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A P P U N T I

STUDENTE : Guardini

MATERIA : Analisi Matematica I - Inglese
Prof.

FUNCTIONS

- f defined on X with values in Y

  to each $x \in X \to$ at most one element $y \in Y$

- if $Y = \mathbb{R}$ the function is called real or real-valued

  $X = \mathbb{R}$ one real variable

ex.

i) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ (a, b real coefficients)

ii) $f: \mathbb{R} \setminus \{0\} \subset \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x}$

iii) $f: [0,3] \to \mathbb{R}$

$$f(x) = \begin{cases} 3x & \text{if } 0 \le x \le 1 \\ 4 - x & \text{if } 1 < x \le 2 \\ x - 1 & \text{if } 2 < x \le 3 \end{cases}$$ PIECEWISE FUNCTION

- Different kind of functions:

  - ABSOLUTE VALUE:

    $$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

  - SIGN:

    $$f: \mathbb{R} \to \mathbb{Z}, \quad f(x) = \text{sign}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$ increasing

  - INTEGER PART:

    $f: \mathbb{R} \to \mathbb{Z}$, $f(x) = [x]$ = the greatest integer $\le x$ increasing

    $$[x] \le x \le [x] + 1$$

  - MANTISSA:

    $f: \mathbb{R} \to \mathbb{R}$, $f(x) = M(x) = x - [x]$ $\quad 0 \le M(x) < 1$ not strictly increasing

SURJECTIVE & INJECTIVE FUNCTIONS; INVERSE FUNCTION

- SURJECTIVE: each $y \in Y$ is the image of one element $x \in X$

  ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ $a \neq 0$

  $$x = \frac{y-b}{a}$$

- ONE-TO-ONE: every $y \in \operatorname{im} f$ is the image of a unique element $x \in \operatorname{dom} f$

  $$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$
  $$x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$$

- INJECTIVE: each element y has an unique x in the domain with $f(x) = y$

- INVERSE FUNCTION: has the image of f as its domain and the domain as its range

  $$\operatorname{dom} f^{-1} = \operatorname{im} f \quad \operatorname{im} f^{-1} = \operatorname{dom} f$$

MONOTONE FUNCTIONS

- INCREASING: given x_1, x_2 in I with $x_1 < x_2$, one has $f(x_1) \leq f(x_2)$

  $$\forall x_1, x_2 \in I \quad x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

- STRICTLY INCREASING: $\forall x_1, x_2 \in I \quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

- (STRICTLY) MONOTONE: if it is either (strictly) increasing or (strictly) decreasing on I

  : an interval where f is monotone is said interval of monotonicity

ex.

i) $f(x) = ax + b$ $a > 0$ strictly increasing

$a = 0$ constant

$a < 0$ strictly decreasing

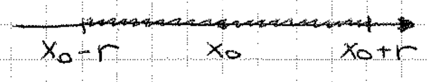
N.B. If f is strictly monotone on its domain, then f is one to one (not the reverse)

LIMITS AND CONTINUITY I

- NEIGHBOURHOOD: $x_0 \in \mathbb{R}$, $r > 0$ real number

$$\Downarrow$$

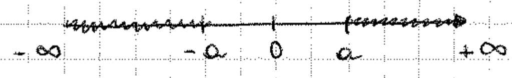
$$I_r(x_0) = (x_0 - r, x_0 + r) = \{x \in \mathbb{R} : |x - x_0| < r\}$$



: $a \geq 0$, $+\infty / -\infty$ as end-point

$$I_a(+\infty) = (a, +\infty)$$

$$I_a(-\infty) = (-\infty, -a)$$



Limit of a sequence

- $a : n \to a$

We fix $\varepsilon > 0$, from n_ε onwards all values a_n approximate 1 with a margin smaller than ε.

$$\Downarrow$$

$$\underbrace{\forall \varepsilon > 0}_{\text{however small}} \quad \exists\, n_\varepsilon : n > n_\varepsilon \Rightarrow |\underbrace{a_n}_{\frac{n}{n+1}} - 1| < \varepsilon$$

$$\frac{n}{n+1} - 1 = \frac{n-n-1}{n+1} = \left| -\frac{1}{n+1} \right| < \varepsilon$$

① So $\lim_{n \to \infty} a_n = \ell$ converges to ℓ if for any real number $\varepsilon > 0$

$$\exists \text{ an integer } n_\varepsilon : \forall n \geq n_0,\ n > n_\varepsilon \Rightarrow |a_n - \ell| < \varepsilon$$

② $\lim_{n \to \infty} a_n = +\infty$: diverges to $+\infty$ if for any real $A > 0$

$$\exists \text{ an } n_A : \forall n \geq n_0,\ n > n_A \Rightarrow a_n > A$$

$$a_n < -A \to = -\infty$$

- $\lim_{x \to x_0} f(x) = +\infty \quad \forall A > 0 \ \exists \delta > 0:$

$$\forall x \in \text{dom} f, \ 0 < |x - x_0| < \delta \Rightarrow f(x) > A$$

DISCONTINUITIES

1. DISCONTINUITY OF THE FIRST KIND:

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x) = [f]_{x_0}$$

2. DISCONTINUITY OF THE SECOND KIND:

no removable disc. points $\lim_{x \to x_0^\pm} f(x) = \pm\infty$

LIMITS OF MONOTONE FUNCTIONS

- f is a monotone funct. defined on a right neighbourhood $I^+(c)$:

$$\lim_{x \to c^+} f(x) = \begin{cases} \inf\{f(x) : x \in I^+(c), \ x > 0\} & f \text{ is increasing} \\ \sup\{f(x) : x \in I^+(c), \ x > 0\} & f \text{ is decreasing} \end{cases}$$

- f monotone on a left neighbourhood $I^-(c)$:

$$\lim_{x \to c^-} f(x) = \begin{cases} \sup\{f(x) : x \in I^-(c), \ x < 0\} & f \text{ is increasing} \\ \inf\{f(x) : x \in I^-(c), \ x < 0\} & f \text{ is decreasing} \end{cases}$$

Corollary

i) if f is increasing

$$\lim_{x \to x_0^-} f(x) \leq f(x_0) \leq \lim_{x \to x_0^+} f(x)$$

ii) if f is decreasing

$$\lim_{x \to x_0^-} f(x) \geq f(x_0) \geq \lim_{x \to x_0^+} f(x)$$

- COMPARISON THEOREM 2nd (INFINITE CASE)

f, g are given and $\lim_{x \to c} f(x) = +\infty$

if $\exists I(c)$ of c where f, g are defined:

$$f(x) \leq g(x) \quad \forall x \in I(c) \setminus \{c\}$$

then $\lim_{x \to c} g(x) = +\infty$

- SUBSTITUTION THEOREM

$\lim_{x \to c} f(x) = \ell$ admits a finite or not limit

Let g be defined on $I(\ell)$:

i) if $\ell \in \mathbb{R}$ is continuous at ℓ

ii) if $\ell = \pm\infty$, the $\lim_{y \to \ell} g(y)$ $\exists$

then $g \circ f$ $\lim_{x \to c} g(f(x)) = \lim_{y \to \ell} g(y)$

$$= g\left(\lim_{x \to c} f(x)\right)$$

GLOBAL FEATURES OF CONTINUOUS MAPS

- ZERO: of a real-valued function f is a point $x_0 \in \text{dom} f$ at which the function vanishes.

- EXISTENCE OF ZEROES THEOREM

f continuous on a closed, bounded interval $[a, b]$. If

$$f(a) f(b) < 0 \quad \text{i.e}$$

if the images of the endpoints under f have different signs f admits a zero within the open interval (a, b)

If f is strictly monotone on $[a, b]$ the zero is unique.

LOCAL COMPARISON OF FUNCTIONS. NUMERICAL SEQUENCES AND SERIES.

Landau symbols

- THE SAME ORDER OF MAGNITUDE AS g for x tending to c

  $f \asymp g, \; x \to c$

- f EQUIVALENT TO g (if $\ell=1$) for x tending to c

  $f \sim g, \; x \to c$

- f is NEGLIGIBLE WITH RESPECT to g ($\ell=0$) for x tending to c

  $f = o(g), \; x \to c$

The algebra of "little o's"

i) $x^n, \; x \to 0$

- $x^n = o(x^m), \; x \to 0 \iff n > m$
- $\lim_{x \to 0} \frac{x^n}{x^m} = \lim_{x \to 0} x^{n-m} = 0$ if and only if $n-m>0$
  - the biggest of two powers of x is negligible.

ii) $x^n, \; x \to \pm\infty$

- $x^n = o(x^m), \; x \to \pm\infty \iff n < m$
  - the smallest of two powers of x is negligible.

$\sin x \sim x \quad x \to 0$

$1-\cos x \sim \frac{1}{2}x^2 \quad x \to 0$

$\log(1+x) \sim x \quad x \to 0$

$\log x \sim x-1 \quad x \to 1$

$e^x - 1 \sim x \quad x \to 0$

$(1+x)^\alpha - 1 \sim \alpha x \quad x \to 0$

DIFFERENTIAL CALCULUS

- $f: \text{dom} f \subseteq \mathbb{R} \to \mathbb{R}$
- $x_0 \in \text{dom} f$, f defined in $I_r(x_0)$ with $x \in I_r(x_0)$ $x \neq$

$$\Delta x = x - x_0$$

the increment of the independent variable between x_0 and x

$$\Delta f = f(x) - f(x_0)$$

the increment of the dependent variable

$$\Downarrow$$

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$f'(x_0) = \lim_{x \to x_0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

FIRST DERIVATIVE OF f AT x_0

- TANGENT LINE: $y = t(x) = f(x_0) + f'(x_0)(x - x_0)$

- DERIVATIVE OF THE INVERSE FUNCTION:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

$$Dx^\alpha = \alpha x^{\alpha - 1}$$

$$D \sin x = \cos x \qquad D \cos x = -\sin x$$

$$D \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$D \arcsin x = \frac{1}{\sqrt{1 - x^2}} \qquad D \arccos x = -\frac{1}{\sqrt{1 - x^2}} \qquad D \arctan x = \frac{1}{1 + x^2}$$

$$D\, a^x = (\log a)\, a^x$$

$$D \log_a |x| = \frac{1}{(\log a) x} \qquad D \log |x| = \frac{1}{x}$$

$$D |x| = \text{sign}(x) \quad \forall x \neq 0$$

CONVEXITY AND INFLECTION POINTS

- f CONVEX (smile) at x_0 if $\exists I_r(x_0) \subseteq \text{dom} f$:
  $\forall x \in I_r(x_0)$, $f(x) \geq t(x)$
- f STRICTLY CONVEX if $f(x) > t(x)$, $\forall x \neq x_0$

changing $\leq$ and $<$ you have the definition of CONCAVE.

- INFLECTION POINT : $\exists I_r(x_0) \subseteq \text{dom} f$ where:

$$\forall x \in I_r(x_0) \begin{cases} \text{if } x < x_0, & f(x) \leq t(x) \\ \text{if } x > x_0, & f(x) \geq t(x) \end{cases}$$

or

$$\forall x \in I_r(x_0) \begin{cases} \text{if } x < x_0, & f(x) \geq t(x) \\ \text{if } x > x_0, & f(x) \leq t(x) \end{cases}$$

- THEOREM: - f diff on I :
  a) if f is convex on I, then f' is increasing on I
  b) if f' is increasing on I, then f is convex
  c) if f' is strictly increasing, then f is strictly convex

Corollary

- f is twice differentiable around x_0
  1) if x_0 is an inflection point, then $f''(x_0) = 0$
  2) if f'' changes sign crossing x_0, then x_0 is an inflection point (ascending if $f''(x) \leq 0$ at the left, descending if $f''(x) \geq 0$ at its right)

THEOREM OF DE L'HÔPITAL

- f, g defined on $I(c) \setminus \{c\}$:

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L \qquad L = 0, +\infty \text{ or } -\infty$$

- if f and g are differentiable around c, with $g' \neq 0$ and if

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$$\exists \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

TAYLOR EXPANSIONS AND APPLICATIONS

- $f(x) = T f_0, x_0 (x) + o(1) \quad x \to x_0$

$\Downarrow$ with f cont. and differentiable at x_0

$f(x) = T_{f_1, x_0}(x) + o(x - x_0) \rightarrow T f_{1, x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$

- THEOREM OF TAYLOR FORMULA:

$n \geq 0$ and f diff. at x_0

$$f(x) = T f_{n, x_0}(x) + o((x - x_0)^n), \quad x \to x_0$$

- $e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^k}{k!} + \dots + \frac{x^n}{n!} + o(x^n)$
- $\log(1+x) = x - \frac{x^2}{2} + \dots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5} - \dots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots + \binom{\alpha}{n} x^n + o(x^n)$
- $\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^n x^n + o(x^n)$
- $\sqrt{1+x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 + o(x^3)$
- $\tan x = x + \frac{x^3}{3}$

COMPLEX NUMBERS - INTRODUCTION

- A complex number z is an ordered pair (x,y) with $x,y \in \mathbb{R}$

  $\mathbb{C} \overset{def}{\equiv} \{\text{complex numbers}\}$

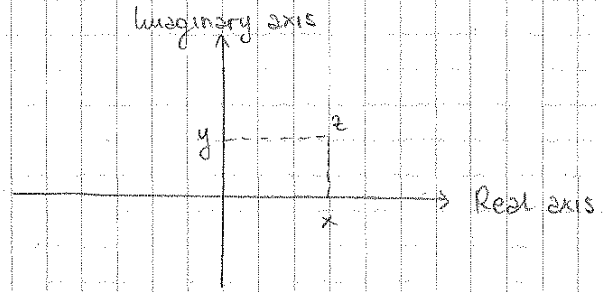
  = can be identified with $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

- Cartesian form or algebraic form of $z = (x,y)$

  $(x,y) \to x + iy$ where $i = (0,1)$ is called the IMAGINARY UNIT

  x = Real part of z $\quad x = Re(z)$

  y = Imaginary part of z $\quad y = Im(z)$



ARGAND-GAUSS PLANE

ALGEBRAIC OPERATIONS

$z_1 = x_1 + iy_1 \quad (x_1, y_2) \qquad z_2 = x_2 + iy_2 \quad (x_2, y_2)$

$$z_1 = z_2 \iff \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

$\alpha \in \mathbb{R}$

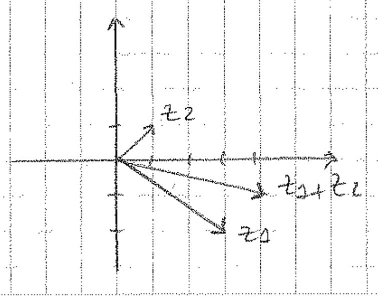
$$\alpha z_1 \overset{def}{\equiv} \alpha x_1 + i \alpha y_1$$

<u>SUM</u>: $z_1 + z_2 \overset{def}{\equiv} (x_1 + x_2) + i(y_1 + y_2)$

ex.

$z_1 = 3 - 2i \quad z_2 = 1 + i$

$z_1 + z_2 = 4 - i$



$$y = -\frac{b}{a} \frac{a}{a^2+b^2} = -\frac{b}{a^2+b^2}$$

$$\Rightarrow w = \frac{1}{z} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

5 QUOTIENT: $z_2 \neq (0,0)$ $\quad \frac{z_1}{z_2} \stackrel{def}{=} z_1 \frac{1}{z_2}$

6 POWERS OF i: $\quad i^2 = -1 \quad i^3 = i^2 \cdot i = -i \quad i^4 = 1$

Modulus

$|z| = \sqrt{x^2+y^2}$ distance from the origin

- PROPERTIES:

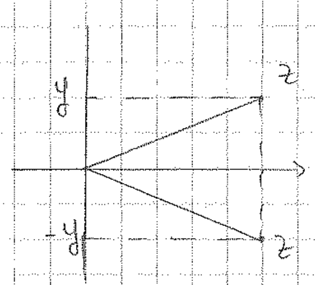
$|z| \geq 0, \quad |z| = 0 \Leftrightarrow z = 0$

$|z_1 z_2| = |z_1||z_2|$

$|z_1 + z_2| \leq |z_1| + |z_2|$

Complex conjugate

$z = x + iy \qquad \bar{z} = x - iy$



PROPERTIES:

- $z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2+y^2 = |z|^2$

- $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

- $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

- $z = x + iy$, $\bar{z} = x - iy$ $\qquad 2x = z + \bar{z} \qquad \underset{Re(z)}{x} = \frac{z+\bar{z}}{2}$

- $2iy = z - \bar{z} \qquad y = \frac{z-\bar{z}}{2i} \qquad Im(z) = \frac{z-\bar{z}}{2i}$

ALGEBRAIC OPERATIONS - FORMULAS

- TRIGONOMETRIC FORM: $z = r(\cos\vartheta + i \operatorname{sen}\vartheta)$

- CARTESIAN FORM: $z = x + iy$
- EXPONENTIAL FORM: $e^{i\vartheta} = \cos\vartheta + i \operatorname{sen}\vartheta$

EXAMPLE

$z = 1 + i \qquad |z| = \sqrt{2}$

$Re = x = 1$

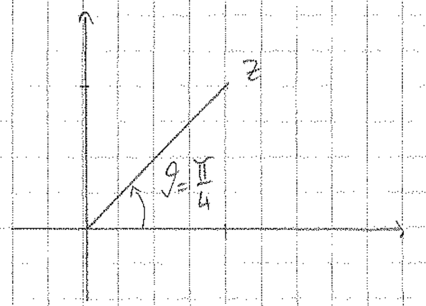
$\Rightarrow |z| = \underbrace{\sqrt{1+1}}_{x^2+y^2} = \underbrace{\sqrt{2}}_{r} \qquad r = \sqrt{2}$

$Im = y = 1$

$\cos\vartheta = \underbrace{\frac{1}{\sqrt{2}}}_{\frac{x}{r}} = \frac{\sqrt{2}}{2} \qquad \sin\vartheta = \underbrace{\frac{1}{\sqrt{2}}}_{\frac{y}{r}} = \frac{\sqrt{2}}{2} \qquad \Rightarrow \vartheta = \frac{\pi}{4}$

$z = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ TRIG. FORM



- ROOT OF COMPLEX NUMBER: $z^n = w$

$w = s(\cos\varphi + i\sin\varphi)$

$zw = rs(\cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi))$

- DE MOIVRE FORMULA: $z^4 = r^4(\cos 4\vartheta + i\sin 4\vartheta)$

$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

Complex roots using exponential form

$z = r(\cos\vartheta + i\sin\vartheta) = re^{i\vartheta}$

$z^n = r^n e^{in\vartheta}$

exercise

$z^2 + |z|^2 = i$

$z^2 = (x+iy)^2 = x^2 + (iy)^2 + 2ixy$

$= x^2 - y^2 + 2ixy$

$|z|^2 = (\sqrt{x^2+y^2})^2 = x^2+y^2$

$\Rightarrow x^2 - y^2 + 2ixy + x^2 + y^2 = i$

$2x^2 + 2ixy = i$

$\begin{cases} 2x^2 = 0 \\ 2ixy = 1 \end{cases} \Longrightarrow$ find the solutions $\Rightarrow \begin{cases} x = 0 \\ 0 = 1 \text{ IMP.} \end{cases}$ NO SOL

$\boxed{\text{N.B}}$

$|z| = 1 \Rightarrow$ circle

$|z| < 1 \Rightarrow$ points in the circle

$|z| \geq 1 \Rightarrow$ " " " " and on it.

Fundamental theorem of algebra

- Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ (poly of degree n) with $a_0, a_1, \dots, a_n \in \mathbb{C}$, $a_n \neq 0$

- THEOREM
  - $\exists$ $m \leq n$ distinct complex numbers $z_1, \dots, z_m$ (roots)
  - $\exists$ $m \neq 0$ natural numbers $\mu_1 \dots \mu_m$ (multiplicity) :

  $p(z)$ factorizes as $p(z) = a_n (z-z_1)^{\mu_1} (z-z_2)^{\mu_2} \cdot \dots \cdot (z-z_m)^{\mu_m}$

- THEOREM:

If f admits a primitive F on I then any primitive of f on is

$$F(x)+c \qquad c \in \mathbb{R}$$

① If $F(x)$ is a primitive of f in I

$\Rightarrow F(x)+c$ is a primitive

$$\begin{cases} F'(x) = f(x) \\ F(x) \text{ DIFF.} \end{cases} \rightarrow \begin{cases} (F(x)+c)' = f(x) \\ F(x)+c \text{ DIFF.} \end{cases}$$

② Suppose $G(x)$ is a primitive $\rightarrow G(x) = F(x)+c$

$$G'(x) = f(x) \qquad F'(x) = f(x)$$

- $H(x) = G(x) - F(x) \qquad H'(x) = (G(x) - F(x))' = G'(x) - F'(x)$

$$= f(x) - f(x) = 0$$

- $H'(x) = 0 \rightarrow H(x) = c$ } on interval

$$\Downarrow$$

$$G(x) - F(x) = c$$

$$G(x) = F(x) + c$$

- Given I int. the set of all primatives of $f(x)$ in I

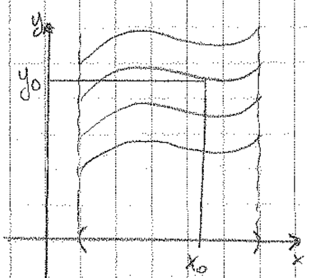
$\int f(x)\,dx$ INDEFINITE INTEGRAL OF $f(x)$ in I

$\int f(x)\,dx = \{F(x) + c \, ; \, c \in \mathbb{R}\}$ set of inf. nb. functions

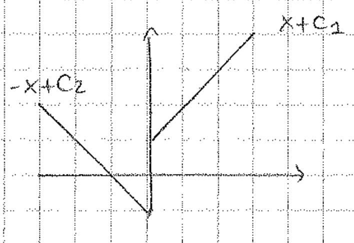
il grafico trasla

DUMMY VARIABLE (muta)

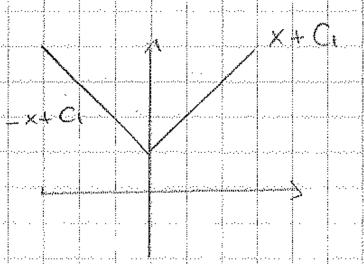
Graph



$F(x)$ primitive of $f(x)$ in $I \rightarrow$ the function cover completely the interval.



$F'(x) = \text{sign}\,x \quad \forall x \neq 0$



cont. at x_0 (except 1), not diff. in zero

$$\lim_{x \to 0^+} x + c_1 = \lim_{x \to 0^-} -x + c_2$$

$$c_1 = c_2$$

There are infinitely many functions

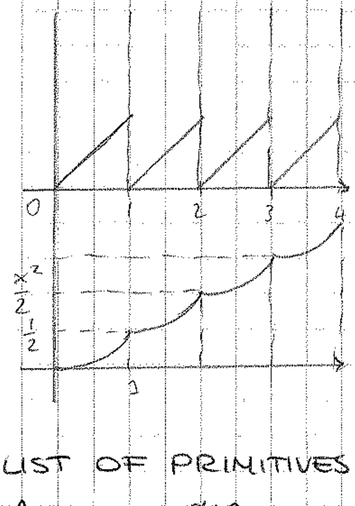
$$F(x) = \begin{cases} x + c_1 & x \geq 0 \\ -x + c_1 & x < 0 \end{cases}$$ that are generalized primitives of sign$x$ in $\mathbb{R}$

$F(0) = 0 \Rightarrow F(x) = |x|$ ('cause $c = 0$, so $x > 0$ and $x < 0$)

$$\int \text{sign}\, dx = |x| + c$$

example ②

- Mantissa function



$\left(1, \frac{1}{2}\right)$ = CORNER POINT

- LIST OF PRIMITIVES:

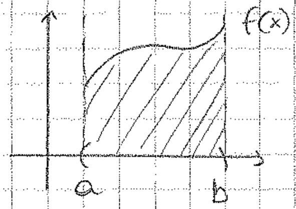
$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c$ $\quad$ $\int e^x dx = e^x + c$ $\quad$ $\int \cos x\, dx = \sin x + c$

$\int \frac{1}{x} dx = \log|x| + c$ $\quad$ $\int \frac{1}{1+x^2} dx = \arctan x + c$

$\int \sin x\, dx = -\cos x + c$ $\quad$ $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$

DEFINITE INTEGRAL

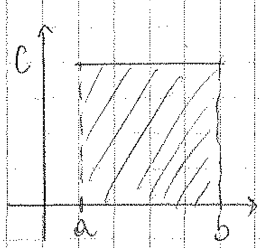
- Area of particular subsets of the plane ($\mathbb{R}^2$)



→ CAUCHY f. continuous

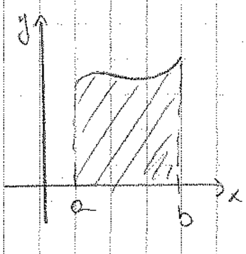
RIEMANN f. bounded

LEBESGUE f. measurable function

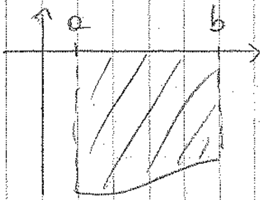


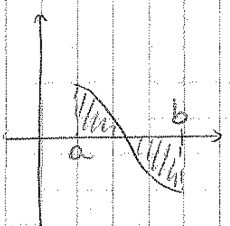
$C(b-a)$

- f defined in $[a,b]$
- f bounded on $[a,b]$

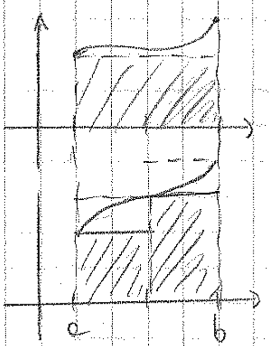


- $T(f;a,b)$ TRAPEZOIDAL REGION
- $\left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} a \le x \le b \\ 0 \le y \le f(x) \ \text{in} \\ f(x) \le y \le 0 \end{array} \right\}$



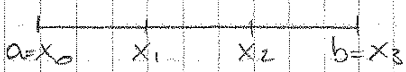


- $T(f;a,b)$ → NUMBER
- $f(x) \ge 0$ → area of $T(f;a,b)$



APPROX

$[a,b]$ $\quad a = x_0 < x_1 < \dots < x_n = b$

- 

PARTITION OF $[a,b]$

- $[x_k, x_{k+1}]$

- UPPER INTEGRAL of f:

$$\overline{\int_I} f = \cancel{\inf}\ \sup \left\{ \int_I h : h \in S^+ f \right\}$$

- LOWER INTEGRAL of f:

$$\underline{\int_I} f = \cancel{\sup}\ \inf \left\{ \int g : g \in S^-_f \right\}$$

$$\Downarrow$$

THEOREM: $\underline{\int} f \leq \overline{\int} f$

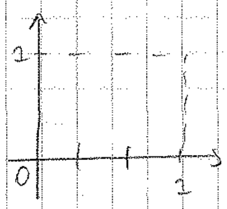
- DEFINITE INTEGRAL:

If $\underline{\int_I} f = \overline{\int_I} f \Rightarrow$ f IS INTEGRABLE OVER $I = [a,b]$

and $\int_I f = \underline{\int_I} f = \overline{\int_I} f$

- There are functions that aren't riemann integrals:

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$$



$h \in S^+_f \Rightarrow \int h \geq 1$

$g \in S^- f \Rightarrow \int g \leq 0$

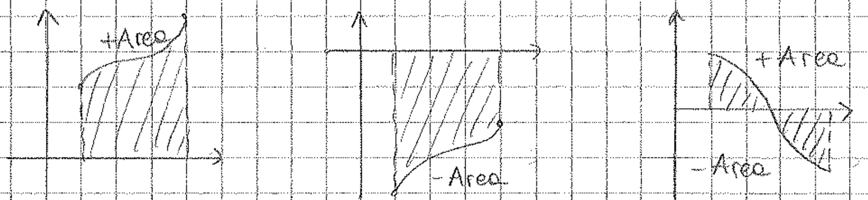
$\overline{\int_{[a,b]}} f = 1 \qquad \underline{\int_{[a,b]}} f = 0$

$\underline{\int_{[a,1]}} f < \overline{\int_{[a,1]}} f$

$0 < 1$

GEOMETRICAL INTERPRETATION OF RIEMANN ORIENTED INT.

$$\int_{[a,b]} f \quad \begin{cases} \text{AREA of } T(f;a,b) & \text{if } f(x) \geq 0 \\ \text{AREA } \text{"} \quad \text{"} & \text{if } f(x) \leq 0 \end{cases}$$

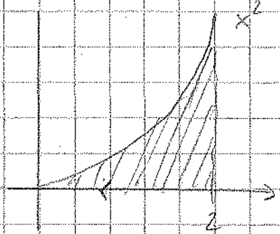


example ①

$$\int_2^0 x^2\,dx = -\int_0^2 x^2\,dx$$

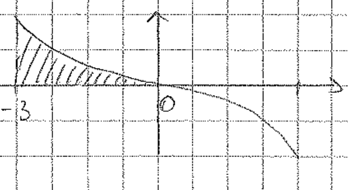
$$= -\int_{[0,2]} x^2\,dx$$

$$= -\text{Area}$$



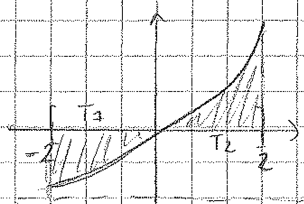
example ②

$$\int_{-3}^0 -x^3\,dx = \text{area}$$



example ③

Area of $T(f;a,b)$ $\quad f(x) = e^x - 1 \quad I = [-2,2]$



$$\int_{-2}^2 (e^x - 1)\,dx = \text{Area}(T_2) - \text{Area}(T_1)$$

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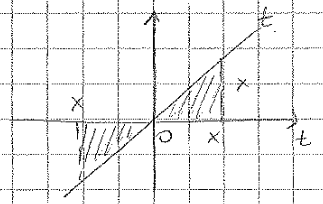
Corollary

$F_{x_0}(x) = \int_{x_0}^{x} f(t)\,dt$ INTEGER FUNCTION

- $F_0(x) = \int_0^x t\,dt$ $x \in \mathbb{R}$

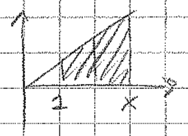


$x=0$ $F_0(0) = \int_0^0 t\,dt = 0$

$x>0$ $F_0(x) = \frac{x^2}{2}$ ('cause it's a triangle)

$x<0$ $F_0(x) = \int_0^x t\,dt = -\int_x^0 t\,dt = -(-\text{Area}\,T) = \text{Area}\,T = \frac{(-x)(-x)}{2} = \frac{x^2}{2}$

- $F_1(x) = \int_1^x t\,dt$



$x>1$ $F_1(x) = \int_1^x t\,dt = \frac{(x+1)(x-1)}{2} = \frac{x^2-1}{2} = \frac{x^2}{2} - \frac{1}{2}$

$x=1$ $F_1(1) = 0$

$0<x<1$ $F_1(x) = \int_1^x t\,dt = -\int_x^1 t\,dt = \frac{-(1+x)(1-x)}{2} = -\frac{1-x^2}{2} = \frac{x^2}{2} - \frac{1}{2}$

$x<0$ $\int_0^x t\,dt = -\int_x^1 t\,dt = -\left[\int_x^0 t\,dt + \int_x^1 t\,dt\right] = -\left[-\frac{x^2}{2} + \frac{1}{2}\right] = \frac{x^2}{2} - \frac{1}{2}$

example ①

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx \qquad \alpha > 0$$

- $\alpha = 1$ $\quad \displaystyle\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{c\to+\infty} \int_1^c \frac{1}{x}\,dx = \lim_{c\to+\infty} \Big[\ln x\Big]_1^c = \lim_{c\to+\infty} \ln c = +\infty$

- $\alpha \neq 1$ $\quad \displaystyle\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \lim_{c\to+\infty} \int_1^c x^{-\alpha}\,dx = \lim_{c\to+\infty} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^c$

$$= \lim_{c\to+\infty} \frac{c^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}$$

$$= \begin{cases} -\alpha+1 > 0 \Rightarrow \alpha < 1 & +\infty \\ -\alpha+1 < 0 \Rightarrow \alpha > 1 & \dfrac{1}{\alpha-1} \end{cases}$$

- $\displaystyle\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ $\begin{cases} \text{DIVERGES IF } \alpha \leq 1 \\ \text{CONVERGES IF } \alpha > 1 \end{cases}$

example ②

$f(x) \geq 0 \qquad F_a(c) = \displaystyle\int_a^c f(x)\,dx$

- $F'_a(c) = f(c) \geq 0 \Rightarrow F_a(c)$ MON. INCREASING IN $[a, +\infty)$

$\displaystyle\int_a^{+\infty} f(x)\,dx$ $\begin{cases} \text{CONVERG.} \\ \text{DIVERG. (to } +\infty) \end{cases}$ $\Leftarrow \displaystyle\lim_{c\to+\infty} F_a(c)$ $\begin{cases} \ell \in \mathbb{R} \\ +\infty \end{cases}$

In this case $g(x) = \frac{1}{x^\alpha} \,//\, \frac{1}{x^\beta}$

1) $f(x) \overset{\cup}{\sim} \frac{1}{x^\alpha} \Rightarrow \lim\limits_{x \to +\infty} \frac{f(x)}{\frac{1}{x^\alpha}} = l > 0$

loc. bound th $\Rightarrow$ $\frac{f(x)}{1/x^\alpha} \leq \frac{l}{2}$ $(x \to +\infty)$ $f(x) \leq \frac{l}{2} \frac{1}{x^\alpha} \Rightarrow \int f(x)\,dx$ CONV.

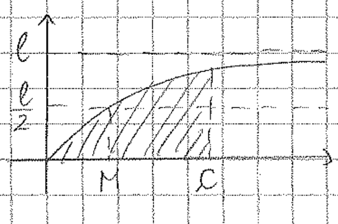
$f(x) = o\left(\frac{1}{x^\alpha}\right) \Rightarrow \lim\limits_{x \to +\infty} \frac{f(x)}{\frac{1}{x^\alpha}} = 0$

loc. bo. th. $\Rightarrow$ $\frac{f(x)}{\frac{1}{x^\alpha}} \leq k > 0 \to f(x) \leq k \frac{1}{x^\alpha}$

• THEOREM:

$f \in R_{loc}([a, +\infty))$: $\lim\limits_{x \to +\infty} f(x)$ $\exists$ and is $\neq 0$

$\Rightarrow \int_a^{+\infty} f(x)\,dx$ DIVERGES



$\lim\limits_{x \to +\infty} f(x) = l > 0$ $(+\infty)$

$\Downarrow$ loc. bound. theorem

$\lim\limits_{x \to +\infty} f(x) = l > 0$

$\exists\, M > 0 : f(x) > \frac{l}{2} \quad \forall x > M$

$$\int_0^{+\infty} f(x)\,dx = \underbrace{\int_0^M f(x)\,dx}_{\text{NUMBER}} + \int_M^{+\infty} f(x)\,dx$$

- PROP. OF INT. AVERAGE THEOREM:

1. f. INT. FUNC. on $[a,b]$

$$\inf_{x\in[a,b]} f(x) \le m(f;a,b) \le \sup_{x\in[a,b]} f(x)$$

2. f. CONT. FUNCT. on $[a,b]$

$$\exists\, z \in [a,b] : \quad m(f;a,b) = f(z)$$

- PROOF

1. $$\underbrace{\inf_{x\in[a,b]} f(x)}_{if} \le f(x) \le \underbrace{\sup_{x\in[a,b]} f(x)}_{sf} \qquad \forall x \in [a,b]$$

$$\int_a^b if\,dx \le \int_a^b f(x)\,dx \le \int_a^b sf\,dx \qquad \text{MONOTONICITY}$$

$$if(b-a) \le \int_a^b f(x)\,dx \le sf\,(b-a)$$

divide $(b-a)$

$$if \le m(f;a,b) \le sf$$

2. sf continuous, I apply the Weierstrass theorem:

the absolute minimum $\le m(f;a,b) \le$ the absolute maximum

$$m \le m(f;a,b) \le M$$

$$x+h \leq z(h) \leq x \quad || \quad x \leq z(h) \leq x+h$$

- $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(z(h))$ composition of function h gives me the position of z, composed with f.

$= f\left(\lim_{h \to 0} z(h)\right)$

$= f(x)$ 'cause the second comp. th.

- TORRICELLI - BARROW TH:

① Suppose that $G(x)$ is any primitive of f on I

② Find the primitive of f that vanishes at x_0

$$\underset{\text{is zero}}{\int_{x_0}^{x} f(t)dt} \quad || \quad \underbrace{G(x) - G(x_0)}_{\text{= only one primitive}}$$

$a, b \in I \qquad \int_a^b f(t)dt = G(b) - G(a)$ DEFINITE INTEGRAL

$$\left[G(x)\right]_a^b = G(x)\Big|_a^b$$

Corollary

- $f \in C^1(I) \Rightarrow f' \in C^0(I)$

$$\int_{x_0}^{x} f'(t)dt = \underset{\text{at } x}{f(x)} - \underset{\text{at } x_0}{f(x_0)}$$

$\Delta f = f'(x_0) \cdot \Delta x + o(\Delta x)$ (PEANO)

$\Delta f = f(t)\, \Delta x$ (LAGRANGE)

- How to compute the costants A, B, C? by using the principle of identity of polynomials:

two polynomials of the same degree $m-1$ coincide if and only if either of the next conditions hold:

•) the coefficients of the corresponding monomials coincide

•) the polynomials assume the same values at m distinct points

Example ①

$$\int \frac{x+5}{x^2+x-2}\,dx$$

$$Q(x) = x^2+x-2 = (x+2)(x-1)$$

$$\frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{A(x-1)\quad B(x+2)}{(x+2)(x-1)}$$

Compute A and B

Method ①: $A(x-1)+B(x+2) = (A+B)x + 2B-A = x+5$

$$\begin{cases} A+B=1 & 3B=6 \quad B=2 \\ 2B-A=5 & A=-1 \end{cases}$$

Method ②: $A(x-1) + B(x+2) = x+5$

$x=1$: $3B=6 \quad B=2$

$x=-2$: $-3A=-2+5=3 \quad A=-1$

$$\frac{x+5}{(x+2)(x-1)} = -\frac{1}{(x+2)} + \frac{2}{(x-1)}$$

$+ \; y''=0 \iff y=ax^2+bx+c$

- Diff. eq. must be satysfied in an interval, not in a point
- A function $\varphi(x)$ differentiable in an interval I is a solution f d. eq. I order

$$F(x,y,y')=0 \quad \text{if}$$
$$F(x,\varphi(x),\varphi'(x))=0 \quad \forall x \in I$$

① $y'=xy \qquad y'-xy=0 \qquad I=\mathbb{R}$

$y(x)=e^{x^2/2}$ is a sol. in $\mathbb{R}$

$\left(e^{x^2/2}\right)=x\,e^{x^2/2} \quad \forall x \in \mathbb{R}$

$x\,e^{x^2/2}=x\,e^{x^2/2} \quad \forall x \in \mathbb{R}$ 'couse D.e. are defined in an interval

② $y'=xy$ if $y=e^x$

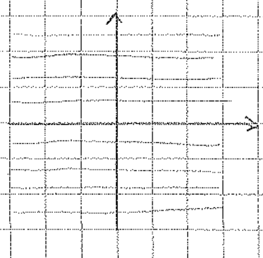
$(e^x)'=xe^x \qquad e^x=xe^x \qquad e^x(1-x)=0$ true just for $x=1$

The set of all solutions of diff. eq. in an interval is called the general integral of D.E in the interval I.

The general integral contains infinitely many functions, it depends on one arbitrary constant.

So how can I choose the value? I need the function that in a given number has a determinated value.

$y'=0 \rightarrow y=c$



- CAUCHY PROBLEM for a 1^{st} order D.E.

$$\begin{cases} y'=f(x,y) & \forall x \in I \\ y(x_0)=y_0 & x_0 \in I \end{cases}$$

$y>0 \quad y(x) = e^c \cdot e^{x^2/2}$

$y<0 \quad y(x) = -e^c e^{x^2/2}$

$\Big\} \; y(x) = k e^{x^2/2}$

$k \to \pm e^c$ (never zero)
can be both positive and negative

$$y(x) = \begin{cases} y(x) = 0 \\ y(x) = k e^{x^2/2} & k \neq 0 \end{cases}$$

$$y(x) = k e^{x^2/2} \qquad k \in \mathbb{R}$$

$$\begin{cases} y' = xy \\ y(3) = 0 \end{cases} \Rightarrow y(x) = 0$$

$$\begin{cases} y' = xy \\ y(1) = 3 \end{cases} \qquad y = k e^{x^2/2}$$

$3 = k e^{1/2} \Rightarrow k = 3e^{-1/2}$

$\Rightarrow y(x) = 3e^{-1/2} e^{x^2/2}$

ex.

$y' = h(x)\, g(y)$
$\qquad\;\; x \qquad y$

① solution of the eq. $g(y) = 0$

if $y_1 \dots y_p$ solution of alg. eq.

$\Rightarrow$ the function $y(x) = y_1$ ... $y(x) = y_p$ } if they ∃, accettano queste soluzioni (costanti)

$(y_1)' = h(x)\, g(y_1)$

$0 = h(x) \cdot 0$

$y' = x(1-y)^2 \; \begin{cases} y(x) = 1 \\ y(x) = -1 \end{cases}$

HOMOGENEOUS EQUATION

$y' = \varphi\left(\frac{y}{x}\right) \longrightarrow z = \frac{y}{x} \longrightarrow z(x) = \frac{y(x)}{x} \longrightarrow y = xz$

$y' = z + xz'$

$z + xz' = \varphi(z)$

$xz' = \varphi(z) - z$

$z' = \dfrac{\varphi(z) - z}{\frac{1}{x}}$ separable var. eq.

→ function of z

→ function of x

- $y' = y - y^2$ logistic equation

$g(y) = y - y^2 \qquad h(x) = 1$

$\displaystyle\int \frac{dy}{y - y^2} = \int \left(\frac{1}{y} + \frac{1}{-1+y}\right) dy = \log|y| - \log|y-1|$

$\log|y| - \log|y-1| = x + c$

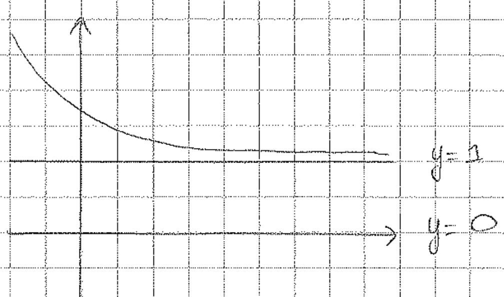
$\log\left|\frac{y}{y-1}\right| = x + c$

$\left|\frac{y}{y-1}\right| = e^x \cdot e^c$

$\frac{y}{y-1} = ke^x \qquad (k \neq 0)$

$y = ke^x y - ke^x \qquad y(1 - ke^x) = -ke^x$

$$\begin{cases} y = \dfrac{ke^x}{ke^x - 1} \\ y = 0 \\ y = 1 \end{cases}$$



so this function is continuous!

$$(z(x))' = (\mathrm{Re}\, z(x))' + i\,(\mathrm{Im}\, z(x))'$$

$$z'(x) = 1 + i2x$$

EULER FORMULA

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2}$$

$$\begin{aligned}(e^{ix})' &= (\cos x + i\sin x)' \\ &\overset{!}{=} (\cos x)' + i(\sin x)' \\ &\overset{!}{=} -\sin x + i\cos x \\ &\overset{!}{=} i(\cos x + i\sin x) = ie^{ix}\end{aligned}$$

i^2

$(e^{ix})' = ie^{ix}$

LINEAR DIFF. EQUATIONS

I^{st} order: $a_1(x)y' + a_0(x)y = b(x)$

II^{nd} order: $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$

III^{rd} order: $a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$

HOM. $b(x) = 0$

$a_0(x), a_1(x) \dots \in C^0(I)$

$a_1(x) \neq 0$ I ord.

$a_2(x) \neq 0$ II ord.

- The general integral of a linear diff. eq. is given by
  1. the general integral of the corresponding homogeneous equation

+

- SOLUTION:

$$Ke^{-A(x)} + e^{-A(x)} \underbrace{B(x)}_{\text{primitive of } b(x)e^{-A(x)}}$$

$$e^{-A(x)} (K + B(x))$$

$$y(x) = e^{-A(x)} \int b(x) e^{A(x)} dx$$

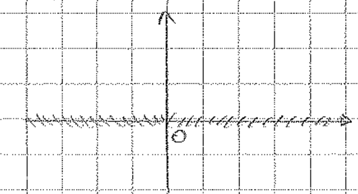
example 1

$xy' + y = x^2$ $\qquad$ $y' + a(x) y = b(x)$

$\Downarrow$

$x \neq 0$

$y' + \frac{1}{x} y = x$



$x > 0$ or $x < 0$

$A(x) = $ primitive of $\frac{1}{x} = \log|x|$ $\quad \Big\|$ $e^{-A(x)}$

$e^{-\log|x|} = e^{\log|x|^{-1}} = \frac{1}{|x|}$

$$\int x e^{\log|x|} dx = \int x |x| dx$$

$$= \begin{cases} \int x^2 dx = \frac{x^3}{3} dx & x > 0 \\ \int -x^2 dx = -\frac{x^3}{3} dx & x < 0 \end{cases}$$

$$y(x) = \begin{cases} \frac{1}{x} \left(c + \frac{x^3}{3}\right) & x > 0 \\ -\frac{1}{x} \left(c - \frac{x^3}{3}\right) & x < 0 \end{cases}$$

$y(x) = \frac{c}{x} + \frac{x^2}{3}$ $\quad x > 0$ $\quad \Big\|$ $\quad y(x) = \frac{c}{x} + \frac{x^2}{3}$ $\quad x < 0$

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$a_0(x), a_1(x), a_2(x)$

$y'' + ay' + by = 0$

$y = e^{\lambda x} \quad y' = \lambda e^{\lambda x} \quad y'' = \lambda^2 e^{\lambda x}$

$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$

$e^{\lambda x}(\lambda^2 + a\lambda + b) = 0 \Rightarrow \lambda^2 + a\lambda + b = 0$

① $\Delta > 0 \quad \lambda_{1/2} \in \mathbb{R} \quad e^{\lambda_1 x}, e^{\lambda_2 x}$

② $\Delta = 0 \quad \lambda_1 = \lambda_2 \in \mathbb{R} \quad e^{\lambda x}, x e^{\lambda x}$

③ $\Delta < 0 \quad \lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R} \quad e^{\lambda x}, e^{\bar{\lambda} x}$

① $y_{hom}(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

② $y_{hom}(x) = C_1 e^{\lambda x} + C_2 e^{\lambda x}$ $\Big] - C_1, C_2 \in \mathbb{R}$

③ $y_{hom}(x) = K_1 e^{\lambda x} + K_2 e^{\bar{\lambda} x}$ $\Big] - K_1, K_2 \in \mathbb{C}$

example ①

$$\begin{cases} y'' - 3y' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$y'' + ay' + by = 0$ it could have a bounded solution: zero

$\lambda^2 + a\lambda + b = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2)$

$y(x) = C_1 e^x + C_2 e^{2x}$

$y'(x) = C_1 e^x + 2C_2 e^{2x}$

$C_1 = 0 \, (C_2)$

$y(0) = 1 \Rightarrow \begin{cases} C_1 + C_2 = 1 \\ C_1 + 2C_2 = 0 \end{cases} \quad \begin{cases} -C_2 = 1 \\ C_1 = 2C_2 \end{cases} \quad \begin{cases} C_2 = -1 \\ C_1 = 2 \end{cases}$

$y'(0) = 0 \Rightarrow$

$y(x) = 2e^x - e^{2x}$

GENERAL INTEGRAL

$$y(x; c_1, c_2) = y_0(x; c_1, c_2) + y_p(x)$$

gen solution of the hom eq. — arb. part integral

$y'' + ay' + by = p_n(x)e^{\alpha x}$ $\alpha \in \mathbb{C}$ $p_n(x)$ polynomial of degree $n \geq 0$

look for $y_p(x) = q_n(x)e^{\alpha x}$ same degree

$N \geq n$

- let $\chi(\lambda) = \lambda^2 + a\lambda + b$:

① if $\chi(\alpha) \neq 0$: can take $N = n$

② if $\chi(\alpha)$ is a simple root we choose $N = n+1$

$q_{n+1}(x) = x q_n(x)$ $\chi'(\alpha) \neq 0$ $\chi(\alpha) = 0$

③ if $\chi(\alpha)$ is a double root we choose $N = n+2$

$q_{n+2}(x) = x^2 q_n(x)$ $\chi(\alpha) = \lambda$

example ①

$y'' + 3y' - 10y = 3 + 2t$ $p_n(x)e^{\alpha x}$ $\alpha = 0$

① put a zero on the right-hand side

$y'' + 3y' - 10y = 0$

$\lambda^2 + 3\lambda - 10 = 0$ characteristic equation

$\Leftrightarrow (\lambda + 5)(\lambda - 2)$

$$y_0(t; c_1, c_2) = c_1 e^{-5t} + c_2 e^{2t}$$

② first case

$y_p(t) = A + Bt$

$y'_p(t) = B$

$y''_p(t) = 0$

$y''_p + 3y'_p - 10y_p = 3 + 2t$

$\Rightarrow 0 + 3B - 10A - 10Bt = 3 + 2t$

Theory

$y'' + ay' + by = P_n(x)\, e^{\mu x} \cos \vartheta x$

$y'' + ay' + by = P_n(x)\, e^{\mu x}$ sen ϑx

choose $y_p(x) = x^m e^{\mu x} \left(q_{1,n}(x) \cos \vartheta x + q_{2,n}(x) \sin \vartheta x \right)$

m or zero or 1

① if $\alpha = \mu + i\vartheta$ is not a root of χ ($\chi(\alpha) \neq 0$) then $m = 0$

② if $\vartheta = 0$, μ is a simple root ($\chi(\alpha) = 0$, $\chi'(\mu) \neq 0$) then $m = 1$

③ if $\alpha = \mu + i\vartheta$ is a root of χ, choose $m = 1$

example ①

$y'' + 9y = 4 \cos 3t$ (q_1 = costant)

- $\lambda^2 + 9 = 0$ $\quad \lambda^2 = \sqrt{-9}$

$\lambda = -i3 \quad \lambda = +i3 \qquad (y_0, c_1, c_2) = C_1 \cos 3t + C_2 \sin 3t$

- $\alpha = 0 + 3i\vartheta$

$y_p = t(A \cos 3t + B \sin 3t)$

$y'_p = \ldots$

$y''_p = \ldots$

$$\begin{cases} -6A = 0 & A = 0 \\ 6B = 4 & B = \frac{2}{3} \end{cases}$$

! If $g = g_1 + g_2$: $y'' + ay' + by = g_1 + g_2$

then $y_p = y_{p1} + y_{p2}$, where

y_{p1} solves $y'' + ay' + by = g_1$

y_{p2} " $y'' + ay' + by = g_2$