NUMERO : 374

DATA : 17/10/2012

A P P U N T I

STUDENTE : Gemello

MATERIA : Analisi II

Prof. Mazzi

TOPOLOGIA IN $\mathbb{R}^n$

$\mathbb{R}^n = \{(x_1, ..., x_n)\ \ x_i \in \mathbb{R},\ i = 1, ..., n\}$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ PIANO

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ SPAZIO

DISTANZA

$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ $\quad (X, Y) \to$ DIST. TRA I 2

1) DOM $d = \mathbb{R}^n \times \mathbb{R}^n$

2) $d(X,Y) = d(Y,X)$

3) $d(X,Y) \geq 0 \quad \forall X, Y \in \mathbb{R}^n$

4) $d(X,Y) = 0 \iff X = Y$

5) $\forall X, Y, Z \quad d(X,Y) \leq d(X,Z) + d(Y,Z)$

DISEG. TRIANGOLARE

DISTANZA EUCLIDEA

$n = 1$ SU $\mathbb{R}$ $\quad d(X,Y) = |Y - X|$ $\quad X \ \ Y$

$n = 2$ SU $\mathbb{R}^2$ $\quad d(X,Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$n > 2$ SU $\mathbb{R}^n$ $\quad d(X,Y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$

$d(X,Y) = |X - Y| = \|X - Y\| \leftarrow$ NORMA EUCLIDEA

$d(X,0) = \|X\| \leftarrow$ NORMA DI X

$= \sqrt{x_1^2 + ... + x_n^2}$

INTORNI SFERICI $\to$ INTERVALLI SOLO IN $\mathbb{R}$

$\bar{X} \in \mathbb{R}^n$, INTORNO DI $\bar{X}$ DI RAGGIO $\rho > 0$

$\{X \in \mathbb{R}^n : |X - \bar{X}| < \rho\}$

A APERTO $\Leftrightarrow$ $\complement A$ CHIUSO

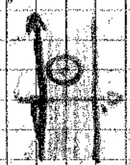
<u>CHIUSURA</u> DI $A = \bar{A} = A \cup \partial A$

$\bar{A}$ È IL + PICCOLO CHIUSO CHE CONTIENE A

$Q\ [0,1] \times [0,1] = \{(x,y) \in Q,\ x, y \in \mathbb{Q}\}$

$D = \{(x,y) \in Q : x \in \mathbb{Q},\ y \in [0,1]\}$

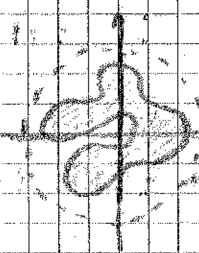
D NON HA P. INTERNI, HA SOLO P. FRONTIERA



INSIEME <u>LIMITATO</u> IN $\mathbb{R}^n$

$\exists M > 0,\ A \subseteq B(0,M)$

$\exists M > 0 \quad x \in A \Rightarrow \|x\| < M$



SE A NON È LIMITATO $\Rightarrow$ A <u>ILLIMITATO</u>

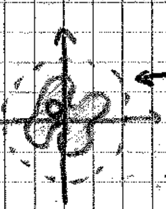
$\forall M > 0 \quad \exists\ x \in A,\ \|x\| > M$

ES. $\{(x,y) \in \mathbb{R}^2 : y = x^2\} \quad \|x\| \to +\infty$

$K \subseteq \mathbb{R}^n$ È <u>COMPATTO</u> SE CHIUSO E LIMITATO

$\to \partial K \subseteq K$ CHIUSO

$\to$ LIMITATO

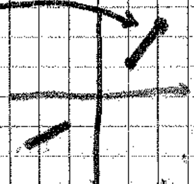


$\leftarrow$ IN $\mathbb{R}^2$

CHIUSO

LIMITATO

COMPATTO



$F : \mathbb{R}^n \to \mathbb{R}^m$

CONTINUA SU K COMPATTO

$\Rightarrow$ F HA UN MAX E UN MIN ASS SU K

CALCOLO DIFFERENZIALE

$F: \mathbb{R}^n \to \mathbb{R}^m$

F CONTINUA IN $\bar{x} \in \text{DOM}F$ SE

$\forall B(F(x), \varepsilon) \; \exists \; B(\bar{x}, \delta)$

$x \in B(\bar{x}, \delta) \cap \text{DOM}F \Rightarrow F(x) \in B(F(\bar{x}, \varepsilon))$

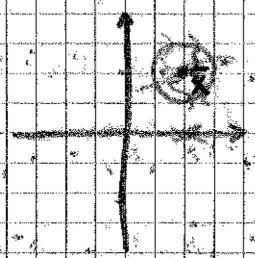
$\forall \varepsilon > 0 \; \exists \; \delta > 0 \quad x \in \text{DOM}F \;\; e \;\; \|x - \bar{x}\| < \delta$

$\Rightarrow \|F(x) - F(\bar{x})\| < \varepsilon$

$F: \mathbb{R}^n \to \mathbb{R}^m$ CONT. IN $\bar{x}$

$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots F_m(x_1, \dots, x_n))$

$\Leftrightarrow F_1, \dots, F_m$ SONO CONT. IN $\bar{x}$



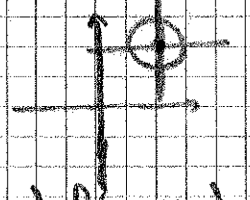
$f: \mathbb{R} \to \mathbb{R}$

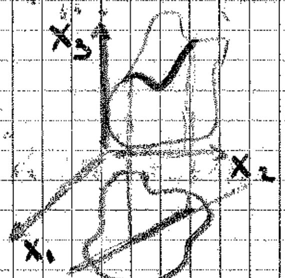
$\bar{x}$ INTERNO A DOM f

$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x})}{h} = f'(\bar{x}) \leftarrow$ DERIVATA

SE $\exists$ FINITO

$f: \mathbb{R}^2 \to \mathbb{R}$





$\lim_{h \to 0} \frac{f(x_1+h, x_2) - f(x_1, x_2)}{h}$

DERIVATA PARZIALE

$= \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2) = g'(\bar{x}_1)$

SE $\exists$ D. PARZ. NON NECESSARIAM. f DERIVABILE

$g(x_1) = f(x_1, \bar{x}_2)$

$\vec{v} = (v_1, v_2) \quad \|\vec{v}\| = 1$

$x_1 = \bar{x}_1 + t v_1 \qquad \lim_{t \to 0} \frac{f(\bar{x}_1 + t v_1, \bar{x}_2 + t v_2) - f(\bar{x}_1 - \bar{x}_2)}{t}$

$x_2 = \bar{x}_2 + t v_2$

$F: \mathbb{R}^n \to \mathbb{R}^m$

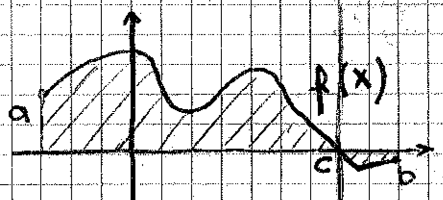
$(x_1, \dots, x_n) \to (F_1(x_1, \dots, x_n) + F_m(\dots))$

$\dfrac{\partial F_i}{\partial x_j}(\bar{x}) \quad i = 1, \dots, m \quad j = 1, \dots, n$

$$J_{\bar{x}} F = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\bar{x}) & \dots & \dfrac{\partial f_1}{\partial x_n}(\bar{x}) \\ \vdots & & \\ \dfrac{\partial f_m}{\partial x_1}(\bar{x}) & \dots & \dfrac{\partial f_m}{\partial x_n}(\bar{x}) \end{pmatrix} = \begin{pmatrix} \nabla f_1(\bar{x}) \\ \dots \\ \nabla f_m(\bar{x}) \end{pmatrix}$$

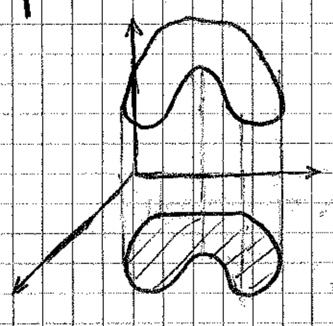
MATRICE JACOBIANA

INTEGRALI DOPPI



$\int_a^c f(x) \quad \left| \int_c^b f(x) \right|$

$f: \mathbb{R}^2 \to \mathbb{R}$



$A \subseteq \mathbb{R}^2$

A limitato

$\mathring{A} \neq \emptyset$

① AREA DI F.G. PIANE

② VOLUME DI SOLIDI TRA $0 \leq z \leq f(x,y) \quad (x,y) \in A$

a) QUALI SONO f BUONE

b) QUALI SONO DOM BUONI

SE ANCHE MODIFICO f SU ARCHI DI CURVA REGOLARI f NON CAMBIA

f LIMITATA SU R

$H^+ = \{ h$ A SCALA SU R, $f(x,y) \leq h(x,y) \}$ MAGGIORANTI

$H^- = \{ g$ A SCALA SU R, $f(x,y) \geq g(x,y) \}$ MINORANTI

$\exists \sup f(x,y) = M \in \mathbb{R} \quad (x,y) \in R \qquad h(x,y) = M \in H^+$

$\exists \inf f(x,y) = m \in \mathbb{R} \qquad\qquad g(x,y) = m \in H^-$

$\iint_R h \geq \iint_R g$

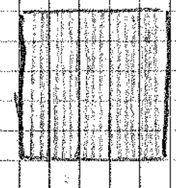
$\inf \{ \iint_R h,\ h \in H^+ \} = \overline{\int}_R f$ INTEGRALE SUP.

$\sup \{ \iint_R g,\ g \in H^- \} = \underline{\int}_R f$ INT. INF.

f INTEGRABILE SECONDO RIEMANN SE $\overline{\int}_R f = \underline{\int}_R f$

FUNZ. DIRICHLET

$f(x,y) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$



NON E' INTEGRABILE SECONDO RIEMANN

* TUTTE LE f CONTINUE SONO INTEGRABILI

* SE f CONT SU R (RETTANGOLO)
$\Rightarrow$ f E' RIEMANN-INTEGRABILE SU R

TEOREMA (FORMULE DI RIDUZIONE)

f INTEGR. SU $R = [a,b] \times [c,d]$

1) $\forall y \in [c,d]$, $\exists\, g(y) = \int_a^b f(x,y)\,dx$
ED E' INTEGR. SEC. RIEMANN

* $f(x,y) = x\,e^{xy} \quad (x,y) \in [0,1] \times [0,2]$

$\int_0^2 \left(\int_0^1 x e^{xy} dx \right) dy = \int_0^1 \left(\int_0^2 x e^{xy} dy \right) dx$

1) $\int_0^2 x e^{xy} dy = x \int_0^2 e^{xy} dy \qquad \tau = xy \quad d\tau = \frac{\partial xy}{\partial y} dy = x\,dy$

$y = 0 \to \tau = 0x = 0$
$y = 2 \to \tau = 2x$
$= \int_0^{2x} e^{\tau} d\tau = e^{\tau}\Big]_0^{2x} = e^{xy}\Big]_0^2 = e^{2x} - e^0 = e^{2x} - 1$

$\int_0^1 e^{2x} - 1\, dx = \frac{1}{2} e^{2x} - x \Big]_0^1 = \frac{e^2}{2} - 1 - \frac{1}{2} - 0 = \frac{1}{2} e^2 - 3/2$

2) $\int_0^1 x e^{xy} dx \qquad \tau = xy \quad d\tau = y\,dx$

$f' = e^{xy} \qquad f = \frac{1}{y} e^{xy}$
$g = x \qquad g' = 1$
$= \frac{x}{y} e^{xy} \Big]_0^1 - \frac{1}{y} \int_0^1 e^{xy}$

$= \frac{e^y}{y} - 0 - \left[\frac{1}{y} \cdot \frac{e^{xy}}{y} \right]_0^1 = \frac{e^y}{y} - \frac{e^y}{y^2} + \frac{1}{y^2}$

$\int_0^2 \frac{e^y}{y} - \frac{e^y}{y^2} + \frac{1}{y^2} dy = \ldots = \frac{1}{2} e^2 - \frac{3}{2} \quad (1 = 2)$

PROP: $R = [a,b] \times [c,d]$
$f(x,y) = g(x)\,h(y)$ CONTINUA $\Rightarrow \int_R f = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right)$

$\int_R g(x) h(y) = \int_c^d \left(\int_a^b g(x) h(x) dx \right) dy =$

$= \int_c^d h(y) \cdot \underbrace{\left(\int_a^b g(x) dx \right)}_{\kappa \text{ (NUMERO)}} dy = \int_c^d \kappa\, h(y) dy$

* $f(x,y) = x \cos y \quad (x,y) \in [0,\pi] \times [0, \pi/2]$

$\int_R x \cos y = \left(\int_0^{\pi} x\,dx \right) * \left(\int_0^{\pi/2} \cos y\, dy \right) = \frac{x^2}{2}\Big]_0^{\pi} \cdot \left(\sin y \Big]_0^{\pi/2} \right) = \frac{\pi^2}{2} \cdot 1$

TEOREMA

Ω E' MISURABILE $\iff |\partial\Omega| = 0$

UN SEGM HA MISURA NULLA

1) SE Ω HA MISURA NULLA $\Rightarrow$ TUTTI I SUOI SOTTOINSIEMI HANNO MISURA NULLA

2) # (NUM.) FINITO DI PUNTI

3) SEGMENTI

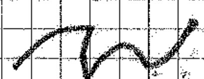
4) $\cup$ FINITO DI INS. DI MISURA NULLA

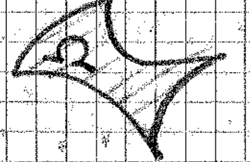
5) $\{(x, f(x)), x \in [a,b], y = f(x), f$ INT SU $[a,b]\}$

$\{(f(y), y), y \in [c,d], x = f(y), f$ INT. SU $[c,d]\}$

6) SOSTEGNI DI ARCHI DI CURVA REGOLARI A TRATTI

$\hookrightarrow |\partial\Omega| = 0 \to \Omega$ MISURABILE





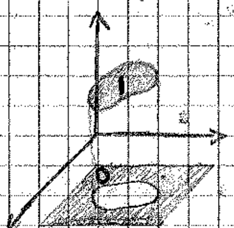
$\{a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}$

VERTICALE CONNESSO

TEOREMA

Ω MISURABILE, R RETTANGOLO T.C. $\Omega \in R$

$$\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$ FUNZ. CARATTERISTICA DI Ω



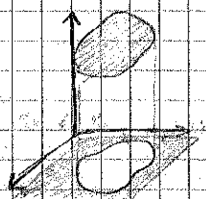
$\int_R \chi_\Omega =$ VOLUME CILINDRO $= |\Omega| \cdot 1$

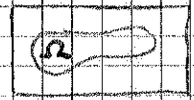
Ω MISURABILE $\iff \chi_\Omega$ E' R-INTEGRABILE SU R

INTEGRALE DOPPIO SU INSIEMI MISURABILI

$f: \Omega \to \mathbb{R}$ LIMITATA Ω MISURABILE $\Omega \subseteq R$ (RETTANGOLO)

$$\tilde{f}(x) = \begin{cases} f(x) & \text{SE } x \in \Omega \\ 0 & \text{SE } x \notin \Omega \end{cases}$$



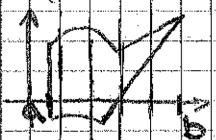


f E' INTEGR. SU Ω SE
$\tilde{f}$ E' INTEGR. SU R E

$$\int_\Omega f \overset{DEF}{=} \int_R \tilde{f} \qquad \forall R,\ \Omega \subseteq R,\ \int_R \tilde{f} \text{ E' UGUALE}$$

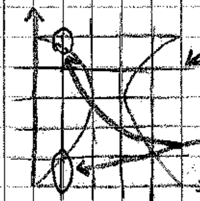
PROP Ω MISURABILE
SE f CONT SU $\Omega \setminus C$, DOVE C E' UN SOTTOINSIEME DI Ω
DI MISURA NULLA
$\Rightarrow f$ E' INTEGRABILE SU Ω

INSIEME VERTICALMENTE CONVESSO (RISPETTO A y) SE
$a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, g_1, g_2 CONT



$\forall$ RETTA VERTICALE $r \to r \cap \Omega = \emptyset \vee$ PUNTO $\vee$ SEGM.

INSIEME ORIZZONTALMENTE CONNESSO (RISPETTO A x) SE
$c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$



TEOREMA
SIA Ω UN INSIEME VERT. CONNESSO, SIA f CONT SU Ω
(TRANNE AL + SU C, $|C| = 0$) $\Rightarrow f$ E' INTEGR. SU Ω E

$$\int_\Omega f = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$$

PROPRIETÀ INTEGRALI DOPPI

f, g INTEGRABILI SU Ω MISURABILE

1) LINEARITÀ

$f+g$, $\forall \alpha \in \mathbb{R}$

$$\left.\begin{array}{l}\int_\Omega f+g = \int_\Omega f + \int_\Omega g \\ \int \alpha f = \alpha \int f\end{array}\right\} \rightarrow \int(\alpha f + g) = \alpha \int f + \int g$$

2) POSITIVITÀ

SE $f \geq 0$ SU $\Omega \setminus C$, $|C| = 0 \Rightarrow \int_\Omega f \geq 0$

$\int f$ = VOLUME DELLA PARTE DI SPAZIO T.C.
$0 \leq z \leq f(x,y)$ E $(x,y) \in \Omega$

$f \geq 0$ SU Ω, $|\Omega| \neq 0$ E f CONT SU Ω $\int f = 0 \Leftrightarrow f = 0$ SU Ω

3) CONFONTO

SE $f \leq g$ SU $\Omega \setminus C$, $|C| = 0 \Rightarrow \int_\Omega f \leq \int_\Omega g$

DIM: $g - f \geq 0 \rightarrow \int_\Omega g - \int_\Omega f = \int_\Omega g - f \geq 0$

4) SE f INTEGR. SU $\Omega \Rightarrow |f|$ È INTEGR. SU Ω E

$$\left|\int_\Omega f\right| \leq \int_\Omega |f| = \text{VOLUME} \begin{cases} 0 \leq z \leq f(x,y) & \text{SE } f(x,y) \geq 0 \\ f(x,y) \leq z \leq 0 & \text{SE } f(x,y) \leq 0 \end{cases}$$

5) TEOREMA DELLA MEDIA

Ω CONVESSO E MISURABILE

$m = \inf_{(x,y)\in\Omega} f(x,y)$ $\qquad M = \sup_{(x,y)\in\Omega} f(x,y)$

$m \leq \frac{1}{|\Omega|} \int_\Omega f \leq M$ → MEDIA INTEGRALE / VALORE MEDIO

$$\int_{x^2}^{\sqrt{x}} x+2y\,dy = xy+\frac{2y^2}{2}\Big]_{x^2}^{\sqrt{x}} = x\sqrt{x}+\frac{2x}{2}-x\cdot x^2-\frac{2x^4}{2}=$$
$$=x\sqrt{x}+x-x^3-x^4$$

$$\int_0^1 (x\sqrt{x}+x-x^3-x^4)\,dx=\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^5}{5}=\frac{2}{5}+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}=\int_{\Omega_1} f$$

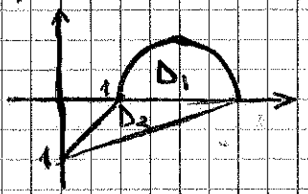
$$\int_{\Omega_2} f=\int_1^2 f=-\int_2^1 f=\int_1^2(-x\sqrt{x}-x+x^3+x^4)\,dx=$$
$$=-\frac{x^{5/2}}{5/2}-\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^5}{5}\Big]_1^2=-\frac{8}{5}\sqrt{2}+3+\frac{34}{5}-\frac{1}{2}$$

2) $D=D_1\cup D_2$ $\quad \iint_D f(x,y) \quad f(x,y)=xy$

$D_1=\{(x,y): 1\le x\le 3 \quad 0\le y\le\sqrt{1-(x-2)^2}$

$D_2=\{(x,y): -1\le y\le 0 \quad y+1\le x\le 3y+3$

$$y=\sqrt{1-(x-2)^2}\rightarrow y^2=1-(x-2)^2\rightarrow (x-2)^2+y^2=1 \begin{cases} R=1 \\ C=(2,0)\end{cases}$$



CONVIENE D_2 X ORIZZ.

$$\int_D f=\int_{D_1} f+\int_{D_2} f-\cancel{\int_{D_1\cap D_2} f} \qquad |D_1\cap D_2|=0$$

$$\int_1^3\left(\int_0^{\sqrt{1-(x-2)^2}} xy\,dy\right)dx=\int_1^3\left(\frac{xy^2}{2}\Big]_0^{\sqrt{1-(x-2)^2}}\right)dx=$$

$$=\int_1^3\frac{1}{2}\left(x\sqrt{1-(x-2)^2}^{\,2}\right)dx=\int_1^3-3x-x^3+4x^2=$$

$$=\frac{1}{2}\left(-\frac{27}{8}-\frac{81}{4}+4\cdot 9+\frac{3}{2}+\frac{1}{4}-\frac{5}{4}\right)$$

$$\int_{D_2} xy=\int_{-1}^0\left(\int_{y+1}^{3y+3} xy\,dx\right)dy=\int_{-1}^0 y\,\frac{x^2}{2}\Big]_{y+1}^{3y+3}=\int_{-1}^0 4(y^3+2y^2+y)$$

$$=4\left(\frac{y^4}{4}+2\frac{y^3}{3}+\frac{y^2}{2}\right)\Big]_{-1}^0=4\,\frac{-3+8-6}{12}$$

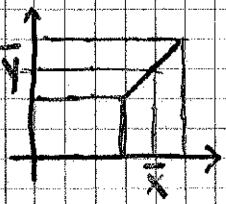
TEOREMA DELL'INVERTIBILITÀ LOCALE

$f: I \to J \subseteq \mathbb{R}$ $\quad I, J$ INTERVALLI

f DERIV. DI CLASSE C^1 SU I

SE $f(\bar{x}) \in I$, $f'(\bar{x}) \neq 0 \Rightarrow \exists B(\bar{x}), \exists B(\bar{y} = f(\bar{x})), \exists f^{-1}$

$B(\bar{y}) \to B(\bar{x})$



f^{-1} È DI CLASSE C^1

$(f^{-1})'(\bar{y}) = \dfrac{1}{f'(\bar{x})} \quad \bar{x} = f^{-1}(\bar{y})$

$F: \mathbb{R}^n \to \mathbb{R}^n$ DEF SU $\Omega \subseteq \mathbb{R}^n$

$F \in C^1(\Omega)$, $\bar{X} \in \Omega$ T.C. $J_x F$ INVERTIBILE ($\det J_x F \neq 0$)

$\Rightarrow \bar{y} = F(\bar{x}), \exists B(\bar{x}), \exists B(\bar{y}), \exists F^{-1}: B(\bar{y}) \to B(\bar{x})$

F^{-1} È DI CLASSE C^1 SU $B(\bar{y})$ E $J_x F^{-1} = (J_x F)^{-1}$

CAMBIAMENTI DI VARIABILE

Ω, Ω' T.C. $\begin{cases} \mathring{\Omega} \neq \emptyset, \ \Omega \text{ CONVESSO}, \ \Omega \text{ MISURABILE} \\ \mathring{\Omega}' \neq \emptyset, \ \Omega' \text{ CONVESSO}, \ \Omega' \text{ MISURABILE} \end{cases}$

$\Omega, \Omega' \in \mathbb{R}^n$

$\phi: \Omega' \to \Omega$ È UN CAMBIAM. DI VARIABILI SE:

1) ϕ È BIUNIVOCA (BIETTIVA)

2) ϕ È DI CLASSE C^1 SU Ω'

3) $\forall P \in \Omega'$, $J_P \phi$ È INVERTIBILE

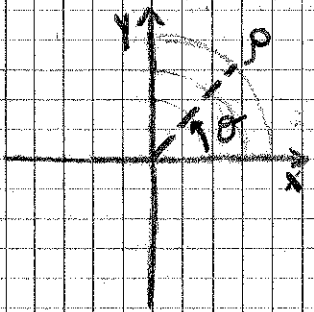
NE SEGUE CHE:

- $\phi(\mathring{\Omega}') \subseteq \Omega$ È UN INS. APERTO
- $\phi(\partial\Omega') \subseteq \partial\Omega$

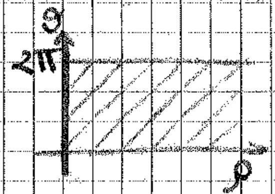
POICHE CONTINUA

DET $J_P \phi$ HA LO STESSO SEGNO $\forall P \in \mathring{\Omega}'$ ← TEOR. PERM. SEGNO

CAMBIAMENTO DI VARIABILI A COORD. POLARI



$$\begin{cases} 0 \le \vartheta \le 2\pi \\ \rho \ge 0 \end{cases}$$



SE $\rho = 0$ NON È BIUNIVOCA
↓
SEGM.
↓
$|\text{SEGM}| = 0$ → NON CI DISTURBA NELL'INTEGRAZ.

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases}$$

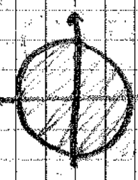
$$J\phi(\rho,\vartheta) = \begin{pmatrix} \cos\vartheta & -\rho\sin\vartheta \\ \sin\vartheta & \rho\cos\vartheta \end{pmatrix} = \begin{pmatrix} x\,d\rho & x\,d\vartheta \\ y\,d\rho & y\,d\vartheta \end{pmatrix}$$

$$\det J = |\rho\cos^2\vartheta + \rho\sin^2\vartheta| = |\rho(\cos^2\vartheta + \sin^2\vartheta)| = |\rho| = \rho \ge 0$$

$$\det J\phi(\rho,\vartheta) = 0 \iff \rho = 0$$

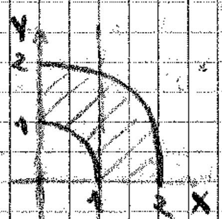
$$* \ |\Omega| = \int_\Omega 1\,dx\,dy = \int 1\cdot\rho\,d\rho\,d\vartheta = \left(\int_0^{2\pi} 1\,d\vartheta\right)\left(\int_0^r \rho\,d\rho\right)$$

$$\Omega = \{(x,y) : x^2+y^2 = r^2\} \qquad = 2\pi \cdot \rho^2 \cdot \frac{1}{2}\Big]_0^r = \pi r^2$$



ESEMPI

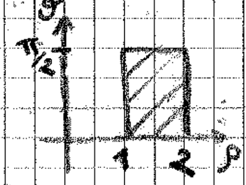
① $\displaystyle\int \frac{xy^2}{x^2+y^2}\,dx\,dy$



$$\Omega = \{(x,y) : 1 < x^2+y^2 < 4,\ x > 0,\ y > 0\}.$$

$$\begin{cases} 0 \le x \le 1 \to \sqrt{1-x^2} \le y \le \sqrt{4-x^2} \\ 1 \le x \le 2 \to 0 \le y \le \sqrt{4-x^2} \end{cases}$$

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases} \qquad \Omega' = \begin{cases} 1 \le \rho \le 2 \\ 0 \le \vartheta \le \pi/2 \end{cases}$$



$\det J\phi = \rho$

DET $J\phi$

$$\int_\Omega \frac{xy^2}{x^2+y^2}\,dx\,dy = \int_{\Omega'} \frac{\cancel{\rho}\cos\vartheta\,\rho^2\sin^2\vartheta}{\cancel{\rho^2}}\,\cancel{\rho}\,d\rho\,d\vartheta$$

NOTARE CHE IN $\begin{cases} x=0 \\ y=0 \end{cases}$ LA FUNZ. NON ERA IN DOM (A DENOM)

$\downarrow$ MA $\lambda = \dfrac{DEG(NUM)}{DEG(DEN)} = \dfrac{2}{1} > 1 \rightarrow$ SEMPRE PROLUNGABILE (IN O)

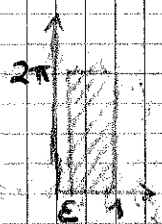
SE $\lambda > 1 \rightarrow$ SEMPRE PROLUNGABILE $\rightarrow$ IN COORD. POLARI SI SEMPLIFICA

SE $\lambda \leq 1 \rightarrow$ POTREBBE NON SEMPLIF. IN COORD. POLARI

4) $\displaystyle\int_\Omega \log\sqrt{x^2+y^2}\,dx\,dy$

$\Omega = \{(x,y) : \varepsilon^2 < x^2+y^2 < 1,\ \varepsilon \neq 0\}$

$\Omega' = \{(x,y) : \varepsilon^2 < \rho^2 < 1,\ \varepsilon \neq 0,\ 0 \leq \vartheta < 2\pi\}$



$\displaystyle\int_{\Omega'} \log\rho \cdot \rho \, d\rho \, d\vartheta = \left(\int_\varepsilon^1 \rho \log\rho \, d\rho\right)\left(\int_0^{2\pi} d\vartheta\right)$

$f = \log\rho \quad f' = 1/\rho$

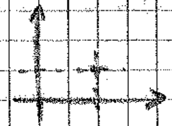
$g' = \rho \quad g = \rho^2/2$

$\displaystyle\left(\log\rho \frac{\rho^2}{2} - \int \frac{1}{\cancel{\rho}} \frac{\rho^{\cancel{2}}}{2} d\rho\right)\Bigg]_\varepsilon^1 \cdot (2\pi - 0) = \left(\cancel{\frac{1}{2}\log 1} - \frac{1}{4} + \frac{\varepsilon^2}{4} - \frac{\varepsilon^2}{2}\log\varepsilon\right) 2\pi$

$\displaystyle = 2\pi\left(-\frac{\varepsilon^2}{2}\log\varepsilon - \frac{1}{4} + \frac{\varepsilon^2}{4}\right)$

COORD. POLARI TRASLATE

$\begin{cases} x = x_0 + \rho\cos\vartheta \\ y = y_0 + \rho\sin\vartheta \end{cases}$



$J = \begin{pmatrix} \cos\vartheta & -\rho\sin\vartheta \\ \sin\vartheta & \rho\cos\vartheta \end{pmatrix} \rightarrow |J| = \rho$

COORD. ELLITTICHE (POLARI SCHIACCIATE)

$\begin{cases} x = \rho a \cos\vartheta \\ y = \rho b \sin\vartheta \end{cases}$

$a > 0 \quad b > 0$

$\rho \geq 0$

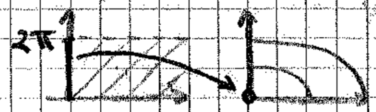
$0 \leq \vartheta < 2\pi$

$\begin{cases} x = a\rho\cos\vartheta \\ y = b\rho\sin\vartheta \end{cases}$ $\quad 1 \ge \rho \ge 0$

$$\frac{\rho^2 a^2\cos^2\vartheta}{a^2} + \frac{\rho^2 b^2\sin^2\vartheta}{b^2} = \rho^2(\cos^2\vartheta + \sin^2\vartheta) = \rho^2 \le 1$$

$\rho = 0 \to \forall\vartheta \to (0,0)$

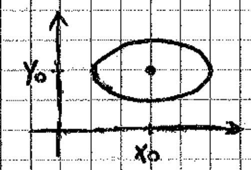
MA MISURA NULLA



$$\Psi = \begin{cases} x = a\rho\cos\vartheta & 0 \le \vartheta \le 2\pi \\ y = b\rho\sin\vartheta & \rho \ge 0 \end{cases}$$

DET $J\Psi = a\cdot b\cdot\rho \ge 0$

COORD. ELLITTICHE TRASLATE



$$\begin{cases} x = x_0 + a\rho\cos\vartheta & \rho > 0 \\ y = y_0 + b\rho\sin\vartheta & \vartheta \in [0, 2\pi) \end{cases}$$

AREA ELLISSE E DI SEMIASSI a,b

$$\iint_E 1\,dx\,dy = \int_0^{2\pi}\int_0^1 1\cdot ab\rho\,d\rho\,d\vartheta = ab\int_0^1 \rho\,d\rho\int_0^{2\pi} d\vartheta = ab\left.\frac{\rho^2}{2}\right]_0^1 2\pi = ab\pi$$

ESERCIZI

1) $\iint y^2\,dx\,dy \qquad \{(x,y)\in\mathbb{R}^2 \ 4(x-3)^2 + 9y^2 \le 36,\ y \ge 0\}$

$\frac{4(x-3)^2}{36} + \frac{9y^2}{36} \le 1 \qquad \frac{(x-3)^2}{3^2} + \frac{y^2}{2^2} \le 1 \qquad \begin{cases} x = 3 + 3\rho\cos\vartheta \\ y = 2\rho\sin\vartheta \end{cases}$

$$\int_0^{\pi}\int_0^1 2^2\rho^2\sin^2\vartheta\, 3\cdot 2\rho\,d\rho\,d\vartheta =$$

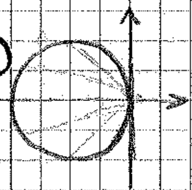
$$\int_0^1 24\rho^3\,d\rho\cdot\int_0^{\pi}\sin^2\vartheta = 24\left.\frac{\rho^4}{4}\right]_0^1\cdot\left(\left[\frac{-\sin\vartheta\cos\vartheta + \vartheta}{2}\right]_0^{\pi}\right) =$$

$$= \frac{16}{3}R^3\int_0^{\pi/2}(1-\sin^2\vartheta)\cos\vartheta\, d\vartheta = \frac{16}{3}R^3\int(1-t^2)\,dt =$$

$t = \sin\vartheta \qquad dt = \cos\vartheta\, d\vartheta$

$$= \frac{16}{3}R^3\left(\sin\vartheta - \frac{\sin^3\vartheta}{3}\right)\Bigg]_0^{\pi/2} = \frac{16}{3}R^3\left(1-\frac{1}{3}\right) = \frac{32R^3}{9}$$

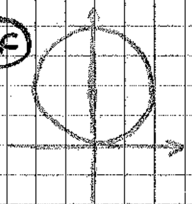
2B



$(x+R)^2 + y^2 \le R^2$

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases} \qquad \begin{cases} 0 \le \rho \le \text{[illegible]} \\ \frac{\pi}{2} \le \vartheta \le \frac{3\pi}{2} \end{cases}$$

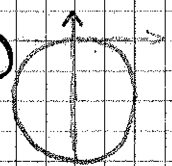
2C



$(x)^2 + (y-R^2) \le R^2$

$$\begin{cases} 0 \le \rho \le R\cdot 2\cdot\sin\vartheta \\ 0 \le \vartheta \le \pi \end{cases}$$

2D

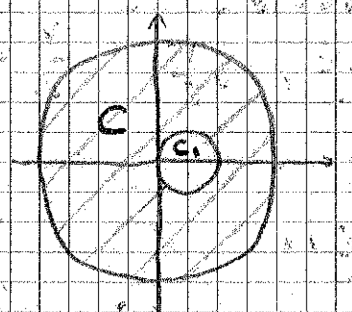


$x^2 + (y+R)^2 \le R^2$

$$\begin{cases} 0 \le \rho \le 2R\sin\vartheta \\ \pi \le \vartheta \le 2\pi \end{cases}$$

3) $\iint_\Omega (x^2+y^2)^{3/2}\,dx\,dy$

$$\Omega = \left\{(x,y) : \begin{cases} x^2+y^2 \le 4 \\ x^2+y^2-x \ge 0 \end{cases}\right.$$

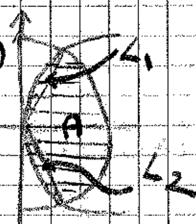


$x^2 - x + \frac{1}{4} + y^2 \ge \frac{1}{4} \longrightarrow \left(x-\frac{1}{2}\right)^2 + y^2 \ge \left(\frac{1}{2}\right)^2$

$C = C_1 \cup \Omega \qquad |C_1 \cap \Omega| = 0$

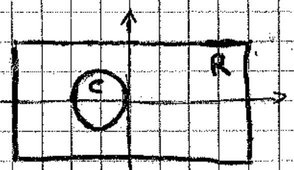
$$\iint_C R = \iint_{C_1} R + \iint_\Omega R \longrightarrow \iint_\Omega R = \iint_C R - \iint_{C_1} R$$

$x^2+y^2 \le 4 \qquad x^2+y^2-x \le 0$

5) 

$A = \begin{cases} 0 \le \rho \le 1 \\ -\frac{\pi}{3} \le \vartheta \le \frac{\pi}{3} \end{cases}$

$L_2 = \begin{cases} 0 \le \rho \le 2\cos\vartheta \\ -\frac{\pi}{2} \le \vartheta \le -\frac{\pi}{3} \end{cases}$ $\quad L_1 = \begin{cases} 0 \le \rho \le 2\cos\vartheta \\ \frac{\pi}{3} \le \vartheta \le \frac{\pi}{2} \end{cases}$

6) 

$R = \begin{cases} -2 < x < 2 \\ -1 < y < 1 \end{cases}$ $\quad C: \left(x+\frac{1}{2}\right)^2 + y^2 \le \frac{1}{4}$

BARICENTRO DI $R \setminus C \rightarrow \rho(x,y) = 1$

$\frac{1}{M}\iint x \, dx dy = x_B$

$M = \iint_{R \setminus C} dx dy = 4 \cdot 2 - \frac{\pi}{4} = 8 - \frac{\pi}{4} = \frac{32-\pi}{4}$

$\frac{1}{M}\iint y \, dx dy = y_B$

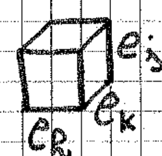
$x_B = \frac{1}{M}\left[\iint_R x \, dx dy - \iint_C x \, dx dy\right] = 0 - 0 = 0$

$y_B = \frac{1}{M}\left[\iint_R y \, dx dy - \iint_C y \, dx dy\right] = \frac{1}{M}\left[\underbrace{\int_{-2}^{2} dx \int_{-1}^{1} y \, dy}_{0} - \iint_C y \, dx dy\right]$

$\iint_C y \, dx dy = \int_{\frac{3\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\cos\vartheta} \rho^2 \sin\vartheta \, d\rho \, d\vartheta$

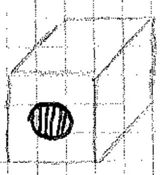
$= \frac{1}{M}\left[0 - \frac{1}{2}\right] = -\frac{2\pi}{32-\pi}$

INTEGRALI TRIPLI

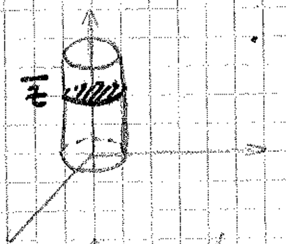
R_{hkj} 

$f(x,y,z) = c_{hkj}$ SU R_{hkj} $\quad f$ LIMITATA

$\iiint f = \sum c_{hkj} e_h e_k e_j \quad R = \bigcup_{h,k,j} R_{hkj}$

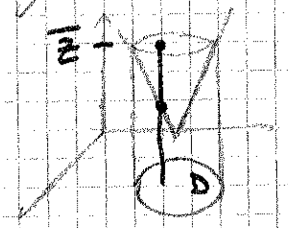


$x \in Q$, $(x,y) \in \mathbb{R}$



$\pi = \bar{z}$ $\quad C \cap \pi = C_{\bar{z}}$

$a \le z < b \quad (x,y) \in C_{\bar{z}}$



$(x,y) \in D$ MISURABILE

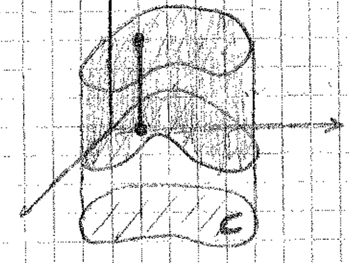
$\alpha(x,y) \le z \le \beta(x,y)$

INTEGRAZIONE PER FILI

$C \subseteq \mathbb{R}^2$ MISURABILE, $|C| \neq 0$

$\alpha, \beta : C \to \mathbb{R}$ CONTINUE E LIMITATE

$\Omega = \{(x,y,z) \in \mathbb{R}^3 \quad (x,y) \in C, \ \alpha(x,y) \le z \le \beta(x,y)$



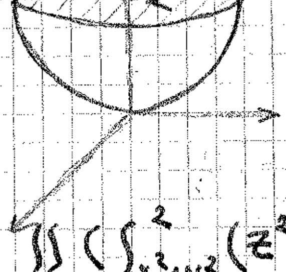
Σ SEGMENTI

f(CONTINUA) $: \mathbb{R}^3 \to \mathbb{R}$

$$\iiint_{\Omega} f = \iint_{C} \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz \right) dx\,dy$$

ESEMPIO

$$\begin{cases} z \ge x^2+y^2 \\ 0 \le z \le 2 \end{cases} = \Omega \qquad f(x,y,z) = z^2 x$$



$C\{x^2+y^2 \le 2\} \subseteq \mathbb{R}^2$

$x^2+y^2 \le z \le 2$

$\Omega = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in C \quad x^2+y^2 \le z \le 2\}$

$$\iint_C \left(\int_{x^2+y^2}^{2} (z^2 x)\,dz \right) dx\,dy$$

$$\begin{cases} x = \rho\cos\theta & 0 \le \theta \le 2\pi \\ y = \rho\sin\theta & 0 \le \rho \le \sqrt{2} \end{cases}$$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \rho\cos\theta\,\rho\,d\rho\,d\theta = \left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_0^{\sqrt{2}}\rho^2\,d\rho\right) =$$

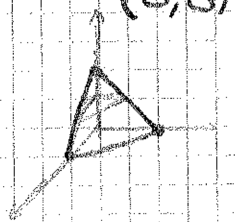
$$\sin\theta\Big]_0^{2\pi} \cdot \rho^3\Big]_0^{\sqrt{2}} \cdot 3^{-1} = \frac{1}{3}\cdot 2\sqrt{2}\cdot 0 = 0$$

$$\int_0^2 z^2 \cdot 0\,dz = \int_0^2 0\,dz = 0$$

<u>ESERCIZI</u>

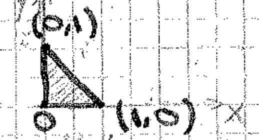
1) $\iiint x\,dx\,dy\,dz$

Ω = TETRAEDRO DI VERTICI $(1,0,0), (0,1,0), (0,0,1), (0,0,0)$



1) PER STRATI $0 \le z \le 1$ Ω_z TRIAGOLI

2) PER FILI $\rightarrow$ C = TRIANGOLO



$(x,y \in C)$

$0 \le z \le$ PIANO

$$\begin{vmatrix} x-1 & y & z \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 0 \qquad x-1+y+z=0 \;\;\hookrightarrow z = x+y-1$$

$$\iint\left(\int_0^{x+y-1} x\,dz\right)dx\,dy = \iint_C x\,(x+y-1)\,dx\,dz$$

$$C = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le -x+1 \end{cases}$$

$$\int_0^1\int_0^{-x+1} x^2+xy-1\,dy\,dx = \int_0^1 x^3y + x\frac{y^2}{2} - y\Big]_{y=0}^{-x+1}$$

$$= \int_0^1 x^3(-x+1) + x\frac{(-x+1)^2}{2} + x - 1\,dx$$

$$= \int_0^1 -x^4 + x^3 + \frac{1}{2}(x^3) - x^2 + \frac{x}{2} + x - 1\,dx = \int_0^1 -x^4 + \frac{3}{2}x^3 - x^2 + \frac{3}{2}x - 1\,dx =$$

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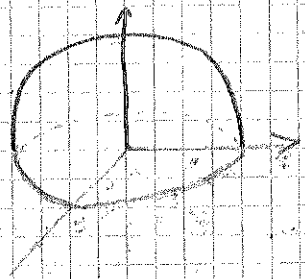
3) $\{(x,y,z): x^2+y^2+z^2 \leq R^2,\ z \geq 0\} \rightarrow$ EMISFERA POSITIVA

$f(x,y,z) = x^2 z$



a) PER STRATI

$$\begin{cases} 0 \leq z \leq R \\ x^2+y^2 \leq R^2-z^2 \rightarrow \rho^2 = R^2 - z^2 \rightarrow 0 \leq \rho \leq \sqrt{R^2-z^2} \end{cases}$$

$$\int_0^R \iint_E x^2 z\, dx\,dy\,dz = \int_0^R z\left(\int_0^{2\pi}\left(\int_0^{\sqrt{R^2-z^2}} \rho^2 \rho\, d\rho\right)\cos^2\theta\, d\theta\right) dz$$

$$= \frac{\cancel{2}\pi}{2}\int_0^R z\,\frac{(R^2-z^2)^2}{\cancel{4}\,2} = \frac{\pi}{2}\int_0^R \frac{zR^4}{2} - \frac{z^3R^2}{1} + \frac{z^5}{2}$$

$$= \frac{\pi}{2}\left(\frac{z^2R^4}{4} - \frac{z^4R^2}{4} + \frac{z^6}{12}\right)\Bigg]_0^R = \frac{\pi}{2}\left(\frac{1}{4} - \frac{1}{4} + \frac{1}{12}\right)R^6 = \frac{\pi}{24}R^6$$

b) PER FILI $\rightarrow C: x^2+y^2 \leq R^2$

$$\iint_C x^2\left(\int_0^{\sqrt{R^2-(x^2+y^2)}} z\, dz\right) dx\,dy = \iint \frac{R^2-(x^2+y^2)}{2}\, x^2\, dx\, dy$$

$$= \iint \frac{R^2-\rho^2}{2}\,\rho^2\cos^2\vartheta\,\rho\, d\rho\, d\vartheta = \int_0^{2\pi}\cos^2\vartheta\, d\vartheta \int_0^R \rho^3R^2 - \rho^5\, d\rho \cdot \frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{\vartheta + \cos\vartheta\sin\vartheta}{2}\right)\Bigg]_0^{2\pi} \cdot \frac{1}{2}\left(\frac{\rho^4R^2}{4} - \frac{\rho^6}{6}\right)\Bigg]_0^R = \frac{\pi}{2}\left(\frac{1}{4} - \frac{1}{6}\right)R^6$$

$$= \frac{\pi}{2}\cdot\frac{1}{12}R^6 = \frac{\pi}{24}R^6$$

PROPRIETÀ DEGLI INTEGRALI TRIPLI

1) LINEARITÀ

$$\iiint af + g = \iiint g + a\iiint f$$

2) POSITIVITÀ $\rightarrow$ SE $f(x,y,z) \geq 0$ SU $\Omega \Rightarrow \iiint f \geq 0$

VOLUME DI UN CILINDRO DI BASE C E ALTEZZA 3

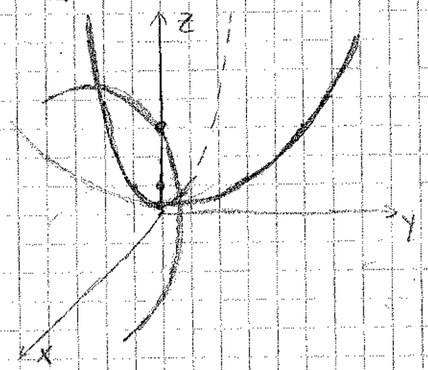
V = AREA BASE $\cdot h = |C| \cdot 3$

$$\iint C = \iint_{C_1} + \iint_{C_2} = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} d\theta \int_0^4 \rho\, d\rho + \frac{\cancel{2}}{\cancel{2}}\left| r\cdot \sin\frac{3\pi}{2} \cdot r \cos\frac{3\pi}{2}\right|$$

$$\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right)\frac{\rho^2}{2}\Big]_0^4 + 16\,\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{4\pi}{3}\cdot\frac{\cancel{16}^{8}}{\cancel{2}} + 4\sqrt{3} = \frac{32\pi}{3} + 4\sqrt{3}$$

$$\iiint dx\,dy\,dz = 3\left(\frac{32\pi}{3} + 4\sqrt{3}\right) = 32\pi + 12\sqrt{3}$$

3) $\displaystyle\int_\Omega \frac{1}{3-z} \qquad \Omega \begin{cases} 9z \le 1 + y^2 + 9x^2 \\ 0 \le z \le \sqrt{9-(y^2+9x^2)} \end{cases}$



CAMBIAMENTI DI VARIABILI

$\phi : \Omega' \subseteq \mathbb{R}^3 \to \Omega \subseteq \mathbb{R}^3$

1) ϕ BIUNIVOCA SU Ω'

2) ϕ DI CLASSE C^1

3) DET $J\phi \neq 0$

Ω E' MISURABILE $\Rightarrow \phi^{-1}(\Omega) = \Omega'$ E' MISURABILE

$\displaystyle\int_\Omega f(x,y,z)\, dx\,dy\,dz \qquad \underset{(u,v,w)}{\Omega'} \longrightarrow \underset{(x,y,z)}{\Omega}$

$\displaystyle\int_\Omega f(\phi(u,v,w)\,|\text{DET } J\phi(u,v,w)|)\, du\, dv\, dw$

COORD. CILINDRICHE ELLITTICHE

$$\begin{cases} x = a\rho\cos\vartheta \\ y = b\rho\sin\vartheta \\ z = \tau \end{cases} \qquad \det J\phi = ab\rho \geq 0$$

ESERCIZI

$$\Omega = \begin{cases} x^2 + 9y^2 \leq 1 \\ 3 \leq z \leq 5 \end{cases} \qquad \int_\Omega x z^2 \, dx\,dy\,dz$$

$$\begin{cases} x = \rho\cos\vartheta \\ y = \frac{1}{3}\rho\sin\vartheta \end{cases}$$

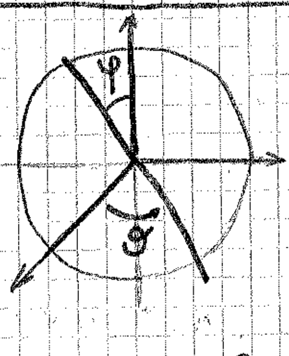
$$\int \tau^2 \rho\cos\vartheta \, \frac{1}{3}\rho \, d\vartheta\, d\rho\, d\tau$$

$$\begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \vartheta \leq 2\pi \\ 3 \leq \tau \leq 5 \end{cases}$$

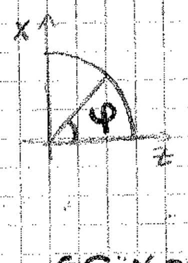
$$\frac{1}{3}\left(\int_3^5 \tau^2 d\tau\right)\left(\int_0^{2\pi} \cos\vartheta \, d\vartheta\right)\left(\int_0^1 \rho^2 d\rho\right)$$

$$\frac{1}{3} \cdot \frac{\tau^3}{3}\Big]_3^5 \; \underbrace{\sin\vartheta\Big]_0^{2\pi}}_{=0} \; \frac{\rho^3}{3}\Big]_0^1 = 0$$

COORD. SFERICHE



$0 \leq \varphi \leq \pi$



SE $y = 0 \rightarrow \vartheta = 0$ $\qquad$ SE $x = 0 \rightarrow \vartheta = \pi/2$

$$\begin{cases} x = \rho\sin\varphi \\ z = \rho\cos\varphi \end{cases} \qquad \begin{cases} y = \rho\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

$$\Omega = \begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \\ z = z \end{cases} \qquad \begin{cases} (r,z) \in S \\ 0 \le \vartheta \le 2\pi \end{cases} \qquad \det J = r \ge 0$$

VOLUME

$$\iiint_{\Omega} dx\,dy\,dz = \iiint_{\Omega'} r\,dr\,d\vartheta\,dz = \int_0^{2\pi} \Big(\iint_{(r,z)\in \Omega'} r\,dr\,dz \Big) d\vartheta$$

$$= 2\pi \iint_{S} x\,dx\,dz = 2\pi \iint_{S'} y\,dy\,dz$$

S' ∈ PIANO (y,z) RUOTATA DI $\pi/2$

$$\frac{1}{|S|} \iint_{S} x\,dx\,dz = x_{GS}$$

$|S|$ → M → SE $\rho = 1$ (DENSITÀ) → $M = |S| \cdot 1$ VOLUME

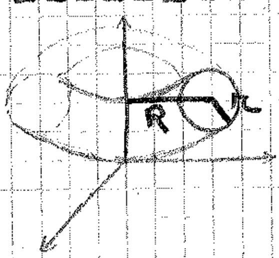
$$|S|\,x_{GS} = \iint x\,dx\,dz$$

$$\mathrm{vol}(\Omega) = |\Omega| = 2\pi |S|\, x_{GS} = (2\pi x_{GS})\,|S|$$

1° TEOREMA DI GULDINO

IL VOLUME DEL SOLIDO DI ROTAZ. Ω OTTENUTO RUOTANDO UNA SUP. PIANA INTORNO ALL'ASSE z È UGUALE ALL'AREA DI S MOLTIPLICATA PER LA LUNGH. DELLA CIRC. DESCRITTA DAL BARICENTRO DI S NELLA ROTAZ. ATTORNO ALL'ASSE z

VOLUME TORO



$$y_{GS} = R$$

$$|S| = \pi r^2$$

$$V = 2\pi R \pi r^2 = 2\pi^2 R r^2$$

$$\text{VOL }\Omega_z = 2\pi \iint_D x\,dx\,dz$$
$$\parallel$$
$$\text{VOL }\Omega_x = 2\pi \iint_D z\,dz\,dx$$

$$R: \iint_D x\,dx\,dz = \iint_D z\,dz\,dx$$

VOLUMI SONO UGUALI SE $X_{GS} = Z_{GS}$ $\quad X_{GS}$ SU: $\begin{cases} y=0 \\ x=z \end{cases}$

$$(2\pi X_G)|D| = 2\pi z_G |D| \Rightarrow X_G = Z_G$$

$$\begin{cases} z = 2x^2 \\ z = R \\ x \geq 0 \end{cases} \qquad 2x^2 = z = R \quad x = \sqrt{\frac{R}{2}} \qquad \begin{cases} 0 \leq x \leq \sqrt{\frac{R}{2}} \\ 2x^2 \leq z \leq R \end{cases}$$

$$\int_0^{\sqrt{R/2}} x \int_R^{2x^2} dz\,dx = \int_0^{\sqrt{R/2}} \int_R^{2x^2} z\,dz\,dx$$

$$\Omega_x: \int_0^{\sqrt{R/2}} \int_{2x^2}^{R} z\,dz\,dx = \frac{1}{2}\int_0^{\sqrt{R/2}} R^2 - 4x^4\,dx$$
$$= \frac{1}{2}\left[R^2\sqrt{\frac{R}{2}} - \frac{\cancel{4}}{5}\frac{R^2}{\cancel{4}}\frac{\sqrt{R}}{2}\right] = \frac{4}{10}R^2\sqrt{\frac{R}{2}}$$

$$\Omega_z: \int x(R-2x^2)\,dx = R\frac{x^2}{2} - \frac{2x^4}{4}\Big]_0^{\sqrt{R/2}} = \frac{R^2}{4} - \frac{R^2}{8} = \frac{R^2}{8}$$

$$\frac{\cancel{R^2}}{8} = \frac{2}{5}\cancel{R^2}\sqrt{\frac{R}{2}} \rightarrow \frac{1}{8} = \frac{2}{5}\sqrt{\frac{R}{2}}$$

$$\sqrt{\frac{R}{2}} = \frac{5}{16} \rightarrow \frac{R}{2} = \frac{25}{4^4} \rightarrow R = \frac{25}{2^7} \rightarrow R = \frac{25}{128}$$

<u>BARICENTRO</u> $\rightarrow$ DENSITÀ $\rho(x,y,z)$

$$M = \text{MASSA}(\Omega) = \iiint \rho\,dx\,dy\,dz$$

$$X_G = \frac{1}{M}\iiint x \cdot \rho(x,y,z)\,dx\,dy\,dz \qquad Y_G = \ldots \qquad Z_G = \ldots$$

DEF: γ È UNA CURVA REGOLARE SE
$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$

- γ È DI CLASSE C^1 SU I
- IL VETTORE TANGENTE $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t)) \neq \vec{0} \ \forall t$

$\downarrow$

NON POSSO CAMBIARE VERSO

CURVE E INTEGRALI

$\gamma : [a, b] \to \mathbb{R}^n$

$t \to (\gamma_1(t), \dots, \gamma_n(t))$ SOSTEGNO DI $\gamma = \{\gamma(t), t \in [a,b]\} \subseteq \mathbb{R}^n$

1) γ INIETTIVA (SEMPLICE) $t_1 \neq t_2 \Rightarrow \gamma(t_1) \neq \gamma(t_2)$

2) γ REGOLARE
- $\gamma_1, \dots, \gamma_n$ DI CLASSE $C^1 \Rightarrow \exists\ \gamma_i'(t)$ CONT. $\forall i, \forall t \in [a,b]$
- $\vec{\gamma}'(t) \neq \vec{0} \ \forall t$

CURVA ORIENTATA OGNI CURVA REGOLARE
$\downarrow$
CURVA RETTIFICABILE $\Rightarrow$ MISURABILE

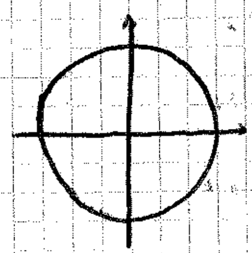
$\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$

$\|\gamma'(t)\| = \sqrt{(\gamma_1')^2(t) + \dots + (\gamma_n')^2(t)}$

$L = \int_a^b \|\gamma'(t)\| dt$

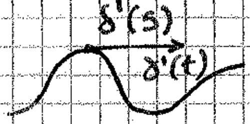
ESEMPIO

CIRCONF. DI RAGGIO R



$\begin{cases} \gamma_1(t) = r\cos t \\ \gamma_2(t) = r\sin t \end{cases} \quad 0 \leq t \leq 2\pi$

$\gamma'(t) = (-r\sin t; r\cos t)$



$\delta(s) = \gamma(\alpha(s)) = \gamma(t) \qquad t = \alpha(s)$

SE $\alpha'(s) > 0 \longrightarrow$ STESSA DIREZIONE ($\neq$ LUNGH. $\to \neq$ VELOC)
STESSO ORIENTAMENTO

SE $\alpha'(s) < 0 \longrightarrow$ STESSA DIREZIONE
ORIENTAMENTO OPPOSTO

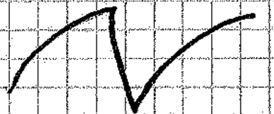
$$\begin{cases} \alpha(c) = b \\ \alpha(d) = a \end{cases} \qquad \begin{matrix} \delta(c) = \gamma(\alpha(c)) = \gamma(b) \\ \delta(d) = \gamma(\alpha(d)) = \gamma(a) \end{matrix}$$

SE $\alpha'(s) > 0 \to$ EQUIVALENTI

$$\vec{\delta'}(s) = \alpha'(s)\, \vec{\gamma'}(\alpha(s))$$

$$* \begin{cases} \gamma_1 = \pi \cos s \\ \gamma_2 = \pi \sin s \end{cases} \qquad \begin{cases} \delta_1 = \pi \cos t^3 \\ \delta_2 = \pi \sin t^3 \end{cases} \qquad \begin{matrix} s = t^3 \\ s' = 3t^2 \end{matrix}$$

<u>CURVA REGOLARE A TRATTI</u>



SE NUM. FINITO DI CURVE REGOLARI

$$\gamma^1(b_1) = \gamma^2(a_1)$$

$$\begin{matrix} \gamma^1 : [a_1, b_1] \to \mathbb{R}^n \\ \gamma^2 : [a_2, b_2] \to \mathbb{R}^n \end{matrix} \qquad \gamma = \gamma^1 + \gamma^2$$

INTEGRALE CURVILINEO $\to$ INTEGRALE DI LINEA DI
1 SPECIE

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$

A APERTO, CONVESSO

$\gamma : [a, b] \to A \subseteq \mathbb{R}^n$ CURVA REGOLARE

$$\int_\gamma f\, ds$$

APPROX CON SEGM.
$f(\gamma(a_i))$

SE $\alpha'(s)<0 \to \int_d^c f(\delta(s))\left(-\|\delta'(s)\|\right)ds$

$$= \int_c^d f(\delta(s))\,\|\delta'(s)\|\,ds$$

$$\|\delta'(s)\| = |\alpha'(s)| \cdot \|\gamma'(\alpha(s))\| = -\alpha'(s)\,\|\gamma'(\alpha(s))\|$$

ESERCIZI

1) LUNGH. ELICA CILINDRICA $\longrightarrow f(\gamma(t))=1$

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = a t \end{cases} \qquad 0 \le t \le 4\pi$$

$$\gamma'(t) = (-\sin t, \cos t, a)$$

$$\int_0^{4\pi} \sqrt{(\sin^2 t + \cos^2 t + a)}\,dt = \int_0^{4\pi} \sqrt{1+a}\,dt = \sqrt{1+a}\,(4\pi)$$

2) $\begin{cases} x = \sqrt{2}\sin t\cos t \\ y = 4\sin t \end{cases} \qquad 0 \le t \le \frac{\pi}{2}$

$$\gamma'(t) = \left(\sqrt{2}\cos^2 t - \sqrt{2}\sin^2 t,\ 4\cos t\right) =$$
$$= \left(\sqrt{2}(2\cos^2 t - 1),\ 4\cos t\right)$$

$$\|\gamma'(t)\|^2 = 2\left((\cos^2 t)2 - 1\right)^2 + 16\cos^2 t$$
$$= 8\cos^4 t + 2 - 8\cos^2 t + 16\cos^2 t$$
$$= 8\cos^4 t + 8\cos^2 t + 2 = 2(2\cos^2 t + 1)^2$$

$$\int_\gamma ds = \int_0^{\pi/2} \sqrt{2}\,(2\cos^2 t + 1)\,dt$$
$$= \int_0^{\pi/2} \sqrt{2}\,dt + \sqrt{2}\cdot\cancel{2}\left[\frac{\cancel{8}4\cos t\sin t}{\cancel{2}}\right]_0^{\pi/2} = \sqrt{2}\cdot\frac{\pi}{2} + \sqrt{2}\,\frac{\pi}{2} - 0$$
$$= \pi\sqrt{2}$$

$* \ \gamma_3 = \begin{cases} x = 1 \\ y = t \end{cases} \quad 0 \le t \le 1 \qquad \gamma_3'(t) = (0,1) \qquad \|\gamma_3'(t)\| = \sqrt{1} = 1$

$\int_0^1 1 \, dt = 1$

$* \ \gamma_1 = \begin{cases} x = 1/2 + 1/2 \cos t \\ y = 1/2 \sin t \end{cases} \quad 0 \le t \le \pi$

$\gamma_1'(t) = \left(-\frac{1}{2}\sin t ; \frac{1}{2}\cos t\right) \quad \|\gamma_1'(t)\| = \sqrt{\frac{1}{4}(\cos^2 t + \sin^2 t)} = \frac{1}{2}$

$\int_{\gamma_1} x \, ds = \int_0^\pi \left(\frac{1}{2} + \frac{1}{2}\cos t\right)\frac{1}{2} dt = \int_0^\pi \frac{1}{4}(1 + \cos t) = \frac{\pi}{4}$

$\int_\gamma x \, ds = \frac{5\sqrt{5} - 1}{12} + \frac{\pi}{4} + 1$

$x_G = \frac{1}{L(\gamma)} \int_\gamma x \, ds \ne x_G(\gamma_1) + x_G(\gamma_2) + x_G(\gamma_3)$

$L(\gamma_1) = \frac{\pi}{2} \qquad L(\gamma_3) = 1$

$L(\gamma_2) = \int_0^1 \sqrt{1 + 4t^2} \, dt = \int \sqrt{1 + (2t)^2} \, dt$

$\cosh^2 s - \sinh^2 s = 1 \Rightarrow \cosh^2 s = 1 + \sinh^2 s \qquad 2t = \sinh s$

$\int \sqrt{1 + \sinh^2 s} \, \frac{1}{2} \cosh s \, ds = \qquad 2dt = \cosh s \, ds$

$\frac{1}{2} \int \cosh^2 s \, ds$

$\int \cosh^2 s \, ds = \int \cosh s \cdot \cosh s \, ds = \cosh s \sinh s - \int \sinh^2 s$

$f = \cosh s \qquad g' = \cosh s$

$f' = \sinh s \qquad g = \sinh s$

$\int \cosh^2 s = \cosh s \sinh s - \int \cosh^2 s \, ds + \int 1 \, ds$

$\int \cosh^2 s = \frac{\cosh s \sinh s + s}{2}$

$$\|\gamma'(t)\| = \sqrt{t^2 + \sin^2 t + \pi^2 + \pi^2\cos^2 t - 2t\sin t + 2\pi t - 2t\pi\cos t + 2\pi\sin t\cos t - 2\pi^2\cos t - 2\pi\sin t + \sin^2 t\,\pi^2 + 1 + \cos^2 t + 2\pi\sin t - 2\sin t\cos t - 2\cos t} =$$

$$\sqrt{2}\,\pi\sqrt{1-\cos t}$$

$$\cos^2\frac{t}{2} - \sin^2\frac{t}{2} = \cos t$$

$$1 - 2\sin^2\frac{t}{2} = \cos t \longrightarrow 2\sin^2\frac{t}{2} = 1-\cos t$$

$$\|\gamma'(t)\| = \sqrt{2}\,\pi\sqrt{2\sin^2\frac{t}{2}} = 2\pi\left|\sin\frac{t}{2}\right|$$

$$x_G = \frac{1}{L(\gamma)}\int x\,\|\gamma'(t)\|\,dt =$$

$$\int_0^{2\pi} \pi(t-\sin t)\,2\pi\left|\sin\frac{t}{2}\right|dt$$

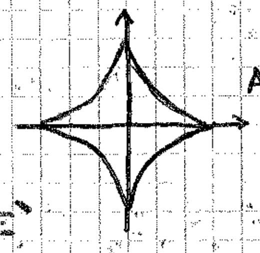
SI NOTI CHE LA γ E' SIMM. RISPETTO ALL'ASSE VERTICALE PER CUI $t=\pi$

$\downarrow$

$$x_G = \pi(\pi - \sin\pi) = \pi\pi$$

<u>ESERCIZIO</u>

$$\begin{cases} x = (\cos\vartheta)^3 \\ y = (\sin\vartheta)^3 \end{cases} \quad 0 \le \vartheta \le 2\pi$$



ASTROIDE

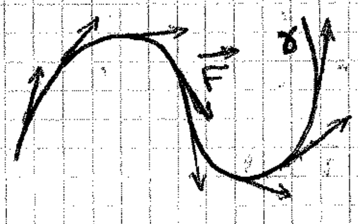
$\rho = 1 \ \forall (x,y)$

$\left.\begin{array}{l} x_G = 0 \\ y_G = 0 \end{array}\right\}$ POICHE' LA CURVA E' SIMMETRICA $\forall$ ASSE PASSANTE PER $(0,0)$

INTEGRALI DI LINEA (DI 2' SPECIE)

$F: \mathbb{R}^n \to \mathbb{R}^n$

$\quad x \to (F_1(x), \dots, F_n(x))$



SE γ E δ DIFFERISCONO X CAMBIAM. PARAMETRIZZAZ. E INVERTONO ORIENTAMENTO

$$\int_\gamma F \cdot dP = -\int_\delta F dP \qquad \int_{-\gamma} F dP = -\int_\gamma F dP$$

ESERCIZIO

$$F(x,y,z) = \left(\frac{x}{x^2+y^2}; \frac{y}{x^2+y^2}; -1\right) \qquad \gamma = \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$

$0 \le t \le 3\pi$

$$\gamma'(t) = (-\sin t, \cos t, 1)$$

$$F(\gamma(t)) = (\cos t; \sin t; -1)$$

$$\int_0^{3\pi} F(\gamma(t))\, \gamma'(t)\, dt$$

$$F(\gamma(t))\gamma'(t) = (\cos t, \sin t, -1)(-\sin t, \cos t, 1)$$
$$= -\cos t \sin t + \sin t \cos t - 1 = -1$$

$$\int_0^{3\pi} -1\, dt = -3\pi$$

PROPRIETA'

$F: \mathbb{R}^n \to \mathbb{R}^n$ CONTINUA

$\gamma: [a,b] \to \mathbb{R}^n$ REGOLARE $\Rightarrow \int_\gamma F dP = \int_a^b F(\gamma(t))\, \gamma'(t)\, dt$

· INT. LINEARE

* ADDITIVITA' RISPETTO AL DOM: $\int_{\gamma = \gamma_1 + \gamma_2} F dP = \int_{\gamma_1} F dP + \int_{\gamma_2} F dP$
* γ E δ EQUIV $\Rightarrow \int_\gamma F dP = \int_\delta F dP$

ESERCIZI

1) $F: \mathbb{R}^n \to \mathbb{R}^n$

$F = (-y, x) \qquad \int_\gamma F dP$: CIRCONF. DI CENTRO O E RAGGIO 2 IN SENSO ORARIO

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$$\begin{cases} x' = -\sin\vartheta \\ y' = \cos\vartheta \\ z' = -4\sin\vartheta\cos\vartheta \end{cases} \qquad \vec{\gamma}'(\vartheta) \neq \vec{0} \quad \forall\vartheta$$

$$\int F\,dP = \int_0^{2\pi} 2(\cos\vartheta+1)^2\sin\vartheta, -(\cos\vartheta+1)2\sin^2\vartheta, -(\cos\vartheta+1)) \cdot (-\sin\vartheta, \cos\vartheta, -4\sin\vartheta\cos\vartheta)\,d\vartheta$$

$$= \int_0^{2\pi} -2(\cos\vartheta+1)^2\sin^2\vartheta - \cos\vartheta(\cos\vartheta+1)2\sin^2\vartheta + 4\sin\vartheta\cos\vartheta(\cos\vartheta+1)\,d\vartheta$$

$$\int_0^{2\pi} 2(\cos\vartheta+1)\left[-\cos^2\vartheta\sin^2\vartheta - 2\cos\vartheta\sin^2\vartheta - \sin^2\vartheta + -\cos\vartheta\sin^2\vartheta + 2\sin\vartheta\cos\vartheta\right]d\vartheta$$

3) $F(z, x, y)$

$$\begin{cases} 2x^2 + y^2 - 6x = 0 \\ x + z = 3 \end{cases}$$

$$2\left(x^2 - 2\frac{3}{2}x + \frac{9}{4} - \frac{9}{4}\right) + y^2 = 0 \qquad \frac{\left(x-\frac{3}{2}\right)^2}{\frac{1}{2}\cdot\frac{9}{2}} + \frac{y^2}{\frac{9}{2}} = 1$$

$$2\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{2}$$

$$\begin{cases} x = \frac{3}{2}\cos\vartheta + \frac{3}{2} \\ y = \frac{3}{\sqrt{2}}\sin\vartheta \\ z = 3 - \frac{3}{2}\cos\vartheta - \frac{3}{2} \end{cases} \qquad \begin{cases} x' = -\frac{3}{2}\sin\vartheta \\ y' = \frac{3}{\sqrt{2}}\cos\vartheta \\ z' = \frac{3}{2}\sin\vartheta \end{cases}$$

$\sigma : A \subseteq \mathbb{R}^2 \to \mathbb{R}^3$

σ REGOLARE SE $\sigma \in C^1(A)$ A APERTO

$\partial\sigma(u,v)$ HA RANGO MAX $\forall (u,v) \in A$

σ E' SEMPLICE, SE INIETTIVA

$\vec{N}(u,v) = \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v}$

PIANO TG. $= (x-\sigma_1(u,v); y-\sigma_2(u,v); z-\sigma_3(u,v)) \cdot$
$\left(\frac{\partial\sigma}{\partial u}(\bar{u},\bar{v}) \wedge \frac{\partial\sigma}{\partial v}(\bar{u},\bar{v})\right) = 0$

PRODOTTO MISTO

ESEMPI

1) $x^2+y^2=1$ $\sigma : (\vartheta,\tau) \to \mathbb{R}^3$

$$\begin{cases} x = \cos\vartheta \\ y = \sin\vartheta \\ z = \tau \end{cases} \qquad \begin{cases} \tau \in \mathbb{R} \\ \vartheta \in (0, 2\pi) \end{cases}$$

IN MODO CHE SIA APERTO

SE C^1 SU UN APERTO + GRANDE

POSSO AGGIUNGERE BORDO $\to \vartheta \in [0, 2\pi)$

$A \subseteq \mathbb{R}^2$ $[0, 2\pi) \times \mathbb{R}$ $\sigma \in C^1(\mathbb{R}^2)$

$$\partial\sigma(\vartheta,\tau) = \begin{vmatrix} -\sin\vartheta & 0 \\ \cos\vartheta & 0 \\ 0 & 1 \end{vmatrix}$$

σ SEMPLICE

$\begin{matrix} \| & \| \\ \frac{\partial\sigma}{\partial\vartheta} & \frac{\partial\sigma}{\partial\tau} \end{matrix}$

LIN. INDIP $\to$ RANGO MAX $\forall(\vartheta,\tau)$

σ REGOLARE

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* $x = g(y, z)$

$$\vec{N}(x,y,z) = \left(1; -\frac{\partial g}{\partial y}; -\frac{\partial g}{\partial z}\right) = (1; -\partial_y g; -\partial_z g)$$

CALOTTA SUPERFICIALE REGOLARE

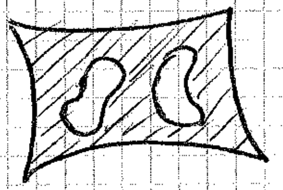
$\sigma : A \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ A APERTO σ REGOLARE

$K : \mathring{K} \neq \emptyset$ $K \subseteq A$

K COMPATTO (CHIUSO/LIMITATO)

$\partial K'$

∂K È UN INSIEME DI UN NUM. FINITO DI ARCHI DI CURVA REGOLARE CHIUSA E SEMPLICE (A TRATTI)

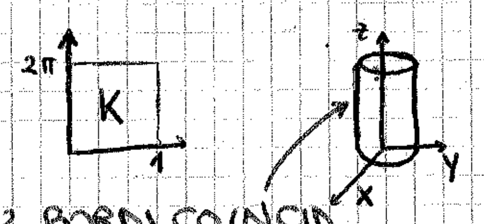


$\sigma(K)$ CHIUSO, LIMITATO

ESEMPIO

$K = [0,1] \times [0, 2\pi] \to \mathbb{R}^3$

$$\begin{cases} x = \cos\vartheta \\ y = \sin\vartheta \\ z = \tau \end{cases}$$



CALOTTA MA 2 BORDI COINCID.

OSS

$$\begin{cases} x = \cos(2\varphi) \\ y = \sin(2\varphi) \\ z = s^3 \end{cases} \qquad \begin{cases} x = \cos\vartheta \\ y = \sin\vartheta \\ z = \tau \end{cases}$$

$0 \le \varphi \le \pi$ $\qquad$ $0 \le \vartheta \le 2\pi$

$0 \le s \le 1$ $\qquad$ $0 \le \tau \le 1$

STESSO SOSTEGNO
↓
PUÒ CAMBIARE VERSO, LUNGH. DI $\vec{N}$, MA NO DIREZ. (STESSO π_{TG})

$$\text{AREA}(\sigma(K)) = \iint_K \left\| \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\| du\,dv = \int_\sigma 1\, d\sigma$$

$f: \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ CONTINUA SU Ω $\quad \sigma(K) \subseteq \Omega$

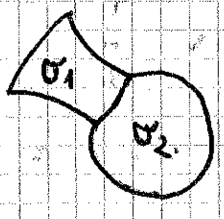
$$\int f\, d\Omega = \iint_K f(\sigma(x,y)) \| N(u,v) \| du\,dv$$

1) $\int f\, d\sigma$ E' LINEARE

$$\forall a \in \mathbb{R}, \int_\sigma a f\, d\sigma = a \int_\sigma f\, d\sigma$$

$$f, g: \int f + g\, d\sigma = \int f\, d\sigma + \int g\, d\sigma$$

2) $\sigma = \sigma_1 + \sigma_2 \Rightarrow \int_\sigma f = \int_{\sigma_1} f + \int_{\sigma_2} f$



$\sigma(K)$ LAMINA, ρ DENSITA'

$$M(\sigma(K)) = \int_\sigma \rho\, d\sigma$$

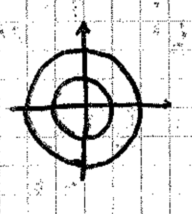
$$X_G = \frac{1}{M(\sigma(K))} \int_\sigma x \rho\, d\sigma$$

<u>TEOREMA</u>

L'INTEGRALE SUP. E' INDIP. DALLA PARAM. DELLA CALOTTA SUP.

<u>ESEMPIO</u>

$S: \begin{cases} z = x^2 + y^2 \\ 1 \leq z \leq 2 \end{cases} \quad \begin{cases} x = \rho \cos \vartheta \\ y = \rho \sin \vartheta \\ z = \rho \end{cases} \quad \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \vartheta \leq 2\pi \end{cases}$

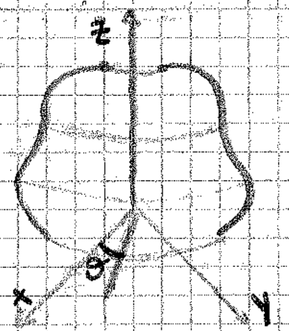


$\text{AREA}(\sigma(K)) = \int_\sigma 1 \, d\sigma$

$\text{MASSA} = M(\sigma(K)) = \int_\sigma \rho \qquad \rho$ DENSITÀ

$x_G = \frac{1}{M} \int_\sigma x \rho$

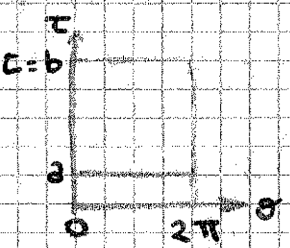
SUPERFICI DI ROTAZIONE



$\gamma : [a,b] \to \mathbb{R}^3$

$\tau \to (\gamma_1(\tau), 0, \gamma_3(\tau)) \quad (\text{SE } \vartheta = 0)$

$\begin{cases} a \le \tau \le b \\ 0 \le \vartheta \le 2\pi \\ (\vartheta, \tau) \in K \end{cases}$



γ REGOLARE

$$J\sigma(\vartheta,\tau) = \begin{vmatrix} -\gamma_1(\tau)\sin\vartheta & \gamma_1'(\tau)\cos\vartheta \\ \gamma_1(\tau)\cos\vartheta & \gamma_1'(\tau)\sin\vartheta \\ 0 & \gamma_3'(\tau) \end{vmatrix}$$

$$-\gamma_1(\tau)\gamma_1'(\tau)\sin^2\vartheta - \gamma_1(\tau)\gamma_1'(\tau)\cos^2\vartheta$$
$$= -\gamma_1(\tau)\gamma_1'(\tau) \neq 0$$

$\gamma_1(\tau)\gamma_3'(\tau)\cos\vartheta$

TRANNE CHE NEI PUNTI TANG. A ASSE Z SE UNO SI ANNULLA L'ALTRO È $\neq \vec{0}$

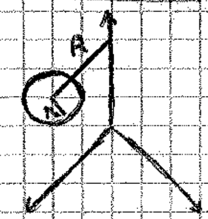
$\frac{\partial\sigma}{\partial\vartheta} \wedge \frac{\partial\sigma}{\partial\tau}$

$$\begin{vmatrix} i & j & \\ -\gamma_1(\tau)\sin\vartheta & \gamma_1(\tau)\cos\vartheta & 0 \\ \gamma_1'(\tau)\cos\vartheta & \gamma_1'(\tau)\sin\vartheta & \gamma_3'(\tau) \end{vmatrix} =$$

$A(\sigma) = 2\pi x_G L(\gamma)$ 2° TEOREMA DI GULDINO

$A(\sigma) = 2\pi \int_\gamma x \, ds$

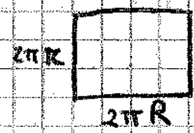
ESERCIZI

1) 

$x_G = R$

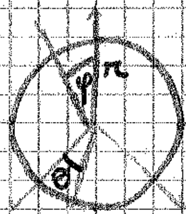
SUPERFICIE
TORO RAGGIO ROTAZ. R,
RAGGIO SEZ. r

$A(T) = L(\gamma)\, 2\pi R = 2\pi r\, 2\pi R = 4\pi^2 r R$



2) SFERA

$\begin{cases} 0 \le \varphi \le \pi \\ 0 \le \vartheta \le 2\pi \end{cases}$



$\begin{cases} x = r \sin\varphi \cos\vartheta \\ y = r \sin\varphi \sin\vartheta \\ z = r \cos\varphi \end{cases}$

CON QUESTA PARAM. I POLI NON SONO REGOLARI

$\gamma(\varphi) = \begin{cases} x = r \sin\varphi \\ y = 0 \\ z = r \cos\varphi \end{cases}$

$\gamma'(\varphi) = (r\cos\varphi, -r\sin\varphi)$ $\|\gamma'(t)\| = r$

A SFERA $= 2\pi \int_\gamma x =$

$= 2\pi \int_0^{2\pi} r \sin\vartheta \cdot r \, d\vartheta = 2\pi r^2 \int_0^\pi \sin\varphi$

$= 2\pi r^2 \left[-\cos\varphi\right]_0^\pi = 4\pi r^2$ AREA SUP. SFERICA

ESERCIZI

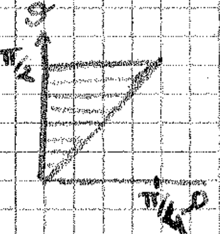
1) AREA DELLA PARTE DI SUP

$z = \sqrt{x^2+y^2}$

CHE STA AL DI SOTTO DEL PIANO $z = \frac{1}{\sqrt{2}}(y+2)$

$\begin{cases} x = x \\ y = y \\ z = \sqrt{x^2+y^2} \end{cases}$

$$\begin{cases} 0 \le \rho \le \frac{1}{2}\vartheta \\ 0 \le \vartheta \le \pi/2 \end{cases}$$



$$\begin{vmatrix} i & j & k \\ \cos\vartheta & \sin\vartheta & \cancel{\tfrac{1}{2}} Sh(2\rho)\cancel{2} \\ -\rho\sin\vartheta & \rho\cos\vartheta & 0 \end{vmatrix}$$

$$-\rho\cos\vartheta\, Sh(2\rho)\,\vec{i} - \rho\sin\vartheta\, Sh(2\rho)\,\vec{j} + \rho\vec{k} = N(\rho,\vartheta)$$

$$\|N(\rho,\vartheta)\| = \sqrt{\rho^2 Sh^2(2\rho)(\cos^2\vartheta + \sin^2\vartheta) + \rho^2} =$$

$$= \sqrt{\rho^2 Sh^2(2\rho) + \rho^2} = \rho\sqrt{Sh^2(2\rho)+1}$$

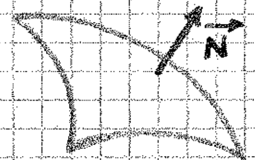
$$= \rho\sqrt{Ch^2(2\rho)} = \rho\, Ch(2\rho)$$ ← NON METTO MODULO XKE' $Ch > 0\ \forall\rho$

$$A(\rho,\vartheta) = \int_0^{\pi/2}\int_0^{\frac{1}{2}\vartheta} \rho\, Ch(2\rho)\, d\rho\, d\vartheta =$$

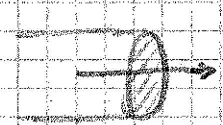
$$= \frac{1}{4}\left(\frac{\pi}{2} Ch\frac{\pi}{2} - 2\, Sh\frac{\pi}{2}\right)$$

INTEGRALE DI FLUSSO

$\sigma : K \subseteq A \subseteq \mathbb{R}^2 \to \mathbb{R}^3$



$\int_\sigma \vec{v}\cdot\vec{n}$



PORTATA

$F : \mathbb{R}^3 \to \mathbb{R}^3$

DEF. ALMENO SU $\sigma(K)$ E CONTINUO

$$\Phi = \int_\sigma \vec{F}\cdot\vec{n} = \iint_K (\vec{F}\cdot\vec{n})\, \|\vec{N}(u,v)\|\, du\, dv$$

$$\vec{n} = \frac{\vec{N}}{\|\vec{N}\|}$$

$$\Phi = \iint \vec{F}\,\frac{\vec{N}}{\|\vec{N}\|}\,\|\vec{N}\|\, du\, dv = \iint_\sigma \vec{F}\cdot\vec{n} = \iint_K F(\sigma(u,v))\cdot\vec{N}(u,v)$$

$\vec{F}\cdot\vec{n} = -(\vec{F}\cdot(-\vec{n}))$ AL CAMBIARE ORIENTAM. DI σ, Φ CAMBIA SEGNO

$$\begin{vmatrix} i & j & k \\ -R\sin\varphi\sin\vartheta & R\sin\varphi\cos\vartheta & 0 \\ R\cos\varphi\cos\vartheta & R\cos\varphi\sin\vartheta & -R\sin\varphi \end{vmatrix} =$$

$$= (-R^2\sin^2\varphi\cos\vartheta)\vec{i} + (-R^2\sin^2\varphi\sin\vartheta)\vec{j} +$$
$$+ (\underbrace{R^2\cos\varphi\sin\varphi\sin^2\vartheta - R^2\cos^2\vartheta\sin\varphi\cos\varphi}_{\parallel})\vec{k}$$
$$(-R^2\cos\varphi\sin\varphi)\vec{k}$$

PROVO IN UN PUNTO:

$\varphi = \frac{\pi}{2};\ \vartheta = 0 \longrightarrow \vec{N}\left(0, \frac{\pi}{2}\right) = (-R^2, 0, 0)$ ENTRANTE

CAMBIO ORIENTAM.

$$-\vec{N} = (R^2\sin^2\varphi\cos\vartheta, R^2\sin^2\varphi\sin\vartheta, R^2\sin\varphi\cos\varphi)$$

$$\phi = \int_\sigma \vec{F}(-\vec{N})\,d\sigma$$

$$F(\sigma(\vartheta,\varphi)) = (R\sin\varphi\cos\vartheta, 0, R\sin\varphi\sin\vartheta)$$

$$\phi = \iint_K R^3\sin^3\varphi\cos^2\vartheta + 0 + R^3\sin^2\varphi\sin\varphi\cos\varphi =$$
$$= R^3\left(\int_0^\pi \sin^3\varphi\,d\varphi\right)\left(\int_0^{2\pi}\cos^2\vartheta\,d\vartheta\right) +$$
$$+ R^3\left(\int_0^\pi \sin^2\varphi\cos\varphi\,d\varphi\right)\left(\int_0^{2\pi}\sin\vartheta\,d\vartheta\right)$$

$$\phi = \int F(\sigma(\varphi,\vartheta))\cdot(-N(\varphi,\vartheta))\,d\varphi\,d\vartheta$$

IL 2° ADDENDO $= 0$

$$\int_0^\pi \sin^2\varphi\cos\varphi\,d\varphi = \int_0^\pi \sin^2\varphi\,d\sin\varphi = \frac{\sin^3\varphi}{3}\Big]_0^\pi = 0$$

$$\int_0^{2\pi}\cos^2\vartheta\,d\vartheta = \frac{\vartheta + \sin\vartheta\cos\vartheta}{2}\Big]_0^{2\pi} = \pi$$

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$$\Phi = \iint_K x - 6zy + 2y^3 z \, dy\, dz$$

$$\iint x \, dy\, dz + \iint_K -6zy + 2y^3 z \, dy\, dz$$

$$\|$$

$$\left(\pi \cdot 1 \cdot \sqrt{3} - \pi \frac{1}{\sqrt{3}}\right) x \qquad z = \sqrt{3}\rho$$

$$\int_0^{2\pi}\int_{1/\sqrt{3}}^{1} \left(-6\sqrt{3}\rho^2 \cos\theta \sin\theta + 2\sqrt{3}\rho^4 \cos^3\theta \sin\theta\right) \rho\sqrt{3}\, d\rho\, d\theta$$

$$\iint \left(-18\rho^3 \cos\theta \sin\theta + 6\rho^5 \cos^3\theta \sin\theta\right) d\rho\, d\theta$$

$$-18\left(\int \rho^3 d\rho\right)\left(\int_0^{2\pi} \cos\theta \sin\theta\, d\theta\right) + 6\left(\int \rho^5 d\rho\right)\left(\int_0^{2\pi} \cos^3\theta\, d\cos\theta\right)$$

$$6\frac{\rho^6}{6}\Big]_{1/\sqrt{3}}^{1} \left(-\cos^4\theta\right)\Big|_0^{2\pi} = 0 \quad (*)$$

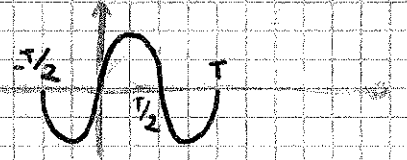
$$-18\int \rho^3 d\rho \int_0^{2\pi} \cos\theta\, d\cos\theta = 0 \quad (*)$$

$$\Phi = x\left(\pi\sqrt{3} - \pi\frac{1}{\sqrt{3}}\right) = 0 \iff x = 0$$

OSSERVAZIONE

f PERIODICA DI PERIODO T

$$\int_0^T f(x)\, dx = \int_{-T/2}^{T/2} f(x)\, dx$$



$$f \text{ DISPARI} \Rightarrow \int_0^T f = \int_{-T/2}^{T/2} f = 0$$

$$f \text{ PARI} \Rightarrow \int_0^T f = \int_{-T/2}^{T/2} f = 2\int_0^{T/2} f$$

$$(*) \rightarrow \int_0^{2\pi} \cos^3\theta \sin\theta\, d\theta = \int_{-\pi}^{\pi} \cos^3\theta \sin\theta\, d\theta = 0$$

$$(f \text{ PARI})^3 \cdot (f \text{ DISPARI}) = f \text{ DISPARI}$$

$$f(\bar{x}+\bar{v}) - f(\bar{x}) = \nabla f(\bar{x}) \cdot \vec{v} + \underset{\vec{v}\to 0}{o}(\|\vec{v}\|) = d_x f(\vec{v}) + o(\|\vec{v}\|)$$

* $f(x) = x = id$

$\phi_{x_0}(id)(h) = 1 \cdot x \qquad (d_{x_0} x)(h) = h \to dx$

$d(id) = dx \qquad d_{x_0} f = f'(x_0) h = f'(x_0) dx$

$df : x_0 \to$ APPLICAZ. LINEARE

$x_0 \to [f'(x_0)](dx)$

$df = f'(x) dx$

$$\int \sin x \underbrace{\cos x \, dx}_{d \sin x}$$

* CONVENZIONE DI LEIBNIZ

$$\frac{df}{dx}(x_0) = f'(x_0)$$

$$\Updownarrow$$

$$df(x_0) = f'(x_0)\, dx$$

$$d_{\bar{x}} f(\vec{v}) = \nabla f(x_0) \cdot \vec{v} = \frac{\partial f}{\partial x_1}(x_0)\, \vec{v_1} + \ldots + \frac{\partial f}{\partial x_n}(x_0)\, \vec{v_n}$$

$$= \frac{\partial f}{\partial x_1}(x_0) \cdot (v_1, 0, \ldots, 0) + \ldots + \frac{\partial f}{\partial x_n}(x_0) \cdot (0, \ldots, 0, v_n)$$

$$= \frac{\partial f}{\partial x_1}(x_0)\, dx_1 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

$$d_{x_0} f = \frac{\partial f}{\partial x_1}(x_0)\, dx_1 + \ldots + \frac{\partial f}{\partial x_n}(x_0)\, dx_n$$

$\vec{x} \mapsto (d_x f : \vec{v} \mapsto$ NUMERO$)$

$= \int_\gamma f(\gamma(t))\, \gamma'(t)\, dt = \int_\gamma f\, dx$

$f\, dx + 0\, dy$

* $F = (f_1 ; f_2)$

$\int_\gamma F\, dP = \int_\gamma (f_1\, dx + f_2\, dy) = \int_\gamma \omega$

* $\int_\gamma f_2\, dy = \int_\gamma (0\, dx + f_2\, dy)$

$= \int_a^b 0 \cdot \gamma_1'(t)\, dt + f_2(\gamma(t))\, \gamma_2'(t)\, dt$

TEOREMA DI GREEN

PREMESSE

* $A \subseteq \mathbb{R}^2$

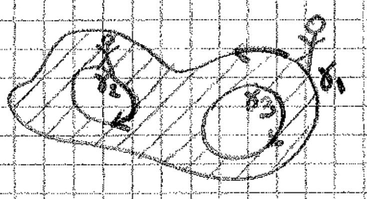
A APERTO $\quad A \neq \emptyset$

∂A INSIEME DI CURVE REG. SEMPLICI CHIUSE

$A \cup \partial A = \bar{A}$

APERTO CON BORDO DI $\mathbb{R}^2$

* ∂A è ORIENTATA POSITIVAMENTE SE UN OSSERV. CHE CAMMINA SU ∂A VEDE A A SX



TEOREMA DI GREEN

$A \subseteq \mathbb{R}^2$ APERTO CON BORDO

∂A ORIENTATA POSITIVAMENTE

$F : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $F \in C^1(\Omega)$

ESERCIZI

1) $F(x,y) = (xy^2, x^2y)$ su γ $A(0,1)$ $B(1,1)$ $C(1,2)$ $D(0,2)$

$$\int_\gamma F\,dP = \sum_{i=1}^{4} \int_{\gamma_i} F\,dP = \iint_Q \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2xy - 2xy = 0$$

$$\iint_Q \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint 0\,dx\,dy = 0$$

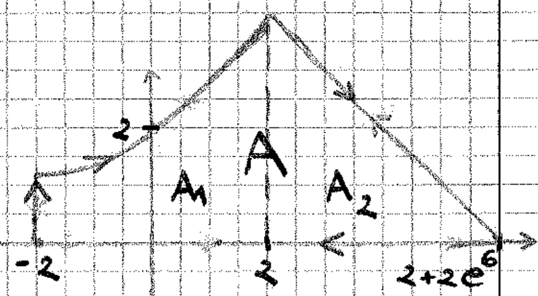
2) $\vec{F} = (-3y; 2x) \in C^\infty(\mathbb{R}^2)$

$\gamma_1: y = 2e^{3x}$ $\quad -2 \le x \le 2$

$\gamma_2: x = -2$ $\quad 0 \le y \le 2e^{-6}$

$\gamma_3: y = 0$ $\quad -2 \le x \le 2e^6 + 2$

$\gamma_4: x + y = 2 + 2e^6$ $\quad 2 \le x \le 2 + 2e^6$



CALCOLARE $\int_\gamma F \cdot dP$ $\quad \gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

X APPLICARE GREEN DEVE ESSERE POSITIVA (CON A A SX)

$$-\int F\,dP = -\iint_A \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dx\,dy =$$

$$= -\iint_A 2 + 3\,dx\,dy = -5 \iint_A dx\,dy =$$

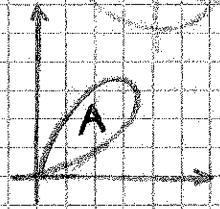
$$= -5\left[\underbrace{\int_{-2}^{2} 2e^{3x}\,dx}_{\text{AREA } A_1 \text{ (INT. SEMPLICE)}} + \underbrace{\frac{2e^6(2e^6+2-2)}{2}}_{\text{AREA TRIANG.}}\right] = -5\left[2\frac{e^{3x}}{3}\Big|_{-2}^{2} + 2e^{12}\right] =$$

$$= -10\left[\frac{e^6}{3} - \frac{e^{-6}}{3} + e^{12}\right]$$

$$\mu(A) = \iint_A 1\,dx\,dy = \int_{\partial A} F \cdot dP = \begin{cases} \int_{\partial A} x\,dy \\ \int_{\partial A} -y\,dx \\ \int_{\partial A} \frac{1}{2}(-y\,dx + x\,dy) \end{cases}$$

ESERCIZIO

1) $\gamma(t) = (t^3 - 3t^2 + 2t,\ t - t^3) \qquad 0 \leq t \leq 1$



$\gamma(0) = (0,0)$
$\gamma(1) = (0,0)$ CURVA SEMPLICE CHIUSA

$\gamma'(t) = (3t^2 - 6t,\ 1 - 3t^2)$

$\gamma'(0) = (2;1) \qquad \gamma'(1) = (-1;-2)$

SCELGO UNO DEI 3 TERMINI

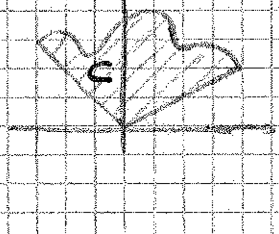
$$\text{AREA}(A) = \int_{\partial A} x\,dy = \int_\gamma x\,dy = \int_0^1 \gamma_1(t) \cdot \gamma_2'(t)\,dt =$$

$$= \int_0^1 (t^3 - 3t^2 + 2t)(1 - 3t^2)\,dt = \int_0^1 t^3 - 3t^2 + 2t - 3t^5 + 9t^4 - 6t^3\,dt =$$

$$= \frac{t^4}{4} - t^3 + t - \frac{t^6}{2} + 9\frac{t^5}{5} - 6\frac{t^4}{4}\Bigg|_0^1 = \frac{1}{4} - \cancel{1} + \cancel{1} - \frac{1}{2} + \frac{9}{5} - \frac{6}{4} = \frac{5 - 10 + 36 - 30}{20} = \frac{1}{20}$$

AREE DI REGIONI: $\rho = \rho(\vartheta)$

$$\begin{cases} \vartheta_1 \leq \vartheta \leq \vartheta_2 \\ 0 \leq \rho \leq \rho(\vartheta) \end{cases}$$



$$\begin{cases} \gamma_1 = \rho\cos\vartheta \\ \gamma_2 = \rho\sin\vartheta \end{cases} \qquad \begin{cases} \gamma_1' = -\rho\sin\vartheta \\ \gamma_2' = \rho\cos\vartheta \end{cases}$$

$$\mu(C) = \iint_C dx\,dy$$

$$\frac{1}{2}\int_{\partial C} -y\,dx + x\,dy = \frac{1}{2}\int_{\vartheta_1}^{\vartheta_2} -\rho\sin\vartheta(-\rho\sin\vartheta) + \rho\cos\vartheta\,\rho\cos\vartheta\,d\vartheta =$$

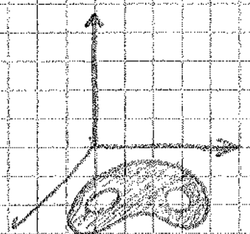
$$= \frac{1}{2}\int_{\vartheta_1}^{\vartheta_2} \rho^2(\sin^2\vartheta + \cos^2\vartheta)\,d\vartheta = \frac{1}{2}\int_{\vartheta_2}^{\vartheta_1} \rho^2(\vartheta)\,d\vartheta$$

TEOREMA DI GREEN (VISTO COME STOKES IN 2D)

$$F \in \mathcal{C}^1(\mathbb{R}^3) \Rightarrow \int_{\partial K} F\,dP = \iint_K (\partial_x F_2 - \partial_y F_1)\,dx\,dy$$

3° ELEMENTO DEL ROTORE

$(x, y, 0)$ $\quad F = (F_1, F_2, 0) = (F_1(x,y), F_2(x,y), 0)$



$$\mathrm{ROT}F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} =$$

$$= 0\,\vec{i} + 0\,\vec{j} + (\partial_x F_2 - \partial_y F_1)\,\vec{k}$$

$$\int_{\partial K} F\,dP = \int_K \mathrm{ROT}F \cdot \vec{n} = \int (0, 0, \partial_x F_2 - \partial_y F_1)\cdot(0,0,1)\,dx\,dy$$

$$= \iint_K \partial_x F_2 - \partial_y F_2\,dx\,dy$$

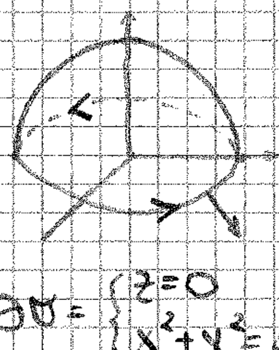
ESERCIZI

1) $\begin{cases} x^2+y^2+z^2=4 \\ z \geq 0 \end{cases}$

$$F = \left(-\frac{1}{8}x^2 y, \frac{1}{8}x^3 + z^2, \mathrm{ARCTG}\, e^{x+y+z}\right)$$

ORIENTATA SECONDO I VETTORI USCENTI DALLA SUP.

CALCOLARE IL FLUSSO DEL ROTORE: $\int_\sigma \mathrm{ROT}F \cdot \vec{n} = ?$



$$\int_\sigma \mathrm{ROT}F \cdot \vec{n} = \int_{\partial K} F\,dP$$

$\partial\sigma = \begin{cases} z=0 \\ x^2+y^2=4 \end{cases}$ $\quad \gamma: \begin{cases} x = 2\cos\vartheta \\ y = 2\sin\vartheta \\ z = 0 \end{cases}$ $\quad 0 \leq \vartheta \leq 2\pi$

$\sin^5\vartheta = \sin\vartheta(\sin^4\vartheta) = \sin(1-\cos^2\vartheta)^2$

$$\int F\,dP = \frac{3}{2}\pi^4 \int_0^{2\pi} \cos^7\vartheta\,(1-\cos^2\vartheta)^2 \underbrace{\sin\vartheta\,d\vartheta}_{-d(\cos\vartheta)}$$

$$= \frac{3}{2}\pi^4 \int_1^1 t^7(1-t^2)^2\,dt = 0$$

SI POTEVA DEDURRE DAL FATTO CHE:

$$\left.\begin{array}{l}\sin^5\vartheta \to \text{PERIODICA, DISPARI, PERIODO } 2\pi \\ \cos^7\vartheta \to \text{PERIODICA, PARI, PERIODO } 2\pi\end{array}\right\} f(\vartheta)\ \text{DISPARI}$$

$$\int_0^{2\pi} f(\vartheta)\,d\vartheta = \int_{-\pi}^{\pi} f(\vartheta)\,d\vartheta = 0$$

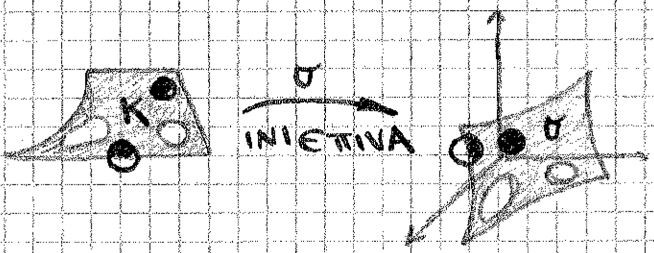
CALOTTE SUPERFICIALI CON BORDO

$\sigma: \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ $\quad \sigma \in C^1(\Omega)$ $\quad J_p\sigma$ RANGO MAX Ω

$A \subseteq \mathbb{R}^2$ APERTO, NON VUOTO, LIMITATO

∂A E' L'UNIONE DI CURVE CHIUSE, SEMPLICI, REGOLARI A TRATTI

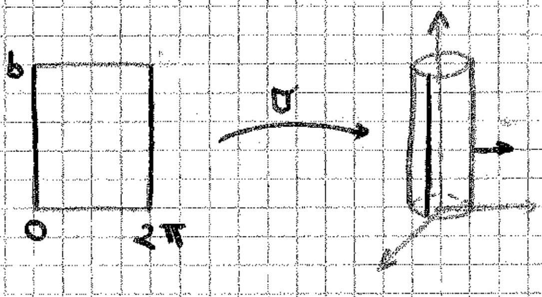
$\Rightarrow K = A \cup \partial A = \bar{A} \subseteq \Omega$



SE SUL BORBO
↳ BALL ∩ σ → SEMICIRC.

SE NON SUL BORBO
↳ BALL ∩ σ → CIRC.

$\partial\sigma = \sigma(\partial K)$



$\vartheta = 0 \rightarrow \vec{N} = (1,0,0) \rightarrow$ DIRETTO COME VOGLIAMO NOI

$\text{ROT}F \cdot \vec{n} = (-xy; 0; yz)(\cos\vartheta; \sin\vartheta; 0)$
$= -\cos\vartheta \cdot \sin\vartheta \cdot \cos\vartheta$

$\int_S \text{ROT}F \cdot \vec{n} = \iint (-\cos^2\vartheta \sin\vartheta)\, dt\, d\vartheta$

$= \int_0^{2\pi} -\left(3 - \cos\vartheta - \frac{1}{2}\sin\vartheta\right) \cos^2\vartheta \sin\vartheta \, d\vartheta =$

$= \int_0^{2\pi} +3\cos^2\vartheta \sin\vartheta + \cos^4\vartheta \sin\vartheta + \frac{1}{2}\cos^2\vartheta \sin^2\vartheta \, d\vartheta$

$= \int_{-\pi}^{\pi} \underbrace{3\cos^2 \sin\vartheta}_{0} + \underbrace{\cos^4\vartheta \sin\vartheta}_{0} + \frac{1}{2}\cos^2\vartheta \sin^2\vartheta \, d\vartheta =$

$= \int_{-\pi}^{\pi} \frac{1}{2} \sin^2\vartheta \cos^2\vartheta \, d\vartheta = 2\frac{1}{2}\int_0^{\pi} \sin^2\vartheta \cos^2\vartheta \, d\vartheta =$

$2\sin\vartheta\cos\vartheta = \sin 2\vartheta$

$\sin^2\vartheta \cos^2\vartheta = \left(\frac{1}{2}\sin^2 2\vartheta\right)^2 = \frac{1}{4}\sin^2(2\vartheta)$

$\int_0^{\pi} \frac{1}{4}\sin^2(2\vartheta)\, d\vartheta = \int_0^{2\pi} \frac{1}{8}\sin^2 s \, ds \qquad s = 2\vartheta$
$\qquad ds = 2\, d\vartheta$

$= \frac{1}{8} \left. \frac{s - \sin s \cos s}{2} \right]_0^{2\pi} = \frac{\pi}{8}$

FRONTIERA DEI SOLIDI

$A \subseteq \mathbb{R}^3$

∂A FRONTIERA = UNIONE SUPERFICI
↓
VETTORE NORMALE ORIENTATO IN SENSO USCENTE
↓
IL PRIMO PEZZO DEL VETTORE NON DEVE INTERSECARE SOLIDO
↓
$\vec{N}$ ORIENTATA IN SENSO USCENTE DA A SE IL VETTORE

$F: \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3 \qquad F \in C^1(\Omega),\ A \subseteq \Omega$

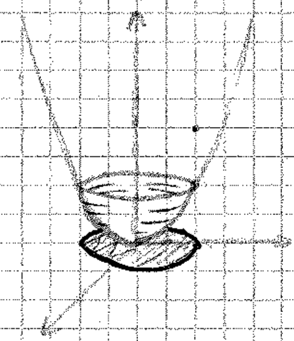
$$\Rightarrow \iint_{\partial A} F \cdot \vec{n} = \iiint_A \text{DIV} F \, dx\,dy\,dz$$

ESERCIZI

1) $E = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z \leq 1\}$

$F = (xy^2; z^2 y; x^2 z)$

FLUSSO DI F USCENTE DA E?



$$\iint_{\partial E} F \cdot \vec{n} = \iint_{\substack{\text{CIRC.}\\ \text{SUP.}\\ (C)}} F \cdot \vec{n} + \iint_{\substack{\text{SUP. LAT.}\\ \text{PARABOLOIDE}\\ (P)}} F \cdot \vec{n}$$

$$\iint_C F \cdot \vec{n} = \iint_C \vec{F} \cdot \vec{N} \, dx\,dy \qquad C = \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = 1 \end{cases}$$

$\vec{N} = (0,0,1)$

$\vec{F} \cdot \vec{N} = (xy^2; z^2 y; x^2 z) \cdot (0,0,1) = x^2 z$

$$\iint_C x^2 z \, dx\,dy = \int_0^{2\pi}\int_0^1 \rho^3 \cos^2\theta \, d\theta\, d\rho = \frac{1}{4} \cdot \frac{\theta + \cos\theta\sin\theta}{2} = \frac{\pi}{4}$$

$$\iint_E (xy^2; z^2 y; x^2 z)(+2x; +2y; -1) \, dx\,dy$$

$$\iint 2x^2 y^2 + z^2 y^2$$

$$\iint$$

$$\iint F \cdot \vec{n} = \iiint_E (\text{DIV} F) \, dx\,dy\,dz$$

$$\text{DIV} F = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = y^2 + z^2 + x^2$$

$$\iiint_E x^2 + y^2 + z^2 \, dx\,dy\,dz$$

$$\iiint_K \operatorname{DIV} F = \int_E F\cdot\vec{n} + \int_C F\cdot\vec{n}$$

$$\int_E F\cdot\vec{n} = \iiint \operatorname{DIV} F - \int_C F\cdot\vec{n}$$

$$\operatorname{DIV} F = z+1+0 = z+1$$

$$\iiint_K z+1 \, dx\,dy\,dz$$

$$z^2 = 1-\frac{x^2+y^2}{4} \qquad \begin{cases} 0\le z\le \sqrt{1-\dfrac{x^2+y^2}{4}} \\ 0\le x^2+y^2\le 4 \end{cases}$$

$$\iint_C \left(\int_0^{\sqrt{1-\frac{x^2+y^2}{4}}} (z+1)\,dz\right) dx\,dy =$$

$$\iint_C \frac{1}{2}\left(1-\frac{x^2+y^2}{4}\right)+\sqrt{1-\frac{x^2+y^2}{4}}\,dx\,dy$$

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \qquad \begin{cases} 0\le\rho\le 2 \\ 0\le\theta\le 2\pi \end{cases}$$

$$\int_0^{2\pi}\int_0^2 \left(\frac{1}{2}\left(1-\frac{\rho^2}{4}\right)+\sqrt{1-\frac{\rho^2}{4}}\right)\rho\,d\rho\,d\theta =$$

$$2\pi\int_0^2 \frac{\rho}{2}-\frac{\rho^3}{8}+\sqrt{1-\frac{\rho^2}{4}}\,\rho\,d\rho =$$

$$= 2\pi\left[\frac{\rho^2}{2\cdot 2}-\frac{\rho^4}{4\cdot 8}-2\int\sqrt{1-\frac{\rho^2}{4}}\,d\left(1-\frac{\rho^2}{4}\right)\right]_0^2 =$$

$$= 2\pi\left[\frac{4}{4}-\frac{4^2}{32}-2\,\frac{\left(1-\frac{1}{4}\rho^2\right)^{3/2}}{3/2}\right]_0^2 = \qquad 1-\frac{\rho^2}{4}=s \qquad ds=\frac{2\rho}{4}$$

$$= 2\pi\left[1-\frac{1}{2}-\frac{4}{3}(0-1)\right] =$$

$$= 2\pi\left[1-\frac{1}{2}+\frac{4}{3}\right] = \frac{11\pi}{3}$$

PROPOSIZIONE

SE F È CONSERVATIVO SU Ω E f È UN POTENZ. DI F SU Ω, ALLORA $\forall k \in \mathbb{R}$, $f(x)+k$ È ANCORA UN POTENZIALE DI F SU Ω

DIM

$\nabla f(x) = F(x) \quad \forall x \in \Omega$

$\nabla(f(x)+k) = \nabla f(x) = F(x)$ POICHÉ DERIV. DI UNA COST. È UGUALE A ZERO

PROP

F CAMPO CONTINUO CONSERVATIVO SU Ω CONNESSO

DATA $\gamma : [a,b] \to \Omega$ REGOLARE T.C. $\gamma(a)=A \quad \gamma(b)=B$

SIA $g(x)$ UN POTENZ. DI F SU Ω

$\Rightarrow \int_\gamma F \cdot dP = g(B) - g(A)$ NON CONTA IL PERCORSO

DIM $\qquad 1 \Rightarrow 2$

$F = (f_1, \dots, f_n)$

$\int_\gamma F \cdot dP = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$

$h(t) = g(\gamma(t))$

$h'(t) = \nabla g(\gamma(t)) \cdot \gamma'(t) = F(\gamma(t)) \cdot \gamma'(t)$

$$\int_\gamma F \cdot dP = \int_a^b h'(t)\, dt = h(t)\Big]_a^b = h(b) - h(a)$$
$$= g(\gamma(b)) - g(\gamma(a)) = g(B) - g(A)$$

COROLLARIO

F CONSERV. CONTINUO SU Ω AP. CONNESSO E γ È UNA CURVA CHIUSA REGOLARE (A TRATTI) $\qquad 1 \Rightarrow 3$

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