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A P P U N T I

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MATERIA : Dynamic Design + Esercizi
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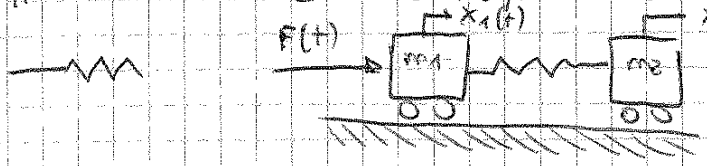
VIBRATION OF NATURAL SYSTEMS

oscillation: motion periodical (kinematic)
vibration: (dynamic)

There's a difference between oscillation and vibration. The vibration motion is part of the oscillation. The oscillation is a motion where it's not involved the internal dynamic of the system.



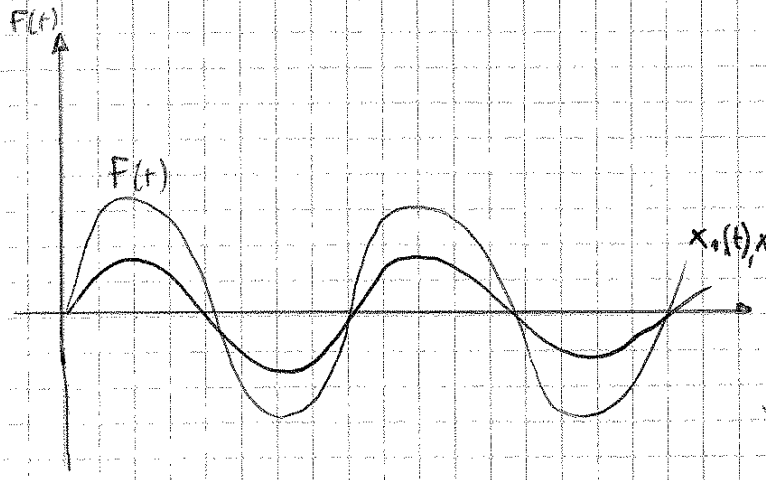
x displacement

This system is free to move as there're not constrains. The two masses have the same motion, so they're oscillating.

$$F(t) = F_0 \sin(\Omega t)$$

natural frequency Ω frequency

Ω is much lower than the internal frequency of the system.



x_1, x_2 move with the same frequency

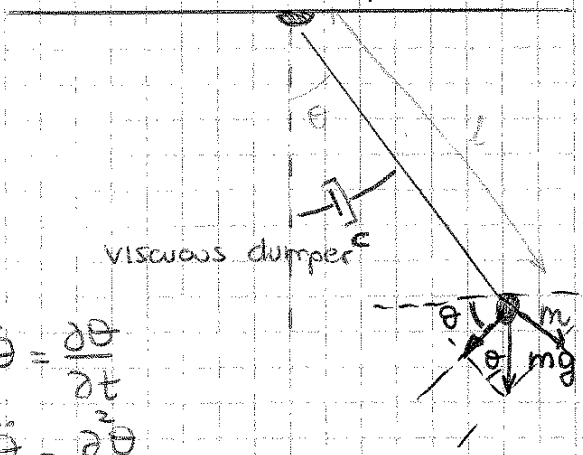
If we increase the frequency close to the natural frequency

The two subsystems exchange energy each other
↝ the two systems are vibrating as the internal part is involved.

We will focus on linear systems.

EXAMPLE OF A 1 DOF SYSTEM ANALYSIS

It's a preliminary analysis even for complex systems, simplifying them.
We consider a pendulum. If we consider the frequency is lower than the natural frequency we consider the bar rigid and simplify by lumped parameters.
There's contribution of potential energy, stiffness and inertia.
The dof is the angle θ.
The forces which are involved are 2



$$M_\theta = mg \sin\theta \, l$$ torque

$$M_c = c\dot{\theta}$$ (reaction) damping torque

$$M_i = ml^2\ddot{\theta}$$ inertial torque

$$\dot{\theta} = \frac{\partial\theta}{\partial t}$$

$$\ddot{\theta} = \frac{\partial^2\theta}{\partial t^2}$$

$$F_i = F_e \rightsquigarrow ml^2\ddot{\theta} = -c\dot{\theta} - mgl\sin\theta$$

I can write the motion equation in canonical form

$$\underbrace{ml^2\ddot{\theta}}_{\text{① inertial}} + \underbrace{c\dot{\theta}}_{\text{② damping}} + \underbrace{mgl\sin\theta}_{\text{potential ⑤}} = 0$$

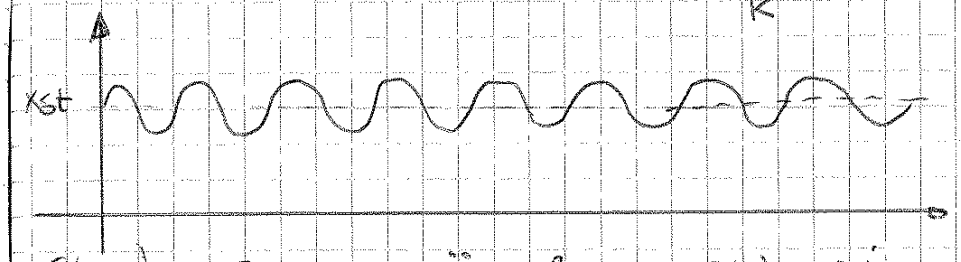
$\rightsquigarrow m\ddot{x} + c\dot{x} + kx = k(x_A + l_0) + c\dot{x}_A + F(t) - mg$

If the system is not vibrating

$\ddot{x} = \dot{x} = \dot{x}_A = F(t) = 0$ in a static condition

$kx = k(x_A + l_0) - mg$

So we can find $x_{st} = x_A + l_0 - \frac{mg}{k}$ is a deflection due to weight



Starting from $m\ddot{x} + c\dot{x} + kx = F(t) + c\dot{x}_A + kx_A$

it's convenient to introduce $x_R = x - x_A$ so that

$$m\ddot{x} = F(t) - c(\dot{x} - \dot{x}_A) - k(x - x_A)$$

$x = x_R + x_A \rightsquigarrow \dot{x} = \dot{x}_R + \dot{x}_A \rightsquigarrow \ddot{x} = \ddot{x}_R + \ddot{x}_A \rightsquigarrow$

$m(\ddot{x}_R + \ddot{x}_A) = F(t) - c\dot{x}_R - kx_R$

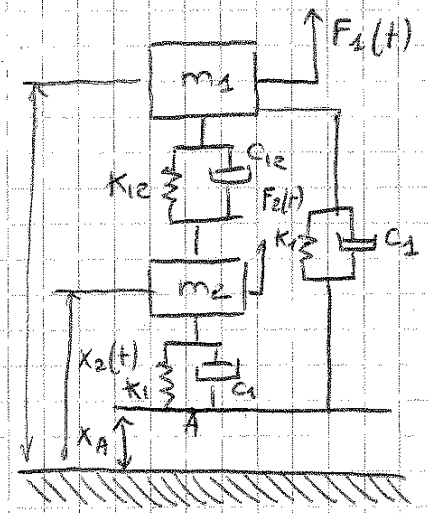
We can write an equation of motion as $m\ddot{x}_R + c\dot{x}_R + kx_R = F(t) - m\ddot{x}_A$

It's not an inertial reference frame, we have an additional term of the acceleration of the frame body

If the dof is rotational

x	$\rightsquigarrow$	θ
m [Kg]	$\rightsquigarrow$	J [Kg m^2]
k [N/m]	$\rightsquigarrow$	χ [Nm/rad]
c [N s/m]	$\rightsquigarrow$	Γ [Nms/rad]
$F(t)$ [N]	$\rightsquigarrow$	M [Nm]

$m\ddot{x} + c\dot{x} + kx = F(t) + c\dot{x}_A + kx_A \rightsquigarrow J\ddot{\theta} + \Gamma\dot{\theta} + \chi\theta = M(t) + \Gamma\dot{\theta}_A + \chi\theta_A$



EXAMPLE OF 2 DOF SYSTEM

If we apply the Newton law $F_i = F_e$

$$m_1\ddot{x}_1 = F_1(t) - k_{12}(x_1 - x_2) - c_{12}(\dot{x}_1 - \dot{x}_2) + \\ - c_1(\dot{x}_1 - \dot{x}_A) - k_1(x_1 - x_A)$$

$$m_2\ddot{x}_2 = -c_{12}(\dot{x}_2 - \dot{x}_1) - k_{12}(x_2 - x_1) + \\ - c_2(\dot{x}_2 - \dot{x}_A) - k_2(x_2 - x_A) + F_2(t)$$

It's convenient to write them with matrixes

Generally speaking, matrices [M] and [K] are symmetrical matrices of order n, where n is the number of d.o.F. of the system.

The symmetry of the matrices can be destroyed if some equations are substituted by linear combinations of the equations or are just multiplied by a constant. The equation of motion can thus be written in forms in which the relevant matrixes are not symmetrical.

Generally matrices M and K are positive semidefinite, but in many cases they're positive definite. The mass matrix is positive defined when a non-vanishing mass is associated to all d.o.f. The stiffness matrix is positive definite when no rigid body motion is allowed.

Devices un constrained are characterized by singular stiffness matrices.

In state space $\dot{z} = Az + Bu$ matrix A is the dynamic matrix and depend on only the characteristics involved in free motion of the system, while B is the gain matrix

[A] has dimension $2n \times 2n$

[B] has dimension $2n \times$ dimension of input vector

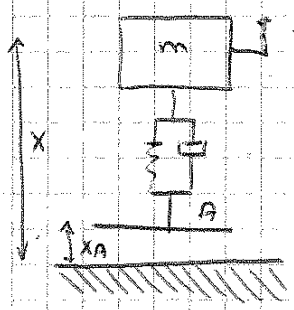
Out put gain matrix has $2n$ columns and number of rows depending on the number of outputs

If we have n dof, the matrixes have size n x n
Looking to the model, [M] is diagonal but [C] and [K] are simmetric but they're not always like that.
Natural systems are systems in which the structure is vibrating, where [C] and [K] are simmetric.
IF we have no natural systems, we split the simmetric part from the not simmetric one.

$$[M]\{\ddot{X}\} + ([C^S] + [G^{\not S}])\{\dot{X}\} + ([K^S] + [H^{\not S}])\{X\} = \{F(t)\}$$

Practically, rotating machines considering gyroscopic are not natural systems.
This is a equation of motion written in configuration space.
If we have a mechatronic system, we have to write in a different form, in the state space.

EXAMPLE OF 1 DOF MECHATRONIC SYSTEM



$$m\ddot{x} + c\dot{x} + Kx = F_1(t) + Kx_A + c\dot{x}_A$$ configuration space

$$\begin{cases} \ddot{x} = -\frac{c}{m}(\dot{x}-\dot{x}_A) + \frac{K}{m}(x-x_A) + \frac{F_1(t)}{m} \\ \dot{x} = \dot{x} \end{cases}$$ state configuration (position and velocity)

I introduce a trivial equation and I write in matrix

$$\begin{Bmatrix} \ddot{x} \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} -\frac{c}{m} & -\frac{K}{m} \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix} + \begin{bmatrix} \frac{1}{m} & \frac{K}{m} & \frac{c}{m} \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} F(t) \\ x_A \\ \dot{x}_A \end{Bmatrix}$$

It's only an algebrical transformation, so they're the same.
So generally

$$\{\dot{z}\} = [A]\{z\} + [B]\{u(t)\}$$

We pass from a 2nd order differential equation to a 1st order differential equation.
$\{z\}$ the state vector, [B] input gate matrix, [A] dynamic matrix
So we need 2 boundering information for the equation in the state space, only one for the configuration space (only displacement)
[A] and [B] has different rules. [A] represents the internal energy account of the system, while [B] links the state with the input, gives no info about what the system is doing.
The eigenvalues of [A] represent the vibration frequency of our system. The eigenvectors represent instead the mode of vibrating of the system.
The vector $\{z\}$ has generally $(2n \times 1)$, [A] is $(2n \times 2n)$ and [B] is $(2n \times m)$ and $\{F(t)\}$ is $(m \times 1)$
[A] has size twice the [M] [C] [K]
Generally the outputs are not equal to states. For instance $F_s = K(x - x_A)$

$$\rightsquigarrow \begin{Bmatrix} x \\ F \end{Bmatrix}_{(r\times 1)} = \underset{(2r\times 2n)}{\begin{bmatrix} 0 & 1 \\ 0 & K \end{bmatrix}} \underset{(2n\times 1)}{\begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}} + \underset{(2r\times m)}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -K & 0 \end{bmatrix}} \underset{(m\times 1)}{\begin{Bmatrix} F(t) \\ x_A \\ \dot{x}_A \end{Bmatrix}}$$

$$\underset{\text{Vector of outputs}}{\{Y\}} = \underset{\substack{\text{OUTPUT GAIN} \\ \text{MATRIX}}}{[C]}\{z\} + \underset{\substack{\text{DIRECT} \\ \text{LINK MATRIX}}}{[D]}\underset{\text{Vector of inputs}}{\{u\}}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} - \frac{d}{dt}\frac{\partial V}{\partial \dot{x}_i} - \frac{\partial A}{\partial x_i} + \frac{\partial D}{\partial \dot{x}_i} + \frac{\partial}{\partial x_i} = \frac{\partial \delta L}{\partial \delta x_i}$$

$\frac{\partial V}{\partial \dot{x}_i} \to 0$ as the system is natural; $\frac{\partial A}{\partial x_i} \to 0$

For a natural system we can use a simplyfied form of Lagrangian equation.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial U}{\partial x_1} = K_{12}(x_1 - x_2) + K_1(x_1 - x_A)$$

$$\frac{\partial \tilde{R}}{\partial \dot{x}_1} = c_{12}(\dot{x}_1 - \dot{x}_2) + c_1(\dot{x}_1 - \dot{x}_A)$$

$$\frac{\partial \delta L}{\partial \delta x_1} = F_1(t)$$

The choice of generalized coordinates is in a way arbitrary, and different sets of generalized coordinates can be devised for a given system. However, this choice is not immaterial and the complexity of the mathematical model can strongly depend on it

$$\rightsquigarrow m_1 \ddot{x}_1 + K_{12}(x_1 - x_2) + K_1(x_1 - x_A) + c_{12}(\dot{x}_1 - \dot{x}_2) + c_1(\dot{x}_1 - \dot{x}_A) = F_1(t)$$

That's the same equation we obtain by Newton approach

RESPONCE OF A 1 DOF SYSTEM

Demostraction

$$m\ddot{x} + c\dot{x} + Kx = F(t)$$

$$x(t) = x_0 + x_F$$

We start from solving the omogeneuous equation

$$m\ddot{x} + c\dot{x} + Kx = 0 \rightsquigarrow x = x_0 e^{st}$$

$$\dot{x} = x_0 s e^{st}, \quad \ddot{x} = x_0 s^2 e^{st}$$ (derive)

s laplace variable complex $S = s_R + i s_1$

$$(ms^2 + cs + K) x_0 e^{st} = 0$$ (trivial solution $= 0$)

characteristic equation $= 0 \rightsquigarrow s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

$$\rightsquigarrow x(t) = x_{0_1} e^{s_1 t} + x_{0_2} e^{s_2 t}$$

x_{0_1}, x_{0_2} are defined by boundening conditions.

IF $c^2 > 4km \rightsquigarrow$ the root square is real value $\rightsquigarrow s_1, s_2 \in \mathbb{R}$

$$\rightsquigarrow x(t) = x_{0_1} e^{s_{1R} t} + x_{0_2} e^{s_{2R} t}$$

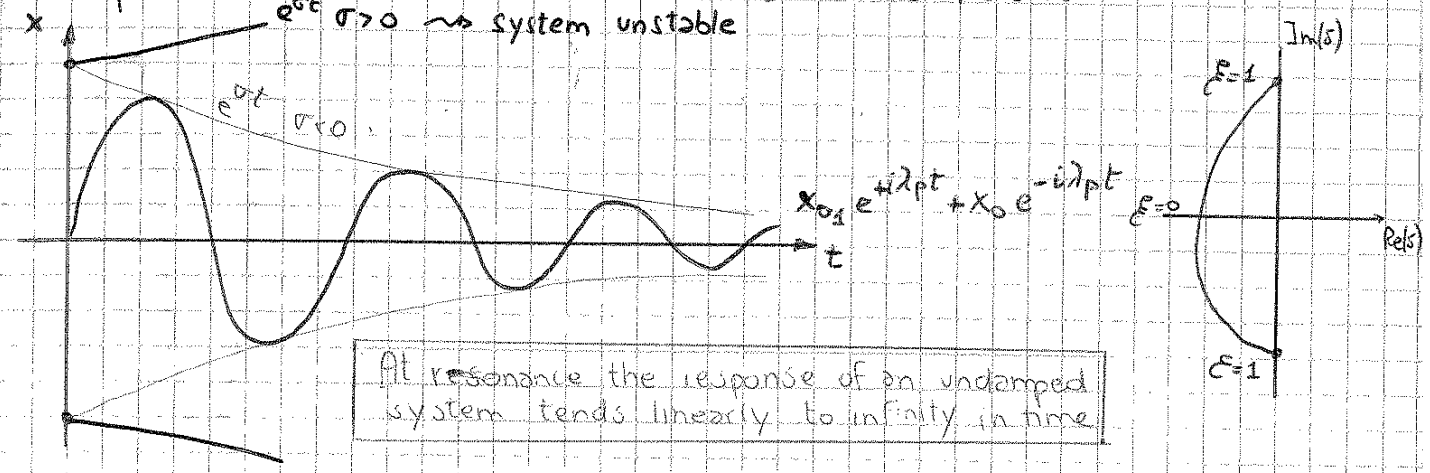
The answer is somewhat that reduces in time.

The amount of dumping is so large we don't have vibration in the system

IF $c^2 = 4km$ is the limit conolution $c_{cr} = \sqrt{4km}$ critical dumping

IF $c^2 < 4km$ we have a vibrating sistem so $s_1, s_2 \in \mathbb{C}$

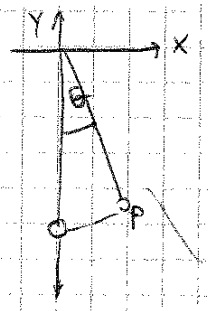
If I plot that function in time I obtain this result



At resonance the response of an undamped system tends linearly to infinity in time

LINEARIZATION OF A PENDULUM

Importance of damping can be neglected in computing the natural frequency



$P \begin{Bmatrix} x \\ y \end{Bmatrix}$ $x^2 + y^2 = \ell$

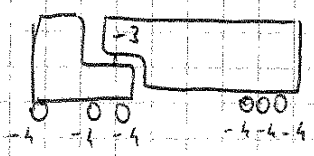
We need to add a constrain equation.

$\begin{cases} m\ddot{x} + \dots \\ m\ddot{y} + \dots \\ x^2 + y^2 = \ell \end{cases}$ DAE (differential algebric equation)

← it's an algebric equation, contains the position and not velocity (holonomic). They're for now and not for future.

$mr^2\ddot{\theta} + \dots$ ODE (ordinary differential equation)

EXAMPLE

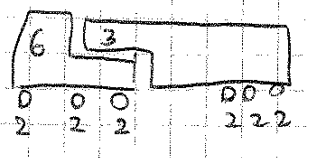


8 bodies with 6 d.o.f each ⇝ 48 d.o.f.
But these are linked together, for istanche the fifth wheel, the suspensions

$\begin{cases} 48 \text{ DOEs} \\ 27 \text{ AE} \end{cases}$ 27 constraint equations

21 d.o.f

If we follow the DAE approach we have the situation explained, but we can consider



⇝ 21 ordinary differential equations.

21 is the minimum d.o.f required by the problem. If we follow the first approach, is general and the equations are very simple and the constrains equation can be classified in many ways. The MULTIBODIES CODES are based on that approach. It's quite heavy in computational time. On the other hand, the second approach, it's more complex to write the model but it's easier to solve.
DAE have the algebric loop problem, they're more difficult to solve than ODE. Algebric equations have no dynamics, they link only the coordinates as they'le in a instant t. You can introduce a trick, a delay in to the systems to make them differential.

We have to find the output $X(t)$. To solve the liner differential equation the total order is $2n$, we have to find $2n$ solution. The phisical meaning is that all systems act with what it wont to do and what others want it to do.
We start with the free behavior by the homogeneous solution

$$M\ddot{x} + Kx = 0 \quad \rightsquigarrow \quad x = x_0 e^{st}$$
$$\ddot{x} = s^2 x_0 e^{st}$$

It's more convenient to say $x = x_0 \sin\omega t$, $\ddot{x} = -x_0\omega^2 \sin\omega t$
I can put the solution in the equation and obtain

$$(-\omega^2 M + K) x_0 \sin\omega t = 0$$

If we say that this is true for everytime, we can concel $\sin\omega t$ and so

$$(-\omega^2 M + K) x_0 = 0$$

There're 2 cases.
* M singular the only solution is $x_0 = 0$ Trivial solution, the system doesn't move.
* M singular, some (x_0) satisfy the equation with $x_0 \neq 0$ and it's what we're looking for

$\det(-\omega^2 M + K) = 0$ must be satisfied to have M singular.
Here we don't work in time domain anymore, we have to look at ω to have a singular M matrix.
They're EIGENPROBLEMS and ω^2 is eigenvalues and ω is eigenvector.
That's not in a canonical form, which is

$$\det(-\omega^2 I + A) = 0 \qquad I: \text{identity matrix}$$

The matrixes are lambda matrixes. We can trasform it in a canonical form by multiplying every term for M^{-1}

$$\det(-\omega^2 I + \underbrace{M^{-1}K}_{\text{eig}(\text{inv}(M)\times K)}) = 0 \rightarrow \text{Dynamic matrix in the configuration space } D$$

In matlab

We can instead moltiply everything for K^{-1}

$$\det(-\omega^2 K^{-1} M + I) = 0$$

and we if divide everything for $-\omega^2$

$$\det\left(-\frac{1}{\omega^2} I + K^{-1}M\right) \rightarrow \text{Dynamic matrix } D$$

Which one is more usefull. Either M or K must be singular, so it depends on the problem.
We are interested in the lowest values of ω. If we use the second we get easily the highest values of $\frac{1}{\omega^2}$, so the lowest ω.
That was prefered once it was difficult to find them, not now with calculators.
ω is the frequency of the motion. Circular frequency [rad/s] but [Hz]
f ω

But K is simmetric, so $K = K^T \rightsquigarrow (q_i K q_j)^T$ but a transpose of a number has no meaning, $\rightsquigarrow q_i K q_j$

$\rightsquigarrow \omega_i^2 q_j^T M q_i - \omega_j^2 q_i^T M q_j = 0$

We can do the same for M and so

$$(\omega_i^2 - \omega_j^2) q_i^T M q_j = 0$$

The product of two numbers is 0, and it's true if one of the two is zero. if $i \neq j$, $q_i^T M q_j = 0$; if $i = j$, $q_i^T M q_j \neq 0 = \bar{M}_i$

M-ortogonality

Operating in the same way we get the same property for K

$i \neq j \quad q_i^T K q_j = 0$

$i = j \quad q_i^T K q_j \neq 0 = \bar{K}_i$ K-ortogonality

Pay attention that $q_i^T \cdot q_j \neq 0$, they're not in general ortogonal. It's $q_i^T q_j = 0$ if $M = c \cdot I$ so they're ortogonal (For instances a costant cross section beam divided in n masses with lumped parameters)
Eigenvectors can be used to make a coordinate transformation, they're the base for the space.
I can build a matrix of eigenvectors

$$\Phi = [\{q_1\}\ \{q_2\} \dots \{q_n\}]$$

It's a square matrix. Based on what we have said, Φ it's never singular and it's demostrable that $\det \Phi \neq 0$. If eigenvalue have more than two multiplicity, the eigenvector are different

$\omega_1 = \omega_2$

$q_1 \neq q_2$

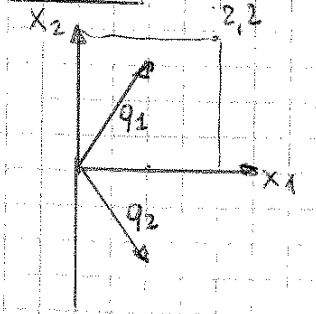
$a q_1 + b q_2$

Any linear combination of eigenvector is an eigenvector itself
As Φ it's not singular we can write another coordinate system called MODAL COORDINATES

$$X = \Phi \eta$$

> Eigenvectors are orthogonal with respect to the mass and the stiffness matrices, but they're not orthogonal to each other. $\Phi^T I$ does not yeld to a diagonal matrix, and $\Phi^{-1} \neq \Phi^T$

EXAMPLE



$\begin{Bmatrix} 1 \\ 1.5 \end{Bmatrix} \quad \begin{Bmatrix} 1 \\ -1,2 \end{Bmatrix}$

$1 \quad 1,15 \quad \begin{matrix} 1 \\ -1,2 \end{matrix} = 1 - 1,8 = -0,8 \neq 0$

$X = \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} \qquad \eta = \begin{Bmatrix} 2,4 \\ 0.6 \end{Bmatrix}$

$$\Phi = \begin{bmatrix} 1 & 1 \\ 1.5 & -1.2 \end{bmatrix}$$

That's in general a not ortogonal reference frame.

The solution of an homogenuous equation is not totaly definited, any eigenvector multiplied by a costant is it's itself an eigenvector the lenght of the vector doesn't matter. The sense is completely arbitrary. In the same example the

All the system have some damping, so we have to consider $\bar{C}$ which is not diagonal. IF $C = \alpha M + \beta K$, so it's a linear combination of the mass and stiffness matrixes so

$$\bar{C} = \alpha \bar{M} + \beta \bar{K}$$ PROPORTIONAL DAMPING

So now $\bar{C}$ it's diagonal and I can do what we have done before. This is a sufficient condition to uncoupling but it's not necessary. We have to introduce a GENERAL PROPORTIONAL DAMPING. Unfortunaty systems in real life are not proportional damped. Mechanical systems are small damped they vibrate. So $\bar{C}$ it's not very large and I can uncouple the system in an approximate way. We through away all that it's out of the main diagonal.

$$\bar{C} = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$$

If the terms x are smaller than the terms on the main diagonal, we can deleted them, but it's not what we have said, C must be small

We can introduce a reduced modal transformation

$$\Phi^* = [\{q_1\}\{q_2\}] \quad m << n$$

$$\underset{m\times n}{\Phi^{*T}} \underset{n\times n}{M} \underset{n\times m}{\Phi^*} = \underset{m\times m}{\bar{M}}$$ REDUCED MASS MATRIX

$$\underset{n}{x} \cong \underset{n\times m}{\Phi^*} \underset{m}{\eta}$$

The fact that a system has a proportional damping is sufficient to apply a modal decoupling

I cannot invert the reduced matrix 'cause it's not a square matrix, It's an approximation. So I use the pseudo-reverse

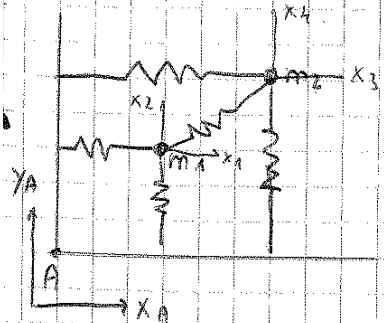
$$x \cong \Phi^{*+} \eta_m$$

We have another chance.

$$\eta \cong \bar{M}^{-1} \Phi^{*T} M x$$

It's an approximation too.

EXCITATION DUE TO THE MOTION OF THE CONSTRAINS



The Frame can move. When I write the inertial forces in an inertial reference

$$\ddot{x}_{inertial} = \ddot{x} + \delta_x \ddot{x}_A + \delta_y \ddot{y}_A + \delta_z \ddot{z}_A$$

In this case we don't have rotation, so we don't have Coriolis or centrifugal acceleration. It's typical for earthquakes.

$$\delta_x = \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad \delta_y = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} \quad \delta_z = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

they're direction cosines of the local motion respect to global motion.

$H(\omega)$ frequency response

$$\mathbb{R}(H) = K\frac{K-m\omega^2}{(K-m\omega^2)^2+c^2\omega^2} = \frac{1-\left(\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_N}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_N}\right)^2}$$

COMPONENT OF THE RESPONSE THAT IS IN PHASE WITH THE EXCITATION

$$\mathbb{J}(H) = K\frac{-c\omega}{(K-m\omega^2)+c^2\omega^2} = \frac{-2\xi\frac{\omega}{\omega_N}}{\left[1-\left(\frac{\omega}{\omega_N}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_N}\right)^2}$$

COMPONENT IN QUADRATURE (ANGLE PHASE 90°)

$$|H| = \frac{K}{\sqrt{(K-m\omega^2)^2+c^2\omega^2}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_N}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_N}\right)^2}}$$

MAGNIFICATOR FACTOR

$$\Phi = \operatorname{arctg}\left(\frac{-c\omega}{K-m\omega^2}\right) = \operatorname{arctg}\left[\frac{-2\xi\left(\frac{\omega}{\omega_N}\right)}{1-\left(\frac{\omega}{\omega_N}\right)^2}\right]$$

PHASE

In a continuous system modal partecipation factors are in theory infinite but in general we can get a rather good approximation using a discrete number

The sentence "in a damped system the frequency corresponding to the maximum amplitude does not depend on the damping" is an approximation with the hypotesis of small damping

Today we use a different notation which is the exponential notation

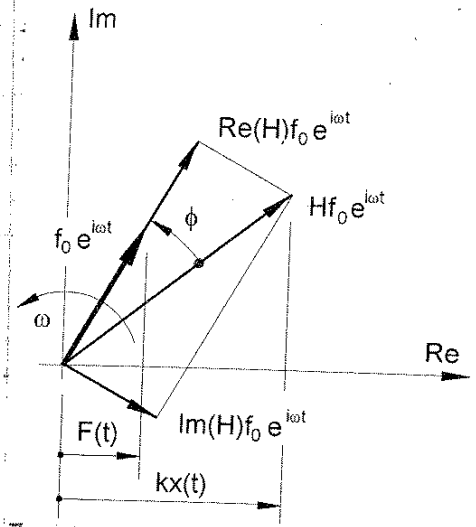
$x = x_0 e^{i\omega t}$ (We mean $X = Re[x_0 e^{i\omega t}]$

$F = f_0 e^{i\omega t}$

$X_A = x A_0 e^{i\omega t}$

I put inside the equation the terms and so

$\underbrace{(-m\omega^2 + i\omega c + K)}_{\text{costant which is a stiffness (you have to moltyply something for a displacement to get a force)}} x_0 = f_0$



DYNAMIC STIFFNESS K_{dyn}

I can rewrite $K_{dyn} = K\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2i\xi\left(\frac{\omega}{\omega_n}\right)^2\right]$

Dynamic stiffness in a n d.of system is a real or complex matrix

If the system is damped, K_{dyn} is a complex number.

1/ K_{dyn} = DYNAMIC COMPLIANCE, but 1 over a complex number is somewhat we don't want to use, cause

$\frac{1}{a+ib} = \alpha + i\beta$

$\alpha = \frac{1 - \left(\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left(2\xi\frac{\omega}{\omega_n}\right)^2}$

$\beta = \frac{-2\xi\frac{\omega}{\omega_n}}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left(2\xi\frac{\omega}{\omega_n}\right)^2}$

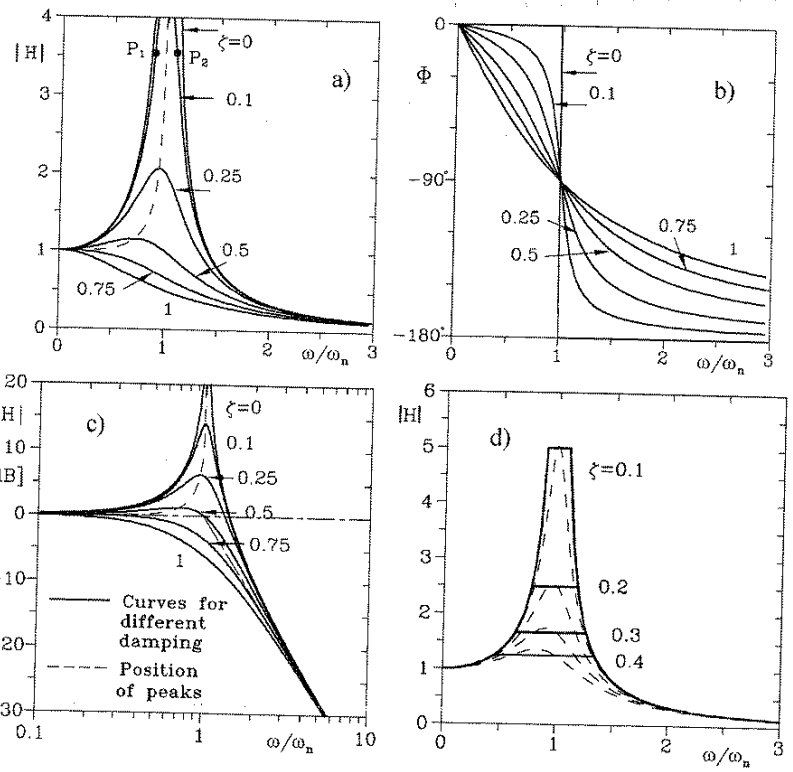
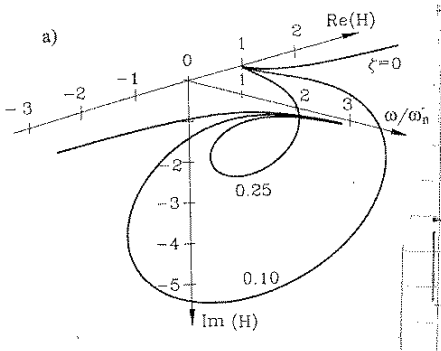


FIGURE 7.2. Bode diagram, i.e., magnification factor and phase as functions of the forcing frequency: (**a**) and (**b**) linear scales, (**c**) logarithmic scale for frequency and dB scale for amplitudes. (**d**) Frequency response of lightly damped systems approximated by using the response of the undamped system and shaving the peak at the value expressed by Eq. (7.7)

Bode diagram represent amplitude in function of the frequency

Frequency response of the dynamic stiffness with $\lambda \to 0$ tends to the stong stiffness

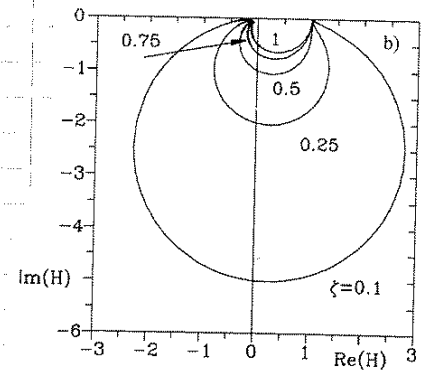


Instead of give Φ and H we can give Re and Im. It's not so common. It's a 3D plot. It's the master plot, all the others come from that plot.

In a 1 d.o.f damped system phase delay at natural frequency is 90°

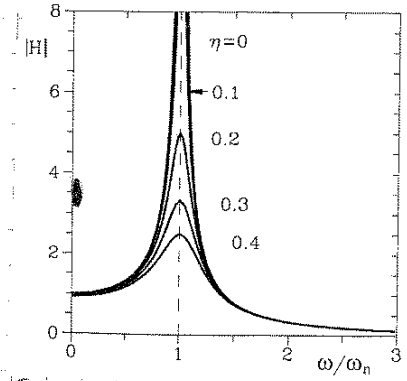
The frequency response in 1 d.o.f system at resonance is dominated by damping

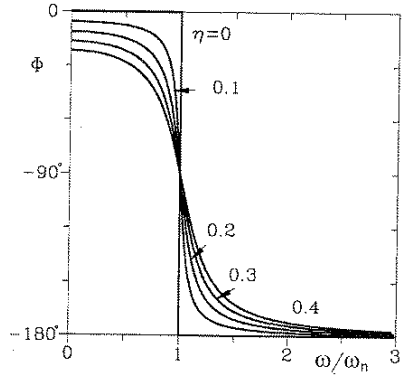
Nyquist diagram shows the frequency response of the system



The Nyquist plot represents real part and immaginary part and the dumping is the parameter.
The advantage is that you can sum up everything in a single pat.
The disadvantage is that the frequency is implicit. Another disadvantage is that it's difficult to get sperimentaly you get well the upper part while few results in the loop.
If the plot is almost circular it means the system is fairly linear viceversa more distorted is not linear.

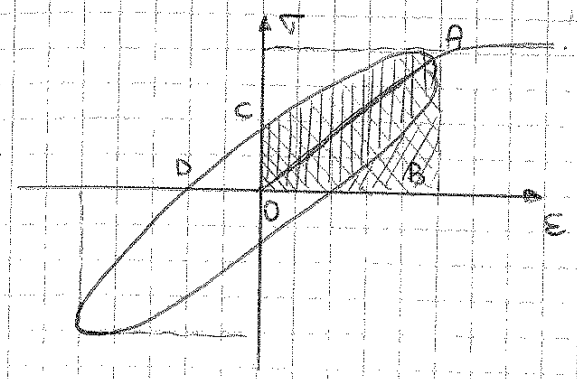
The Bode plot is the most used





Sometimes we measure velocity or acceleration so we can get other frequency responses. Sometimes they're called TRANSFER FUNCTIONS.
The difference is amplitude/frequency and amplitude/laplace but they're strictly linked.

STRUCTURAL DAMPING



Instead of a linear, we get an hysteretic cycle

given /// get /// dissipated energy

In engineering materials the area of the eclipse is so small, but as it's for second you can dissipate a lot. Calculating damping is so difficult as at every cicle we dissipate a small area.

> In structural damping the loss factor is the elastic modulus in quadrature over the one in phase

$$\begin{cases} \sigma = \sigma_0 \cos(\omega t) = \sigma_0 e^{i\omega t} \\ \varepsilon = \varepsilon_0 \cos(\omega t - \phi) = \varepsilon_0 e^{-i\phi} e^{i\omega t} \end{cases}$$

← the deformation comes with a small delay. That's what causes damping.

$$\frac{\sigma}{\varepsilon} = \frac{\sigma_0}{\varepsilon_0}\left[\cos(\phi) + i\sin(\phi)\right] = \underbrace{E'}_{\text{STORED MODULUS}} + i\underbrace{E''}_{\text{LOSS MODULUS}}$$

stiffness — damping

IN PHASE MODULUS — IN QUADRATURE MODULUS

> Many materials, when subjected to cyclic loading, exhibit a type of internal damping causing energy losses per cycle that are proportional to the square of the amplitude and independent of the frequency

We can do the same for the stiffness of objects

$$\frac{F}{x} = k' + ik'' \rightsquigarrow \eta = \frac{k''}{k'} = \operatorname{arctg}\left(\frac{\phi}{2}\right)$$

It's the phase between stress and strain

LOSS FACTOR

Those formulae are used for hi-frequency materials. In engineers materials $E' >> E''$ and $k'' >> k'$ ⇝ ϕ very small so η very small

$$\rightsquigarrow k' \approx k \qquad \eta \approx \frac{\phi}{2}$$

$$k^* \approx k(1+i\eta)$$
$$E^* \approx E(1+i\eta)$$

In a first approximation we can consider η as a costant we can find on books.

⚠ IT'S WRONG IN EVERY CASES WHICH IS NOT HARMONIC MOTION.

ϕ, and so η, is assumed constant of the material, depending also on heat treatment. ϕ increases with fatigue, depends on state, stress

We can use ψ RELATIVE DAMPING, the percentuality of energy dissipated

$$\psi \cong 2\pi\frac{\phi}{2} \cong 2\pi\eta$$

> Is defined as the ratio between the energy dissipated in a cycle and the elastic energy stored in the system in the condition of maximum amplitude

If I write $m\ddot{x} + c\dot{x} + kx = f(t)$, I ~~m running the equation of motion~~ in time domain. I cannot write $m\ddot{x} + k(1+i\eta)x = f(t)$. Apparently is correct, but it's wrong 'cause you cannot use structural damping in time domain. If you try to solve that equation, you obtain absurde result. The system starts moving even before excitation.

If I write instead in frequency domain

All elastomerics have a rubbery zone at low frequencies. At high frequency it's glassy. In between the material comes more and more brittel.
In --- it's loss factor.
This plot it's influenced by temperature, the curve shifted to right in increasing temperature and viceversa.

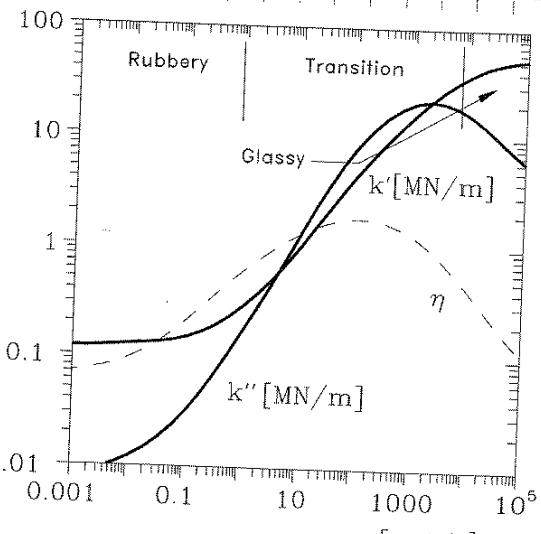
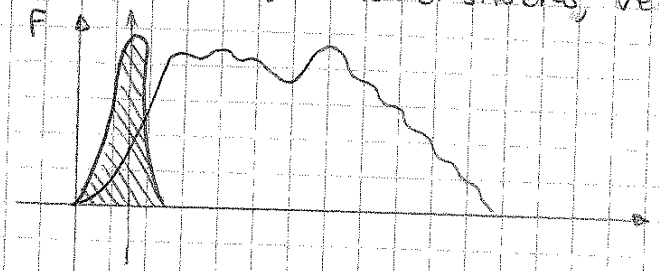


FIGURE 3.9. In-phase and in-quadrature stiffness and loss factor of an elastomeric spring as a function of the frequency.

RESPONCE TO NON-HARMONIC EXCITATION

IMPULSE EXCITATION

It's used to model shocks, very high forces in very short time



Example of the forces on the bonnet in a crash test. Time is measured in few milliseconds, so it's a short time.
I prefer to plot line in red

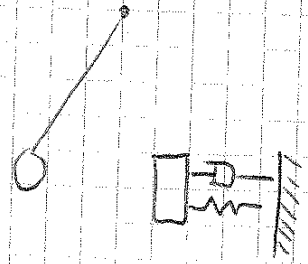
$$I = \int F dt$$

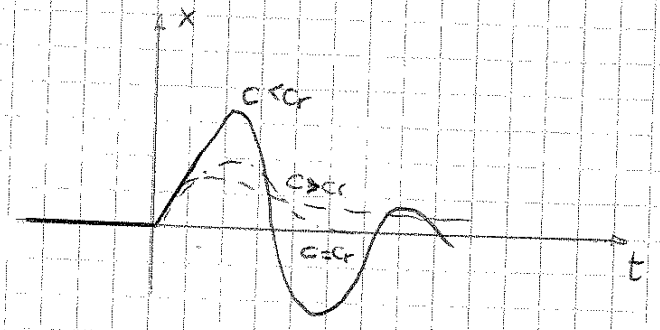
Lenght → 0, Amplitude → ∞. The area is 0^{∞}, so it's indeterminate.

$$\delta(t) = 0 \quad t \neq 0$$
$$\delta(t) = \infty \quad t = 0$$
$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1$$
$$I = m(v_2 - v_1) = m\dot{x}$$

Usually the impulse function is called DIRAC-δ

To compute, we can assume that after the peak the system is in free motion

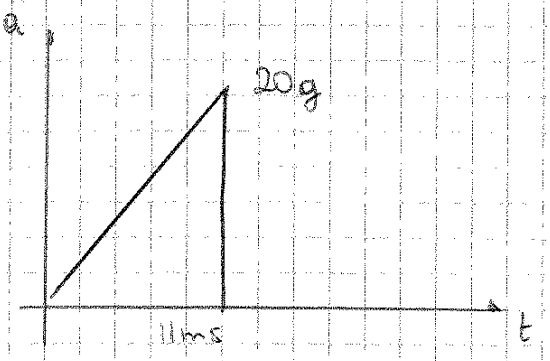




h(t)

If th shock time is different from zero, impulse function is suitable if the period of the motion is much larger than the time of the impulse. In mechanical parts usually $\omega_{N_1} > 50$ Hz, so $T_1 < 20$ ms. The duration of the impulse in rigid objects it's about 10 m s and so it's not true $T >> t_{imp}$ usually we can't use the impulse analysis, even thaught it's a conservative method.

Shocks are standardized by MIL-STD



In a shock, if dtime is nearly equal to the one of free obsallation of the system, this shock is described with more precision by Duhamel's integral

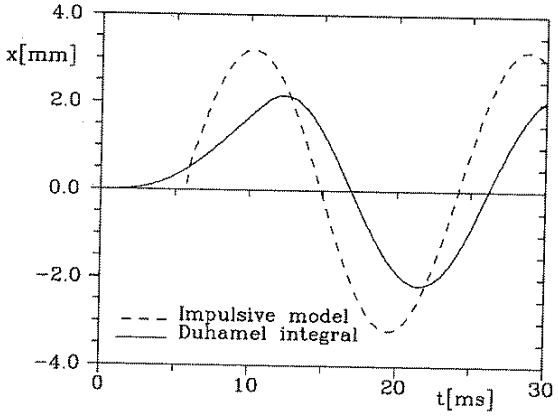


FIGURE 8.5. Time history of the response of the system. Impulsive model and results obtained through Duhamel's integral.

To compute shocks we don't include damping. If there's damping, the peaks will be lower, so we're safe.

STEP EXCITATION

$$\begin{cases} \mu = 0 & t < 0 \\ \mu = 1 & t > 1 \end{cases}$$

It's practically very important it models few situations. It's also very easy to compute, we can consider free motion after time > 0.
The response is

$$x(t) = \frac{f_0}{K} g(t)$$

$$\begin{cases} g(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right], & c < c_r \\ g(t) = 1 - (1-\omega_n t)e^{-\omega_n t}, & c = c_r \\ g(t) = 1 - \dfrac{1}{2}\left\{-e^{-\left[\zeta+\sqrt{1-\zeta^2}\right]\omega_n t} + e^{-\left[\zeta-\sqrt{1-\zeta^2}\right]\omega_n t}\right\}. & c > c_r \end{cases}$$

RANDOM VIBRATION

hint: square amplify

There're many cases where the forcing function acting on a dynamic system has a very complex time history, that cannot be reduced to a simple periodic pattern and cannot be defined in a closely deterministic way. In all these cases the time history of the excitation can be measured, and if enough experimental data are taken, it's possible to perform a statistical analysis

When studying random excitation, a first assumption is that the phenomenon is STATIONARY. Another assumption is that of ERGODICITY (every spec is representative for all specs)
Under this assumption the average, the r.m.s value, the variance and all other statistical parameters can be considered independent of the particular sample used for their computation. These are oversimplifications of a more complex phenomenon, but in most cases they allow for results that are in close accordance with experimental ones to be obtained.

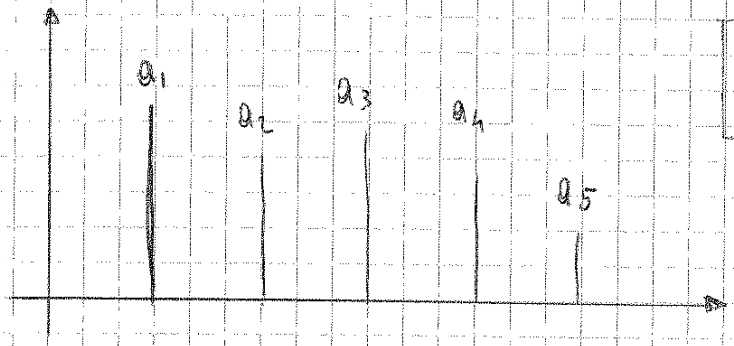
We take in account the positive and negative sign, but square amplify high-values and it's very handy. It's called VARIANCE

$$\sigma^2 = \frac{1}{T}\int_0^T F^2(t)\,dt$$

Instead of variance we can use ROOT MEAN SQUARE VALUE

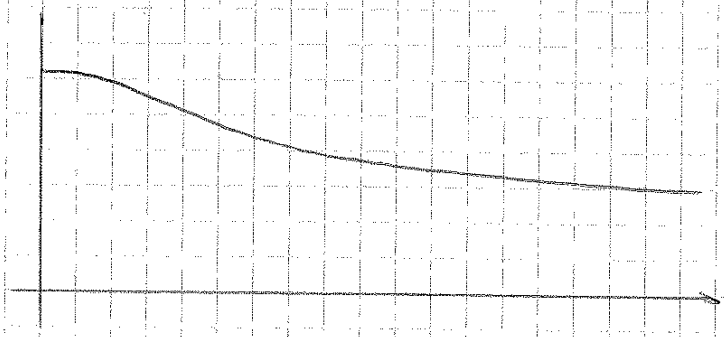
$$F_{rms} = \sqrt{\frac{1}{T}\int_0^T F^2(t)\,dt}$$

We need now something that gives use how fast changes happens and we have to use Fourier analysis to get a f... spectrum



Power spectral density is the Fourier transform of the autocorrelation function

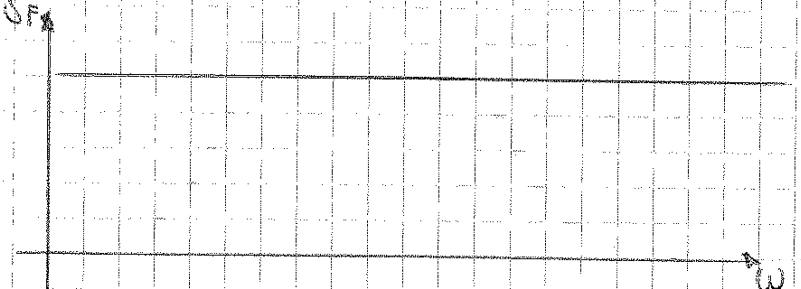
The square of the spectrum is called powerspectrum
Doubling and doubling the lenght of the signal I densify the spectrum, it's to limit proces.



By The Fourier Transformation I get the POWER SPECTRAL DENSITY

S

Gives me the frequencies involved in the phenomenon.
The WHITE NOISE is a excitation with costant powerspectral density



The area under the curve it's the F_{rms}^2, in this case would be infinity large, so it's impossible, it's only ideal

LIMITED WHITE NOISE

PSD are usually safe, but sometimes it makes impossible to design
For the designer it's an input

As you have a random input, you have to compute the random output.
As the system is linear, we can use superposing principle

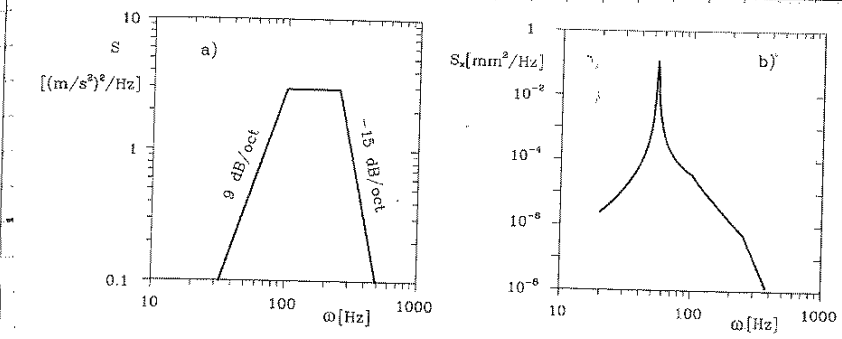


FIGURE 9.6. Power spectral density of the excitation (**a**) and the response (**b**) of the system of Example 9.1.

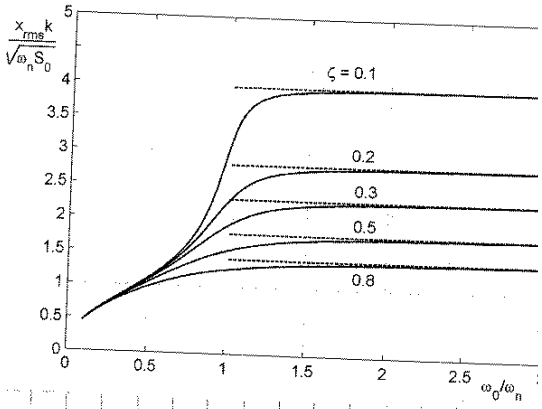
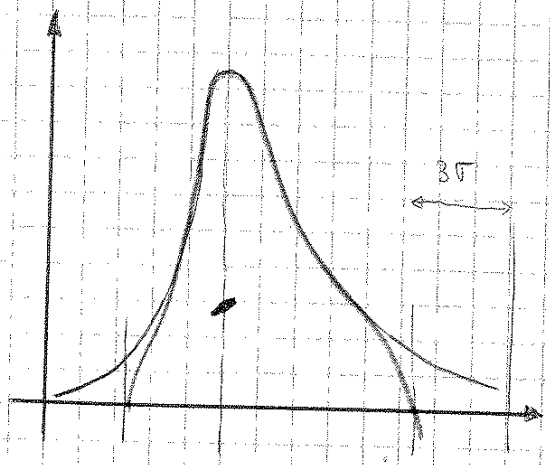


FIGURE 9.7. Nondimensional r.m.s. value of the response to a band-limited white noise as a function of the ratio ω_0/ω_n. The *dashed lines* are the asymptotes for $\omega_0/\omega_n \to \infty$, i.e., for an ideal white noise.

We have the problem of the TALES OF THE DISTRIBUTION, we have to cut it at 3σ

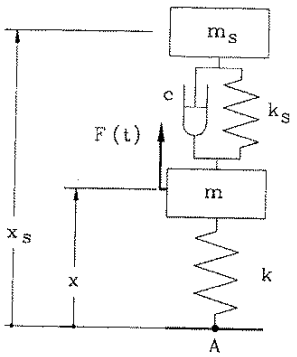


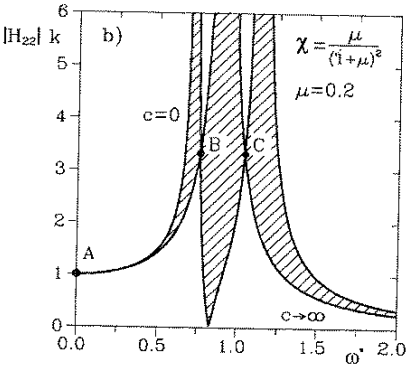
Ingegneria è progettare ed essere responsabile di una soluzione di cui si hanno pochi dati

Even though the probability of failure of an object is low, it can't be enough.

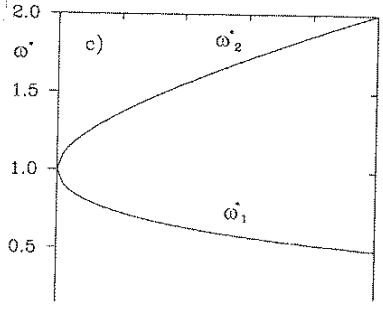
FIGURE 7.12. Vibration absorber applied to a system with a single degree o freedom: (**a**) sketch of the system; (**b**) limiting cases (damping tending to zero and infinity) for systems with optimum tuning; (**c**) natural frequencies of the undamped system as functions of the mass ratio μ; (**d**) amplitude of the response of the system in (**b**), but with three different values of damping.

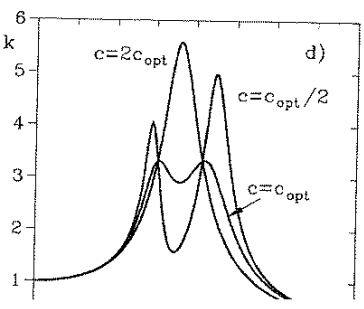
DYNAMIC VIBRATION ABSORBER





A dynamic vibration absorber is basically a spring-mass-damper system that is added to any vibrating system with the aim of reducing the amplitude of the vibration of the latter





Applying a dynamic absorber made up of a mass-spring system, amplitude of the main system will be reduced because the mass of the damper exerts a force on the main system and so reduction of amplitude depends on the entity of the first

The unidimensional nature of beam allows simplification of the study: each cross section is considered as a rigid body whose thickness in the axial direction is vanishingly small; it has 6 d.o.f, 3 translational and 3 rotational. The problem is then reduced to a unidimensional problem, in the sense that a single coordinate, namely the axial coordinate, is required.

IF the system is linear, D is linear too. We have to write also the boundary condition

$$U\left[u(x,y,z,t)\right]_B = 0$$

U is an algebric operator, is a very peculiar boundary condition, it's holonomic and homogeneous.
The boundary is not a 3D, but 4D as it's boundary in time. The boundary are in space and in time (initial and final conditions)

We can solve the problem by writing

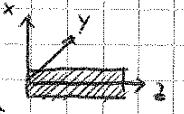
$$u(x,y,z,t) = D^{-1}\left[f(x,y,z,t)\right]$$

We are almost not able to solve that, except for some particular cases. Now we have to deal with beams.

BEAMS AND BARS

Beams have one dimension which is dominant respect to the others it's straight and have particular proprieties, so that we can split up in 4 behaviours

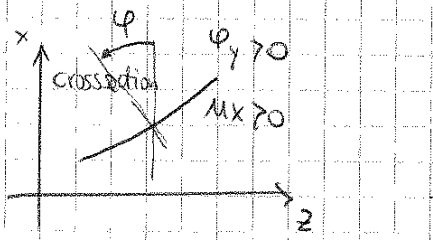
* AXIAL ⇝ BAR	dof u_z	generated force	F_z
* TORTIONAL	Φ_z		M_z
* FLEXURAL (xz plane)	u_x, Φ_y		F_x, M_y
* FLEXURAL (yz plane)	u_y, Φ_x		F_y, M_x

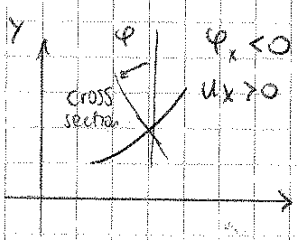


↑ convention

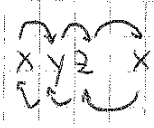
We neglect the cross action, as rigid body.

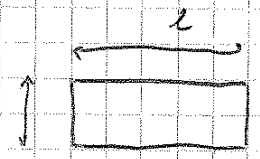
Let's take the beam and bent it





The equation for bending are not equal in different planes.



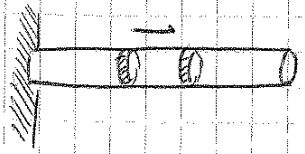


Euler-Bernoulli

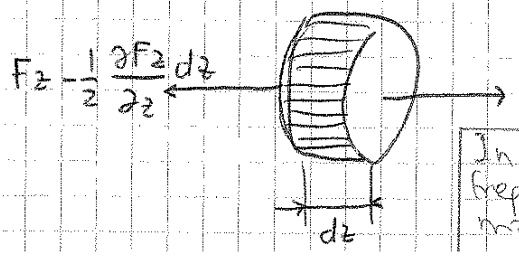
The basic assumption is that cross section is perpendicular to the beam.
The Timoshenko theorie takes in count that the cross section may not be perpendicular.
We use the first theory when the beam is slender. IF S/L is more than 50,

EULER-BERNOULLI

> In the beam element torsional behavior can be decoupled from the flexional behavior if the section has two symmetry planes



We can write the equilibrium equation



> In a beam flexural and axial natural frequency are coupled is center of mass and shear center are not the

How is possible, for all values of z, t, two function of different variable are the same? It's only possible if the two function are the same constant

$$\frac{1}{\eta}\frac{d^2\eta}{dt^2} = cost = -\omega^2$$

$$v_s^2 \frac{1}{q}\frac{d^2q}{dz^2} = cost = -\omega^2$$

Let's take the function of time

$$\frac{d^2\eta}{dt^2} = -\omega^2\eta \qquad \text{if } \eta = \cos(at),\ \ddot{\eta} = -a^2\cos(at)$$

$$\rightsquigarrow -a^2\cos(at) = -\omega^2\cos(at) \rightsquigarrow a^2 = \omega^2 \rightsquigarrow a = \omega$$

It will move with harmonic motion in ω frequency. All points will move in an harmonic function and they're all in phase. Discrete system and continuous system, without damping, the vibrate harmonically and in phase, this makes them similar.

So $u = q\cos\omega t$

Now we have to compute the other function

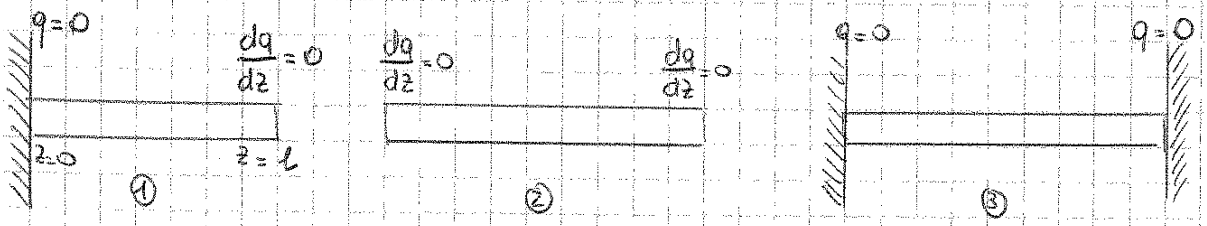
$$\frac{v_s^2}{q}\frac{d^2q}{dz^2} = -\omega^2 \rightsquigarrow \frac{d^2q}{dz^2} = -\frac{\omega^2}{v_s^2}q$$

It's an another harmonic function, so I can write $q = q_0\cos(az)$

$$-a^2\cos(az) = -\frac{\omega^2}{v_s^2}\cos(az) \rightsquigarrow a^2 = \frac{\omega^2}{v_s^2}$$

To compute the frequency, I must state boundary condition. I must take in account the phase for q

$$q = c_1\cos(az) + c_2\sin(az)$$



Let's solve case ①

$0 = c_1\cos 0 + c_2\sin 0 \rightsquigarrow c_1 = 0$

$\frac{dq}{dz} = 0 = c_2 a\cos(a\ell)$ that's an eigen problem. $c_2 = 0$ it's a trivial solution.

$a = 0$ I'm not interested, it's a trivial solution too.

TORSIONAL VIBRATION

$$\rho I_z \frac{d^2 \Phi_z}{dt} = \frac{\partial}{\partial z}\left[G I_p' \frac{\partial \Phi_z}{\partial z} \right] + m_z$$

It's that the same as Euler-Bernoulli, provided

$$\begin{aligned} u_z &\longrightarrow \Phi_z \\ \rho A &\longrightarrow \rho I_z \\ EA &\longrightarrow G I_p' \\ f_z(z,t) &\rightarrow m_z(z,t) \end{aligned}$$

TORSIONAL VIBRATION

$$\rho I_z \frac{d^2 \Phi_z}{dt} = \frac{\partial}{\partial z}\left[G I_p' \frac{\partial \Phi_z}{\partial z} \right] + m_z$$

It's the same as Euler-Bernoulli, provided that

$u_z \longrightarrow \Phi_z$
$\rho A \longrightarrow \rho I_z$
$EA \longrightarrow G I_p'$
$f_z(z,t) \rightarrow m_z(z,t)$

And so I can write, if the beam is slammed

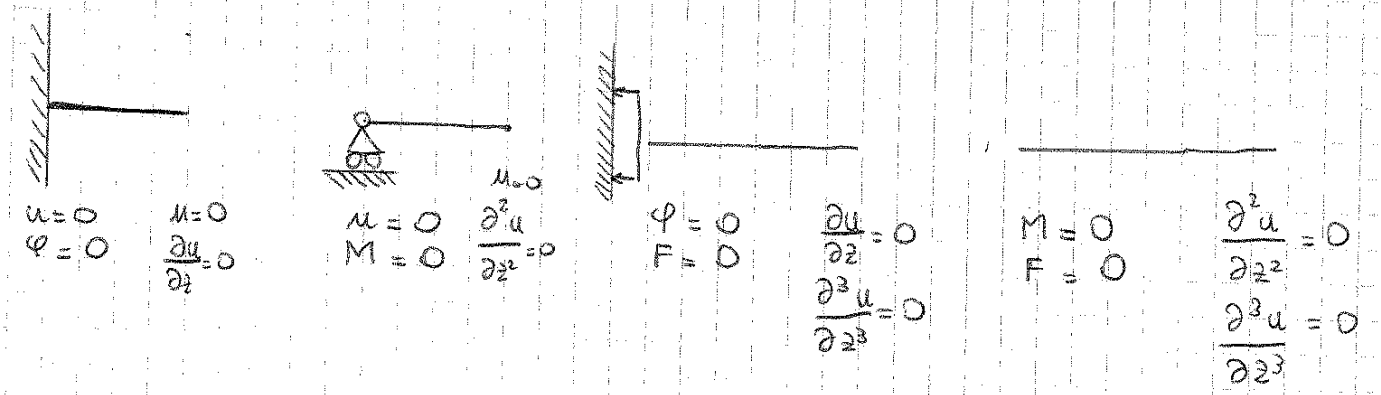
$$\rho A \ddot{u}_x + EI_y \frac{\partial^4 u_x}{\partial z^4} = f_x(z,t)$$

It can be solved in the same way shown before.

$$\omega_i = \frac{\beta^2 f_i}{l^2}\sqrt{\frac{EI_y}{\rho A}}$$

TABLE 12.3. Values of constants $\beta_{f_i} = a_i l$ for the various modes with different boundary conditions.

Boundary condition	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i>4$
Free–free	0	4.730	7.853	10.996	14.137	$\approx (i+1/2)\pi$
Supported–free	0	1.25π	2.25π	3.25π	4.25π	$(i+1/4)\pi$
Clamped–free	–	1.875	4.694	7.855	10.996	$\approx (i-1/2)\pi$
Supported–supported	–	π	2π	3π	4π	$i\pi$
Supported–clamped	–	3.926	7.069	10.210	13.352	$\approx (i+1/4)\pi$
Clamped–clamped	–	4.730	7.853	10.996	14.137	$\approx (i+1/2)\pi$



If we have an harmonic equation, I need 4 boundary conditions

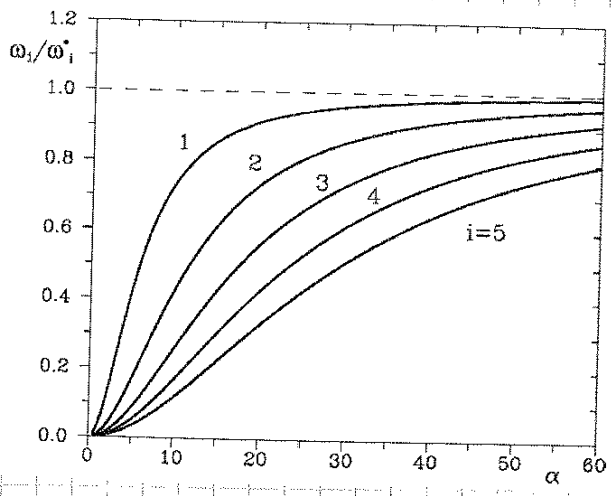
$$q = c_1 \sin(az) + c_2 \cos(az) + c_3 e^{az} + c_4 e^{-az}$$

It's more handy to write instead of e, sum and difference of e which are hyperbolic functions

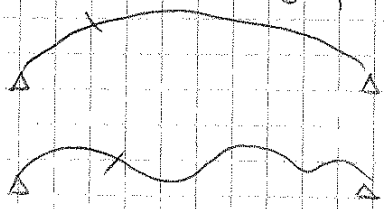
$$q = c_1 \sin(az) + c_2 \cos(az) + c_3 \sinh(az) + c_4 \cosh(az)$$

Sometimes the boundary conditions are not solvable in a closed form

Euler-Bernoulli beam can only be used in case of slender beam, Timoshenko beam can be used for rather squat beams

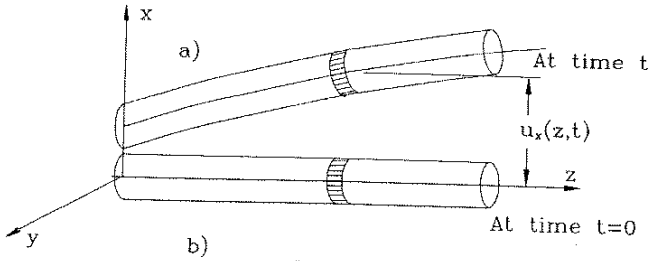


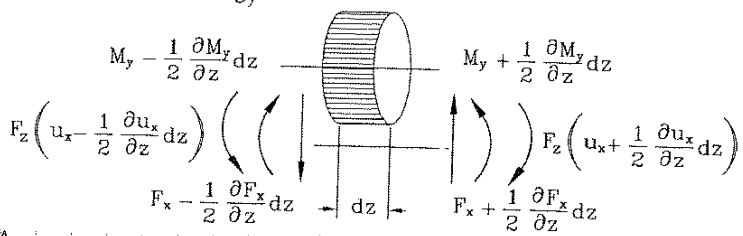
You have the slenderness on x. $\alpha = 60$ it's not very slender. When the result is 1 Eulero o Timoshenko are the same. Timoshenko gives a result lower than Eulero Bernoulli. When the slenderness is low, there's no phisical meaning, i are natural frequencies



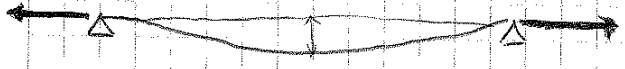
Rotation inertia at high frequency is much more important, so Eule Bernoulli gives a bad result.

AXIAL FORCES - BENDING BEHAVIOUR INTERACTION





$$F_x\,dz + \frac{\partial M_y}{\partial z}dz - F_z\frac{\partial u_x}{\partial z}dz = 0$$



It's not a linear problem. By neglecting the stiffness, we can use this approach to strings

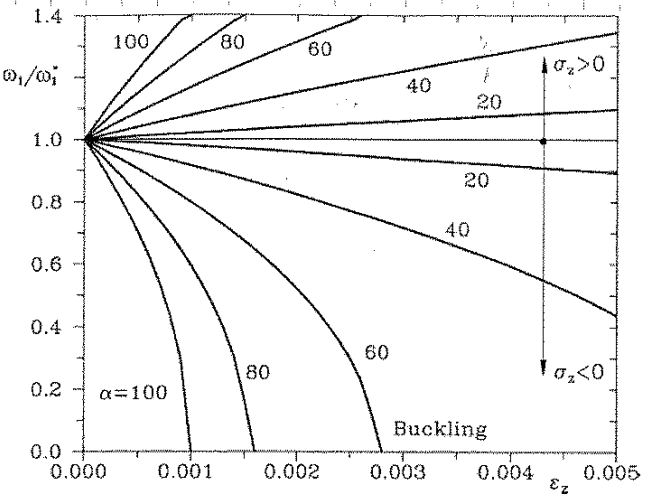
$$\rho A\frac{d^2u_x}{dt^2} + \frac{\partial^2}{\partial z^2}\left[EI_y\frac{\partial^2 u_z}{\partial z^2}\right] - F_z\frac{\partial^2 u_x}{\partial z^2} = f(z,t)$$

The characteristic equation is

$$\lambda_i^2\rho A = EI_y\left(\frac{i\pi}{l}\right)^4 + F_z\left(\frac{i\pi}{l}\right)^2$$

We can relation the beam with and without axial forces

$$\frac{\lambda_i}{\lambda_i^*} = \sqrt{1 + \frac{F_z}{E}\frac{\alpha^2}{i^2\pi^2}}$$



In a oscillating rope natural frequencies λ_n depend on axial load and traction load applied

ε_z is done using Area and Young modulus. Up 1, it's tensile configuration. For higher slenderness the effect is higher. Tensile forces increase natural frequency. In case of compression the natural frequency decreses and the effect is higher for higher slenderness.
A natural frequency that goes to zero means that you're loosing stiffness, you get buckling, falls in elastic instability.

DISCRETIZATION METHODS

We can have 3 different approaches for discretization
* ASSUMED-MODES METHOD is the oldest method.
* LUMPED PARAMETERS METHOD we keep the stiffness distributed, but the mass is concentrated. it can be used for FEM analysis
* FEM METHOD to compute stiffness and inertia. Today it's the most common approach. You use approximate function for displacements. You assume parameters as unknown, displacement is unknown.

FEM

$u(x,y,z,t) = N(x,y,z)\,q(t)$ general equation for time.

N are arbitrary function, while u are displacements and $q(t)$ is what we are looking for.

$$\begin{Bmatrix} u_x(x,y,z,t) \\ u_y(x,y,z,t) \\ u_z(x,y,z,t) \end{Bmatrix} = \begin{bmatrix} N(x,y,z) & 0 & 0 \\ 0 & N(x,y,z) & 0 \\ 0 & 0 & N(x,y,z) \end{bmatrix} \begin{Bmatrix} q_x(t) \\ q_y(t) \\ q_z(t) \end{Bmatrix}$$

N has only one row and as many columns as the number of the nodes of the element
The element is isotropic. N are called SHAPE FUNCTIONS. They must be simple, must allow rigid motion, patch test, borders must move in consistent way.
Usually N are polinomial.
If the material is linear, we can write

$$\varepsilon(x,y,z,t) = B(x,y,z)\,q(t)$$

B is made of derivatives of N matrix.

TORSIONAL BEHAVIOUR

$$N = \begin{bmatrix} 1-\xi & \xi \end{bmatrix}$$

$$f(t) = \frac{1}{2} l\, m_z(t) \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Each cross section has dof, so we have 12 dof totally.

$$u_z = N \begin{Bmatrix} u_{z1} \\ u_{z2} \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ u_{z2} \end{Bmatrix}$$

$$u_z = a_0 + a_1 \frac{z}{l} + a_2 \left(\frac{z}{l}\right)^2 + a_3 \left(\frac{z}{l}\right)^3 + \dots = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + \dots$$

Boundary conditions

$z = 0 \quad u_z = u_{z1}$

$z = l \quad u_z = u_{z2}$ ⇝ the polinomial must be linear, we can find only a_0 and a_1.

⇝ $\boxed{u_{z1} = a_0}$, $u_{z1} = a_0 + a_1$ ⇝ $\boxed{u_{z2} - u_{z1} = a_1}$

⇝ $u_z = u_{z1} + (u_{z2} - u_{z1})\xi$ and so I can write the shape function $u_z = \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \begin{Bmatrix} u_{z1} \\ u_{z2} \end{Bmatrix}$

$$\varepsilon = \frac{d}{dz}(u_z) = \frac{d}{dz} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \begin{Bmatrix} u_{z1} \\ u_{z2} \end{Bmatrix} = \frac{1}{l} \frac{d}{d\xi} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \begin{Bmatrix} u_{z1} \\ u_{z2} \end{Bmatrix} =$$

$$= \underbrace{\frac{1}{l} \begin{bmatrix} -1 & 1 \end{bmatrix}}_{[B]} \begin{Bmatrix} u_{z1} \\ u_{z2} \end{Bmatrix}$$

$$[K] = \int B^T E B \, dV = \frac{1}{l^2} \int \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} E \begin{bmatrix} -1 & 1 \end{bmatrix} A\, l\, d\xi = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M] = \int_V \rho N^T N \, dV = \rho A l \int \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} d\xi = \underline{\rho A l} \begin{bmatrix} 2 & 1 \end{bmatrix}$$

EFFECT OF AXIAL FORCES

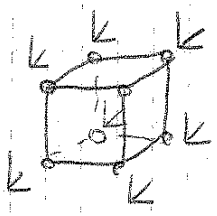
formulae (ΔL, dz, $\Delta\Omega$, t)

pg 375

MASS ELEMENT $M = diag(m, m, m, J_x, J_y, J_z)$

SPRING ELEMENT $K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

Let's consider a general solid element, a brick element



Each element has 3 displacement. Total is $8 \times 3 = 24$ displacement

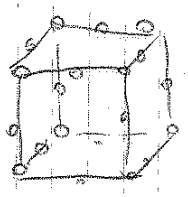
$$\left\{ \right\}^{(24\times1)}$$

So when I write $\underset{(3\times1)}{u} = \underset{(3\times24)}{N}\ \underset{(24\times1)}{q}$

When I compute $K = \int_V B^T E B \, dV$, $[K]$ will be a (24×24). The same the mass matrix.

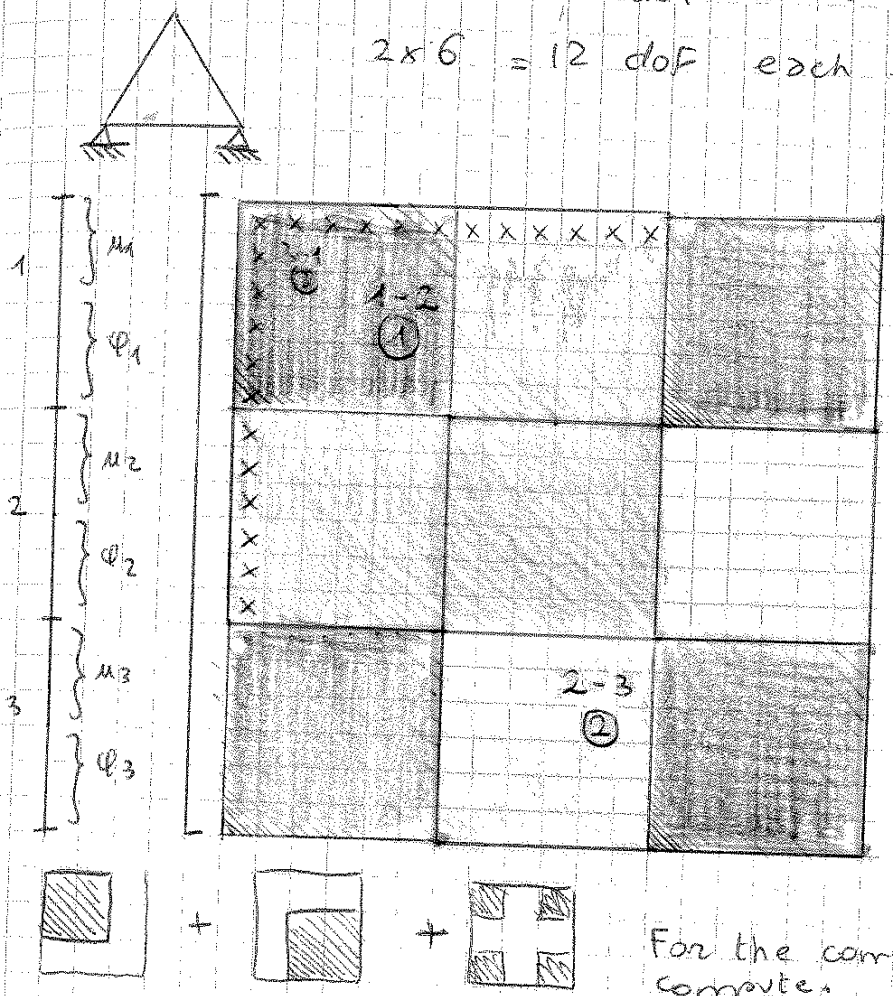
The best solution is the brick

Where K is (60×60)



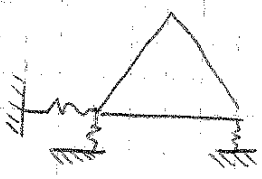
$3 \times 6 = 18$ dof → 18x18 structure

$2 \times 6 = 12$ dof each element



For the computer it's very easy to compute.

Talking about constrains, we can consider the structure in this way



We have to write $\frac{1}{2} k_i \mu_i^2$ adding it on the main diagonal

EXAMPLE

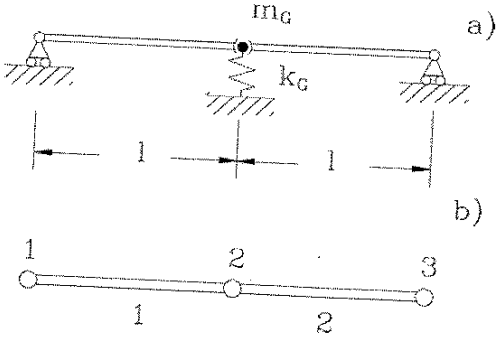


FIGURE 15.6. (**a**) Sketch of the system and (**b**) model based on two beam elements.

ISOPARAMETRIC ELEMENTS

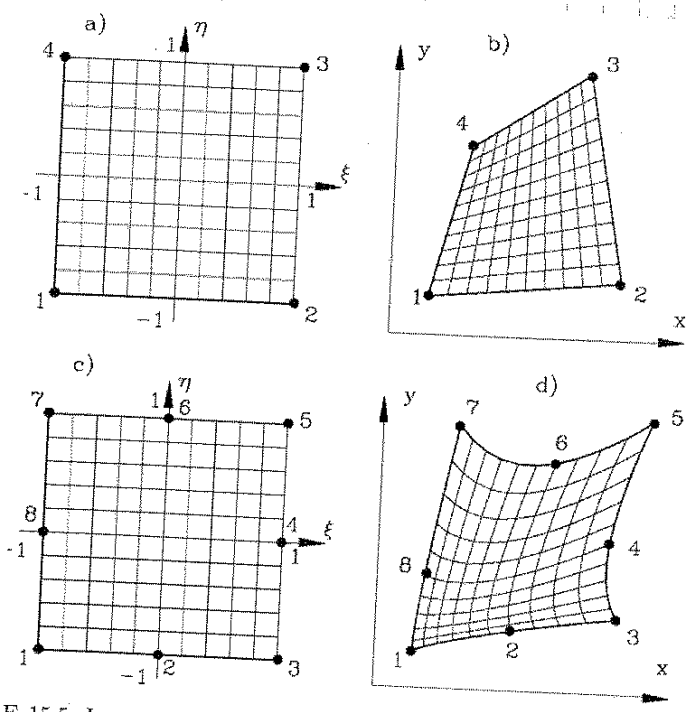


FIGURE 15.5. Isoparametric elements obtained by distorting a 4-node and an 8-node rectangular element.

Nearly all the elements used are isoparametric. The only problem can come up during calculating the matrix K

$$[k] = \int_V B^T E B \, dV$$

The integration must be done on the actual volume of the element, not in a), so

$$[k] = \int_{V'} B^T E B |J| dV'$$

we have to introduce the determinant of the Jacobian matrix.

det(J) can be printed as an average of the element, it gives the idea how much the element is distorted.

DYNAMIC STIFFNESS METHOD

It concerns precision of finite element method.

We can use the solution of the dynamic matrix to build a shape function to get the exact result.

> The finite element method yelds approximi results because the shape functions cannot give way to the exact inflected shape. To overcome this limitation, an exact method has been developed for beam elements.

In case of beam

$$q(t) = C_1 \sin(\alpha t)$$

By developing in series we get

$$K_{dyn} = K_{d_0} + \lambda^2 K_{d_2} + \lambda^4 K_{d_4} + \dots$$

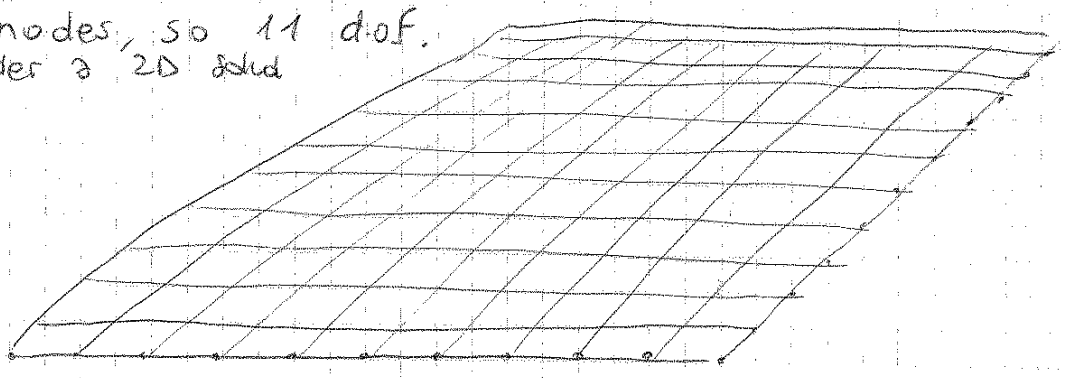
At first and second term coincide with finite element. It works but computations are so heavy, so it's better to use classic finite element adding more elements. You get the same precision but with less computations.
Dynamic stiffness method it's still proposed, but it's old and not used.

FINITE ELEMENT IN TIME

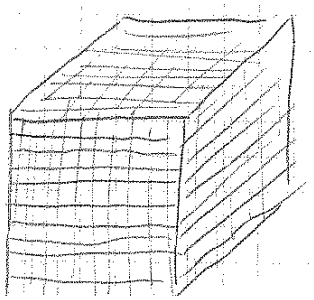
Instead of writing $u = N(x,y,z)\,q(t)$, we can write $u = N(x,y,z,t)\,q$
We transform dynamic in space into static in space time.
Apparently the advantage is huge, sometimes we can make a mesh that changes in time, useful for some problems in which points change in time (for example contact problems)
There's a big problem that limits it

Let's consider a beam

It has 11 nodes, so 11 d.o.f.
If we consider a 2D solid



It has 121 nodes so 242 d.o.f.
If we consider a solid



It has 1331 nodes, so 3993 d.o.f.
If we are in time, we have

14641 nodes

and multiplying for 4 dof for each node we have

58564 dof

This approach is worthwhile for some problems, for example deformation similar to the first mode. That's the so called RITZ METHOD.

Example 14.1 *Compute the first natural frequency of a prismatic, homogeneous simply supported beam using the Rayleigh method. Compare the result with that obtained using the beam theory. Neglect the rotational inertia of the cross-sections and shear deformation and assume a parabolic assumed-mode shape. Using the reference frame shown in Fig.* 14.1, *the parabolic assumed mode is expressed by the formula*

$$q(\zeta) = \zeta(\zeta - 1) \ ,$$

where $\zeta = z/l$. *The kinetic and potential energies corresponding to the deflected shape* $aq(z)$ *can be easily computed:*

$$\mathcal{T} = \frac{1}{2}\int_0^l \rho A\,[\dot{a}q(z)]^2\,dz\,, \qquad \mathcal{U} = \frac{1}{2}\int_0^l EI_y\left[a\frac{\partial^2 q(z)}{\partial dz^2}\right]^2 dz\,.$$

Rayleight method gives different values of the first natural frequency due to the choosen modal shape

The mass and stiffness of the system with a single degree of freedom and its natural frequency are

$$m = \int_0^l \rho A\,[q(z)]^2\,dz = \frac{\rho A l}{30}\,, \qquad k = \int_0^l EI_y\left[\frac{\partial^2 q(z)}{\partial dz^2}\right]^2 dz = \frac{4EI_y}{l^3}\,,$$

$$\omega = \sqrt{\frac{k}{m}} = \frac{10.95}{l^2}\sqrt{\frac{EI_y}{\rho A}}\,.$$

The value computed through the beam theory is

$$\omega = \frac{\pi^2}{l^2}\sqrt{\frac{EI_y}{\rho A}} = \frac{9.87}{l^2}\sqrt{\frac{EI_y}{\rho A}}\,.$$

The Rayleigh method leads to a value 9.9% in excess of the correct one. Different results would have been obtained if a different assumed-mode shape had been used.

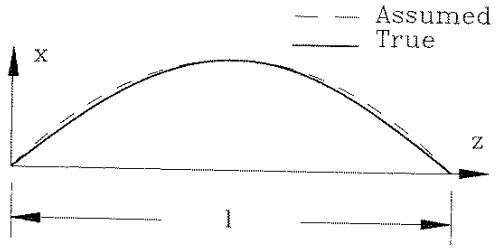


FIGURE 14.1. A simply supported beam; first mode shape and parabolic assumed shape. The two shapes have been normalized in order to have the same amplitude.

The aim of the Rayleight method is to compute an approximate value of the first natural frequency by transforming the original system into a system with only 1 d.o.f. Only one arbitrary mode shape is thus assumed. It must be choosen in such a way that it approximates the actual mode shape corresponding to the natural frequency that is looked for, usually the lowest natural frequency, and must be compatible with the constraints and the boundary conditions. Increasing the stiffness causes the relevant natural frequency to grow, so this method yelds a value for the lowest natural frequency which is always higher than the correct one. The closes the assumed mode is to the actual mode shape, the lower the value of the natural frequency computed from this method is.

Modal method come together to the first natural frequency

The aim of these methods is replacing the equation of motion consisting of partial derivatives differential equations (with derivatives with respect to time and space coordinates) with a set of linear ordinary differential equations containing only derivatives with respect to time. The resulting set of equations, generally of the second order, is of the same type as seen for discrete systems (hence the terme discretization)

Dunkerley's formula allows the computation approximated of the first natural frequency

If the second natural frequency is near to the first natural frequency, Dunkerley's gives bad results, while Stodola good result but with more iterations

The finite elements works as $\{F_R\} = [K]\{u_R\}$ Hooke's law

The transfer matrix method works in different way

$$\{u_R, F_R\} = [T]\{u_L, F_L\}$$

STATE VECTOR TRANSFER MATRIX STATE VECTOR

The two method conceptually are equivalent, so you get the same results. The equations tells that the system is linear.

The mass is lumped in stations, so we have inertia forces

So we can write $q_{R_{Hi}} = T_{Hi}\, q_{L_{Hi}}$

you put here the inertia forces

$$q_{L_{Fi}} = T_{Fi}\, q_{R_{F_{i-1}}}$$

$$S_n = T_n T_{n-1} T_{n-2} \cdots T_2 T_1 S_0 = T_G S_0$$

All the transfer matrix are small, is its best advantage.

HOLTZER METHOD

state vector of the Holtzer method contains torsional angle and torsional moment

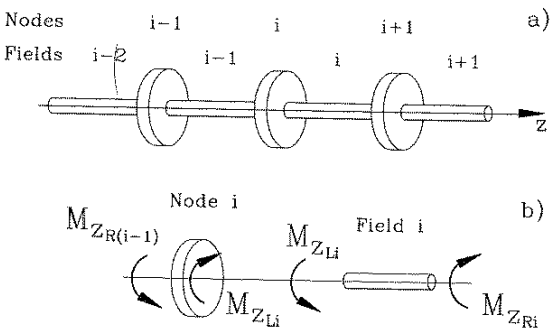


FIGURE 14.3. Holtzer method: (a) sketch of the system, (b) moments acting on the ith node and on the ith field.

The field transfer matrices are easily computed by considering that the moment at the left is equal to the moment at the right because no moment acts along the field, while the rotation at the right end is equal to the rotation at the left end increased by the twisting of the field

$$\begin{cases} \phi_{z_{R_i}} = \phi_{z_{R_i}} + \Delta\phi_{z_i} = \phi_{z_{R_i}} + \dfrac{l_i}{G_i I'_{P_i}} M_{z_{L_i}} \\ M_{z_{L_i}} = M_{z_{R_i}} . \end{cases} \tag{14.10}$$

The field transfer matrix is thus

$$\begin{Bmatrix} \phi_z \\ M_z \end{Bmatrix}_{R_i} = \begin{bmatrix} 1 & \dfrac{l_i}{G_i I'_{P_i}} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \phi_z \\ M_z \end{Bmatrix}_{L_i} . \tag{14.11}$$

MYKLESTADT METHOD

In Myklestadt method, the state vector contains displacement, rotation, shear force, flexural moment

Transfer matrix used in the Myklestadt method comes from the stiffness matrix of a beam element

$$\begin{Bmatrix} u_x \\ \phi_y \\ F_x \\ M_y \end{Bmatrix}_{R_i} = \begin{bmatrix} 1 & l_i & -\frac{l_i^3}{6EI_y} + \frac{l_i\chi}{GA} & \frac{l_i^2}{2EI_y} \\ 0 & 1 & -\frac{l_i^2}{2EI_y} & \frac{l_i}{EI_y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -l_i & 1 \end{bmatrix} \begin{Bmatrix} u_x \\ \phi_y \\ F_x \\ M_y \end{Bmatrix}_{L_i}$$

$$\begin{Bmatrix} u_x \\ \phi_y \\ F_x \\ M_y \end{Bmatrix}_{L_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\omega^2 m_i + k_c & 0 & 1 & 0 \\ 0 & -\omega^2 J_{i_y} + \chi_c & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_x \\ \phi_y \\ F_x \\ M_y \end{Bmatrix}_{R_{i-1}}$$

Example 14.4 *Compute the natural frequencies of the beam shown in Fig. 14.5a using Myklestadt's method. The beam has an annular cross-section with inner and outer diameters of 80 mm and 100 mm, respectively, and the dimensions shown in the figure are $l_1 = l_3 = 200$ mm and $l_2 = 800$ mm. The main characteristics of the material are $E = 2.1 \times 10^{11}$ N/m^2, $\rho = 7,810$ kg/m^3, and $\nu = 0.3$.*

The mass of the beam is lumped in seven stations (4.417 kg in all nodes except the first and last ones where there is a mass of 2.209 kg). In the second and sixth stations there are two supports. The rigid supports are modeled using elastic constraints with a stiffness of 1×10^{11} N/m. The computation of the global transfer matrix involves the multiplication of six field transfer matrices and seven node transfer matrices.

Because both ends are free, the plot of the determinant $T_{31}T_{42} - T_{41}T_{32}$ of the reduced global transfer matrix as a function of ω is reported in Fig. 14.5b, for a range up to the third natural frequency.

The natural frequencies, up to the fifth, computed by searching the points at which the plot of Fig. 14.5b (extended to higher frequencies) crosses the frequency axis, are

$\omega_1 = 2,158$ rad/s $= 343.4$ Hz, $\omega_2 = 5,091$ rad/s $= 810.2$ Hz,
$\omega_3 = 6,780$ rad/s $= 1,079.1$ Hz, $\omega_4 = 10,469$ rad/s $= 1,666.2$ Hz, and
$\omega_5 = 15,858$ rad/s $= 2,523.9$ Hz .

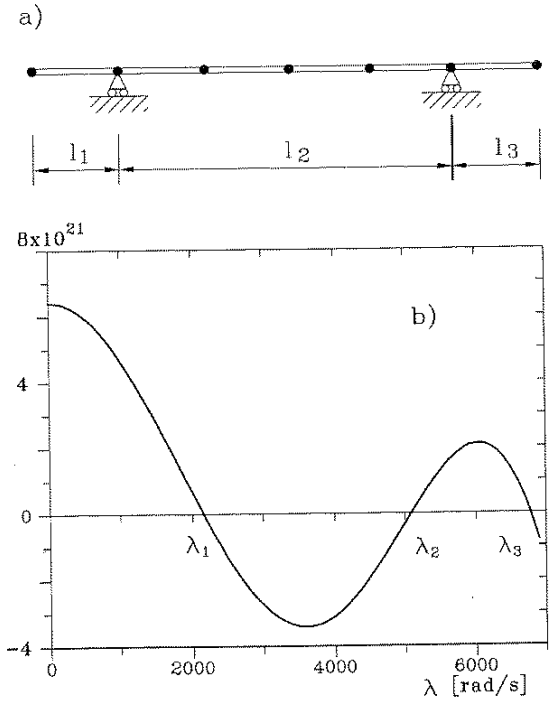


FIGURE 14.5. (a) Sketch of the beam of Example 14.4. (b) Plot used for the graphical computation of the natural frequencies using Myklestadt's method.

Guyan reduction considers secondary d.o.f and move the mass matrix not singolar.

Guyan reduction produces an error due to the fact that relation between the d.o.f are purely elastic

Guyan reduction involves a reduction of the number of natural frequency computable

COMPONENT MODE SYNTHESIS

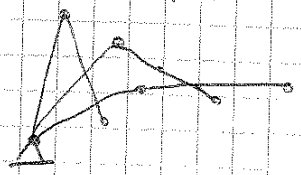
it's very similar to Guyon, but you also take in account the mode of deformation of slave.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ -K_{22}^{-1}K_{21}x_1 + \Phi\eta_2 \end{Bmatrix} = \begin{bmatrix} I & 0 \\ -K_{22}^{-1}K_{21} & \Phi \end{bmatrix} \begin{Bmatrix} x_1 \\ \eta_2 \end{Bmatrix} = \Psi \begin{Bmatrix} x_1 \\ \eta_2 \end{Bmatrix}.$$

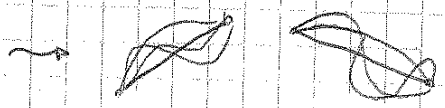
$$M^* = \Psi^T M \Psi, \quad K^* = \Psi^T K \Psi,$$
$$C^* = \Psi^T C \Psi, \quad f^* = \Psi^T f.$$

The advantage is that in some cases the configuration is variable, and this approach can save a lot of computation.

For example, let's take a robot arm



The way of vibration depend on configuration



You make the modal analysis and you get the natural frequency

The computation is more simple in this way.

COMPUTATION OF THE EIGENVALUES

They're computed by the computer (for example Matlab).
There are 3 common approaches:
* Compute only the first eigenvalue
* Compute a certain number of eigenvalues (the lowest)
* Compute all eigenvector.

The first approach was used in the past, where computers were not powerfull enough.
The algoritms for the second and third approach are completely different
The third is faster than the second. If you have to compute more than 20% of the eigenvalues, it's better to compute them all or it's faster.

Example 23.1 *A case in which the natural frequencies of the system are strongly influenced by the spin speed is that of the rotating pendulum, a pendulum attached to the outer radius of a disc rotating at a constant angular velocity (Fig. 23.1a).*
As the angular velocity Ω of the disc is imposed, the system only has 2 degrees of freedom, and angles θ (between the projection of line PC on the plane of the disc and radius OC) and ϕ (between line PC and the mentioned plane) can be assumed to be generalized coordinates.

All other force fields except the centrifugal field are neglected.
The position of point P is

$$(\overline{P\text{-}O}) = \left\{ \begin{array}{c} r\cos(\Omega t) + l\cos(\phi)\cos(\Omega t + \theta) \\ r\sin(\Omega t) + l\cos(\phi)\sin(\Omega t + \theta) \\ l\sin(\phi) \end{array} \right\}.$$

By differentiating the expressions of the coordinates with respect to time, the velocity of point P is readily obtained

$$\vec{V}_P = \left\{ \begin{array}{c} -\Omega r\sin(\Omega t) - \dot{\phi} l\sin(\phi)\cos(\Omega t + \theta) - l(\Omega + \dot{\theta})\cos(\phi)\sin(\Omega t + \theta) \\ \Omega r\cos(\Omega t) - \dot{\phi} l\sin(\phi)\sin(\Omega t + \theta) + l(\Omega + \dot{\theta})\cos(\phi)\cos(\Omega t + \theta) \\ \dot{\phi} l\cos(\phi) \end{array} \right\}.$$

The kinetic energy of the mass located in point P is simply

$$\mathcal{T} = \frac{1}{2} m |\vec{V}_P|^2 = \frac{1}{2} m \big[\Omega^2 r^2 + \dot{\phi}^2 l^2 + l^2(\Omega + \dot{\theta})^2 \cos^2(\phi) +$$

$$-2\Omega r l \dot{\phi}\sin(\phi)\sin(\theta) + 2\Omega r l(\Omega + \dot{\theta})\cos(\phi)\cos(\theta)\big].$$

The equations of motion can be easily obtained by resorting to Lagrange equations:

$$\left\{ \begin{array}{l} l\ddot{\theta}\cos^2(\phi) - 2l(\Omega + \dot{\theta})\dot{\phi}\cos(\phi)\sin(\phi) + \Omega^2 r\cos(\phi)\sin(\theta) = 0, \\ l\ddot{\phi} + l(\Omega + \dot{\theta})^2\cos(\phi)\sin(\phi) + \Omega^2 r\sin(\phi)\cos(\theta) = 0\,. \end{array} \right.$$

The equations of motion are nonlinear, but can be linearized in the study of the small oscillations of the pendulum about the static equilibrium position

$$\left\{ \begin{array}{l} l\ddot{\theta} + \Omega^2 r\theta = 0, \\ l\ddot{\phi} + \Omega^2(r + l)\phi = 0\,. \end{array} \right.$$

The linearized equations can also be obtained directly from an expression of the kinetic energy truncated after quadratic terms. By introducing the series for the sine and cosine and neglecting products of the generalized coordinates in which terms of order greater than two are contained, the kinetic energy can be written as

$$T = \frac{1}{2} m \big[\Omega^2 (r + l)^2 + \dot{\phi}^2 l^2 + \dot{\theta}^2 l^2 - \Omega^2 l(r + l)\phi^2 - \Omega^2 r l\theta^2 + 2\Omega l(r + l)\dot{\theta}\big].$$

The expression of the kinetic energy can be subdivided into three terms:

$$\mathcal{T}_0 = \frac{1}{2} m \big[\Omega^2(r + l)^2 - \Omega^2 l(r + l)\phi^2 - \Omega^2 r l\theta^2\big]$$

is independent of the generalized velocities.
Apart from a constant term, whose derivatives are nil, it yields the so-called geometric stiffness terms in the equation of motion. As usual in rotating systems, they constitute a centrifugal stiffening and are proportional to the square of the spin speed:

$$\mathcal{T}_1 = m\Omega l(r + l)\dot{\theta}$$

is linear in the generalized velocities. However, this term is independent of the displacements, and its derivatives in the equations of motion are nil; there is no gyroscopic term in the equations of the rotating pendulum:

$$\mathcal{T}_2 = \frac{1}{2} m \big[\dot{\phi}^2 l^2 + \dot{\theta}^2 l^2\big]$$

is quadratic in the generalized velocities and yields the inertia terms of the equations of motion, which are also present in natural systems.
The motion in the rotation plane xy is uncoupled, within the validity of the linearization of the equations of motion, from the motion in axial direction z. The former is the equation of motion of a pendulum whose length is l within a constant force field whose acceleration is $r\Omega^2$, and the latter is the equation motion of the same pendulum within a constant force field whose acceleration is $(r + l)\Omega^2$.

The spin speeds at w
one of the forcing fre
has a frequency coinc
with one of the natu
frequency of the syst
are usually referred
as critical speeds a
can be identified
the Campbell diagram
by intersecting the cu
related to the natura
frequencies with thos
related to the forci
frequencies

The natural frequencie
of a rotating shaft
depend on the spin
speed

Lot of machines works in supercritical range (for ex. washing machines)

INSTABILITY RANGE is not related to critical speed, it's a self excited vibration. A rotor has kinetich energy, if IT goes in exciting vibration. It has 20 times the force need to break a ring. It's a very danger condition, the faster you go, worse is the danger. All rotors have an instability range over the last critical frequency, but you cannot go behind the instability range.
It's a very easy predictable phenomenon. If s has a real part >0 or $Im(\lambda) < 0$. In many cases s is not easily computed so the model is difficult.
It occours rapidly and without any warning

SYMMETRY

* Global axial symmetry, holle the machine has an axis simmethy, but the rotor and stator, it's the simple case. I can use any reference frame
* Axial simmetry of the rotor (inertial frame to stator)
* Axial symmetry of the stator (rotor fixed frame)
* no axial symmetry, very complex equation that cannot be solved in closed form.

⚠ No real world is fully isotropic.

Axial symmetric means that a body that rotates in every angle and it's always itself. If the body is equal to itself at certain angles it's cycling symmetric.
If simmetry has got order of symmetry more than 3, it's like an axis symmetric object (which are called infinity order cycling simmetry)

MATHEMATICAL MODELLING

* frequency domain → analytical study / numerical study
* time domain → numerical study.

FREQUENCY DOMAIN

SIMPLE MODELS < JEFFCOTT (2 d.o.f)
RIGID ROTORS (4 d.o.f)

CONTINUOUS MODEL / BEAM MODELS → TRANSMITION SHAFT

MULTI D.O.F MODELS < ASSUMED MODES METHODS
LUMPED PARAMETHERS METHODS
FEM

JEFFCOTT ROTOR

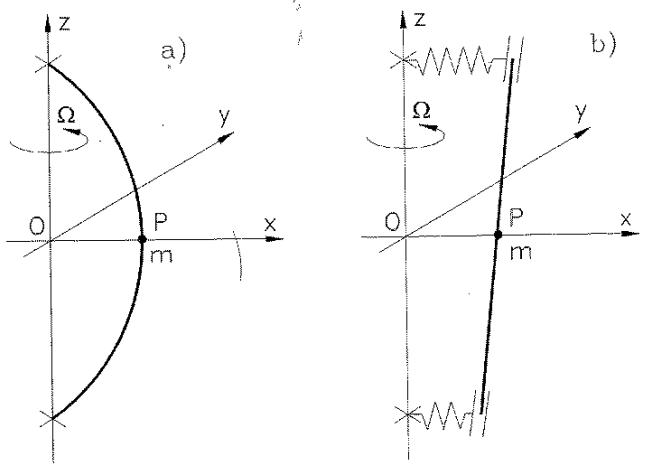


FIGURE 23.3. Sketch of a Jeffcott rotor. The model is sketched in its deformed configuration at the time in which point P crosses the xz-plane; (a) flexible shaft on stiff supports, (b) stiff shaft on compliant supports.

Real coordinates $\begin{cases} m\ddot{x} + Kx = 0 \\ m\ddot{y} + Ky = 0 \end{cases}$ Complex coordinates $z = x + iy \rightsquigarrow m\ddot{z} + Kz = 0$

$$\omega_{cr} = \sqrt{\frac{K}{m}}$$

$$z = z_1 e^{i\sqrt{\frac{K}{m}}t} + z_2 e^{-i\sqrt{\frac{K}{m}}t}$$

We can work both in x-z plane and in complex plane. The physical meaning of complex plane is completely different from what seen before. It's the projection of a rotating vector, it's a trick to describe such a vector.

z is a combination of forward and backward rotation. Real coordinates are more powerfull, as they compute orbits and vibration.

whirls

The motion z is the superimposition of a circular forward or direct motion and a circular backward motion. They both occur at an angular velocity, often called WHIRL SPEED, equal to the natural frequency of the nonrotating system

Critical speed in a Jeffcot rotor is the same critical speed of the non rotating system

Immaginary part of a rotor type $e^{\lambda t}$ represents the natural frequency

The Campbell diagram of an undamped Jeffcott rotor has two lines parallel to ω axis

The immaginary part of the natural frequency of a rotor type $e^{i\omega t}$ represents the decay rate (positive for stability)

The unbalance is proportional to $\dot{\theta}^2 - (\ddot{\theta}$ and m.

The acceptable ε is in order of μm.

We neglect the contribution of accelleration.

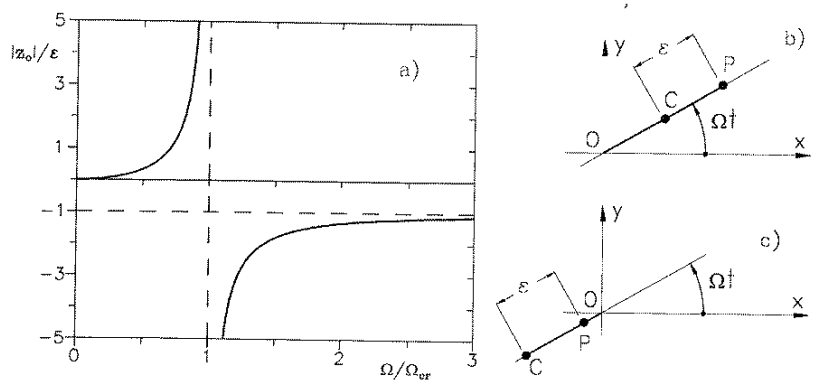


FIGURE 23.5. (a) Unbalance response of an undamped Jeffcott rotor. Nondimensional amplitude as a function of the nondimensional spin speed. Configuration of the system in (b) subcritical conditions and (c) supercritical conditions.

$$(-m\Omega^2 + k)z_0 = m\epsilon\Omega^2,$$

which yields

$$z_0 = \epsilon\frac{m\Omega^2}{k - m\Omega^2} = \epsilon\frac{\Omega^2}{\Omega_{cr}^2 - \Omega^2},$$

You build supercritical machines as they're self-balanced. The faster the better is self-centring.

I need to introduce damping to avoid the effect of critical frequency.

The sign of the solution determines the equilibrium configurations. When the solution is positive, in the subcritical fields, point O C P are alligned in the mentioned order and the centre of mass of the rotor lies outside the deformed configuration of the shaft. In the supercritical field, however, point P lies between point C and point O, and when the speed tends to infinity the amplitude z_0 tends to $-\varepsilon$ or point P tends to point O. This phenomenon is usually refered to as SELF-CENTERING because the rotor tends to rotate about it's center of mass instead of its geometrical center.

Rotor self centers at a spin speed practically infinite.

At speed higher than ω_{cr} a rather good self-centering happens.

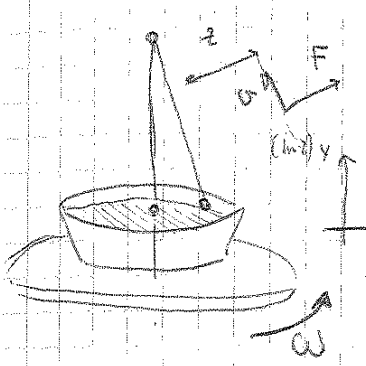
$$\vec{F}_{r_{xy}} = \vec{F}_{r_{\xi\eta}} e^{i\theta} = -c_r(\dot{z} - i\dot{\theta}z).$$

and so I get

c_n non rotating damping
c_r rotating damping

$$m\ddot{z} + (c_r + c_n)\dot{z} + (k - ic_r\dot{\theta})z = m\epsilon(\dot{\theta}^2 - i\ddot{\theta})e^{i\theta} = m\epsilon^2\omega^2$$

Let's make an example why this term come out.



A dish full of soup is put on a CD player and there's a pendulum.
There's a force due to density of the soup.

$$F = -\dot{z}\, c_r$$

We have a drag due to the fact that the soup rotates

$$v = \omega z\, c_r$$

It is 90° from z, so to rotate by 90° we multiply by i

$$\rightsquigarrow v = i\omega z\, c_r$$

Introducing the solution $z = z_0 e^{i\lambda t}$ we get

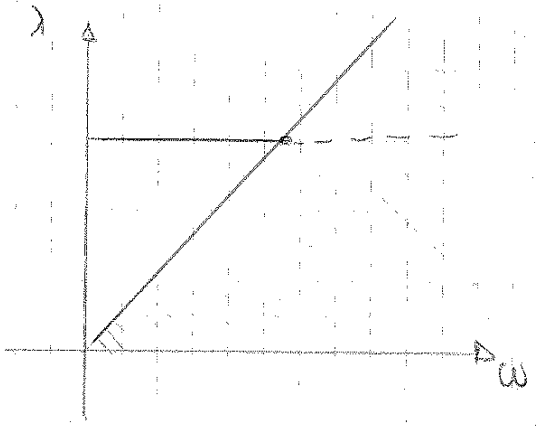
$$\left[-m\lambda^2 + i\lambda(c_n + c_r) + (k - ic_r\omega)\right] z_0 = 0$$

Solving the characteristic equation

$$-m\lambda^2 + i\lambda(c_n + c_r) + k - i\omega c_r = 0$$

$$-m\lambda^2 + i\lambda c_n + k + \underbrace{ic_r(\lambda - \omega)} = 0$$

IF $\omega > \lambda$ this is NEGATIVE! Physically it's excitation and danger of instability. Brings energy into the system from the energy field due to rotation.



Damping has no influence on the crossing of critical speeds

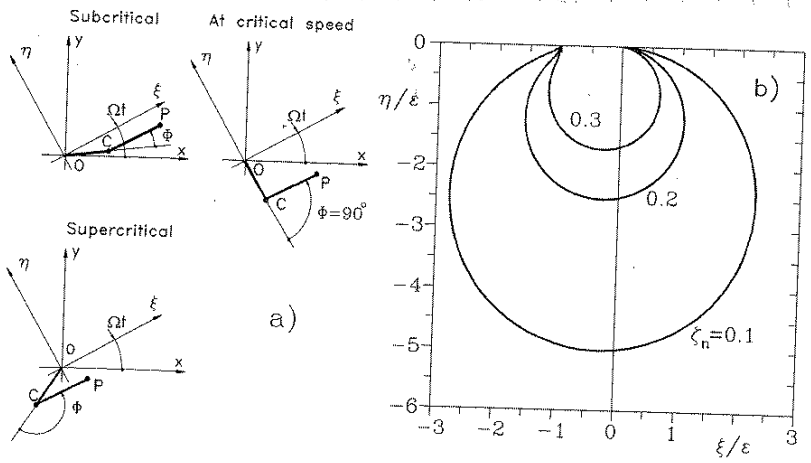


FIGURE 23.11. (a) Situation in the xy -plane in subcritical conditions, at the critical speed, and in supercritical conditions; (b) trajectories of point C in the $\xi\eta$-plane expressed in noudimensional form.

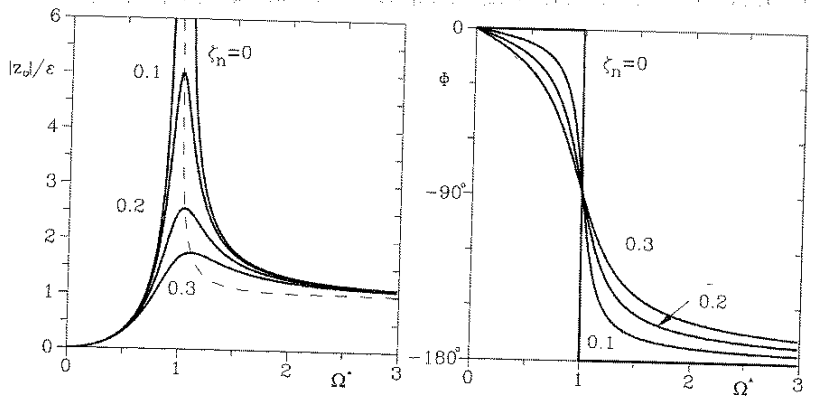
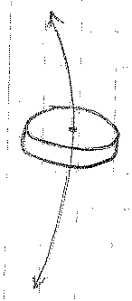


FIGURE 23.10. Nondimensional amplitude and phase of the unbalance response for three different values of nonrotating damping. Full line indicates curves for different values of damping; dashed line indicates line connecting the peaks.

MODEL WITH 4 D.O.F - GYROSCOPIC EFFECT

The Jeffcott has a mass attached at the shaft, but it's not enough, we need a solid body with its inertia.

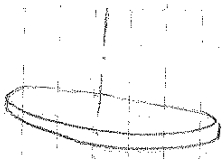


A gyroscopic model has $J_x = J_y$ and J_z

$J_x = J_y$: transversal moment of inertia; J_z: polar moment of inertia

The ratio $\frac{J_p}{J_t}$ it's basilar for the dynamic of the rotor.

A rotor can be a very thiker disk



$$J_z = J_x + J_y \rightsquigarrow \frac{J_p}{J_t} = 2$$

while a body long and narrow

$$\frac{J_p}{J_t} = 0$$

In the case of a sphere $J_z = J_x = J_y$

$$\frac{J_p}{J_t} = 1$$

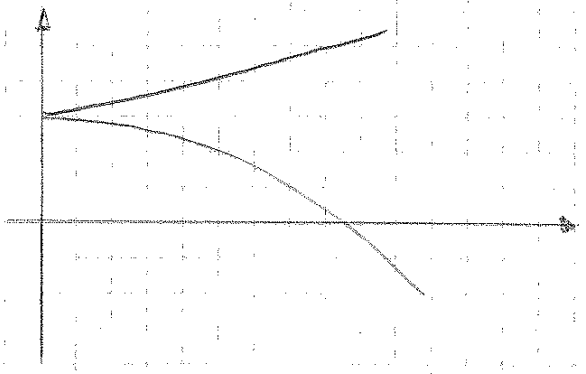
So the ratio is $0 \leq \frac{J_p}{J_t} \leq 2$. The extreme values are only theorical. For a thick disk 2 means no stiffness, so really is 1,8 while for the beam 0,2 instead of 0.

1,8 | disc rotors
1 |

In Jeffcott model, when studying the acceleration through the critical speed, the moment of inertia about the axis of rotation was introduced, but this had no effect on the flexural behaviour. Actually, the rotor's moments of inertia can considerably influence it's dynamic behaviour since they're responsible for the gyroscopic moments that cause the natural frequencies of bending modes to depend on the spin speed and the Campbell diagram to be different from a number of straight lines running in a horizontal direction

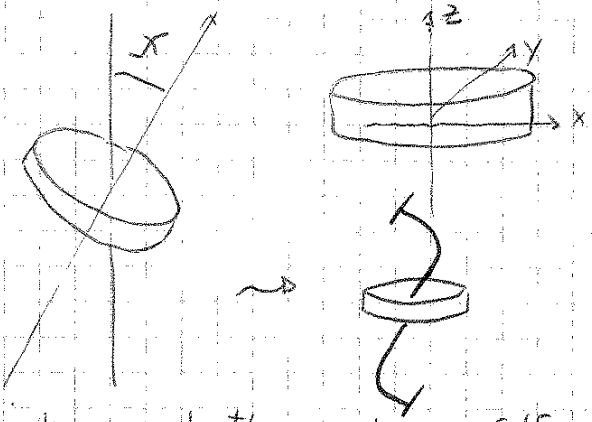
Mathematical, you always have 2 solution, but in case 1 we have 2 real sowtions, while in second case we have 1 real solution and 1 immaginary solution.

In case of presence of dumping



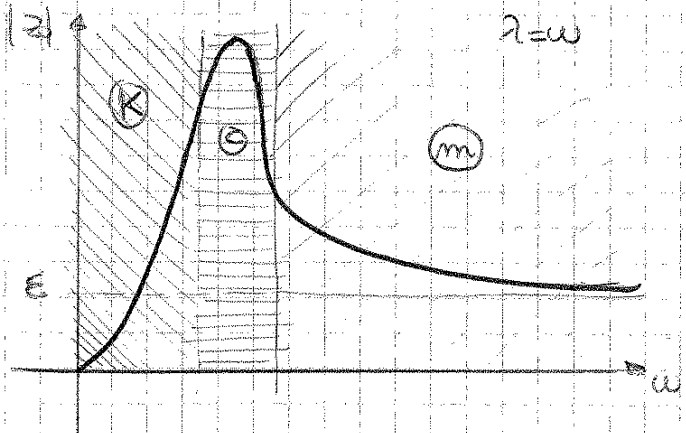
It's not changed from what we have seen in campbell diagram.

In Jeffcott we got only one mass and so only one type of unbalance. In presence of a rigid body is diffrent, it has pnncupal axys of Inertia



χ generates COUPLE UNBALANCE while ε gives STATIC UNBALANCE.

At high speed the system self-centres, but it'd be spectable.

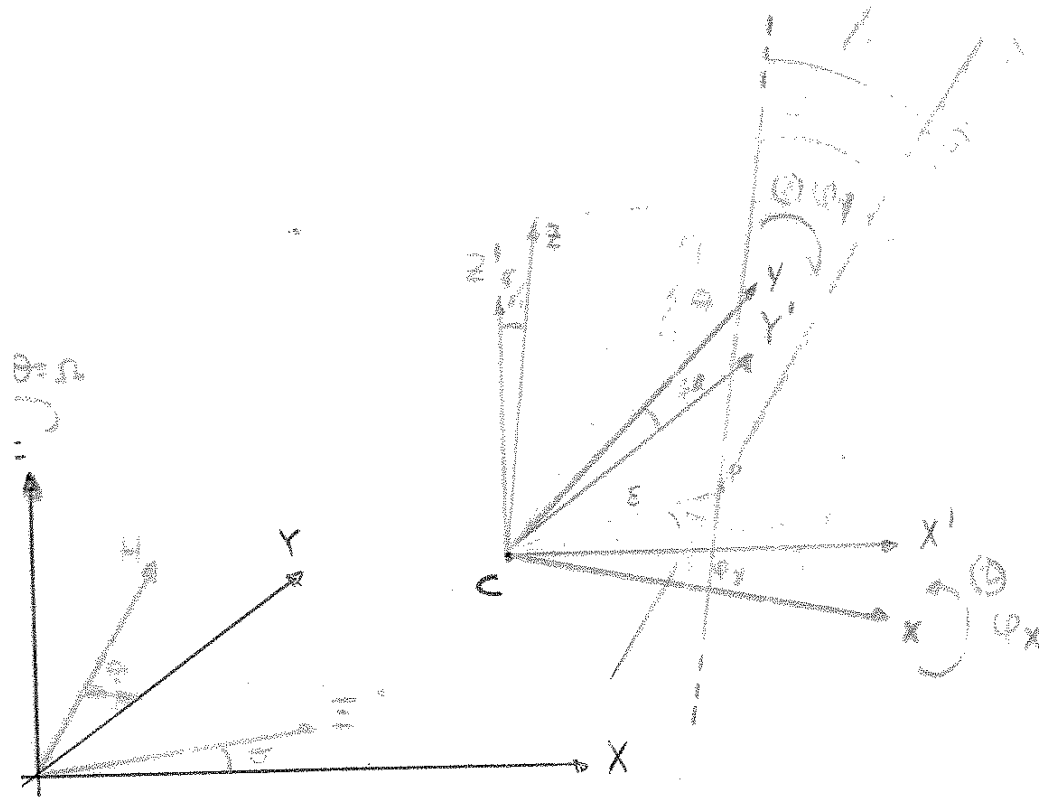


When it rotates in a range dominated by mass, the system tends to rotate about the mass centre.

At low speed it tends to rotate about the geometric axys, while at high speed it tends to rotate abot the principal axys of inertia

In a n d.o.f rotor self centes happens at a value of the response to unbalance of Π

In this case self centes happens always after the higher critical speed



We assume that rotations have small angles in order to linearize. Θ is not a small degree, only φ and χ.

As P don't coincide we have an eccentricity ε, so the rotor is not balanced. The rotors doesn't rotate exactly around its axis of inertia. R_4 is used for that.
ε is consider small, the rotor is balanced enough to work.

The position of point P is position of C plus vector $\overline{CP}$, and α is a phase, we have to add α in R_3 to get $R_{3'}$

$$\overline{P-O} = \left\{ \begin{matrix} X \\ Y \\ Z \end{matrix} \right\} + R_1^T R_2^T R'^T_3 \left\{ \begin{matrix} \varepsilon \\ 0 \\ 0 \end{matrix} \right\}$$

It's the position P in the inertial frame. R are ortonormal, so R^T and R^{-1} are the same. Doing the computation we get

$$\overline{P-O} = \left\{ \begin{matrix} X + \varepsilon \cos(\Theta + \alpha) \\ Y + \varepsilon \sin(\Theta + \alpha) \\ Z + \varepsilon \left[\frac{1}{2} \chi' \sin(\Theta + \alpha) - \frac{1}{2} \chi \cos(\Theta + \alpha) \right] \end{matrix} \right\}$$

Pay attention to the fact that I can linearize. $\overline{P-O}$ is strictly related to small unbalanced assumption.
The derivative brings us to V_P

$$V_P = \left\{ \begin{matrix} \dot{X} - \epsilon\dot{\theta}\sin(\theta+\alpha) \\ \dot{Y} + \epsilon\dot{\theta}\cos(\theta+\alpha) \\ \dot{Z} + \epsilon\left[(\dot{\theta}\phi_{X'} - \dot{\phi}_y)\cos(\theta+\alpha) + (\dot{\theta}\phi_y + \dot{\phi}_{X'})\sin(\theta+\alpha)\right] \end{matrix} \right\}$$

And so I can write kinetich energy

$$T_t = \frac{1}{2} m V_P^2 = \frac{1}{2} m \left\{ \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 + \right.$$

$$\left. + \epsilon^2\dot{\theta}^2 + \epsilon\dot{\theta}\left[-\dot{X}\sin(\theta+\alpha) + \dot{Y}\cos(\theta+\alpha) \right] \right\}.$$

It's very similar to Jeffcott rotor.

Let's write the total velocity of the rotor

$$\boldsymbol{\Omega} = \mathbf{R}_4\mathbf{R}_3\mathbf{R}_2 \left\{ \begin{matrix} \dot{\phi}_{X'} \\ 0 \\ 0 \end{matrix} \right\} + \mathbf{R}_4\mathbf{R}_3 \left\{ \begin{matrix} 0 \\ \dot{\phi}_y \\ 0 \end{matrix} \right\} + \mathbf{R}_4 \left\{ \begin{matrix} 0 \\ 0 \\ \dot{\theta} \end{matrix} \right\}$$

While $\overline{P-O}$ is written in fixed reference frame Ω it's written in the rotor reference frame. Computating it, by neglecting terms with order higher than 2,

$$\boldsymbol{\Omega} = \left\{ \begin{matrix} \dot{\phi}_{X'}\cos(\theta) + \dot{\phi}_y\sin(\theta) - \chi\dot{\theta} \\ -\dot{\phi}_{X'}\sin(\theta) + \dot{\phi}_y\cos(\theta) \\ \dot{\phi}_{X'}\left[\chi\cos(\theta) + \phi_y\right] + \dot{\phi}_y\chi\sin(\theta) + \dot{\theta} \end{matrix} \right\}.$$

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$$\begin{cases} m\ddot{X} = m\epsilon\Omega^2\cos(\Omega t+\alpha) + F_X\,, \\ m\ddot{Y} = m\epsilon\Omega^2\sin(\Omega t+\alpha) + F_Y\,, \\ J_t\ddot{\phi}_{X'} + J_p\Omega\dot{\phi}_y = -\chi\Omega^2(J_t-J_p)\sin(\Omega t) + M_{x_s}, \\ J_t\ddot{\phi}_y - J_p\Omega\dot{\phi}_{X'} = \chi\Omega^2(J_t-J_p)\cos(\Omega t) + M_y\,. \end{cases}$$

$$\begin{Bmatrix} F_X \\ M_y \end{Bmatrix} = -\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}\begin{Bmatrix} X \\ \phi_y \end{Bmatrix},$$

$$\begin{Bmatrix} F_Y \\ M_x \end{Bmatrix} = -\begin{bmatrix} K_{11} & -K_{12} \\ -K_{12} & K_{22} \end{bmatrix}\begin{Bmatrix} Y \\ \phi_{X'} \end{Bmatrix}$$

In x direction the rotation is ξ_Y,

$$\begin{cases} m\ddot{z} + K_{11}z + K_{12}\phi = m\epsilon\Omega^2 e^{i(\Omega t+\alpha)}\,, \\ J_t\ddot{\phi} - i\Omega J_p\dot{\phi} + K_{12}z + K_{22}\phi = \chi\Omega^2(J_t-J_p)e^{i\Omega t}\,, \end{cases}$$

Gyroscopic effects cause a stiffening of the system, which increase with increasing speed

that can be written in matrix form

$$\mathbf{M}\ddot{\mathbf{q}} - i\Omega\mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \Omega^2\mathbf{f}e^{i\Omega t}\,.$$

$$\mathbf{q} = \begin{Bmatrix} z \\ \phi \end{Bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & J_t \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0 & 0 \\ 0 & J_p \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}, \qquad \mathbf{f} = \begin{Bmatrix} m\epsilon e^{i\alpha} \\ \chi(J_t-J_p) \end{Bmatrix}.$$

All matrices are symmetric when using the complex coordinate notation, while when using real coordinates the gyroscopic matrix is skew symmetric

The gyroscopic matrix is real and symmetric, but gyroscopic term is immaginary.

To get the free whirling, we can write

$$\begin{cases} z_0\left(-m\omega^2 + K_{11}\right) + \phi_0 K_{12} = 0\,, \\ z_0 K_{12} + \phi_0\left(-J_t\omega^2 + J_p\omega\Omega + K_{22}\right) = 0\,. \end{cases}$$

and get the eigenproblem

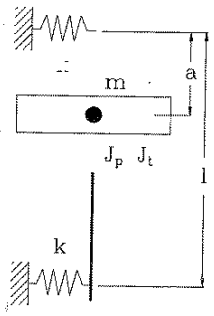
The characteristic equation allowing computation of the whirl frequency is

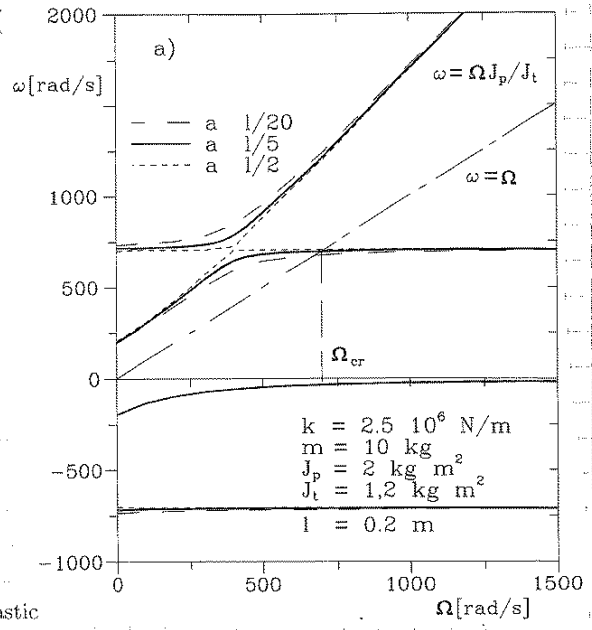
$$\det\begin{bmatrix} -m\omega^2 + K_{11} & K_{12} \\ K_{12} & -J_t\omega^2 + J_p\omega\Omega + K_{22} \end{bmatrix} = 0\,,$$

which yields

$$\omega^4 - \Omega\omega^3\frac{J_p}{J_t} - \omega^2\left(\frac{K_{11}}{m} + \frac{K_{22}}{J_t}\right) + \Omega\omega\frac{K_{11}J_p}{mJ_t} + \frac{K_{11}K_{22} - K_{12}^2}{mJ_t} = 0\,.$$

b)





URE 24.2. Campbell diagram of a system made by a rigid rotor on elastic ports.

The Jeffcott rotor and the model with 4 d.o.f. allow at least a qualitative understanding of the dynamic behavior of linear axi-symmetrical rotors, modelled as a point mass or a rigid body connected to a massless elastic shaft running in massless elastic bearings. These models are, however, inadequate to supply quantitative information on the behaviour of complex rotating systems, and more accurate models are needed in the design stage. The next step is modelling rotors as beam-like objects: the degree of freedom are the lateral displacements and the rotation of the cross section. This approach is usually referred to as 1D modelling, since the property of the rotor depend on one coordinate only, the axial coordinate z.
The 1 1/2 D is an approach in which the rotor a beamlike object, but the flexibility of the disks is accounted for in a simplified way. There's also 3D approach.
In the CONTINUUM MODELS, we can consider a rotor tha can be modelled as a beam slender enough to use the standard beam theory. If the support can be assumed to be rigid, there's no difficulty computing the natural frequencies at standstill. If the polar moment of inertia of the cross section can be neglected, gyroscopic moments do not significantly affect the dynamic behavior of the system when rotating and the natural frequencies are independent of the spin speed. The continuum approach is, however, feasible only in the case of very simple geometrical shapes such as shafts with costant cross-section. Dealing with more complicated shapes using the continuum aproach leads to analythcn difficulties, and today the most popular methods are based on some discretization techniques.

There's no difficulty in adding the contribution of rotation to the kinetic energy of any type into the FEM. The mass matrix that result from this approach is coincident with that related to the nonrotating model. Also, the stiffness and damping matrices are not affected by the fact that the system rotates, apart from a possible geometrical effect due to the stiffening of the element that can be ascribed to centrifugal stressing. This effect, which is usually neglected in the formulation of beams elements used to model rotating shafts, must be considered when dealing with beam elements in a direction perpendicular to the rotation axis. If centrifugal stiffening must be considered, it can be generally taken into account by adding a term of the type $\Omega^2 K_\Omega$ to the stiffness matrix, where K_Ω is a matrix of constants that can be computed at the element level and then assembled in the usual way. The greatest difference between a finite element model for a rotor and that for