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APPUNTI

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MATERIA : Geometria
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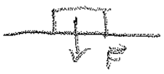
Geometry

Vectors are abstraction we don't see them in reality we go from reality to abstraction (math) and then we go back to reality

Vectors (1

we have a body sitting on the table;



the force can be described as a vector

The direction to go from a point to another is a vector

We have a vector $\vec{v}$ when we have:

1) Direction: the line containing the vector.

2) Verse: it tells in which way we are going

3) Lenght or norm $= \|\vec{v}\| \Rightarrow$ it tells how much we are going

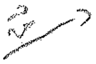
N.B. each direction has 2 verses. the verse is the point of the arrow.

<u>Free vector have no application point</u>

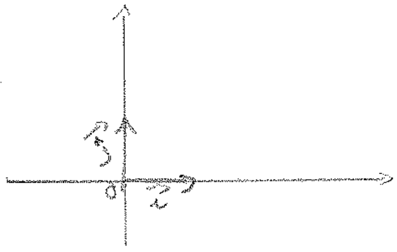
an applied vectors are applied vectors plus one point.

There are two different approaches. to code a vector

Geometric approach

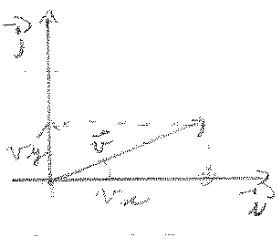


Algebraic approach (coordinates)



unitary vectors $\|\vec{i}\| = \|\vec{j}\| = 1$

we can describe the vector $\vec{v}$ using $\vec{i}, \vec{j}$

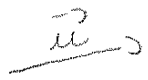


$$\vec{v} = v_x \vec{i} + v_y \cdot \vec{j}$$

③

Sum $\vec{u} + \vec{v}$

geometric way -> rule of parallelogram



We make tail coincide

the direction is the diagonal at the parallelogram

II) Concatenation

Algebraic way

$\vec{u} = u_x \vec{i} + u_y \vec{j} + u_z \vec{K}$

$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{K}$

$\vec{u} + \vec{v} = (V_x + V_x)\vec{i} + (u_y + V_y)\vec{j} + (u_z + V_z)\vec{K}$

① Dot product (scalar product)

from $\vec{u} \cdot \vec{v}$ i want to get a number $\in \mathbb{R}$ (scalar)

N.B. the angle is the smallest one $0 \le \alpha \le \pi$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos \alpha$ -> corrective term

$\vec{i}$ è un vettore unitorio

Ex: $\vec{i} \cdot \vec{i} = \|\vec{i}\| \cdot \|\vec{i}\| \overset{1}{\overset{\|}{\cos}} 0 = \cancel{1}\|\vec{i}\|^2 = 1$

$\vec{i} \cdot \vec{j} = \|\vec{i}\| \cdot \|\vec{j}\| \cdot \underset{0}{\underset{\|}{\cos}} \frac{\pi}{2} = 0$

Note

$\vec{v} \cdot \vec{u} = 0$

$\Updownarrow$ orthogonal to $\vec{v}$

$\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\vec{u} \perp \vec{v}$

Ex. $(\vec{i} + \vec{j}) \cdot (\vec{i} + \vec{k}) = ?$

In order to ~~make~~ do that we need the follow

result: ->

Ex

$\vec{u}\cdot\vec{v} = u_x \vec{w}_x + V_y \cdot W_y + u_z W_z$

$\parallel$

$(u_x\vec{i} + u_y\vec{j} + u_z\vec{k})\cdot(w_x\vec{i} + W_y\vec{j} + W_z\vec{k})$

note

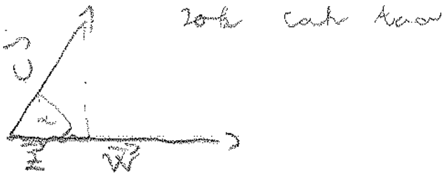
When I have A and B points we can construct a vector $\vec{AB}$ once we know the coordinates it's [illegible] easy to make the vector:

$A = (x_a ; y_a) \quad B = (x_B ; y_B)$

$\overline{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$

$\|\vec{AB}\| = \sqrt{\vec{AB}\cdot\vec{AB}} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

Orthogonal Projections:

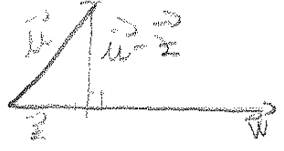


$\frac{\vec{u}\cdot\vec{W}\,\vec{W}}{\|\vec{w}\|^2} \Rightarrow$ orthogonal projection of $\underset{\vec{v}}{\vec{u}}$ on $\vec{W}$

$\vec{u}\cdot\vec{W} = \|\vec{u}\|\cos\alpha\,\|\vec{w}\|$

$\frac{\vec{u}\cdot\vec{w}}{\|\vec{w}\|}\cdot\frac{\vec{W}}{\|\vec{W}\|} = \frac{(\vec{u}\cdot\vec{w})\vec{w}}{\|\vec{W}\|^2} = \vec{z}$

Ex

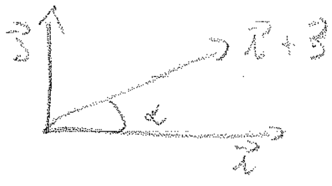


$(\vec{v}-\vec{z})\cdot\vec{W} = 0$

↳ check !!

Compute angles:

Es: $\vec{i}+\vec{j}$



$(\vec{i}+\vec{j})\cdot\vec{i} = 1+0 = 1$

$(\vec{i}+\vec{j})\vec{i} = \underbrace{\|\vec{i}+\vec{j}\|}_{\substack{\sqrt{(\vec{i}+\vec{j})(\vec{i}+\vec{j})} \\ \sqrt{2}}}\,\underbrace{\|\vec{i}\|}_{1}\cos\alpha$

$1 = \sqrt{2}\cos\alpha$

$\alpha = \frac{\pi}{4}$

→ x casa

Compute the angle $\vec{i}$ and $\vec{i}+\vec{j}+\vec{k}$

(7)

Per fare il prodotto vettoriale

$\vec{u} = u_x \vec{i} + u_y \vec{j} + u_z \vec{k}$

$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$

$$\text{matrix} \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_x & \cancel{v_y} & v_z \\ v_x & v_y & v_z \end{pmatrix} \qquad \vec{z} = \vec{u} \times \vec{v}$$

$$z = (A)\vec{i} + (B)\vec{j} + (C)\vec{k}$$

Compute A → take the vector cross the lines in which they appear
A will be the determinant of the left matrix

$A = u_y v_z - \cancel{u} z\, v_y$

B → Be carefull you have to put a minus in front of it!

$B = -(u_x v_z - u_z v_x)$

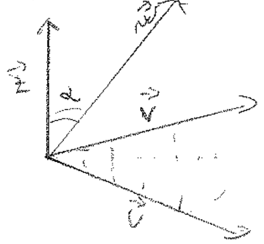
$C = (u_x v_y - u_y v_x)$

Mixed product

$\vec{u} \times \vec{v} \cdot \vec{w}$ → we get a scalar

? do we need parenthesis! No! there is only one possible way we have to make the cross product first!

$\vec{z} = \vec{u} \times \vec{v}$



box

$\|\vec{z}\|$ = area of the parallelogram

moving the parallelogram along $\vec{w}$ we get the box

$\vec{z} \cdot \vec{w} = \|\vec{z}\| \, \|\vec{w}\| \cos\alpha$ $\qquad$ $\|\vec{w}\| \cos\alpha$ = height of the box

the mixed product is the volume of the box; we can have negative signs; so we say that the absolute value of the mixed product is absolute value

2) Proof of n° 2

$(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$

If I want to show $a = b$ i can show that $a - b = 0$

I set $\vec{z} = (\vec{u}+\vec{v})\times\vec{w} - (\vec{u}\times\vec{w} + \vec{v}\times\vec{w}) = 0$

$\vec{z} = 0 \Leftarrow \vec{z}\cdot\vec{z} = 0$ -> try to show

Matrices

We deal with linear system of equation

Ex $\begin{cases} \textcircled{1}x + \textcircled{1}y = \textcircled{1} \\ \textcircled{1}x - \textcircled{1}y = \textcircled{1} \end{cases} \Leftrightarrow \begin{cases} x+y=1 \\ 2x+0=2 \end{cases} \Rightarrow \begin{cases} y=0 \\ x=1 \end{cases}$

It's important to the everything in place

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 1 \end{array}\right) \xrightarrow{+1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 2 \end{array}\right)$$

-> we have to make a row in which everything before the dashed line is 0 besides ~~the~~ one object in this example I can add one

$A \in \mathbb{R}^{n,m}$ -> a matrix with $\overset{\text{orizzontale}}{n}$ rows and $\overset{\text{verticale}}{m}$ columns

$R_i \in \mathbb{R}^{1,m}$

$$\begin{pmatrix} & \vdots & \\ \cdots & a_{ij} & \\ & \vdots & \end{pmatrix}$$

$R_1 = (1\ 1\ 1)$ -> 1 row and 3 column

$C_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ -> 2 ~~colum~~ rows and 1 column

$A \in \mathbb{R}^{n,m}$ transpose of A

$A^T = {}^tA \in \mathbb{R}^{m,n}$

$A = (a_{ij})$

$A^T = (b_{lk})$

Ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\underset{\substack{\Uparrow \\ \mathbb{R}^{3,2}}}{A^T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$A \in \mathbb{R}^{2,3}$

How make transpost:

1) take A and imagine it as a collection of rows:

$$A = \begin{pmatrix} \overline{R_1} \\ \vdots \\ \overline{R_n} \end{pmatrix} \to A \in \mathbb{R}^{n,m}$$

$$A^T = \left(R_1 | R_2 | \cdots R_n \right) \text{ in } \mathbb{R}^{m,n}$$

Operations on matrices

Multiplication by a constant (scalar)

The ingredients are: $A \in \mathbb{R}^{n,m}$ and $c \in \mathbb{R}$

$cA = (c\,a_{ij}) \to$ multiply all entries by c

Ex:

$\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k} \iff A = (1 \quad 2 \quad 3)$

$c = 3$

$3\vec{v} = 3\vec{i} + 6\vec{j} + 9\vec{k}$ $3A = (3 \quad 6 \quad 9)$

Note

I) $1A = A$

II) $0 \cdot A = 0_{n \times m}$

III) $0 \in \mathbb{R}^{n,m} = n \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

chiede come vediamo che è 1,1

Example

$A = (1 \quad 2 \quad 3)$ $\quad B = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$: $A \in B^{1,3}$ $\quad B \in \mathbb{R}^{3,1}$

$$\underset{1\times 3}{AB} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \overset{3\times 1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} = c_{11} = 1\cdot 1 + 2\cdot 0 + 3\cdot(-1) = (-2)$$

notice that

$A \longleftrightarrow \vec{i} + 2\vec{j} + 3\vec{k} = \vec{v}$ $\qquad \vec{v}\cdot\vec{u} = 1\cdot 1 - 2\cdot 0 + 3(-1) = -2$

$B \longleftrightarrow \vec{i} \qquad - \vec{k} = \vec{u}$

Note: $A \quad n \times m$, $B \quad m \times p$ $\Rightarrow AB \in \mathbb{R}^{n,p}$ It is always true

Ex: $A = (1 \quad 2 \quad 3)$ $\boxed{①\times 3}$ $\qquad B = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $3\times①$

$$BA = \overset{3\times 3}{\begin{pmatrix} 1\cdot 1 & 1\cdot 2 & 1\cdot 3 \\ 0\cdot 1 & 0\cdot 2 & 0\cdot 3 \\ -1\cdot 1 & -1\cdot 2 & -1\cdot 3 \end{pmatrix}}$$

$B = 3\times 1$

$A = 1\times 3$

we can make it

Note: in general $AB \neq BA$

Ex

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\qquad B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

$AB = 2\times 2$ $\qquad\qquad BA = 2\times 2$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ; \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Note: 1) Matrix multiplication is not commutative

2) In real number $x\cdot y = 0 \Rightarrow x$ or y are equal to zero

$A \neq 0$ and $B \neq 0$

then A and B can be 0

Square Matrices

$A, B \ \mathbb{R}^{n,n} \Rightarrow \exists AB \quad \exists BA$

The Identity Matrix is a special case of diagonal matrix

Def a square matrix i diagonal

$A = (a_{ij}) \quad i \neq j \Rightarrow a_{ij} = 0$

es $A = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{pmatrix}$ It's all zero beside the diagonal

Def: A $n \times n$ is upper triangular
if $i > j \Rightarrow a_{ij} = 0$
A $n \times n$ is lower triangular
if $i < j \Rightarrow a_{ij} = 0$

Simmetric Matrices:

Def: A $n \times n$ is symmetric if $A = A^T$

Es $n = 1$
all in $\mathbb{R}^{1,1}$ are symmetric

Es $n = 2$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Find symm matrices in $\mathbb{R}^{2,2} \Leftrightarrow A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

note:

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$
dimostrarlo

$A \ n \times m$, $A^T \ m \times n$
$B \ m \times p$ $B^T \ p \times m$
$AB \ n \times p$ $(AB)^T = p \times n$

Ex: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $|A| = 1$ find $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$

s.t. $XA = I = AX$

$$XA = \begin{pmatrix} x & x+y \\ z & z+t \end{pmatrix} = I$$

Check $AX = I$

$$\begin{cases} x = 1 \\ x+y = 0 \\ z = 0 \\ z+t = 1 \end{cases}$$

Thm A 2×2 remember $A^{-1}A = I = AA^{-1}$

$\exists A^{-1} \iff |A| \neq 0$

note

$A \neq 0$, $B \neq 0$ if $AB = 0 \Rightarrow \nexists A^{-1} \nexists B^{-1}$ I prove by contradic

$\exists A^{-1}$

$$A^{-1}AB = A^{-1}0$$
$$IB = 0 \Rightarrow B = 0 \text{ not possible}$$

We prove that $\exists A^{-1} \iff |A| \neq 0$ I have double implication, I have to prove both part,

First I prove $\exists A^{-1} \Longleftarrow |A| \neq 0$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ → Then I claim $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$

$= \frac{1}{|A|}\begin{pmatrix} |A| & 0 \\ 0 & |A| \end{pmatrix}$ We need that $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Now I prove the $\exists A^{-1} \Rightarrow |A| \neq 0$ part

Contradiction if $|A| = 0$ $\overset{\text{we}}{}AB = 0 \Rightarrow \nexists A^{-1}$

3×3 matrices

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ We can compute ~~matrix~~ determinant

1) Sarrus rule (only for 3×3 matrices)

2) Recursive (also for bigger matrices)

$$\overset{m\times n}{A}\underset{n\times 1}{x} = \underset{m\times 1}{b}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ a_{m1} & \cdot & a_{mn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$Ax = \Big($$

Example

$$Ax = B$$

A is square

$$\begin{cases} x - y = 0 \\ x + y = 1 \end{cases} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x = A^{-1}b$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}b$$

~~so~~

$$A^{-1}b = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Cramer's rule

$$x = \frac{\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}}{|A|}$$

determinant on the bottom, on top I put determinant of the matrix substituting the vector b to the 1st column if I want x or to the 2nd column if I want y

~~matrix~~

$$AB = \big(c_1 | c_2 | \cdots | c_n\big)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 c_1 + b_2 c_2 + \dots$$

This is a linear combination

How to solve linear system of equation

$$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$R_2 \to -\frac{1}{2} R_2$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_2 + \underset{\text{free parameter}}{\boxed{x_3}} = 1 \\ x_2 = -1 \end{cases}$$

There will be free parameter

$$\begin{cases} x_1 = 2 - x_3 \\ x_2 = -1 \end{cases} \quad \forall \; [illegible] \qquad (x_1, x_2, x_3) = ([illegible])$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

Elementary row operations

i) $R_i \to R_i + a R_j \quad i \neq j \quad a \in \mathbb{R}$

ii) $R_i \to c R_i \quad c \neq 0$

iii) $R_i \longleftrightarrow R_j$

$\exists$ a matrix B $\quad BA = A'$

ex $R_1 \longleftrightarrow R_2$

$$\overset{A}{\left(\begin{matrix} 1 & 1 & 1 \end{matrix}\right.} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \overset{B}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \overset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

2×2 -> identity matrix with the same switch

$R_2 \to R_2 + R_1$

$$\overset{B}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} \overset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

N.B.
$*$ ⟷ non zero entry
0 ⟷ zero
$?$ ⟷ either zero or not

How to apply ERO in order to get a simpler linear system?

Reduced matrices -> Good matrices to solve the system

Def: Marker

A marker is a non zero entry of A if it is the non zero from the left

Ex $\begin{pmatrix} \textcircled{1} & 2 \\ \textcircled{3} & 4 \end{pmatrix}$ marker $\quad \begin{pmatrix} 0 & \textcircled{1} \\ \textcircled{2} & 0 \end{pmatrix}$ marker -> this is correct

the idea is that a column with a marker is good a column with more markers is not so good.

~~Conditions~~

M1) Each column has at most one marker -> it does not require any marker to exist

~~Using ERO's~~ Given A and using ERO of type (i)

A -> A' has (M1)

$A \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow R_2 \rightarrow R_2 - R_1 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{pmatrix}$

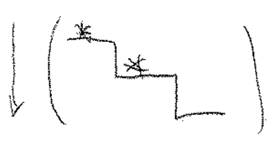
or

$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow R_1 \Rightarrow R_1 - R_2 \rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

M_4) Each marker is the number one
M_5) Zero rows at the bottom } ? BAZINGA

$(M_1)\ (M_4)\ (M_5)$ is step reduced

M2)



when we go down markers go from left to right

M3) All the zero rows (if they exist) are at the bottom?

Step reduced matrices

Ex $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ -> step reduced

-> not step reduced, the problem is M_2

Ex $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Gaussian Elimination

- Skip zero columns
- Switch rows in order to have the first marker at row 1
- Use that marker to reach M1

$$A = \begin{pmatrix} 0 & 0 & * & \\ & \vdots & 0 & * \\ \vdots & \vdots & & \\ 0 & 0 & & \end{pmatrix}$$

Thm

$\forall A \quad \exists A'$ (step reduced) $: A \sim A'$

$v_1, v_2 \dots v_k$ vectors $\in \mathbb{R}^n = \begin{cases} \mathbb{R}^{1,n} \\ \mathbb{R}^{n,1} \end{cases}$

Linear combination of vectors.

$a_i \in \mathbb{R} \qquad a_1v_1 + a_2v_2 + \dots + a_kv_k$

Linear span is the set of all possible linear combination.

Given $v_1 \dots v_k$

$$\mathcal{L}(v_1 \dots v_k) = \{a_1v_1 + \dots a_kv_k : a_i \in \mathbb{R}\}$$

$\vec{0}$ is an element of $\mathcal{L}(v_1, \dots v_k)$ since $0 = 0v_1 + 0v_2 + \dots 0v_k +$

Ex

$$A = \begin{array}{c} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \end{array} \begin{array}{l} v_1 \\ v_2 \end{array}$$

Solve $a_1v_1 + a_2v_2 = 0$ with $a_1, a_2 \neq 0$

$a_1v_1 + a_2v_2 = (a_1, 2a_1) + (0, 3a_2) = (a_1, 2a_1 + 3a_2) = (0, 0)$

$\begin{cases} a_1 = 0 \\ 2a_1 + 3a_2 = 0 \end{cases} \qquad \begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases} \qquad$ not possible

- $\{(1,0), (0,1), (2,3)\}$

$2(1,0) + 3(0,1) + (-1)(2,3) = 0$ i.e. $(2,3)$ is a linear comb. of the other 2 vector

$(0,0,0)$ is not the only sol.

Th. non scrivo enunciato

$a_1 v_1 + \dots a_k v_k = 0$ if not L.I. $\Rightarrow v_1 = -\frac{1}{a_1}(a_2 v_2 \dots + a_k v_k)$

Prop. If A' is step reduced, then the non-zero rows of A' are L.I.

proof:
$$A = \begin{pmatrix} * & \dots & \dots & \dots \\ & * & & \\ & & * & \\ & & & \ddots \\ 0 & \dots & \dots & 0 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ \\ \\ R_k \end{matrix}$$

$a_1 R_1 + a_2 R_2 + \dots + a_k R_k = 0$

$$\begin{cases} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_k = 0 \end{cases}$$

Prop. if A is step reduced, then the markers columns are L.I.

Prop $\Rightarrow$ # markers = # of linearly independ. rows = L.I. columns

Def: Given A step red its rank is the number of marker

~~Ex~~ $r(A) = rk(A) = \rho(A) =$ # of markers

Ex $r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$ $r\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1$ $r\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$r(A) \leq \min\{m, n\}$ given $A \in \mathbb{R}^{m,n}$ step red.

Theorem : Rouché - Capelli

1) ∃ solutions $\iff r(A) = r(A|b)$

2) $r(A) = r(A|b) \implies \exists\, \infty^{n-r}$ solution
$n - r$

$(A|b) \overset{\text{super. red}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & & 0 & \cdots \\ 0 & 1 & & & \\ & 0 & & & \end{pmatrix}$

Ex $\begin{cases} x + y = 2 \\ x + z = 2 \\ y - z = 0 \end{cases}$ inhomogeneous system

$$(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right) \quad R_2 \to R_2 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \quad R_3 \to R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$r(A) = 2 = r(A|b) \implies \exists$ solution,
$\infty^{3-2} = \infty^{1}$ solution 1 free parameter

The free parameter will be t

$$\begin{cases} x + y = 2 \\ -y + z = 0 \\ 0 = 0 \end{cases} \to \begin{cases} x = 2 - t \\ y = t \\ z = t \end{cases}$$

I start from the bottom for super [illegible]

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{matrix} R_2 = -R_2 \\ R_3 = \frac{1}{2} R_3 \end{matrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{ERO} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$$ the identity matrix

A is invertible : by definition

$\exists A^{-1}$ s.t. $A A^{-1} = I$

$A \overset{ERO}{\rightsquigarrow} A'$

$EA = A'$

The idea is: if I want to compute the inverse of a matrix I have to super reduce it; but super-reducing I may not get the identity; this happens the result is not as big as possible.

How to get the inverse.

I build

$B = (A \mid I_n)$ $\qquad B \overset{Ero's}{\rightsquigarrow} (I \mid E) = FB$

- Step_red A
- $r(A)$
- If $r(A) < n \Rightarrow \nexists A^{-1}$

$FB = \begin{pmatrix} FA & \mid & F \end{pmatrix}$

$FA = I \qquad F = A^{-1}$

Example

$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

now I step reduce $\qquad$ super red

$$\left(\begin{array}{cc|cc} \overset{A}{1} & 1 & \overset{I_2}{1} & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \Rightarrow R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{2} R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$-2 = |A|$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

N.B the product of the number of the diagonal ~~[illegible]~~ of the step reduced matrix is the determinant

Usato come definizione di determinante

Subspace of solution.
The idea is to take a linear system of equation
$V = \{x : Ax = 0\}$

Theorem $V = \{x : Ax = 0\}$ is a subspace
N.B. If the system is not homogeneous $\Rightarrow$ V is not a sub space
Proof:
$x \in V$, $a \in \mathbb{R}$
S2) ? $ax \in V$!
$ax \in V \iff \underset{\shortparallel}{A}(ax) = 0$
$a \underbrace{(Ax)}_{0} = 0$

Some important remarks $\quad \exists A^{-1} \Rightarrow r(A) = n$
$\exists A^{-1} \Leftarrow r(A) = n$
$EA = A^{-1} = \begin{pmatrix} * & & & & \\ 0 & * & & & \\ 0 & 0 & * & & \\ 0 & 0 & 0 & * & \\ 0 & 0 & \cdot & 0 & * \end{pmatrix}$
Then I super reduce
$\underbrace{E'EA}_{A^{-1}} = I_n$
Now I prove that
$\exists A^{-1} \Rightarrow r(A) = n$
Contradiction $r(A) = n$
$E\cancel{A}I_n = A' = \begin{pmatrix} & ? & \\ 0 & \cdots & 0 \end{pmatrix}$
If $\exists A^{-1}$
$EAA^{-1} = \begin{pmatrix} & ? & \\ 0 & — & 0 \end{pmatrix} A^{-1}$ → the last row will always be 0

I can also construct another subspace

$\{x : \; Ax = 0\}$ space $\Rightarrow$ Ker(A) null

The set of subspace containing the solutions

The row space and the Kernal can tell us a lot about the matrix

Thm

$A \sim B \iff \text{Row}(A) = \text{Row}(B) \iff \text{Ker}(A) = \text{Ker}(B)$

$\underset{\text{Es}}{\overset{\text{ERO}}{E}}A = B \quad \Rightarrow \quad \text{Row}(A) \geqslant \text{Row}(B)$

Row(A) = Row(B)

$A = E'B \quad \Rightarrow \quad \text{Row}(A) \leq \text{Row}(B)$

To prove Ker (A) = Ker (B) it's easy since 2 implies 1

now I have to go from 3 to 1

$\text{Ker}(A) = \{x : \; Ax = 0\}$

$\{x : \; Rx = 0 \quad R \in \text{Row}(A)\}$

when I look for null space I look for the element killing the Rows

$\{x : Rx = 0 \quad R \in \text{Row}(A)\}$

$R \in \mathcal{L}\{R_1, \ldots, R_m\}$

$R = \alpha_1 R_1 + \ldots + a_m R_m$

$Rx = \alpha_1 R_1 + a_2 R_2 + \ldots$

The conclusion is that $\text{Ker}(A) \longleftrightarrow \text{Row}(A)$

$\text{Row}(A) = \mathcal{L}\{R_1, \ldots, R_m\}$

$\{R : \; Rx = 0 \quad x \in \text{Ker}\, A\}$

$\text{Row}(A) = \mathcal{L}\{R_1, \ldots R_m\}$

$\text{Row}(A) = \{R : \quad Rx = 0 \quad x \in \text{Ker}\, A\}$

$$\mathbb{R}^n \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots e_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

$x = (x_1, x_2$

$$(e_1 | e_2 | \dots e_n) = I_n \qquad r(I_n) = n$$

these are called canonical basis

Es 1: How can I say whether 2 vectors are basis or not!
I've to check if they generate and are LI
So I find sol. to $x_1 V_1 + x_2 V_2 + x_3 V_3 = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

They are L.I if and only if the rank of the matrix is 3

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{not LI}$$

Es 2 are $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ basis?

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}^{A^1} \Rightarrow r = 3 \rightarrow \text{they are L.I}$$

Now I want to know if they generate. This happens if
$\mathcal{L}\{V_1, V_2, V_3\} = \mathbb{R}^3 \Longleftrightarrow r(A) = r(A|V) \ \forall v \in \mathbb{R}^3$
i: for every $V \in \mathbb{R}^3$ can I find $x_1, x_2, x_3 \in \mathbb{R}$ s.t.
$x_1 V_1 + x_2 V_2 + x_3 V_3 = V$

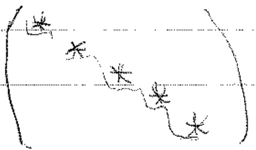
$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = V \Rightarrow r(A) = r(A|V)$$

Since A^1 exist I can find all x_1, x_2, x_3

$\begin{pmatrix} v_1 \\ \vdots \\ v_s \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix}$ $\mathcal{L}\{v_1, \dots, v_s\} = V = Row($

$\mathcal{L}\{u_1, \dots, u_k\} = V = Row(B)$

$EA =$



S non zero rows

$= E'B$

K non zero rows

$\Rightarrow A \sim B$

$\Rightarrow r(A) = r(B)$

A zero subspace is peculiar: the problem is that the
vector is linearly dependent

$V = \{0\}$ basis $\emptyset$ $\dim\{0\} = 0$

Basis and dimensions

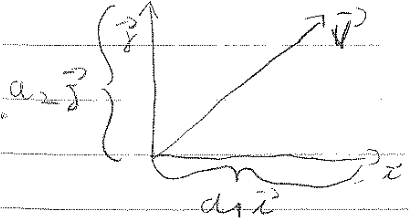
A vector space $\{V_1, \dots, V_K\}$ is a basis of V if and only if

1) They generate : $\mathcal{L}\{V_1, \dots, V_K\}$

2) They are L.I.

The consequences are that for each $v \in V$

$V = a_1V_1 + \dots + a_kV_k$ $a_i \in \mathbb{R}$ in a unique way



Given a generator we can find the basis:

$V = \mathcal{L}\{u_1, \dots, u_m\}$

We construct a matrix using the vector as rows $A = \begin{pmatrix} u_1 \\ u_2 \\ \overline{u_n} \end{pmatrix}$

I step reduce it: $A \rightsquigarrow A'$

Then the non zero rows of A' are a basis of V

i) they are L.I.

ii) $V = Row(A) = Row(A')$

Vector Spaces

The name vector ~~from~~ now on is going to be everything has the characteristic of a vector.

We need a field ⇒

$F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

V is a set ~~of~~

We say that V is an F vector space if I have addition and multiplication by a scalar with some properties

•) $u, v \in V \Rightarrow u + v \in V$

Properties of addition.

•) The sum must be associative: $(u+v)+w = u+(v+w) = u+v+$

•) Commutative: $u + v = v + u$

•) $\exists 0 \in V ; 0 + v = v \quad \forall v \in V$ → It has an identity

•) opposite $\exists (-v)$: $v + (-v) = 0, \ \forall v \in V$

Properties of product

•) $(a+b)v = av + bu$

•) $(ab)v = a(bv)$

•) $a(v+u) = av + au$

•) $1 \in F \quad 1 \cdot v = v$

What about 0 in F $\quad 0 \cdot v$?

$$\underset{F}{\underset{\cap}{0}} \cdot \underset{V}{\underset{\cap}{V}} \Rightarrow \in V \quad = (0+0)v = 0 \cdot v + 0 \cdot v$$

$$0_v = 0 \cdot v - \cancel{0} \cdot v = \cancel{0}\, v + 0_v - 0 \cdot v$$

$$0_v = 0 \cdot v + 0_v = 0 \cdot v$$

Ex. $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots \mathbb{R}^n$ are $\mathbb{R}$-vector spaces

$v = (v_1, \ldots, v_n)$

$u = (u_1, \ldots, u_n)$

$v + u = (v_1 + u_1, \ldots, v_n + u_n)$

$0 = (0, \ldots, 0)$ Identity

The sum is OK) have Ass / Com / identity / opp

Corollary

In $\mathbb{R}^n$: $\{v_1, \dots v_n\}$ is a basis

$$\Updownarrow$$

$\mathcal{L}\{v_1, \dots v_n\} = \mathbb{R}^n$

$$\Updownarrow$$

$\{v_1, \dots v_n\}$ L.I.

Ex: $v_1 = (1, 1, 1)$
$v_2 = (1, 2, 3)$
$v_3 = (-1, -7, -10)$
} Is this a basis of $\mathbb{R}^3$

L.I. $\Leftrightarrow r(A) = 3$

$\dim \mathcal{L}\{v_1, v_2, v_3\} = r(A^*)$

Back to vector spaces

$F = \mathbb{R}$ field

$\mathbb{R}$ - vector spaces

1) $\mathbb{R}^n$

2) $\mathbb{R}^{n,m}$

3) $\mathbb{R}[x] \leftarrow$ The set of polinomial in x with real coeffic

$\mathbb{R}[x] = \{ P(x) = a_0 + a_1 x + \dots + a_d x^d$ s.t. $a_i \in \mathbb{R}$

1 deg 0

$1 + 2x + 3x^3$ deg 3

$\mathbb{R}$ vector space $\rightarrow$ I define sum and m

1) $P(x) + g(x)$

$P, g \in \mathbb{R}[x]$

$\Rightarrow P(x) + g(x) \in \mathbb{R}[x]$

$P(x) = a_0 + a_1 x + a_2 x^2 + a_d x^d$

$g(x) = b_0 + b_1 x + b_2 x^2 + b_t x^t$

$P(x) + g(x) = a_0 + b_0$

Ex: subspace - - polinomial

$V = \mathbb{R}_2[x]$

$W = \{ p(x) \in V \text{ s.t. } p(0) = 0 \}$

W is a subspace

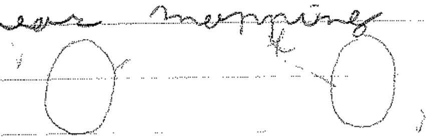
S1) $p(x), g(x) \in W$

? $p(x) + g(x) \in W$? $\Longleftrightarrow$ $p(0) + g(0) = 0$

Ex $\mathbb{R}^{2,2} = V$ $W = \{ M : \text{rk}(M) \leq 1 \}$ is a subspace

not! es $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix}$

Linear mapping



in

$\mathbb{R}^{2,2} \longleftrightarrow \mathbb{R}^4$ - - - ~~switch from~~ $\mathbb{R}^{2,2}$ ~~to~~ $\mathbb{R}^4$

V; W F - vector space

$f : V \longrightarrow W$ map

$v \longmapsto f(v)$

f is a linear map if and only if

(LM1) $v_1, v_2 \in V \Rightarrow f(v_1 + v_2) = f(v_1) + f(v_2)$

(LM2) $a \in F, v \in V \Rightarrow f(av) = a f(v)$

note:

$(LM1) + (LM2) \Rightarrow f(av) = 0_W$

Ex

$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto x+1$ → this is not a linear map!

$x \mapsto 2x$

LM_1 $f(x_1 + x_2) = 2(x_1 + x_2)$ ✓

LM_2 $f(ax) = 2ax = a(2x)$ ✓

$x \mapsto x^2$

$$f(ax) \qquad a f(x)$$
$$\| \qquad\qquad \|$$
$$(ax)^2 \neq ax^2$$
$$\forall a \in \mathbb{R}$$

$f: V \to W$ is linear, injective, surjective $\Rightarrow$
$\Rightarrow$ f is an isomorfismo

Thm V $\mathbb{R}$-vector space s.t.
$V = \mathcal{L}\{v_1, \dots v_n\}$ L.I.
then
$V \simeq \mathbb{R}^n$ $\to$ V is isomorphic to $\mathbb{R}^n$

$\mathbb{R}^n \xrightarrow{f} V$
$(a_1, \dots a_n) \longmapsto a_1 v_1 + \dots + a_n v_n$
$f(a_1, \dots a_n) = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$
check if the map is
1) linear
2) injective
3) surjective
1) $\to$ holds trivially
3) $\to$ [illegible] linear span [illegible] V
2) By basis

V is finitely generated
$V = \mathcal{L}\{v_1, \dots, v_n\}$ L.I. $\Rightarrow V \xleftarrow{f} \mathbb{R}^n$ iso

$\dim V = n$
$\{e_1, \dots e_n\}$ basis of $V \mathbb{R}^n$ then $\{f(e_1), \dots, f(e_n)\}$ basis
$\mathbb{R}^{2,2} = \mathcal{L}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$
$\mathbb{R}^4 \to \mathbb{R}^{2,2}$
$(a_1, a_2, a_3, a_4) \longmapsto f(a_1, a_2, a_3, a_4) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ $\quad f(x, y) = ?$

$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \\ 5x + 6y \end{pmatrix}$ → then we write it as a row

$f(x\ y) = (x + 2y,\ 3x + 4y,\ 5x + 6y)$

now we want to go in the opposite direction

$f \rightsquigarrow A$? To do this we need to know something more about linear maps. In order to do this we need to chose a base

Linear maps and bases

Lemma: V, W vector spaces

$B = \{v_1, \dots, v_m\}$ basis of V

giving $f(v_1) = w_1$

$\quad\quad f(v_m) = w_m$ (is equivalent)

~~giving $f: V \to W$~~

Proof

given $v \in V$ want to compute $f(v)$

write $v = a_1 v_1 + \dots + a_n v_n$

$f(v) = f(a_1 v_1) + f(a_2 v_2) + \dots + f(a_n v_n)$

$f(v) = a_1 f(v_1) + a_2 f(v_2) + \dots$ ~~+an~~ $f(v_n)$

given $f: V \to W$

and B - basis of V $\quad B'$ basis of W

matrix associated to f using B and B'

M_f or $M(f)$

Example

$f: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad B, B'$ canonical bases

$B = \{(1, 0) ; (0 ; 1)\}$ $\quad B' = \{(1, 0, 0) ; (0\ 1\ 0) ; (0, 0, 1)\}$

$f(x, y) = (x + y,\ x - y,\ 0)$ $\quad M = ?$

$f(0, 1) = (1\ {-1}\ 0)$ $\quad f(1, 0) = (1, 1, 0)$

$(1\ {-1}\ 0) = 1(1\ 0\ 0) + (-1)(0\ 1\ 0) + 0(0\ 0\ 1)$

$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}$

$$f(0\ 1) \overset{(0\ 1\ 0)}{=} a_{12}w_1 + a_{22}w_2 + a_{32}w_3 = \begin{pmatrix} a_{12} + a_{22} + a_{32}, \\ a_{22} + a_{32},\ a_{32} \end{pmatrix}$$

$$\begin{cases} a_{12} + a_{22} + a_{32} = 0 \\ a_{22} + a_{32} = 1 \\ a_{32} = 0 \end{cases}$$

13/04/2012 review:

$f: V \to W$ V and W are vector spaces of the same field

$v \mapsto f(v)$ → Linear map

The idea is to fix bases for associate a matrix

B basis of V

B' basis of W

$$\left.\begin{matrix} f \\ B \\ B' \end{matrix}\right\} \rightsquigarrow M_f$$

$B = \{b_1, \dots, b_n\}$

$v = a_1 b_1 + \dots + a_n b_n$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = M_f \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$B' = \{d_1, \dots d_m\}$

$c_1 d_1 + \dots + c_m d_m = f(v)$

example

$$\left.\begin{matrix} f: \mathbb{R}^2 \to \mathbb{R}^3 \\ (x, y) \to (x+y,\ x+2y,\ x-y) \end{matrix}\right\}$$ Produce M_f in order to this I chose canonical bases

$B = (1\ 0)\ (0\ 1)$

$B' = (1\ 0\ 0)\ ;\ (0\ 1\ 0)\ ;\ (0\ 0\ 1)$

$\underset{3\times1}{f(x, y)} = \underset{3\times2}{M_f} \underset{2\times1}{\begin{pmatrix} x \\ y \end{pmatrix}}$ → This is the way in which the matrix is

$S_2)$ $v_1, v_2 \in f^{-1}(0)$

? $v_1 + v_2 \in f^{-1}(0)$?

N.B. $f^{-1}(0) \Rightarrow$ the v's such that $f(v) = 0$

$v_1 + v_2 \in f^{-1}(0) \Leftrightarrow f(v_1 + v_2) = 0 \Leftrightarrow \overset{0}{\overset{\shortparallel}{f(v_1)}} + \overset{0}{\overset{\shortparallel}{f(v_2)}} = 0$

Prove S_1 at home

2) $\underline{\text{Im}}\, f \subset W = \{ w : \exists v \in V \;.\; f(v) = w \}$

We prove the product property:

$\forall a \in \mathbb{R} \quad \forall w \in \text{Im} f \Rightarrow$? $aw \in \text{Im} f$?

$w = f(v)$

notice that $f(av) = aw$

Remark

$f : \mathbb{R}^n \to \mathbb{R}^m$

$\to f^{-1}(0) = \{ v : Av = 0 \} = \text{Ker} A$

$\text{Im} f \subset \mathbb{R}$? Col A

Using canonical bases $f(v) = Av$

$A = \left(f(e_1) | \cdots | f(e_n) \right)$ $\left(\underline{\text{Im} f} = \{ f(v) ; v \in \mathbb{R}^n \} \right)$

$\overset{\shortparallel}{\mathcal{L}\{ f(e_1), \ldots, f(e}$

$\overset{\shortparallel}{\text{Col} A}$

$v \in \mathbb{R}^n$ is a L.C. of $\{ e_1, \ldots, e_n \}$

$v = x_1 e_1 + \cdots + x_n e_n$

$f(v) = x_1 f(e_1) + \cdots + x_n f(e_n)$

~~to summarize~~

$f : \mathbb{R}^n \to \mathbb{R}^m$ ~~linear~~

$A = M_f$ canonical

1) $f^{-1}(0) = \text{Ker} A$ dim $f^{-1}(0) = n - r$

2) $\text{Im} f = \text{Col} A \to \text{rank}(A) = \dim \text{Im} f$)

Def $f : V \to W$

the Kernel of f

is $\text{Ker} f = f^{-1}(0)$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear -> Endomorphism

Composition of maps

$$w \longmapsto g(w) = Bw$$

$$\mathbb{R}^{n,1} \xrightarrow{f} \mathbb{R}^{m,1} \xrightarrow{g} \mathbb{R}^{p,1}$$

$$v \longmapsto Av = f(v)$$

$A \quad m \times n$ ↗ the associated matrix to $g \circ f$

$B \quad p \times m \qquad BA \quad p \times n$

$$\mathbb{R}^{n,1} \longrightarrow \mathbb{R}^{p,1}$$

Maps

$U \xrightarrow{f} V \xrightarrow{g} W$

$\mathcal{B}_U \quad \mathcal{B}_V \quad \mathcal{B}_W$

Bases

$M_{g \circ f} = M_g M_f$

Ex

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$(x, y, z) \longmapsto (x + y + z, 2x + 2y + 2z)$] = ker f , Im f

The associated matrix is 2x3

$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ -> check if $f(x, y, z) = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

ker $\{v : Av = 0\}$ $\qquad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$r(M) = 1$

∞^{3-1} sol

$$\ker f = \{(-y-z, y, z) \text{ s.t. } y, z \in \mathbb{R}\} =$$

$$= \mathcal{L}\{(-1, 1, 0) \underbrace{,}_{\text{Bases}} (-1, 0, 1)\}$$

2) These are usually not L.I. so we have to make an eliminato in order to make it a basis

$V, U \subset \mathbb{R}^p$ p columns

$$n+m \begin{pmatrix} u_1 \\ \\ v_m \end{pmatrix} = M$$

Bases for $V \cap U$

$U, V \subset \mathbb{R}^p$

$u_1, u_2 \quad v_1 \; v_m \qquad x_1 u_1 + \dots + x_n u_n = y_1 v_1 + \dots + y_m v_m$

Ex

$V = \mathcal{L}\{(1,1,0,0);(0,0,1,1)\}$

$U = \mathcal{L}\{(1,0,1,0),(0,1,0,1)\}$

$u+V$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim$$ [illegible]

$$U \cap V: \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Determinant

Recursive method

$$A = \begin{pmatrix} \boxed{\overset{+}{a_{11}} \quad \overset{-}{a_{12}} \quad \overset{+}{\ldots}} \\ \\ \\ \end{pmatrix}$$

A_{11} is a cofactor = det of the n-1 × matrix after we cut away element 1, 1

$$|A| = a_{11} + A_{11} - a_{1,2} A_{1,2} + \ldots +$$

Ex:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} = -3$$

$$A_{12} = \begin{vmatrix} 0 & 0 \\ -1 & -1 \end{vmatrix} = 0$$

$$A_{1,3} = \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} = 3$$

$$|A| = 1 \cdot (-3) + 2 \cdot 3 = 3$$

I can also take cofactors of the second line

A way to get the inverse

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{3,3} \end{pmatrix}$$

Binet theorem

If $A, B \in \mathbb{R}^{n,n}$ then

$$|AB| = |A||B|$$

Ero's and determinant

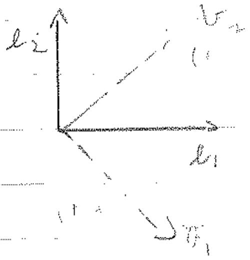
- Ero of type I) do not change the determinant
- Ero of type II) multiply the determinant by a scalar
- Ero of type III) change the sign of the determinant

Ese

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad |B| = 1 \cdot A_{33} = 3$$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $f(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = Eigen vecto

& $\lambda = 2$ = Eigenva



$\{v_1, v_2\}$

$M_f = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

Eigen values

$Mv = \lambda v$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

$Mv - \lambda v = 0$

$Mv - \lambda I_2 v = 0 \Rightarrow (M - \lambda I)v = 0$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\exists$ non-zero solution $\Longleftrightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 = 1$ $1 - \lambda = 1 \Longleftrightarrow \lambda = 0$

$1 - \lambda = -1 \Longleftrightarrow \lambda = 2$

$(M - \lambda I)v = 0$

$\lambda_1 = 0 \quad \lambda_2 = 1$

$Mv = 0 \Longleftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Longleftrightarrow \{(t, -t) : t \in \mathbb{R}$

$\Longleftrightarrow \mathcal{L}\{(1,-1)\}$

Ex 2×2

$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ we have to take out λ from the value th are not zero in the identity matrix

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda)(3-\lambda) = 0 \qquad \lambda = 3 \qquad \lambda = 1$$

Definition: The characteristic polynomial $p(x) = |A -$

Eigen values $\Longleftrightarrow$ Solution of $p(x) = 0$

Properties of the characteristic polynomial

$p(x) = |A - xI|$

$p(x) = (-1)^n x^n + \dots + a_1 x + a_0$

.) A $n \times n \Rightarrow \deg p(x) = n$

..) $a_0 = p(0) = |A|$

...) $a_{n-1} x^{n-1}$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad p(x) = |A - xI| = (1-x)(2-x)(3-$

$$(-1)x^3 + \underbrace{(3+2+1)}_{\text{trace of } A} x^2 + \dots x + \underbrace{6}_{\text{determ}}$$

Trace of a matrice = sum diagonal elements

Note $n = 2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad p(x) = |Ax - xI| =$

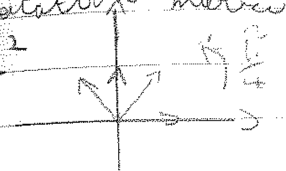
$$= \underbrace{x^2}_{(-1)^2} - \underbrace{(a+d)}_{tr(A)} x + \underbrace{(ad - bc)}_{\det A}$$

Lemma

A is invertible $\Longleftrightarrow$ 0 is not an Eigen values ~~is not~~ for A

It can be that Eigenvalue they might be not real
if we are working in real we say no real ~~soluti~~ Ei
if we are working in the $\mathbb{C}$ field we compute.

Rotation matrix
$\mathbb{R}^2$



$f : \mathbb{R}^2 \to \mathbb{R}^2$ rotation of $\frac{\pi}{4}$

$$\left(f(e_1) \,\middle|\, f(e_2) \right) = M_f \quad \to \text{canoni}$$

$$f(e_1) = \cos\frac{\pi}{4} e_1 + \sin\frac{\pi}{4} e_2$$

$$M_f = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

these are not real Eigenvectors
Eigen value

Ex: $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$p(x) = |A - xI| = \begin{vmatrix} 1-x & -1 \\ 1 & 1-x \end{vmatrix}$$
$$= x^2 - 2x + 2 \Rightarrow \text{complex Eigenva}$$

Prop A
$v_1, \dots, v_k$ Eigenvectors
$\lambda_1, \quad, \lambda_k$ Eigenvalues
$\lambda_i \neq \lambda_j \quad i \neq j \quad \Rightarrow v_1, \dots, v_k$ are Linearly a

Proof (idea)
$k = 2 \quad v_1, v_2$ eigen vectors
$\lambda_1 \neq \lambda_2$ eigen values
want to show v_1, v_2 are L.I.
$a_1 v_1 + a_2 v_2 = 0$ multiplying by A $\nearrow$ Eq II
$a_1 A v_1 + a_2 A v_2 = 0 \Rightarrow a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0$
i) if $\lambda_1 = 0$ complete

↗ l'esponen $(1-x)^2$

Eigen values : $\lambda_1 = 1$ mult $(\lambda_1) = 2$

$\lambda_2 = 2$ mult $(\lambda_2) = 1$

↘ since $(2-x)$

Notes

$p(x) = (\lambda_1 - x)^{m_1} \cdots (\lambda_r - x)^{m_r}$

mult $(\lambda_i) = m_i$

! $A \in \mathbb{R}^{n,n}$

$\sum$ mult $(\lambda_i) = n$ → dimensione della matrice

The multiplicity gives us a bound

Thm

$0 < \dim E_\lambda < \text{mult}(\lambda)$

$E = \{v_1, v_2\}$ → a Basis

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$$

$v_1 \longmapsto \lambda_1 v_1$ $\quad M_f = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$v_2 \longmapsto \lambda_2 v_2$

Bases and Eigenvectors

Thm

$A \in \mathbb{R}^{n,n}$

$\underset{1}{\text{mult}(\lambda_i)}$ $\left[\lambda_1, \dots, \lambda_n \right.$ Distinct eigenvalues

$\Rightarrow \exists$ basis of Eigenvectors of $\mathbb{R}^n$

Proof $v_1, \dots, v_n$ eigenvectors

$\lambda_1, \dots, \lambda_n$ distinct

$\Rightarrow \{v_1, \dots, v_n\}$ L.I

← richiami

$AP = P\Delta$

? P invertible ? if 3 are distinct Eigen values → is invertible
it can be invertible even if the 2 values are not distinct

Def
$A, B \in \mathbb{R}^{n,n}$
A is similiar to B
iff $\exists\, P \in \mathbb{R}^{n,n}$ invertible
such that $P^{-1}AP = B$

Note equivalence relation

Def $A \in \mathbb{R}^{n,n}$ is diagonalizable iff
1) all Eigen values are real
2) A is similiar to Δ, $\Delta \in \mathbb{R}^{n,n}$ is diagonal

note
$A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$ is it diagonizable? → Yes

1) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3 \in \mathbb{R}$
2) $I^{-1}AI$ → in complete
Is A diagonalizable ?

Prop $A \in \mathbb{R}^{n,n}$ is diagonalizable $\Longleftrightarrow$ $\exists$ basis of $\mathbb{R}^n$ of eigenvec and all the Eigenvalues are real
$\Longleftarrow$) want to show that given the hyptoses the matrix is diagonalizable ?
Proof (idea)
? P ? invertible such that $AP = P\Delta$
$\lambda_1 \dots \lambda_n$ eigen values
$\{v_1, \dots, v_n\}$ eigenvector

$$\begin{array}{lcl} \dim E_{\lambda_1} < mult(\lambda_1) & = & m_1 + \\ \dim E_{\lambda_2} & & m_2 + \\ \hline m & & n \end{array} \quad =$$

$$m < n$$

There are situation in which matrices are not dia
nalizable :

Ex

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonalizable ?

1) Find eigen values $p(x) = (1-x)^2 \Rightarrow \lambda = 1$ mult(λ

2) $\dim E_\lambda = ?$ to find E_λ we solve

$(A - \lambda I) v = 0$
2x2 by rouché-capelli

$2 - r(A - I) \Rightarrow 2 - r\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow$ the $\dim E_\lambda = 2 - 1 = 1$

$\dim E_\lambda <$ mult $\Rightarrow A$ is not diagonalizable

Note:

repeated eigenvalues $(mult(\lambda_i) > 1)$
$\not\Downarrow$
not diagonalizable

ex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\lambda = 1$ $mult(\lambda) = 2$
but A is diagonalizable.

Def:

$f: \mathbb{R}^n \to \mathbb{R}^n$ linear is simple if M_f is diagona
(any basis)

Rotation in higher dimensional spaces

Orthogonal matrices

Def $v_1, v_2 \in \mathbb{R}^{n,1}$

$\mathbb{R} \ni v_1 \cdot v_2 = (v_1)^T \cdot v_2 \quad \in \mathbb{R}^{1,1}$

Diagonalization

Criterion to determine if a matrix is diagonalizable:

$A \in \mathbb{R}^{n,n}$ is diagonalizable

$\Updownarrow$

1) All real Eigen values $\lambda_1, \dots \lambda_r$

2) $mult(\lambda_i) = \dim E_{\lambda i} \quad \forall i = 1, \dots, r$

$$p(x) = |A x - I| = (\lambda_1 - x)^{m_1} \dots (\lambda_r - x)^{m_r}$$

$$\dim E_{\lambda i} = n - r(A - \lambda_i I) \sim \bar{e} \text{ ver}$$

Now that I know that A is diagonalizable I have to find P and Δ

$$P^{-1} A P = \Delta$$

$$P = ((v_1 | v_2 | \dots | v_n) \leftarrow$$

I can chose them at random but when I put them in the matrix it must be in order

$\{v_1, \dots, v_n\}$ L.I Eigen vectors

$$\Delta = \begin{pmatrix} \lambda_1 & 0 & 0 & \\ 0 & \lambda_2 & 0 & \\ 0 & 0 & \lambda_2 & \\ \dots & & & \lambda_n \end{pmatrix}$$

Symmetric matrices

A matrix is symmetric $\Longleftrightarrow S^t = S$

Theorem

$S \in \mathbb{R}^{n,n}$ if $S^t = S \Rightarrow S$ is diagonalizable

moreover:

$$P^{-1} S P = \Delta$$

and P is orthogonal

Hence

$$P^t S P = \Delta$$

Proposition: $S \in \mathbb{R}^{n,n}$ symmetric

v_1, v_2 eigenvectors $\lambda_1 \neq \lambda_2$ eigen values

Then

$v_1 \cdot v_2 = 0$ orthogonal

7.5

and Eigen spaces $E\lambda_1 = \mathcal{L}(1,1)$ also $\mathcal{L}(3,3)$: doesn't matter)
$E\lambda_2 = \mathcal{L}(1,-1)$

? P, Δ ?

$\begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$ pick the one with det > 0

normalize: $\frac{v}{\|v\|}$ has lenght 1

$$\frac{(1,-1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1,-1)$$

$$\frac{(3,3)}{\sqrt{9+9}} = \frac{(3,3)}{3\sqrt{2}} = \frac{1}{\sqrt{2}}(1,1)$$

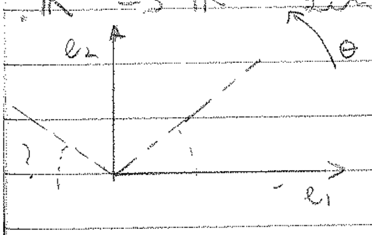
$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ orthogonal

$$P^{t}AP = \Delta = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

Rotation

Convention: The rotations are always counter clock wise.
Rotation is a endomorphism; it doesn't change geometrical shape

$\mathbb{R}^2 \to \mathbb{R}^2$ linear map



$$\mathbb{R}_\theta \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
2x2

Note $(R_\theta)^t R_\theta = I_2$
R_θ is orthogonal

Quadratic Form (preview)

DEF $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$v \longmapsto vP(v)$

$P \in \mathbb{R}^{n,n}$

$n=1$ $P=(a)$

$f(x) = x(a)x = ax^2$

$n=2$ $P\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$f(v) = v(Pv)$ $v\begin{pmatrix} x \\ y \end{pmatrix}$

$f(x\ y) = \begin{pmatrix} x \\ y \end{pmatrix} \cdot P\begin{pmatrix} x \\ y \end{pmatrix} = (x\ y)\ P\begin{pmatrix} x \\ y \end{pmatrix}$

$= (x\ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$= ax^2 + (b+c)xy + dy^2$

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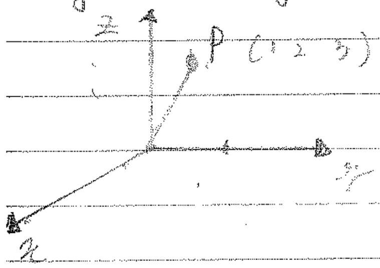
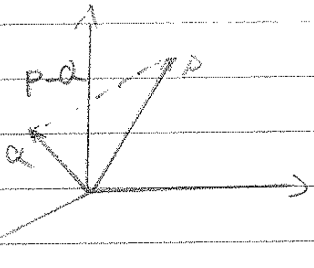
Geometry



$\vec{OP} = \vec{P} = 1\vec{i} + 2\vec{j} + 3\vec{k}$ :: position vector

What happens if we take two points!



$\vec{P} - \vec{Q}$ is the vector going from $\vec{Q}$ to $\vec{P}$

$\vec{P} - \vec{Q} = \vec{QP}$ it is also called displacement vector

We want to distinguish between points and vectors

Idea $\quad \underset{\vec{n}\cdot\vec{v}}{\underset{\|}{ax+by+cz}}=0$

$$\vec{n}=(a,b,c)$$
$$\vec{v}=(x,y,z)$$

Solve $\boxed{\vec{n}\,\vec{v}=d}$

$A=(a,b,c) \qquad A^{+}=(a\ b\ c\,|\,d)$

$\left.\begin{array}{l} r(A)=1 \\ r(A^{+})=1 \end{array}\right\}$ since a,b,c are not all zero; the system is consistent and we have ∞^{2} solutions

Take v_0 such that $n\cdot v_0=d$

The system

$$\vec{n}\cdot\vec{v}=d=\vec{n}\cdot\vec{v_0}$$
$$\boxed{\vec{n}\cdot(\vec{v}-\vec{v_0})=0}\quad(1)$$

The solution set is the vector space of dimension 2

$\mathcal{L}\{\vec{p},\vec{q}\}\Rightarrow$ This the solution of system (1)

Thus

$$\vec{v}-\vec{v_0}=s\vec{p}+t\vec{q}\Rightarrow\vec{v}=s\vec{P}+t\vec{q}+\vec{V_0}$$

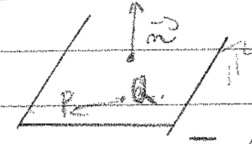
NOTE

I) $ax+by+cz=d\Rightarrow$ Cartesian equation

II) $\vec{n}=(a,b,c)$ normal vector

The normal vector is orthogonal to all vectors sitting in the plane

Ex) take the plane π



$$\pi: ax+by+cz=d$$

$$(\vec{P}-\vec{Q})\cdot\vec{n}=\vec{p}\cdot\vec{n}-\vec{q}\cdot\vec{n}=(x_p,y_p,z_p)\cdot\vec{n}=$$
$$=ax_p+by_p+cz_p$$

$$r(A)=1 \Rightarrow A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}=\begin{pmatrix} \vec{n}_1 \\ \vec{n}_2 \end{pmatrix} \Rightarrow \pi_1 \parallel \pi_2$$

$$r(A^+)=\begin{cases} 1 \Rightarrow \pi_1 \equiv \pi_2 \\ 2 \Rightarrow \pi_1 \cap \pi_2 = \emptyset \end{cases}$$

~~if~~

$$r(A)=2 \Rightarrow r(A^+)=2 \quad \Rightarrow r(A^+) \text{ cannot be } 3$$

$$\Rightarrow \exists \ \infty^1 \text{ solution}$$

The intersection of 2 planes that are not parallel and do not coincide is a line

Def The intersection of ~~planes~~ π_1 and π_2 such that π_1 and π_2 are not parallel $\pi_1 \not\parallel \pi_2$ is called a line

Note

I) $$l: \begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \end{cases} \quad \text{Cartesian equation}$$

II)

The parametric equation of a line

a) I solve the system, it has 1 free parameter

$$t\vec{v}+\vec{v}_0 \qquad t\in\mathbb{R} \leftarrow \text{particular solution of the homogeneous system}$$

Example

π_1 $x=y$ $\qquad \vec{n}_1(1,-1,0)$

π_2 $y=z$ $\qquad \vec{n}_2(0,1,-1)$

Solve the system $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & \boxed{-1} \end{pmatrix}$ $\quad y=t$

solution $\{(t,t,t) : t\in\mathbb{R}\}$

3 points

$P=(0,0,1)$

$Q=(0,0,0)$

$R=(1,1,1)$

$$\begin{cases} c = a \\ d = 0 \\ a+b+c=d \end{cases}$$

∞^1 solutions $\Rightarrow \exists 1$ plane

$x - y = 0$

Note

I) $\pi \ni P \iff$ solve one equation

II) given 3 points not on a line $\Rightarrow \exists$ only one plane

III) given 4 general points there is no plane containing them

Planes condition for parallel and orthogonal

I) $\pi_1 // \pi_2 \iff \vec{n}_1 // \vec{n}_2$

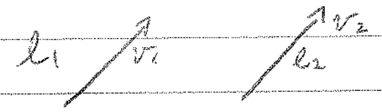
$\iff \vec{n}_1, \vec{n}_2$ L.D.

II) $\pi_1 \perp \pi_2 \iff \vec{n}_1 \perp \vec{n}_2$

$\iff \vec{n}_1 \cdot \vec{n}_2 = 0$

Lines conditions for being parallel and orthogonal

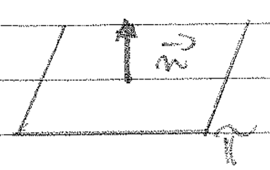
I) $l_1 // l_2 \iff \vec{v}_1 // \vec{v}_2$



II) $l_1 \perp l_2 \iff \vec{v}_1 \perp \vec{v}_2$

Lines and plane

I) $\vec{v}$ l $\quad l // \pi \iff \vec{v} \perp \vec{n}$



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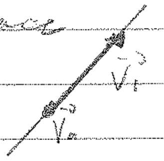
Lines: Two lines in space

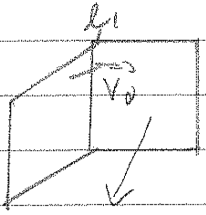
direction position vector

$l_1 : \vec{v}_1 t + \vec{v}_0$



$l_2 : \vec{v}_2 t + \vec{u}_0$

$\vec{v}_1 = \vec{n}_1 \times \vec{n}_2$



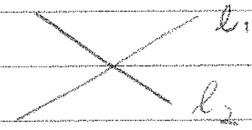
I) Two lines can be parallel

$l_1 // l_2 \Longleftrightarrow \vec{v}_1 // \vec{v}_2$

If the direction vector are proportional then the lines are parallel; if the position vector are the same they coincides

II) $l_1 \cap l_2 \neq \emptyset$

solve 4 x 3 linear system



III) In space two lines can neither be parallel nor intersect → we speak about skew lines

1) l_1, l_2 not parallel

2) $l_1 \cap l_2 = \emptyset$

Skew lines do not lie on a common plane.

So:

If $l_1 // l_2$ or $l_1 \cap l_2 \neq \emptyset$

$\Rightarrow \exists$ A plane such that $\pi \supset l_1$, $\pi \supset l_2$

⓪

The idea is to compute the direction vector → to see if they are parallel; then) solve the linear system, ~~&~~ Decide whether the lines intersect or they are skew

Example

$l_1: \; x = y = z \iff \begin{cases} x = y \\ y = z \end{cases} \rightarrow$ I need to get the direction vector v_1

$l_2: \; \underbrace{\boxed{(1,0,1)}}_{v_2} t \iff \begin{cases} x = t \\ y = 0 \\ z = t \end{cases}$

to find v_1

solve $\begin{cases} x = y \\ y = z \end{cases}$ $\quad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad y = s$

$\begin{cases} x = s \\ y = s \\ z = s \end{cases} \iff \underbrace{(1,1,1)}_{v_1} s + (0,0,0)$

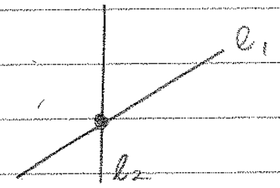
$v_1 = (1\,1\,1)$ and $v_2\,(1\,0\,1)$ are not proportional

so l_1 is not parallel to l_2

points of l_i are solution of the linear system

$P \in l_1 \iff P(s, s, s)$

$Q \in l_2 \iff Q(t, 0, t)$



find $s, t \in \mathbb{R}$

$P \equiv Q$

$\begin{cases} s = t \\ s = 0 \\ s = t \end{cases} \iff s = t = 0$

They are not skew, they intersect

to find the plane in which they lie

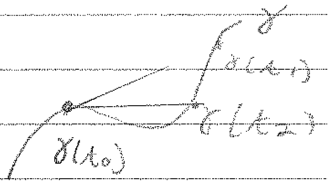
$\vec{n} = \vec{v_1} \times \vec{v_2} = (1,1,1) \times (1,0,-1) = \alpha (1,0,-1)$

Plane $\quad \vec{n} \cdot (x, y, z) = d$

$x - z = d$

$\Pi \ni$ complete

Idea



$$\frac{\gamma(t_1)-\gamma(t_0)}{t_1-t_0}$$ ↗ this is a vector

$$\|$$

$$\left(\frac{x(t_1)-x(t_0)}{t-t_0}, \quad y \quad z\right)$$

limit $t_1 \to t_0$

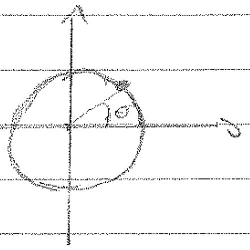
Ex

$\gamma:[0,2\pi] \to \mathbb{R}^2$

$t \to (\cos t, \sin t)$

is this regular?

$\gamma'(t) = (x'(t), y'(t)) = (-\sin t; \cos t)$



$\|\gamma'(t)\| = 1 \neq 0 \quad \forall t \Rightarrow$ Regular

$\gamma'(0) = (0,1)$ $\quad\quad \gamma(0) = (1,0)$

$\gamma'\left(\frac{\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right)$ $\quad\quad \gamma(\pi/4) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

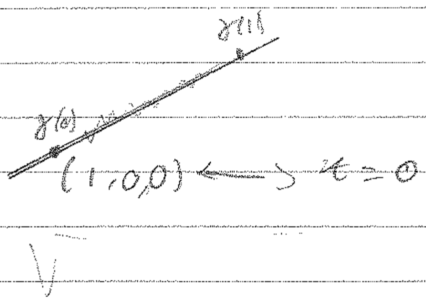
Example

$\gamma: \mathbb{R} \to \mathbb{R}^3$

$t \longmapsto (t+1, 2t, 3t)$

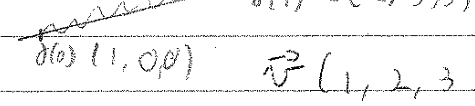
$$\|$$

$t(1,2,3) + (1\,0\,0)$



$\gamma'(t) = (1,2,3)$

$\|$



$\vec{v}(1,2,3)$

$\|\vec{v}\| \cdot \underset{t_1-t_0}{(1-0)} = \sqrt{1+2^2+3^2} \cdot 1 = \sqrt{14}$

$F : D \subseteq \mathbb{R} \to \mathbb{R}$ campo elettrico

$(x, y) \longmapsto F(x, y)$

$\gamma : [a, b] \in \mathbb{R} \longrightarrow \mathbb{R}^2$ fili percorsi da corrente

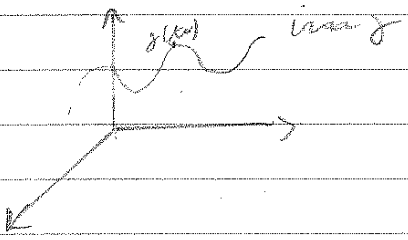
$t \longmapsto (x(t), y(t))$

$$\int_\gamma F = \int_a^b F(x(t), y(t)) \, \|\gamma'(t)\| \, dt$$

Back to curves

$\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$

$t \longmapsto (x(t), y(t), z(t))$

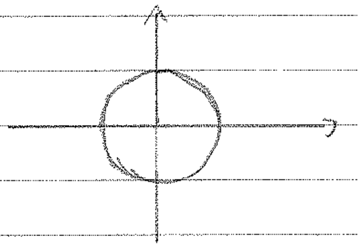


γ regular

tangent line ~~to~~ γ in $\gamma(t_0)$

$\gamma'(t_0)s + \gamma(t_0)$

Ex



$$\gamma'(t_0)s \underset{\|}{+} \gamma(t_0)$$

$$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)s + \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\begin{cases} x = -\frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2} \\ y = \frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2} \end{cases}$$

$x + y = \sqrt{2}$

↓

Tangent line

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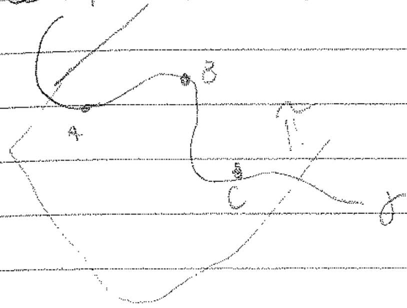
Example on plane curve



We want to decide whether γ is plane
I want to find the plane containing the 3 points; N.B if there are many planes then the 3 point lie on a line

Ex

$A(1, 0, 0)$
$B(2, 0, 1)$
$C(0, 1, 1)$

Find $\pi \ni A, B, C$

method I

$\pi: ax + by + cz = d$

Find $a, b, c, d \in \mathbb{R}$

solve the linear system

$$\begin{cases} \pi \ni A & a \cdot 1 + b \cdot 0 + c \cdot 0 = d \\ \pi \ni B & a \cdot 2 + b \cdot 0 + c \cdot 1 = d \\ \pi \ni C & a \cdot 0 + b + 1 + c \cdot 1 = d \end{cases}$$

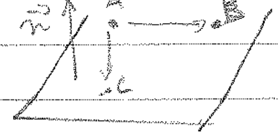
$$\underset{\pi}{\underset{\Uparrow}{r(A)=2}} \quad \underset{\parallel}{\underset{\Uparrow}{r(A|)=3}}$$

The system can have ∞^1 or ∞^2 solutions

If is ∞^1 ⇒ I have one plane passing through the 3 points (there can be multiples)

If is ∞^2 ⇒ the points lie on a line

method 2



$\vec{n} = (\vec{B} - \vec{A}) \times (\vec{C} - \vec{A}) =$

$= (2-1, 0-0, 1-0) \times (-1, 1, 1) =$

$= (1, 0, 1) \times (-1, 1, 1)$ = take determinant $\vec{n} = (-1, -2, 1)$

now we ask π to pass trought a point for ex. A

$\pi: \vec{n} \cdot (x, y, z) = d$

$A \in \pi \Longleftrightarrow \vec{n} \cdot (1, 0; 0) = d.$

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Functions

$F: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$

I) $n=1$ $m=1$ $F(x) \in \mathbb{R}$

II) $n=1$ $m=2,3$ $F(t) = \gamma(t) \to$ curve

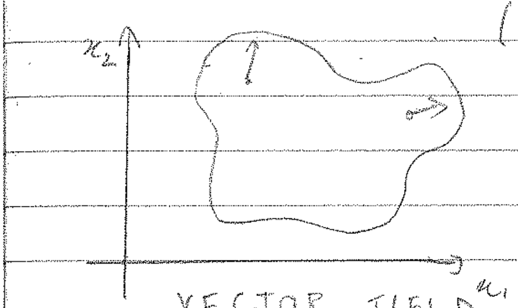
III) $n=2$ $m=2$

$F: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$

$(x_1, x_2) \to \mathbb{F}(x_1, x_2)$

$\| $

$(F_1(x_1, x_2), F_2(x_1, x_2))$



VECTOR FIELD

IV) $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$(x_1, \dots, x_n) \longmapsto A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

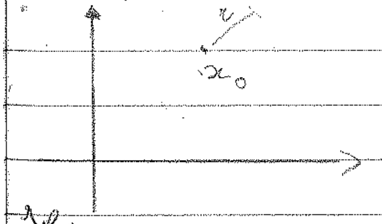
Linear map

$A \in \mathbb{R}^{m,n}$ $m \times n$

note We are interested in $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^1$ $n=2,3$

Continuity $\to$ Continuous functions

DEF



$x_0 \in \mathbb{R}^n$

$r \in \mathbb{R}$ $r > 0$

the neighborhood of x_0 of radius r

$I_r(x_0) = \{ x \in \mathbb{R}^n : \|x - x_0\| < r \}$

If:

$n=2$ disc of center x_0 and radius r

$n=3$ Ball of center x_0 and radius r