NUMERO : 179

DATA : 03/11/2011

A P P U N T I

STUDENTE : Gemello

MATERIA : Geometria
Prof. Caire

VETTORI NELLO SPAZIO

1) GRANDEZZE SCALARI → NUMERI REALI

2) GRANDEZZE VETTORIALI → VETTORI

↓

SEGMENTO ORIENTATO

$\underline{v} = \vec{v} = \overrightarrow{OP}$

$(\overrightarrow{OP} \neq \overrightarrow{PO})$

MODULO; DIREZIONE; VERSO

↓

LUNGHEZZA DI UN VETTORE

↓

$|\underline{v}| = \|\underline{v}\| = $ LUNGH. $OP \in \mathbb{R}^+$

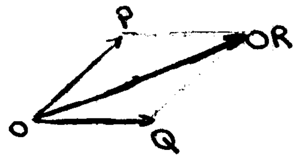
VERSORE = VETTORE DI LUNGH. 1

SOMMA DI VETTORI

REGOLA PARALLELOGRAMMA

$\underline{v} = \overrightarrow{OP}$

$\underline{w} = \overrightarrow{OQ}$



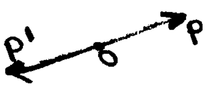
$\underline{v} + \underline{w} = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$

VETTORE NULLO = VETTORE $\overrightarrow{OO}$ (MODULO NULLO) $= \underline{o} = \vec{O}$

$\forall \vec{v}, \ \vec{v} + \vec{O} = \vec{O} + \vec{v} = \vec{v}$

INDICHIAMO $-\underline{v}$ L'OPPOSTO DI $\underline{v}$ / $(-\underline{v}) + (\underline{v}) = O$

$\underline{v} = \overrightarrow{OP}$



$-\underline{v} = \overrightarrow{OP'}$

DISUGUAGLIANZA TRIANGOLARE

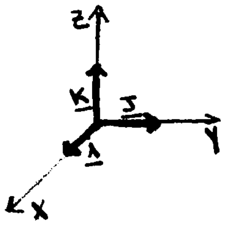
DATI $\underline{w}, \underline{v}$

$|\underline{v} + \underline{w}| \leq |\underline{v}| + |\underline{w}|$

DIFFERENZA TRA VETTORI

DATI $\underline{u}$ E $\underline{w}$ LA LORO DIFF. È DATA DALLA SOMMA TRA $\underline{u}$ E $-\underline{w}$

COMPONENTI DI UN VETTORE



VERSORI FONDAMENTALI

$\underline{i}, \underline{j}, \underline{k}$

$\underline{i}\ (1,0,0)$

$\underline{j}\ (0,1,0)$

$\underline{k}\ (0,0,1)$

TUTTI I VETTORI NELLO SPAZIO SONO COMBINAZ. LINEARI DEI VERSORI FONDAMENTALI

$\forall \underline{v}, \quad \underline{v} = v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k} \qquad v_1, v_2, v_3 \in \mathbb{R}$

$\underline{v} = (v_1, v_2, v_3)$

$\underline{w} = (w_1, w_2, w_3)$

$\underline{v} + \underline{w} = (v_1 + w_1, w_2 + v_2, v_3 + w_3)$

$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

ESEMPIO

$\underline{v} = (1,2,3) = \underline{i} + 2\underline{j} + 3\underline{k}$

$\underline{w} = (0,2,4) = 2\underline{j} + 4\underline{k}$

SE $\underline{v} = \overrightarrow{OP} \rightarrow$ COORD $P = (1,2,3)$
O → ORIGINE

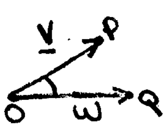
$\underline{v} + \underline{w} = (1,4,7) = \underline{i} + 4\underline{j} + 7\underline{k} \qquad \|\underline{v}\| = \sqrt{1+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$

$\forall \alpha \in \mathbb{R} \Rightarrow \alpha \underline{v} = (\alpha v_1, \alpha v_2, \alpha v_3)$

$|\alpha \underline{v}| = \sqrt{\alpha^2 v_1^2 + \alpha^2 v_2^2 + \alpha^2 v_3^2} = |\alpha| \sqrt{v_1^2 + v_2^2 + v_3^2} = |\alpha| |\underline{v}|$

PRODOTTO SCALARE $\vec{v} \cdot \vec{w}$

INDICHIAMO L'ANGOLO TRA $\underline{v}$ E $\underline{w}$ CON $\widehat{\underline{v}\,\underline{w}}$



SI DEFINISCE PRODOTTO SCALARE

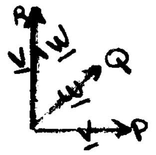
$\underline{v} \cdot \underline{w} = |\underline{v}| |\underline{w}| \cos \widehat{\underline{v}\,\underline{w}} \rightarrow$ RISULTATO È UNO SCALARE

PRODOTTO VETTORIALE $\underline{v} \wedge \underline{w}$

È IL VETTORE CON MODULO $|\underline{v}|\cdot|\underline{w}|\,\text{sen}\,\widehat{\underline{v}\;\underline{w}}$

DIREZIONE ORTOGONALE AL PIANO FORMATO DAI 2 VETTORI

VERSO USCENTE DAL PUNTO O



SE $\underline{v} \parallel \underline{w} \Rightarrow \underline{v} \wedge \underline{w} = 0$

PROP

1) $\forall\ \underline{v}, \underline{w} \qquad \underline{v} \wedge \underline{w} = -\underline{w} \wedge \underline{v}$ (NO PROP. COMMUT.)

2) $\forall\ \underline{v}, \underline{w},\ \forall \alpha \in \mathbb{R} \qquad \alpha(\underline{v} \wedge \underline{w}) = \alpha\underline{v} \wedge \underline{w} = \underline{v} \wedge \alpha\underline{w}$

3) NO PROP. ASSOCIATIVA

4) $\forall\ \underline{v}, \underline{w}, \underline{z} \qquad \underline{v} \wedge (\underline{w} + \underline{z}) = \underline{v} \wedge \underline{w} + \underline{v} \wedge \underline{z}$

PR. VETTORIALE IN COMPONENTI

$\underline{v} = (v_1, v_2, v_3) \qquad \underline{w} = (w_1, w_2, w_3)$

$$\underline{v} \wedge \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \underline{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \underline{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \underline{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \underline{i}(v_2 \cdot w_3 - v_3 \cdot w_2) - \underline{j}(v_1 w_3 - v_3 w_1) + \underline{k}(v_1 w_2 - v_2 w_1)$$

$$= (v_2 \cdot w_3 - v_3 \cdot w_2 \,;\, v_3 w_1 - v_1 w_3 \,;\, v_1 w_2 - v_2 w_1)$$

OSS $\underline{i} \wedge \underline{j} = \underline{k}$

$\underline{j} \wedge \underline{i} = -\underline{k}$

$\underline{i} \wedge \underline{k} = \underline{j} \qquad \underline{k} \wedge \underline{i} = -\underline{j} \qquad \underline{j} \wedge \underline{k} = \underline{i} \qquad \underline{k} \wedge \underline{j} = -\underline{i}$

PRODOTTO MISTO

$$\underline{v} \cdot (\underline{w} \wedge \underline{z}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = v_1(w_2 z_3 - w_3 z_2) - v_2(w_1 z_3 - w_3 z_1) + v_3(w_1 z_2 - w_2 z_1)$$

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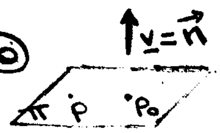
CONDIZ. PER DETERMINARE UN PIANO NELLO SPAZIO

I) 3 PUNTI A, B, C NON ALLINEATI

II) RETTA r E UN PUNTO $P_0 \notin r$

III) 2 RETTE NON SGHEMBE

0) PIANO PER $P_0 \perp$ AL VETTORE $\underline{v}$

0) 

$$P \in \pi \Leftrightarrow \overrightarrow{P_0P} \perp \underline{v} \longleftrightarrow (P-P_0)\cdot \underline{v} = 0$$

$$\pi = \{P \in \mathbb{R}^3 : (P-P_0)\cdot \underline{v} = 0\}$$

$P_0 = (x_0, y_0, z_0)$

$P = (x, y, z) \in \mathbb{R}^3$ $\qquad \vec{n} = \underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

$\uparrow$ NORMALE $(\perp)$

$$P - P_0 = (x-x_0)\underline{i} + (y-y_0)\underline{j} + (z-z_0)\underline{k}$$

$$P \in \pi \Longleftrightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

EQ. CARTESIANA DEL PIANO PER $P_0 \perp \pi$

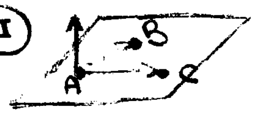
$$ax + by + cz + d = 0 \longrightarrow d = -(ax_0 + by_0 + cz_0)$$

** $P_0 = (1, 5, -1)$ $\quad \underline{v} = \underline{i} - 5\underline{j} + \underline{k}$ $\quad 1(x-1) - 5(y-5) + 1(z+1)$

$x - 5y + z + 25 = 0$

** $x + y + 5z = 0$ $\quad \vec{n} = (1, 1, 5)$

$A(2, 0, 0) \in \pi$ $\qquad B\left(-2\lambda, -2\lambda, \frac{2}{5}\right)$

I) 

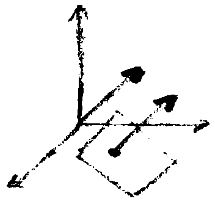
$$\underline{n} = (B-A) \wedge (C-A)$$

$P_0 = A$

* $A = (1, 2, 3)$ $\quad B = (4, 2, 1)$ $\quad C = (0, -1, 1)$

CONTROLLO CHE NON GIACCIANO SULLA STESSA RETTA

$B - C = (4, 3, 0)$ $\qquad A - C = (1, 3, 2)$



$\pi: x+y+2z-3=0$

$\underline{n}_\pi: \underline{i}+\underline{j}+2\underline{k}=(1,1,2)$

$P_0=(3,0,0)$ $\quad P_1=(1,0,1)$

$P_0 \in \pi$ $\quad P_2=(0,3,0)$

ESEMPIO (I)

$A(1,0,+1)$

$B(0,1,-1)$

$C(1,1,-2)$

$B-A=(-1,1,-2)$

$C-A=(0,1,-3)$

$$\vec{n}_\pi=(B-A)\wedge(C-A)=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 1 & -2 \\ 0 & -1 & -3 \end{vmatrix}$$

$\vec{n}_\pi=-\underline{i}-3\underline{j}-k$

$\vec{n}_\pi=\underline{i}+3\underline{j}+k$

$\vec{PA} \in \pi \Leftrightarrow (P-A) \perp \vec{n}_\pi$

$(P-A)\cdot\vec{n}_\pi$

$P(x,y,z) \in \pi \Leftrightarrow (P-A)\cdot(B-A)\wedge(C-A)$

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_B-x_A & y_B-y_A & z_B-z_A \\ x_C-x_A & y_C-y_A & z_C-z_A \end{vmatrix}=0$$

$1(x-1)+3(y-0)+1(z-1)=0$

$x+3y+z-2=0$

OPPURE APPLICO SUBITO PRODOTTO MISTO

$A=(1,0,1)$

$B=(0,1,-1)$

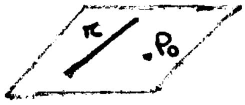
$C=(1,1,-2)$

$$\begin{vmatrix} x-1 & y & z-1 \\ -1 & 1 & -2 \\ 0 & 1 & 3 \end{vmatrix}=0$$

$$(x-1)\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}-y\begin{vmatrix} -1 & -2 \\ 0 & 3 \end{vmatrix}+(z-1)\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix}=(x-1)(-1)-y(3)+(z-1)(-1)$$

$$=-x-3y-z+2$$

II) RETTA r, PUNTO ESTERNO P_0



SCEGLGO 2 PUNTI $\rightarrow$ CASO (I)

SU r $\searrow \vec{AB}\wedge\vec{AP_0}=\vec{n} \rightarrow$ CASO (0)

ESEMPIO (II)

$r: \begin{cases} y=0 \\ z=0 \end{cases} \rightarrow$ ASSE X $\rightarrow \vec{AB}=\vec{i}$ $\qquad P_0(1,1,1)$

③ $\pi_1: ax+by+cz+d=0$
$\pi_2: a'x+b'y+c'z+d'=0$ $\quad (a,b,c) \neq K(a',b',c')$

$r = \pi_1 \cap \pi_2$

$\begin{cases} \pi_1: ax+by+cz+d=0 \\ \pi_2: a'x+b'y+c'z+d'=0 \end{cases}$ EQ. CARTESIANA DI r

SE PRENDIAMO $\vec{n}_{\pi_1}$ E $\vec{n}_{\pi_2}$ E APPLICHIAMO $\vec{n}_{\pi_1} \wedge \vec{n}_{\pi_2}$
OTTENIAMO UN VETTORE $/\!/ \, r$
PRENDENDO $P_0 \in \pi$ CI RICONDUCIAMO A CASO (1)

PASSARE DA EQ. CART. A EQ. PARAM.

$r = \begin{cases} 3x-y+5z-1=0 & \rightarrow \vec{n}_1 = (3,-1,5) \\ 2x-z+3=0 & \quad \vec{n}_2 = (2,0,-1) \end{cases}$

$\left.\begin{array}{l} r \in \pi_1 \Rightarrow r \perp \vec{n}_1 \\ r \in \pi_2 \Rightarrow r \perp \vec{n}_2 \end{array}\right\} \Rightarrow r /\!/ \vec{n}_1 \wedge \vec{n}_2$

$\vec{n}_1 \wedge \vec{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & 5 \\ 2 & 0 & -1 \end{vmatrix} = \vec{i} + 13\vec{j} + 2\vec{k} = \vec{v} /\!/ r \qquad \underline{v} = (1,13,2)$

TROVO UN PUNTO $P \in r$, OVVERO CHE SODDISFI IL SISTEMA

$\begin{cases} x=0 \\ z = 2\cdot 0+3 = 3 \\ y = 3\cdot 0 + 5\cdot 3 - 1 = 14 \end{cases} \qquad P_0(0,14,3)$

$\begin{cases} x = 0 + 1\cdot t \\ y = 14+13t \\ z = 3+2t \end{cases}$ EQ. PARAMETRICA DI r

DA EQ. PARAMETRICHE A CARTESIANE

$r: \begin{cases} x = x_0 + lt \\ y = y_0 + mt \\ z = z_0 + nt \end{cases}$

RICAVO IL PARMETRO DA 1 DELLE 3: $\begin{cases} t = (x-x_0)/l \\ y = y_0 + m(x-x_0)/l \\ z = z_0 + n(x-x_0)/l \end{cases}$

LE EQ. NORMALI NON SI POSSONO SCRIVERE SE UNA COMPONENTE DI $\underline{v}$ È NULLA

$\downarrow$

ESEMPIO:

$P(1,2,-5)$ $\vec{v}=\vec{i}-3\vec{k}$

$$\cancel{\frac{x-1}{1}=\frac{y-2}{0}=\frac{z+5}{-3}} \longrightarrow \begin{cases} y=-2 \\ \dfrac{x-1}{1}=\dfrac{z+5}{-3} \end{cases} \qquad \begin{cases} y=2 \\ 3x+z=-2 \end{cases}$$

PARALLELISMO E ORTOGONALITÀ

$\pi_1 \perp \vec{n}_1$ $\qquad r \parallel \vec{u}$

$\pi_2 \perp \vec{n}_2$ $\qquad s \parallel \vec{w}$

$*\ \pi_1 \parallel \pi_2 \longleftrightarrow \vec{n}_1 \parallel \vec{n}_2 \longleftrightarrow \vec{n}_1 \wedge \vec{n}_2 = \underline{0}$

$\pi_1 \perp \pi_2 \longleftrightarrow n_1 \perp n_2 \longleftrightarrow \vec{n}_1 \cdot \vec{n}_2 = 0$

$*\ r \parallel s \longleftrightarrow \underline{u} \parallel \underline{w} \longleftrightarrow \underline{u} \wedge \underline{w} = \underline{0}$

$r \perp s \longleftrightarrow \underline{u} \perp \underline{w} \longleftrightarrow \underline{u} \cdot \underline{w} = 0$

$*\ \pi_1 \parallel r \longleftrightarrow \underline{n}_1 \perp \underline{u} \longleftrightarrow \underline{n}_1 \cdot \underline{u} = 0$

$\pi_1 \perp r \longleftrightarrow \underline{n}_1 \parallel \underline{u} \longleftrightarrow \underline{n}_1 \wedge \underline{u} = \underline{0}$

INTERSEZIONI

TRA 2 PIANI

- SE $\pi_1 \cap \pi_2 = \emptyset \rightarrow$ PARALLELI DISTINTI
- SE $\pi_1 \cap \pi_2 = \pi_1 \rightarrow$ PARALLELI COINCIDENTI
- SE $\pi_1 \cap \pi_2 =$ RETTA $r \rightarrow$ INCIDENTI

TRA UN PIANO E UNA RETTA

- SE $\pi_1 \cap r = r \rightarrow$ COINCIDENTI
- SE $\pi_1 \cap r =$ UN PUNTO $\rightarrow$ INCIDENTI
- SE $\pi_1 \cap r = \emptyset \rightarrow$ PARALLELI DISTINTI

TRA 2 RETTE

SE $r \cap s =$ UN PUNTO $\rightarrow$ INCIDENTI

$$r: \begin{cases} x+2y+1=0 \\ 3y-2z=0 \end{cases} \qquad \Phi_r = \lambda(x+2y+1)+\mu(3y-2z)=0$$

a) PASSANTE PER $P_0(0,0,1) \rightarrow P_0 \in \Phi$

$$\lambda(1)+\mu(-2)=0 \rightarrow \lambda = 2\mu \rightarrow \begin{cases} \lambda=2 \\ \mu=1 \end{cases}$$

$$2(x+2y+1)+3y-2z=0$$

$$\pi: 2x+7y-2z+2=0$$

b) $\pi \in \Phi_r$, $\pi \perp \alpha \leftrightarrow \vec{n}_\Phi \perp \vec{n}_\alpha \leftrightarrow \vec{n}_\Phi \cdot \vec{n}_\alpha = 0$

$$\alpha: x+y+z=0 \rightarrow \vec{n}_\alpha = (1,1,1) \qquad \vec{n}_\Phi = (\lambda, 2\lambda+3\mu, -2\mu)$$

$$\underline{n}_\alpha \cdot \underline{n}_\Phi = \lambda+2\lambda+3\mu-2\mu=0$$

$$-\mu = 3\lambda \;\; \begin{cases} \mu=-3 \\ \lambda=1 \end{cases} \qquad \pi: x+2y+1+(-3)(3y-2z)=0$$

c) $\pi \in \Phi_r$ // AB

$$\underline{n}_\pi \perp \vec{AB} \leftrightarrow \vec{n}_\pi \cdot \vec{AB}=0 \qquad A(1,2,0) \quad B(1,3,-1)$$

$$B-A=(0,1,-1)$$

$$2\lambda+3\mu+2\mu=0$$

$$\lambda=5 \quad \mu=2$$

FASCIO DI PIANI //

$$\pi: x-3y+5z+1=0$$

$$\pi': 2x-6y+10z+23=0$$

$$\lambda(x-3y+5z+1)+\mu(2x-6y+10z+23)=0$$

$$\lambda(x-3y+5z+1)+2\mu\left(x-3y+5z+\frac{23}{2}\right)=0$$

$$(\lambda+2\mu)(x-3y+5z)+\lambda+ [illegible]\,\mu=0$$

$$x-3y+5z+K=0$$

$$K=\frac{\lambda+23\mu}{\lambda+2\mu}$$

DISTANZE

1) DISTANZA TRA 2 PUNTI

$A=(x_1, y_1, z_1) \qquad B=(x_2, y_2, z_2)$

$\text{DIST}(A,B)=\|B-A\|=\|A-B\|=\sqrt{(x_A-x_B)^2+(y_A-y_B)^2+(z_A-z_B)^2}$

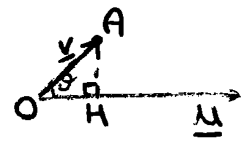
2) DISTANZA DI UN PUNTO DA UN PIANO

$P_0=(x_0, y_0, z_0)$

$\pi: ax+by+cz+d=0$

$$\text{DIST}(P_0, \pi)=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

3) DISTANZA PUNTO-RETTA



$OH = \text{COMP}_{\vec{u}}(\vec{v})$ LUNGHEZZA CON SEGNO

$= \overline{OA} \cdot \cos\vartheta$

$$|\underline{v}| \cdot \frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \vec{v} \cdot \frac{\vec{u}}{|\underline{u}|}$$

$\overrightarrow{OH} = \overrightarrow{pr_{\underline{u}}(\underline{v})}$

$= (\text{COMP}_{\underline{u}}(\underline{v}))\,\text{VERS}\,\underline{u}$

$\text{COMP}_{\underline{u}}(\underline{v}) = \vec{v} \cdot (\text{VERS}\,\vec{u}) = \overrightarrow{OH}$

APPLICO POI DIST TRA 2 PUNTI TRA A, H

ESERCIZIO

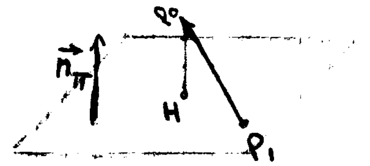
DATO $\underline{u}=(2,-1,2)$

TROVARE I VETTORI DI MODULO 5 CON DIREZ UGUALE AD $\underline{u}$

$$\text{VERS}\,\vec{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{2\underline{i}-\underline{j}+2\underline{k}}{3}$$

$$\vec{v} = \pm 5\left(\frac{2\underline{i}-\underline{j}+2\underline{k}}{3}\right)$$

DIMOSTRAZIONE (2)



$\forall P_1 \in \pi \qquad P_1(x_1, y_1, z_1)$

$ax_1+by_1+cz_1=0$

$\text{DIST}(P_0, H) = \text{COMP}_{\underline{n}}(P_0-P_1)$

ESEMPIO (3)

DIST P_0, π $\quad P_0 \notin \pi$

H PASSANTE PER RETTA $\perp \pi$ INCIDENTE

↳ SENNO' ∞

$H = \pi \cap \Pi \quad \Pi$ PER $P_0 \perp \pi$

$DIST(P_0, \pi) = DIST(P_0, H)$

$P_0(1,2,-1)$

$$\pi: \begin{cases} 2x - y + z + 1 = 0 \\ x - 3y + 5z - 7 = 0 \end{cases} \quad \vec{\pi} = (2,-1,1) \wedge (1,-3,5) = (+2,+9,+5)$$

Π PER $P_0(1,2,-1) \perp \vec{\pi} = 2(x-1) + 9(y-2) + 5(z+1) = 0$

$= 2x + 9y + 5z - 15 = 0$

$H: \Pi \cap \pi$

$R \in \pi$

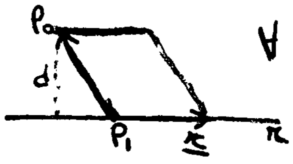
$$\begin{cases} z = 0 \\ 2x - y + 1 = 0 \\ x - 3y - z = 0 \rightarrow -5x = 10 \end{cases} \rightarrow R: \begin{cases} x = -2 \\ y = -3 \\ z = 0 \end{cases}$$

$$\pi: \begin{cases} x = 2t - 2 \\ y = 9t - 3 \\ z = 5t \end{cases} \quad 4t - 4 + 81t - 27 + 25t - 15 = 0$$

$$110t = 46 \rightarrow t = \frac{23}{55}$$

$$H\left(\frac{46}{55} - 2; \frac{207}{55} - 3; \frac{23}{11}\right) \rightarrow DIST(P_0, H)$$

METODO VETTORIALE (3)



$\forall P_1 \in \pi \quad d =$ ALTEZZA PARALLELOGR. $= DIST(P_0, \pi)$

$$d = \frac{AREA\ (PARALL.)}{BASE\ (PARALL.)} = \frac{\|(P_0 - P_1) \wedge \vec{\pi}\|}{\|\vec{\pi}\|} = \|\overrightarrow{P_0P_1} \wedge vers\, \vec{\pi}\|$$

ESEMPIO:

$P_0(1,-2,5)$

SPAZI VETTORIALI

* $V_3 / \mathbb{R}$

+ (SOMMA) $\underline{v}, \underline{w} \in V_3$ $\quad \underline{v} + \underline{w} \in V_3$

$V_3 \times V_3 = V_3$

S_1) PROP. ASSOCIATIVA $\quad \underline{v} + \underline{w} (+ \underline{z}) = \underline{v} + (\underline{w} + \underline{z})$

S_2) PROP. COMMUTATIVA $\quad \underline{v} + \underline{w} = \underline{w} + \underline{v}$

S_3) $\exists\ \underline{0}\ /\ \forall\ \underline{v},\ \underline{v} + \underline{0} = \underline{v}$ (ESISTENZA NEUTRO)

S_4) $\exists\ -\underline{v}$ (OPPOSTO) $/\ \forall\ \underline{v},\ \underline{v} + (-\underline{v}) = \underline{0}$

· (PRODOTTO PER UNO SCALARE) $\underline{v} \in V,\ \alpha \in \mathbb{R},\ \alpha \cdot \underline{v}$

$\mathbb{R} \times V \to V$

P_1) FILTRAZIONE SCALARE $\to (\alpha \cdot b)\underline{v} = (\alpha \underline{v}) b = \alpha (b \underline{v})$

P_2) $1 \cdot \underline{v} = \underline{v}$ ($\exists$ DELL'UNITA')

D_1) $\forall\ \alpha, b \in \mathbb{R},\ \forall\ \underline{v} \in V \to (\alpha + b)\underline{v} = \alpha \underline{v} + b \underline{v}$

D_2) $\forall\ \alpha \in \mathbb{R},\ \forall\ \underline{v}, \underline{w} \in V \to \alpha(\underline{v} + \underline{w}) = \alpha \underline{v} + \alpha \underline{w}$

} DISTRIBUTIVE

SPAZIO VETTORIALE $\Leftrightarrow$ GODE DI QUESTE 8 PROP.

V_3 E' UNO SPAZIO VETTORIALE IN $K = \mathbb{R} \to V_{3\,\mathbb{R}}$

V_3 E' UN $\mathbb{R}$-SPAZIO VETTORIALE

CAMPO VETT. $\to$ USIAMO $\mathbb{R}, \mathbb{C}$

* $\mathbb{R}_{\mathbb{R}}$

S_1) P. ASSOC $\to (a+b)+c = a+(b+c)$

S_2) COMMUTATIVA $\to a+b = b+a$

S_3) $\exists$ NEUTRO $\to a+0 = a$

S_4) $\exists$ OPPOSTO $\to a+(-a) = 0$

P_1) ASSOC $\to a(b+c) = ab+ac$

P_2) COMMUT $\to a \cdot b = b \cdot a$

P_3) $\exists$ NEUTRO $\to a \cdot 1 = a$

P_4) RECIPROCO $\to a \cdot (a^{-1}) = 1$

PRODOTTO DISTRIBUTIVO RISPETTO ALLA SOMMA

~~SE PROVASSI A DEF. UN ALTRO PRODOTTO~~

~~$\lambda \in \mathbb{R}$ $(a,b) \in \mathbb{R}^2$~~ ↳ SOMMA TENGO LA STESSA

~~$\lambda(a,b) = (|\lambda| a, |\lambda| b)$~~ ↳ VALGONO S1, S2, S3, S4

~~P1) $\lambda[\mu(a,b)] = (\lambda \cdot \mu)(a,b)$ VALE~~

~~$\lambda[|\mu| a, |\mu| b] = [|\lambda||\mu| a, |\lambda||\mu| b] = [|\lambda \cdot \mu| a, |\lambda \cdot \mu| b] =$~~
~~$= (\lambda \cdot \mu)(a,b)$~~

~~D1) $(\lambda+\mu)(a,b) \overset{?}{=} \lambda(a,b) + \mu(a,b)$ NON VALE~~ → SE NON VALGONO

~~$(|\lambda+\mu| a, |\lambda+\mu| b) \neq ((|\lambda|+|\mu|)a, (|\lambda|+|\mu|)b)$~~ TUTTE LE PROP. NON VA BENE

* $V = \mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \ \ x_i \in \mathbb{R}\}$
  $\mathbb{K} = \mathbb{R}$

$+ : (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$

$\cdot : \lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$

$\mathbb{R}^n$ È UN $\mathbb{R}$- SPAZIO VETTORIALE (CON 2 OPERAZ. SOPRA DEF)

* $V = \mathbb{C} \to (a + ib)$
  $\mathbb{K} = \mathbb{R}$

$+ : (a+ib) + (c+id) = (a+c) + i(b+d)$

$\cdot : \lambda \in \mathbb{R}, \ \ \lambda(a+ib) = (\lambda a + i\lambda b) = \lambda a + i(\lambda b)$

GODE DELLE 8 PROP.

$\underline{0} = 0 + i0 \qquad -(a+ib) = -a - ib$

$\mathbb{C}_{\mathbb{R}} \underset{\text{ISOMORFO}}{\cong} \mathbb{R}_{\mathbb{R}} \qquad \mathbb{C} = \{a+ib\}$

$\qquad \qquad \qquad \qquad \quad \ \mathbb{R}^2 = \{(a,b)\}$

MATRICI $m \times n$ A COEFF. REALI

$\mathbb{R}^{m,n} \rightarrow m$ RIGHE ; n COLONNE $\qquad \mathbb{K}^{m,n}$

$A \in \mathbb{R}^{m,n} \qquad B \in \mathbb{R}^{m,n}$

$$A = \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} \dots & a_{mn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{21} \dots & b_{1n} \\ b_{21} & \dots \dots & \dots \\ \dots & \dots \dots & \dots \\ b_{m1} & \dots \dots & b_{mn} \end{pmatrix}$$

$$A+B = \begin{pmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & \dots & \dots \\ \dots & & \dots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

$$a \in \mathbb{R} \rightarrow aA = \begin{pmatrix} a \cdot a_{11} & \dots a \cdot a_{1n} \\ \dots & \dots \\ a \cdot a_{m1} & \dots a \cdot a_{mn} \end{pmatrix}$$

$$-A = \begin{pmatrix} -a_{11} & \dots -a_{1n} \\ \dots & \dots \\ -a_{m1} & \dots -a_{mn} \end{pmatrix} \qquad O_V = \begin{pmatrix} 0 & \dots & 0 \\ \dots & & \dots \\ 0 & \dots & 0 \end{pmatrix}$$

$$* \mathbb{R}^{2,3} = \left\{ \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}, a,b,c,a',b',c' \in \mathbb{R} \right\} \cong \mathbb{R}^6$$

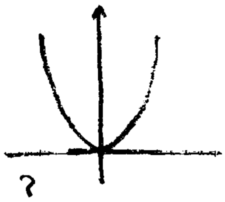
$A \longleftrightarrow (a,b,c,a',b',c')$

$\mathbb{R}^{m,n} \cong \mathbb{R}^{m \cdot n}$

PROP. ELEMENTARI SPAZI VETTORIALI

- UNICITÀ NEUTRO ($\underline{0}$)
- UNICITÀ OPPOSTO ($-\underline{v}$)
- ANNULLAMENTO PRODOTTO $\longrightarrow a\underline{v} = \underline{0} \Leftrightarrow a = \underline{0}_K \vee \underline{v} = \underline{0}_K$

* $W = \{(a, a^2), a \in \mathbb{R}\}$



0) $0_V \in W \to$ OK

1) $\forall \lambda \in \mathbb{R} \Rightarrow \lambda(a, a^2) \in W$?
$(\lambda a, \lambda a^2) \notin W \quad (\lambda a^2 \neq (\lambda a)^2) \to$ NO $\to W$ NON È S.S. DI $\mathbb{R}^2$

* $W = \{(a, 2a), a \in \mathbb{R}\}$

0) $0_V = (0,0) = (0, 2 \cdot 0) \to$ OK

1) $\lambda \in \mathbb{R} \Rightarrow \lambda(a, 2a) \in W$?
$\lambda(a, 2a) = (\lambda a, \lambda 2a) \in W \to$ OK $\qquad W$ S.S. DI $\mathbb{R}^2$

2) $(a, 2a) + (b, 2b) \in W)$?
$(a, 2a) + (b, 2b) = (a+b, 2a+2b) = ((a+b), 2(a+b)) \in W \to$ OK

* TUTTE E SOLE LE RETTE PASSANTI PER L'ORIGINE

$V = \mathbb{R}^2$
$W \subseteq \mathbb{R}^2$ È UN S.S. DI $V \Leftrightarrow W_m = \{(a, ma), a \in \mathbb{R}\}$
FISSATO $m \in \mathbb{R}$

* $W = \{(2a - 3b, 5a + 7b) \ \forall a, b \in \mathbb{R}$
$W \subseteq \mathbb{R}^2 \quad V = \mathbb{R}^2$

0) $0_V \in W$

1) $\lambda \in \mathbb{R} \Rightarrow \lambda(2a - 3b, 5a + 7b) = (2\lambda a - 3\lambda b, 5\lambda a + 7\lambda b)$
$= 2(\lambda a) - 3(\lambda b), 5(\lambda a) + 7(\lambda b) \quad \in W$

2) $(2a - 3b, 5a + 7b) + (2a' - 3b', 5a' + 7b') \in W$

$(2a - 3b, 5a + 7b) = (x, y)$
$$\begin{cases} x = 2a - 3b \\ y = 5a + 7b \end{cases}$$

DIPENDENZA LINEARE

COMBINAZIONI LINEARI

$\vec{v} \in V$ È C.L. DI $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ SE $\exists$ $a_1, a_2, \dots, a_n \in \mathbb{K}$ /

$\underline{v} = a_1 \underline{v_1} + a_2 \underline{v_2} + \dots + a_n \underline{v_n} \rightarrow \underline{v} = \mathcal{L}(\underline{v_1}, \underline{v_2}, \dots, \underline{v_n})$

* IN $V_2 \Rightarrow$ OGNI ELEM. IN V_2 È C.L. DI $\underline{i}, \underline{j}$ (SCOMPOSIZIONE)

* $V = \mathbb{R}^3$

$v_1 = (1,-1,2)$ $\quad v_2 = (3,0,1)$ $\quad \underline{v} = (1,5,4)$ $\quad \underline{v} \in \mathcal{L}(\underline{v_1}, \underline{v_2})$?

$\exists\, a,b \in \mathbb{R} / \underline{v} = a v_1 + b v_2$

$(1,5,4) = a(1,-1,2) + b(3,0,1) = (a,-a,2a) + (3b,0,b)$

$$\begin{cases} 1 = a + 3b \rightarrow 1 = -5 + 3b \rightarrow b = 2 \\ 5 = -a \rightarrow a = -5 \\ 4 = 2a + b \rightarrow 4 \neq -10 + 2 \end{cases}$$

$\underline{v}$ NON È C.L. DI v_1, v_2 $\quad \underline{v} \notin \mathcal{L}(v_1, v_2)$

$\underline{v} \in \mathcal{L}(v_1, v_2) \Longleftrightarrow \underline{v} \in$ PIANO $(v_1, v_2) \Longleftrightarrow \underline{v}, \underline{v_1}, \underline{v_2}$ SONO COMPLANARI

* $\underline{v_1} = (1,-1,2)$ $\quad \underline{v_2} = (3,0,1)$ $\qquad V = \mathbb{R}^3$

$\underline{u} = (-5,-1,0) = -5\underline{i} - \underline{j}$

$-5\underline{i} - \underline{j} = a(\underline{i} - \underline{j} - 2\underline{k}) + b(3\underline{i} + 0\underline{j} + \underline{k})$

$$\begin{cases} -5 = a + 3b \rightarrow b = -2 \\ -1 = -1 \cdot a \rightarrow a = 1 \\ 2a + b = 0 \rightarrow 0 = 0 \rightarrow \text{OK} \rightarrow \underline{u} \text{ C.L. DI } (\underline{v_1}, \underline{v_2}) \rightarrow \underline{u} = \underline{v_1} - 2\,\underline{v_2} \end{cases}$$

$\underline{u} \in \mathcal{L}(\underline{v_1}, \underline{v_2})$

POSSO FARE PRODOTTO MISTO $\leftarrow$ $\underline{u}$ È COMPLANARE A $\underline{v_1}, \underline{v_2}$

↳ PERÒ NON TROVO a,b

$$\begin{vmatrix} -5 & -1 & 0 \\ 1 & -1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = -5(-1) + 1(1-6) = 0 \rightarrow \text{OK}$$

TEOREMA

$U = \mathcal{L}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n) \to$ INSIEME C.L.

È UN SOTTOSPAZIO DI $V \to$ C.L. RIMANGONO NEL PIANO INDIVIDUATO

U È IL + PICCOLO S.S. DI V CONTENENTE $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$

$\Updownarrow$

$\forall W$ S.S. DI V CHE CONTIENE $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n \Rightarrow U \subseteq W$

INSIEME DI GENERATORI

* $V_3 = \mathcal{L}(\underline{i}, \underline{j}, \underline{k})$
* $\mathbb{R}^3 = \mathcal{L}(\underline{e}_1, \underline{e}_2, \underline{e}_3) = \mathcal{L}((1,0,0),(0,1,0),(0,0,1))$ VERS. FONDAM.
* $\mathbb{R}^n = \mathcal{L}(\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n) = \mathcal{L}((1,0,\dots,0),(0,1,\dots,0),(0,0,\dots,1))$
* $\mathbb{R}_2[x] = \mathcal{L}(1 + x + x^2)$
* $\mathbb{R}^{2,3}$ $\left\{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\right\} = \mathcal{L}(M_1, M_2, \dots, M_6)$

$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad M_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

SIA V UN $\mathbb{K}$-SPAZIO VETT., SE $\exists (v_1, v_2, \dots, v_n) \in V$ CHE GENERANO V SI DICE CHE V È FINITAMENTE GENERATO

INSIEMI LIBERI

INDIPENDENZA LINEARE

$v_1, v_2, \dots, v_n$ SI DICONO LINEARM. INDIPENDENTI SE

$a_1 v_1 + \dots + a_n v_n = 0 \Leftrightarrow a_1 = \dots = a_n = 0$

SE NON SONO L.I. DICIAMO CHE SONO LINERM. DIPENDENTI

PROP: $v_1, v_2, \dots, v_n$ SONO L.D SE QUALUNQUE DI ESSI È ESPRIMIBILE COME C.L. DEI RIMANENTI

BASI

INSIEME GENERATORI L.I.

$\mathcal{B} = (v_1, v_2, \dots, v_n)$ È BASE DI $V \iff V$ SI PUÒ ESPRIMERE COME C.L. DI ELEMENTI DI $\mathcal{B}$ IN MODO UNICO

- $\mathcal{B}$ È ORDINATO
- $\mathcal{B}$ È LIBERO
- $\mathcal{B}$ GENERA V

ESEMPI

* $V_3 \to \mathcal{B} = (\underline{i}, \underline{j}, \underline{k})$ BASE STANDARD

* $\mathbb{R}^n \to E = (e_1, e_2, e_3, \dots, e_n)$ BASE CANONICA DI $\mathbb{R}^n$

* $\mathbb{R}_2[x] \to \mathcal{B} = (1, x, x^2)$ BASE STANDARD

COMPONENTI RISPETTO A UNA BASE

SE $(\underline{v_1}, \underline{v_2}, \dots, \underline{v_n})$ È UNA BASE DI V E $\underline{v} \in V$, $\underline{v} = a\,\underline{v_1} + b\,\underline{v_2} + \dots + c\,\underline{v_3}$

I COMPONENTI SONO $a, b, \dots, c$ PRESI NELL'ORDINE

* $\mathbb{R}_3[x] \to \mathcal{B} = (1, x, x^2, x^3)$

$p(x) = 2x - 3x^2$

$\text{COMP}_{\mathcal{B}}(p) = (0, 2, -3, 0)$

* $\mathbb{R}^2$ $\underline{v} = (-2, 4)$ $E(\underline{e_1}, \underline{e_2})$

$\text{COMP}_E(\underline{v}) = (-2, 4)$

$(-2, 4) = -2\,\underline{e_1} + 4\,\underline{e_1}$

ES

* $\mathcal{B} = (\underline{e_1}, \underline{e_2})$ $\underline{e_1} = (1,1)$ $\underline{e_2} = (5,-1)$ BASE DI $\mathbb{R}^2$

1) SOLO L.I. $\iff \nexists\, K \in \mathbb{R} \,/\, \underline{e_1} = K \cdot \underline{e_2} \to$ OK

2) GENERANO $\mathbb{R}^2 \iff \forall\, \underline{v} \in \mathbb{R}^2\ \exists\, \alpha, \beta \in \mathbb{R} \,/\, \underline{v} = \alpha\,\underline{e_1} + \beta\,\underline{e_2}$

$\forall\, a, b \in \mathbb{R}, \exists\, \alpha, \beta \in \mathbb{R} \,/\, (a, b) = \alpha(1,1) + \beta(5,-1)$

ESEMPIO

$\mathbb{R}^3$ $\quad \underline{v}_1=(1,2,1) \quad \underline{v}_2=(0,1,2) \quad \underline{v}_1,\underline{v}_2$ L.I. $\Rightarrow I=(v_1,v_2)$ LIBERO

PRENDIAMO LA BASE CANONICA $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$

$G=(\underline{v}_1,\underline{v}_2,\underline{e}_1,\underline{e}_2,\underline{e}_3)$

$=((1,2,1),(0,1,2),(1,0,0),(0,1,0),(0,0,1))$

APPLICHIAMO REGOLA DEGLI SCARTI:

$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \neq 0 \to$ NON SONO COMPL. $\to e_1$ L.I. $\to$ LO TENIAMO

$(0,1,0)=a(1,2,1)+b(0,1,2)+c(1,0,0)$

$\begin{cases} 0=a+c \to c=-2/3 \\ 1=2a+b \to 1=-4b+b \Rightarrow b=-1/3 \\ 0=a+2b \to a=2/3 \end{cases}$ $\quad \underline{e}_2 \in \mathcal{L}(v_1,v_2,e_1)$

$\underline{e}_2$ L.D. $\to$ LO ESCLUDIAMO

$(0,0,1)=a(1,2,1)+b(0,1,2)+c(1,0,0)$

$\begin{cases} 0=a+c \to c=-a=1/3 \\ 0=2a+b \to b=-2a \to -2\cdot -1/3=b=2/3 \\ 1=a+2b \to 1=a-2(2a)=-3a \to a=-1/3 \end{cases}$ $\quad \underline{e}_3$ L.D. $\to$ LO ESCLUDIAMO

$B=(\underline{v}_1,\underline{v}_2,\underline{e}_1)=((1,2,1),(0,1,2),(1,0,0))$ BASE DI $\mathbb{R}^3$

PROP: SIA V UN K-SPAZIO VETT. FINITAMENTE GENERATO, OGNI INSIEME LIBERO DI ELEMENTI DI V È CONTENUTO IN UNA BASE

MATRICI

$\mathbb{R}^{m,n}$ $\quad m$ RIGHE $\quad n$ COLONNE

* MATRICE RIGA $\to m=1$
* MATRICE COLONNA $\to n=1$
* MATRICE QUADRATA $\to m=n$

TRASPOSTA

$A \in K^{m,n} \quad {}^tA \in K^{n,m}$ $\quad$ SCAMBIO RIGHE/COLONNE

METODO DEGLI SCARTI

$R_A = \mathcal{L}(R_1, R_2, R_3) \subseteq \mathbb{R}^4$ → MAX RANGO CERCHEREI 3

$\vec{R}_1 = (1, 3, -1, 0)$

$\vec{R}_2 = (2, 5, 7, 1)$ $\qquad g = \{\vec{R}_1, \vec{R}_2, \vec{R}_3\}$ GENERANO R_A

$\vec{R}_3 = (5, 13, 13, 2)$ $\qquad \vec{R}_1 \neq \vec{0}, \vec{R}_2 \neq \vec{0}, \vec{R}_3 \neq \vec{0}$

• $\vec{R}_2 \neq k\vec{R}_1$ → OK

• $\vec{R}_3 = a\vec{R}_1 + b\vec{R}_2$? $(\exists ? \; a, b \in \mathbb{R})$

$$\Longleftrightarrow \begin{cases} a + 2b = 5 \rightarrow a = 1 \\ 3a + 5b = 13 \rightarrow 13 = 13 \text{ OK} \\ -a + 7b = 13 \rightarrow 13 = 13 \text{ OK} \\ b = 2 \end{cases}$$

$\vec{R}_3 = \vec{R}_1 + 2\vec{R}_2$ → $\vec{R}_3$ ... → $\vec{R}_3 \in \mathcal{L}(\vec{R}_1, \vec{R}_2)$ SI SCARTA

BASE DI $R_A = \{\vec{R}_1, \vec{R}_2\}$ → DIM $R_A = 2$

SE PROVO CON LE COLONNE:

• $\vec{C}_2 \neq k\vec{C}_1$

• $\vec{C}_3 \stackrel{?}{=} a\vec{C}_1 + b\vec{C}_2$

$$\begin{cases} a + 3b = -1 \rightarrow a = 26 \\ 2a + 5b = 7 \rightarrow b = -9 \\ 5a + 13b = 13 \rightarrow 13 = 13 \text{ OK} \end{cases} \rightarrow \vec{C}_3 \in \mathcal{L}(\vec{C}_1, \vec{C}_2) \text{ LO SCARTO}$$

• ALLO STESSO MODO $\vec{C}_4 \in \mathcal{L}(\vec{C}_1, \vec{C}_2)$ → $\vec{C}_4 = 3\vec{C}_1 - \vec{C}_2$ SCARTO

DIM $C_A = 2$ (= DIM R_A)

MATRICI RIDOTTE

UNA MATRICE È RIDOTTA PER RIGHE SE $\forall$ RIGA NON NULLA $\exists$ ALMENO UN ELEM. NON NULLO (ELEMENTO SPECIALE) AL DI SOTTO DEL QUALE CI SONO SOLO ZERI

RIDOTTA PER COLONNA SE $\forall$ COLONNA... ALLA DESTRA DEL QUALE...

$* B = \begin{pmatrix} 1 & 2 & 5 & 1 \\ 2 & 2 & 8 & 5 \end{pmatrix}$ NON r.r. NE' r.c.
CERCO DI r.r.

$$B = \begin{pmatrix} 1 & 2^* & 5 & 1 \\ 2 & \boxed{2} & 8 & 3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} B_1^{(r)} = \begin{pmatrix} 1 & 2^* & 5 & 1 \\ 1 & \boxed{0} & 3 & 2 \end{pmatrix} \text{ r.r.}$$

$$R_B = \mathcal{L}(R_1, R_2) = \mathcal{L}(R_1, R_2 - R_1) \quad \rho(B^{(r)}) = \rho(B) = 2$$

$$* A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \text{ NON r.r } \xrightarrow{R_4 \leftrightarrow R_1} \begin{pmatrix} 2 & 2 & 4 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_4}$$

$$\begin{pmatrix} 2^* & 2 & 4 & 2 \\ \boxed{0} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_4 \to R_4 - \frac{1}{2}R_1} \begin{pmatrix} 2^* & 2 & 4 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_2}$$

$$\begin{pmatrix} 2^* & 2 & 4 & 2 \\ 0 & 1^* & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & \boxed{0} & 0 & 2 \end{pmatrix} \text{ r.r.} = A^{(r)} \to \dim A^{(r)} = \dim A = \rho(A) = 3$$

BASE DI $R_A = \{E_1, E_2, E_3\}$
$E_1 = (2, 2, 4, 2)$
$E_2 = (0, 1, 1, 0)$
$E_3 = (0, 0, 0, 2)$

$* \mathbb{R}^3$

$\underline{v_1} = (1, 1, 2)$ $\underline{v_2} = (1, 0, [illegible])$ $\underline{v_3} = (2, 1, -1)$ $\underline{v_4} = (1, 0, 0)$
$W = \mathcal{L}(\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4})$ DIM W = ? BASE W = ?

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & [illegible] & -1 \\ 2 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1^* & 2 \\ 1 & \boxed{0} & - \\ 1 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to 2R_3 + 2R_2} \begin{pmatrix} 1 & 1^* & 2 \\ 1 & \boxed{0} & 1^* \\ 0 & 0 & \boxed{0} \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_4}$$

$W = \langle M, M_1, M_2 \rangle \subseteq \mathbb{R}^{3,2}$

Uso base standard

$E = \{E_{11}, E_{12}, E_{21}, E_{22}, E_{31}, E_{32}\}$ $\quad E_{11} = $ [illegible] ...

$M_1 = $ [illegible] $E_{11} + 1 E_{12} + 2 E_{21} + 1 E_{22} + 0 E_{31} + 1 E_{32}$

$\rightarrow$ ([illegible], 1, 2, 1, 0, 1)

A matrice che ha per righe M, M_1, M_2 in coordinate.

$A = $ [illegible] $\xrightarrow{R_3 \to R_3 - R_2}$ [illegible]

$\forall h \in \mathbb{R}$ dim R_A = dim W = [illegible]

Base di $W = \{M, M_1, M_2\}$

* $V = \mathbb{R}_3[x]$

$f_1(x) = $ [illegible] $\quad f_2(x) = $ [illegible] $\quad f_3(x) = $ [illegible] $+ x^3 \quad f_4(x) = $ [illegible]

$W = \langle f_1, f_2, f_3, f_4 \rangle$

Uso base $E = \{1, x, x^2, x^3\}$

A matrice che ha per righe le coordinate di f_1, f_2, f_3, f_4 nella base [illegible]

$A = $ [illegible] $\xrightarrow{R_1 \leftrightarrow R_2}$ [illegible]

Se $h \neq 0 \wedge h \neq 1 \rightarrow A'' \rightarrow$ dim W = 4 $\rightarrow$ base di $W = \{f_1, f_2, f_3, f_4\}$

Se $h = 0$

$A = $ [illegible] $\xrightarrow{R_1 \leftrightarrow R_2 \quad R_4 \to R_4 - R_2}$ $A'' =$ [illegible]

dim W = 3 base di W = [illegible]

[illegible] $x^2, x, x + x^3$ [illegible]

$$B=\begin{pmatrix} \boxed{\begin{matrix}1\\0\\1\end{matrix}} & \begin{matrix}2\\1\\3\end{matrix} & \begin{matrix}4\\4\\4\end{matrix} & \begin{matrix}-1\\0\\0\end{matrix}\end{pmatrix} \qquad C=AB=\begin{pmatrix} 4 & 13 & 24 & -1 \\ -1 & -1 & 0 & 1 \end{pmatrix}$$

$$c_{11}=1\cdot 1+0\cdot 2+3\cdot 1=4 \qquad C\in\mathbb{R}^{2,4}$$

SISTEMI LINEARI

$$\begin{cases} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2 \\ \dots \qquad \dots \qquad \dots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=b_m \end{cases}$$

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$ MATRICE COEFFICIENTI

$$B=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$ MATRICE COLONNA TERMINI NOTI

$$X=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$ MATRICE COLONNA INCOGNITE

$$A\cdot X=B$$

$$\begin{pmatrix} \boxed{a_{11} \; a_{12} \dots a_{1n}} \\ a_{21} \; a_{22} \dots a_{2n} \\ \dots \quad \dots \quad \dots \\ a_{m1} \; a_{m2} \; a_{mn} \end{pmatrix} \boxed{\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

$$\begin{pmatrix} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n \\ \dots \quad \dots \quad \dots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$

$$* \begin{cases} 3x-y=2 \\ x-3y=5 \\ y=1 \end{cases} \qquad A\in\mathbb{R}^{2,3} \qquad A=\begin{pmatrix} 3 & -1 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \qquad X=\begin{pmatrix} x \\ y \end{pmatrix} \qquad B=\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$AX=B \qquad (3,2)\cdot(2,1)$$

RIDUZIONE DI UN SISTEMA

TRASFORMAZ. COMPLETA DI $AX=B$, CHE DIVENTA $A'X=B'$ (EQUIVALENTE)

* $\begin{cases} x+y=1 \\ x-y=1 \end{cases}$ $(A|B)=\begin{pmatrix} 1 & 1 & | & 1 \\ 1 & -1 & | & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2+R_1} \begin{pmatrix} 1 & 1 & | & 1 \\ 2 & 0 & | & 2 \end{pmatrix}$ RIDOTTA $A'X=B'$

$\exists!$ SOLUZ: $(1,0) \leftarrow \begin{cases} x+y=1 \to y=1-1=0 \\ 2x=2 \to x=1 \end{cases}$

* $\begin{cases} 2x+2y=2 \\ 3x-3y=3 \\ 3x+y=3 \end{cases}$ $(A|B)=\begin{pmatrix} 2 & 2 & | & 2 \\ 3 & -3 & | & 3 \\ 3 & 1 & | & 3 \end{pmatrix} \xrightarrow[R_2 \to 2R_2+3R_1]{R_3 \to 2R_3-R_1}$

$(A'|B')=\begin{pmatrix} 2 & 2 & | & 2 \\ 12 & 0 & | & 12 \\ 4 & 0 & | & 4 \end{pmatrix} \xrightarrow{R_2 \to R_2:3-R_2} \begin{pmatrix} 2 & 2 & | & 2 \\ 0 & 0 & | & 0 \\ 4 & 0 & | & 4 \end{pmatrix}$ SI PUÒ TOGLIERE PERCHÉ TUTTI 0 (ANCHE IN B)

$A'X=B'$: $\begin{cases} 2x+2y=2 \to y=0 \\ 4x=4 \to x=1 \end{cases}$ $\begin{cases} x=1 \\ y=0 \end{cases}$ → RANGO 2

* $\begin{cases} 2x-y=2 \\ x-y=-1 \\ ax+2y=11 \quad a \in \mathbb{R} \end{cases}$ $(A|B)=\begin{pmatrix} 2 & -1 & | & 2 \\ 1 & -1 & | & -1 \\ a & 2 & | & 11 \end{pmatrix} \xrightarrow[R_3 \to R_3+2R_1]{R_2 \to R_2-R_1}$

$(A'|B')=\begin{pmatrix} 2 & -1 & | & 2 \\ -1 & \boxed{0} & | & -3 \\ a+4 & 0 & | & 15 \end{pmatrix} \xrightarrow{R_3 \to R_3+(a+4)R_2} \begin{pmatrix} 2 & -1 & | & 2 \\ -1 & 0 & | & -3 \\ 0 & 0 & | & 15-3(a+4) \end{pmatrix}$ $= 3-3a$

SE $a \neq 1$ SIST. INCOMPATIBILE

SE $a=1$ $\begin{pmatrix} 2 & -1 & | & 2 \\ -1 & 0 & | & -3 \end{pmatrix}$ $\begin{cases} 2x-y=2 \to y=4 \\ -x=-3 \to x=3 \end{cases} \Rightarrow \exists!$ SOLUZ. $(3,4)$

IN $\mathbb{R}^2$ (PIANO), 3 RETTE PASSANTI X UN PUNTO (FASCIO DI RETTE)

SE $a=1 \to \rho(A)=2=\rho(A|B)$

SE $a \neq 1 \to \rho(A)=2 \quad \rho(A|B)=3$ NO SOLUZIONE

1) INS. SOLUZ. $S \in \mathbb{K}^n$

2) SE $\rho = \rho(A) \to$ LA DIM. DELLE SOLUZ. È $\dim(S) = n - \rho$ (NUMERO INC. LIBERE)

3) SI PUÒ TROVARE UNA BASE DI S DANDO ALLE I.L. I VALORI DELLE $(n-\rho)$-UPLE $(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,0,\dots,1)$

IL SIST HA SOLO 1 SOLUZ. BANALE $\Longleftrightarrow \rho(A) = n$

$\hookrightarrow A \in \mathbb{K}^{m,n} \Rightarrow m = n$

$$* \begin{cases} x + 3y + 2z = 0 \\ 2x - y + z = 0 \\ 3x + 2y + 3z = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 1 \\ 3 & 2 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 2R_2]{R_2 \to 3R_2 + R_1} \begin{pmatrix} 1 & 3 & 2 \\ 7 & 0 & 5 \\ 7 & 0 & 5 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 3 & 2 \\ 7 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rho(A) = 2 < 3$ $\quad \dim S = 3 - 2 = 1 \to \infty^1$ SOLUZ.

$\hookrightarrow$ SPAZIO SOLUZ. $\to$ S.S. DI $\mathbb{R}^3$

SIST. RIDOTTO EQ.

$$\begin{cases} x + 3y + 2z = 0 \\ 7x + 5z = 0 \end{cases}$$

z LIBERA $\to$ POICHÉ ALTRE COLONNE L.I.

$$\begin{cases} x = -\frac{5}{7} z \\ -\frac{5}{7} z + 3y + 2z = 0 \to y = -\frac{3}{7} z \end{cases}$$

$$S = \left\{ \left(-\frac{5}{7} z, -\frac{3}{7} z, z \right), z \in \mathbb{R} \right\}$$

BASE DI S: $\mathcal{B} = (\underline{b})$ $\quad \underline{b} = \left(-\frac{5}{7}, -\frac{3}{7}, 1 \right) = (-5, -3, 7)$

INTERPRETAZ. GEOMETRICA:

π_1, π_2, π_3 FORMANO UNA RETTA

$$r = \begin{cases} x = -5\tau \\ y = -3\tau \\ z = 7\tau \end{cases}$$

* SI CONSIDERANO I S.S. $U, V \in \mathbb{R}^3$

$U = \mathcal{L}(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ $\quad V = \{(x,y,z,t) \in \mathbb{R}^4\}$

$$\begin{cases} \underline{u}_1 = (1,1,-1,2) \\ \underline{u}_2 = (-1,0,0,1) \\ \underline{u}_3 = (2,1,-1,1) \end{cases} \qquad \begin{cases} x+y+z=0 \\ x+2y-z-t=0 \end{cases}$$

a) DIM / BASI U
b) DIM / BASI V
c) DIM / BASI $U \cap V$
d) DIM / BASI $U+V$

a) $$\begin{pmatrix} 1 & 1 & -1 & 2 \\ -1 & 0 & 0 & 1 \\ 2 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 2 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 2 \\ -1 & 0 & 0 & 1 \end{pmatrix} \to \text{DIM } U = 2$$

$\mathcal{B}_U = (\underline{u}_1, \underline{u}_2) = (\underline{e}_1, \underline{e}_2)$

b) $$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \rho = 2 \to \text{DIM } V = 2$$
$\hookrightarrow 2$ I.L. $\to x, y$

$$\begin{cases} z = -x-y \\ t = 2x+3y \end{cases} \qquad (x, y, -x-y, 2x+3y)$$

$\mathcal{B}_V = (\underline{e}_3, \underline{e}_4)$

$x=1, y=0 \quad \underline{e}_3 = (1,0,-1,2)$
$y=1, x=0 \quad \underline{e}_4 = (0,1,-1,3)$

d) GENERATORI : $S = (\underbrace{\underline{e}_1, \underline{e}_2}_{\mathcal{B}_U}, \underbrace{\underline{e}_3, \underline{e}_4}_{\mathcal{B}_V})$

$$\begin{array}{c} \begin{pmatrix} 1 & 1 & -1 & 2 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{pmatrix} \begin{matrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \\ \underline{e}_4 \end{matrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_4 \to R_4 - R_1 - R_2} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 0 & -1 & 2 \\ -1 & 0 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \begin{matrix} \underline{b}_1 \\ \underline{b}_3 \\ \underline{b}_2 \\ \underline{b}_4 - \underline{e}_2 - \underline{e}_1 = 0 \end{matrix} \end{array}$$

$\rho = 3$
DIM $(U+V) = 3$

BASE DI $U+V = (\underline{e}_1, \underline{e}_3, \underline{e}_2)$

c) RELAZ. DI KRASMANN $\to$ DIM $(U+V) =$ DIM $U +$ DIM $V -$ DIM $(U \cap V)$

SIST. A INCOGNITE VETTORIALI

$$\begin{cases} a_{11}\vec{X_1}+a_{12}\vec{X_2}+\dots+a_{1n}\vec{X_n}=\vec{B_1} \\ a_{21}\vec{X_1}+a_{22}\vec{X_2}+\dots+a_{2n}\vec{X_n}=\vec{B_2} \\ \dots \\ a_{m1}\vec{X_1}+a_{m2}\vec{X_2}+\dots+a_{mn}\vec{X_n}=\vec{B_m} \end{cases} \qquad A\in \mathbb{K}^{m,n}$$

$$\begin{aligned} \vec{X_1}&=(x_{11},x_{12},x_{13},\dots,x_{1p}) \\ \vec{X_2}&=(x_{21},x_{22},x_{23},\dots,x_{2p}) \\ \vec{X_n}&=(x_{n1},x_{n2},x_{n3},\dots,x_{np}) \end{aligned} \qquad X\in \mathbb{K}^{n,p}$$

$$\begin{aligned} \vec{B_1}&=(b_{11},b_{12},\dots,b_{1p}) \\ \vec{B_2}&=(b_{21},b_{22},\dots,b_{2p}) \\ \vec{B_m}&=(b_{m1},b_{m2},\dots,b_{mp}) \end{aligned} \qquad B\in \mathbb{K}^{m,p}$$

$$\star\begin{cases} \vec{X_1}+2\vec{X_2}=(2,1,3) \\ 3\vec{X_1}+2\vec{X_2}=(4,7,1) \end{cases} \qquad \begin{aligned}\vec{B_1}&=(2,1,3)\\ \vec{B_2}&=(4,7,1)\end{aligned} \qquad \begin{aligned}\vec{x_1}&=(a,b,c)\\ x_2&=(a',b',c')\end{aligned}$$

$$(a,b,c)+2(a',b',c')=(2,1,3)$$
$$3(a,b,c)+2(a',b',c')=(4,7,1)$$

$$\begin{cases} (a+2a',b+2b',c+2c')=(2,1,3) \\ (3a+2a',3b+2b',3c+2c')=(4,7,1) \end{cases}$$

$$\begin{cases} a+2a'=2 \\ b+2b'=1 \\ c+2c'=3 \\ 3a+2a'=4 \\ 3b+2b'=7 \\ 3c+2c'=1 \end{cases}$$

$$AX=B \qquad \underset{(2,2)}{A}=\begin{pmatrix}1&2\\3&2\end{pmatrix} \qquad \underset{(2,3)}{X}=\begin{pmatrix}a&b&c\\a'&b'&c'\end{pmatrix} \qquad B=\begin{pmatrix}2&1&3\\4&7&1\end{pmatrix}$$

$$AX_{(2,3)}=\begin{pmatrix}1&2\\3&2\end{pmatrix}\begin{pmatrix}a&b&c\\a'&b'&c'\end{pmatrix}=\begin{pmatrix}a+2a'&b+2b'&c+2c'\\3a+2a'&3b+2b'&3c+2c'\end{pmatrix}=\begin{pmatrix}2&1&3\\4&7&1\end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ (TUTTI 0, CON 1 SULLA DIAGONALE)

$AX = I \quad \forall A \in \mathbb{R}^{n,n} \qquad AI = A$

$IA = A$

$xa = 1 \rightarrow x = a^{-1}$

$\forall a \in \mathbb{R}, a \neq 0 \qquad AX = I \Rightarrow X = A^{-1} \qquad I \in \mathbb{R}^{n,n}$

IDENTICA

$\forall A \in \mathbb{R}^{n,n}, |A| \neq 0 \Leftrightarrow \rho(A) = n$

$AX = I \Leftrightarrow \exists \ X = A^{-1}$

$$\star A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \exists A^{-1} ?$$

$$AX = I \longrightarrow I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad \rho(A) = \rho(A|B) = 3$$

$$\begin{cases} \vec{X_2} = (0,0,1) \\ 2\vec{X_1} = (0,1,0) - (0,0,1) \rightarrow \vec{X_1} = (0, \frac{1}{2}, -\frac{1}{2}) \\ \vec{X_3} = \vec{X_1} + 3\vec{X_2} - (1,0,0) \longrightarrow \vec{X_3} = (-1, \frac{1}{2}, \frac{5}{2}) \end{cases}$$

$$X = A^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ -1 & \frac{1}{2} & \frac{5}{2} \end{pmatrix}$$

MATRICE INVERSA DI $A \qquad A \cdot A^{-1} = I = A^{-1} \cdot A$

$$\star A = \begin{pmatrix} 1 & 0 & h \\ 0 & -1 & 0 \\ 0 & 0 & K \end{pmatrix} \quad h, K \in \mathbb{R} \qquad \text{DISCUTERE } \exists A^{-1}$$

SE QUADRATA:

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 4 & 5 \end{pmatrix} \qquad A_{2,3}=(-1)^{2+3}\left|\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\right|$$

SE SOMMA DISPARI - / PARI +

COMPLEMENTO ALGEBRICO

$$A_{3,1}=(-1)^{3+1}\left|\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\right| \qquad A_{R,K} \text{ DI } a_{R,K}$$

TEOREMA DI LAPLACE SE $A \in \mathbb{K}^{n,n}$

$$|A| = a_{R1}A_{R1}+a_{R2}A_{R2}+\dots+a_{Rn}A_{Rn}$$
$$= a_{1K}A_{1K}+a_{2K}A_{2K}+\dots+a_{nK}A_{nK}$$

$$\star A=\begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 2 \\ \boxed{0 \quad 0 \quad 3} \end{pmatrix}$$

POSSO SCEGLIERE RIGA/COLONNA + COMODA (CON + ZERI

$$\det A = 0\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 3(-1-4) = -15$$

$$\det A = 1(-1)^{1+1}\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} + \underbrace{2(-1)^{2+1}}_{-2}\begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} + 0 = -15$$

PER MATRICI GRANDI APPLICO IN MODO RICORSIVO

$$\star A=\begin{pmatrix} \boxed{1 \quad 0 \quad -1 \quad 0} \\ 2 \quad 1 \quad 0 \quad 1 \\ 3 \quad 2 \quad 2 \quad 1 \\ 1 \quad 5 \quad 0 \quad 2 \end{pmatrix} = 1\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 5 & 0 & 2 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 5 & 2 \end{vmatrix} - 0 =$$

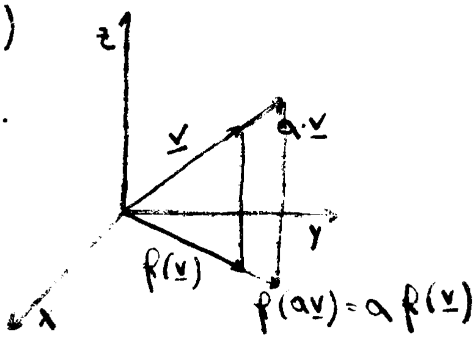
$$= \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 0 + \begin{vmatrix} 2 & 2 \\ 5 & 0 \end{vmatrix} - 1\left(2\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}\right) =$$

$$(4-1)+(0-10)-1\left(2(4-5)-(6-1)+(15-2)\right) = 3-10-(-2-5+13) = -15$$

1) $f(\underline{v}+\underline{w}) = f((x\underline{i}+y\underline{j}+z\underline{k})+(x'\underline{i}+y'\underline{j}+z'\underline{k})) =$

$= f((x+x')\underline{i}+(y+y')\underline{j}+(z+z')\underline{k}) := (x+x')\underline{i}+(y+y')\underline{j} =$

$(x\underline{i}+y\underline{j})+(x'\underline{i}+y'\underline{j}) := f(\underline{v})+f(\underline{w})$



$f: V_3 \to V_3 \quad f(x,y,z)=(x,y,0)$ È LIN.

$f: V_3 \to V_2$ È LIN. $f(x,y,z)=f(x,y)$

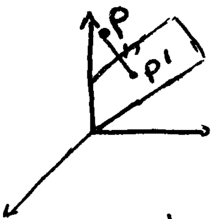
* $f: \mathbb{K}^n \to \mathbb{K}^m$

SONO FUNZ. LIN. SOLO LE FUNZ

$f(x_1, x_2, \ldots, x_n) = f(y_1, y_2, \ldots, y_m)$

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ y_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \ldots \\ y_m = a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{cases}$$

* $f: \mathbb{R}^3 \to \mathbb{R}^3 \quad P \xrightarrow{f}$ PROIEZ. DI P SUL PIANO $\pi: x-2y+2z=0$



n: RETTA $\perp \pi$ PASSANTE PER $P(a,b,c)$

TROVARE L'EQ DI f

$$n: \begin{cases} x = a+t \\ y = b-2t \\ z = c+2t \end{cases}$$

$P': \pi \cap n \quad (a+t)-2(b-2t)+2(c+2t)=0$

$$\frac{\cancel{9}t}{\cancel{9}} = \frac{-a+2b-2c}{9}$$

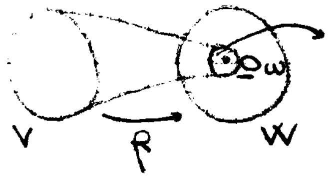
$$P' = \left(\frac{8a+2b-2c}{9}, \frac{2a+5b+4c}{9}, \frac{-2a+4b+5c}{9}\right)$$

$$(1,1,2) \xrightarrow{f} \left(\frac{2}{9}, \frac{5}{9}, \frac{4}{9}\right)$$

id_V APPL. LIN. IDENTICA $V \to V$

$id_V(v) = v \quad \forall v \in V$

IMMAGINE DI f



$\mathrm{Im} f \subseteq W$

$\underline{0}_W \in \mathrm{Im} f$

$\mathrm{Im} f : \{ \underline{w} \in W \setminus \exists\, \underline{v} \in V \Rightarrow f(\underline{v}) = \underline{w} \}$

1) $\mathrm{Im} f$ È UN SOTTOSPAZIO

2) $B = (e_1, \dots, e_n)$ È UNA BASE DI V, ALLORA $f(B)$ È UNA BASE DI $\mathrm{Im} f$

1) $\forall\, \underline{w}_1, \underline{w}_2 \in \mathrm{Im} \quad \forall\, a \in K$

$\underline{w}_1 + \underline{w}_2 = f(\underline{v}_1) + f(\underline{v}_2) = f(\underline{v}_1 + \underline{v}_2) \Rightarrow w_1 + w_2 \in \mathrm{Im}$

$a f(v_1) = a\, \underline{w}_1 = f(a\, \underline{v}_1) \Rightarrow a\, \underline{w}_1 \in \mathrm{Im}$

3) SE $G = (\underline{v}_1, \dots, \underline{v}_n)$ È INS. DI GENERATORI DI V ALLORA $(f(\underline{v}_1), \dots, f(\underline{v}_n))$ È UN INSIEME DI GEN. DI $\mathrm{Im} f$

$\forall\, \underline{w} \in W \quad \underline{w} \in \mathrm{Im} f$

$\exists\, x_1, x_2, \dots, x_n \in \mathbb{K} : \ f(\underline{v}) = \underline{w}$

$\underline{v} = x_1 \underline{v}_1 + x_2 \underline{v}_2 + \dots + x_n \underline{v}_n$

$\underline{w} = f(\underline{v}) = f(x_1 \underline{v}_1 + \dots + x_n \underline{v}_n) = x_1 f(\underline{v}_1) + \dots + x_n f(\underline{v}_n)$

$w \in \mathcal{L}(f(\underline{v}_1), \dots, f(\underline{v}_n))$

OPERAZIONI CON A.L.

$(af) : V \to W \quad (af)(\underline{v}) = f(a\underline{v})$

$(f+g) : V \to W \quad (f+g)(\underline{v}) = f(\underline{v}) + g(\underline{v})$

SONO LINEARI SE f LINEARE (E g)

IN ANALISI I SONO LINEARI SOLO LE RETTE PASSANTI X L'ORIG.

$f(x) = ax$

* $V = \mathcal{C}^{(1)}(\mathbb{R})$ $f: \mathbb{R} \to \mathbb{R}$ DERIVABILI CON f' CONTINUA

$f+g$
$af, a \in \mathbb{R}$ $\{V, +, \cdot\}$ È UN $\mathbb{K}$-SP. VETT

$D: V \to V$ $D(f) = D'$ D È LINEARE $\begin{cases} D(f+g) = D(f) + D(g) \\ D(af) = a \cdot D(f) \end{cases}$

$\mathrm{Ker}\, D = \{$TUTTE LE FUNZ. COSTANTI $\mathbb{R} \to \mathbb{R}\}$

$\mathrm{Ker}\, D \neq \{\underbrace{\theta}_{\text{FUNZ. NULLA}}\} \Rightarrow$ NON È INIETTIVA

$\mathrm{Im}\, D = \{f \in V : \exists\, g \in V : D(g) = f,\ g' = f\}$

DOBBIAMO VEDERE SE È INTEGRABILE

$\{f$ INTEGRABILI SU $\mathbb{R}\}$ $\to$ f CONTINUE

$\mathrm{Im}\, D = V \Rightarrow D$ SURIETTIVA

* $V = \mathcal{C}^{(0)}([a,b])$ $I: V \to \mathbb{R}$ $I: f \to \int_a^b f(x)\,dx$

I È UN A.L.

NON È INIETTIVA

$\mathrm{Ker}\, I = \{$TUTTE LE FUNZ. CON PARTE POSIT = PARTE NEG.$\} \neq \{\theta\}$

$\mathrm{Im}\, I = \{$QUALUNQUE NUM (BASTA PRENDERE f COST + NUM)$\} =$
$= \mathbb{R} \Rightarrow$ SURIETTIVA

A.L. E DIPENDENZA LINEARE

1) $f(v_1), f(v_2)$ L.I. $\Rightarrow v_1, v_2, \dots$ L.I.

2) f INIETTIVA E $v_1, v_2, \dots$ L.I. $\Rightarrow f(v_1), f(v_2), \dots$ L.I.

3) $V = \mathcal{L}(v_1, v_2, \dots) \Rightarrow \mathrm{Im}\, f = \mathcal{L}(f(v_1), f(v_2), \dots)$

4) f SURIETTIVA E $V = \mathcal{L}(v_1, v_2, \dots) \Rightarrow W = \mathcal{L}(f(v_1), f(v_2), \dots)$

SE f È ISOMORFA $\to$ BIETTIVA $\to$ INIETTIVA $\wedge$ SURIETTIVA

I1) $v_1, v_2, \dots \Longleftrightarrow f(v_1), f(v_2), \dots$

1) φ È LINEARE

2) φ INIETTINA $\Leftrightarrow (v_1, v_2, \dots, v_n)$ SONO L.I.

$\Updownarrow$

$\mathrm{Ker}\,\varphi(\underline{0})$ $\quad \varphi(x_1, \dots, x_n) = \underline{0}_V \Leftrightarrow (x_1, \dots, x_n) = (0, \dots, 0) \Leftrightarrow v_1, \dots, v_n$ L.I.

3) φ SURIETTIVA $\Leftrightarrow (v_1, v_2, \dots, v_n)$ GENERANO V

4) φ ISOMORFISMO $\Leftrightarrow (v_1, v_2, \dots, v_n)$ SONO UNA BASE DI V

DATO UN $\mathbb{K}$-SP. VETT. UNA SUA BASE È UN ISOMORFISMO $\varphi: V \to K^n$

I RISULTATI PRECEDENTI VALGONO SE SP. VETT. FINITAM. GENERATI

$\Updownarrow$ VS

$\exists$ SP. VETT. NON FINIT. GENERATI

ES: SE DEG $p(x) >$ MAX DEG (g_i)

$\mathbb{R}[x]$ $\quad \mathcal{B}$ BASE DI $\mathbb{R}[x]$

$\mathcal{B} = (1, x, x^2, \dots, x^n, \dots)$

SIN X NON SI RIESCE A ESPRIMERE CON UNA BASE DI $\mathbb{R}[x]$

VEDI MC-LAURIN SENZA o-PICCOLI $\to$ SOMMA ∞

$\hookrightarrow$ SERIE

MATRICI ASSOCIATE ALLE A.L.

* $f: \mathbb{K}^n \to \mathbb{K}^m$ LINEARE $\quad f(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_m)$

$y_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$

$\dots \qquad \dots$

$y_m = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

* $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f(x,y) \to (x-2y, 3x-y, 5x+y)$

$$A_f = \begin{pmatrix} 1 & -2 \\ 3 & -1 \\ 5 & 1 \end{pmatrix}$$

E BASE CANONICA DI $\mathbb{R}^2$

$E = (e_1, e_2)$ $\quad f(e_1) = (1, 3, 5)$

$\qquad f(e_2) = (-2, -1, 1)$

$C_1 = f(e_1)$

COLONNE DELLE MATR. A_f SONO LE Im DEGLI ELEM. DI E

$f(3+5\sin x-\cos x) = f(3-\cos x+5\sin x)$

INSIEME ORDINATO

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} \rightarrow 5\cos x + \sin x$$

$\rho(A) = \dim \operatorname{Im} f$

$f: V_n \rightarrow W_n$

f SURIETTIVA $\Leftrightarrow \rho(A) = m$

$\operatorname{Im} f$ E SP. COLONNE DI A

$\rho(A) = \dim C_A = \dim \operatorname{Im} f$

$C_A = \mathcal{L}(c_1, c_2, \dots)$

$\operatorname{Im} f = \mathcal{L}(f(b_1), f(b_2), \dots)$ ISOMORFI

X TROVARE BASE DI $\operatorname{Im} f$ BASTA RIDURRE X COLONNE

CONTROIMM. E SIST. $AX = W$

$W = w_1 d_1 + \dots + w_n d_n$

$V = x_1 b_1 + \dots + x_n b_n$

$f(a_{11} x_1 + a_{12} x_2 + \dots)$

$B_V = (e_1, e_2, \dots)$

$w_1, w_2, \dots, w_n$ QUALUNQUE VETT. DI W

$\exists!$ A.L. $f: V \rightarrow W$

$f(b_1) = w_1$

$f(b_2) = w_2$

$f(b_n) = w_n$

$v = x_1 b_1 + \dots$

$f(v) = x_1 w_1 + x_2 w_2 + \dots$

$\ast\ V_3 \xrightarrow{f} \mathbb{R}_3[x]$

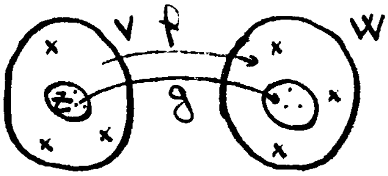
$B_{V_3} = \{\underline{u}, \underline{v}, \underline{w}\}$ $\quad \underline{u} = \underline{i}$ $\quad \underline{v} = \underline{j}$ $\quad w = \underline{i} - \underline{k} \rightarrow$ L.I.?

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \neq 0 \Rightarrow \text{L.I.}$$

TROVARE A.L. $f: V_3 \rightarrow \mathbb{R}_3[x]$ /

$\underline{u}, \underline{v} \in \operatorname{Ker} f$, $f(\underline{w}) = x - x^3$

ESTENSIONE DI UN A.L.



$G_Z(g_1, g_2, \dots, g_z) \rightarrow B_V$

SI COMPLETI G_Z A BASE B_V

$f(g_1) = g(g_1)$
$f(g_r) = g(g_r)$
$f(g_{r+1}) = w_1$
$f(g_n) = w_{n-r}$

$w_1, \dots, w_{n-r}$ SONO QUALUNQUE VETT. DI W

OPERAZIONI

$B_V = (e_1, e_2, \dots)$ $D_W = (d_1, d_2, \dots, d_n)$

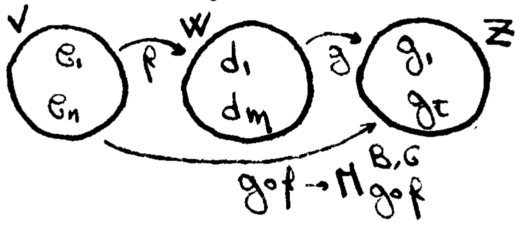
* $A = M_f^{B,D} \longrightarrow a M_f = a A \quad \forall a$

* SI POSSONO COMPORRE V, W, Z, CON BASI RISP. B_V, D_W, G_Z

$f: V \rightarrow W \rightarrow A = M_f$

$g: W \rightarrow Z \rightarrow B = M_g$ $\quad g \circ f: V \rightarrow Z \Rightarrow BA$

$M_{g \circ f}^{B,G} = M_g^{DG} \cdot M_f^{BD}$ SI MANTIENE ORDINE!!



$M_{g \circ f}^{BG} = M_g^{DG} \cdot M_f^{BD}$

* $V = \mathbb{R}_3[x]$ $\quad W = \mathbb{R}_1[x]$

SS. $Z \leq V$ $\quad Z = \mathcal{L}(x + x^2, 1 + 2x)$

A.L. $g: Z \rightarrow W$ $\quad g$ DERIVATA PRIMA

ESTENDERE g A $f: V \rightarrow W$ / $\mathrm{Ker} f = \mathcal{L}(1, x^3)$

$f(1) = \underline{0} = (0,0)$ $\quad (1, x+x^2, x)$ L.I. $\rightarrow \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$

$f(x+x^2) = \underline{0} = (0,0)$

$f(x) = (1,1)$

$f(x^3) = \; ? \leftarrow f(x^3) = (a,b) \; \forall (a,b) \neq K(1,1) \rightarrow$ ES: $f(x^3) = (0,1)$

$D(1, x+x^2, x, x^3)$ BASE DI $\mathbb{R}_3[x]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \rho = 4$$

MATRICE DELL'A.L. INVERSA

$$A = M_f^{B,S} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \qquad S = (i, j, K) \qquad f \text{ ISOMORF} \Leftrightarrow |A| \neq 0$$

$$|A| = 2$$

$$M_{f^{-1}}^{S,B} = A^{-1} = \frac{1}{2}\begin{pmatrix} 0 & +2 & 0 \\ +1 & -1 & +1 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \\ 1 & -1 & 0 \end{pmatrix}$$

$f^{-1}(\vec{i}) = 1/2(x+x^2) + (x^2 - 3x)$

$f^{-1}(\vec{j}) = 1 - 1/2(x+x^2) - (x^2 3x)$

$f^{-1}(\vec{K}) = 1/2 x + 1/2 x^2$

METTO I COMPL. ALGEBRICI DEI TRASPOSTI)

$f^{-1}(\vec{i}) = 5/2 x + 3/2 x^2$

$f^{-1}(\vec{j}) = 1 + 5/2 x - 3/2 x^2$

$f^{-1}(\vec{K}) = 1/2 x + 1/2 x^2$

CAMBIO DI BASE

B_V, D_V 2 BASI DI V

$d_1 = p_{11} b_1 + p_{21} b_2 + \ldots + p_{n1} b_n$

$d_n = p_{1n} b_1 + p_{2n} b_2 + \ldots + p_{nn} b_n$ $\qquad P = (p_{hk})$

$* V = \mathbb{R}^2 <$ $\mathcal{E}(\underline{e}_1, \underline{e}_2)$

$D(\underline{u}, \underline{v}) <$ $\underline{u} = (1,-2) = \underline{e}_1 - 2\underline{e}_2$; $\underline{v} = (3,5) = 3\underline{e}_1 + 5\underline{e}_2$ $\qquad P = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

P MATRICE ASSOC. $\mathcal{E} \rightarrow D$

$\mathcal{B} = (1, x+x^2, x^2-3x)$ BASE DI $\mathbb{R}_2[x]$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & 1 \end{pmatrix} \neq 0 \to \text{OK}$$

MATRICI DI PASSAGGIO

1) $f(b_1) = d_1 = p_{11} b_1 + p_{21} b_2 + \ldots + p_{n1} b_n$
$f(b_n) = d_n = p_{1n} b_1 + p_{2n} b_2 + \ldots + p_{nn} b_n$ $\quad P = M_f^{B,D}$

2) $id_V : V \to V$
$id(d_1) = d_1 = p_{11} b_1 + p_{21} b_2 + \ldots$ $\quad P = M_{id}^{D,B}$
$id(d_n) = d_n = \ldots$

TEOREMA

DATE 2 BASI $\neq$ $\mathcal{B}$ E $\mathcal{B}'$ DI V, P SIA LA MATR. DI PASSAGGIO DA $\mathcal{B}$ A $\mathcal{B}'$, ALLORA

1) P INVERTIBILE, P^{-1} MATR. PASS. $\mathcal{B}' \to \mathcal{B}$

2) $v \in V$, $(x_1, \ldots, x_n)$ COMP. RISPETTO A $\mathcal{B}$, $(y_1, \ldots, y_n)$ COMP. A $\mathcal{B}'$
$^t(x_1, \ldots, x_n) = P\, ^t(y_1, \ldots, y_n)$
$^t(y_1, \ldots, y_n) = P^{-1}\, ^t(x_1, \ldots, x_n)$

* $E = (e_1, e_2, e_3)$ $\quad F = (f_1, f_2, f_3)$ $\quad f_1 = e_1 - e_2$ $\quad f_2 = e_1 - e_3$ $\quad f_3 = e_2 + e_3$
E, F BASI DI $\mathbb{R}^3$

$$\underset{E,F}{P} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$ (X COLONNE COMP. f_1, f_2, f_3)

$$\underset{F,E}{Q} = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$ $\quad e_1 = \frac{1}{2} f_1 + \frac{1}{2} f_2 + \frac{1}{2} f_3$

$P \cdot Q = 1 \Rightarrow Q = P^{-1}$

AUTOVALORI E AUTOVETTORI

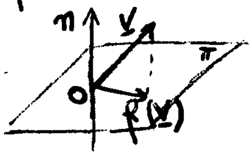
ENDOMORFISMO → A.L. DI UNO SP. VETT. IN SE STESSO

$$\varphi : V \to V$$

$\lambda \in \mathbb{K}$ È UN <u>AUTOVALORE</u> $\Leftrightarrow \exists\ \underline{v} \in V, v \neq 0 \ / \ f(\underline{v}) = \lambda \underline{v}$

* $f: V_3 \to V_3$

f: PROIEZ. SU π $\quad \pi$ = PIANO x O



$f(\underline{v}) = \lambda \underline{v}$?

$\cdot\ w \in \pi \Rightarrow f(\underline{w}) = \underline{w} \to \lambda = 1$

$\cdot\ w \in n_\pi \Rightarrow f(\underline{w}) = \underline{0} = 0 \cdot \underline{w} \to \lambda = 0$

→ AUTOVALORI DI f

* $f: V_3 \to V_3$ $\quad f$: SIMMETRICO RISPETTO A π (PIANO x O)

$\cdot\ v \in n \Rightarrow f(\underline{v}) = -\underline{v} = -1 \cdot \underline{v} \to \lambda = -1$ AUTOVALORE → $v \in n$ AUTOVETT.

$\cdot\ v \in \pi \Rightarrow f(\underline{v}) = \underline{v} = 1 \cdot \underline{v} \to \lambda = +1$ AUTOVALORE → $v \in \pi$ AUTOVETT.

SE NON CI FOSSE $\underline{v} \neq 0$ OGNI λ SAREBBE AUTOVALORE

<u>AUTOVETTORE</u> → VETTORE ASSOCIATO A UN AUTOVALORE
(CHE SODDISFANO CONDIZ.)

$\exists\ \underline{v} \neq 0 \mid f(\underline{v}) = 0 \Leftrightarrow f$ INIETTIVA

$\lambda = 0$ È UN AUTOVALORE

AUTOSPAZI

$V_\lambda := (\underline{v} \in V : f(\underline{v}) = \lambda \underline{v})$

↳ AUTOSPAZIO RELATIVO ALL'AUTOVALORE λ

↳ INSIEME AUTOVETTORI

PROP

AUTOVETT. (NON NULLI) $\in$ IN AUTOSPAZI $\neq$ SONO L.I.

$\lambda_1,\dots,\lambda_s$ AUTOVALORI DISTINTI. $\lambda_1 \neq \dots \neq \lambda_s$

$\underline{v}_1 \in V_{\lambda_1},\dots,\underline{v}_s \in V_{\lambda_s}$ SONO L.I. $(\underline{v}_i \neq 0)$

DIM X ASSURDO

$\underline{v}_s \neq \underline{0}$ $\quad \underline{v}_1,\dots,\underline{v}_{s-1}$ L.I. $\quad \underline{v}_s \in \mathcal{L}(\underline{v}_1,\dots,\underline{v}_{s-1})$

$\underline{v}_s = a_1\underline{v}_1+\dots+a_{s-1}\underline{v}_{s-1} \quad a_i \in \mathbb{K}$

$f(\underline{v}_s) = f(a_1\underline{v}_1+\dots+a_{s-1}\underline{v}_{s-1})$

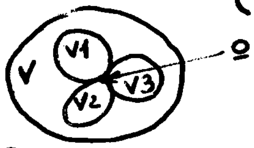
$\lambda_s\underline{v}_s = a_1 f(\underline{v}_1)+\dots+a_{s-1}f(\underline{v}_{s-1})$

$\lambda_s(a_1\underline{v}_1+\dots+a_{s-1}\underline{v}_{s-1}) = a_1(\lambda_1\underline{v}_1)+\dots+a_{s-1}(\lambda_{s-1}\underline{v}_{s-1})$

$a_1(\lambda_s-\lambda_1)\underline{v}_1+\dots+a_{s-1}(\lambda_s-\lambda_{s-1})\underline{v}_{s-1}=0$

$$\underline{v}_1,\dots,\underline{v}_s \text{ L.I.} \rightarrow \left.\begin{cases} a_1(\lambda_s-\lambda_1)=0 \rightarrow a_1=0 \\ a_{s-1}(\lambda_s-\lambda_{s-1})=0 \rightarrow a_{s-1}=0 \end{cases}\right\} \rightarrow a_1=\dots=a_{s-1}=0 \Rightarrow \underline{v}_s=\underline{0}$$

ASSURDO



ALLORA $\underline{v}_s$ L.I. DA PRECEDENTI

PROP: LA SOMMA TRA AUTOSPAZI E' DIRETTA

AUTOSPAZI E Ker DI A.L.

$f(\underline{v})=\lambda\underline{v} \quad (\underline{v}\neq\underline{0})$

$f(\underline{v})=\lambda\underline{v} \Leftrightarrow f(\underline{v})-\lambda\underline{v}=0 \Leftrightarrow (f-\lambda id)(\underline{v})=0 \Leftrightarrow$

$\Leftrightarrow \underline{v} \in Ker(f-\lambda id)$

$f_\lambda = f-\lambda_{id}$

$\lambda \in \mathbb{K}$ E' UN AUTOVALORE $\Leftrightarrow Ker\, f_\lambda$ CONTIENE VETT. NON NULLI

$\Leftrightarrow f_\lambda$ NON E' INIETTIVA

λ AUTOVALORE DI $f \Leftrightarrow Ker\, f_\lambda \neq \{0\} \Leftrightarrow f_\lambda$ NON E' INIETTIVA

SE $\lambda=0 \Rightarrow f_\lambda = f-0\cdot id = f \Leftrightarrow Ker\, f \neq \{0\} \Leftrightarrow f$ NON E' INIETTIVA

$V_\lambda = Ker\, f_\lambda \quad f_\lambda : f-\lambda\cdot id \quad f(\underline{v})-\lambda\underline{v}=0 \Leftrightarrow V_\lambda \equiv Ker\, f_\lambda$

POLINOMIO CARATT. DI f = P.C. DI UNA QUALUNQUE $M_f^{B,B}, \forall B$

$\lambda \in \mathbb{K}$ È UN AUTOVALORE $\Longleftrightarrow$ RADICI DI $P(t)$

PROP

1) V_λ È ISOMORFO A SP. SOLUZ. DI $(A-\lambda I)X=0$

2) DIM $V_\lambda = n - \rho(A-\lambda I)$

3) BASE V_λ = BASE $\operatorname{Ker} f_\lambda$ = BASE SP. SOLUZ. $(A-\lambda I)X=0$

* $A=\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ (VEDI ES. PREC.)

$V_0 = \operatorname{Ker} f \Rightarrow AX=0$

$V_4 = \operatorname{Ker} f_4 \Rightarrow (A-4I)X=0$

• $AX=0 \to \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \to \operatorname{Ker} f = \{2x-y=0 \to y=-2x \to \text{DIM}=1$

• $(A-4I)X=0 \to \begin{pmatrix} 2-4 & 1 \\ 4 & 2-4 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \to \operatorname{Ker} f_4 = \{y=2x$

$$A-\lambda I = \begin{pmatrix} a_{11}-\lambda_1 & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda_2 & \dots & a_{2n} \\ \dots & & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda_n \end{pmatrix}$$

ESEMPI

* $D: C^{(\infty)}(\mathbb{R}) \to C^{(\infty)}(\mathbb{R})$

$D(f) := f'$

• AUTOVALORI DI D

$\lambda \in \mathbb{R} / \exists f \neq 0 : D(f) = \lambda f \to f' = \lambda f$

$\left.\begin{array}{l} \lambda=1,\ f(x)=e^x \\ \quad f(x)=Ke^x \ (K \in \mathbb{R}) \end{array}\right\} \to V_{(1)} = \mathcal{L}(e^x)$

$\lambda=-1,\ f'(x)=-f(x)=-Ke^{-x}$

* $f(a,b,c) = (a+b, 2b, -a-b+2c)$

1) A
2) A INVERTIBILE?
3) AUTOVALORI DI A (DI f)
4) AUTOSPAZI DI A
5) DETERM. $D, P \in \mathbb{R}^{3,3}$ CON D DIAGONALE E P INVERTIBILE / $P^{-1}AP = D$.

1) $A = M_f^{\mathcal{E}\mathcal{E}} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad \mathcal{E} = (e_1, e_2, e_3)$

2) $|A| = 2 \cdot 2 = 4 \neq 0 \rightarrow A$ INVERTIBILE $\rightarrow$ NO $\lambda = 0$
$\hookrightarrow A$ INIETTIVA $\rightarrow \operatorname{Ker} f = 0$

3) $P_A(t) = |A - tI|$

$$|A - tI| = \begin{vmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 0 \\ -1 & 1 & 2-t \end{vmatrix} \rightarrow p(t) = (2-t)(2-t)(1-t)$$

$\lambda_1 = \lambda_2 = 2 \quad \lambda_3 = 1$

$V_2 = \operatorname{Ker} f_2 \quad f_2 = f - 2\,id$
$(A - 2I)X = 0$

$$(A-2I) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow x_1 = x_2$$

SOL $S = V_{(2)} = \{(x_1, x_1, x_3), \forall x_1, x_3\}$

$\dim S = \dim V_{(2)} = 3 - \rho(A - 2I) = 3 - 1 = 2$

BASE DI $V_{(2)} = B(\underline{e}_1, \underline{e}_2)$ con $\underline{e}_1 = (1,1,0)$, $\underline{e}_2 = (0,0,1)$ $\quad V_{(2)} = \mathcal{L}(\underline{e}_1, \underline{e}_2)$

$$(A - I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \rho(A - I) = 2 \rightarrow \dim V_{(1)} = 3 - 2 = 1$$

$$\begin{cases} x_2 = 0 \\ -x_1 + x_2 + x_3 = 0 \rightarrow x_1 = x_3 \end{cases}$$

$S = \{(x_1, 0, x_1) \ \forall x_1 \in \mathbb{R}\}$
$B_{V_{(1)}} = ((1,0,1)) \rightarrow V_{(1)} = \mathcal{L}((1,0,1))$

② f SEMPLICE $\Leftrightarrow$ $\dim V_{\lambda_1} \oplus \ldots \oplus \dim V_{\lambda_s} = \dim V$

③ SE f SEMPLICE $\Rightarrow$ B_V SI OTTIENE $B_{V_{\lambda_1}} \cup B_{V_{\lambda_2}} \cup \ldots \cup B_{V_{\lambda_s}}$

$$a = B_1 \cup B_2 \cup B_3 = \{\underbrace{\underline{a}_1, \underline{a}_2}_{B_1}, \underbrace{\underline{a}_3, \underline{a}_4}_{B_2}, \underbrace{\underline{a}_5, \underline{a}_6}_{B_3}\}$$

$f(a_1) = \lambda_1 a_1$
$f(a_2) = \lambda_1 a_2$
$f(a_3) = \lambda_2 a_3$
$f(a_4) = \lambda_2 a_4$
$f(a_5) = \lambda_3 a_5$
$f(a_6) = \lambda_3 a_6$

$$A = M_f^{a,a} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

CRITERIO SEMPLICITÀ

$\lambda_1, \lambda_2, \lambda_3$ AUTOVALORI
m_1, m_2, m_3 MOLTEPLICITÀ

f SEMPLICE $\Leftrightarrow$ $\dim(V_{\lambda_k}) = m_k$

$$n - \rho(A - \lambda_k I) = m_k \qquad n = \dim V$$
$$\rho(A - \lambda_k I) = n - m_k$$

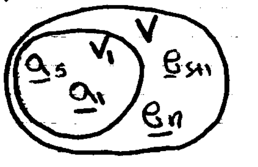
$1 \leq \dim V_\lambda \leq n$
$1 \leq \dim V_\lambda \leq m_\lambda$ $\qquad s = \dim V_\lambda$

a_1 = BASE DI $V_\lambda = \{\underline{a}_1, \ldots, \underline{a}_s\}$
COMPLETO a_1, A BASE DI V (B_V)

$B_V = (\underline{a}_1, \ldots, \underline{a}_s, \underline{e}_{s+1}, \underline{e}_n)$

$f(\underline{a}_1) = \lambda \underline{a}_1$
$f(\underline{a}_s) = \lambda \underline{a}_s$
$f(\underline{e}_{s+1}) = \underline{w}_{s+1}$
$f(\underline{e}_n) = \underline{w}_n$

$$A = \left(\begin{array}{ccc|c} \lambda & 0 \cdots & 0 & \\ 0 & \lambda \cdots & 0 & ? \\ \vdots & \vdots & & \\ 0 & 0 \cdots & \lambda & \\ \hline 0 & 0 & 0 & Q \end{array}\right) = M_f^{B,B}$$



A, D SIMILI $\Rightarrow \exists P \,/\, D = P^{-1} A P$ ↳ MATR. PASSAGGIO $\mathcal{B} \to \mathcal{A}$

A DIAGONALIZZABILE $\Leftrightarrow f$ SEMPLICE

$\star\ f: \mathbb{R}^3 \to \mathbb{R}^3$ VEDI ES. PREC.

$$A = M_f^{\mathcal{E}\mathcal{E}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \longrightarrow A - tI = \begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 0 \\ 1 & 1 & 2-t \end{pmatrix}$$

$\lambda_3 = 1 \to \mathcal{L}(\underline{e}_3) \to \mathcal{L}((1,0,1))$

$\lambda_1 = \lambda_2 = 2 \to \mathcal{L}(\underline{e}_1, \underline{e}_2) = \mathcal{L}((1,1,0),(0,0,1))$

$\mathcal{A} = (\underline{e}_1, \underline{e}_1, \underline{e}_3)$ L.I.

$$D = M_f^{a,a} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} f(b_1) = 2b_1 \\ f(b_2) = 2b_2 \\ f(b_3) = b_3 \end{matrix}$$

$D = P^{-1} A P$ ↳ M. BASE $\mathcal{E} \to \mathcal{A} \Rightarrow$ COLONNE DI COMP. DI $\mathcal{A}$ RISPETTO A $\mathcal{E}$

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$\mathcal{A}_1 = (e_1, e_2, e_3)$

$\mathcal{A}_2 = (e_3, e_1, e_2)$

...

$$D_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & & 1 \end{pmatrix} \quad D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad D_3 = \ldots$$

<u>METODO PRATICO</u>

1) SCRIVO $p_A(t) \to$ SE HA TUTTE LE RADICI IN $\mathbb{K}$ (A)

2) SE $\lambda \in \mathbb{K} \to f$ SICURAMENTE SEMPLICE SE $\forall \lambda, m_\lambda = 1$
$(1 \leq \dim V_\lambda \leq 1) \to \dim V_\lambda = 1$

3) SE $m_\lambda > 1$, CONTROLLO CHE $\dim V_\lambda = m_\lambda$
$n - \rho(A - \lambda I) = \dim V_\lambda = m_\lambda$

PRODOTTO SCALARE

$\underline{u} \cdot \underline{v}$ $(\underline{u}, \underline{v})$ $<\underline{u}, \underline{v}>$

$p: V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$

1) BILINEARE $p(\underline{u}, \underline{v}) \in \mathbb{K}$
SU 2 ELEM.

BIN LIN $\Leftrightarrow$
- LIN. 1° ELEMENTO $p(a\underline{u} + b\underline{u}', \underline{v})$
- LIN 2° ELEMENTO $p(\underline{u}, c\underline{v} + d\underline{v}')$

$(a\underline{u} + b\underline{u}') \cdot \underline{v} = a(\underline{u} \cdot \underline{v}) + b(\underline{u}' \cdot \underline{v})$

2) SIMMETRICA

$p(\underline{u}, \underline{v}) = p(\underline{v}, \underline{u})$

3) $p(\underline{u}, \underline{u}) \geq 0$ $p(\underline{u}, \underline{u}) = 0 \Leftrightarrow \underline{u} = 0$

ESEMPI

1) PRODOTTO SCALARE ORDINARIO IN V_3

$\underline{u} = (u_1, u_2, u_3)$
$\underline{v} = (v_1, v_2, v_3)$

$\underline{u} \cdot \underline{v} := u_1 v_1 + u_2 v_2 + u_3 v_3$

$\underline{u} \cdot \underline{u} = u_1^2 + u_2^2 + u_3^2 \geq 0$

$\underline{u} \cdot \underline{u} = 0 \Leftrightarrow u_1^2 + u_2^2 + u_3^2 = 0 \Leftrightarrow u_1 = u_2 = u_3 = 0 \Leftrightarrow \underline{u} = 0$

2) P.S. EUCLIDEO

$V = \mathbb{R}^n$ $(x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n) := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

3) P.S. FUNZIONALE

$V = \mathcal{C}^{(0)}([a,b])$ $[a,b] \in \mathbb{R}$ $f, g \in V$

$(f, g) := \int_a^b f(x)\, g(x)\, dx$

$(\cos kx, \sin hx) = \int_{-\pi}^{\pi} (\cos kx)(\sin hx)\,dx = 0 \quad \forall x$

$(\sin kx, \sin hx) = \int_{-\pi}^{\pi} \sin^2 x\,dx = (\cos kx, \cos hx) = \begin{cases} 0 & \text{se } h \neq k \\ \pi & \text{se } h = k \end{cases}$

$(\cos hx, \cos hx) = \pi$

$(\sin x, \sin x) = \int_{-\pi}^{\pi} \sin^2 x\,dx = 2\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx = \left| x - \frac{\sin 2x}{2} \right|_0^{\pi} = \pi$

$\|\sin x\| = \sqrt{\pi}$

$\|\sin hx\| = \|\cos hx\| = \sqrt{\pi}$

3) $O_{2\pi} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \dots, \frac{1}{\sqrt{\pi}}\cos kx, \frac{1}{\sqrt{\pi}}\sin kx, \dots \right\}$

INSIEME ORTONORMALE ← NORMALIZZATO

COPPIE VETT ≠ = 0

COPPIE VETT = = 1 → NORMA = 1

NORMA

$\|u\| = \sqrt{\underline{u} \cdot \underline{u}}$

1) $\|v\| \geq 0$, $\|v\| = 0 \Leftrightarrow \underline{v} = 0$

2) $|a|\,\|\underline{v}\| = \|a v\|$

3) $\|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$ DISUGUAGLIANZA TRIANGOLARE

IN $V_3 \rightarrow \|\underline{v}\|$ = MODULO DI $\underline{v}$ → RISPETTA LE 3 PROP.

$\|f\| = (f,f)^{1/2} = \sqrt{\int_a^b f^2(x)\,dx}$

NORM $\underline{v} = \frac{\underline{v}}{\|\underline{v}\|}$

BASI ORTONORMALI → VERSORI → ⊥ TRA LORO, $\|\underline{v}\| = 1$

$\delta_{hk} = \begin{cases} 1 & \text{se } h = k \\ 0 & \text{se } h \neq k \end{cases}$

SE $\underline{u} \neq \underline{v} \rightarrow \underline{u} \cdot \underline{v} = 0$

$\|\underline{u}\| = 1 \quad \forall \underline{u}$

BASI ORTONORMALI

$\underline{y} = \{v_1, v_2, \dots, v_n\}$ ORTONORMALE

$$v_h \cdot v_k = \begin{cases} 0 & \text{se } h \neq k \\ 1 & \text{se } h = k \end{cases}$$

$a_1 \underline{v_1} + \dots + a_n v_n = 0 \rightarrow \underline{v_1} \cdot (a_1 v_1 + \dots + a_n v_n) = \underline{v_1} \cdot \underline{0}$

$a_1 (v_1 \cdot v_1) + a_2 (v_1 \cdot v_2) + \dots = 0$

$a_1 + 0 = 0 \rightarrow a_1 = 0 \rightarrow$ ALLO STESSO MODO DIMOSTRO CHE

$a_1 = a_2 = \dots = a_n = 0$

$\forall h = 1, 2, \dots, n \rightarrow y$ E' LIBERA

UN INSIEME ORTOGONALE NON SI PUÒ DIRE LIBERA POICHE'
CI POTREBBE ESSERE $\underline{0}$ ($\perp$ A QUALSIASI $\underline{v}$)

$\downarrow$

INS. ORTOGONALE SENZA $\underline{0}$ E' LIBERO

$\underline{o}_1 = \text{VERS}(\underline{v}_1)$

$\underline{o}_2 = \text{VERS}[v_2 - (v_2 \cdot o_1)\underline{o}_1]$

$\underline{o}_3 = \text{VERS}[v_3 - [(v_3 \cdot o_1)\underline{o}_1 + (v_3 \cdot o_2)\underline{o}_2]]$

$\underline{o}_4 = \text{VERS}[v_4 - [(v_4 \cdot o_1)\underline{o}_1 + (v_4 \cdot o_2)\underline{o}_2 + (v_4 \cdot o_3) \cdot \underline{o}_3]]$

$O = (o_1, o_2, \dots, o_n) =$ BASE O.N. DI V, $\underline{v} \in V$, $v = v_1 o_1 + v_2 o_2 + \dots + v_n o_n$

$\Downarrow$

$v_k = \underline{v} \cdot \underline{o}_k$

$\star\ V = \mathbb{R}_2[x] \begin{cases} ① \simeq \mathbb{R}^3 \rightarrow \text{P. S. EUCLIDEO} \\ ② \ \mathcal{C}^{(0)}([0,1]) \rightarrow \text{P. S. FUNZIONALE} \end{cases}$

① $(a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2) = a_0 b_0 + a_1 b_1 + a_2 b_2$

$O_1 =$ BASE O.N. $= (1, x, x^2)$

ES

$p(x) = 1 - 5x + 8x^2$

COMP DI $p(x)$ RISPETTO A: $\begin{cases} \mathcal{O}_1 \rightarrow \text{P.S. EUCLIDEO} \\ \mathcal{O}_2 \rightarrow \text{P.S. FUNZIONALE} \end{cases}$

$a = \underline{p} \cdot \underline{o}_1 \to a = \int_0^1 (1 - 5x + 8x^2) dx = \left| x - 5\frac{x^2}{2} + 8\frac{x^3}{3} \right|_0^1 = \frac{7}{6}$

$b = \underline{p} \cdot \underline{o}_2 \to b = \int_0^1 (1 - 5x + 8x^2)(2\sqrt{3}x - \sqrt{3})\, dx = \ldots$

$c = \underline{p} \cdot \underline{o}_3 \to c = \int_0^1 (1 - 5x + 8x^2)(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5})\, dx = \ldots$

MATRICI ORTOGONALI $P \in \mathbb{R}^{n,n}$

$\Leftrightarrow P \cdot {}^tP = I$

$P^{-1}(P\,{}^tP) = P^{-1} I \to (P^{-1}P)\,{}^tP = P^{-1} I \to I\,{}^tP = P^{-1} I \to {}^tP = P^{-1}$

$P^{-1} \exists \to |P| \neq 0$

$|P \cdot {}^tP| = |I| \to |P||{}^tP| = 1 \to |P| \neq 0$

FANNO PASSARE DA UNA BASE ORTONORMALE AD UN'ALTRA BASE ORTONORMALE

$O_V = (\underline{o}_1, \ldots, \underline{o}_n)$ O.N. DI V

M = M.P. DA O_V A BASE $D_V \longrightarrow P_\perp \Leftrightarrow D$ O.N.

PROP

1) $P \cdot {}^tP = I \Rightarrow {}^tP = P^{-1}$

2) ${}^tP \cdot P = I \Rightarrow P^{-1} P = I$

3) ${}^tP$ ORTOGONALE

4) LE RIGHE DI P FORMANO UNA BASE O.N. DI $\mathbb{R}^n$ EUCLIDEO

5) LE COLONNE DI P FORMANO UNA BASE O.N. DI $\mathbb{R}^n$ EUCLIDEO

* $P = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ $|P| = 5$

${}^tP = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \neq P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \to$ NON $\perp$ POICHE' ${}^tP \neq P^{-1}$

$P \in \mathbb{R}^{2,2}$ ORTOG. SPECIALE $\rightarrow$ ROTAZIONE ANTIORARIA DEL SIST. DI RIFERIMENTO

TEOREMA

OGNI M. SIMM. REALE È DIAGONALIZZABILE ATTRAVERSO UNA MATR. ORTOGONALE

$A \sim D$ A DIAGONALIZZ. $\Leftrightarrow \exists$ P INVERTIBILE, D DIAGONALE /

$A = M_f^{\mathcal{E},\mathcal{E}}$ $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ $D = P^{-1}AP = {}^tPAP$ SE P ORTOGONALE

$P: \mathcal{E} \rightarrow \mathcal{A}$ (BASE DI AUTOVETTORI)

$* A = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$ $\begin{vmatrix} 8-t & 2 \\ 2 & 5-t \end{vmatrix} = 0$ $p_A(t) = 40 - 5t - 8t - 4 + t^2$
$= t^2 - 13t + 36 \quad t = \begin{cases} 9 \\ 4 \end{cases}$

$\lambda_1 = 9 \rightarrow V_9 : (A - I9)X = 0$

$A - 9I = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow x = 2y \rightarrow \mathcal{L}((2,1))$

$\underline{a}_1 = (2, +1)$

$\lambda_2 = 4 \rightarrow V_4 : (A - 4I)X = 0$

$A - 4I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \rightarrow V_4 = \{(x, -2x), x \in \mathbb{R}\} = \mathcal{L}((1,-2))$

$\underline{a}_2 = (1, -2)$

BASE DI $\mathbb{R}^2$ FORMATA DA AUTOVETTORI

$\mathcal{A} = (\underline{a}_1, \underline{a}_2)$ $M_f^{\mathcal{A},\mathcal{A}} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} = D$ $\underline{a}_1 \cdot \underline{a}_2 = 0 \rightarrow$ ORTOGONALI

P = M. PASS. $\mathcal{E} \rightarrow \mathcal{A}$ $P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ $D = P^{-1}AP$

$\underline{o}_1 = \dfrac{a_1}{\|a_1\|} = \dfrac{(2,1)}{\sqrt{5}} = \left(\dfrac{2}{\sqrt5}, \dfrac{1}{\sqrt5}\right)$

$\underline{o}_2 = \dfrac{a_2}{\|a_2\|} = \left(\dfrac{1}{\sqrt5}, \dfrac{-2}{\sqrt5}\right)$

$\mathcal{O} = (\underline{o}_1, \underline{o}_2)$ È UNA BASE O.N. DI AUTOVETTORI

Q : M. PASS. $\mathcal{E} \rightarrow \mathcal{O}$

$Q = \begin{pmatrix} 2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & -2/\sqrt5 \end{pmatrix}$ $D = Q^{-1}AQ = {}^tQAQ$

$a_{11} = 1 \qquad xy \to a_{12} = a_{21} = 0$

$a_{22} = -1 \qquad xz \to a_{13} = a_{31} = -5/2 \qquad A = \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & -1 & 1/2 \\ -5/2 & 1/2 & 1 \end{pmatrix}$

$a_{33} = 1 \qquad yz \to a_{23} = a_{32} = 1/2$

$q(x_1, x_2, \dots, x_n) = (x_1, \dots, x_n)\, A\, {}^t(x_1, \dots, x_n)$

$q(X) = {}^tX A X \quad X \in \mathbb{R}^{n,1} \to$ MATRICE COLONNA

$\forall$ M. SIMM. REALE SI PUÒ ASSOCIARE UNA F.Q. $\to \rho(A) = \rho(q_A)$

$* A = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} \to q_A(x,y) = 8x^2 + 4xy + 5y^2$

<u>CAMBIAMENTO LINEARE DI VARIABILI</u>

$X = {}^t(x_1, \dots, x_n) \quad Y = {}^t(y_1, \dots, y_n) \qquad X = PY \quad P \in \mathbb{R}^{n,n}$ INVERTIBILE

$\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} = P \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix} \qquad r(Y) = q(PY) = q(X)$

$q(X) = {}^tX A X \to q(PY) = \underset{({}^tY \cdot {}^tP)}{{}^t(PY)} A (PY) = {}^tY \underset{B}{({}^tPAP)} Y$

$r(Y) = {}^tY B Y$

${}^tB = {}^t({}^tPAP) = {}^tP \cdot {}^tA \cdot {}^t({}^tP) = {}^tPAP = B \to {}^tB = B \to B$ SIMM. REALE

$r(Y)$ F.Q.

A E B EQUIVALENTI $\Leftrightarrow \exists P / B = {}^tPAP$

M.A. q — M.A. r — HANNO LO STESSO RANGO

<u>FORMA CANONICA</u>

F.Q. SI DICE IN F.C. SE COMPAIONO SOLO TERMINI DI 2° GRADO

M. ASSOC. È UNA M. DIAGONALE