NUMERO : 448

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A P P U N T I

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MATERIA : Analisi Matematica I
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Logic and sets

When we can establish whether a statement is true or false we say that this statement is a formula.

The negation of a formula P is denoted by the symbol $\neg P$. The formula P is true when P is false and viceversa

Starting with 2 or several formulas we can build new formulas using connectives.

The conjunction of P and Q ($P \wedge Q$ = P and Q) is true when both formulas are true, false in the other cases

The disjunction of P and Q ($P \vee Q$ = P or Q) is false when both formulas are false, and it is true in the other cases.

The conjunction $P \wedge (\neg P)$ is always false

The disjunction $P \vee (\neg P)$ is always true

P	$\neg P$
T	F
F	T

P	Q	P and Q ($\wedge$)	P or Q ($\vee$)
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

(cosa con i circuiti?)

Implication $P \Rightarrow Q$ (if P then Q) is F if P is T and Q is F

it is true in the other cases

P is called hypothesis or assumption

Q is called consequence or conclusion

We can state that $P \Rightarrow Q$ also saying that

- P is said to be a sufficient condition for Q
- Q is said to be a necessary condition for P

We obtain a formula in different way using a quantifier

1) The universal quantifier $\forall$ (for all)

$\forall x : p(x)$ means that the property is true for all x

2) The essential quantifier $\exists$ (there exist a)

$\exists x : p(x)$ means that the property $p(x)$ is true for at least one x.

The negation of one universal proposition is a counter example ~~not~~

We can negate a quantified predicate by changing the quantifier and by negating the property. PDF

$\neg(\forall x : p(x)) \Longleftrightarrow (\exists x : \neg p(x))$

$\neg(\exists x : p(x)) \Longleftrightarrow (\forall x : \neg p(x))$

A predicate with two or more variables is called also a relation. We consider here predicate with 2 variables $p(x; y)$. Using quantifiers we get 8 examples

$\exists x, \exists y : p(x; y)$ $\quad\quad$ $\exists y, \exists x : p(x, y)$

$\exists x\ \forall y : p(x, y)$ $\quad\quad$ $\forall y\ \exists x : p(x, y)$

$\forall x\ \exists y : p(x, y)$ $\quad\quad$ $\exists y\ \forall x : p(x, y)$

$\forall x, \forall y : p(x, y)$ $\quad\quad$ $\forall y\ \forall x : p(x, y)$

These 8 statements have different meanings.

The negation of a multiply quantified predicate is obtained by changing the quantifier and by negating the relation. For istance:

$\neg(\forall x, \exists y) : p(x; y) \Longleftrightarrow \exists x, \forall y : \neg(p(x, y))$

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Given 2 sets A and B we define the following sets:

The union set = $A \cup B$
which is the set of all x belonging to A or B

The intersection set = $A \cap B$
which is the set of all x belonging to A and belonging to B

The difference = $A \setminus B$
the set of all x that belong to A and do not belong to B.

The symmetric difference = $A \Delta B$
the set of the elements that belong to A and do not belong to B or belong to B and do not belong to A.

$x \in A$	$x \in B$	$A \cap B$
T	T	T
T	F	F
F	T	F
F	F	F

Given a set $A \subseteq M$, the complement of A in M ($\bar{A}$ or CA) is the set $M \setminus A$

De Morgan laws show the relation between complement, union and intersection.

$$C(A \cup B) = CA \cap CB$$
$$C(A \cap B) = CA \cup CB$$

Lesson (slides 2) → Algebraic operations in $\mathbb{N}$

In $\mathbb{N}$ the sum has the following properties:
- Associative: $\forall x, y, z \in \mathbb{N}: x + (y+z) = (x+y) + z$
- It is Commutative: $\forall x, y \in \mathbb{N}: x+y = y+x$
- The number 0 is the identity element or neutral element

In the product,
- Is associative: $\forall x, y, z \in \mathbb{N}: x(yz) = (xy)z$
- Is commutative: $\forall x, y \in \mathbb{N}: xy = yx$
- The number 1 is the neutral element

The product distributes over the addition
$\forall x, y, z \in \mathbb{N}: (x+y)z = xz + yz$

In other terms the is no rational number $\frac{p}{q}$ such that $\frac{p^2}{q^2} = 2$

proof

$\frac{p^2}{q^2} = 2$, $p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even

$p = 2K \rightarrow 4K^2 = 2q^2 \Rightarrow 2K^2 = q^2$

q^2 is even $\rightarrow q$ is even.

The contradiction is tha p and q must be relativelly prime so there's not rational number x such that $x^2 = 2$; $x = \sqrt{2}$

$\mathbb{R}$ is the numerical set that allows us to define a one to one relation between the points and the line

1) $\forall x \in \mathbb{R}$ $\exists p$ in the line corr. to x

2) $\forall$ point in the line there is one x corr. to P

$\mathbb{R}$ is complete

Rational numbers: have an finite decimal expansion or an infinite periodic expansion.

Non rational numbers: have an infinite non periodical decimal expansion.

- Given two rational numbers q_1 and q_2 there are infinitelly many rational numbers and infinite many irrational numbers between them. The same for 2 irrational numbers.
- We can approximate an irrational number as well as we please with rational numbers and viceversa.
- It is impossible to find a non-empty interval of $\mathbb{R}$ containi only rational or only irrational numbers.

Ordering

In $\mathbb{R}$ it is possible to define an ordering or order relatio its notation is $\leq$ and its properties are:

1) It is a total ordering: given 2 different numbers x e y it always possible to say if we have $x < y$ or $y < x$ →

Using the absolute value we can describe some subsets of $\mathbb{R}$
We fix $a > 0$

- $\{x \in \mathbb{R} : |x| = a\} = \{-a ; a\}$ set of points having distance a from the origin.
- $\{x \in \mathbb{R} : |x| < a\} (-a, a)$ is the set of points having distance smaller than a from the origin
- $\{x \in \mathbb{R} : |x| \leq a\} = [-a ; a]$ is the set of points having distance smaller or equal to a from O
- $\{x \in \mathbb{R} \ |x| > a\} = (-\infty, -a) \cup (a ; +\infty)$ is the set of points having distance greater than a from the origin
- $\{x \in \mathbb{R} : |x| \geq a\} = (-\infty -a] \cup [a +\infty)$ is the set of points having distance greater or equal to a from the origin.

Remarks

- An upper or lower bound does not necessarily belong to X
- If X admits a maximum then is bounded from above and the maximum X_m is an upper bound of X
- If X admits a minimum then is bounded by below and the min X_m is ~~a~~ lower bound ~~of~~ for X.

Examples

a) $X = \mathbb{N}$ is bounded by below, $\mathbb{N} \subseteq [0, +\infty)$, e $\min \mathbb{N} = 0$

b) $X = (-\infty, 1]$ is bounded from above and $\max X = 1$

c) $X = (-5, 12]$ is bounded, $\max X = 12$ and $\nexists \min X$

d) $X = \left\{ \frac{n}{n+1} ; n \in \mathbb{N} \right\}$ is bounded, $\min X = 0$ $\nexists \max X$

The supremum of a set

Def. Let $X \subset \mathbb{R}$ be a set bounded from above.
The least upper bound or supremum of X is the smallest upper bound of X.

$S = \sup X$ = supremum of X

In other words

- $\forall x \in X \quad x \leq S$, (i.e. S is an upper bound for X)
- $\forall z < S, \ \exists x \in X : x > z$ (any number smaller than S is not an upper bound)

Def. If X is not bounded from above one defines: "$\sup X = +\infty$"

The infimum

Def. Let $X \subseteq \mathbb{R}$ be a set bounded by below. The greatest lower bound or infimum of X is the largest lower bound of X

$s = \inf X$ = infimum of x

i.e.:

- $\forall x \in X, \ x \geq s$ (i.e. s is an ~~upper~~ lower bound for X)
- $\forall z > s, \ \exists x \in X \quad x < z$ (any number larger than s is not a lower bound, i.e. s is the greatest lower bound)

Def. If X is not bounded by below we say that $\inf X = -\infty$

A set of relations is the function.
Function is a law that associate to each value x at most one y.

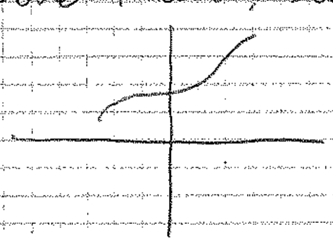
The element y associate to x is called the ~~image~~ image of x by f or under f

Domain of f = dom f $\{x \in X : \exists y \in Y$ such that $y = f(x)\}$
Range of f = im f $\{y \in Y : \exists x \in X$ such that $y = f(x)\}$
Domain → where I start from ; range → where I arrive
Graph = $\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in \text{dom } f\}$
We have that

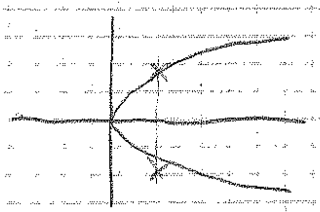
$$\text{dom } f \subseteq X \qquad \text{im } f \subseteq Y$$

The gr

The graph of a real-valued function of a real value is a subset of the plane, the opposite is not necessarily true.

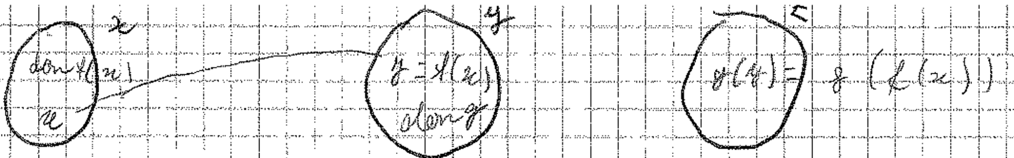


function



relation not a function

Basic functions
- Constant functions $y = c \quad c \in \mathbb{R}$
- Powers $y = x^n$
- Exponentials $y = a^x$
- trig. $y = \sin x, \cos x$



$x \rightarrow g(f(x))$ Composition of f and g

$(g \circ f)(x) = g(f(x))$

$x \in \mathbb{R} \xrightarrow{f} x^2+1 \xrightarrow{=} y \xrightarrow{g} \sqrt{y}$

$x \xrightarrow{f} x^2+1 \xrightarrow{g} \sqrt{x^2+1}$ $\quad$ dom $(g \circ f) = \mathbb{R}$

The value $f(x)$ must b compatible with g

- $f(x) \in$ im f
- $f(x) \in$ dom g

$(g \circ f)(x)$ is $\neq \emptyset$ if and only im $f \cap$ im $g \neq \emptyset$

example

$x \rightarrow 1+x^2$ $\quad$ dom $f = \mathbb{R}$

im $f = [1 ; +\infty)$

im $f \subset$ dom g $\quad$ OK

remark

It is possible to define composition if dom g and im f have a point in common.

we have that

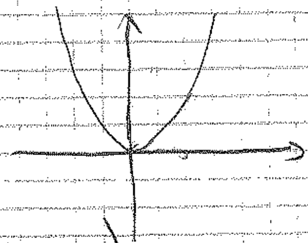
$x \in g \circ f \rightarrow$ vedi

~~ese~~

$x \xrightarrow{g} \sqrt{x} \xrightarrow{f} 1+(\sqrt{x})^2 = 1+x$ $\quad$ dom $f \circ g = x \geq 0$

with even value

$y = x^2$



dom $f = \mathbb{R}$

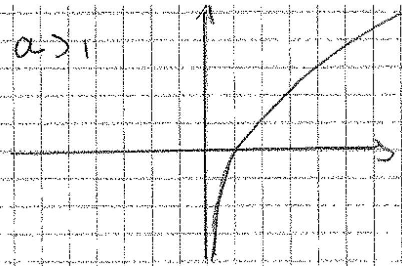
im $f = [0 ; +\infty)$

not inj in $\mathbb{R}$

Chose $[0 ; +\infty)$ dom $f = [0 ; +\infty)$

The symbol $\sqrt{9}$ is not the solution of $x^2 = 9$

Logarithmic function



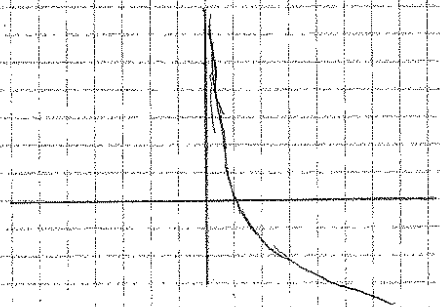
$a > 1$

$a^0 = 1 \rightarrow \log_a 1 = 0$

$a^1 = a \rightarrow \log_a a = 1$

$\text{Im } f = \mathbb{R}$

$\text{Dom } f = (1 \, ; +\infty)$

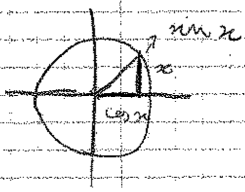


THE $\log_1 x$ does not exist

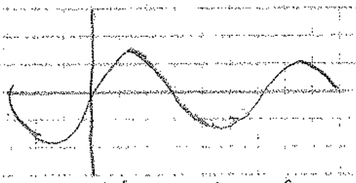
Trig functions

· If $x \in \text{dom } f$ then $x \pm p \in \text{dom } f \quad \forall x \in \text{dom } f$

Then the function is said to be periodic



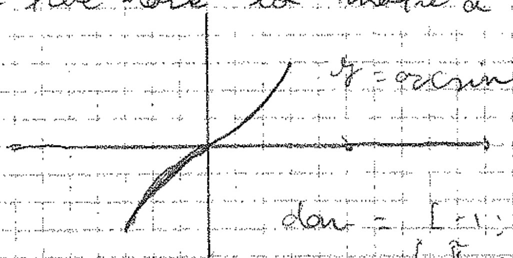
$y = \sin x$



$y = \sin x$ is not injective so we have to make a restriction

$\mathbb{R} \rightarrow [-1 \, ; 1]$ restriction

$\left[-\frac{\pi}{2} \, ; \frac{\pi}{2}\right] \rightarrow [-1 \, ; 1]$



$\text{dom} = [-1 \, ; 1]$

$\text{im} = \left[-\frac{\pi}{2} \, ; \frac{\pi}{2}\right]$

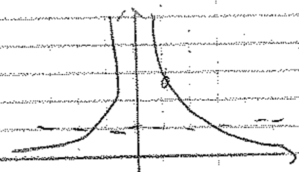
same for the cosine → LOOK IT UP

Infimum of a function

Def: the number $C \in \mathbb{R}$ is an upper bound for f in I if $\forall x \in I$ $f(x) \leq C$, If the function f admits an upper bound

Ex

$y = \frac{1}{x^2}$



$(0, +\infty)$ $\sup \frac{1}{x^2} = +\infty$ $\max \frac{1}{x^2} \nexists$

$\inf \frac{1}{x^2} = 0$ $\min \frac{1}{x^2} \nexists$

$[0, 1]$ $(0, 1)$ $\sup = +\infty$ $\max \nexists$

$\inf = 1$ $\min \nexists$

$\sup = 1$ $\max = 1$

$\inf = 0$ $\min \nexists$

If a fun. is monotone it is not injective because it can be constant

If a function is strictly monotone then it is injective

Proof

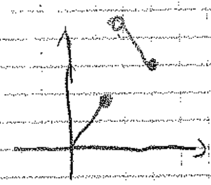
Suppose that $f(x)$ is strictly monotone

If the function is strictly increasing

$\forall x_1, x_2$ $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

or $x_2 < x_1 \Rightarrow f(x_2) < f(x_1)$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$



The opposite is not always true

Limits at infinity

Extended real line: $\mathbb{R}^* = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$

$-\infty$ ——— 0 — 1 ——— $\to +\infty$

$\mathbb{R}$

Ordering relation: $-\infty < +\infty$

$-\infty < x < +\infty \quad \forall x \in \mathbb{R}$

No algebraic operation; it is impossible to define some particular symbols, all

Ex $+\infty + (-\infty) = +\infty$ no because $-\infty = 0$ same reason

$+\infty + (-\infty) = -\infty$

$+\infty + (-\infty) = 0$

$+\infty + a = +\infty$

$+\infty + (-\infty) + a = +\infty + (-\infty)$

$a = 0$ no!

$+\infty$ and $-\infty$ are symbols and not numbers

Neighbourhood

1) $x_0 \in \mathbb{R} \quad r > 0$

$x_0 - r \quad x_0 \quad x_0 + r$

$I_r(x_0) = (x_0 - r, x_0 + r) =$ x_0 = centre, r = radius

$= \{x \in \mathbb{R} : |x - x_0| < r\}$

$a > 0 \quad A = \{x \in \mathbb{R} : |x| = a\} = \{-a; a\}$

$B = \{x \in \mathbb{R} : |x| < a\} = \quad -a \quad a \quad = (-a; a)$

$= \{x \in \mathbb{R} : -a < x < a\}$

So if I write

$|x - x_0| < r \Rightarrow -r < x - x_0 < r \Rightarrow x_0 - r < x < x_0 + r$

2) $+\infty$ $I_a(+\infty) = (a; +\infty)$ a smaller neigh. at $+\infty$ is a larger value of a

3) $-\infty$ $I_a(-\infty) = (-\infty; a)$

$$\lim_{x \to -\infty} f(x) = +\infty$$

$\forall A > 0 \quad \exists B \geq 0$

$\forall x : x \in \text{dom } f$ and $x < -B \Rightarrow f(x) > A$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$\forall A > 0 \quad \exists B \geq 0$

$\forall x \, . \, x \in \text{dom } f$ and $x < -B \Rightarrow f(x) < -A$

Ex. $\lim_{x \to +\infty} x^3 = +\infty$

$\forall A > 0 \quad \exists B \geq 0$

unknown

$\forall x : x > B \Rightarrow x^3 > A$ → given value

since dom $f = \mathbb{R}$

$x^3 > A \Longleftrightarrow x > \sqrt[3]{A} \qquad B = \sqrt[3]{A}$

Ex 2 $\lim_{x \to -\infty} \left(\frac{1}{2}\right)^x = +\infty$

$\forall A > 0 \quad \exists B > 0$

$\forall x : x < -B \Rightarrow \left(\frac{1}{2}\right)^x > A$

$\left(\frac{1}{2}\right)^x > A \Rightarrow \log_{\frac{1}{2}} x < \log_{\frac{1}{2}} A \qquad x < \log_{\frac{1}{2}} A = -\log_2 A$

$B = \log_2 A$

$\lim_{x \to +\infty} x^n = +\infty$ for $n = 1, 2, 3, 4, 5$

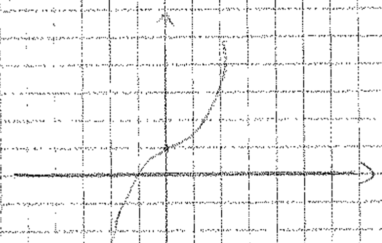
$\lim_{x \to -\infty} x^n = -\infty \quad n = 1, 3, 5$

$\lim_{x \to -\infty} x^n = +\infty \quad n = 2, 4, 6$

1) $f(x) = 1 + x^3 \qquad x_0 = 0$

2) $g(x) = \text{sing}(x^2)$

3) $h(x) = \dfrac{\sin x}{x}$



1) $f(x) = 1 + x^3$

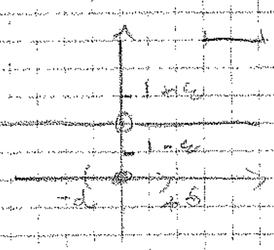
$$\lim_{x \to 0} 1 + x^3 = 1$$

$\forall \varepsilon > 0 \quad \exists \delta > 0$

$\forall x \in \text{dom } f$ and $0 < |x| < \delta \Rightarrow |1 + x^3 - 1| < \varepsilon$

$|x^3| = |x|^3 < \varepsilon \iff |x| < \sqrt[3]{\varepsilon} \qquad \delta = \sqrt[3]{\varepsilon}$

2) $g(x) = \text{sign}(x^2)$



$x = 0 \Rightarrow \text{sign}(0^2) = 0$

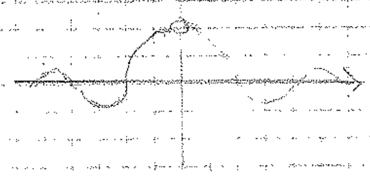
$x \neq 0 \Rightarrow x^2 > 0 \Rightarrow \text{sign}\, x^2 = 1$

$\lim_{x \to 0} \text{sing}(x^2) = 1$ → the limit look in the neigh. not in the point

$\forall \varepsilon > 0 \quad \exists \delta > 0$

$\forall x : x \in \text{dom } f$ and $0 < |x| < \delta \Rightarrow |\text{sign}(x^2) - 1| < \varepsilon$

3) $h(x) = \dfrac{\sin(x)}{x} \qquad x \neq 0$

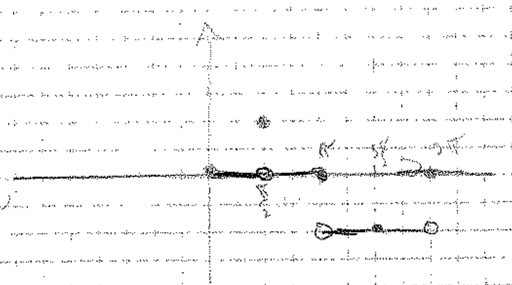


$$\lim_{x \to 0} \frac{\sin}{x} = 1$$

$\nexists\, h(x_0)$

Example

$y = [\sin x]$



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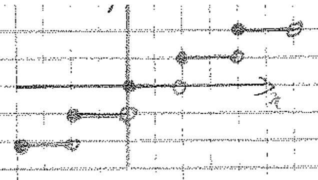
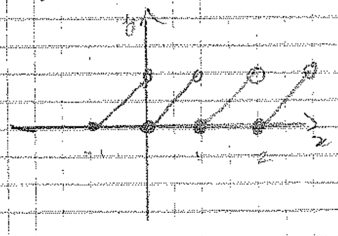
$y = [x]$



$\lim_{x \to 1^+} [x] = 1$ right continuous

$\lim_{x \to 1^-} [x] = 0$

$y = M(x)$



$\lim_{x \to 2^+} M(x) = 0$

$\lim_{x \to 2^-} M(x) = 1$

Jump $= 0 - 1 = -1$

$\lim_{x \to x_0} f(x) = l$ <
- $\exists f(x_0)$ and $f(x_0) = l$ → continuous
- $\nexists f(x_0)$

$\nexists \lim_{x \to x_0} f(x)$

$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$ — x_0 = Jump disc, 1^{st} kind disc

they are both

All other cases — x_0 = disc of 2^{nd} kind

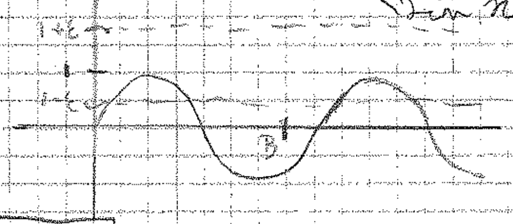
$\lim_{x \to +\infty} \sin x$ is not 1

$\exists \varepsilon > 0 \quad \forall B > 0$

$\exists x : \quad (x > B)$ and $(|\sin x - 1| \geq \varepsilon)$

$$\left\langle \begin{array}{l} \sin x - 1 \geq \varepsilon \\ \sin x - 1 \leq -\varepsilon \end{array} \right.$$

$$\left\langle \begin{array}{l} \sin x \geq 1+\varepsilon \text{ impossible} \\ \sin x \leq 1-\varepsilon \end{array} \right.$$



Theorem

$$\lim_{x \to x_0} f(x) = \left\langle \begin{array}{l} l \\ +\infty \\ -\infty \end{array} \right. \iff \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \left\langle \begin{array}{l} -\infty \\ l \\ +\infty \end{array} \right.$$

$\Rightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; \ldots \quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$

$\lim_{x \to x_0^+}$ $\qquad 0 < x - x_0 < \delta$

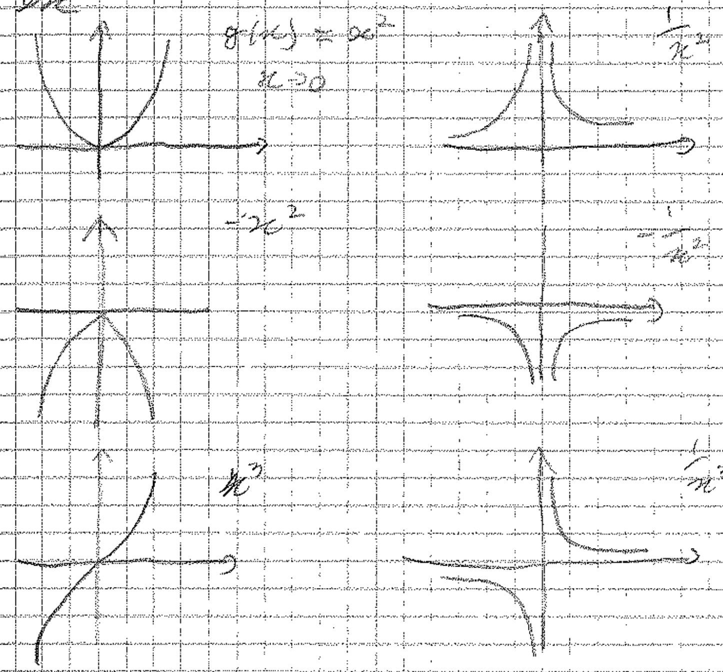
$\lim_{x \to x_0^-}$

$P \Rightarrow Q \quad | \quad \neg Q \Rightarrow \neg P$

$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x) \Rightarrow \lim_{x \to x_0} f(x) \nexists$

$g(x)$	$\frac{1}{g(x)}$
$m \neq 0$	$\frac{1}{m}$
∞	0
0	

ex



In this case we have to look at one-sided limit

$$\lim_{x \to \gamma} \frac{f(x)}{g(x)} = \lim_{x \to \gamma} f(x) \cdot \frac{1}{g(x)}$$

$\left.\begin{array}{l} f(x) \to 0 \\ g(x) \to \infty \end{array}\right\} 0$

$\left.\begin{array}{l} f(x) \to 1 \\ g(x) \to 0 \end{array}\right\} +\infty$

$\left.\begin{array}{l} f(x) \to 0 \\ g(x) \to 0 \end{array}\right\}$ $f(x) \cdot \frac{1}{g(x)}$ so $\frac{0}{0}$ is an indeterminate form.
$0 \cdot \infty$

$\left.\begin{array}{l} f(x) \to \infty \\ g(x) \to \infty \end{array}\right\}$ $f(x) \cdot \frac{1}{g(x)}$ so $\frac{\infty}{\infty}$ is an indeterminate form
$\infty \cdot 0$

I.F.

$\infty - \infty$	$0 \cdot \infty$
$\frac{\infty}{\infty}$	$\frac{0}{0}$

$$\lim_{x\to 0} \frac{\cos x - 1}{x \sin x} \quad \frac{0}{0}$$

$$\lim_{x\to 0} \frac{\cos x - 2}{x \sin x} \quad \frac{-1}{0} = -\infty$$

Limiti che tendono all ∞ di un polinomio

$$\lim_{x\to \pm\infty} P(x) \qquad \lim_{x\to +\infty} x^n = +\infty$$

$$\lim_{x\to -\infty} x^n = \begin{cases} -\infty & \text{if } n \text{ is odd} \\ +\infty & \text{if } n \text{ is even} \end{cases}$$

$$\lim_{x\to -\infty} 3x^3 + 2x^2 + 4 = \infty - \infty$$

$$\lim_{x\to -\infty} 3x^3 \left(1 + \frac{2}{3}\frac{1}{x} + \frac{4}{3x^3} \right) = -\infty \times 1 = -\infty$$

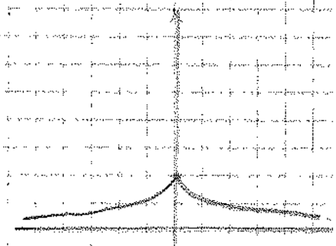
a similar result can be found for rational function

ex

$$\lim_{x\to +\infty} \frac{-x^3+3}{2x^2+x+1} = \lim_{x\to +\infty} \frac{-x^3}{2x^2} = \lim_{x\to +\infty} -\frac{1}{2}x = -\infty$$

The substitution theorem

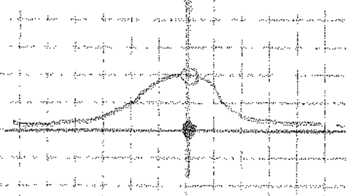
$$\lim_{x\to 0} 2^{x^2} = 2^{\lim_{x\to 0} x^2} = 2^0 = 1$$



$$\lim_{x\to 0} M\left(\frac{1}{1+x^2}\right) = M\left(\lim_{x\to 0} \frac{1}{1+x^2}\right) = M(1) = 0$$

by graphing the function we realize that

$$\lim_{x\to 0} M\left(\frac{1}{1+x^2}\right) = 1$$



ex

$f(x) = \log_2(1-x^3)$; $\lim_{x\to -3} f(x)$

$x \xrightarrow{f} 1-x^3$

$y \xrightarrow{g} \log_2 y$

it requires $1-x^3>0$: $x<1$; dom $f = (-\infty, 1)$

$$\lim_{x\to -3} \log_2(1-x^3) = \log_2 28$$

$\lim_{x \to \infty} f(x) = l$ if dist at l — in this case is better to look at the complete function rather than the components

limit notation
- γ = limit point ($x_0 \in \mathbb{R}$, $+\infty$, $-\infty$)
- λ = the limit (l ; $+\infty$; $-\infty$)

$I(\gamma)$ = a neighborhood of the limit point

$\lim_{x \to \gamma} f(x) = \lambda$ general definition

$\forall I(\lambda) \quad \exists I(\gamma)$

$\forall x : x \in \text{dom} f$ and $x \in I(\gamma) \setminus \{\gamma\} \Rightarrow f(x) \in I(\lambda)$

$\lim_{x \to +\infty} f(x) = l$

$\gamma = +\infty \Rightarrow I(\gamma) = (A ; +\infty)$

$\lambda = l \Rightarrow I(l) = (l - \varepsilon ; l + \varepsilon)$

$\forall \varepsilon > 0 \quad \exists A > 0$

$\forall x \in \text{dom} f$ and $x \in (A ; +\infty) \Rightarrow f(x) \in (l - \varepsilon ; l + \varepsilon)$

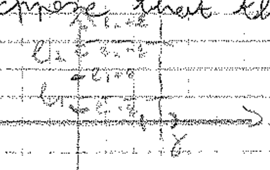
General property of limits

Uniqueness of the limit

Proof by contradiction → suppose that the limit is not unique

- $\lim_{x \to \gamma} f(x) = l_1$
- $\lim_{x \to \gamma} f(x) = l_2$

$\neq$



$\forall \varepsilon > 0 \quad \exists I(\gamma)$

$\forall x : x \in \text{dom} f$ and $x \in I(\gamma) \setminus \{\gamma\} \Rightarrow |f(x) - l_1| < \varepsilon$

$\varepsilon_0 < \frac{|l_1 - l_2|}{2}$ → then means: choose value ε smaller than the distance

$\varepsilon_0 \quad \exists I_1(\gamma) : \forall x \ \ldots \quad |f(x) - l_1| < \varepsilon$

$\varepsilon_0 \quad \exists I_2(\gamma) : \forall x \ \ldots \quad |f(x) - l_2| < \varepsilon$

In $I_1(\gamma) \cap I_2(\gamma)$ both the condition are true

If I chose $x \in I_1 \cap I_2$ I have that $f(x)$ $\in I_{\varepsilon_0}(l_1)$, $\in I_{\varepsilon_0}(l_2)$ — AB surd! since the intersection of 2 neigs is empty!

Corollary

$$\left.\begin{cases} f(x) \geq 0 \text{ in } I(\gamma) \setminus \{\gamma\} \\ \lim_{x \to \gamma} f(x) \ \exists \end{cases}\right] \Rightarrow \lim_{x \to \gamma} f(x) \begin{cases} l \geq 0 \\ +\infty \end{cases}$$

Proof by contradiction

$$\lim_{x \to \gamma} f(x) \begin{cases} l < 0 \\ -\infty \end{cases}$$

$\Rightarrow \exists I_2$ s.t. in $I_2\gamma / \{\gamma\}$ $\boxed{f(x) < 0}$

$f(x) < 0$ $f(x) \geq 0$ in the intersection the function is negative and positive at the same time

The comparison theorem (1)

Suppose f and g defined in $I(\gamma) \setminus \{\gamma\}$ and

- $\lim_{x \to \gamma} f(x) = \lambda$ and $\lim_{x \to \gamma} g(x) = M$
- $\exists I_2(\gamma)$ s. t. $f(x) \leq g(x)$ in $I_2(\gamma) \setminus \{\gamma\}$

Proof

$h(x) = f(x) - g(x) \leq 0$

$\lim h(x) \underset{\leq 0}{=} \lim (f(x) - g(x))$ } vero solo se non c'è la forma indeterminata

$= \lim f(x) - \lim g(x)$

$= \lambda - M \leq 0 \Rightarrow \lambda \leq M$

Comparison theorem (2)

- f, g, h $I_1(\gamma) / \gamma$
- $\lim_{x \to \gamma} f(x) = \lim_{x \to \gamma} h(x) = l$
- $f(x) \leq g(x) \leq h(x)$ in $I_1(\gamma) \setminus \{\gamma\}$

$\Rightarrow \lim_{x \to \gamma} g(x) = l$

Proof

$\lim_{x \to \gamma} f(x) = l$ ε_0

$\exists I_2$ $\forall x \in \mathrm{dom} f$ $x \in \mathrm{dom} f$ $x \in I_2(\gamma) \setminus \{\gamma\}$ $\Rightarrow |f(x) - l| < \varepsilon_0$

$l - \varepsilon_0 < f(x) < l + \varepsilon_0$

II comparison theorem in the infinite case

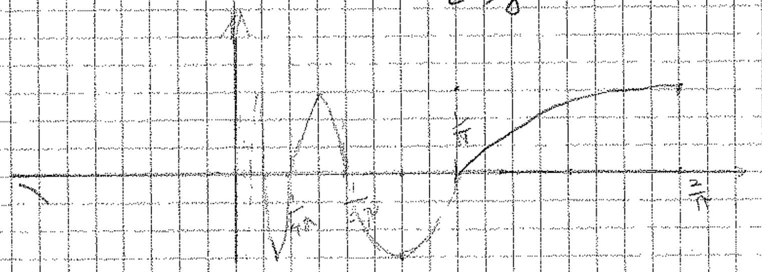
- $f, g \quad I(\gamma) \setminus \{\gamma\}$
- $f(x) \leq g(x)$
- if $\lim_{x \to \gamma} f(x) = +\infty \Rightarrow \lim_{x \to \gamma} g(x) = +\infty$

if $\lim_{x \to \gamma} g(x) = -\infty \Rightarrow \lim_{x \to \gamma} f(x) = -\infty$

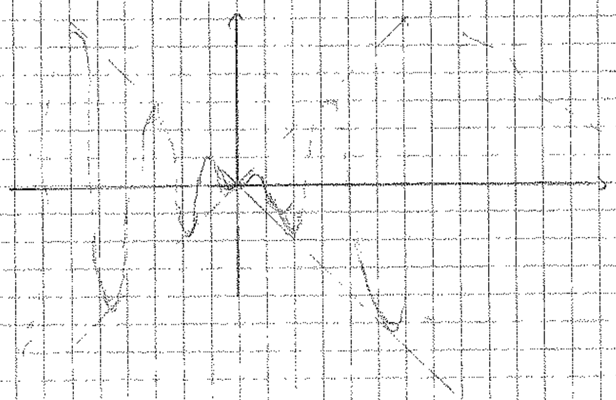
$y = \sin \frac{1}{x}$

$y = 0 \Leftrightarrow \frac{1}{x} = k\pi$

$x = \frac{1}{k\pi}$



$y = x \sin \frac{1}{x}$



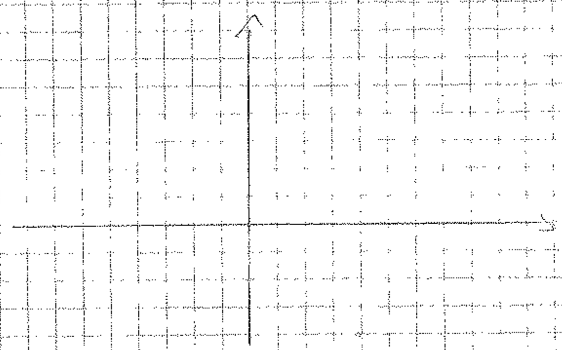
the function $x \sin \frac{1}{x}$ is bounded between $-x$ and $+x$ so the limit is between the lim of x and $-x$
2nd comparison theorem

$f(x) = x\,(3 + \sin x)$

$\lim_{x \to +\infty} x\,(3 + \sin x)$

$x > 0 \quad 2 \leq 3 + \sin x \leq 4$

$$\underset{+\infty}{\underset{\downarrow}{2x}} \leq \underset{+\infty}{\underset{\downarrow}{x(3 + \sin x)}} \leq \underset{+\infty}{\underset{\downarrow}{4x}}$$



X [illegible] matematica alle 8:30

Limits

Sequences

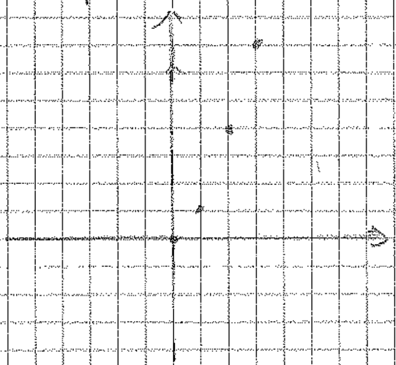
a sequence is a function having domain $\mathbb{N}$

$\{n \in \mathbb{N} : n \geq n_0\}$

$f(n)$ the notation is not used we use f_n [illegible] since f is a function

we write a_n ; b_n

- the sequence $\{a_n\}$



$n \to n^2$

$\{(0,0)\ (1,1)\ (2,4)\ (3,9)\}$

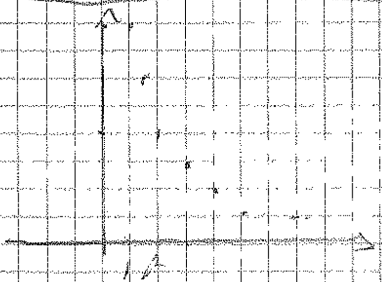
$a_n : \mathbb{N} \to \mathbb{R}$

ese

$n \to \frac{1}{\log_2 n}$ is defined for $n \geq 2$

The problem is to study the limit of a sequence

$\lim_{n \to +\infty} a_n = \lim_{n \to \infty} a_n$



$n \to \frac{1}{n} \quad n \geq 1$

$\lim_{n \to \infty} \frac{1}{n} = 0$

$\lim_{x \to +\infty} f(x) = l$

$\forall \varepsilon > 0 \quad \exists B > 0$

$\forall x \in \text{dom} f$ and $x > B \Rightarrow |f(x) - l| < \varepsilon$

$\lim_{n \to \infty} a_n = l$

$\forall \varepsilon > 0 \quad \exists n_\varepsilon > 0$

$\forall n, \ n \geq n_0 \qquad n > n_\varepsilon \Rightarrow |a_n - l| < \varepsilon$

$\lim_{n\to\infty} n^2 + (-1)^n$

$+\infty \quad \nexists$

$-1 \leq -1^n \leq 1$

$n^2 - 1 \leq n^2 + -1^n \leq n^2 + 1$

$+\infty \qquad +\infty$

The geometric sequence

$a \in \mathbb{R} \quad n \to a^n$

$1, a, a^2, a^3, a^4, a^5, a^6$

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{a^n} = a$$

$\boxed{a > 0} \quad a_n = a^n \longleftrightarrow f(x) = a^x$

$a > 1 \quad \lim a^n = +\infty$

$a = 1 \quad \lim a^n = 1$

$0 < a < 1 \quad \lim a^n = 0$

$\boxed{a = 0} \quad \lim a^n = 0$

$\boxed{-1 < a < 0}$

$a = -\frac{1}{2} \qquad 1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}$

$-|a|^n \leq a^n \leq |a|^n$

$a \leq -1 \quad \nexists \lim$

ex

$$\lim_{n\to\infty} \frac{1+3^n}{5^n} = \lim_{n\to\infty} \frac{1}{5^n} + \frac{3^n}{5^n} = \lim_{n\to+\infty} \left(\frac{1}{5}\right)^n + \left(\frac{3}{5}\right)^n = +\infty$$

Monotonicity is a sufficient condition for the existance of the limit.

The limit of the seq. is the supremum of the values of the sequence

$$\lim_{x\to 0} \log_a(1+x)^{\frac{1}{x}}$$

$$\log_a \lim_{x\to 0} (1+x)^{\frac{1}{x}} = \log_a e$$

We can also write

$$\boxed{\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e}$$

$$\boxed{\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1}$$

$$\lim_{x\to 0} \frac{a^x - 1}{x}$$

$y = a^x - 1$

$a^x = y + 1$

$x = \log_a y + 1$

$$\lim_{y\to 0} \frac{y}{\log_a(y+1)} = \frac{1}{\log_a e} \to \log a$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

Composition

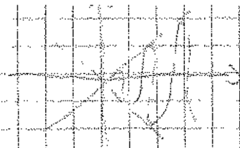
$$\lim_{x\to g}$$

$$\lim_{n\to\infty} a_n \lambda$$

- g in cont at $\lambda = l$
- $\exists \lim_{x\to\infty} g(y) \,//\, \exists \lim_{y\to-\infty} g(y)$ when $\lambda = \pm\infty$

$\Rightarrow$

$$\lim_{x \to 0} \operatorname{sign} \left| x \sin \frac{1}{x} \right|$$



$$0 \qquad \nexists \qquad -x < x$$

$\lim_{x \to 0^+}$ Chicu 6 - 9x

if now the derivative exist so I write

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \cdot \lim_{x\to x_0} (x-x_0)$$

$\Downarrow$ a number, $\Downarrow$ 0 — so the result is zero

since

$P \Rightarrow Q \qquad \neg Q \Rightarrow \neg P$

f diff at $x_0 \Rightarrow$ f cont at x_0
f not cont $\Rightarrow$ f not diff.
} contrapositive statement

The opposite is not true

1) $f(x) = c \Rightarrow f'(x) = 0$

2) $y = ax + b \qquad f'(x) = a$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{ax+b-(ax_0-b)}{x-x_0} = \lim_{x\to x_0} \frac{a\cancel{(x-x_0)}}{\cancel{(x-x_0)}} = a$$

3) $y = x^n = n x^{n-1}$

$$\lim_{x\to x_0} \frac{x^n - x_0^n}{(x-x_0)} = \lim_{x\to x_0} \frac{\cancel{(x-x_0)}\left(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \dots + x x_0^{n-2} + x_0^{n-1}\right)}{\cancel{(x-x_0)}}$$

$= n x_0^{n-1}$

4) $y = x^\alpha \to \alpha x^{\alpha-1}$

5) $y = \sin x \quad y' = \cos x$

$\boxed{x_0 = 0}$
$$\lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$\boxed{x_0 \neq 0}$
$$\lim_{h\to 0} \frac{\sin(x_0+h) - \sin x_0}{h} = \lim_{h\to 0} \frac{\sin x_0 (\cos h - 1)}{h} + \cos x_0 \frac{\sin h}{h}$$

$$= \lim_{h\to 0} \sin x_0 \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x_0 \lim_{h\to 0} \frac{\sin h}{h}$$

$\Downarrow$ 0 $\qquad \Downarrow \qquad \Downarrow$ 1

Derivative of the reciprocal

diff quotient

$$\frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \frac{g(x_0) - g(x)}{g(x)\, g(x_0)} \cdot \frac{1}{x - x_0} = -\frac{g(x) - g(x_0)}{x - x_0} \cdot \frac{1}{g(x)g(x_0)}$$

$$\to -g'(x) \cdot \frac{1}{(g(x))^2} = -\frac{g'(x)}{g(x)^2}$$

quotient

$$\left(f \cdot \frac{1}{g}\right)' = f\left(-\frac{g'}{g^2}\right) + \frac{f'}{g} = \frac{f'g - fg'}{g^2}$$

$$D \tan x = D\,\frac{\sin x}{\cos x} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} =$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \begin{cases} 1 + \tan^2 x \\ \dfrac{1}{\cos^2 x} \end{cases}$$

$\tan x = \frac{\sin}{\cos}$ ...

Derivative of inverse function

f cont and invertible in $I(x_0)$

- f diff at x_0 $f'(x_0) \neq 0$

$\Rightarrow f^{-1}(y)$ diff at $y_0 = f(x_0)$ and $(f^{-1})'|_{y_0} = \dfrac{1}{f'(x_0)}$

ese

$f(x) = \tan x \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R} \Rightarrow f'(x) = 1 + \tan^2 x$

$y = \tan x_0 \qquad x = \arctan y$

$$D \arctan y \Big| = \frac{1}{D \tan x \big|_{x_0}} = \frac{1}{1 + \tan^2 x}$$

← can't have y and x ... ... coefficient

since $y_0 = \tan x_0$

$$= \frac{1}{1 + y_0^2}$$

$$D \arctan = \frac{1}{1 + x^2}$$

$\ln|x| = \frac{1}{x}$

$$\ln|x| \begin{cases} \ln \Rightarrow \frac{1}{x} & x>0 \\ \ln(-x) & x<0 \end{cases}$$

$x \xrightarrow{f} -x$

$y \xrightarrow{g} \ln y \qquad -\frac{1}{x}(-1) = \frac{1}{x}$

der of even function?

. f diff on a symm interval $(-a; a)$

. f Even

$$f(-x) = f(x) \qquad x^3 \to 3x^2 \to 6x \to$$

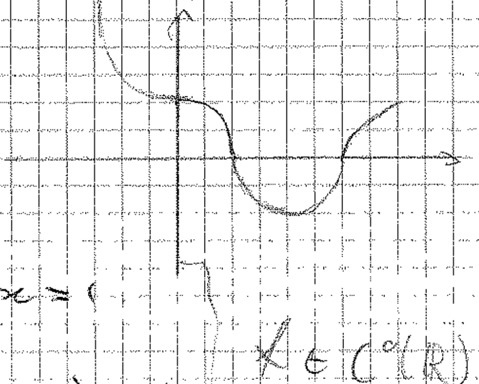
$$-f'(-x) = f'(x)$$

$$f'(-x) = -f'(x) \qquad \cancel{not}\ f' \text{ is odd}$$

and also
if f is odd f' is even

A function f is differentiable if ~~it is diffe~~ right and left derivatives and are the same

$$f(x) \begin{cases} 1+x^2 & x \leq 0 \\ \cos x & x>0 \end{cases}$$



$\cdot \lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} \cos x = 1$

$\cdot \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1+x^2 = 1$

$f \in C^0(\mathbb{R})$

non serve l'uguale!

$$f'(x) \begin{cases} 2x & x<0 \\ -\sin x & x>0 \end{cases}$$

Right D: $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\cos x - 1}{x} = 0$

left D $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{1+x^2-1}{x} = 0$

$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$ | $f'_-(x_0)$ → the same

1) $f'_+(x_0) = f'_-(x_0) = l \implies f$ diff at x_0

2) $l_1 = f'_+(x_0) \neq f'_-(x_0) = l_2 \implies f$ not diff → CORNER POINT

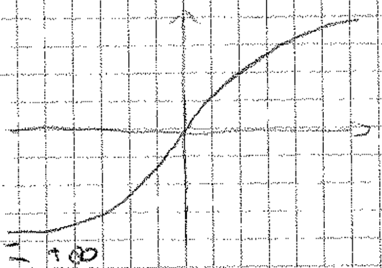
3) $f'_+(x_0) = f'_-(x_0) = \begin{cases} +\infty \\ -\infty \end{cases}$ vertical tangent

4) $f'_+(x_0) = +\infty$ cusp point

$f'_-(x_0) = -\infty$

5) one finite corner point
one inf

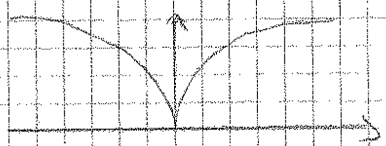
ex (3) $f(x) = \sqrt[3]{x}$



$\lim_{x \to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x \to 0} \sqrt[3]{\frac{x}{x^3}} = +\infty$

ex (4)

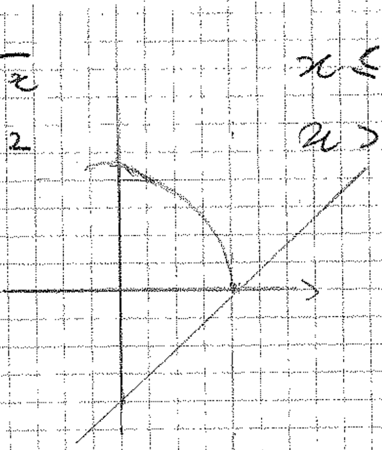
$f(x) = \sqrt{|x|}$



$\lim_{x \to 0} \frac{\sqrt{|x|}}{x} = \begin{cases} \lim_{x \to 0^+} \frac{\sqrt{x}}{x} = \lim_{x \to 0^+} \sqrt{\frac{1}{x}} = +\infty \\ \lim_{x \to 0^-} \frac{\sqrt{-x}}{x} = \lim_{x \to 0^-} -\sqrt{-\frac{x}{x^2}} = \lim_{x \to 0^-} -\sqrt{-\frac{1}{x}} = -\infty \end{cases}$

$l_1 \neq$ (5)

$f(x) \begin{cases} \sqrt{2-x} & x \leq 2 \\ x - 2 & x > 2 \end{cases}$

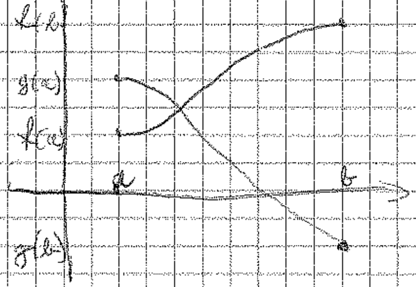


application

- f, g cont on $[a, b]$

$$\begin{cases} f(a) < g(a) \\ f(b) > g(b) \end{cases} \quad \text{OR} \quad \begin{cases} f(a) > g(a) \\ f(b) < g(b) \end{cases}$$

$\Longrightarrow \exists x_0 \in (a, b) \qquad f(x_0) = g(x_0)$



Proof

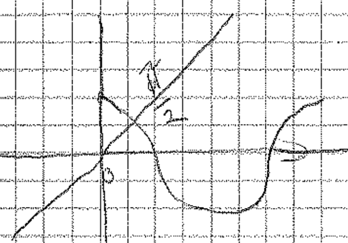
$h(x) = f(x) - g(x)$

- $h(x)$ cont on $[a; b]$
- $h(a) = f(a) - g(a) < 0$
- $h\ b = f(b) - g(b) > 0$

$\Rightarrow \exists x_0 \in (a, b) \Rightarrow$ existence of zeros

example

$x = \cos x$



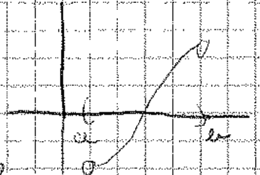
the zero is unique since the 2 functions are both monotone with different monotonicity $\Rightarrow$ the difference function is monotone

$f(x) = x$

$g(x) = \cos x$

	$f(x)$	$g(x)$
$a = \frac{\pi}{6}$	$\sim 0{,}5$	$\frac{\sqrt{3}}{2} \sim 0{,}8$
$b = \frac{\pi}{3}$	$\sim 1{,}05$	$0{,}5$

- f cont on (a, b)
- $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-}$ have diff signs
- $\exists x_0 \in (a, b)$



we can prove this using the sign and limit theorem the f is negative in a neigh of a and positive in $I(b)$ so we take 2 new point x_1 and x_2 in which the function exists

EVERY POLINOMIAL FUNCT. OF ODD DEGREE HAS AT LEAST ONE ZERO

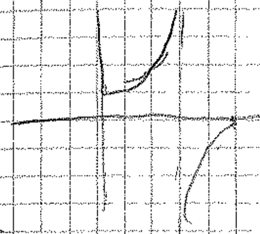
I closed and bounded I $\Rightarrow$ f(I) closed and bounded

WEIERSTRASS THEOREM

- f cont on [a ; b]

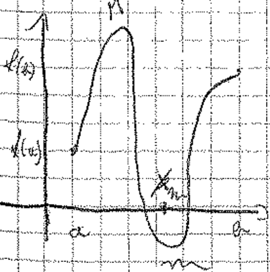
$\Rightarrow$ • f is bounded
- f admits minimum (absolute) $m = \min f$ on $[a, b]$
- or f admits maximum $M = \max f$ $[a, b]$

Counter examples
f non cont on [a, b)



With continuous function this does not happen

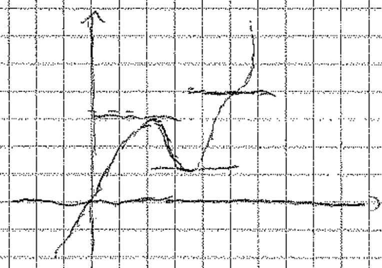
An immediate conclusion of WEIER TH is the II version of intermediate value theorem.
If f is cont on [a, b]
$\Rightarrow$ f assumes all the values between m and M



Proof: from Weier th we know
$\exists X_m \in [a, b] \Rightarrow f(X_m) = m$
$\exists X_M \in [a, b] \Rightarrow f(X_M) = M$
I consider the closed interval with end points X_m and X_M (i.e. $[X_m ; X_M]$ or $[X_M ; X_m]$) and apply the intermediate value TH

So we can say that the range of the function $f([a ; b]) = [m, M]$

Let f be differentiable at x_0
$f'(x_0) = 0$ $\Rightarrow$ x_0 is a critical point

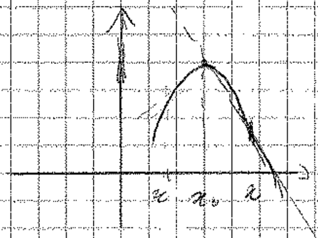


Fermat's theorem

- f Diff At x_0
- x_0 is an rel extremum point

$\Rightarrow f'(x_0) = 0$

Proof



Suppose x_0 be a Max point

I study $\dfrac{f(x)-f(x_0)}{x-x_0}$

$x > x_0$ $\quad f(x) - f(x_0) < 0$ $\quad x - x_0 > 0$ $\quad \dfrac{f(x)-f(x_0)}{x-x_0} \leq 0$

$\lim\limits_{x \to x_0^+} \dfrac{f(x)-f(x_0)}{x-x_0}$ $\exists$ this lim exist since the f is diff so the lim of diff exist

from corollary of sign and lim theorem we get:

$f'_+(x_0) \leq 0$

$x < x_0$ $\quad f(x) - f(x_0) \leq 0$ $\quad x - x_0 < 0$ $\quad \dfrac{f(x)-f(x_0)}{x-x_0} \geq 0$

$\lim\limits_{x \to x_0^-} \dfrac{f(x)-f(x_0)}{x-x_0} = f'_-(x_0) \geq 0$

since f is diff at x_0

$\Rightarrow f'_+(x_0) = f'_-(x_0) \Rightarrow$ it must be $f'_+(x_0) = f'_-(x_0) = 0$

$f'(x_0) = 0$

Fermat's theorem

- f is diff at x_0

if x_0 is an EXTR point $\Rightarrow$ x_0 critical point.

the opposite is not true

2) At least one of the point X_m and X_M is not an end point

$\Downarrow$

At least one of the point X_m and X_M $\in (a, b)$

$\Downarrow$

f is diff at this point $\begin{pmatrix} x_m \\ x_M \end{pmatrix}$

$\Downarrow$ Fermat theorem

$$f'(X_m) = 0 \quad // \quad \text{or} \quad f'(X_M) = 0$$

Rolle's theorem was generalized by Lagrange

Lagrange or Mean value theorem

$\left.\begin{array}{l} f \text{ cont on } [a ; b] \\ f \text{ diff at } (a ; b) \end{array}\right\} \exists x_0 \in (a ; b) \quad f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$

RMK if $f(b) = f(a) \Rightarrow$ Rolle's theorem

there is a point in which the tangent is parallel to the secant

Proof

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right]$$

$g \to$ Cont on a, b since f is continuous on $[a ; b]$ and 1st degree polinomial are continuous everywhere.

$g(a) = f(a) - f(a) = 0$

$g(b) = f(b) - [f(a) + f(b) - f(a)] = 0$

I can apply Rolle's theorem on $g(x)$

$\exists x_0 \in (a , b) \; : \; g'(x_0) = 0$

$g'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a} \qquad g'(x_0) = 0$

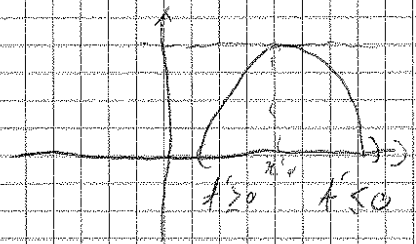
$f'(x_0) - \dfrac{f(b) - f(a)}{b - a} = 0 \; ; \; f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$

f Diff on an open interval I

- f(x) incr $\iff f'(x) \geq 0$
- f(x) strictly incr $\Longleftarrow f'(x) > 0$

Classification of critical points

f diff on open int I
$x_0 \in I$ critical $f'(x_0) = 0$
Classify x_0



Cont funct on an ~~interval~~ interval
How do I find the extremum points?

- $f'(x) = 0 \Rightarrow$ the f should be diff

1) Points where f is diff and ~~f' is diff~~ $f' = 0$
2) Points where f is not diff
3) If there exist the end point of the interval

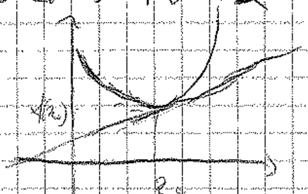
Convex and concave functions

Convex function in a point
1) f Diff at x_0
2) $t(x) = f(x_0) + f'(x_0)(x - x_0) \to$ tg at x_0

DEF

f is convex at x_0 if $\exists I(x_0) : \forall x \in I(x_0)$ $f(x) \geq f(x_0) + f'(x_0)(x-x_0)$
the graph of f is above the tg

the f is concave if we have $\leq$
$f(x) \leq t(x)$

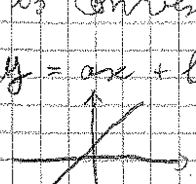


Strictly convex if $f(x) > t(x)$

f is diff in an open int I

f is convex in I $\iff$ f is convex in x_0 $\forall x_0 \in I$

$y = ax + b$



is convex but not strictly
is concave " " "

RMK if $\lim \frac{f(x)}{g(x)}$

ex $\lim_{x\to+\infty} \frac{2x-\sin x}{3x-\cos x} = \frac{+\infty}{+\infty} = \lim_{x\to+\infty} \frac{x\left(2+\frac{\sin x}{x}\right)}{x\left(3+\frac{\cos x}{x}\right)} = \frac{2}{3}$

try to use de l'Hôpital

$\lim_{x\to+\infty} \frac{2+\cos x}{3+\sin x} \quad \nexists$

$a_n = 2\pi n \rightarrow \frac{2+\cos(2\pi n)}{3+\sin(2\pi n)} = 1$

$b_n = \frac{\pi}{2} + 2\pi n \rightarrow \frac{2+\cos\left(\frac{\pi}{2}+2\pi n\right)}{3+\sin\left(\frac{\pi}{2}+2\pi n\right)} = \frac{1}{2}$

example

$\lim_{x\to 0^+} x \ln x \qquad 0\cdot\infty$

$\lim x \left\langle \begin{array}{l} \frac{x}{\frac{1}{\ln x}} = \frac{0}{0} \\ \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty} \end{array} \right.$

$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0$

$\lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}} = \lim_{x\to 0^+} \frac{1}{(\ln x)^{-2}\cdot\frac{1}{x}} = \lim_{x\to 0^+} -x \ln^2 x$

infinite LOOP

THEOREM

- f cont in $I(x_0)$
- f diff in $I(x_0)\setminus\{x_0\}$ I don't know what in the point so I find it out

$\lim_{x\to x_0} f'(x) = l \Rightarrow f$ diff at x_0 and $f'(x_0) = l$

PROOF

apply De l'Hôpital

$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f'(x)}{1} = l$

RMK → Check always the continuity
since a f is not cont. is not differentiable

esempio

$h(x) = f(x)^{g(x)}$

$f(x) > 0$

$h(x) = f(x)^{g(x)} = e^{g(x) \ln [f(x)]}$

$h(x) = x^x = e^{x \ln x}$

$\lim_{x \to \gamma} f(x)^{g(x)} = \lim_{x \to \gamma} e^{g(x) \ln f(x)} = e^{\lim_{x \to \gamma} g(x) \ln f(x)}$ $\boxed{OK}$

Con che hanno indeterminate forme?

$g(x)$	$\ln f(x)$
0	∞
∞	0

$f(x) \to 0$ — 0^0

$f(x) \to \infty$ — ∞^0

$f(x) \to 1$ — 1^∞

$\lim_{x \to 0^+} (\sin x)^x =$

$= \lim_{x \to 0^+} e^{x \ln \sin x} = e^{\lim_{x \to 0^+} x \ln \sin x}$? $0 \cdot -\infty$

$\lim_{x \to 0^+} \dfrac{\ln \sin x}{\frac{1}{x}} =$

$\lim_{x \to 0^+} \dfrac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} - \lim_{x \to 0^+} \dfrac{-x}{\sin} \; x \cdot \cos x = 0$

$\lim_{x \to 0^+} (\sin x)^x = \lim_{x \to 0^+} e^{x \ln \sin x} = e^0 = 1$

$\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$ $\quad 1^\infty$

$e^{\lim_{x \to 0} \frac{\ln \cos x}{x^2}}$ $\quad \lim_{x \to 0} \dfrac{-\frac{\sin x}{\cos x}}{2x} = \lim_{x \to 0} - \dfrac{\sin x}{2 \cos x \cdot x} = -\dfrac{1}{2}$

$\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}} = \dfrac{1}{\sqrt{e}}$

Local comparison of functions

- How the functions tend to a limit?

$\lim_{x \to 0} x^2 = 0 \quad // \quad \lim_{x \to 0} x^4 = 0$

Compare quantitative

Landau's symbol

- f, g are def in $I(\gamma) / \{\gamma\}$
- $g(x) \neq 0$
- $\exists \lim_{x \to \gamma} \frac{f(x)}{g(x)} = \lambda$

1) $\lambda = l \in \mathbb{R} \setminus \{0\}$ $\Rightarrow$ f has the same order of magnitude of g ($f \asymp g$)
f and g have the same order (of magnitude)

2) $\lambda = l = 1$ $\rightarrow$ f is equivalent to g $\quad f \sim g$

3) $\lambda = l = 0$ $\rightarrow$ f is negligible with respect to g
$f\, o(g)$ $\quad$ f is little-o of g

4) $\lim_{x \to \gamma} \frac{f(x)}{g(x)} = l \in \mathbb{R}$ $\quad$ f is controlled by g
$f = O(g)$ $\quad$ f is big-O of g

If $\lambda = \infty$ $\lim_{x \to \gamma} \frac{f(x)}{g(x)} = \infty \Rightarrow \lim_{x \to \gamma} \frac{g(x)}{f(x)} = 0 \Rightarrow$ g negl. w.r.t. to f.
$g = o(f)$

Example

$\begin{cases} f(x) = \sin x \\ g(x) = x \end{cases}$ $\quad \gamma = 0 \quad \lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \sin x \sim x$

$\gamma = +\infty \quad \lim_{x \to +\infty} \frac{\sin x}{x} = 0 \quad \sin x = o(x)$

Example

1 $\sin x$
2 $\tan x$
$x \to \frac{\pi}{2}$

$\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\tan x} = \quad \sin x = o(\tan x) \quad (x \to \frac{\pi}{2})$

$x \to 0 \quad \lim_{x \to 0} \frac{\sin x}{\tan x} \quad \sin x \sim \tan x \quad (x \to 0)$

$0 \quad x \quad +\infty$

$(\quad 1 \quad\quad 5 \quad)$

$f(x)$ is infinite of real order $\alpha > 0$:

$\exists\, \alpha > 0 \quad f(x) \asymp x^\alpha \quad x \to +\infty$

$f(x) = \sqrt{1+x^6}$

$$\lim_{x\to+\infty} \frac{f(x)}{x^\alpha} = l \neq 0$$

$$\lim_{x\to+\infty} \frac{\sqrt{1+x^6}}{x^\alpha} = \lim_{x\to+\infty} \frac{\sqrt{x^6\left(1+\frac{1}{x^6}\right)}}{x^\alpha} = \lim_{x\to\alpha} \frac{x^3\sqrt{1+\frac{1}{x^6}}}{x^\alpha}$$

if $\alpha < 3 \to +\infty$

if $\alpha > 3 \to 0$

if $\alpha = 3 \to$

~~so~~ we conclude

$f(x) = \sqrt{1+x^6} \asymp x^3 \quad \left(\sqrt{1+x^6} \sim x^3\right)$

$\sqrt{1+x^6}$ is an inf of real order 3

There are functions that grow to ∞ in an (higher or lower?) way one others cannot be identified ~~with~~ a natural number α

$$\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty \quad \Bigg\| \quad \lim_{x\to+\infty} \frac{e^{\beta x}}{x^\alpha} = +\infty$$

$$\lim_{x\to+\infty} \frac{\ln x}{x^\alpha} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \lim_{x\to+\infty} \frac{1}{\alpha x^\alpha}$$

$(\ln x)^\gamma \qquad x^\alpha \qquad e^{\beta x}$

$f(x) = x \ln x$

$$\lim_{x\to+\infty} \frac{x \ln x}{x} = +\infty$$

$x \ln x$ inf of bigger order than x

$$\lim_{x\to+\infty} \frac{x \ln x}{x^{1+\varepsilon}} = \lim_{x\to+\infty} \frac{\quad}{\quad}$$

$x \ln x$ inf of lower order than $x^{1+\varepsilon}$ $\forall \varepsilon > 0$

the order of $x \ln x$ $\;>1,\; <1+\varepsilon \quad \forall \varepsilon > 0$

$(\log x)^{\beta}$ $\quad x^{\alpha}$ $\quad e^{x^{\alpha}}$

$\varphi(x) = x$ standard test function

$\varphi^{\alpha}(x) = x^{\alpha}$

$f(x)$

$\exists \alpha \; \exists l \neq 0$ s.t. $f(x) \sim l x^{\alpha} = f(x) \sim l(\varphi(x))^{\alpha}$

ex

$\sqrt{1+x^6} \sim x^3$

1) $\alpha \to$ is the order of the function f w.r.t. the test function $\varphi(x) = x$

2) $l x^{\alpha} = l(\varphi(x))^{\alpha}$ principal part of f

$\boxed{x \to 0^+}$

$\varphi(x) = \frac{1}{x}$ $\quad (\varphi(x))^{\alpha} = \frac{1}{x^{\alpha}}$

ex

$f(x) = \frac{1}{\sin^2 3x}$ $\quad x \to 0^+$ $\quad$ infinite for $x \to 0^+$

$\lim_{x \to 0^+} \frac{\frac{1}{\sin^2(3x)}}{\frac{1}{x^{\alpha}}} = l \neq 0$ $\quad$ can I use l'hospital's rule?

$\lim_{x \to 0^+} \frac{x^{\alpha}}{\sin^2(3x)} = \lim_{x \to 0^+} \frac{\alpha(x^{\alpha-1})}{2\sin(3x)\cdot\cos 3x \cdot 3} = \lim_{x \to 0^+}$

$= \lim_{x \to 0^+} \frac{1}{6\cos(3x)} \cdot \frac{\alpha x^{\alpha-1}}{\sin 3x} = \frac{1}{6} \cdot \lim_{x \to 0^+} \frac{\alpha(\alpha-1)x^{\alpha-2}}{3\cos 3x}$

I can use the l'hospital only on the terms that tend to 0 or ∞

$\exists!\, \alpha = 2$ numerator tends $\to 3$

$\lim_{x \to 0^+} \frac{x^2}{\sin^2(3x)} = \lim_{x \to 0^+} \frac{2x}{2(\sin 3x)(\cos 3x)\,3} = \lim_{x \to 0^+} \frac{3x}{3\sin 3x} \cdot \frac{1}{3\cos 3x} = \frac{1}{9}$

$\lim_{x \to 0^+} \frac{\frac{1}{\sin^2(3x)}}{\frac{1}{x^2}} = \frac{1}{9}$ $\quad \frac{1}{\sin^2 3x} \sim \frac{1}{9}\frac{1}{x^2}$

$f(x) \; \frac{1}{\sin^2(3x)}$ infinite ord 2

P.P. $= \frac{1}{9x^2}$

STUDIA tabella delle standard test function

$f(x) = \sin x - \tan(x)$ $\quad x \to 0$

$$\lim_{x\to 0} \frac{\sin x - \frac{\sin x}{\cos x}}{x^{\alpha}} = \lim_{x\to 0} \frac{\sin x(\cos x - 1)}{\cos x \cdot x^{\alpha}}$$

$$\boxed{\alpha = 3} = \lim_{x\to 0} \frac{1}{\cos} \cdot \frac{\sin}{x} \cdot \frac{\cos x - 1}{x^2} = -\frac{1}{2}$$

$$\lim_{x\to 0} \frac{f(x)}{x^3} = -\frac{1}{2} \qquad f(x) = \sin x - \tan x \approx -\frac{1}{2}x^3$$

Properties of Landau symbols

1) $\lambda \neq 0 \quad x \to \gamma$

$$o(\lambda f) = \lambda o(f) = o(f)$$

$$g = o(\lambda f) \iff g = \lambda o(f) \iff g\ o(f)$$

$$g = o(\lambda f) \Rightarrow \lim_{x\to\gamma} \frac{g}{\lambda f} = 0$$

$$\Updownarrow$$

$$\lambda \lim_{x\to\gamma} \frac{g}{\lambda f} =$$

$$\Updownarrow$$

$$g = o(f)$$

$f \sim g \Rightarrow f = g + o(g)$ $\qquad$ ~~they~~ function are ~~equal~~ equivalent to each other; they are not equal but we can write one as the other plus something

$$\lim_{x\to\gamma} \frac{f(x)}{g(x)} =$$

proof

$$\lim_{x\to\gamma} \frac{f(x)}{g(x)} = 1 \Rightarrow \lim_{x\to\gamma} \frac{f(x)}{g(x)} - 1 = 0$$

$$\lim_{x\to\gamma} \frac{f(x) - g(x)}{g(x)} = 0 \qquad f - g = o(g)$$

2) Rmk

$$f = o(1) \iff \lim_{x\to\gamma} \frac{f(x)}{1} = 0 \iff f \text{ is infinitesima}$$

example

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow \sin x \underset{x\to 0}{\sim} x \qquad \sin x = x + o(x)$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \Rightarrow e^x - 1 \underset{x\to 0}{\sim} x \qquad e^x - 1 = x + o(x)$$

29/11/11

$$\lim_{x\to+\infty} \frac{3^x + x^{10}}{2^x - 3x^8}$$

- x^{10} inf ord 10
- 3^x $\lim_{x\to+\infty} \frac{a^x}{x^\alpha} = +\infty$ $a>1$

$x^{10} = o(3^x)$ // $3x^8 = o(2^x)$

$$\lim_{x\to+\infty} \frac{3^x + x^{10}}{2^x - 3x^8} = \lim_{x\to+\infty} \frac{3^x}{2^x} = \lim_{x\to+\infty} \left(\frac{3}{2}\right)^x = +\infty$$

Remark

If $f \sim f_1$ is not true (in general) that $f \pm g \sim f_1 \pm g$

Brutta forza

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = -\frac{1}{2}$$

- $\sin x \sim x$ $\sin x = x + o(x)$
- $\tan x \sim x$ $\tan x = x + o(x)$

$\lim_{x\to 0} \frac{x - x}{x^3} = 0$ It's wrong, we cannot use eq. terms, but also $o(x)$

$\lim_{x\to 0} \frac{x + o(x) - x - o(x)}{x^3}$ I can't cancel out the $o(x)$ since they are not algebraic expression

$$\begin{cases} f = o(x) \\ g = o(x) \end{cases} \quad f - g$$

$$\lim_{x\to 0} \frac{f - g}{x} = \lim_{x\to 0} \frac{f}{x} - \frac{g}{x} = 0$$

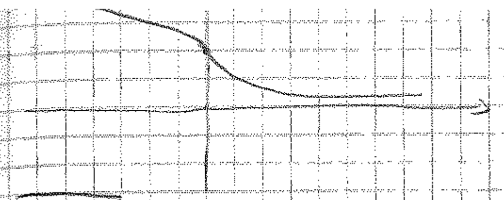
1 $o(x)$ x^2

The slb $x + o(x)$ for sine

$\sin x = x + \underbrace{o(x)}_{\to ???}$ $\sin x - x = o(x)$

$\frac{\sin x - x}{x^2} \to 0$, $\frac{\sin x - x}{x^2} \xrightarrow{\text{Per l'Hôpital}} \frac{\cos x - 1}{2x} = 0$

$\sin x - x = o(x^2)$



$$\varphi(x) = \frac{1}{x}$$

$$\lim_{x\to+\infty} \frac{\frac{\pi}{2} - \arctan x}{\frac{1}{x}}$$

$\alpha = 1$

$$\lim_{x\to+\infty} \frac{\frac{\pi}{2} - \arctan x}{\frac{1}{x}} \overset{H\hat{o}p}{=} \lim_{x\to+\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{x^2}{1+x^2} = 1$$

$$\frac{\pi}{2} - \arctan x \sim \frac{1}{x}$$

$$\frac{\pi}{2} - \arctan x = \frac{1}{x} + o(x)$$

A function f is called asymptotic to a function g for $x \to +\infty$

$$\lim_{x\to+\infty} f(x) - g(x) = 0$$

$f = g + o(1)$

example

$f(x) = x^2 + \frac{1}{x}$ $\qquad f(x) - g(x) = \frac{1}{x} - \frac{1}{x^3} \xrightarrow{x\to+\infty} 0$

$g(x) = x^2 + \frac{1}{x^3}$

$$\begin{cases} f(x) = e^x + e^{-x} \to +\infty \\ g(x) = e^{-x} \to +\infty \end{cases} \qquad x \to -\infty$$

$$f(x) - g(x) = e^x \to 0 \quad (x \to -\infty)$$

$f = g + o(1)$ asymptotic

$\frac{f}{g} \to 1 \quad f \sim g \Rightarrow f = g + o(g)$ Equivalent

$f \to \infty \qquad g \to \infty$

$$\begin{cases} f, g \to \infty \\ f = g + o(1) \end{cases}$$

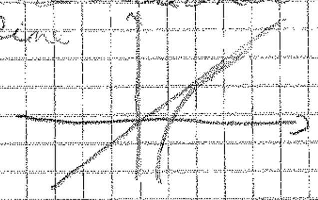
Equivalent, not Asymptotic

$f(x) = x^+$

$g(x) = x$

$$\frac{f(x)}{g(x)} = \frac{x+0}{x} = 1 + \frac{0}{x}$$

$f = \underbrace{\boxed{g}}_{= mx+q} + o(1)$ we are interested in the case in which g is a line



$y = o(x^{n}) \Rightarrow x\, y = o(x^{n+1})$

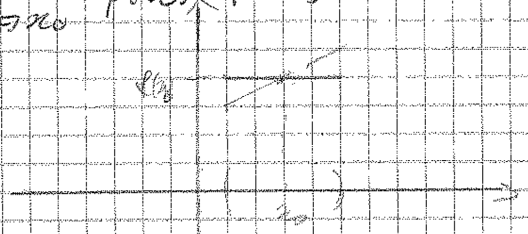
$\lim_{x \to 0} \frac{y}{x^n} = 0$

Taylor Expansion

0) $f(x)$ cont at x_0

$\lim_{x \to x_0} f(x) = f(x_0) \iff \lim_{x \to x_0} f(x) - f(x_0) = 0$

$f(x) = f(x_0) + o(1)$



1) $f(x)$ Diff at x_0

$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$

$\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) = o(1)$ $\to o(x - x_0)$

$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(1)(x - x_0)$

$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$

(tangent line)

(first formula of the finite limits)

$f(x) - \left[f(x_0) + f'(x_0)(x - x_0) \right] = o(x - x_0)$

$\lim_{x \to x_0} \frac{f(x) - \left[f(x_0) + f'(x_0)(x - x_0) \right]}{[x - x_0]^2}$ f Diff in $I(x_0)$

De L'hopital

$\lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{2(x - x_0)} = \frac{1}{2} f''(x_0)$

$\lim_{x \to x_0} \frac{f(x) - [\ldots]}{(x - x_0)^2} - \frac{1}{2} f''(x_0) = o(1)$

$f(x) - [\ldots] - \frac{1}{2} f''(x_0)(x - x_0)^2 = o((x - x_0)^2)$

$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2}_{\text{Parabola}} + o((x - x_0)^2)$

f three diff

esempio

$f(x) = e^x \qquad x_0 = 0 \qquad f(x) \in C^\infty_{\mathbb{R}}$

$f^k(x) = e^x \Rightarrow f^k(0) = 1$

$T_{f n, 0} = 1 + x + \frac{1}{2} x^2 + \dots + \frac{1}{n!} x^n = \sum_{k=0}^{n} \frac{x^k}{k!}$

$f(x) = e^x \quad \boxed{x_0}$

$f^k(x) = e^x \Rightarrow f^{(k)}(x_0) = e^{x_0}$

$T_{f n, x_0} = e^{x_0} + e^{x_0}(x - x_0) + \frac{1}{2} e^{x_0}(x - x_0)^2 + \dots + \frac{1}{n!} e^{x_0}(x - x_0)^n$

$\sum_{k=0}^{n} \frac{e^x}{k!} (x - x_0)^k$

$f(x) = \frac{1}{1-x} \qquad x_0 = 0 \qquad f(x) \in C^\infty (\text{dom} f)$

$f(x) = (1-x)^{-1}$

$f(x) = -1(1-x)^{-2}(-1)$

$f'(x) = -2(1-x)^{-3} \cancel{\ } (-1)$

$f''(x) = 6(1-x)^{-4}$

$f^k(x) = k!(1-x)^{-(k+1)} \qquad f^{(k)}(0) = k!$

$T_{f n, 0} = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} x^k$

$x_0 = 0$

$f(x) = \sin(x)$	0
$f'(x) = \cos(x)$	1
$f''(x) = -\sin x$	0
$f'''(x) = -\cos x$	-1
$f^{IV}(x) = \sin x$	0

$T_{f n, 0} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 \dots$

$\sum_{m=0}^{n} (-1)^m \frac{x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\sin x = x - \frac{x^3}{3!} + o(x^3)$

$\sin x = x - \frac{x^3}{3!} \ o(x^4)$

$f(x) = 3x^2 + x^4 + o(x^4)$

$$\lim_{x\to 0} \frac{f(x)}{3x^2} = \lim_{x\to 0} 1 + \frac{x^2}{3} + \frac{o(x^4)}{x^2} = 1$$

$f \sim 3x^2 \quad x \to 0$

↳ Principal part

The 1^{st} non-zero term of a Taylor expansion is the principal part

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

$f(x)$ Diff n times at x_0

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + o((x-x_0)^n)$$

- $a_0 = a_1 = \dots = a_{m-1} = 0$
- $a_m \neq 0$

$$f(x) = a_m(x-x_0)^m + a_{m+1}(x-x_0)^{m+1} + \dots + a_n(x-x_0)^n + o((x-x_0)^n)$$

$$\frac{f(x)}{a_m(x-x_0)^m} = 1 + \frac{a_{m+1}}{a_m}(x-x_0) + \dots + (x-x_0)^2 + \dots + \frac{a_n}{a_m}(x-x_0)^{n-m}$$

$$\lim_{x\to x_0} \frac{f(x)}{a_m(x-x_0)^m} = 1$$

$f(x) \sim a_m(x-x_0)^m$

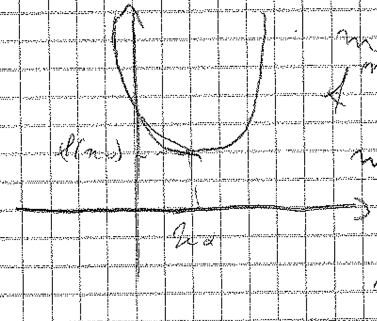
f diff n times at x_0

Critical point $\Rightarrow f'(x_0) = 0$

$$f(x) = f(x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$f''(x_0) = f'''(x_0) = \dots = 0 \qquad f^{(m)}(x_0) \neq 0$

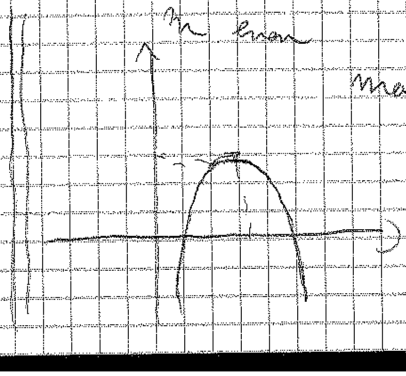
$$f(x) - f(x_0) = \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m + \dots + o((x-x_0)^n)$$



m even

$f^{(m)}(x_0) > 0$

min



m even

max

Example $n=5$

$\sin x : \sin 2h$

$\sin h z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + o(x^5)$

$\sin h(2x) = 2x + \frac{8x^3}{3!} + \frac{32x^5}{5!} + o(x^5)$ — not put the one that end 5

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$

$2x^2 + \frac{8x^4}{3!}$

$\frac{2}{3!}x^4$

$2x^2 + \frac{10}{3!}x^4 + o(x^5)$

now we do with $n=6$

$2x^2 + \frac{8x^4}{3!} \quad \frac{32}{5!}x^6$

$\frac{2x^4}{3!} \quad \frac{8x^6}{3!\cdot 3!}$

$\frac{2}{5!}x^6$

Composition

$h = e^{\sin(x)} \quad x_0 = 0 \quad n = 3$

$z = \sin x \quad x \to 0,\ z \to 0$

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + o(z^3)$

$e^{\sin x} = 1 + \sin x + \frac{1}{2}\sin^2 x + \frac{1}{6}\sin^3 x + o(\sin^3 x)$

a function negligible w.r. to $\sin^3 x$

$\sin^3 x \sim x^3$

$o(\sin^3 x) = o(x^3)$

$= 1 + \left(x - \frac{x^3}{3!} + o(x^3)\right) + o(x^3)$

$+ \frac{1}{2}\left(x - \frac{x^3}{3!} + o(x^3)\right)^2$

$+ \frac{1}{6}\left(x - \frac{x^3}{3!} + o(x^3)\right)^3$

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~~1) proof~~

1) f integr. $\quad \inf_{[a,b]} f \le f(x) \le \sup_{[a,b]} f \quad \forall x \in [a, b]$

Moroton. $\quad \int_a^b \inf f \, dx \le \int_a^b f(x)\, dx \le \int_a^b \sup f \, dx$

$$\inf f (b-a) \le \int_a^b f(x)\, dx \le \sup f (b-a)$$

$$\inf f \le \frac{\int_a^b f(x)\, dx}{b-a} \le \sup f$$

2) f cont

weirst. $\quad m \le m(f; a, b) \le M$

Int value theorem $\quad \exists c \in [a; b] \quad f(c) = m(f; a, b)$

II version

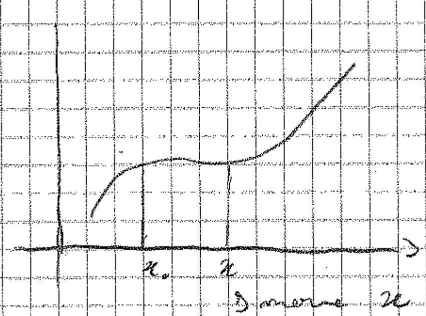
---

f, integrable on I

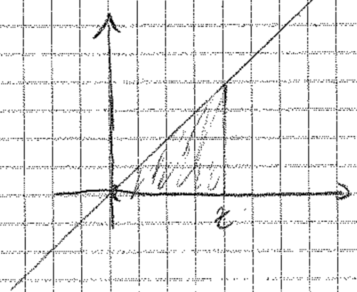
$\underbrace{x_0 \in I}_{\text{fixed}}, x \in I$

$$F(x) = \int_{x_0}^{x} f(t)\, dt$$

integral function of f on I



$$F(x) = \int_0^x t\, dt$$



$x = 0 \quad F(0) = 0$

$x > 0 \quad F(x) = \frac{x^2}{2}$

$x < 0 \quad F(x) = \int_0^x t\, dt = -\int_x^0 t\, dt =$

$-(-\text{Area of } t) = \frac{x^2}{2}$

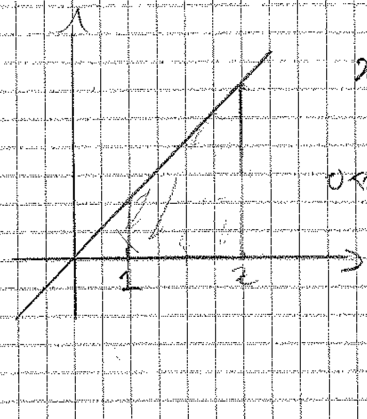
so the integral function would be

$$\int_0^x t\, dt = \frac{x^2}{2}$$

---

now I change the fixed point to 1

$$F(x) = \int_1^x t\, dt$$



$x > 1 \quad \int_1^x t\, dt = \frac{(x+1)(x-1)}{2} = \frac{x^2}{2} - \frac{1}{2}$

$0 < x < 1 \quad F(x) = \int_1^x t\, dt = \frac{x^2}{2} - \frac{1}{2}$

$x > 1 \quad \int_1^x t\, dt = \frac{x^2}{2} - \frac{1}{2}$

$$f(t) = t \begin{cases} F_0(x) = \frac{x^2}{2} \\ F_1(x) = \frac{x^2}{2} - \frac{1}{2} \end{cases}$$

f continuous

$= f(c(h))$

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(c(h)) = f\left(\lim_{h\to 0} c(h)\right)$$

$x \quad c(h) \quad x+h \qquad x+h \quad c \quad x$

$$\begin{array}{ccccc} x & < c(h) & < & x+h \\ \downarrow & \downarrow & & \downarrow \\ x & x & & ? \end{array}$$

AMK

~~$f \in C^1(\mathbb{R})$~~

f function defined on I, $F(x)$ primitive of f on I if f ~~...~~ is differentiable and $F'(x) = f(x)$

IF f is continuous on I then f admits a primitive (infinitely many primitive) and this primitive is the integral function $\int^{x} f(t)\,dt$

$\boxed{\text{TH}}$ f def on I suppose that $F(x)$ is a primitive of f on I, then the set of all primitives is

$$\{F(x)+c,\ c \in \mathbb{R}\}$$

1) $F(x)$ is a primitive $\Rightarrow$ $F(x)+c$ is a primitive

$(F(x)+c)' = F'(x) = f(x)$

2) $\left.\begin{array}{l} F(x) \text{ is a primitive} \\ G(x) \text{ is a primitive}\end{array}\right\}$ $G(x) = F(x)+c$

$$\left.\begin{array}{l} F'(x) = f(x) \\ G'(x) = f(x) \end{array}\right| \quad H(x) = G(x) - F(x)$$

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$$

$H'(x) = 0 \Rightarrow H(x) = c$

$G(x) - F(x) = c$

$G(x) = F(x) + c$

$\int f(x)\,dx \rightarrow$ set of all primitive of f on I $\rightarrow$ indefinite integral

$\int f(x)\,dx = F(x) + c \quad c \in \mathbb{R}$