NUMERO : 424

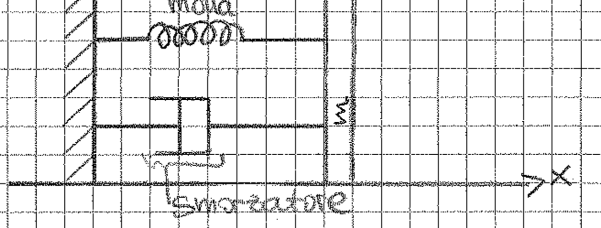
DATA : 10/12/2012

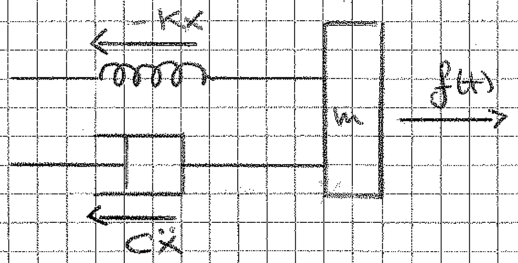
A P P U N T I

STUDENTE : Arlotta

MATERIA : Analisi dei Segnali + esercitazioni
Prof. Visintin

Un **SEGNALE** è una grandezza fisica che varia con il tempo.





<u>K</u> costante di rigidità elastica della molla

<u>c</u> coefficiente di smorzamento

$x(t)$ = posizione della massa al tempo t

$$\dot{x} = \frac{dx(t)}{dt}$$

Definisco <u>$r(t)$</u> la risultante delle forze

$$r(t) = f(t) - Kx(t) - c\dot{x}(t)$$

e per la legge di Newton la forza $= m \cdot a(t)$

$$f(t) - Kx(t) - c\dot{x}(t) = m \cdot a(t)$$

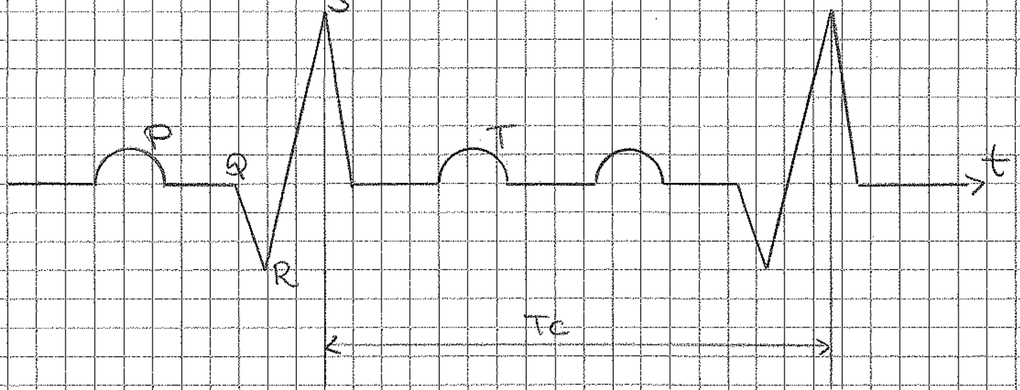
$$f(t) - Kx(t) - c\dot{x}(t) = m \cdot \ddot{x}(t)$$

$$\boxed{f(t) - Kx(t) - c\frac{d}{dt}x(t) = m\frac{d^2}{dt^2}x(t)}$$

immagino $f(t)$ come l'<u>ingresso</u> del sistema, che ne modifica il funzionamento

Voglio conoscere la posizione del sistema, quindi questa sarà <u>l'uscita</u>

SEGNALE CARDIACO

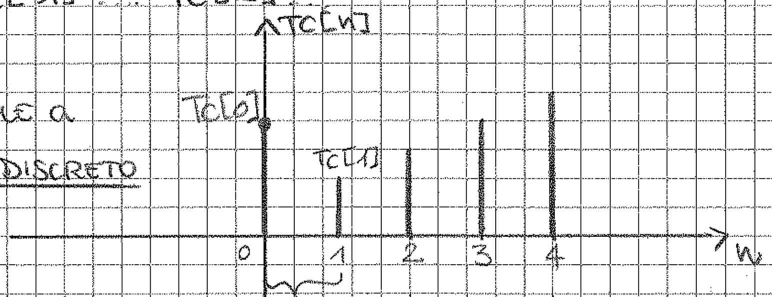


Tc è periodico, ma non costante nel tempo

$Tc[0]$ prima misura di segnale cardiaco

$Tc[1] \dots Tc[2]$

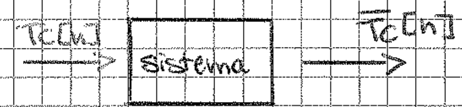
SEGNALE a **TEMPO DISCRETO**



non c'è nulla in mezzo, non sto campionando il tempo.

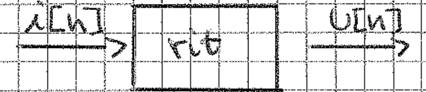
la **MEDIA MOBILE** è la media aritmetica sugli ultimi misurati

$$\overline{Tc}[n] = \frac{Tc[n] + Tc[n-1] + \dots + Tc[n-N+1]}{N}$$



$$\overline{Tc}[n] = \mathcal{T}\{Tc[n]\} \qquad u[n] = \mathcal{T}\{i[n]\}$$

BLOCCHETTO ELEMENTARE $\longrightarrow$ RITARDATORE di un PASSO

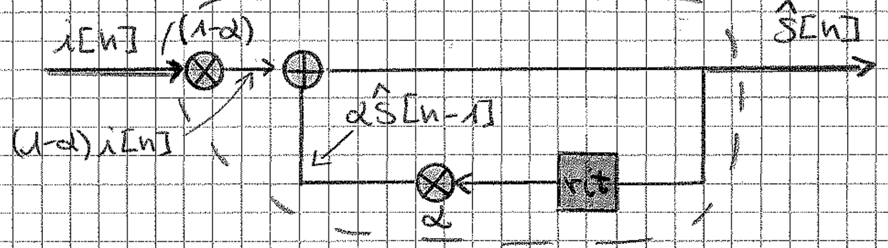


EQUAZIONE alle DIFFERENZE FINITE

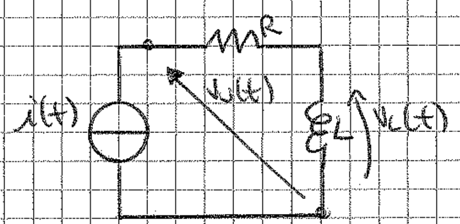
($\wedge$ = Stima)

$$\underbrace{\hat{S}[n]}_{\text{la stima ATTUALE}} = \underbrace{\alpha \hat{S}[n-1]}_{\text{è proporzionale tramite } \alpha \text{ alla STIMA PRECEDENTE}} + \underbrace{(1-\alpha)\, i[n]}_{\text{e proporzionale tramite } (1-\alpha) \text{ alla MISURA ATTUALE}}$$

~▷ SISTEMA a FEEDBACK



~▷ SISTEMA a FOWARD



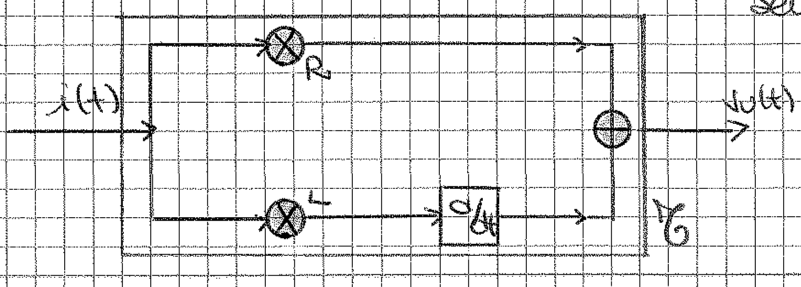
$$v_L(t) = L \frac{d}{dt} i(t)$$

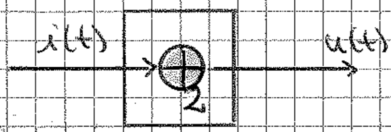
$u(t) = v_L(t)$

Come' $\mathcal{T}$?

Posso invertire l'ordine del moltiplicatore e del derivatore (per i sistemi lineari vale sempre)

$$v_J(t) = R \cdot i(t) + L \frac{d}{dt} i(t)$$



- 

$u(t) = i(t) + 2$

è lineare? NO

$u_1(t) = i_1(t) + 2$
$u_2(t) = i_2(t) + 2$

Se l'ingresso è $x(t) = \alpha_1 i_1(t) + \alpha_2 i_2(t)$ allora all'uscita è $y(t) = x(t) + 2$

$= \alpha_1 i_1(t) + \alpha_2 i_2(t) + 2$

$\alpha_1 u_1(t) + \alpha_2 u_2(t) = \alpha_1 (i_1(t) + 2) + \alpha_2 (i_2(t) + 2)$

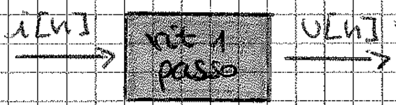
$= \alpha_1 i_1(t) + \underline{2\alpha_1} + \alpha_2 i_2(t) + \underline{2\alpha_2}$

$\neq y(t)$

$\mathcal{T}\{\alpha_1 i_1(t) + \alpha_2 i_2(t)\} \neq \alpha_1 u_1(t) + \alpha_2 u_2(t)$

- il derivatore è un blocchetto LINEARE? SI

la derivata della somma è la somma delle derivate

- 

$v[n] = i[n-1]$

è lineare? SI

$\mathcal{T}\{i[n]\} = i[n-1]$

$i_1[n] \Rightarrow v_1[n] = i_1[n-1]$
$i_2[n] \Rightarrow v_2[n] = i_2[n-1]$

$\mathcal{T}\{\alpha_1 i_1[n] + \alpha_2 i_2[n]\} = \alpha_1 i_1[n-1] + \alpha_2 i_2[n-1] =$

$= \alpha_1 v_1[n] + \alpha_2 v_2[n]$

$u(t-t_0) = 2 + i(t-t_0)$

$v(t) = u(t-t_0)$

il sistema è NON LINEARE ma TEMPO INVARIANTE

1) per l'ingresso $i(t) \to u(t) = i(t) + z(t)$

2) $l(t) = i(t) - i(t_0)$

in uscita? $v(t) = l(t) + z(t)$

$= i(t-t_0) + z(t)$

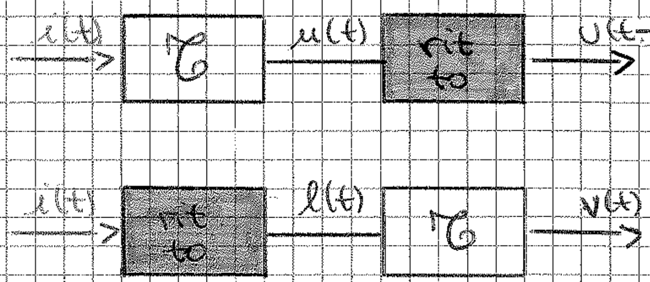
3) Calcolo $u(t-t_0)$

$u(t-t_0) = i(t-t_0) + z(t-t_0)$

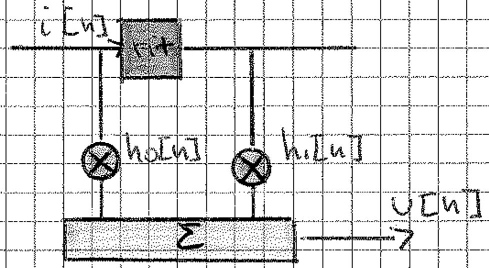
$u(t-t_0) \neq v(t)$

È un sistema TEMPO VARIANTE

REGOLETTA



Il sistema è TEMPO INVARIANTE se posso scambiare T e rit (t_0) senza che l'uscita vari.



Se i coefficienti dei moltiplicatori sono costanti, ovvero indipendenti dal tempo il sistema è TEMPO INVARIANTE.

Se ho $h_0[n]$ e $h_1[n]$ è un sistema TEMPO VARIANTE.

$y[n] = \mathcal{L}\{x[n]\}$

$h[n] = \mathcal{L}\{\delta[n]\}$

$$\mathcal{L}\{x[n]\} = \mathcal{L}\Big\{\sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{coefficiente}}\ \underbrace{\delta[n-k]}_{\text{segnale}}\Big\} =$$

per la linearità $\longrightarrow = \sum_{k=-\infty}^{\infty} x[k]\, \mathcal{L}\{\delta[n-k]\} =$

per le proprietà dei tempi invarianti $\longrightarrow = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

CONVOLUZIONE

$x[n] * h[n]$

$x[n] * \delta[n]$

CONVOLUZIONE DISCRETA

$$x[n] * h[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

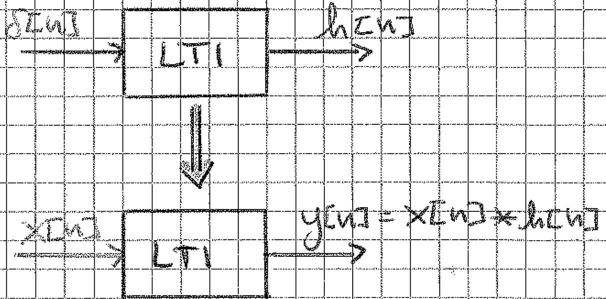
Definisco $n-k = m$

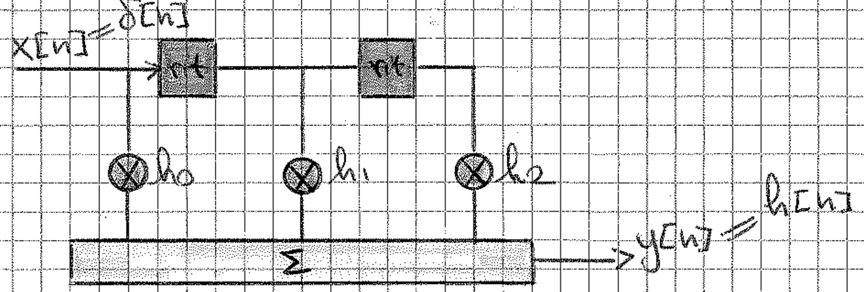
$k = n-m$

$$\sum_{n=-\infty}^{+\infty} x[n-m]\, h[m]$$

la convoluzione gode delle proprietà commutativa

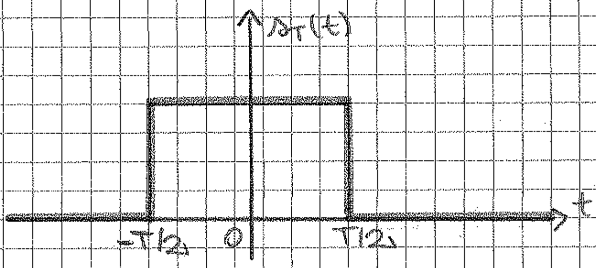
Se ho





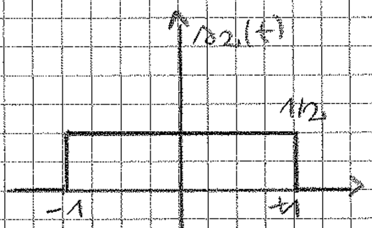
SEGNALE A TEMPO CONTINUO

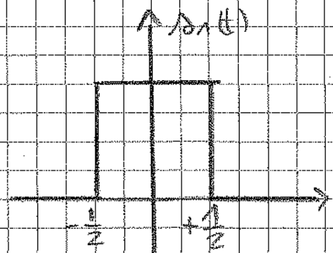
$\delta(t)$ (delta di Dirac) = impulso nel tempo continuo

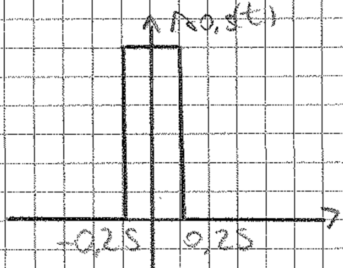


$$s_T(t) = \begin{cases} 1/T & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$\int_{-\infty}^{+\infty} s_T(t)\,dt = 1$$

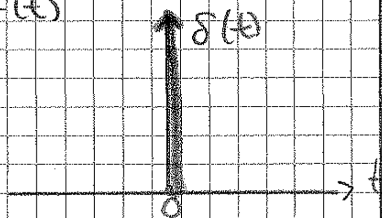


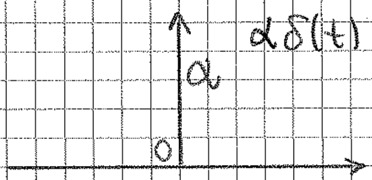


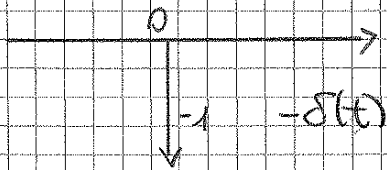


$$\delta(t) = \lim_{T \to 0} s_T(t)$$

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$



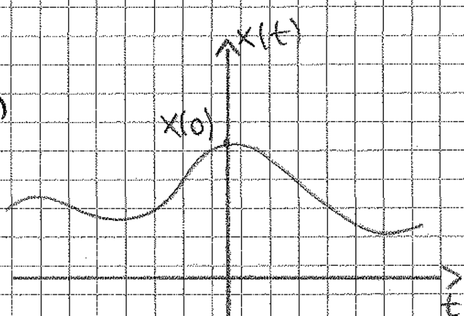


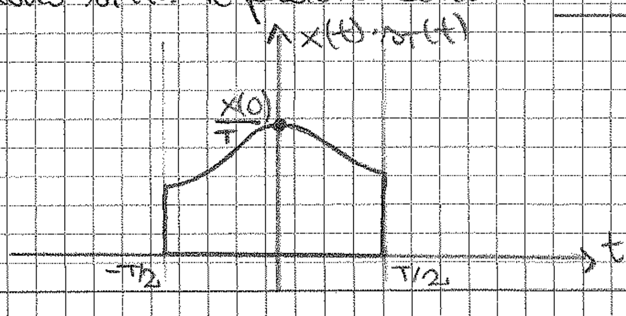


QUANTO FA $x(t) \cdot \delta(t)$?

Prendo una generica funzione $x(t)$

e la funzione che definisce $\delta(t)$, ovvero $s_T(t)$. Il prodotto sarà





$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

o

$$\delta(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ \text{(t continuo)}\end{matrix}} \xrightarrow{\;h(t)\;}$$

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ \text{(t continuo)}\end{matrix}} \xrightarrow{\;y(t)\;}$$

$$y(t) = \mathcal{L}\left\{x(t)\right\} = \mathcal{L}\left\{\int x(\tau)\,\delta(t-\tau)\,d\tau\right\} =$$

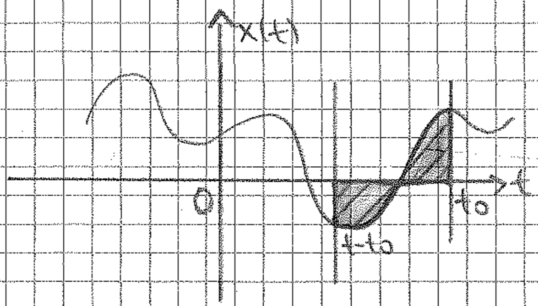
$$= \int x(\tau)\,\mathcal{L}\left\{\delta(t-\tau)\right\} d\tau =$$

$$= \int x(\tau)\,h(t-\tau)\,d\tau \qquad \text{INTEGRALE DI CONVOLUZIONE}$$

$$= x(t) * h(t)$$

o

$$x(t) \longrightarrow \boxed{\int_{t-T}^{t}} \longrightarrow y(t) = \int_{t-T}^{t} x(\tau)\,d\tau$$



Linearità

$$i_1(t) \longrightarrow v_1(t) = \int_{t-T}^{t} i_1(\tau)\,d\tau$$

$$i_2(t) \longrightarrow v_2(t) = \int_{t-T}^{t} i_2(\tau)\,d\tau$$

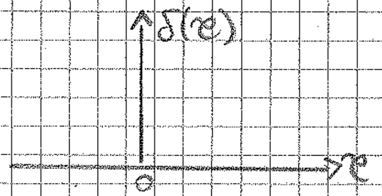
$$x(t) = \alpha_1 i_1(t) + \alpha_2 i_2(t)$$

$$\downarrow$$

$$y(t) = \int_{t-T}^{t} x(\tau)\,d\tau = \int_{t-T}^{t} [\alpha_1 i_1(\tau) + \alpha_2 i_2(\tau)]\,d\tau =$$

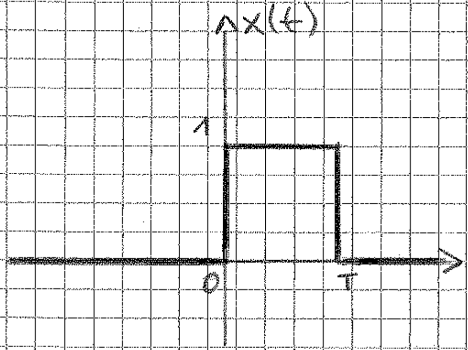
Risposta all'impulso

$$\xrightarrow{\delta(t)} \boxed{\int} \xrightarrow{h(t)}$$



$$h(t) = \int_{t-T}^{t} \delta(\tau)\, d\tau$$

$$= \begin{cases} 0 & t < 0 \\ 1 & t = 0^+ \\ 1 & t < T \\ 0 & t > T \end{cases}$$



BREVE RIASSUNTO

$$\xrightarrow{x(t)} \boxed{\text{sistema}} \xrightarrow{y(t)}$$

$$y(t) = \mathcal{T}\{x(t)\}$$

$$\xrightarrow{x(t)} \boxed{\begin{matrix}\text{Sistema}\\ \text{LTI}\end{matrix}} \xrightarrow{y(t)}$$

lineari tempo invarianti

$$\xrightarrow{\delta(t)} \boxed{\begin{matrix}\text{Sistema}\\ \text{LTI}\end{matrix}} \xrightarrow{h(t)}$$

$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau =$$
$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$\xrightarrow{\delta[n]} \boxed{\begin{matrix}\text{Sistema}\\ \text{LTI}\end{matrix}} \xrightarrow{h[n]}$$

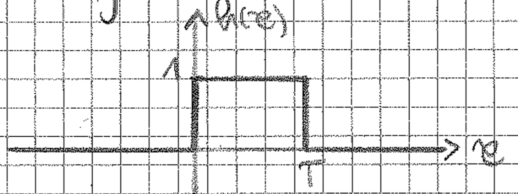
$$y[n] = x[n] * h[n]$$
$$= \sum x[n]\, h[x-k]$$
$$= \sum h[k]\, x[n-k]$$

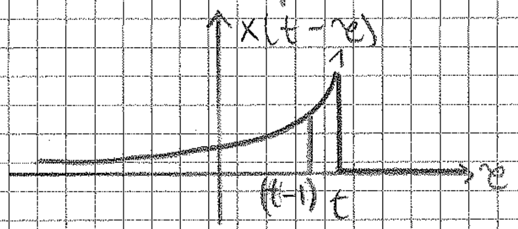
① $= \int h(\tau)\, x(t-\tau)\, d\tau$

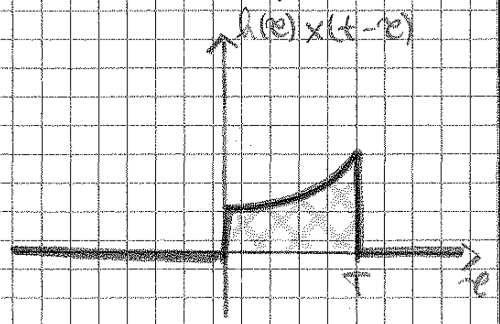
② $= \int x(\tau)\, h(t-\tau)\, d\tau$

! ERRORE ATROCE
$y(t) = \int x(t)\, h(t-\tau)$

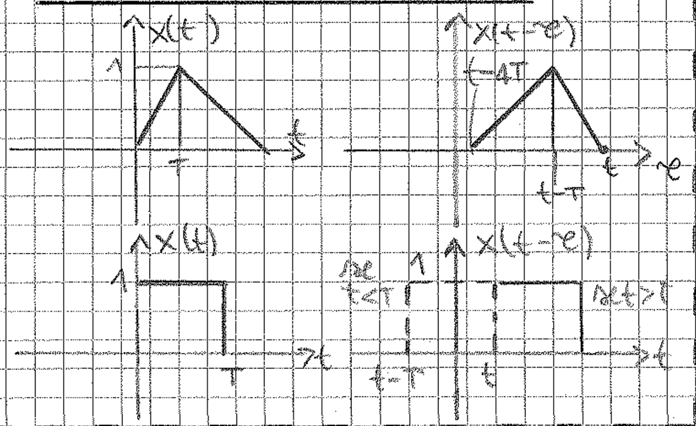
① $y(t) = \int h(\tau)\, x(t-\tau)\, d\tau$ dove $\begin{cases} \tau \text{ variabile} \\ t \text{ parametro} \end{cases}$



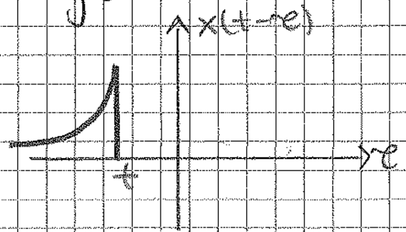




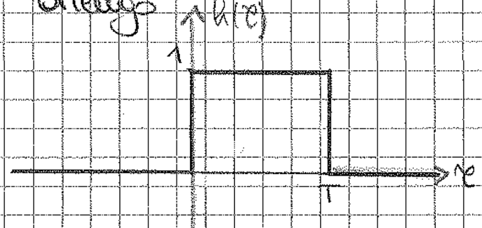
ESEMPI DI FUNZIONI (t-τ)



• per $t \to +\infty$
$y(t) = 0$

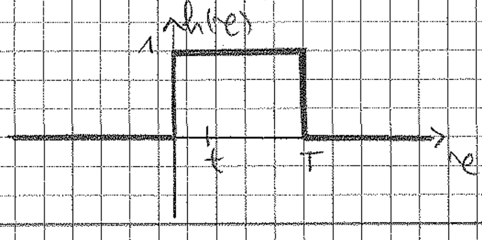


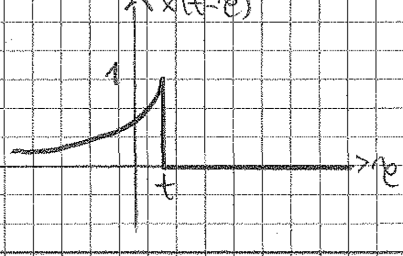
moltiplicando per h(t) ottengo

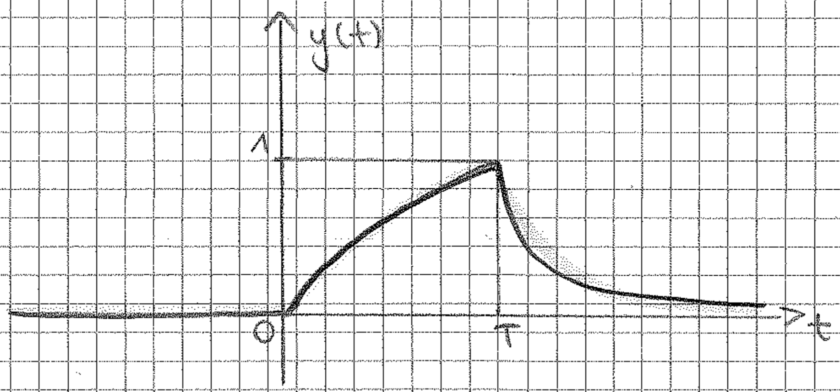


• per $t > 0$ $y(t) = 0$

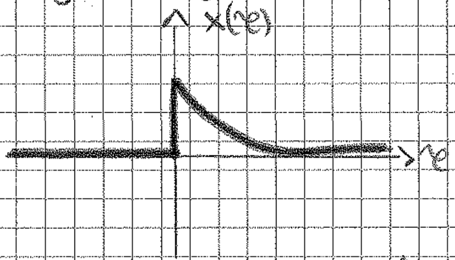
• per $t > 0$ e $t < T$

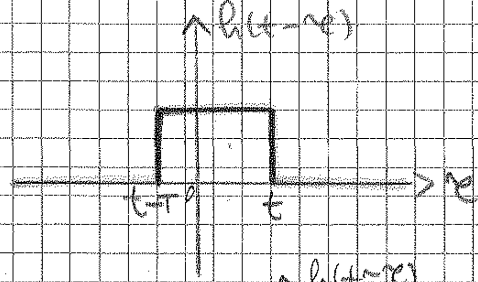






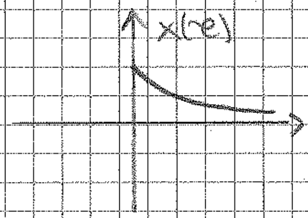
② $y(t) = \int x(\tau)\, h(t-\tau)\, d\tau$

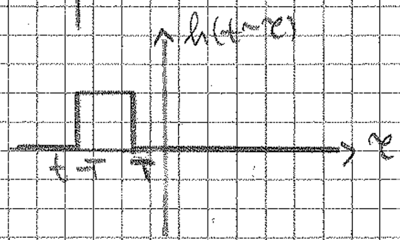




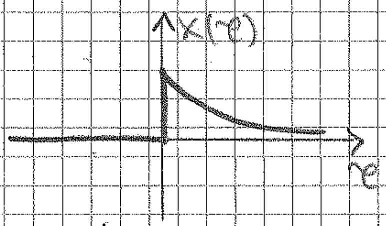
- per $t<0$

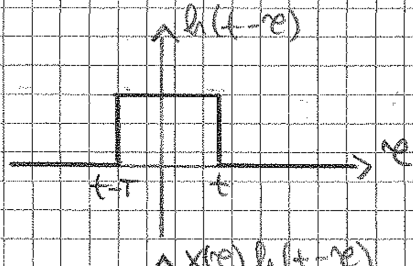
  $y(t)=0$





- per $t \in [0,T]$

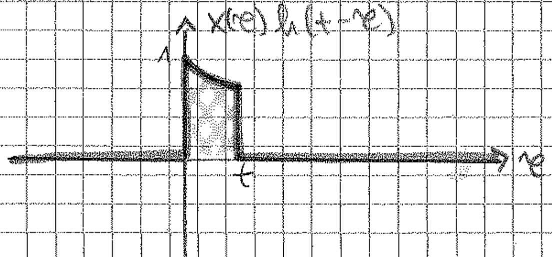




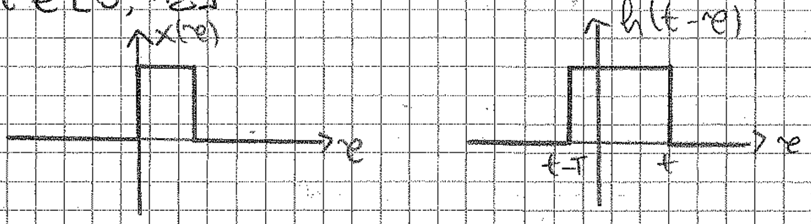
$$y(t) = \int_0^t x(\tau)\, d\tau$$

$$= \int_0^t e^{-\tau}\, d\tau = \left.\frac{e^{-\tau}}{-1}\right|_0^t$$

$$= 1 - e^{-t}$$



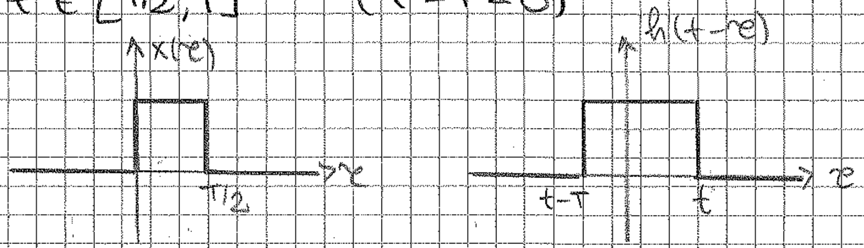
• per $t \in [0, T/2]$



$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t A \; d\tau = At$$

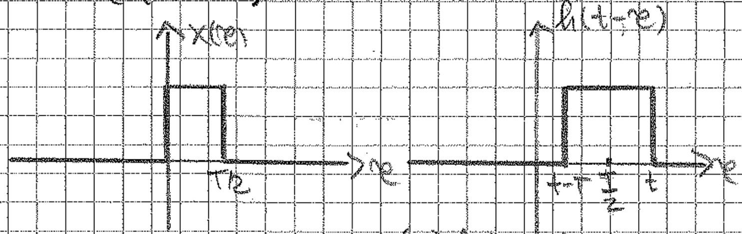
• per $t \in [T/2, T]$ $\quad (t - T < 0)$



$$y(t) = \int_0^{T/2} A\, d\tau = A \cdot \frac{T}{2}$$

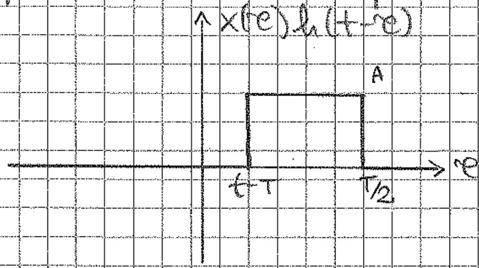
• per $\frac{T}{2} > t - T > 0$ $\quad (t > T/2)$

$\frac{3T}{2} > t > T$

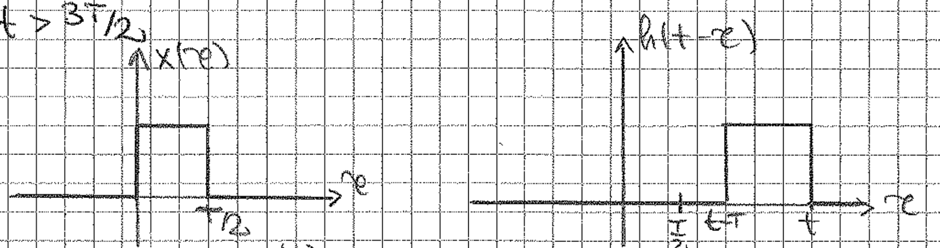


$$y(t) = A\left(\frac{T}{2} - (t-T)\right)$$

$$= A\left(\frac{3T}{2} - t\right)$$



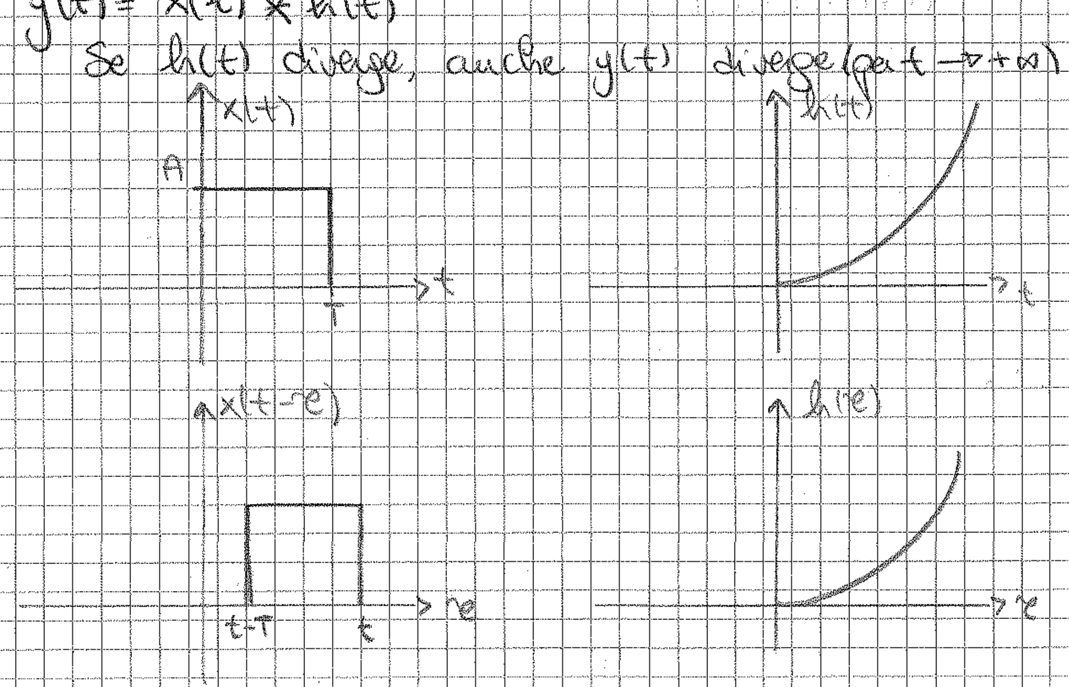
• per $t > 3T/2$



$y(t) = 0$

! $y(t) = x(t) * h(t)$

Se $h(t)$ diverge, anche $y(t)$ diverge (per $t \to +\infty$)



Ad un ingresso limitato in ampiezza, ho un'uscita che aumenta ⟼ C'È UN PROBLEMA di INSTABILITÀ!

STABILITÀ BIBO (Bounded Input Bounded Output)

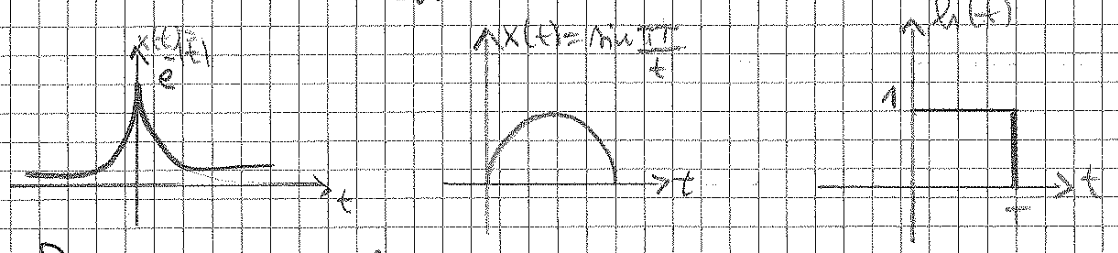
Se $x(t)$ è tale per cui $|x(t)| < A < \infty$, $y(t)$ è sempre tale per cui $|y(t)| < A \cdot B < \infty$

(B è un valore che dipende dal sistema)

↳ teorema sulla stabilità BIBO

Un sistema LTI con risposta all'impulso $h(t)$ è stabile in senso BIBO se e solo se $h(t)$ è modulo-integrabile

$$\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$$



Per un sistema fisicamente realizzabile, $h(t) \in \mathbb{R}e$ e $h(t) = 0$ se $t < 0$ (causalità del sistema)

$$= \int e^{j2\pi f(t-\tau)} h(\tau)\, d\tau =$$

$$= e^{j2\pi f t} \int e^{-j2\pi f \tau} h(\tau)\, d\tau$$

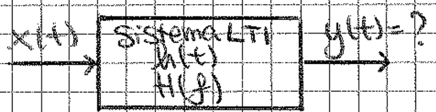
$$= x(t) \underbrace{\int e^{-j2\pi f \tau} h(\tau)\, d\tau}_{H(f)}$$

$$= x(t)\, H(f)$$

$$H(f) \triangleq \int h(t)\, e^{-j2\pi f t}\, dt$$ trasformata di **FOURIER** di $h(t)$

$H(f)$ è la funzione di trasferimento del sistema

ESEMPIO



$$x(t) = A \cos(2\pi f t + \varphi)$$

$$y(t) = \int x(t-\tau)\, h(\tau)\, d\tau =$$

$$= \int A \cos 2\pi f_0 [(t-\tau)+\varphi]\, h(\tau)\, d\tau =$$

uso la formula di EULERO

$$= \int A\, \mathrm{Re}\left\{ e^{j2\pi f_0 [(t-\tau)+\varphi]} \right\} h(\tau)\, d\tau$$

A e $h(t) \in \mathbb{R}$, A è costante

$$= A \int \mathrm{Re}\left\{ e^{j2\pi f_0 [(t-\tau)+\varphi]}\, h(\tau) \right\} d\tau$$

$\int \mathrm{Re} = \mathrm{Re} \int$

$$= A\, \mathrm{Re}\left\{ \int e^{j2\pi f_0 [(t-\tau)+\varphi]}\, h(\tau)\, d\tau \right\}$$

$$= A\, \mathrm{Re}\left\{ e^{j2\pi f_0 (t+\varphi)} \underbrace{\int e^{-j2\pi f_0 \tau} h(\tau)\, d\tau}_{H(f_0)} \right\}$$

$$= A\, \mathrm{Re}\left\{ e^{j2\pi f_0 (t+\varphi)}\, H(f_0) \right\}$$

$$y(t) = x(t) * h(t) = \int x(\tau)\, h(t-\tau)\, d\tau$$

$$Y(f) = ?$$

$$Y(f) = \int_{-\infty}^{\infty} e^{-j2\pi f t}\, y(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi f t} \left(\int x(\tau)\, h(t-\tau)\, d\tau \right) dt$$

$$= \int_t \int_\tau e^{-j2\pi f}\, x(\tau)\, h(t-\tau)\, d\tau\, dt$$

divido e moltiplico per $e^{-j2\pi f \tau}$

$$= \int_t \int_\tau e^{-j2\pi f t}\, x(\tau)\, e^{-j2\pi f \tau}\, e^{j2\pi f \tau}\, h(t-\tau)\, d\tau\, dt$$

$$= \int_\tau \int_t \underline{e^{-j2\pi f (t-\tau)}\, h(t-\tau)}\; \underline{x(\tau)\, e^{-j2\pi f \tau}}\, dt\, d\tau$$

$$t - \tau = u \longmapsto dt = du$$

$$= \int_\tau \int_u e^{-j2\pi f u}\, h(u)\; x(\tau)\, e^{-j2\pi f \tau}\, du\, d\tau$$

$$= \underbrace{\int_\tau x(\tau)\, e^{-j2\pi f \tau}\, d\tau}\; \underbrace{\int_u e^{-j2\pi f u}\, h(u)\, du}$$

$$= X(f)\, H(f)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int x(\tau)\, h(t-\tau)\, d\tau$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f) \cdot H(f)$$

Nel dominio del tempo devo calcolare una convoluzione, in quello della frequenza devo fare il prodotto di due funzioni.

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- $\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt$

  ↳ proprietà di $\delta(t) \Rightarrow \delta(t) \cdot x(t) = \delta(t) \cdot x(0)$

  $$\int_{-\infty}^{\infty} \delta(t)\, \underbrace{e^{-j2\pi f \cdot 0}}_{1}\, dt = 1 \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

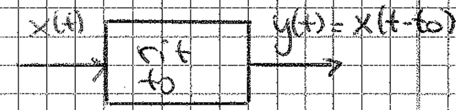
- $\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j2\pi f t}\, dt =$

  $\delta(t-t_0)$ vale ∞ per $t = t_0$

  $$= \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j2\pi f t_0}\, dt =$$

  $$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} \delta(t-t_0)\, dt = e^{-j2\pi f t_0}$$

- $\mathcal{F}\{x(t-t_0)\} =$ l'uscita sarà



Qual'è la risposta all'impulso? $h(t) = \delta(t-t_0)$

$H(f) = e^{-j2\pi f t_0}$

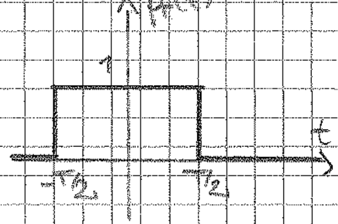
  $$Y(f) = H(f) \cdot X(f)$$
  $$= X(f) \cdot e^{-j2\pi f t_0}$$

  $$y(t) = h(t) * x(t)$$
  $$= \delta(t-t_0) * x(t) =$$
  $$= x(t-t_0)$$

  ↳ proprietà di $\delta(t)$

  $$\delta(t-t_0) * x(t) = x(t-t_0)$$
  $$\#$$
  $$\delta(t-t_0) \cdot x(t) = \delta(t-t_0) \cdot x(t_0)$$

- $\mathcal{F}\{p_T(t)\} = \int_{-\infty}^{+\infty} p_T(t) \cdot e^{-j2\pi f t}\, dt$



  $$= \int_{-T/2}^{T/2} p_T(t) \cdot e^{-j2\pi f t}\, dt$$

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$$= \lim_{T\to+\infty} \left(T \cdot \frac{\sin(\pi f T)}{\pi f T}\right) = \delta(f)$$

° $\mathcal{F}\{\delta(t)\} = 1 \longleftrightarrow \mathcal{F}\{1\} = \delta(t)$

$$\delta(f) = \int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi f t}\, dt = \mathcal{F}\{1\}$$

$$\int_{-\infty}^{\infty} e^{-j2\pi f t}\, dt = \delta(f) = \int_{-\infty}^{\infty} e^{j2\pi f t}\, dt$$

° $\mathcal{F}\{e^{j2\pi f_0 t}\} = \int_{-\infty}^{\infty} e^{j2\pi f_0 t} \cdot e^{-j2\pi f t}\, dt =$

$$= \int_{-\infty}^{\infty} e^{-j2\pi (f-f_0) t}\, dt = \delta(f-f_0)$$

° $\mathcal{F}\{\cos(2\pi f_0 t)\} = \mathcal{F}\left\{\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right\}$

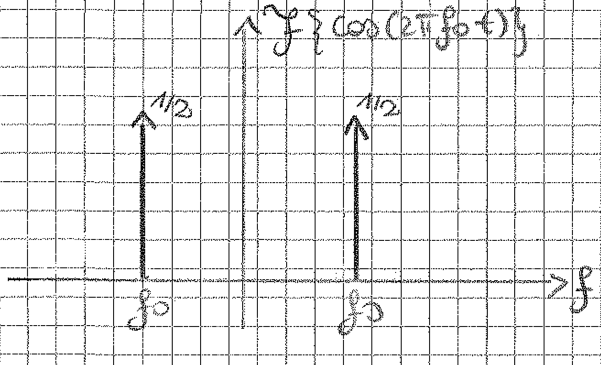
$$= \frac{1}{2}\mathcal{F}\{e^{j2\pi f_0 t}\} + \frac{1}{2}\underbrace{\mathcal{F}\{e^{-j2\pi f_0 t}\}}_{\circledast}$$

$$= \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

$\circledast$ $\mathcal{F}\{e^{-j2\pi f_0 t}\} =$

$= \int_{-\infty}^{\infty} e^{-j2\pi f_0 t}\, e^{-j2\pi f t}\, dt =$

$= \int_{-\infty}^{\infty} e^{-j2\pi (f+f_0) t}\, dt = \delta(f+f_0)$



° $\mathcal{F}\{\sin(2\pi f_0 t)\} = \mathcal{F}\left\{\frac{e^{+j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\right\}$

$$= \frac{1}{2j}\mathcal{F}\{e^{j2\pi f_0 t}\} - \frac{1}{2j}\mathcal{F}\{e^{-j2\pi f_0}\} =$$

$$= \frac{1}{2j}\delta(f-f_0) - \frac{1}{2j}\delta(f+f_0)$$

ALCUNE PROPRIETA'

1) Dualita'

$$\mathcal{F}\{x(t)\} = X(f) = g(f)$$
$$= \int x(t)\, e^{-j2\pi f t}\, dt$$
$$= \int x(\upsilon)\, e^{-j2\pi f \upsilon}\, d\upsilon$$

$$\mathcal{F}\{g(t)\} = \int g(t)\, e^{-j2\pi f t}\, dt$$

$$g(t) = \int x(\upsilon)\, e^{-j2\pi t \upsilon}\, d\upsilon$$

$$\mathcal{F}\{g(t)\} = \int_t \int_\upsilon x(\upsilon)\, e^{-j2\pi t \upsilon}\, d\upsilon\, e^{-j2\pi f t}\, dt$$

$$= \int_\upsilon x(\upsilon) \left[\int_t e^{-j2\pi t (\upsilon + f)}\, dt \right] d\upsilon$$

$$= \int_\upsilon x(\upsilon)\, \delta(\upsilon + f)\, d\upsilon = \int_\upsilon x(-f)\, \delta(\upsilon + f)\, d\upsilon =$$

$$= x(-f) \int_\upsilon \delta(\upsilon + f)\, d\upsilon = x(-f)$$

$$\mathcal{F}\{x(t)\} = X(f)$$
$$\mathcal{F}\{X(t)\} = x(-f)$$

2) Derivazione

IPOTESI: $x(t)$ e' a modulo-integrabile $\mapsto \int |x(t)|\, dt < \infty$

$$\mathcal{F}\{\dot{x}(t)\} = \mathcal{F}\left\{\frac{d}{dt} x(t)\right\} = \int \dot{x}(t)\, e^{-j2\pi f t}\, dt =$$

$$= \underbrace{\cancel{x(t)\, e^{-j2\pi f t}} \Big|_{-\infty}^{\infty}}_{\emptyset} - \int x(t)\, (-j2\pi f)\, e^{-j2\pi f t}\, dt =$$

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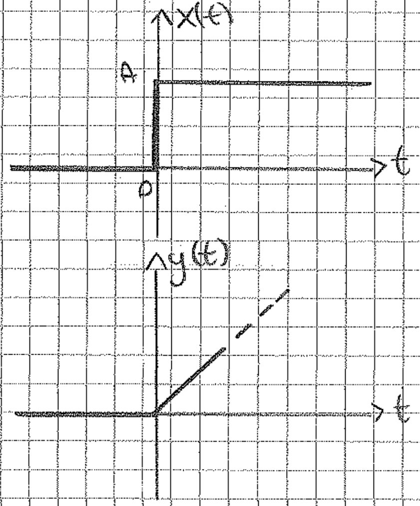


bounded input

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,d\tau \longmapsto \infty$$

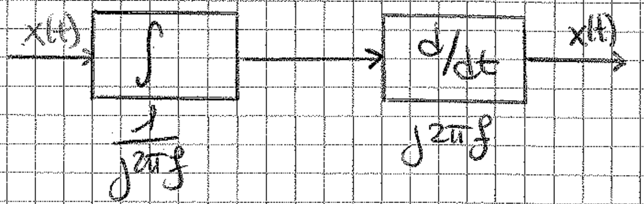
UNbounded output

$\Downarrow$

L'INTEGRATORE <u>non</u> è un sistema <u>stabile</u> in senso <u>BIBO</u>

Normalmente $x(0)=0$ nei sistemi ingegneristici

$$Y(f) = \frac{X(f)}{j2\pi f} \qquad H(f) = \frac{1}{j2\pi f}$$



4) Derivata n-esima di un segnale

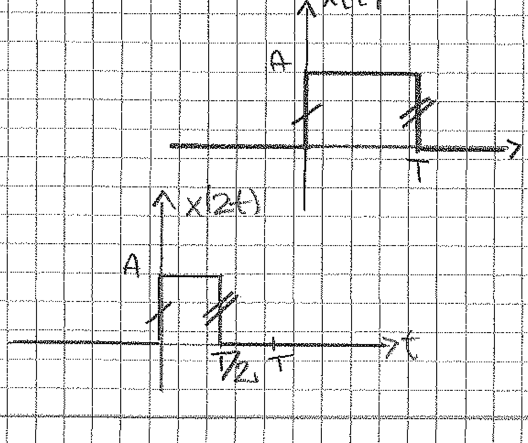
$$\mathcal{F}\left\{\frac{d^n}{dt^n}x(t)\right\} = (j2\pi f)^n X(f)$$

è vista come una cascata di n derivatori

5) Scalamento

$$\mathcal{F}\{x(t)\} = X(f)$$

$$\mathcal{F}\{x(kt)\} = ?$$



$$= \int_t \int_{f_1} X(f_1)\, e^{j2\pi f_1 t}\, df_1 \int_{f_2} y(f_2)\, e^{j2\pi f_2 t}\, df_2\, e^{-j2\pi f t}\, dt =$$

$$= \int_{f_1} \int_{f_2} X(f_1)\, Y(f_2) \underbrace{\left[\int_t e^{j2\pi t (f_1 + f_2 - f)}\, dt \right]}_{\delta(f_1+f_2-f)} df_1\, df_2 =$$

$$= \int_{f_1} \int_{f_2} X(f_1)\, Y(f_2)\, \delta(f_1 + f_2 - f)\, df_1\, df_2 =$$

$$= \int_{f_2} Y(f_2) \left[\int_{f_1} X(f_1)\, \delta(f_1 + f_2 - f)\, \delta f_1 \right] \delta f_2 =$$

$$= \int_{f_2} Y(f_2)\, X(f - f_2)\, df_2 = X(f) * Y(f)$$ convoluzione in frequenza

7) Parità

$\mathcal{F}\{x(t)\}$

$$X(f) = \int x(t)\, e^{-j2\pi f t}\, dt$$

$$X(-f) = \int x(t)\, e^{j2\pi f t}\, dt$$

$$= \left[\int x(t)\, e^{-j2\pi f t}\, dt \right]^* = X^*(f)$$

$$X(f) = R(f) + j\, I(f)$$

$$X(-f) = R(-f) + j\, I(-f) = R(-f) - j\, I(f)$$

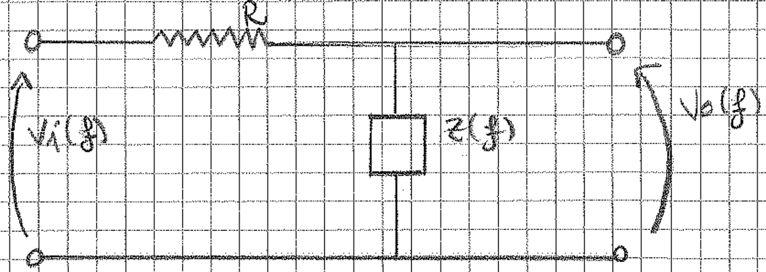
$R(-f) = R(f)$ pari

$I(-f) = -I(f)$ dispari

$$X(f) = \underline{M(f)}\, e^{+j\varphi(f)}$$

$$X(-f) = \underline{M(-f)}\, e^{j\varphi(-f)} = X^*(f) = M(f)\, e^{\underline{-j\varphi(f)}}$$

} modulo pari, fase dispari



$$Z(f) = \frac{1}{j2\pi f C}$$

$$V_o(f) = \frac{Z(f)}{R + Z(f)} V_i(f)$$

$$= \frac{\frac{1}{j2\pi f C}}{R + 1/j2\pi f C} V_i(f) =$$

$$= \frac{1}{1 + j2\pi f RC} V_i(f)$$

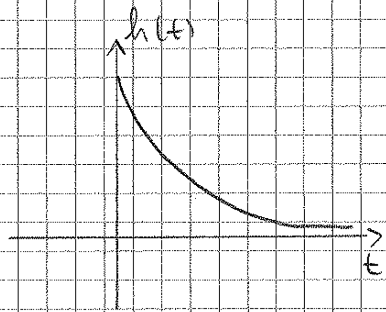
$$H(f) = \frac{1}{1+j2\pi f RC}$$

$$\mathcal{F}\{e^{-at}\mu(t)\} = \frac{1}{a+j2\pi f} = \frac{1/a}{1+j2\pi f/a}$$

con $a = 1/RC$

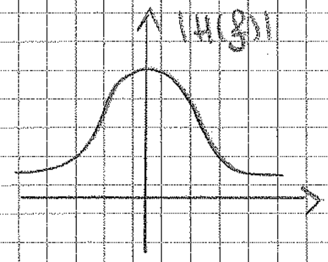
$$H(f) = a\,\mathcal{F}\{e^{-at}\mu(t)\}$$

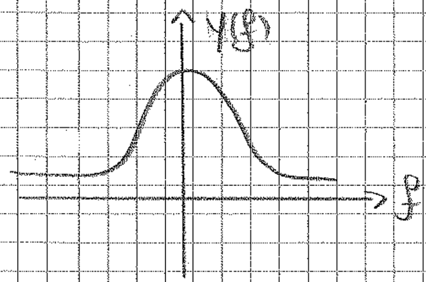
$$h(t) = a\,e^{-at}\mu(t) = \frac{1}{RC} e^{-t/RC}\mu(t)$$



$$H(f) = \frac{1}{1+j2\pi f RC} \cdot \frac{1-j2\pi f RC}{1-j2\pi f RC}$$

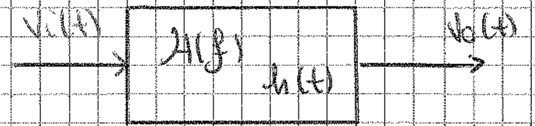
$$= \frac{1-j2\pi f RC}{1+(2\pi f RC)^2}$$





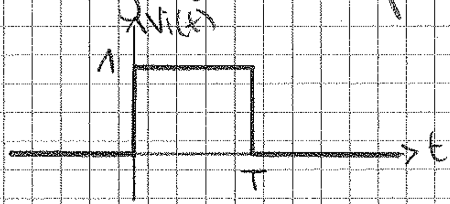
L'uscita contiene solo la parte di X(f) a bassa frequenza.

Il sistema LTI filtra alcune frequenze e ne lascia passare alcune: in questo caso passano le basse frequenze.



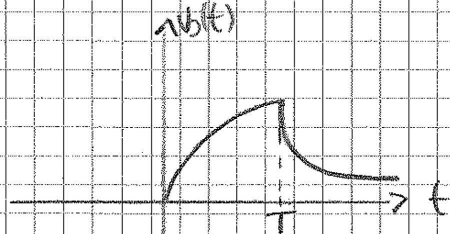
$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Se l'ingresso è una porta?



→ nel dominio del tempo

$$v_0(t) = v_i(t) * h(t)$$



→ nel dominio della frequenza

$$V_0(f) = V_1(f) H(f)$$

$$= \frac{V_1}{2}\left[\delta(f-f_1)H(f_1) + \delta(f+f_1)H(-f_1)\right]$$

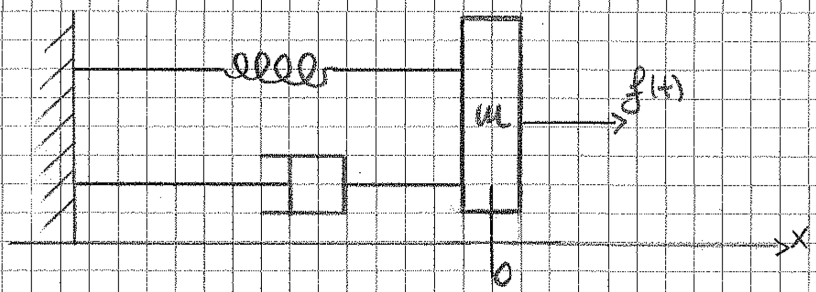
$$\left.\begin{aligned} H(f_1) &= \underbrace{|H(f_1)|}_{\text{modulo pari}} e^{j\varphi(f_1)} \\ H(-f_1) &= |H(f_1)| e^{-j\varphi(f_1)} \end{aligned}\right\} \text{fase dispari}$$

$$v_0(f) = \frac{V_1}{2} H(f_1)\left[\delta(f-f_1)e^{j\varphi(f_1)} + \delta(f+f_1)e^{-j\varphi(f_1)}\right]$$

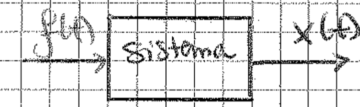
$$= \frac{V_1}{2} H(f_1)\left[e^{j2\pi f_1 t} e^{j\varphi(f_1)} + e^{-j2\pi f_1 t} e^{-j\varphi(f_1)}\right]$$

$$= V_1 \, H(f_1) \cos\left[2\pi f_1 t + \varphi(f_1)\right]$$

MOLLA - MASSA - SMORZATORE



$$m\ddot{x}(t) = f(t) - Kx(t) - C\dot{x}(t)$$



$$x(t) = \frac{1}{K} f(t) - \frac{m}{K}\ddot{x}(t) - \frac{C}{K}\dot{x}(t)$$

$$X(f) = \frac{1}{K} F(f) - \frac{m}{K}(j2\pi f)^2 X(f) - \frac{C}{K}(j2\pi f)X(f)$$

$$X(f)\left[1 + \frac{m}{K}(j2\pi f)^2 + \frac{C}{K}(j2\pi f)\right] = \frac{1}{K}F(f)$$

$$X(f) = \frac{\frac{1}{K}F(f)}{1 + \frac{m}{K}(j2\pi f)^2 + \frac{C}{K}(j2\pi f)} = H(f)F(f)$$

$$H(f) = \frac{1/K}{1 - \frac{m}{K}(2\pi f)^2 + \frac{C}{K}(j2\pi f)}$$

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$h(t)$

$H(f) = \mathcal{F}\{h(t)\}$

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$Y(f) = H(f)X(f)$$
$$= H(f)\,\delta(f-f_0)$$
$$= H(f_0)\,\delta(f-f_0)$$

$$y(t) = H(f_0)\,e^{j2\pi f_0 t}$$

$$x(t) = A\cos 2\pi f_0 t \longrightarrow \boxed{LTI} \longrightarrow y(t) = A\,M(f_0)\cos(2\pi f_0 t + \varphi(f_0))$$

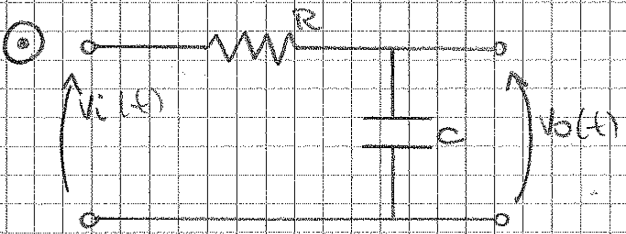
$$H(f_0) = M(f_0)\,e^{j\varphi(f_0)}$$

$$M(f_0) = \frac{\max y(t)}{\max x(t)}$$

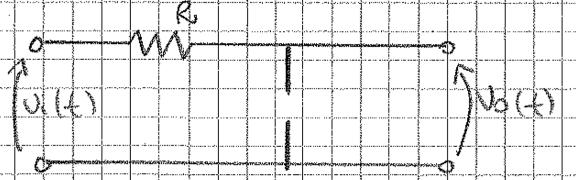
$\varphi(f_0) =$ fase di $y(t)$ - fase di $x(t)$

$$M(f_0) = M(-f_0)$$
$$\varphi(-f_0) = -\varphi(f_0)$$

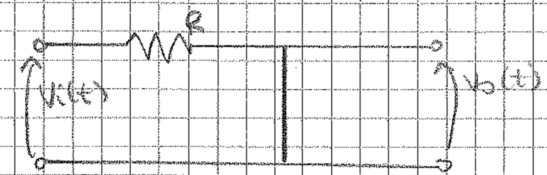
La funzione di trasferimento si dice anche RISPOSTA in FREQUENZA



per $f \to 0$,
C si comporta come un circuito aperto



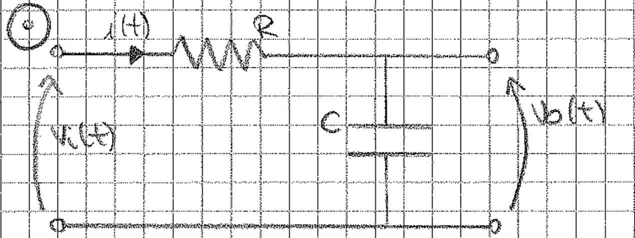
per $f \to \infty$,
C si comporta come un corto circuito



per $f \to 0$ $\quad V_o(t) = V_i(t) \quad i(t) = 0$

per $f \to \infty$ $\quad V_o(t) = 0 \quad i(t) = V_i(t)/R$

Ad alte frequenze, la tensione non passa: si comporta da PASSABASSO. Le correnti si comportano da PASSAALTO, a frequenze infinite riescono a passare.



Un'entrata: $V_i(t)$

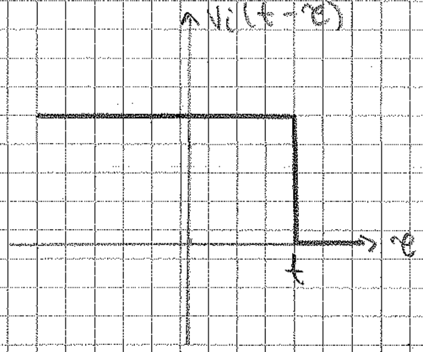
due uscite: $\begin{cases} V_o(t) \\ i(t) \end{cases}$

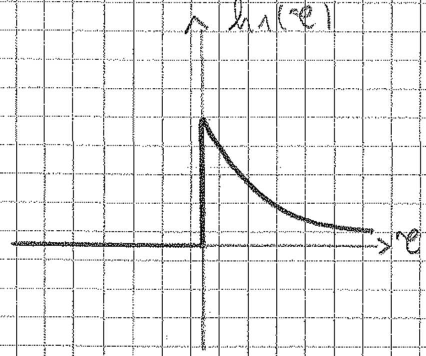
$$V_o(f) = V_i(f) \frac{1/j2\pi fC}{R + 1/j2\pi fC} \Big\} \text{impedenza}$$

$$= V_i(f) \frac{1}{1 + j2\pi fCR}$$

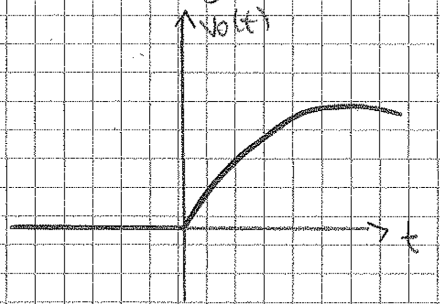
$$H_1(f) = \frac{1}{1 + j2\pi fCR}$$

$$v_o(t) = v_i(t) * h_1(t) = \int h_1(\tau)\, v_i(t-\tau)\, d\tau$$





$$v_o(t) = \int_0^t h_1(\tau) \cdot 1 \, d\tau = \int_0^t h_1(\tau)\, d\tau$$



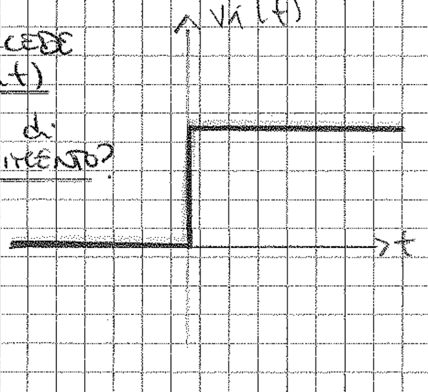
$$= \int_0^t \frac{1}{RC} e^{-t/RC} d\tau = \frac{1}{RC} \left. \frac{e^{-t/R\cdot C}}{-1/RC} \right|_0^t =$$

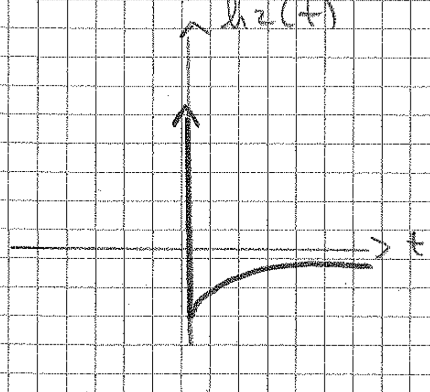
$$= \frac{1}{RC} \frac{e^{-t/RC} - 1}{-1/RC} = \frac{RC}{RC}\left(1 - e^{-t/RC}\right)$$

$$= \left(1 - e^{-t/RC}\right)$$

IL FILTRO PASSA-BASSO ELIMINA LE DISCONTINUITA'

COSA SUCCEDE CON $h_2(t)$ COME FUNZIONE DI TRASFERIMENTO?





$$i(t) = v_i(t) * h_2(t)$$

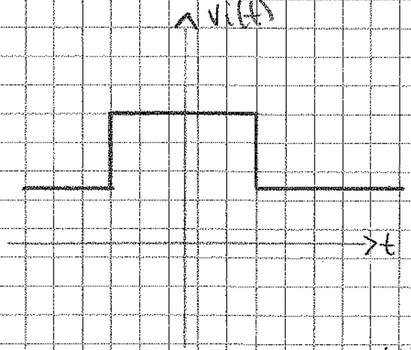
$$= \int_{-\infty}^{t} h_2(\tau)\, v_i(t-\tau)\, d\tau$$

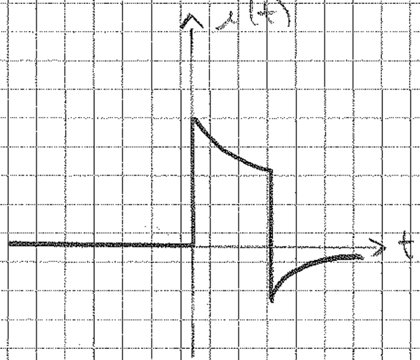
$$t > 0 \qquad i(t) = \int_0^t h_2(\tau)\, d\tau = \int_0^t \left[-\frac{1}{R^2 C} e^{-\tau/RC} + \frac{1}{R}\delta(\tau) \right] d\tau =$$

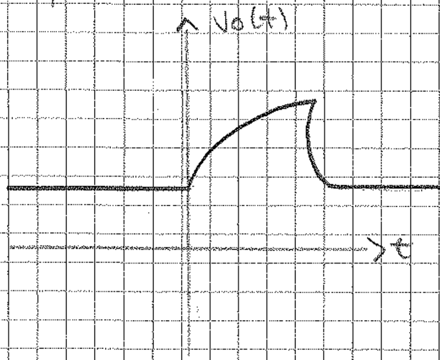
CASO 1: fino a T, $v_o(t)$ cresce come un esponenziale.
Poi $v_i(t)$ va a zero di colpo ma il filtro passa basso ha un'INERZIA: va giù ma in modo lento.
L'uscita di un passabasso è la versione 'addolcita' dell'ingresso.

CASO 2: La risposta è la stessa per il gradino unitario.
Il segnale sale improvvisamente, scende in T, ha discontinuità negativa e va giù della stessa quantità e torna su fino a 0 come esponenziale.

- Se traslo verso l'alto $v_i(t)$







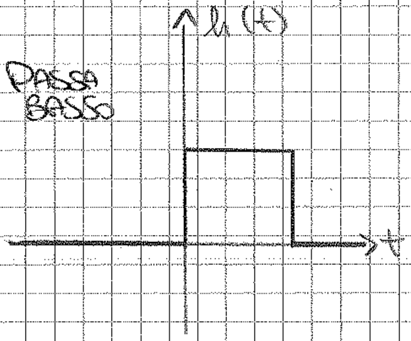
non importa il valore medio (la continua se ne va)

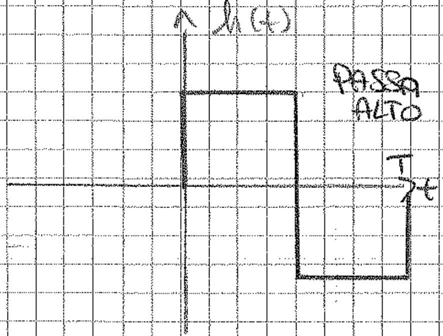
tiene la continua e tira su la funzione di una quantità pari all'offset!

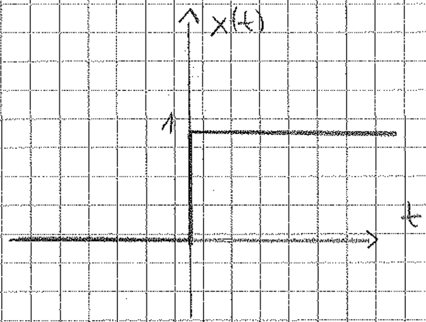
- $v_i(t) \longleftrightarrow V_i(f)$

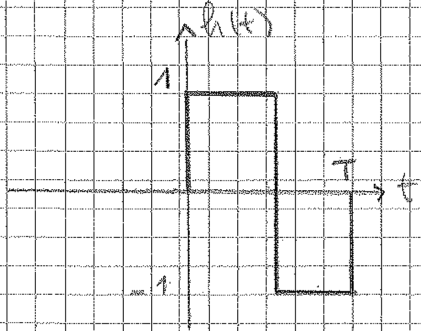
$$v_i'(t) = v_i(t) + A \longleftrightarrow V_i'(f) = V_i(f) + \mathcal{F}\{A\} \qquad \left(\mathcal{F}\{1\} = \delta(f)\right)$$

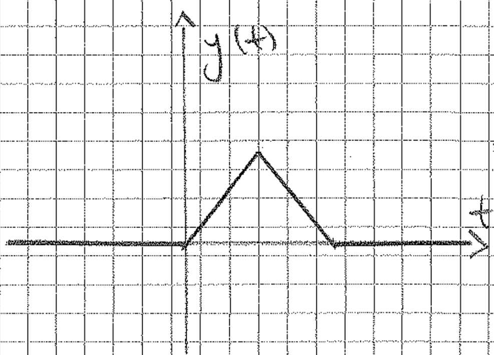
$$= V_i(f) + A\,\delta(f)$$

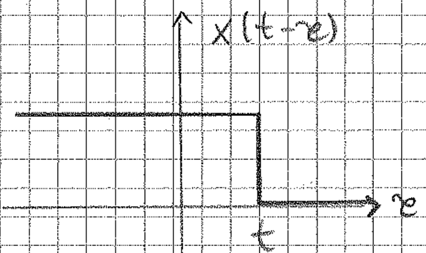




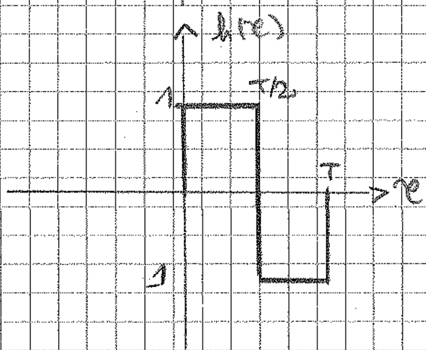








$y(t) = x(t) * h(t)$

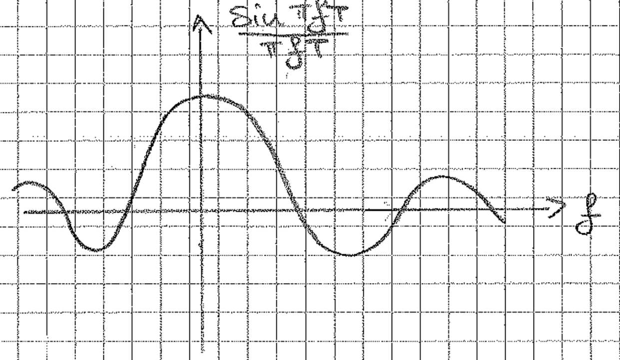


$t<T/2$, il gradino ribaltato entra in h(t), y(t) aumenta in maniera lineare.

$t>T/2$, l'intervallo diminuisce fino a T e poi va a 0. (Sottraggo all'area del quadrato ⊕ l'area del quadrato ⊖).

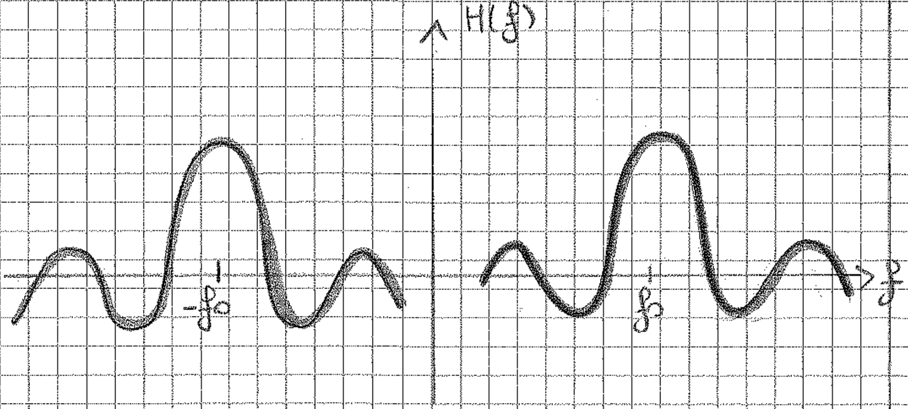
QUESTO PASSAALTO toglie la CONTINUA ma non ha lasciato la DISCONTINUA.

$$= \frac{T}{2}\left[\frac{\sin \pi(f-f_0)T}{\pi(f-f_0)T}\, e^{-j\pi(f-f_0)T} + \frac{\sin \pi(f+f_0)T}{\pi(f+f_0)T}\, e^{-j\pi(f+f_0)T}\right]$$



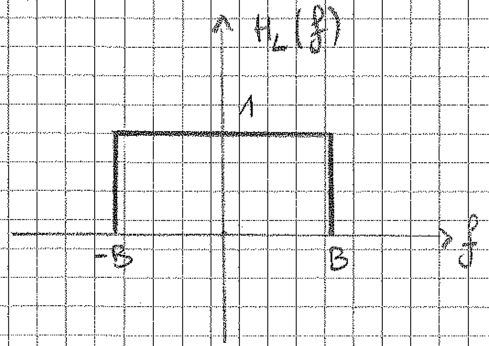
FILTRO PASSA BANDA

passano le bande di frequenza intorno a f_0



FILTRO PASSA BASSO

IDEALE di banda B

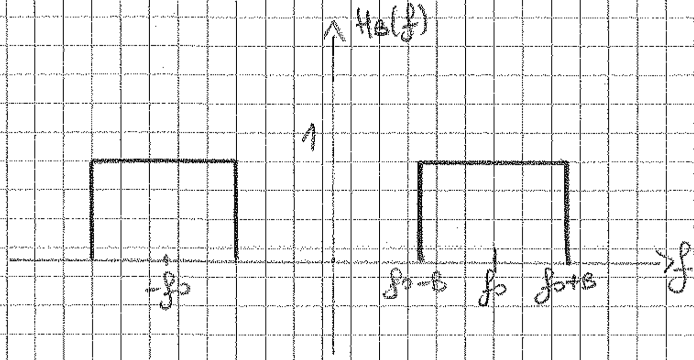


FILTRO PASSA BANDA

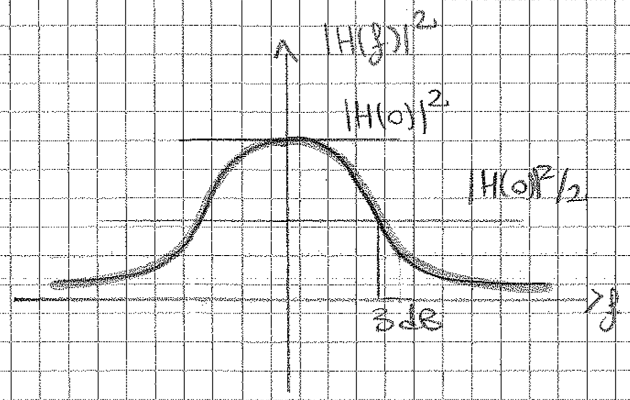
IDEALE di banda 2B

centrato in f_0

intervallo di frequenze che vengono lasciate passare.



BANDA a 3dB
(0 -3dB)



Prendo $|H(0)|^2$ e
cerco per che valore
ho una frequenza
pari a $\frac{|H(0)|^2}{2}$

$$|H(B_{3dB})|^2 = \frac{|H(0)|^2}{2}$$

Perchè 3dB? L'asse è logaritmico
$10 \log_{10}(2) \longrightarrow 3dB$

SEGNALI ad ENERGIA FINITA

SEGNALI TALI PER CUI $E(X) < \infty$

teorema di PARSEVAL

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df}$$

<u>Dim</u> $x(t) \in \mathbb{R}e$

$$X(f) = \mathcal{F}\{x(t)\}$$

$$\mathcal{F}\{x^2(t)\} = X(f) * X(f)$$

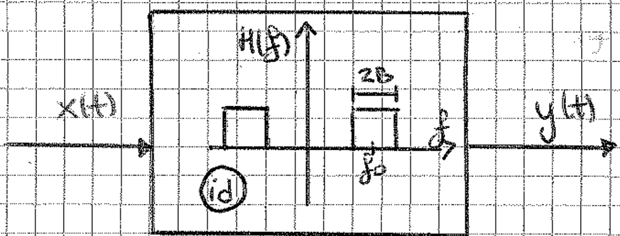
$x^2(t)$ è $x(t) \cdot x(t)$ nel dominio nel tempo; diventa una convoluzione nel dominio delle frequenze

$$= \int X(u)\, X(f-u)\, du$$

$$\int x^2(t)\, dt = \mathcal{F}\{x^2(t)\}_{f=0} = \int X(u)\, X(-u)\, du$$

$$= \int X(u)\, X^*(u)\, du$$

$$= \int |X^2(u)|\, du \quad \blacksquare$$

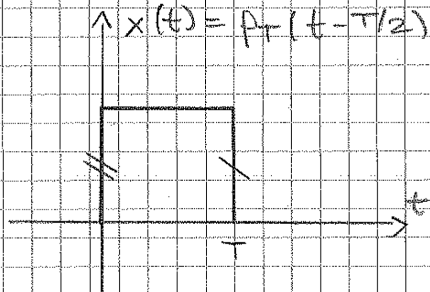


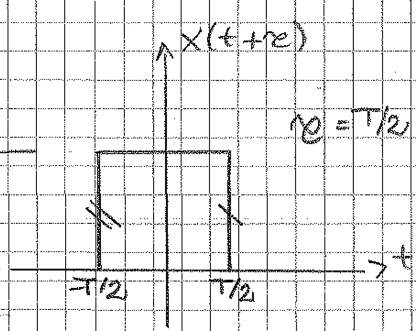
$$E(y) = \int y^2(t)\, dt = \int |Y(f)|^2 df =$$

$$= \int |X(f)\, H(f)|^2 df = \int_{f_0-B}^{+f_0+B} |X(f)|^2 df$$

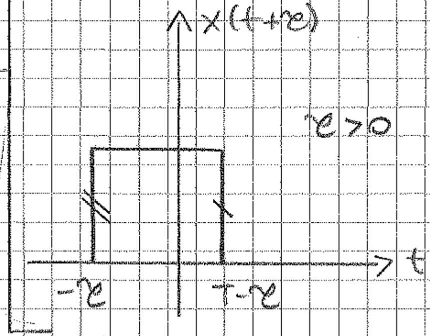
ESEMPIO

$x(t) = p_T(t - T/2)$





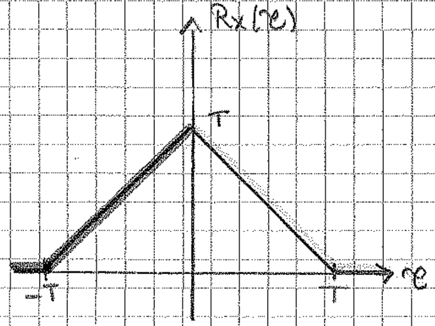
- $R_x(\pm\infty) = 0$

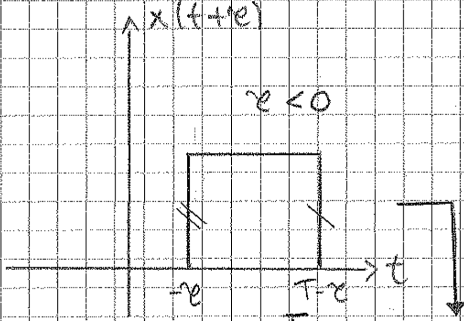


- $\tau < T$ $(\tau > 0)$ $\quad R_x(\tau) \neq 0$

$$R_x(\tau) = \int x(t) \cdot x(t+\tau)\, d\tau$$

$$= \int_0^{T-\tau} 1 \cdot 1\, dt = T - \tau$$





- $R_x(\tau) = \int_{-\tau}^{T} 1 \cdot 1\, dt = T + \tau$

$(-\tau < T,\ \tau > -T)$

ALCUNE PROPRIETA'

1) $R_x(0) = \int |x(t)|^2 dt = \mathcal{E}(x)$

2) $R_x(-\tau) = \int x^*(t)\, x(t-\tau)\, dt$

$t - \tau = u \qquad t = u + \tau$

$$= \int x^*(u+\tau)\, x(u)\, du =$$

5) Se $y(t) = x(t-t_0)$

$$R_y(\tau) = R_x(\tau)$$

Un eventuale ritardo o anticipo del segnale non fa la differenza.

$$Y(f) = X(f)\, e^{-j2\pi f t_0}$$

$$S_y(f) = |Y(f)|^2 = |X(f)|^2 = S_x(f)$$

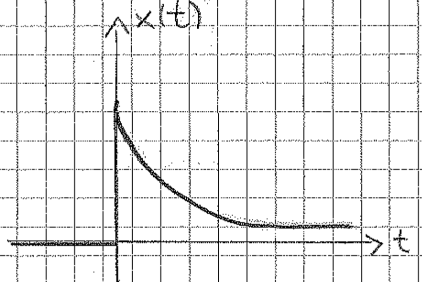
$$\downarrow$$

$$R_y(\tau) = R_x(\tau)$$

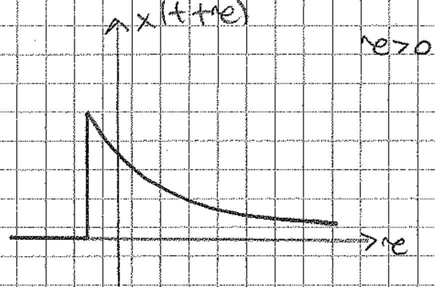
<u>ESEMPIO</u>

Calcolare l'autocorrelazione di

$$x(t) = e^{-t/T} u(t)$$



$$R_x(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$$

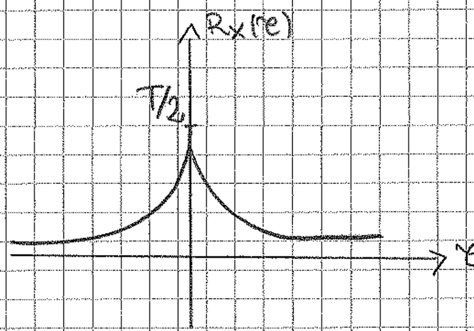


$\tau > 0$

$$= \int_0^{\infty} x(t)\, x(t+\tau)\, dt =$$

$$= \int_0^{\infty} e^{-t/T} e^{-(t+\tau)/T} dt =$$

$$= e^{-\tau/T} \int_0^{\infty} e^{-2t/T} dt = e^{-\tau/T} \left. \frac{e^{-2t/T}}{-2t/T} \right|_0^{\infty} = \frac{T}{2} e^{-\tau/T}$$



$R_x(\tau)$ è pari quindi sarà un esponenziale bilatero

$$\forall \tau \quad R_x(\tau) = \frac{T}{2} e^{-|\tau|/T}$$

$$x(t) \longrightarrow \boxed{h(t), H(f)} \longrightarrow y(t)$$

Se $x(t)$ è a energia finita, anche $y(t)$ lo sarà.

$$Y(f) = X(f)\,H(f)$$

$$S_y(f) = |Y(f)|^2 = |H(f)|^2\,|X(f)|^2 = |H(f)|^2 S_x(f)$$

$$R_y(f) = \mathcal{F}^{-1}\{S_y(f)\} = \mathcal{F}^{-1}\{|H(f)|^2 S_x(f)\} =$$

$$= \mathcal{F}^{-1}\{|H(f)|^2\} * \mathcal{F}^{-1}\{S_x(f)\} =$$

$$= R_h(\tau) * R_x(\tau)$$

funzione di autocorrelazione della risposta all'impulso del sistema LTI

$$R_h(\tau) = \int h^*(t)\,h(t+\tau)\,dt$$

$$R_{xy}(\tau) = \int y^*(t)\,x(t+\tau)\,dt$$

relazione di CROSS-CORRELAZIONE o di correlazione mutua

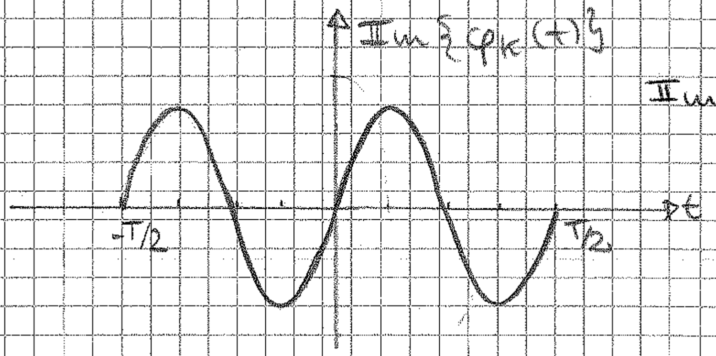
$$S_{xy}(f) = Y^*(f)\,X(f)$$

SPETTRO MUTUO

$$R_{yx}(\tau) = \int x^*(t)\,y(t+\tau)\,dt \qquad S_{yx}(f) = Y(f)\,X^*(f)$$

$$R_{xy}(0) = \int y^*(t)\,x(t)\,dt = \langle x(t), y(t)\rangle$$

se $R_{xy}(0) = 0 \longmapsto x(t)$ è ortogonale a $y(t)$
(ovvero: $x(t)$ e $y(t)$ non hanno nulla da spartire)



$\mathrm{Im}\{\varphi_K(t)\} = \sin(2\pi k t/T)\, p_T(t)$

Verifica dell'ortogonalità dei segnali della base di Fourier

$$<\varphi_K(t), \varphi_l(t)> = \int \varphi_K(t)\, \varphi_l^*(t)\, dt =$$

$$= \int e^{j2\pi k t/T} p_T(t)\, e^{-j2\pi\ t/T} p_T(t)\, dt =$$

Integro una funzione periodica, ogni periodo da contributo nullo.

$$= \int_{-T/2}^{T/2} e^{j2\pi(K-l)t/T}$$

$$= \begin{cases} 0 & k \neq l \\ T & k = l \end{cases}$$

Se $k = l \longmapsto <\varphi_K(t), \varphi_K(t)> = \int \varphi_K(t)\, \varphi_K^*(t)\, dt =$

$$= \int |\varphi_K(t)|^2 dt = \mathcal{E}(\varphi_K(t))$$

BASE ORTONORMALE

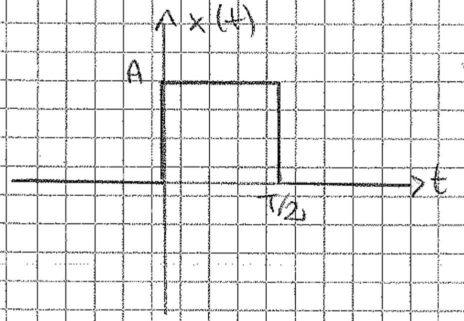
$$\mathcal{B}' = \{\varphi_K(t)\}_{K=1}^{N}$$

$$<\varphi_K(t), \varphi_l(t)> = \begin{cases} 0 & k \neq l \\ 1 & \forall K \end{cases}$$

Base Ortonormale di Fourier

$$\mathcal{B}' = \left\{ \frac{1}{\sqrt{T}}\, e^{j2\pi k t/T}\, p_T(t) \right\}$$

$\mathcal{B}' = \{\psi_1(t), \psi_2(t)\}$

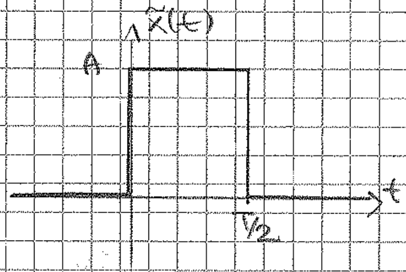


$$x_1(t) = \langle x(t), \psi_1(t) \rangle = \int x(t)\,\psi_1(t)\,dt = \int_0^{T/2} A \cdot \frac{1}{\sqrt{T}}\,dt = \frac{A}{\sqrt{T}} \cdot \frac{T}{2} = \frac{A\sqrt{T}}{2}$$

x_1

$$x_2(t) = \langle x(t), \psi_2(t) \rangle = \int x(t)\,\psi_2(t)\,dt = \int_0^{T/2} A \cdot \frac{1}{\sqrt{T}}\,dt = \frac{A\sqrt{T}}{2}$$

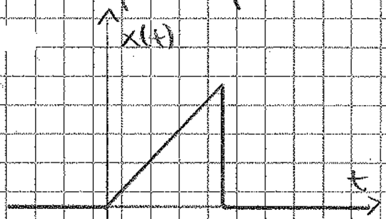
Posso costruire $\tilde{x}(t)$:



$$\tilde{x}(t) = x_1\psi_1(t) + x_2\psi_2(t) = \frac{A\sqrt{T}}{2}\left(\psi_1(t) + \psi_2(t)\right)$$

LA BASE $\mathcal{B}'$ è completa per $x(t)$

- Se $x(t)$ fosse



la base $\mathcal{B}'$ non sarebbe completa.

$\tilde{x}(t) = x_1\psi_1(t) + x_2\psi_2(t)$

$x_1 = \langle x(t), \psi_1(t) \rangle$

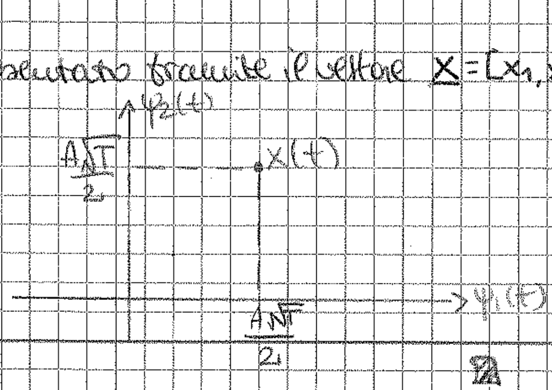
$x_2 = \langle x(t), \psi_2(t) \rangle$

Comunque è la migliore approssimazione di $x(t)$ con le basi, data $\mathcal{B}'$

MIGLIORE = quella che dà $E(e)$ MINIMA

Il segnale $x(t)$ può essere rappresentato tramite il vettore $\underline{X} = [x_1, x_2]$ ma è necessario che la base sia completa.

Lavoro nello spazio di Hilbert



$$= \int x(t) \frac{1}{\sqrt{T}} e^{-j2\pi kt/T} dt$$

$$x(t) = \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \left[\frac{1}{\sqrt{T}} \int x(t) e^{-j2\pi kt/T} p_T(t)\, dt \right] \frac{1}{\sqrt{T}} e^{j2\pi kt/T} p_T(t) =$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{1}{T} \int x(t) e^{-j2\pi kt/T} p_T(t)\, dt \right] e^{j2\pi kt/T} p_T(t) =$$

$$= \sum_{k=-\infty}^{\infty} \mu_k e^{j2\pi kt/T} p_T(t)$$

$$\mu_k \triangleq \frac{1}{T} \int x(t) e^{-j2\pi kt/T} p_T(t)\, dt$$ k-esimo coefficiente della serie di Fourier

$x(t)$ è espresso come combinazione lineare dei segnali di $\mathcal{B}$, con $\mathcal{B} = \left\{ e^{j2\pi kt/T} p_T(t) \right\}_{k=-\infty}^{+\infty}$

$$\mu_k = \frac{1}{T} \int_{-\infty}^{+\infty} x(t) e^{-j2\pi kt/T} p_T(t)\, dt$$

$$= \frac{1}{T} \mathcal{F}\left\{ x(t)\, p_T(t) \right\}_{f=\frac{k}{T}}$$

in termini di trasformata di Fourier, è la trasformata di $x(t)$ per la porta valutato per una frequenza $f = k/T$

Definisco $x_T(t) = x(t)\, p_T(t)$ e riscrivo μ_k

$$\mu_k = \frac{1}{T} \mathcal{F}\left\{ x_T(t) \right\}_{f=\frac{k}{T}} = \frac{1}{T} X_T\left(\frac{k}{T}\right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \mu_k e^{j2\pi kt/T} p_T(t)$$

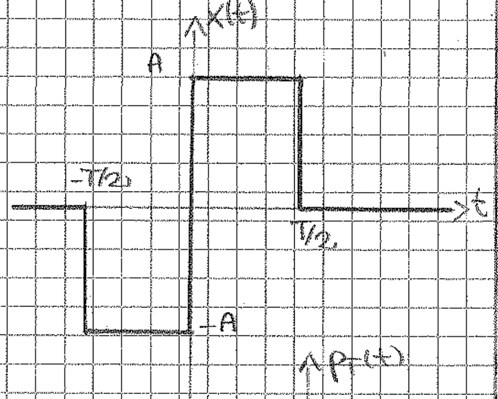
$$\mu_k = \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, p_T(t)\, e^{-j2\pi kt/T} dt = \frac{1}{T} X_T\left(\frac{k}{T}\right)$$

$$\mu_k = \begin{cases} k = 2n+1 & \mu_k = \dfrac{A}{j\pi k}\left[1+1\right] = \dfrac{2A}{j\pi k} \\ k = 2n & \mu_k = \dfrac{A}{j\pi k}\left[\cancel{1-1}\right] = 0 \end{cases}$$

$$\mu_{-k} = \mu_k^*$$

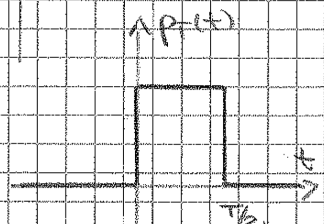
MODO 2

$$\mu_k = \frac{1}{T} X_T(k/T)$$



$$x(t) = x(t)\, p_T(t) = x_T(t)$$

$$X(f) = X_T(f)$$



è una porta ritardata di $T/4$

$$x(t) = \underbrace{A\, p_{T/2}(t - T/4)}_{\text{parte} \oplus} - \underbrace{A\, p_{T/2}(t + T/4)}_{\text{parte} \ominus}$$

$$= A\left[\delta(t - T/4) * p_{T/2}(t) - \delta(t + T/4) * p_{T/2}(t)\right] =$$

$$= A\left[\delta(t - T/4) - \delta(t + T/4)\right] * p_{T/2}(t)$$

$$X(f) = A\left[e^{-j2\pi f T/4} - e^{j2\pi f T/4}\right] \cdot \frac{T}{2}\, \frac{\sin \pi f T/2}{\pi f T/2} =$$

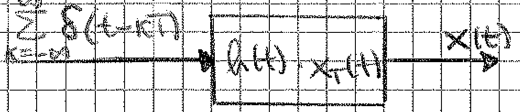
$$= jA\left(-\sin 2\pi f \tfrac{T}{4}\right) \cdot T \cdot \frac{\sin \pi f T/2}{\pi f T/2} = \frac{TA}{j} \cdot \frac{\sin^2 \pi f T/2}{\pi f T/2}$$

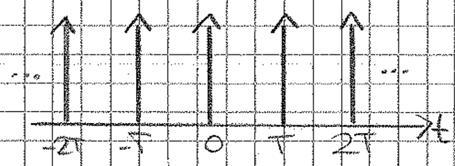
$$\mu_k = \frac{1}{T} X(k/T) = \frac{A}{j} \cdot \frac{\sin^2 \pi \frac{k}{T} \cdot \frac{T}{2}}{\pi \cdot \frac{k}{T} \cdot \frac{T}{2}} = \frac{2A}{j\pi k} \sin\left(\pi \frac{k}{2}\right)$$

$$\mu_k = \begin{cases} k = 2n & \mu_k = 0 \\ k = 2n+1 & \mu_k = \dfrac{2A}{j\pi k} \end{cases}$$

$$= x_T(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$\sum_{k=-\infty}^{\infty} \delta(t-kT)$ è detto treno di delta





È un modello matematico, non pretendo che la risposta all'impulso sia causale.

Quanto vale l'energia del segnale periodico? O è identicamente nullo e va a zero, o è infinita. Non ha senso parlare di energia ma piuttosto di POTENZA.

Per un segnale periodico x(t) di periodo T si definisce la potenza come

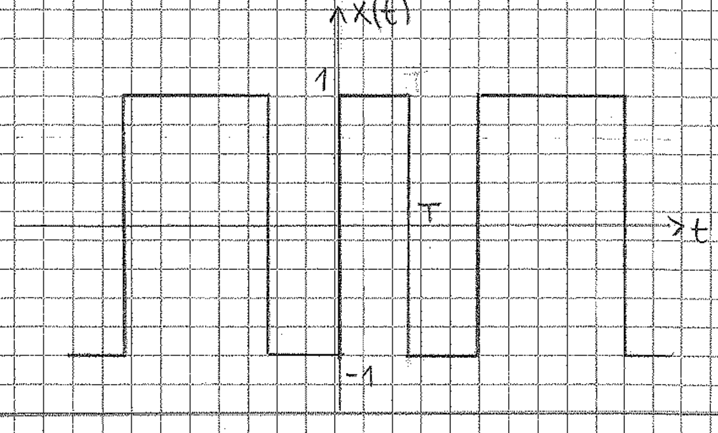
$$p(x) = \lim_{a \to +\infty} \frac{1}{2a} \int_{-a}^{a} |x(t)|^2 dt =$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt =$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \frac{1}{T} E(x_T)$$

il risultato è coerente: la potenza si misura in [W], ovvero [J/s]. L'unità di misura del risultato è

$$\frac{1}{T} E(x) = \frac{1}{[s]} \cdot [J] = [W]$$



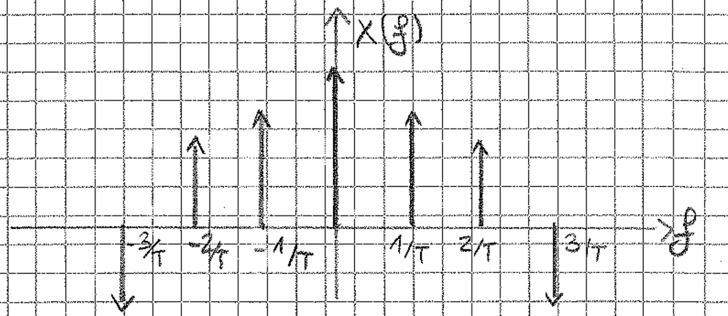
Questo segnale x(t) varia tra -1 e +1 ma lo fa in modo casuale.

$$= \frac{1}{T} X_T\left(\frac{k}{T}\right) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi \frac{k}{T} t}\, dt$$

$X(f) = \mathcal{F}\{x(t)\}$ dove $x(t)$ è il segnale periodico

$$= \mathcal{F}\left\{\sum_{k=-\infty}^{+\infty} \mu_k\, e^{j2\pi k t/T}\right\}$$

$$= \sum_{k=-\infty}^{\infty} \mu_k\, \mathcal{F}\left\{e^{j2\pi k t/T}\right\} = \sum_{k=-\infty}^{+\infty} \mu_k\, \delta(f - k/T)$$



L'intervallo tra due delta è sempre: $\boxed{1/\text{periodo del segnale}}$

$$\mathcal{F}\left\{\sum_{k=-\infty}^{+\infty} \delta(t - kT)\right\} =$$

$$\sum_{k=-\infty}^{+\infty} \delta(t - kT) = \sum_{k=-\infty}^{+\infty} \mu_k\, e^{j2\pi k t/T} \qquad t \in \mathbb{R}$$

$$\mu_k = \frac{1}{T}\int x_T(t)\, e^{-j2\pi k t/T}\, dt =$$

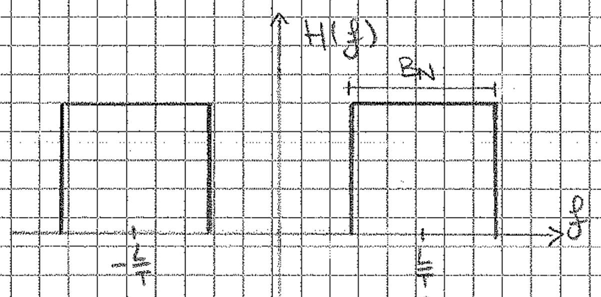
$$= \frac{1}{T}\int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi k t/T}\, dt = \frac{1}{T}\cdot 1 = \frac{1}{T}$$

$$\sum_{k=-\infty}^{+\infty} \delta(t - kT) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j2\pi k t/T}$$

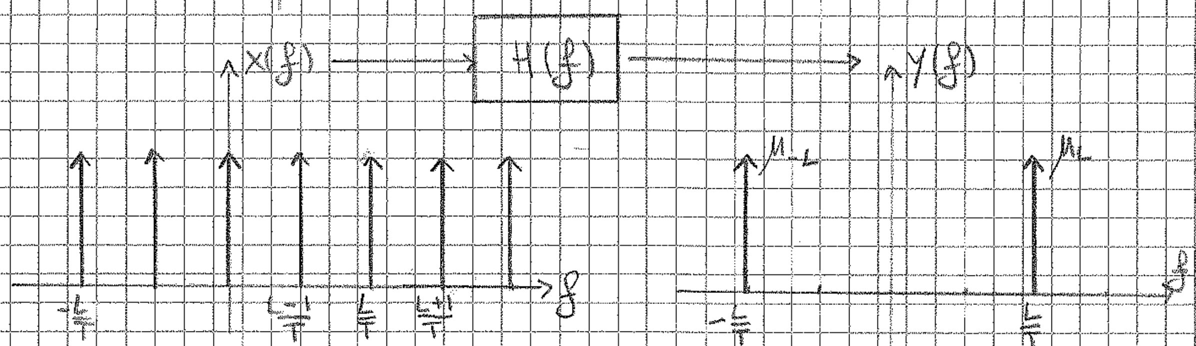
$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} \mathcal{F}\left\{e^{j2\pi k t/T}\right\} = \boxed{\frac{1}{T}\sum_{k=-\infty}^{+\infty} \delta(f - kT)}$$

LA TRASFORMATA È ANCORA UN TRENO DI DELTA.

$$= \sum_{n=-\infty}^{+\infty} \mu_n \, H\left(\frac{n}{T}\right) \delta\left(f - \frac{n}{T}\right)$$



Filtro Passa Banda ideale a guadagno unitario con frequenza centrale L/T e banda inferiore a $\frac{1}{T}$.



Sopravvivono solo due δ

$$P(y) = \sum_{n=-\infty}^{+\infty} |\mu_n|^2 = |\mu_L|^2 + |\mu_{-L}|^2$$

Lo SPETTRO di una grandezza indica come la grandezza in questione è distribuita in frequenza.

$$\mathcal{G}_x(f) = \sum_{n=-\infty}^{+\infty} |\mu_n|^2 \, \delta(f - n/T)$$

$$\int \mathcal{G}_x(f)\, df = \int \sum_{n=-\infty}^{+\infty} |\mu_n|^2 \, \delta\left(f - \frac{n}{T}\right) df =$$

$$= \sum_{n=-\infty}^{+\infty} |\mu_n|^2 \int_{-\infty}^{+\infty} \delta\left(f - \frac{n}{T}\right) = \sum_{n=-\infty}^{+\infty} |\mu_n|^2 = P(x)$$

ESEMPIO

$$X(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} Q(n/T)\, \delta\left(f - \frac{n}{T}\right)$$

$$\mathcal{G}_x(f) = \sum_{n=-\infty}^{+\infty} \left| \frac{1}{T} Q\left(\frac{n}{T}\right)\right|^2 \delta\left(f - \frac{n}{T}\right)$$

$$\phi_x(\tau) = \int G(f)\, e^{j2\pi f\tau} df$$

$$\phi_x(0) = \int G_x(f) dx = P(x)$$

2) $\phi_x(-\tau)$ (per segnali reali)

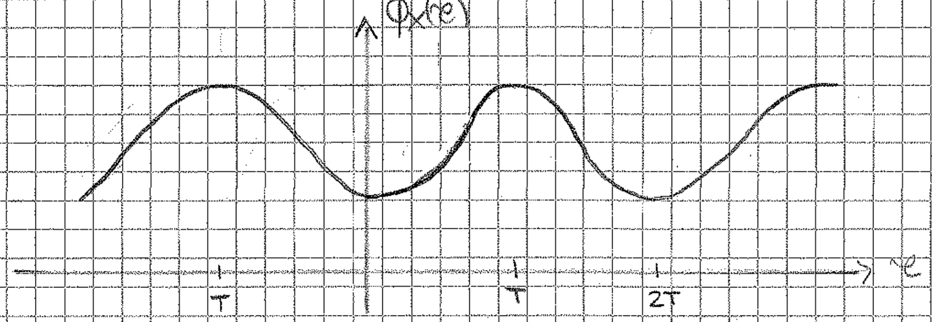
$$\phi_x(-\tau) = \int G_x(f)\, e^{-j2\pi f\tau} df =$$

$$= \left[\int G_x(f) e^{j2\pi f\tau} df\right]^* = \left[\phi_x(\tau)\right]^* = \phi_x^*(\tau)$$

per $x(t) \in \mathbb{R}$, anche $\phi_x(\tau) \in \mathbb{R}$ quindi $\phi_x(\tau) = \phi_x^*(\tau)$

La funzione di autocorrelazione è periodica e pari:

$$\phi_x(T) = \phi_x(0) = P(x)$$



$$\phi_x(\tau) = \sum_{n=-\infty}^{\infty} |\mu_n|^2 e^{j2\pi n t/T}$$

$$\mu_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi n t/T} dt = \frac{1}{T}\int_{-\infty}^{+\infty} x_T(t)\, e^{-j2\pi n t/T} dt$$

$$\phi_x(\tau) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-\infty}^{\infty} x_T(t_1)\, e^{-j2\pi n t_1/T} dt_1 \cdot \frac{1}{T}\int_{-\infty}^{+\infty} x_T(t_2) e^{-j2\pi n t_2/T} dt_2 \cdot$$

$$\cdot\, e^{j2\pi n \tau/T} =$$

$$= \frac{1}{T^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_T(t_1) x^*(t_2) \sum_n e^{j2\pi n(\tau + t_2 - t_1)/T} dt_1 dt_2 =$$

RIASSUNTO di FORMULE

SEGNALI ad ENERGIA FINITA

$$X(f) = \mathcal{F}\{x(t)\} = \int x(t)\, e^{-j2\pi f t}\, dt$$

$$E(x) = \int |x(t)|^2 dt = \int |X(f)|^2 df$$

$$S_x(f) = |X(f)|^2$$

$$R_x(\tau) = \mathcal{F}^{-1}\{S_x(f)\} = \int x^*(t)\, x(t+\tau)\, dt$$

SEGNALI PERIODICI

$$x(t) = \sum \mu_n\, e^{j2\pi n t/T}$$

con $\mu_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi n t/T}\, dt = \frac{1}{T} X_T(n/T)$

$$X(f) = \sum \mu_n\, \delta\left(f - \frac{n}{T}\right)$$

$$G_x(f) = \sum |\mu_n|^2\, \delta(f - n/T)$$

$$\phi_x(\tau) = \mathcal{F}^{-1}\{G_x(f)\} = \frac{1}{T}\int_{-T/2}^{T/2} x^*(\tau)\, x(t+\tau)\, dt$$

$$P(x) = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \int G_x(f)\, df = \phi_x(0)$$

$$x(t) = A\cos(2\pi f_c t + \varphi)$$
$$= A\cos\left(2\pi f_c t + 2\pi f_c \frac{\varphi}{2\pi f_c}\right)$$

$\frac{\varphi}{2\pi f_c}$ si misura in secondi: $\frac{\cancel{rad}}{\cancel{rad}\cdot Hz} = s$

$\downarrow$ lo definisco t_0

$$= A\cos[2\pi f_c(t+t_0)]$$ ho un anticipo/ritardo di $-t_0$

$$= A\cos(2\pi f_c(t+(T-(T-t_0))))$$

$$x(t) \to \boxed{\text{rit } t_0} \to y(t)$$

<u>Se $x(t)$ è ad energia finita:</u>

$$S_y(f) = |Y(f)|^2 = |X(f)H(f)|^2 =$$
$$= |X(f)|^2 |H(f)|^2 = S_x(f)|H(f)|^2$$

Per un ritardatore di t_0, quanto valgono $h(t)$ e $H(f)$?

$$H(f) = e^{-j2\pi f t_0} \longmapsto |H(f)|^2 = 1$$ Sistema PASSA TUTTO

$$h(t) = \delta(t-t_0)$$

$$= |X(f)|^2|H(f)|^2 = S_x(f)$$

quindi $R_x(\tau) = R_y(\tau)$

<u>Se $x(t)$ è a potenza finita, periodico di periodo T:</u>

$$G_y(f) = \sum |\mu_{ky}|^2 \delta(f-k/T)$$
$$G_x(f) = \sum |\mu_{kx}|^2 \delta(f-k/T)$$

$$Y(f) = X(f)H(f) = H_f \sum \mu_{kx}\, \delta(f-k/T)$$
$$= \sum \mu_k H(f)\delta(f-k/T) = \sum \mu_{kx} H(k/T)\, \delta(f-k/T)$$

TEOREMA del CAMPIONAMENTO

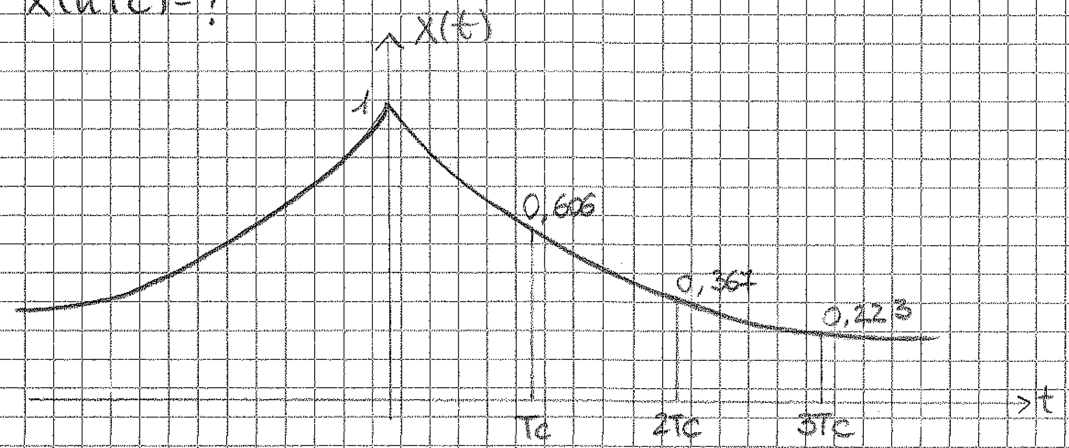
$x(t)$ Segnale a tempo continuo (in elettronica è un segnale ANALOGICO)

$x(t) \longrightarrow x(nT_c)$ con n intero
T_c intervallo di campionamento

$x(t) = e^{-|t|/t_0}$

$T_c = t_0/2$ $\quad n = 0, \pm 1, \pm 2, \pm 3, \dots$

$x(nT_c) = ?$

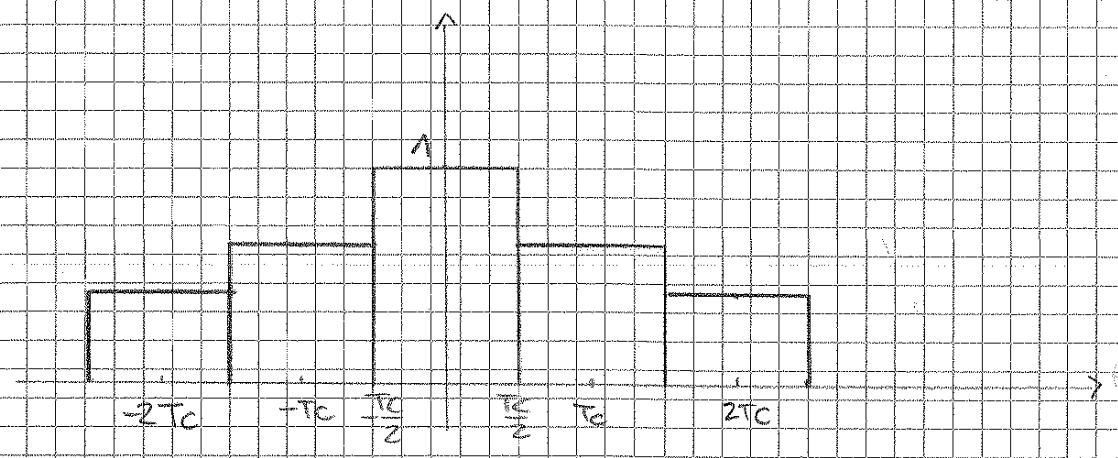


$x(nT_c) = e^{-|n\, t_0/2|/t_0}$

$= e^{-|n|/2}$

n	x(nTc)
0	1
±1	$e^{-1/2} = 0,606$
±2	$e^{-1} = 0,367$
±3	$e^{-3/2} = 0,223$

$$\tilde{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_c)\, \underbrace{g(t-nT_c)}_{\text{Come prendo } g(t)?}$$



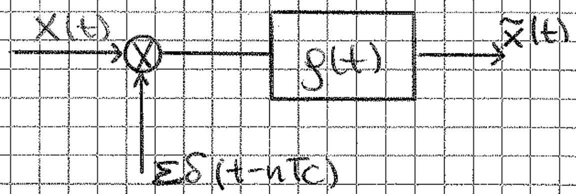
QUALE $g(t)$ È MEGLIO USARE?

$$x(t) = \sum_{n=-\infty}^{+\infty} x(nT_c)\, g(t-nT_c)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT_c)\,\delta(t-nT_c) * g(t)$$

$$x(nT_c)\,\delta(t-nT_c) = x(t)\,\delta(t-nT_c)$$

$$= x(t)\left[\sum_{n=-\infty}^{+\infty} \delta(t-nT_c)\right] * g(t)$$



$g(t)$ = risposta all'impulso di un sistema LTI

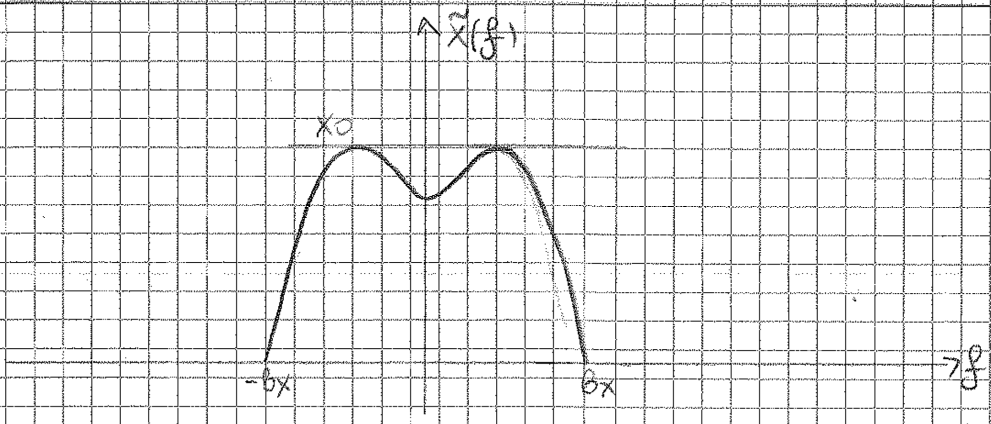
$$X_\delta(f) = \mathcal{F}\left\{x(t)\sum_n \delta(t-nT_c)\right\} =$$

$$= \mathcal{F}\{x(t)\} * \mathcal{F}\left\{\sum \delta(t-nT_c)\right\} =$$

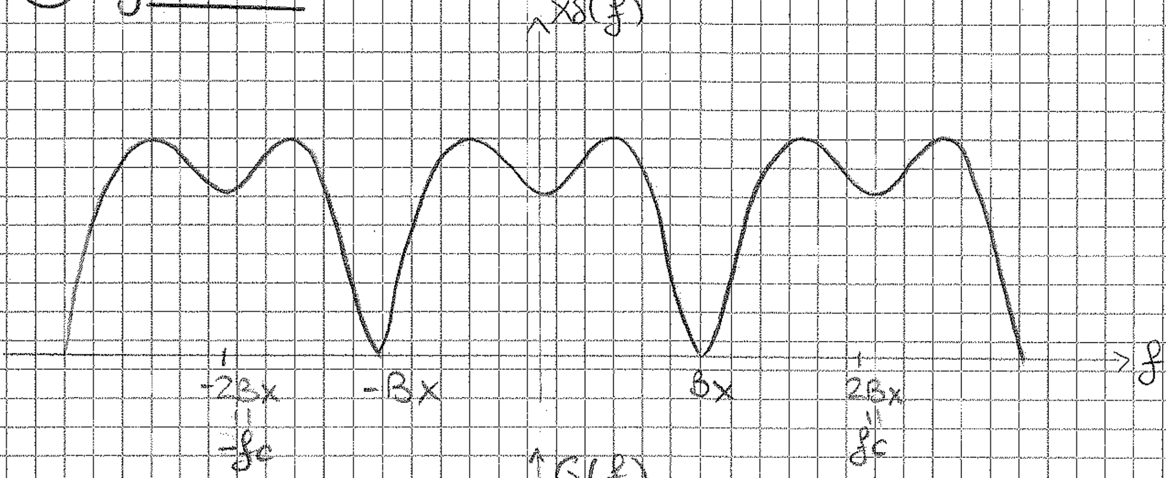
$$= X(f) * \frac{1}{T_c}\sum \delta(f-n/T_c) =$$

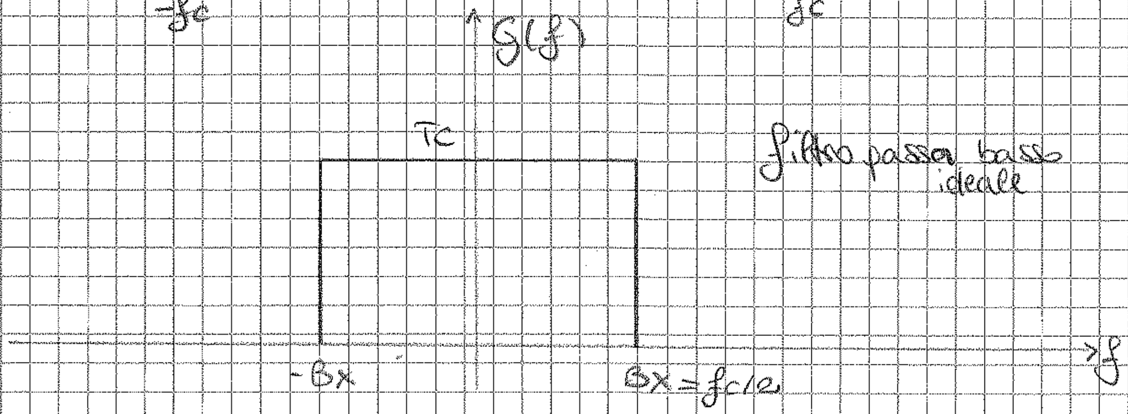
$$= \frac{1}{T_c}\sum_{-\infty}^{\infty} X(f) * \delta(f-n/T_c) =$$

$$= \frac{1}{T_c}\sum_{n=-\infty}^{+\infty} X(f-n/T_c)$$



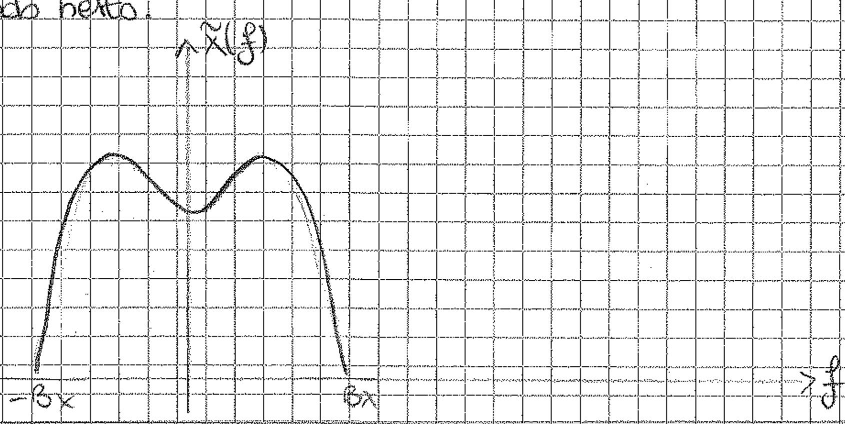
② $\underline{f_c = 2B_x}$





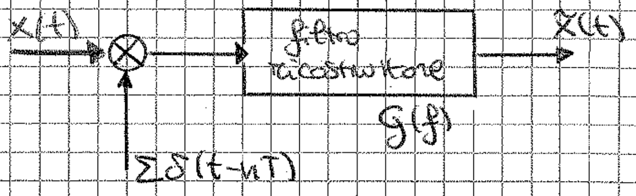
filtro passa basso ideale

$G(f)$ deve essere costante tra $-B_x$ e B_x e andare a zero in modo netto.



Segnale	banda	fr. campionamento
VOCALE	3,6 KHz	8 kHz
MUSICALE	20 kHz	44,2 KHz

• il nostro orecchio ha 20 KHz di banda, la musica più o meno; c'è un po' di margine nel campionamento.



G(f) è quasi sempre un PASSABASSO

$\tilde{x}(t) = x(t)$ se $\frac{1}{T_c} = 2B_x = f_c$

DUALITA' tra CAMPIONAMENTO e SEGNALE PERIODICO

o Segnale PERIODICO

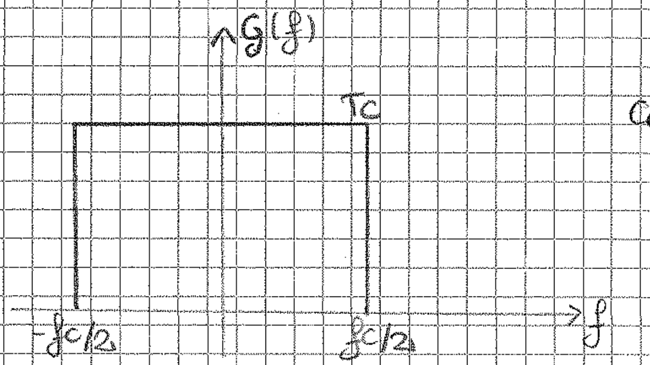
$$x(t) = \sum g(t-nT) =$$
$$= g(t) * \sum \delta(t-nT)$$

la $\mathcal{F}$ è a righe: $X(f) = \frac{1}{T}\sum G(n/T)\,\delta(f-n/T)$

o CAMPIONAMENTO

$$x_\delta(t) = x(t) \cdot \sum \delta(t-nT_c)$$
$$X_\delta(f) = \frac{1}{T_c}\sum X(f-n/T_c)$$

la $\mathcal{F}$ è periodica di periodo $1/T_c$



costante $[-f_c/2 ; f_c/2]$