x}\,dx\,dy = \int_{\partial D} f\,dy$$

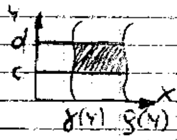
$\downarrow$ quindi

1° TEOREMA nel PIANO $\mathbb{R}^2$

$$\iint_D \frac{\partial f(x,y)}{\partial y}\,dx\,dy = -\int_{\partial D} f(x,y)\,dx$$



$$\iint_D \frac{\partial f(x,y)}{\partial x}\,dx\,dy = \int_{\partial D} f(x,y)\,dy$$



Queste formule valgono per regioni generali, e se vi sono dei buchi, si considerano con ∂D anche i bordi dei buchi

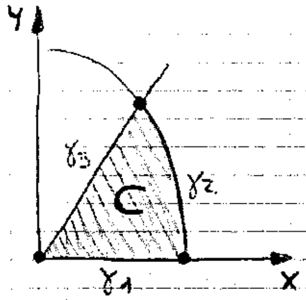
CONSEGUENZA del TEOREMA 1 in un CAMPO VETTORIALE

$$\overline{f}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

$$\iint_D \frac{\partial f_2}{\partial x}\,dx\,dy = \int_{\partial D} f_2\,dy$$

$\longrightarrow$ proviamo a sottrarli membro a membro

$$\iint_D \frac{\partial f_1}{\partial y}\,dx\,dy = -\int_{\partial D} f_1\,dx$$



$$\int_\gamma \overline{F}\cdot dP = \int_\gamma \begin{pmatrix} -2y \\ 3x \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = \int_\gamma (-2y\,dx + 3x\,dy)$$

$$\int_\gamma \overline{F}\cdot d\overline{P} = \int_{\gamma_1} \overline{F}\cdot d\overline{P} + \int_{\gamma_2} \overline{F}\cdot d\overline{P} + \int_{\gamma_3} \overline{F}\cdot d\overline{P}$$

$\downarrow$

① $\overline{\gamma}_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad 0 \le t \le 2 \quad \overline{\gamma}_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \int_{\gamma_1} \overline{F}\cdot d\overline{P} = \int_0^2 \begin{pmatrix} 0 \\ 3t \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = 0$

② $\overline{\gamma}_2(t) = \begin{pmatrix} 2\cos t \\ 4\,\mathrm{sen}\,t \end{pmatrix} \quad 0 \le t \le \frac{\pi}{4} \quad \overline{\gamma}_2' = \begin{pmatrix} -2\,\mathrm{sen}\,t \\ 4\cos t \end{pmatrix}$

$\downarrow$

$$\int_{\gamma_2} \overline{F}\cdot d\overline{P} = \int_0^{\pi/4} \begin{pmatrix} -8\,\mathrm{sen}\,t \\ 6\cos t \end{pmatrix}\begin{pmatrix} -2\,\mathrm{sen}\,t \\ 4\cos t \end{pmatrix} dt = \int_0^{\pi/4} 16\,\mathrm{sen}^2 t + 24\cos^2 t\; dt =$$

$$= \int_0^{\pi/4} 16 - 16\cos^2 t + 24\cos^2 t\; dt = \int_0^{\pi/4} 16 + 8\cdot\frac{1}{2}(1+\cos 2t)\; dt = \ldots = 5\pi + 2$$

③ $\overline{\gamma}_3(t) = \begin{pmatrix} t \\ 2t \end{pmatrix} \quad 0 \le t \le \sqrt{2} \quad \overline{\gamma}_3' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\downarrow$

$$\int_{\gamma_3} \overline{F}\cdot d\overline{P} = -\int_0^{\sqrt{2}} \begin{pmatrix} -4t \\ 3t \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} dt = -\int_0^{\sqrt{2}} (-4t + 6t)\, dt = -2$$

$\downarrow$

$I = 0 + (5\pi + 2) - 2 = 5\pi$ Lavoro compiuto da F lungo γ

$\downarrow$

$$\oint_\gamma \begin{pmatrix} -2y \\ 3x \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = \iint_C \quad \text{(Formula di Green)}$$

$$\iint_C \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_C (3 - (-2))\, dx\,dy = \cancel{5}\iint_C dx\,dy = \cancel{5}\pi$$

$\downarrow$

$$\iint_C dx\,dy = \pi$$

③ $\int_{\Sigma_3} \bar{f}\cdot\bar{n}\, d\sigma$: $\longrightarrow$ parametrizzazione esplicita $\bar{\sigma}(x,y) = \begin{pmatrix} x \\ y \\ \sqrt{2-x^2-y^2} \end{pmatrix}$

$\downarrow$

$$\|\bar{N}\| = \sqrt{1 + \left(\frac{-2x}{2\sqrt{2-x^2-y^2}}\right)^2 + \left(\frac{-2y}{2\sqrt{2-x^2-y^2}}\right)^2} \longrightarrow \bar{N} = \begin{pmatrix} \frac{-x}{\sqrt{2-x^2-y^2}} \\ \frac{-y}{\sqrt{2-x^2-y^2}} \\ 1 \end{pmatrix}$$

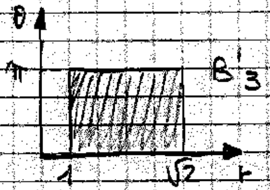
$\downarrow$

$$\iint_{\Sigma_3} \bar{f}\cdot\bar{n}\, d\sigma = \iint_{B_3} \left(2(x^2+y^2)\cdot\sqrt{2-x^2-y^2} + \frac{2}{3}\left(\sqrt{2-x^2-y^2}\right)^3\right) dx\,dy =$$

passo a coordinate $\longrightarrow$ polari $\begin{cases} x = r\cos\varphi \\ y = r\,\mathrm{sen}\,\varphi \end{cases}$ $J = r$

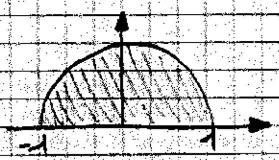
$\downarrow$

$$\iint_{B_3'} \left(2r^2\sqrt{2-r^2} + \frac{2}{3}\left(\sqrt{2-r^2}\right)^3\right) r\,dr\,d\theta =$$



$$= \pi\int_1^{\sqrt{2}} 2r^3\sqrt{2-r^2}\,dr + \int_1^{\sqrt{2}} \pi r\left(\sqrt{2-r^2}\right)^3 dr = \ \underset{\substack{2-r^2=t \\ dt=-2r\,dr}}{\text{sostituisco}} \ = \dots = \frac{16}{15}\pi$$

④ $\int_{\Sigma_4} \bar{f}\cdot\bar{n}\, d\sigma$

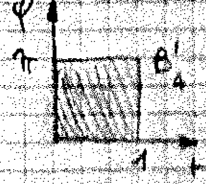


$z = \sqrt{x^2+y^2}$

$$\longrightarrow = \iint_{B_4} \begin{pmatrix} 0 \\ 0 \\ 2(x^2+y^2)\sqrt{x^2+y^2} + \frac{2}{3}\left(\sqrt{x^2+y^2}\right)^3 \end{pmatrix} \cdot \begin{pmatrix} -\frac{\partial z}{\partial x} \\ -\frac{\partial z}{\partial y} \\ 1 \end{pmatrix} dx\,dy =$$

$$= \iint_{B_4} \left(2(x^2+y^2)\sqrt{x^2+y^2} + \frac{2}{3}\left(\sqrt{x^2+y^2}\right)^3\right) dx\,dy = \iint_{B_4'} \left(2r^2\cdot r + \frac{2}{3}r^3\right) r\,dr\,d\varphi =$$

$$= \pi\left[2\frac{r^5}{5} + \frac{2}{3}\frac{r^5}{5}\right]_0^1 = \frac{2}{5}\pi + \frac{2}{15}\pi = \frac{8}{15}\pi$$



$\downarrow$

$$I = 0 + 0 + \frac{16}{15}\pi + \frac{8}{15}\pi = \frac{24}{15}\pi = \frac{8}{5}\pi$$

$$\iiint_V \operatorname{div} \bar{f}(x,y,z)\, dx\,dy\,dz = \int_{\partial V} \bar{f}\cdot\bar{n}\, dS$$ TEOREMA di GAUSS o della DIVERGENZA in $\mathbb{R}^3$

$$\int_S \operatorname{rot} \bar{f}\cdot\bar{n}\, dS = \int_{\partial S} \bar{f}\cdot d\bar{P}$$ TEOREMA di STOKES in $\mathbb{R}^3$

OSSERVAZIONI

Prodotto di funzioni scalari

$$\iint_\Omega \frac{\partial(fg)}{\partial x}\, dx\,dy = \int_{\partial\Omega} fg\, dy$$

$$\iint_\Omega \frac{\partial(fg)}{\partial y}\, dx\,dy = -\int_{\partial\Omega} fg\, dx$$

da queste formule si ricava una formula di integrazione per parti in due variabili

$$\iint_\Omega \frac{\partial f}{\partial x} g\, dx\,dy = \int_{\partial\Omega} fg\, dy - \iint_\Omega f\frac{\partial g}{\partial x}\, dx\,dy$$

$$\iint_\Omega \frac{\partial f}{\partial y} g\, dx\,dy = -\int_{\partial\Omega} fg\, dx - \iint_\Omega f\frac{\partial g}{\partial y}\, dx\,dy$$

• SERIE NUMERICHE

Supponiamo di fare la somma di un numero finito di numeri $a_1, a_2, \dots, a_n$:

numeri interi successivi

$$a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k = \frac{(n+1)\cdot n}{2}$$ FORMULA di GAUSS

Vediamo un altro caso di somme finite con formula:

$$\sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

se

$\lim_{n \to +\infty} S_n = l \in \mathbb{R}$ si dice che la serie CONVERGE, e la somma è l

se

$\lim_{n \to +\infty} S_n = \begin{cases} +\infty \\ -\infty \\ \infty \end{cases}$ si dice che la serie DIVERGE $\begin{cases} + \\ - \\ \text{diverge} \end{cases}$

se

$\lim_{n \to +\infty} S_n =$ NON ESISTE (né finito, né infinito) si dice che la serie è INDETERMINATA o OSCILLANTE

es. 1

$$\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \frac{1}{1-\frac{1}{2}} = 2 \quad \text{CONVERGE}$$

es. 2

$$\sum_{n=0}^{+\infty} 1^n = S_n = n+1 \xrightarrow{n \to +\infty} +\infty \quad \text{DIVERGE positivamente}$$

es. 3

$$\sum_{n=0}^{+\infty} (-1)^n \longrightarrow \text{OSCILLANTE} \longrightarrow S_0 = 1 \quad S_1 = 0 \quad S_2 = 1 \quad S_3 = 0 \quad S_4 = 1 \ldots$$

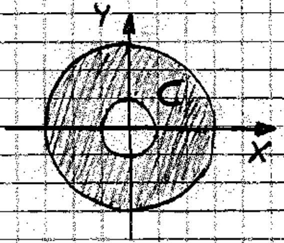
OSSERVAZIONI

$$r \neq 1 \quad \sum_{n=0}^{+\infty} r^n = \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \frac{1-r^{n+1}}{1-r} = \begin{cases} \frac{1}{1-r} & \text{se } |r| < 1 \\ +\infty & \text{se } r \geq 1 \\ \text{oscilla} & \text{se } r = -1 \\ \text{diverge o oscilla} & \text{se } r < -1 \end{cases}$$

(oscilla tra valori sempre più grandi)

$$\pm\overline{N} = \pm\begin{pmatrix} -\frac{\partial}{\partial x}\, 3\sqrt{x^2+y^2} \\ -\frac{\partial}{\partial y}\, 3\sqrt{x^2+y^2} \\ 1 \end{pmatrix}$$ svolgo il +

$$C: \left(\frac{1}{3}\right)^2 \le x^2+y^2 \le \left(\frac{2}{3}\right)^2$$



$$\begin{cases} x = r\cos\theta \\ y = r\,sen\,\theta \end{cases}$$

$$\int_\Sigma \overline{rot}\,\overline{F}\cdot\overline{N} = \iint_C \begin{pmatrix}0\\0\\2x-3y^2\end{pmatrix}\cdot\begin{pmatrix}-\\-\\1\end{pmatrix} dx\,dy = \iint_{C'} \left(2r\cos\theta - 3r^2 sen^2\theta\right) r\,dr\,d\theta =$$

$$= \int_{1/3}^{2/3} 2r^2\,dr\,\underbrace{\cancel{\int_0^{2\pi}\cos\theta\,d\theta}}_{0} - \int_{1/3}^{2/3} 3r^3\,dr \int_0^{2\pi} sen^2\theta\,d\theta = \left[\frac{3}{4}r^4\right]_{1/3}^{2/3}\cdot\left[\frac{1}{2}\theta - sen\theta\cos\theta\right]_0^{2\pi} =$$

$$= -\left[\frac{\cancel{3}^{1}}{4}\cdot\frac{16}{\cancel{81}_{27}} - \frac{\cancel{3}^{1}}{4}\cdot\frac{1}{\cancel{81}_{27}}\right]\pi = -\frac{5}{36}\pi$$

GHE!

posso applicare STOKES e verificare

$$\int_\Sigma \overline{rot}\,\overline{F}\cdot\overline{n}\,d\sigma = \oint_{\partial\Sigma}\overline{F}\cdot d\overline{P} \qquad d\overline{P} = \overline{\tau}$$

$$\gamma_1: \begin{cases} x^2+y^2 = \left(\frac{2}{3}\right)^2 \\ z = 2 \end{cases} \qquad \gamma_2: \begin{cases} x^2+y^2 = \left(\frac{1}{3}\right)^2 \\ z = 1 \end{cases}$$

$$\gamma_1(t) = \begin{pmatrix}\frac{2}{3}\cos t\\ \frac{2}{3}\,sen\,t\\ 2\end{pmatrix} \qquad \gamma_2(t) = \begin{pmatrix}\frac{1}{3}\cos t\\ \frac{1}{3}\,sen\,t\\ 1\end{pmatrix}$$

$$\int_{\partial\Sigma}\overline{F}\cdot d\overline{P} = \int_0^{2\pi}\begin{pmatrix}\left(\frac{2}{3}sen\,t\right)^3\\ \left(\frac{2}{3}\cos t\right)^2\\ 4\end{pmatrix}\cdot\begin{pmatrix}-\frac{2}{3}sen\,t\\ \frac{2}{3}\cos t\\ 0\end{pmatrix}dt - \int_0^{2\pi}\begin{pmatrix}\left(\frac{1}{3}sen\,t\right)^3\\ \left(\frac{1}{3}\cos t\right)^3\\ 1\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{3}sen\,t\\ \frac{1}{3}\cos t\\ 0\end{pmatrix}dt =$$

$$= \int_0^{2\pi}\left(-\left(\frac{2}{3}sen\,t\right)^4 + \left(\frac{2}{3}\cos t\right)^3\right)dt - \int_0^{2\pi}\left(-\left(\frac{1}{3}sen\,t\right)^4 + \left(\frac{1}{3}\cos t\right)^3\right)dt = \dots = -\frac{5}{36}\pi$$

$S_{2^n} \geq n \cdot \frac{1}{2} \xrightarrow{n \to +\infty} +\infty$ la serie DIVERGE

TEOREMA della CONDIZIONE NECESSARIA
(ma non sufficiente)

Sia

$\sum_{n=1}^{+\infty} a_n$ una serie convergente, allora a_n è infinitesimo

cioè $\lim_{n \to +\infty} a_n = 0$

Dim.

$S_n = \sum_{k=1}^{n} a_n \qquad S_{n-1} = \sum_{k=1}^{n-1} a_n \qquad S_n = S_{n-1} + a_n$

$$\underbrace{\underbrace{a_1 + a_2 + \dots + a_{n-1}}_{S_{n-1}} + a_n}_{S_n}$$

$\downarrow$

$\lim_{n \to +\infty} S_n = l \in \mathbb{R} \longrightarrow a_n = S_n - S_{n-1} \xrightarrow{n \to +\infty} l - l = 0$

OSSERVAZIONE

La condizione non è sufficiente, cioè non è vero che

$\lim_{n \to +\infty} a_n = 0 \not\Longrightarrow \sum_{n=1}^{+\infty} a_n$ CONVERGE

CONTROESEMPIO

$\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$

S_n converge perché, essendo a termini positivi, non può oscillare

$$0 \leq S_n \leq T_n \qquad T_n \to +\infty$$

es. $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2} + \dots$

$$\frac{1}{n \cdot n} = \frac{1}{n^2} < \frac{1}{n(n-1)} \qquad n \geq 2$$

$\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n(n-1)} = 1$ SERIE di MENGOLI

$\downarrow$

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \leq 1 + \sum_{n=2}^{+\infty} \frac{1}{n(n+1)} = 2$$

Cosa si intende per DEFINITIVO nelle serie:

aggiungendo un numero finito di termini (positivi o negativi) non si cambia la qualità della serie, cioè il suo comportamento (convergenza, divergenza, oscillanza). Una serie DEFINITIVAMENTE a TERMINI POSITIVI è quella serie che ha i termini $a_n \geq 0$ per $n \geq N$ con N opportuno

Si può applicare il teorema del confronto alle serie definitivamente positive

1) $\displaystyle\sum_{n=1}^{+\infty} a_n \qquad a_n \geq 0$ per $n \geq N$

$\longrightarrow 0 \leq a_n \leq b_n \quad \forall n \geq N$

2) $\displaystyle\sum_{n=1}^{+\infty} b_n \qquad b_n \geq 0$ per $n \geq N$

La convergenza di 2) $\Longrightarrow$ convergenza di 1)

La divergenza di 1) $\Longrightarrow$ divergenza di 2)

CRITERIO del CONFRONTO con un INTEGRALE

Hp: $\sum_{n=1}^{+\infty} a_n \qquad a_n \geq 0$

e esiste una $f(x)$ continua per $x>0$, decrescente e positiva tale che

$f(n) = a_n$

① $\int_1^{+\infty} f(x)\,dx$ CONVERGE

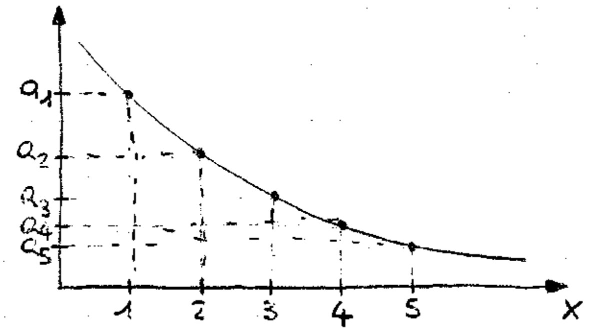
allora anche la serie $\sum_{n=1}^{+\infty} a_n$ CONVERGE → vale anche il contrario

mentre se

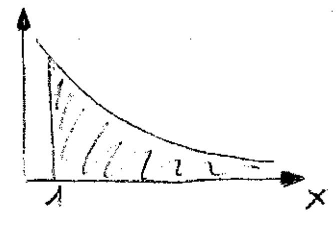
② $\int_1^{+\infty} f(x)\,dx$ NON CONVERGE

allora anche la serie $\sum_{n=1}^{+\infty} a_n$ NON CONVERGE

Dim.



$\int_1^{+\infty} f(x)\,dx$



①

$\sum_{n=2}^{+\infty} a_n$

→ $0 \leq \sum_{n=2}^{+\infty} a_n \leq \int_1^{+\infty} f(x)\,dx$

CONVERGE ⇐ CONVERGE

non può divergere e non può oscillare, perchè è a termini positivi

$\sum_{n=1}^{+\infty} \frac{1}{n^2}$ CONVERGE $\alpha = 2$

Da questi ultimi due casi noti si deduce per confronto che:

$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ CONVERGE se $\alpha \geq 2$ perchè $\frac{1}{n^\alpha} \leq \frac{1}{n^2}$

$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ DIVERGE se $0 \leq \alpha \leq 1$ perchè $\frac{1}{n} \leq \frac{1}{n^\alpha}$ $0 \leq \alpha \leq 1$

Utilizziamo ora il criterio del CONFRONTO INTEGRALE per studiare i casi con

$1 < \alpha < 2$

$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ $f(x) = \frac{1}{x^\alpha}$ $f(n) = \frac{1}{n^\alpha}$ $\alpha > 0$

$\int_1^{+\infty} \frac{1}{x^\alpha}\, dx =$ $\alpha > 0$ e $\alpha \neq 1$

$= \left[\frac{1}{-\alpha+1} x^{-\alpha+1} \right]_1^{+\infty} = \frac{1}{1-\alpha}$ se $\alpha > 1$ $\longrightarrow \sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ CONVERGE se $\alpha > 1$

$= \left[\frac{1}{-\alpha+1} x^{-\alpha+1} \right]_1^{+\infty} = +\infty$ se $0 < \alpha < 1$

es. 1 $\sum_{n=1}^{+\infty} \frac{1}{\sqrt[10]{n}} = \sum_{n=1}^{+\infty} \frac{1}{n^{1/10}}$ DIVERGE $0 < \frac{1}{10} < 1$

es. 2 $\sum_{n=1}^{+\infty} \frac{1}{\sqrt[5]{n^6}} = \sum_{n=1}^{+\infty} \frac{1}{n^{6/5}}$ CONVERGE $1 < \frac{6}{5}$

2° caso $l < 1$

$\forall \varepsilon > 0$ e prendiamo ε in modo che $l + \varepsilon < 1$

$$l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon < 1$$

$$a_{n+1} < q a_n < q^2 a_{n-1} < \ldots < q^k a_N$$

$\sum^{+\infty} q^k a_N$ serie geometrica di $r = q$ $0 < q < 1$

converge $\Longrightarrow$ converge anche la serie

CRITERIO della RADICE

Hp: $\sum_{n=1}^{+\infty} a_n \quad a_n \geq 0$

$$\lim_{n \to +\infty} \sqrt[n]{a_n} = l$$

allora se

$l < 1$ CONVERGE

$l > 1$ DIVERGE

$l = 1$ il criterio non funziona

Dim. Analoga alla precedente, ci si riconduce al confronto definitivo con una serie geometrica

es. $\sum_{n=0}^{+\infty} \frac{n}{e^n} \quad a_n = \frac{n}{e^n}$

$$\lim_{n \to +\infty} \sqrt[n]{\frac{n}{e^n}} = \lim_{n \to +\infty} \frac{\sqrt[n]{n}}{e} = \frac{1}{e} < 1$$ la serie converge

$$\lim_{n \to +\infty} \sqrt[n]{n} = \lim_{n \to +\infty} e^{\log \sqrt[n]{n}} = \lim_{n \to +\infty} e^{\frac{1}{n} \log n} = e^0 = 1$$

se $\sum_{n=1}^{+\infty} (-a_n) = -\sum_{n=1}^{+\infty} a_n \quad a_n \geq 0$

SERIE a SEGNI ALTERNI

$$\sum_{n=0}^{+\infty} (-1)^n a_n = \qquad a_n \geq 0$$

$$= a_0 - a_1 + a_2 + \dots + (-1)^n a_n + \dots$$

CRITERIO di LEIBNIZ per le serie a segni alterni

Hp: $\sum_{n=1}^{+\infty} (-1)^{n+1} a_n \qquad a_n \geq 0$

① $a_n \xrightarrow{n \to +\infty} 0$ condizione necessaria: a_n infinitesima

② $a_{n+1} < a_n \quad \forall n \rightarrow$ decrescita

allora la serie CONVERGE

inoltre, se indichiamo con S la somma della serie si ha:

$$|S - S_n| \leq a_{n+1}$$

↓

$$S_n = \sum_{k=1}^{n} (-1)^{n+1} a_n$$

es. 1 $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n}$ serie armonica a segni alterni

① $\frac{1}{n} \xrightarrow{n \to +\infty} 0$

② $\frac{1}{n+1} < \frac{1}{n}$; termini senza segno sono decrescenti

↓

la serie CONVERGE

$$|S - S_9| = \left| 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - S \right| \leq \frac{1}{10}$$

$$S = \log 2$$

$\rightarrow \int \frac{1}{x^\alpha} = \begin{cases} \log(x) & \text{se } \alpha = 1 \\ \frac{1}{-\alpha+1} \cdot \frac{1}{x^{\alpha-1}} & \alpha \neq 1 \end{cases}$ se $\alpha > 1$ CONVERGE

$\downarrow$

$$\sum_{n=1}^{+\infty} \frac{\sqrt[7]{n^4+3n^3-2n^2+5}}{\sqrt[3]{n^7+8n^2-3}}$$ è a termini positivi

CONFRONTO ASINTOTICO

$$\frac{\dfrac{\sqrt[7]{n^4+3n^3-2n^2+5}}{\sqrt[3]{n^7+8n^2-3}}}{\dfrac{n^{4/7}}{n^{7/3}}} \xrightarrow{n \to +\infty} 1$$

$\downarrow$

$$\frac{n^{4/7}}{n^{7/3}} = \frac{1}{n^{7/3-4/7}} = \frac{1}{n^{37/21}}$$ $\alpha = \frac{37}{21} > 1$ CONVERGE

es. 3

$$\sum_{n=2}^{+\infty} \frac{5}{7n\log^2 n} = \frac{5}{7} \sum_{n=2}^{+\infty} \frac{1}{n\log^2 n}$$

$\downarrow$

$$\lim_{n \to +\infty} \frac{\dfrac{1}{n\log^2 n}}{\dfrac{1}{n^\alpha}} = \frac{n^\alpha}{n\log^2 n}$$

il confronto asintotico non porta da nessuna parte

$\alpha > 1$ ↗ $+\infty$ $\alpha \leq 1$ ↘ 0

qui funziona il confronto con un integrale improprio

$$f(x) = \frac{1}{x\log^2 x} \longrightarrow \int_2^{+\infty} \frac{1}{x\log^2 x}\,dx = \quad t = \log x \quad dt = \frac{1}{x}$$

$$= \int_{\log 2}^{+\infty} \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]_{\log 2}^{+\infty} = \frac{1}{\log 2}$$ CONVERGE $\Longrightarrow$ anche la serie CONVERGE

$$\left(\frac{2n+1}{2n+3}\right)^{2n} = \left(1+\frac{-2}{2n+3}\right)^{2n} = \left(1+\frac{-2}{2n+3}\right)^{2n+3} \underbrace{\left(1+\frac{-2}{2n+3}\right)^{3}}_{n\to+\infty=1} =$$

$$= \left(1+\frac{-2}{2n+3}\right)^{2n+3} = \left[\left(1+\frac{1}{\frac{2n+3}{-2}}\right)^{\frac{2n+3}{-2}}\right]^{-2} \xrightarrow[n\to+\infty]{} \frac{1}{e^2}$$

che è simile a

$$\lim_{n\to+\infty}\left(1+\frac{1}{x}\right)^{1/x} = e$$

$\downarrow$

$$\lim_{n\to+\infty}\frac{a_{n+1}}{a_n} = \frac{1}{e^2} < 1 \quad \text{CONVERGE}$$

es. 6

$$\sum_{n=0}^{+\infty}(-1)^n\frac{n^2+2n+5}{n^3+2n^2+3n+7}$$ serie a segni alterni

$a_n \xrightarrow{n\to+\infty} 0$ $\quad a_n \approx \frac{1}{n}$ per $n\to+\infty$ se non ci fossero i segni alterni divergerebbe

invece

$$\frac{n^2+2n+5}{n^3+2n^2+3n+7} > \frac{(n+1)^2+2(n+1)+5}{(n+1)^3+2(n+1)^2+3(n+1)+7}$$ CONVERGE

es. 7

$$\sum_{n=2}^{+\infty}(-1)^n\frac{1}{\sqrt[2]{n+1}}$$ segni alterni

$a_n \xrightarrow{n\to+\infty} 0$

$$\frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{(n+1)+1}} \quad \forall n \geq 2$$ CONVERGE

INTEGRALI

1) CURVILINEO $\int_\gamma F(\gamma(t)) \cdot \|\gamma'(t)\|$ 2) LINEA $\int_\gamma F(\gamma(t))\, \gamma'(t)$

3) SUPERFICIE $\int_\Sigma dS = \iint_D \underbrace{\left\| \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} \right\|}_{\sqrt{1+\frac{\partial f^2}{\partial x}+\frac{\partial f^2}{\partial y}}} du\,dv$ posso avere una f nell'integral o parametrizzata, o $z = f(x,y)$

$\boxed{\text{NORMALE}}$ $\begin{vmatrix} i & j & k \end{vmatrix}$ o $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \pm 1\right)$

4) FLUSSO: viene dato $F(x,y,z)$ (campo), e una Σ parametrizzata o $z = g(x,y)$

$$\longrightarrow \phi = \iiint_S \mathrm{div}\, F \, dx\,dy\,dz = \iint_\Sigma \overline{F} \cdot \overline{n}\, dS$$

ricordatsi di sostituire nel campo $z = g(x,y)$ o la parametrizzazione

5) FLUSSO DEL ROTORE: dato $F(x,y,z)$ e Σ parametrizzata o $z = g(x,y)$

$$\phi = \iint_\Sigma \mathrm{rot}\, \overline{F} \cdot \overline{n}\, dS = \underbrace{\int_\gamma F\, ds}_{\text{LINEA}}$$

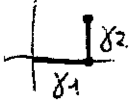
- CAMBIO di COORDINATE:

2D 1) polari $\det|J| = \rho$ $\begin{cases} x = \rho\cos\theta + x_0 \\ y = \rho \operatorname{sen}\theta + y_0 \end{cases}$ 2) ellittiche $\det|J| = ab\rho$ $\begin{cases} x = a\rho\cos\theta + x_0 \\ y = b\rho \operatorname{sen}\theta + y_0 \end{cases}$

3D 3) sferiche $\det|J| = r^2 \operatorname{sen}\varphi$ $\begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho\cos\varphi \end{cases}$ 4) cilindriche $\det|J| = r$ $\begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \\ z = t \end{cases}$

- POTENZIALI

viene dato il campo $F(x,y)$, parametrizzo una generica curva che unisce due punti



$\gamma_1 = \begin{cases} x = t & x' = 1 \\ y = 0 & y' = 0 \end{cases}$ $t \in [0, x]$ $\gamma_2 = \begin{cases} x(t) = x & x' = 0 \\ y = t & y' = 1 \end{cases}$ $t \in [0, y]$

$\underbrace{\int_\gamma F\, ds}_{\text{LINEA}} = \int_{\gamma_1} + \int_{\gamma_2}$ CAMPO CONSERVATIVO: $\mathrm{rot}\, F = 0$, $D = \mathbb{R}^2, \mathbb{R}^3$ semplicemente connesso

LAVORO $L = U(B) - U(A)$

3) FUNZIONI

$\lim_{n \to +\infty} f_n(x) = f(x)$ $\longrightarrow$ $\lim_{n \to +\infty} \sup \left(f_n(x) - f(x) \right) = 0$

converge puntualmente

$= \left(f_n(x) - f(x) \right)' = 0$ trovo una x, verifico che sia MAX, sostituisco in $f_n(x)$ e faccio il lim

se $\lim_{n \to +\infty} \sup \left(f_n(x) \right) = 0$ la serie converge normalmente (punt. e unif.)

WEIERSTRASS : $|f_n(x)| \leq M_n$ serie numerica se $\sum_1^{+\infty} M_n$ converge

$\Downarrow$

$f_n(x)$ converge totalmente

4) POTENZE

criterio del RAPPORTO o RADICE, $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} / \sqrt[n]{a_n} = l$ $R = \frac{1}{l}$

su $(-R, R)$ la serie converge assolut. e punt.

studio poi gli estremi di $(-R, R)$ per vedere se l'insieme è chiuso

$\longrightarrow$ la serie converge uniformemente in $(a, b) \subset (-R, R)$

$0 < a < b < R$

5) FOURIER

data serie a tratti, calcolo la serie di Fourier:

$$S_n(x) = a_0 + \sum_1^{\infty} a_n \cos(nx) + b_n \operatorname{sen}(nx)$$

caso generale:

$$a_0 = \frac{1}{T} \int_0^T f(x)\,dx \qquad a_n = \frac{2}{T} \int_0^T f(x) \cos\left(n \frac{2\pi}{T} x\right) dx \qquad b_n = \frac{2}{T} \int_0^T f(x) \operatorname{sen}\left(n \frac{2\pi}{T} x\right) dx$$

f pari:

$$a_0 = \frac{2}{T} \int_0^{T/2} f(x)\,dx \qquad a_n = \frac{4}{T} \int_0^{T/2} f(x) \cos\left(n \frac{2\pi}{T} x\right) dx \qquad b_n = 0$$

f dispari:

$$a = 0 \qquad a_n = 0 \qquad b_n = \frac{4}{T} \int^{T/2} f(x) \operatorname{sen}\left(n \frac{2\pi}{T} x\right) dx$$

1) TEOREMA FUNZIONI DIFFERENZIABILI

f Differenziabile in x_0 se è (sia di classe C^1) — continua / derivate parziali / " direzionali

p: cioè se $\lim_{\|H\| \to 0} \frac{\omega(x_0+H) - \omega(x_0) - \nabla\omega(x_0)H}{\|H\|} = 0$

Dim.

$$\frac{\partial f}{\partial v} = \lim_{h \to 0} \frac{\omega(x_0 + hv) - \omega(x_0)}{h} \qquad hv = H$$

considero

$$\to \omega(x_0+H) - \omega(x_0) = \nabla\omega(x_0)H + o\|H\|$$

sostituisco nel numeratore del lim:

$$\lim_{h \to 0} \frac{\nabla\omega(x_0)\cdot \cancel{h}v}{\cancel{h}} + \frac{o\|h\|}{h} = \nabla\omega(x_0)v + 0$$

$$\to \frac{\partial f}{\partial v} = \nabla\omega(x_0)v$$

2) LAVORO non dipende dal PERCORSO

Hp: $\exists g(P) / F(P) = \text{grad}\, g(P)$

$D = \text{dom}\, F(P)$ è aperto connesso

$F(P)$ continua

Th: $\int_{\gamma(A,B)} F(P)\, dP = g(B) - g(A)$

$$\int_{\gamma(A-B)} F(P)\, dP = \qquad \gamma(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t \in [t_A, t_B]$$

$$\downarrow$$

$$\int \left[f_1(P(t)) \frac{dx}{dt} dt + f_2(P(t)) \frac{dy}{dt} dt + f_3(P(t)) \frac{dz}{dt} dt \right] =$$

$$= \int_{t_A}^{t_B} \left[\frac{\partial g(P(t))}{\partial x} \frac{dx}{dt} + \frac{\partial g(P(t))}{\partial y} \frac{dy}{dt} + \frac{\partial g(P(t))}{\partial z} \frac{dz}{dt} \right] dt =$$

$$= \int_{t_A}^{t_B} \frac{dg(P \circ \gamma(t))}{dt} dt = \left[g(P \circ \gamma(t)) \right]_{t_B}^{t_B} = g(B) - g(A)$$

5) CRITERIO $\exists$ POTENZIALE

Hp: $F(P) = \text{grad } g(P)$

Th: $\text{rot}\left(F(P)\right) = (0,0,0)$

calcolo il $\text{rot}\left(F(P)\right)$:

$$\text{rot}\left(F(P)\right) = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k} =$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial g(P)}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial g(P)}{\partial y}\right)\right)\hat{i} + \left(\frac{\partial^2 g(P)}{\partial z\,\partial x} - \frac{\partial^2 g(P)}{\partial x\,\partial z}\right)\hat{j} + \left(\frac{\partial^2 g(P)}{\partial x\,\partial y} - \frac{\partial^2 g(P)}{\partial y\,\partial x}\right)\hat{k}$$

per il teorema della inversione delle derivate:

$$\frac{\partial^2 g(P)}{\partial x\,\partial y} = \frac{\partial^2 g(P)}{\partial y\,\partial x} \Longrightarrow \text{rot}\left(F(P)\right) = (0,0,0) = \vec{0}$$

6) TEOREMA del DINI (1)

data $g(x_1, x_2, \dots, x_n) = 0$ $\quad \mathbb{R}^n \to \mathbb{R}$

Hp: $x_0 = (x_1^0, x_2^0, \dots, x_n^0)$

$g(x_0) = 0$

$\dfrac{\partial g(x_0)}{\partial x_i} \neq 0$

Th: $x_i = f_i(x_1, x_2, \dots, x_n)$ $\quad \mathbb{R}^{n-1} \to \mathbb{R}$

c.s. $\nabla g(x_0) \neq \vec{0}$ per esplicitare una funzione

7) TEOREMA del DINI (2)

sia $\nabla g(x_0) \neq \vec{0}$ e $x_j = f(x_1, x_2, \dots, x_n)$

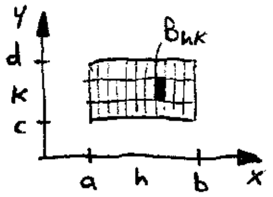
allora

Th: $\dfrac{\partial f_i}{\partial x_j} = -\dfrac{\dfrac{\partial g(x)}{\partial x_j}}{\dfrac{\partial g(x)}{\partial x_i}}$ $\quad j \neq i$

Dim.

$(g \circ f_i)(x_1, x_2, \dots, x_n) = 0 \longrightarrow \dfrac{\partial (g \circ f_i)}{\partial x_j} = \dfrac{\partial g}{\partial x_j} + \dfrac{\partial g}{\partial x_i}\dfrac{\partial f_i}{\partial x_j} = 0 \longrightarrow \dfrac{\partial f_i}{\partial x_j} = -\dfrac{\dfrac{\partial g}{\partial x_j}}{\dfrac{\partial g}{\partial x_i}}$

10) INTEGRALE di RIEMANN



$$B = \{(x,y) \,/\, a \le x \le b \,,\, c \le y \le d\}$$

$$\mathbb{D} : [a,b] \times [c,d] \qquad B_{hk} = I_h \times I_k$$

adesso definisco l'altezza (del volume)

$z = f(x,y)$ LIMITATA   definiamo l'altezza max e min:

$$m_{hk} = \inf_{B_{hk}} f \qquad M_{hk} = \sup_{B_{hk}} f$$

somme sup. e inf.:

$$s = s(\mathbb{D}, f) = \Sigma_k \Sigma_h \, m_{hk} (x_h - x_{h-1})(y_k - y_{k-1})$$

$$S = S(\mathbb{D}, f) = \Sigma_k \Sigma_h \, M_{hk} (x_h - x_{h-1})(y_k - y_{k-1})$$

integrale sup. e inf.:

$$\overline{\int_B} f = \inf_{\mathbb{D}} S(\mathbb{D}, f) \qquad \underline{\int_B} f = \sup_{\mathbb{D}} s(\mathbb{D}, f)$$

$$\underline{\int_B} f \le \overline{\int_B} f$$ se c'è uguaglianza la f si dice Riemann integrabile su B

11) TEOREMA di GULDINO (1) SOLIDI di ROTAZIONE

$$\iiint_{\Omega} dx\,dy\,dz \qquad \text{coordinate cilindriche} \quad \Omega' : \begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \\ z = z \end{cases}$$

$\downarrow$

$$\iiint_{\Omega'} r \, dr\, d\theta\, dz = \iint_A \left(\int_0^{2\pi} r\, d\theta \right) dr\, dz = 2\pi \iint_A r\, dr\, dz \qquad r = x$$

$$= 2\pi \iint_A x\, dx\, dz = 2\pi \underbrace{\frac{\iint_A x\, dx\, dz}{\iint_A dx\, dz}}_{x_G} \iint_A dx\, dz \longrightarrow I = 2\pi x_G \cdot \iint_A dx\, dz$$

dim. ② $\int_{\Omega} \frac{\partial f_3}{\partial z} \, dx\,dy\,dz = \int_D dx\,dy \left[f_3 \right]_{\alpha(x,y)}^{\beta(x,y)} =$

$$= \int_D f_3 (x,y,\beta(x,y))\, dx\,dy - \int_D f_3 (x,y,\alpha(x,y))\, dx\,dy$$

① = ② C.V.D.

14) TEOREMA di GREEN

Hp: $\overline{F} = f_1(x,y)\hat{i} + f_2(x,y)\hat{j} \quad \in C^1$

Ω piana, chiusa, limitata, semiconvessa

γ chiusa, regolare a tratti, orientata

$a \leq x \leq b$

$\alpha(x) \leq y \leq \beta(x)$

Th: $\oint_{\gamma} F\, dP = \int_{\Omega} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy$

$$\underbrace{\oint_{\gamma} f_1(x,y)\,dx + f_2(x,y)\,dy}_{①} = \underbrace{\int_{\Omega} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy}_{②}$$

dim. ① $\int_{\gamma} f_1(x,y)\,dx = \int_{\gamma_1 \,(y=\alpha(x))} f_1(x,y)\,dx + \int_{\gamma_2\,(y=\beta(x))} f_1(x,y)\,dx =$

$$= \int_a^b f_1(x,\alpha(x))\,dx + \int_b^a f_1(x,\beta(x))\,dx =$$

$$= \int_a^b f_1(x,\alpha(x))\,dx - \int_a^b f_1(x,\beta(x))\,dx$$

dim. ② $-\int_{\Omega} \frac{\partial f_1}{\partial y}\, dx\,dy = -\int_a^b \left[f_1(x,y) \right]_{\alpha(x)}^{\beta(x)} dx =$

$$= \int_a^b f_1(x,\alpha(x))\,dx - \int_a^b f_1(x,\beta(x))\,dx$$

① = ② C.V.D.

ε = epsilon
ξ = psi $\Sigma(\sigma)$ → sigma

18) CONFRONTO ASINTOTICO

Hp: $\sum_1^\infty a_n$, $\sum_1^\infty b_n$ $\quad a_n \geq 0, b_n \geq 0 \quad \forall n > N$

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = l \neq 0, \infty \qquad l \in \mathbb{R}$$

Th: $\sum a_n$ e $\sum b_n$ hanno lo stesso comportamento

Dim. $\varepsilon > 0$, $\exists n_0 / \forall n > n_0$, $1-\varepsilon < \frac{a_n}{b_n} < 1+\varepsilon$

$\rightarrow b_n(1-\varepsilon) < a_n < b_n(1+\varepsilon)$

$\rightarrow (1-\varepsilon)\sum_1^\infty b_n < \sum_1^\infty a_n < (1+\varepsilon)\sum_1^\infty b_n$

$(1 \pm \varepsilon)$ è un numero finito e non influisce nel comportamento della serie

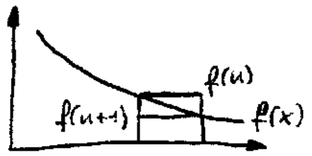
le serie hanno lo stesso comportamento

19) CONFRONTO INTEGRALE

Hp: $\sum_1^\infty a_n$, $a_n > 0 \quad \forall n$

$f(x) / a_n = f(n)$ $\qquad f(x)$ monotona decrescente

Th: $\underset{①}{\sum_1^\infty f(n)}$, $\underset{②}{\int_1^\infty f(x)\,dx}$ $\qquad$ se conv. ②, conv. ①



$\rightarrow f(n) \geq f(x) \geq f(n+1)$

$$\int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

svolgo l'integrale:

$$f_n(\cancel{K}+1-\cancel{K}) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)$$

$$\rightarrow \sum_{K=1}^{+\infty} f(K) \geq \int_1^\infty f(x)\,dx \geq \sum_{K=1}^{+\infty} f(K+1)$$

ripeto lo stesso procedimento da 1 a n+1 per ottenere le $\sum_1^n$ e $\int_1^{n+1}$ e poi applico il $\lim_{n\to+\infty}$

sostituisco $K+1 = m \rightarrow \sum_{K=1}^{+\infty} f(K) \geq \int_1^\infty f(x)\,dx \geq \sum_{m=2}^{+\infty} f(m)$

l'indice di sommatoria è muto

$$\sum_{K=1}^{\infty} f(K) \geq \int_1^\infty f(x)\,dx \geq \sum_{K=2}^{+\infty} f(K)$$

22) TEOREMA ASSOLUTA CONVERGENZA

Hp: $\sum_1^\infty |a_n|$ conv.

Th: $\sum_1^\infty a_n$ conv.

① somma dei termini positivi

② somma dei termini negativi

① $\begin{cases} = a_n & \text{se } a_n > 0 \\ = 0 & \text{se } a_n < 0 \end{cases}$

② $\begin{cases} = a_n & \text{se } a_n < 0 \\ = 0 & \text{se } a_n > 0 \end{cases}$

$$\underbrace{\sum_1^\infty |a_n|}_{\text{conv. per ipotesi}} = \sum_1^\infty \overbrace{\frac{|a_n| + a_n}{2}}^{①} + \sum_1^\infty \overbrace{\frac{|a_n| - a_n}{2}}^{②}$$

$\Rightarrow$ converge il secondo membro e quindi anche $\sum_1^\infty a_n$

23) $f_n(x)$ continue e unif. a $S_n(x)$

Hp: $\sum_0^\infty f_n(x)$ con $f_n(x)$ continue

$\downarrow$

unif. a $S(x)$ in I_n

Th: $S(x)$ continua

formule

Hp:

- $\varepsilon_n > 0, \exists n_0 / \sup_{x \in I_n} |S(x) - s_n(x)| < \varepsilon_n$
- $\varepsilon_c > x_0, \exists \delta / \forall x \in (x_0 - \delta, x_0 + \delta)$, $|f_n(x) - f_n(x_0)| < \varepsilon_c$

Dim.

$|S(x) - S(x_0)| =$ aggiungo e tolgo $s_n(x)$

$= |S(x) + s_n(x) - s_n(x) - S(x_0) + s_n(x_0) - s_n(x_0)| =$

$\rightarrow |(S(x) - s_n(x)) - (S(x_0) - s_n(x_0)) + (s_n(x) - s_n(x_0))| \leq$

$\leq |(S(x) - s_n(x))| + |S(x_0) - s_n(x_0)| + |s_n(x) - s_n(x_0)| \leq$

$\leq 2\varepsilon_n + \varepsilon_c \leq \varepsilon$

24) CRITERIO di WEIERSTRASS

Hp: $\sum_0^\infty f_n(x) / \forall x \in [a,b], \forall n \; |f_n(x)| < M_n$

$\sum_0^\infty M_n$ converge

Th: $\sum_0^\infty f_n(x)$ converge