NUMERO : 163

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APPUNTI

STUDENTE : Stoppa

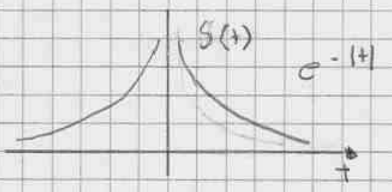
MATERIA : Analisi dei Segnali
Prof. Pagana

Analisi dei Segnali

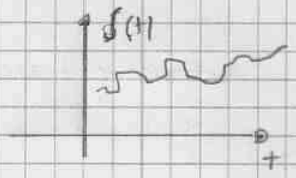
prof. G.Pagana Ing. Biomedica AA 2009/10

Indice:

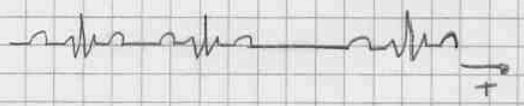
I segnali deterministici sono



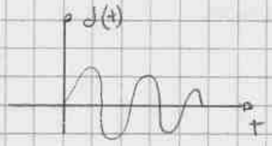
mentre quello stocastico è



un segnale stazionario può essere ad esempio il seno, mentre uno non staziona
varia nel tempo



segnali a energia

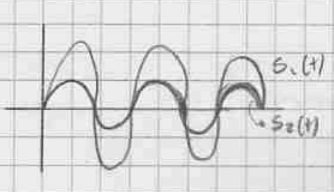


es. la forza sulla mia voce

$$\boxed{E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt}$$

è l'energia di un segnale

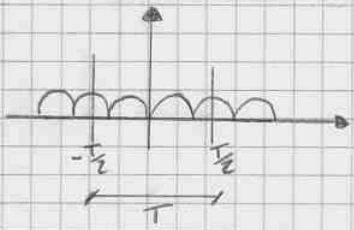
L'energia corrisponde all'area sottesa dal mio segnale



l'energia di una sinusoide è infinita

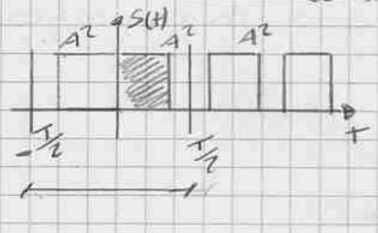
Se il segnale è a energia infinita allora si parla di segnali a potenza.

$$P_x(t) = \lim_{t \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$



prendo un intervallo T e lo dilato fino all'infinito

es. ricevitori



$$\lim_{t \to \infty} \frac{1}{T} A^2 \infty = A^2$$

stesso ordine di infinito

segnale porta

devo definire una base di vettori $\omega_i = \omega_1, \omega_2, \dots$ t.c. $s(t) = \sum_{i=0}^{n} a_i \omega_i$

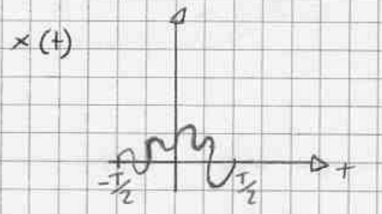
Dati 3 segnali diversi devo fare in modo che tutti abbiano la stessa bas

TRASFORMATA di FOURIER

segnale ad energia finita

26-02

Scomposizione armonica → ricondurre tutto a dei segnali elementari o sinusoidali.



usiamo come segnali elementari gli esponenziali complessi.

$$W_n(t) = e^{j2\pi\frac{n}{T}t} = \cos 2\pi \frac{n}{T} t + j\sin 2\pi \frac{n}{T} t$$

$$x(t) = \sum_{n=\infty}^{+\infty} \mu_n e^{j2\pi \overset{f_n \text{ parametro}}{\boxed{\frac{n}{T}}} t}$$

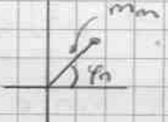
$$\mu_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi\frac{n}{T}t} dt$$

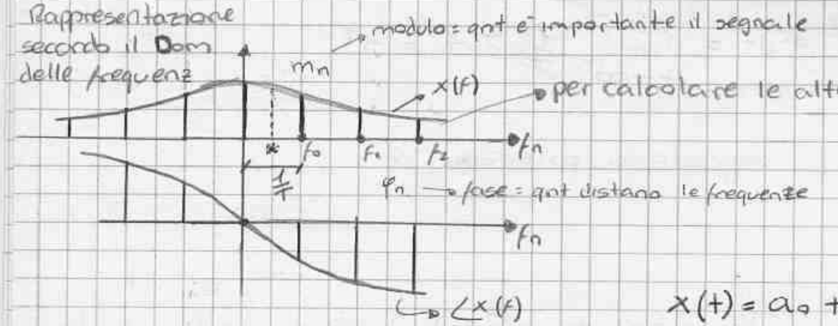
Le informazioni di un segnale, sono contenuti e ricavabili dalla serie di Fourier

$\underset{\text{frequenze}}{\boxed{f_n}}$ $f_n\big|_{n=1} = f_0 = \frac{1}{T}$ fondamentale → dipende dalla finestra di osservazione

$f_n = nf_0$ armoniche

$$\mu_n = |\mu_n| e^{j\varphi_n} = m_m e^{+j\varphi_n}$$





→ per calcolare le altezze bisognerebbe risolvere l'integrale

*fondamentale in $\frac{T}{2}$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_i \cos 2\pi \frac{n}{T} t + \sum_{n=1}^{\infty} b_i \sin 2\pi \frac{n}{T} t$$

Voglio analizzare il segnale tra +∞ e -∞. Più allargo l'intervallo e più piccola ho la fondamentale

$$x(t) = \overset{\int}{\sum_{n=\infty}^{+\infty}} \left[\overset{df}{\boxed{\frac{1}{T}}} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) e^{-j2\pi \overset{f}{\boxed{\frac{n}{T}}} t} dt \right] e^{j2\pi \overset{f}{\boxed{\frac{n}{T}}} t} = \int_{-\infty}^{+\infty} \underbrace{\left[\int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt \right]}_{x(f)} e^{j2\pi f t} d$$

$\Delta f = \frac{1}{T} \Rightarrow df \; \Delta f$ per $T \to \infty$

$f_n = n\Delta f \Rightarrow f$

$$x(t) = \int_{-\infty}^{+\infty} x(f) e^{j2\pi f t} df$$

$x(t) \longleftrightarrow X(f)$

$h(t) \longleftrightarrow H(f)$

$y(t) \longleftrightarrow X(f)H(f)$

$$y(t) = \int_{-\infty}^{+\infty} x(\theta)\, h(t-\theta)\, d\theta =$$ integrale di convoluzione

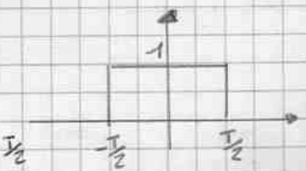
$$= x(t) * h(t)$$ ammette una tr. di Fourier molto semplice.

nel dominio del t abbiamo integrali mentre nel dominio delle f abbiamo moltiplicazioni.

27-02

ESERCIZI

La funzione porta $x(t) = P_T(t) \begin{cases} 1 & \text{se } -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{altrove} \end{cases}$



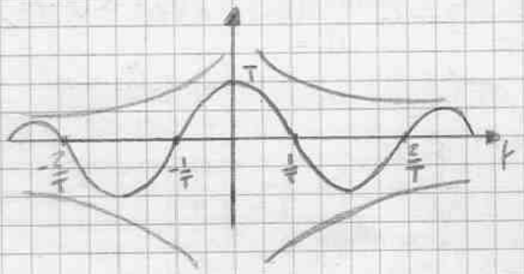
$$X(f) = \int_{-\infty}^{+\infty} P_T(t)\, e^{-j2\pi f t}\, dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} 1\, e^{-j2\pi f t} dt = \left[-\frac{1}{j2\pi f} e^{-j2\pi f t} \right]_{-\frac{T}{2}}^{\frac{T}{2}} =$$

$$= -\frac{1}{j2\pi f} e^{-j2\pi f \frac{T}{2}} + \frac{1}{j2\pi f} e^{-j2\pi f (-\frac{T}{2})} = \frac{1}{j2\pi f} \left[-e^{-j2\pi f \frac{T}{2}} + e^{-j2\pi f (-\frac{T}{2})} \right] = \sin\left(2\pi \frac{T}{2} f\right) \frac{1}{f\pi}$$

↓
funzione seno cardinale SINC

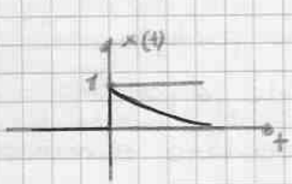
- Studio di funzione SINC

$$\lim_{x \to 0} \frac{\sin(\pi f T)}{\pi f} = T$$ in $-\frac{1}{T}$ e $\frac{1}{T}$ ci sono gli zeri della funzione



- Studio funzione gradino

$x(t) = u(t)\, e^{-kt} \quad k>0 \qquad u(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$



calcolare la ~~trasformata~~ di Fourier

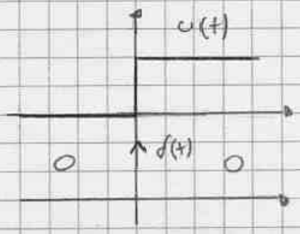
$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t} dt = \int_{-\infty}^{+\infty} e^{-kt}\, u(t)\, e^{-j2\pi f t}\, dt = \int_{0}^{+\infty} e^{-kt} e^{-j2\pi f t} dt =$$

$$= \int_{0}^{+\infty} e^{+t(-k-j2\pi)} dt = \left[\frac{1}{-k-j2\pi f} e^{-t(+k+j2\pi)} \right]_{0}^{+\infty} = \left[0 - \frac{1}{-k-j2\pi f} \right] = \frac{1}{k+j2\pi f}$$

6) $\int e^{\pm j \overset{\text{fase}}{2\pi f} t} = \delta(f)$

la trasformata di un potenziale è una retta

7) $\frac{du(t)}{dt} = \delta(t)$



Quanto vale la trasformata della delta

$$\int_{-\infty}^{+\infty} \delta(t) e^{-j2\pi f t} dt \underset{\text{def. di } \delta}{=} e^{-j2\pi f t}\Big|_{t=0} = 1$$ operatore unitario della trasformata

$x(t) = e^{j2\pi f_0 t}$ $\quad x(f) = \int e^{+j2\pi f_0 t} e^{-j2\pi f t} dt = \int e^{j2\pi (f_0 - f) t} dt = \delta(f - f_0)$

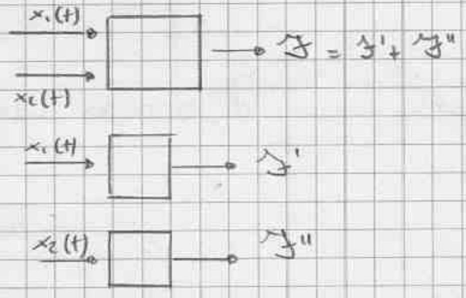
$x(t) = \cos 2\pi f_0 t$

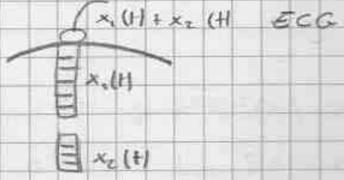
$x(t) = \sin 2\pi f_0 t$ $\quad \int_{-\infty}^{+\infty} e^{j2\pi f_0 t} e^{-j2\pi f t} dt = \delta(f - f_0)$

02-03

$\boxed{1}$ Linearità

$$\mathcal{F}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 \mathcal{F}\{x_1(t)\} + a_2 \mathcal{F}_2\{x_2(t)\}$$



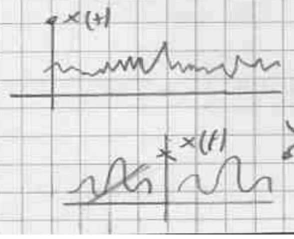


posso applicare una sovrapposizione degli effetti

$\boxed{2}$ Parità

$x(t)$ reale $\rightarrow |x(f)|$ è PARI

$\angle x(f)$ è DISPARI



faccia la trasformata

posso cancellare la parte negativa grazie alla parità

$\boxed{6}$

LA CONVOLUZIONE

$$x_1(t) * x_2(t)$$
$$\updownarrow$$
$$X_1(f) \cdot X_2(f)$$ prodotto tra gli spettri

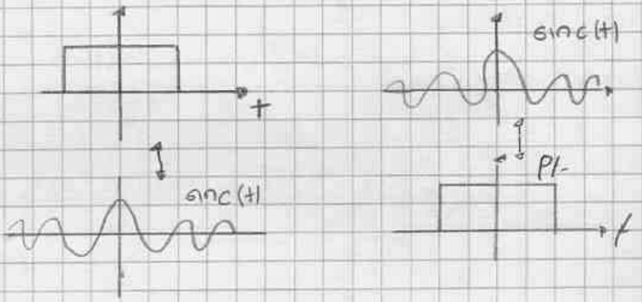
⑦

Derivazione

$$\frac{dx}{dt}(t)$$
$$\updownarrow$$
$$j2\pi f \cdot X(f)$$

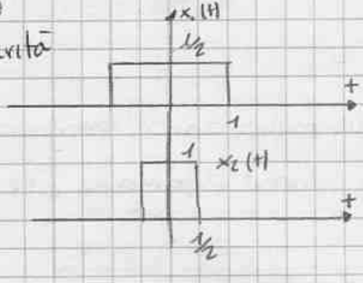
$\boxed{8}$

$$x(t) \longleftrightarrow X(f)$$
$$X(t) \longleftrightarrow x(-f)$$



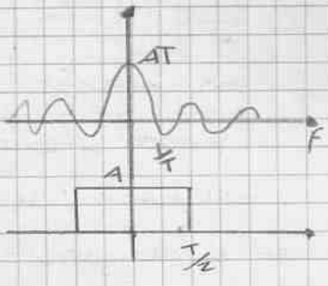
Esempi

① Linearità



$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}$ Fare la trasformata di qst porta

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} e^{-j2\pi f t} dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{3}{2} e^{-j2\pi f t} dt + \int_{\frac{1}{2}}^{1} \frac{1}{2} e^{-j2\pi f t} dt$$



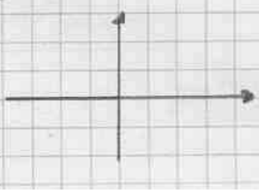
$$X_1(f) = \frac{\sin \pi f 2}{\pi f} \cdot \frac{1}{2}$$

$$X_2(f) = \frac{\sin \pi f 1}{\pi f} \cdot 1$$

la trasformata è praticamente la somma di qst 2 SINC

② Parità

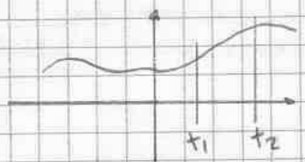
$$x(t) = \cos 2\pi f_0 t$$

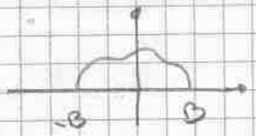


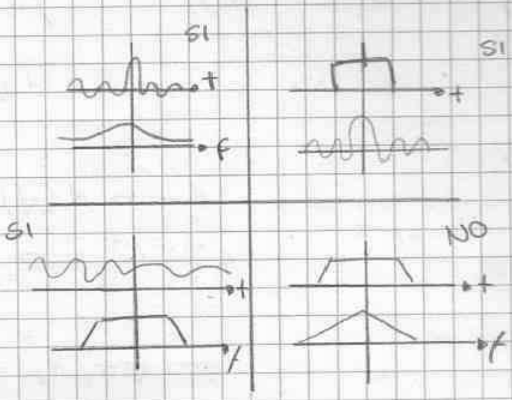
$$= \int \left(e^{j2\pi f_0 t} \cdot \frac{1}{2} + e^{-j2\pi f_0 t}\right) e^{-j2\pi f t} dt =$$

$$= \int \underbrace{e^{+(j2\pi f_0 - j2\pi f)}}_{\text{negativi}} dt + \int \underbrace{e^{+(-j2\pi f_0 - j2\pi f)}}_{\text{positivi}} dt$$

È impossibile che un segnale sia limitato nel tempo e nella frequenza.
Esistono però gli altri 3 casi.







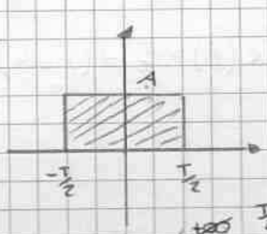
Energia di un segnale

05-03

$$\mathcal{E}_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

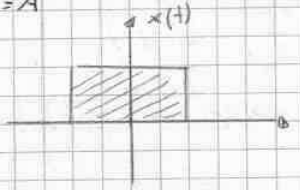
Potenza di un segnale

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$$



$$A p_T(t) \longrightarrow \int_{\cancel{-\infty}\ -\frac{T}{2}}^{\cancel{+\infty}\ \frac{T}{2}} A^2 \, dt = A^2 T$$

$x(t) = A$



$\mathcal{E}_p = A^2 T$ $\quad P = \frac{A^2 \cancel{T}}{\cancel{T}} = A^2$ → potenza

↓

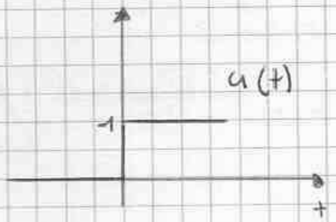
Energia sul periodo

Es

① $x(t) = u(t) e^{-kt}$ $\quad k > 0$

$$\mathcal{E} = \int_{-\infty}^{+\infty} |u(t)| e^{-kt} |^2 \, dt =$$

$$= \int_{0}^{+\infty} e^{-2kt} \, dt =$$



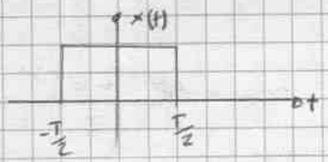
$\int e^{At} dt = \frac{1}{A} e^{At}$

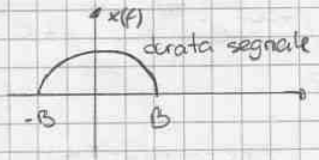
$$= \left[-\frac{1}{2k} e^{-2kt} \right]_0^{+\infty} = 0 + \frac{1}{2k} = \frac{1}{2k}$$

l'energia deve sempre essere positiva

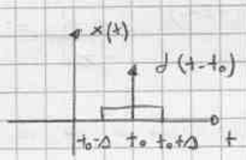
DURATA di un SEGNALE

06-03

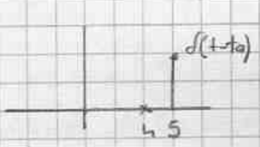




Approfondimento della DELTA di DIRAC



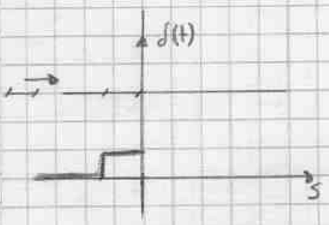
$$\int_{-\infty}^{+\infty} \delta(t-t_0)\,dt = 1 = \int_{t_0-\Delta}^{t_0+\Delta} \delta(t-t_0)\,dt$$



$$\int_{-\infty}^{4} \delta(t-5)\,dt = 0$$

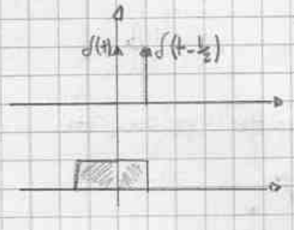
$$\int_{5}^{+\infty} \delta(t-5)\,dt = 1$$

$$\int_{s}^{s+1} \delta(t)\,dt =$$

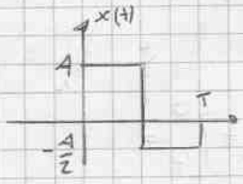


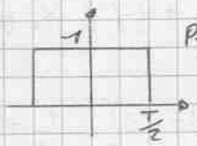
prima dello zero l'integrale è uguale a 0; qnd l'intervallo intercetta la δ allora in qll ~~caso~~ ho un valore 1 per tutta la durata dell'intervallo

$$\int_{s}^{s+1} \delta(t) + \delta(t-\tfrac{1}{2})\,dt$$



Approfondimento ANTICIPO e RITARDO





trasformata

$P_T(t) \longrightarrow T\,\mathrm{sinc}\,(fT)$ $\quad \mathrm{sinc} = \frac{\sin \pi f t}{\pi f}$

$$\int_{-\infty}^{+\infty} x(t)e^{-jf_0 2\pi t}\,dt = /$$

verificare la linearità di $y(t) = 5x(t) + 5$

$x_1 \to (5x_1 + 5)a_1$

$x_2 \to (5x_2 + 5)a_2$ NON è lineare.

$a_1x_1 + a_2x_2 \to 5(a_1x_1 + a_2x_2) + 5 = 5a_1x_1 + 5a_2x_2 + 5$

$y(t) = \int_s^{s+1} x(t)$ è lineare ?

TEMPO e INVARIANZA

$x(t) \to y(t) \to y(t-T)$ ritardo l'uscita $y(t)$

$x(t) \to x(t-T) \to \tilde{y}(t)$

- $y(t) = 5x(t) + 5$ quindi è un sistema non lineare a tempo invariante

  $x(t-T) \to 5x(t-T) + 5$

  $x(t) \to 5x(t) + 5 \to 5x(t-T) + 5$

- $y(t) = \cos(2\pi f_0 t)\, x(t)$ qst sistema NON è a tempo invariante

  $x(t-T) \to \cos 2\pi f_0 t\, x(t-T)$

  $x(t) \to \cos 2\pi f_0 t\, x(t) \to \cos 2\pi f_0 (t-T)\, x(t-T)$

- $y(t) = \boxed{t}\, x(t)$ $\to$ non dipende dal tempo sistema NON a tempo invariante

  $x(t-T) \to t\, x(t-T)$

  $x(t) \to t\, x(t) \to (t-T)\, x(t-T)$

MEMORIA del SISTEMA

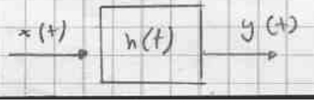
Un sistema che dipende dai sistemi passati si dice a memoria.

$y(t) = 5x(t)$

$y(t) = 5x(t-5)$

SISTEMI LTI

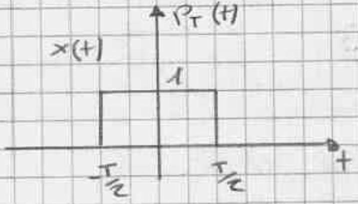
Sistemi lineari a tempo invariante

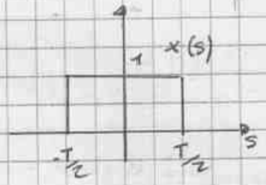


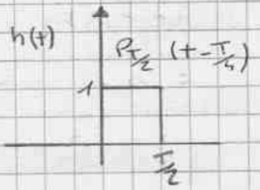
$y(t) = x(t) * h(t)$

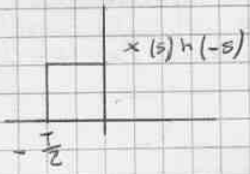
$x(t) = P_T(t)$ ingresso

$x(t) = P_{T/2}(t)$ risposta all'impulso





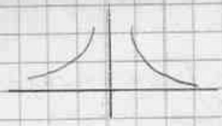




$$\int_{-\infty}^{+\infty} x(s)\, h(-s)\, ds = \frac{T}{2}$$

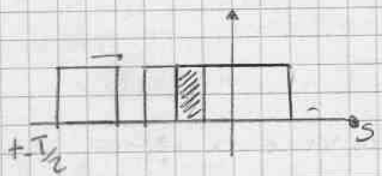
esempio

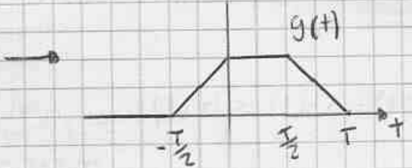
$$\int_{-\infty}^{+\infty} e^{-|t|} dt = 2\int_0^{\infty} e^{-t} dt = 2\left[-e^{-t}\right]_0^{\infty} = 2$$



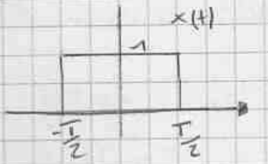
se integro in una variabile con un parametro alla fine il parametro diventa una variabile

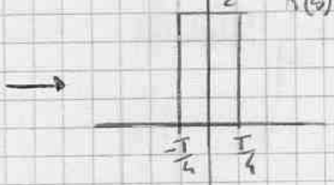
$x(s)$ rimane ferma

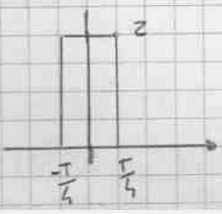


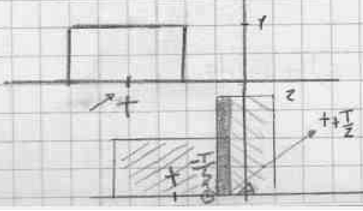


convoluzione di 2 porte







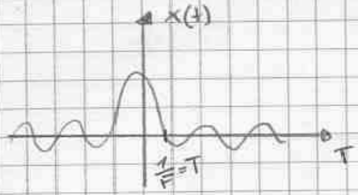


altezza 2 e base 4B

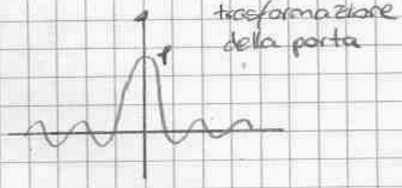
12-03

$$\mathcal{E}_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(f)|^2 df$$

$$\int_{-\infty}^{+\infty} \left| \frac{\sin \pi F t}{\pi F} \right|^2 dt$$ il primo zero è su $\frac{1}{T}$

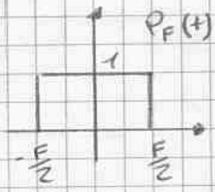


$$\frac{\sin \pi f T}{\pi f}$$



funzione sinc nel tempo

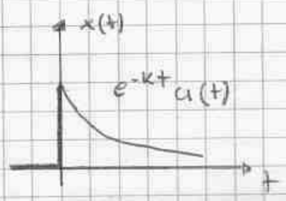
a cui corrisponde una porta in f



Es ①

$x(t) = e^{-kt} u(t)$ $k>0$ Trovare relazione tra k e θ in modo che

$y(t) = x(t) * \frac{\sin \theta t}{\pi t}$ $\mathcal{E}_y = \mathcal{E}_x \frac{1}{2}$



$$\mathcal{E}_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |e^{-kt} u(t)|^2 dt =$$

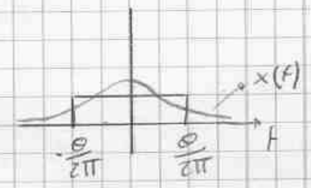
$$= \int_0^{+\infty} |e^{-kt}|^2 dt = \int_0^{+\infty} e^{-2kt} dt = \frac{1}{2k}$$

$$\int_{-\infty}^{\infty} x(s) y(t-s) ds = \int_{-\infty}^{+\infty} y(s) x(t-s) ds$$

$$\int_{-\infty}^{+\infty} e^{-ks} u(s) \frac{\sin \theta (t-s)}{\pi (t-s)} ds = \int_{-\infty}^{+\infty} e^{-k(t-s)} u(t-s) \frac{\sin \theta s}{\pi s} ds$$

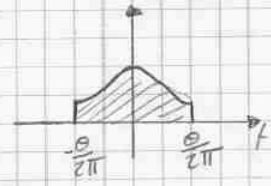
$$Y(f) = x(f) \cdot p_{\frac{\theta}{\pi}}(f)$$

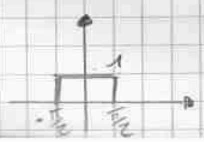
$$\varphi(f) = \frac{1}{-k + j2\pi f}$$



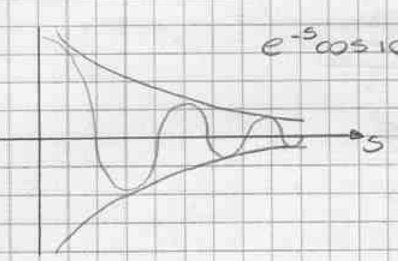
$$\frac{\sin \theta t}{\pi t} \quad \frac{\sin \pi F t}{\pi t}$$

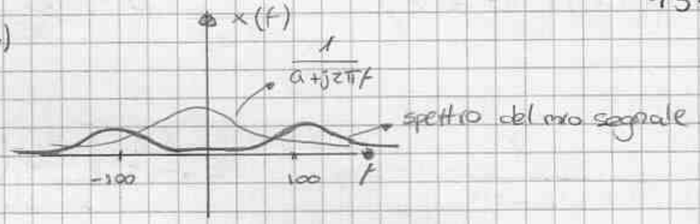
$$\downarrow \quad \theta = \pi F \quad F/2 = \frac{\theta}{2\pi}$$





13-03





$$x(t)\cdot\cos(2\pi\cdot\overset{50}{\cancel{100}}t)$$

$$X(f) * \left[\frac{1}{2}\delta(f-100)+\frac{1}{2}\delta(f+100)\right]$$

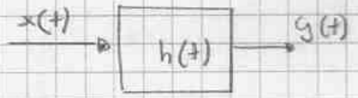
$$\tilde{X}(f) * \left[\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)\right] = \frac{1}{2}\tilde{X}(f-f_0)+\frac{1}{2}\tilde{X}(f+f_0)$$

$s=\frac{t}{2}$

$$=\frac{1}{2}\left[\frac{1}{1+j2\pi(f-f_0)}+\frac{1}{1+2\pi j(f+f_0)}\right]=\tilde{X}(f)$$

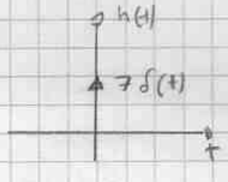
$$e^{-t/2}u(t)\cos 50\cdot 2\pi t \xrightarrow{\mathcal{F}} \frac{1}{2}\left[\frac{1}{\frac{1}{2}+j2\pi(f-50)}+\frac{1}{\frac{1}{2}+j2\pi(f+50)}\right]$$

SEGNALI LINEARI A TEMPO INVARIANTE



$y(t) = x(t) * h(t)$

$y(t) = 7\,x(t) \qquad h(t) = 7\,\delta(t)$



La δ è l'elemento unitario della convoluzione

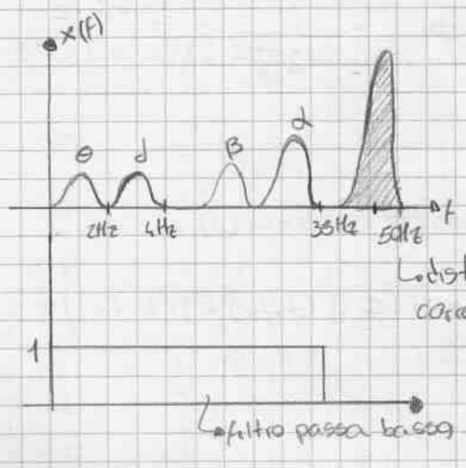
$x(t) * \delta(t) = x(t)$

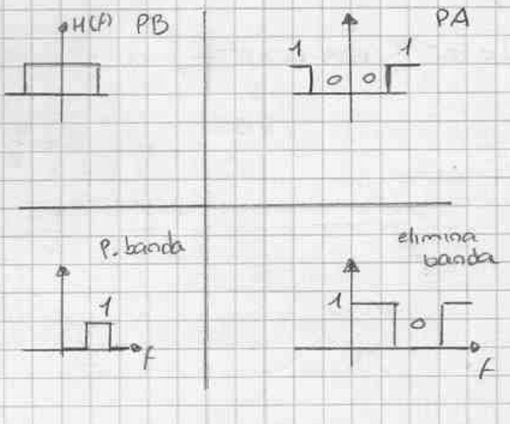
$x(t) * \delta(t-t_0) = x(t-t_0)$

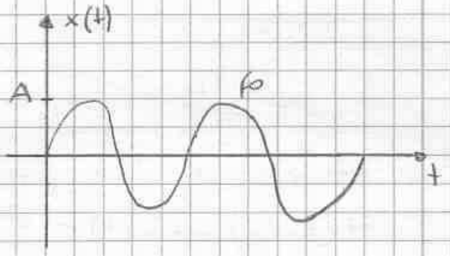
$\mathcal{F}$

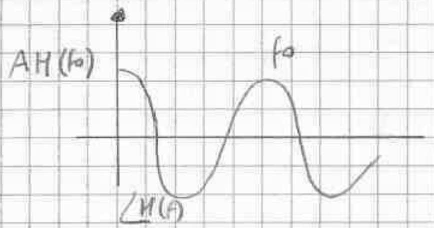
$x(f)\cdot 1 = x(f)$

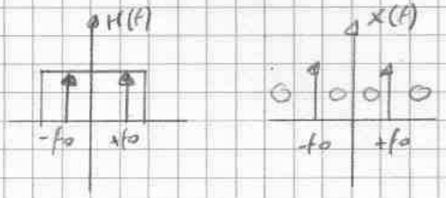
$x(f)\cdot 1\, e^{-j2\pi f t_0}$











⇓

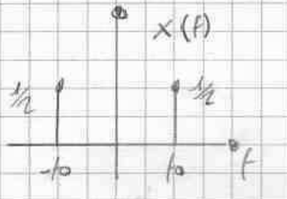
altezza modificata dalla porta

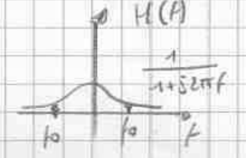
esempio:

$x(t) = \cos 2\pi f_0 t$

$h(t) = e^{-t} u(t)$ se rimango nel dominio del tempo devo fare la convoluzione

$Y(f) = X(f) \cdot H(f)$





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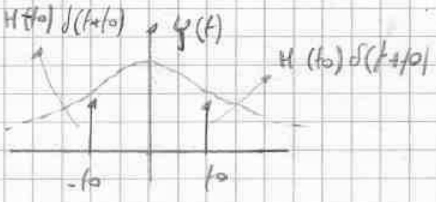
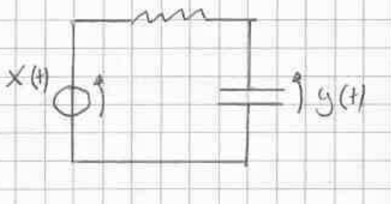


▷ PB

↓

taglia le frequenze alte

a me importa solo H(f0)



$$RC \frac{dy}{dt}(t) + y(t) = x(t)$$

$\downarrow \mathcal{F}$

$$RC\, j2\pi f\, Y(f) + Y(f) = X(f)$$

$$Y(f) = X(f)\, H(f) \qquad H(f) = \frac{Y(f)}{X(f)}$$

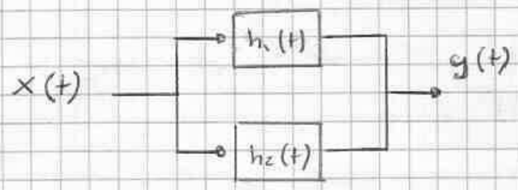
▷ PB

$$Y(f)\left[RC\, j2\pi f + 1\right] = X(f) \qquad H(f) = \frac{Y(f)}{X(f)} = \frac{1}{RC\, j2\pi f + 1}$$

↓

trasformata dell'esponenziale

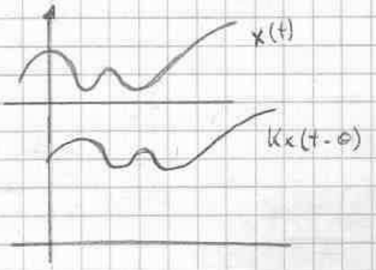
SISTEMI in PARALLELO



SISTEMI DISTORCENTE e NON

Un segnale e' non distorcente se il segnale mantiene una sua forma a meno di una costante

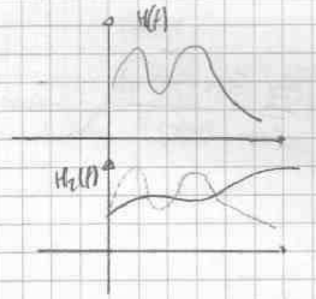
$$y(t) = \underset{\text{amplificazione}}{K}\, x(t-\underset{\text{ritardo}}{\theta})$$



risulta avente una energia più alta e ritardato

$$y(t) = Kx(t-\theta)$$

$$Y(f) = KX(f)e^{-j2\pi f\theta} \qquad H(f) = \frac{Y(f)}{X(f)} = Ke^{-j2\pi f\theta}$$

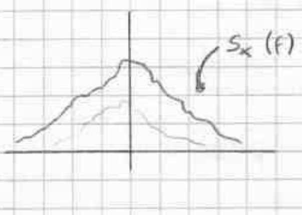


cuore → stetoscopio → orecchio

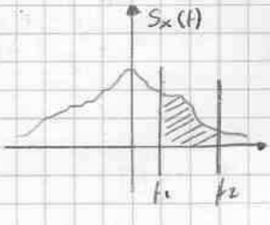
x(t) h(t) y(t)

SPETTRO di ENERGIA di un SEGNALE

$$S_x(f) = |X(f)|^2$$



$$\int_{-\infty}^{+\infty} S_x(f) = \mathcal{E}_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

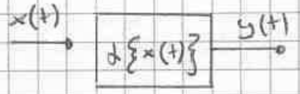


$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$S_y(f) = S_x(f)|H(f)|^2$$

②

$y(t) = x(t) + kx(t-T) \quad k \in \mathbb{R}$



lineare tempo invariante

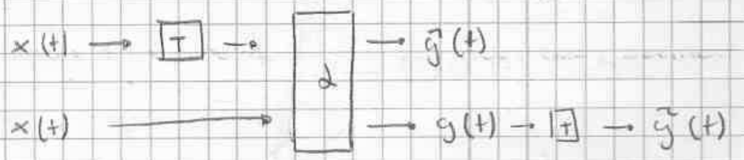
se $\mathcal{L}$ è LTI $\Rightarrow h(t) \mid y(t)$ se $x(t) = \delta(t)$

$y_1 = x_1(t) + kx_1(t-T)\, a_1$

$y_2 = x_2(t) + kx_2(t-T)\, a_2$

$a_1x_1 + a_2x_2 \to (a_1x_1(t) + a_2x_2(t)) + k\,(a_1x_2(t-T) + a_2x_2(t-T))$

$=$ linearità

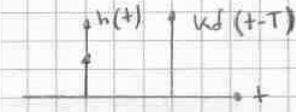


$x(t-T) \to x(t-T) + k\,x(t-T-T)$

$x(t) \to x(t) + kx(t-T)$

$\to x(t-T) + kx(t-2T)$ $\parallel \to$ tempo inv.

risposta impulso $h(t) = \delta(t) + k\,\delta(t-T)$



③

$y(t) = 5x(t) + 5x(t-5T)$

$x(t-T) \to 5(t-T) + 5x(t-T-5T)$

$x(t) \to 5x(t) + 5x(t-5T) \to 5(t-T) + 5x(t-6T)$

$- h(t) = 5\delta(t) + 5\delta(t-5T)$

④

$y(t) = x(t) + 2$

~~$x(t-T) \to x(t-T) + 2$~~ non è lineare

~~$x(t) \to x(t) + 2 \to x(t-T) + 2$~~

h(t) non è definita

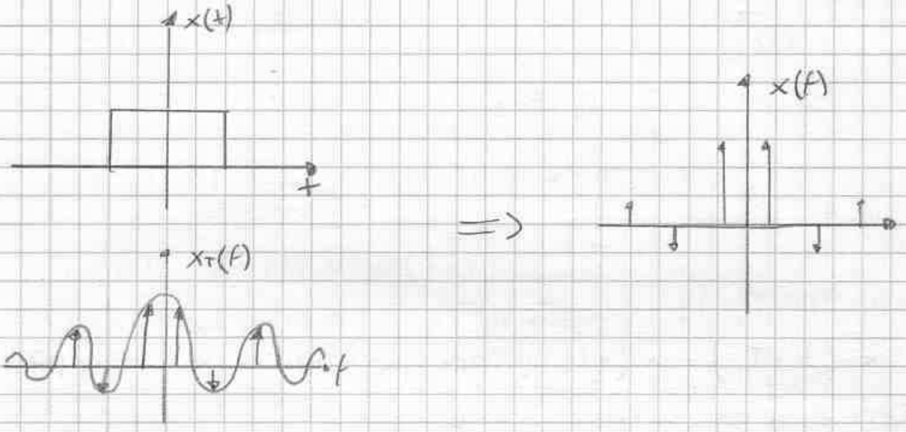
Un segnale periodico ha dunque uno spettro a righe sempre e le righe sono distanti $f_0 = \frac{1}{T}$. Nel caso della porta ripetuta non mi interessa quanto sono larghe ma ogni quanto si ripetono.

Riprendendo la dimostrazione

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j2\pi \frac{n}{T} t}\, dt\, \delta(f - f_0 n) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_T(t)\, e^{-j2\pi \frac{n}{T} t}\, dt\, \delta(f - nf_0) =$$

$$= \sum_{n=-\infty}^{+\infty} X_T(nf_0)\, \delta(f - f_0)$$

Questo risultato mi dice che l'altezza delle righe dello spettro è modulata dall'ampiezza della trasformata del segnale originale



Qualsiasi segnale periodico filtrato (con filtro simmetrico) è una somma di sinusoidi.

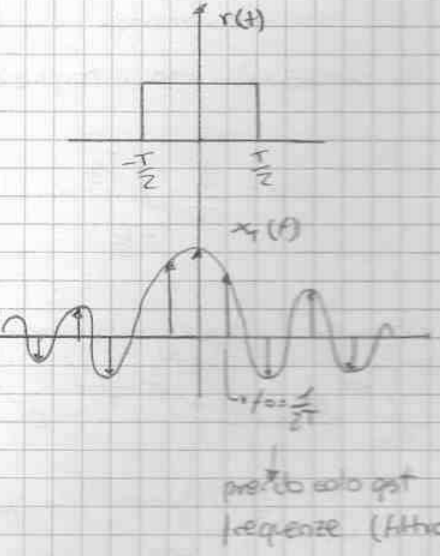
Esempio

$$r(t) = p_T(t)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} r(t - 2nT)$$

$$\sum_{n=-\infty}^{+\infty} X(nf_0)\, \delta(T - nf_0) = \sum \frac{\sin \pi n f_0 T}{\pi n f_0}\, \delta(f - nf_0)$$

con $f_0 = \frac{1}{2T}$



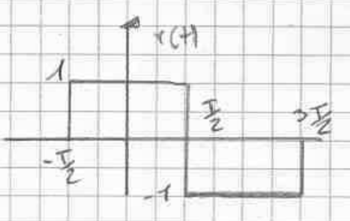
prendo solo qst frequenze (filtro)

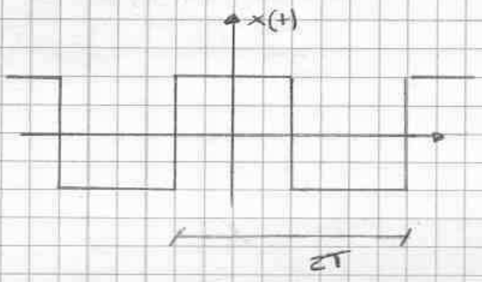
$$= \Sigma \int_{-\infty}^{+\infty} \delta(t - iT)\, e^{-j2\pi f iT}\, dt = \sum_{i=-\infty}^{+\infty} e^{-j2\pi f iT}$$

Scrivere un treno di δ è uguale a scrivere una sommatoria degli esponenziali complessi.

$$x(f) = \frac{1}{T} \Sigma (f - i f_0) = \Sigma e^{-j2\pi f iT}$$

ONDA QUADRA





$$f_0 = \frac{1}{2T}$$

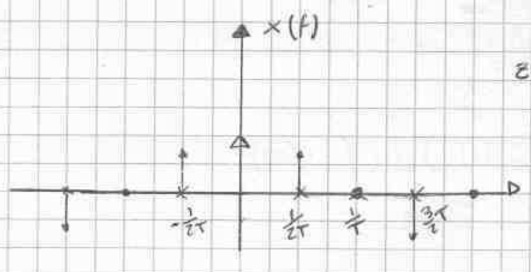
$$x(f) = \frac{1}{T} \sum_{i=-\infty}^{+\infty} R(i f_0)\, \delta(f - i f_0)$$

$$x(t) = \sum_{i=-\infty}^{+\infty} r(t - 2iT) \qquad R(f) = \frac{\mathrm{sinc}(\pi f_0 T)}{\pi f_0} - \frac{\mathrm{sinc}(\pi f_0 T)}{\pi f_0}\, e^{-j2\pi f T}$$

somma di due parte; la prima più la seconda rovesciata traslata.

$$x(f) = \frac{1}{2\cancel{T}} \sum_{i=-\infty}^{+\infty} \frac{\sin \pi \frac{i}{2\cancel{T}} \cancel{T}}{\pi \frac{i}{\cancel{2T}}} \left(1 - e^{-j2\pi \frac{i}{2\cancel{T}} \cancel{T}}\right) \cdot = \delta\left(f - \frac{i}{2T}\right)$$

$$= \sum_{i=-\infty}^{+\infty} \frac{\sin \pi \frac{i}{2}}{\pi i} \left(1 - e^{-j\pi i}\right) \delta\left(f - \frac{i}{2T}\right)$$



zeri $i = 2, 4, 6 ..$ multipli di 2

$$E_y = \int_{-\infty}^{+\infty} |Y(f)|^2 dt = 2\int_0^{\frac{\theta}{2\pi}} \frac{1}{k^2+4\pi^2 f^2} df = \frac{2}{k^2}\int_0^{\frac{\theta}{2\pi}} \frac{1}{1+\frac{4\pi^2 f^2}{k^2}} df =$$

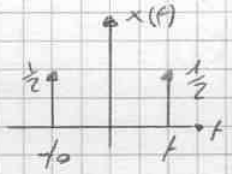
$$x(f) = \frac{1}{k+j2\pi f} \longrightarrow \frac{1}{k^2+4\pi^2 f^2} \qquad = \frac{k2}{2\pi k^2}\left[\mathrm{atg}\left(\frac{2\pi f}{k}\right)\right]_0^{\frac{\theta}{2\pi}}$$

$$E_x = \frac{1}{2k} \qquad E_y = \frac{1}{\pi k}\left[\mathrm{atg}\frac{\theta}{k} - 0\right] \qquad \frac{1}{\pi \cancel{k}}\,\mathrm{atg}\frac{\theta}{k} = \frac{1}{2\cancel{k}}\cdot\frac{1}{2} \qquad \mathrm{atg}\frac{\theta}{k} = \frac{\pi}{4}$$

Es ②

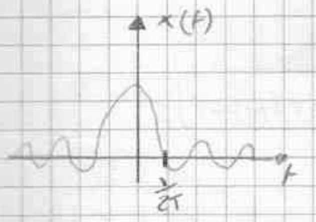
$$x(t) = A\,p_{2T}\,(t-\tau)$$

$$X(f) = A\,\frac{\sin 2\pi T f}{\pi f}\cdot e^{-j2\pi f\tau}$$



trasformata del coseno

$$P = \Sigma|\mu_n|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$



Es ③

- $x(t) = e^{-at}\,u(t)$

$$x(f) = \int_0^{+\infty} e^{-at} e^{-j2\pi f t} dt = \frac{1}{a+j2\pi f}$$

- $x(t) = e^{3t}\,u(-t)$

$$s = -t \quad x(t) = X(f) \longrightarrow x(-t) = X(-f)$$

$$x(-t) = e^{-3t}\,u(t) \longrightarrow X(f) = \frac{1}{3+j2\pi f} \qquad \hat{x}(f) = \frac{1}{3-j2\pi f}$$

- $x(t) = e^{-2t+4}\,u(t-2)$

$$s = t-2 \quad x(f) = \mathcal{F}\left\{e^{-2s}\,u(s)\right\} = \frac{1}{2+j2\pi f}\,e^{-j2\pi f\cdot 2}$$

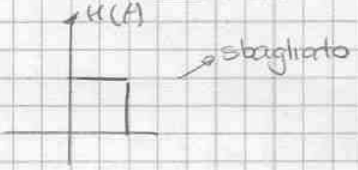
- $x(t) = e^{-t/2}\cos(100\pi t)\,u(t)$

$$\underset{\substack{\downarrow \\ 2\pi 50 t \\ \downarrow \\ f_0}}{} \qquad s = \frac{t}{2} \quad X(s) = e^{-s}\cos 2\pi\, 100 s\cdot u(s)$$

$$\tilde{x} = \frac{1}{1+j2\pi f}\ast h\left(\frac{1}{2}\delta(f-f_0) + \delta(f-f_0)\right)$$

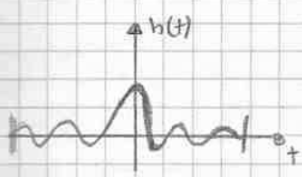
I filtri devono avere i coeff. reali e non complessi

1) se h(t) è reale la sua trasformata è pari.



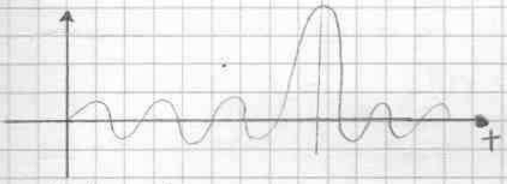
non posso traslare perché
$H(f-f_0) \rightarrow h(t)\,e^{+j2\pi f_0 t}$

2) h(t) reale



se nel tempo moltiplico per una porta, nella frequenza devo fare una convoluzione con una sinc.

3) h(t) causale



$h(t)=0 \quad t<0$

moltiplicazione per un esponenziale nella frequenza

4)
se x(t) è limitato anche y(t) deve esserlo (B.I.B.O)

$$\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$$

Se x(t) è una sinusoide ⇒ y(t) è sinusoidale

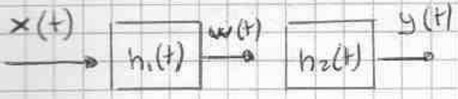
$h(t)$, $|H(f)|$ $\angle H(f)$

$x(t) = A\cos(2\pi f_0 t)$

$y(t) = A|H(f_0)|\cos(2\pi f_0 t + \angle H(f))$ l'uscita ha un cambiamento di fase e di modulo.

SISTEMI IN SERIE

16-03



$w(t) = x(t) * h_1(t)$

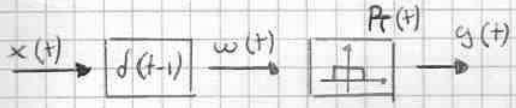
$y(t) = w(t) * h_2(t)$

Li posso vedere come un unico sistema

$h_{eq}(t) = h_1(t) * h_2(t)$ $\qquad y(t) = x(t) * \left(h_1(t) * h_2(t)\right)$ $\qquad x * h = h * x$

$Y(f) = H_2(f) \cdot W(f)$

$W(f) = H_1(f) \cdot X(f) \Rightarrow Y(f) = H_1(f)\, H_2(f)\, X(f)$ $\qquad H_1(f)\, H_2(f) = H_{eq}(f)$



$h_1(t) = \delta(t-1)$

$h_2(t) = P_T(t)$

$Y(f) = X(f)\, H_1(f)\, H_2(f)$

$$Y(f) = X(f) \cdot \underbrace{1\, e^{-j2\pi f} \frac{\sin \pi f t}{\pi f}}_{H_{eq}(f)}$$

$$X(f) = \frac{\sin \pi f t}{\pi f} = x(t)$$

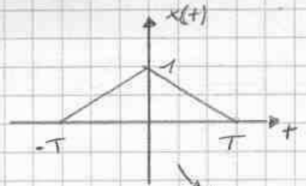
$$Y(f) = \frac{\sin \pi f t}{\pi f} \cdot \frac{\sin \pi f t}{\pi f}\, e^{-j2\pi f} = \frac{\sin^2 \pi f t}{(\pi f)^2}\, e^{-j2\pi f}$$ dalle tavole risulta

↓

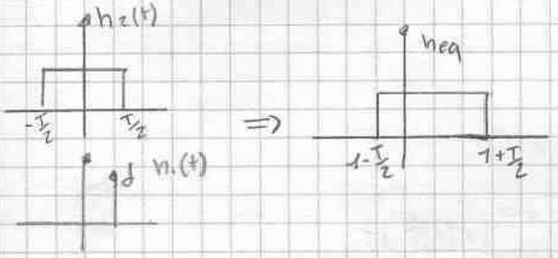
$\frac{\cancel{T}}{T} \frac{\sin^2 \pi f T}{(\pi f)^2}$ trasformata del triangolo

$h_{eq}(t) = h_1(t) * h_2(t)$

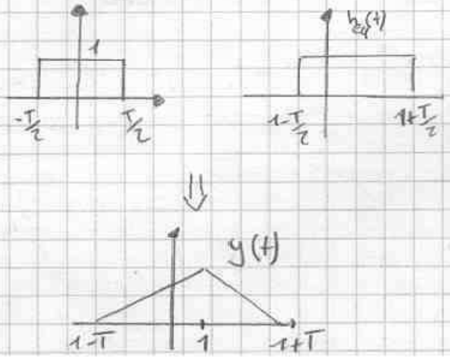
$h_{eq}(t) = \delta(t-1) * P_T(t)$

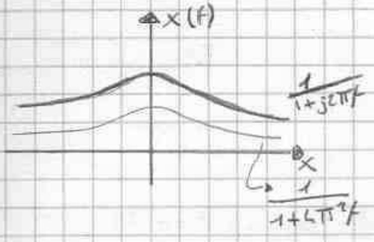


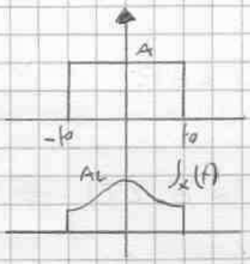
il supporto è doppio



$y(t) = x(t) * h_{eq}(t)$







ESERCITAZIONE

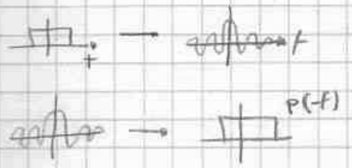
19-03

①

$$x(t) = -\frac{t}{a+j\pi t} = \ast$$

$$\tilde{x}(t) = \frac{t}{a+j\pi t}$$

$x(t) \longrightarrow X(f)$



$$x(kt) \rightarrow \frac{1}{k} X\left(\frac{f}{k}\right)$$

$$j2\pi f\, X(f) \longrightarrow \frac{d}{dt} x(t)$$

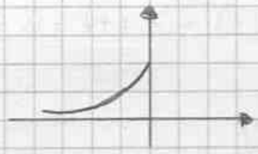
$$\frac{d}{df} x(f) \longleftarrow j2\pi t\, X(t)$$

$$e^{-at}(t)\, u(t) \longrightarrow \frac{1}{a+j2\pi f}$$

$$\frac{1}{a+j2\pi t} \longrightarrow e^{at}\, u(f)$$

$$\frac{1}{a+j2\pi t}\,\frac{2}{2} \longrightarrow 2e^{2af}\, u(-2f)$$

$$\tilde{x}(f) = 2e^{2af}\, u(-f)$$



$$\ast = j2\pi t\,\frac{1}{a+j\pi t} \longrightarrow \frac{d}{df}\, 2e^{2af}\, u(f)$$

$$4ae^{2af}\, u(f) + 2e^{2af}\,\frac{du(f)}{df}$$

$$\downarrow$$

$$-\delta(f)$$

$$x(f) = -\frac{1}{j2\pi}\left(4ae^{2af}\, u(-f) - 2e^{2af}\delta(f)\right)$$

poi $x(f)\,\delta(f) = x(0)\,\delta(f) \Rightarrow$

$$x(f) = -\frac{1}{j2\pi}\left(4ae^{2af}\, u(-f) - \overset{\nearrow x(0)}{2}\,\delta(f)\right)$$

⑤

$$y(t) = |x(t)|$$

$|x(t)|a_1$ Non è lineare → $h(t)$ non è definita

$|x(t)|a_2$

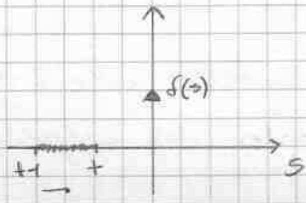
$$\overline{}$$

$|x(t)a_1 + x(t)a_2|$

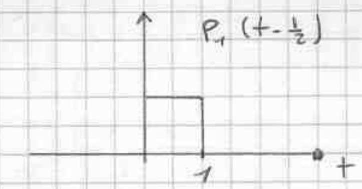
⑥

$$y(t) = \int_{t-1}^{t} x(s)\,ds$$

$$h(t) = \int_{t-1}^{t} \delta(s)\,ds$$



t-1, t si sposta, t incontra la δ



SEGNALI PERIODICI

È un segnale per cui $x(t+T) = x(t)$ quindi periodico nel tempo

Un segnale periodico può essere scritto come

$$x(t) = \sum_{n=-\infty}^{+\infty} x_T(t - nT)$$

quindi somma di un segnale x_T traslato

Devo calcolare la trasformata di F che ha come def. $x(t) = \sum_{n=-\infty}^{+\infty} \mu_n e^{j2\pi\frac{n}{T}}$

quindi somma di sinusoidi con ampiezza μ_n dove μ_n sono:

$$\mu_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,e^{-j2\pi\frac{n}{T}t}\,dt$$

Se devo trasformare un segnale periodico integro

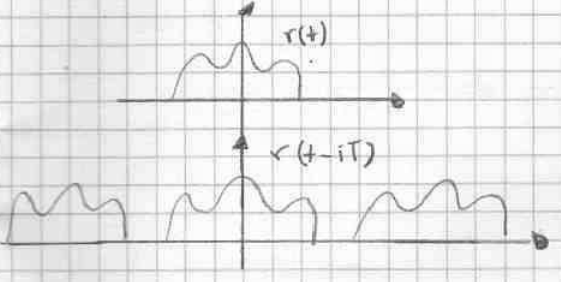
$$X(f) = \int_{-\infty}^{+\infty} \Sigma\mu_n e^{j2\pi\frac{n}{T}t} \cdot e^{-j2\pi ft}\,dt = \Sigma\mu_n \int_{-\infty}^{+\infty} e^{j2n\frac{n}{T}t} \cdot e^{-j2\pi ft}\,dt$$

pongo $f_0 = \frac{1}{T}$ $\quad X(f) = \Sigma\mu_n \int_{-\infty}^{+\infty} e^{j2\pi nf_0t} \cdot e^{-j2\pi ft}\,dt = \Sigma\mu_n \int_{-\infty}^{+\infty} e^{-j2\pi(f-nf_0)t}\,dt$

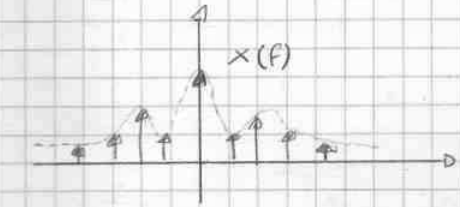
$$= \sum_{n=-\infty}^{+\infty} \mu_n\,\delta(f - nf_0)$$

$$x(t) = \sum_{i=-\infty}^{+\infty} r(t-iT)$$

23/03



$$X(f) = \frac{1}{T} \sum_{i=-\infty}^{+\infty} R(if_0)\, \delta(f-if_0)$$



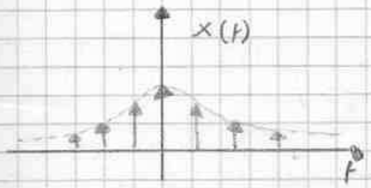
Esempio

①

$$r(t) = e^{-at} u(t) \qquad x(t) = \sum_{i=-\infty}^{+\infty} r(t-iT)$$

$$X(f) = \frac{1}{T} \sum_{i=-\infty}^{+\infty} R(if_0)\, \delta(f-if_0) \qquad f_0 = \frac{1}{T}$$

$$R(f) = \frac{1}{a+2\pi j f} \qquad X(f) = \frac{1}{T} \sum_{i=-\infty}^{+\infty} \frac{1}{a+j2\pi\frac{i}{T}}\, \delta\left(f-\frac{i}{T}\right)$$



②

$$r(t) = \delta(t) \qquad x(t) = \sum_{i=-\infty}^{+\infty} \delta(t-iT)$$

$$X(f) = \frac{1}{T} \sum_{i=-\infty}^{+\infty} R(if_0)\, \delta(f-if_0)$$

$$X(f) = \frac{1}{T} \sum_{i=-\infty}^{+\infty} \delta\left(f-\frac{i}{T}\right)$$

usando la definizione di trasformata $\quad t=iT$

$$X(f) = \int_{-\infty}^{+\infty} \sum_{i=-\infty}^{+\infty} \delta(t-iT)\, e^{-j2\pi f t}\, dt = \sum_{i=-\infty}^{+\infty} \int_{+\infty}^{-\infty} \delta(t-iT)\, e^{-j2\pi f t}\, dt$$

SPETTRO di ENERGIA di un segnale

FATTO

$$S_x(f) = |X(f)|^2$$

Proprietà

1) $S_x(f)$ è ≥ 0

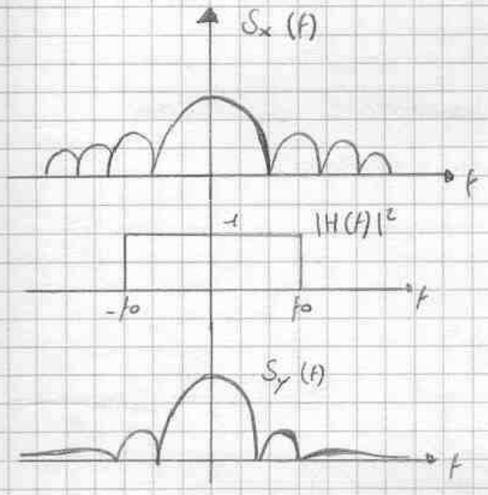
2) $\int_{-\infty}^{+\infty} S_x(f) = \mathcal{E}_x$

3) $x(t) \longrightarrow \boxed{H(f)} \longrightarrow y(t)$

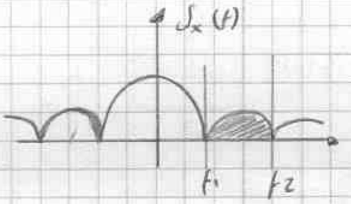
$$S_y(f) = S_x(f)\,|H(f)|^2$$

$$Y(f) = X(f)\,H(f)$$

$$S_y(f) = S_x(f)\,H^2(f)$$



$$S \int_{f_1}^{f} S_x(f)\,df$$



$$R_x(\tau) = \int_{-\infty}^{+\infty} S_x(f)\, e^{j2\pi f\tau}\, d\tau$$

$$\int_{-\infty}^{+\infty} |x^2(f)|\, e^{j2\pi f\tau}\, d\tau = \int x(f)\, \overset{*}{x}(f)\, e^{j2\pi f\tau}$$ → integrale di correlazione

autocorrelazione del segnale

$$x(t) \ast x(t)$$

$$x(f) \cdot x(f)$$

$$\downarrow$$

$$x(t) \ast x(t) = R_x(\tau)$$

$$\int_{-\infty}^{+\infty} x(t)\, x(\tau - t)\, dt = \int_{-\infty}^{+\infty} x(t)\,(t+\tau)\, dt$$

l'antitrasformata è pari!

Proprietà

1) $R_x(\tau)$ è pari

2) $R_x(0) \quad \int x(t) \cdot x(t)\, dt = \mathcal{E}_x$

ESERCITAZIONE 26-03

①

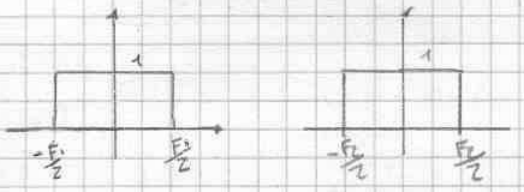
$$x(t) = \frac{1}{2} \sum_{i=0}^{+\infty} (-1)^i r_i(t)$$

$$P(t) = \lim_{aT \to \infty} \frac{1}{aT} \int_{-\frac{aT}{2}}^{\frac{aT}{2}} \left| \frac{1}{2} \sum_{i=0}^{+\infty} (-1)^i r_i(t) \right|^2 dt = \frac{2aT}{2aT} \frac{1}{4} = \frac{1}{4}$$

media temporale nulla → parte positiva si alterna con parte negativa

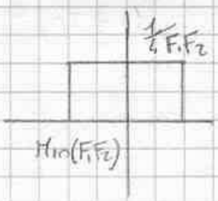
⑤

$$\frac{\sin(2\pi F_1 t)}{2\pi F_1 t} * \frac{\sin(2\pi F_2 t)}{2\pi F_2 t} = A \frac{\sin(2\pi F_A t)}{2\pi F_A t}$$



$\frac{\sin \pi F t}{\pi t}$ fattore $\frac{1}{2F_1}$

$\Downarrow$



$F_A = Min(F_1, F_2)$

$\frac{A}{2F_A} = \frac{1}{4F_1F_2}$ $\qquad \frac{A}{Min(F_1F_2)} = \frac{1}{2F_1F_2}$

$$A = \frac{Min(F_1F_2)}{2F_1F_2} = \frac{\cancel{Min(F_1F_2)}}{2\cancel{Min(F_1F_2)}\,Max(F_1F_2)}$$

$$A = \frac{1}{2} \frac{1}{Max(F_1F_2)}$$

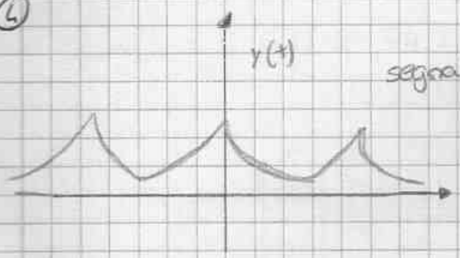
⑦

$$x(t) = -\frac{t-t_0}{a+j\pi(t-t_0)}$$

$s(t) = \frac{1}{a+j2\pi t}$ partiamo da qst

$$= \frac{1}{2}\sum_{i=-\infty}^{\infty}\left(\frac{\sin\frac{\pi n}{2T}\frac{4T}{3}}{\frac{\pi 4 \frac{T}{3}}{2T} n}\right)^2 \delta\left(f-\frac{n}{2T}\right)$$

④



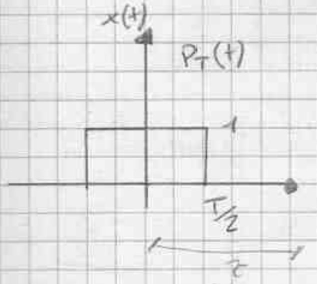
segnale periodicizzato dell'esponenziale bilatera

$$\frac{1}{T}\sum R(n f_0)\,\delta(f - f_0 n)$$

la serie geometrica è $\sum_{0}^{\infty} a^n = \frac{1}{1-a}$ $|a|<1$

AUTOCORRELAZIONE

30-03



è come un Δt ovvero di qnt varia il segnale su Δt

$h(t) = Ax(t)$ $\quad \phi_h(\tau)$ $\quad \mathcal{G}_h(f)$

$\phi_x(\tau) = \int x(t)\, x(t+\tau)\, dt$

- se è pari
- max nell'origine

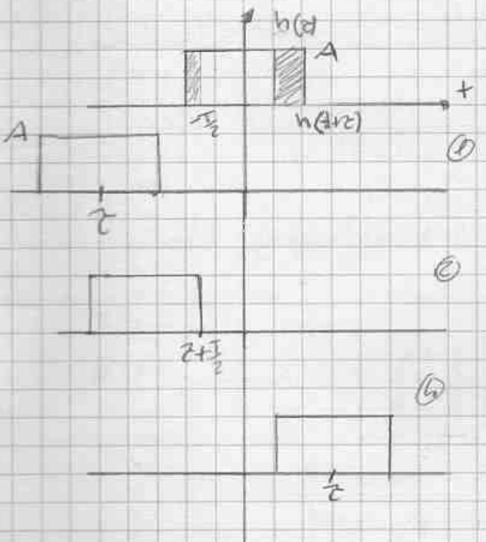
$y_x = \mathcal{F}^{-1}\{\phi_x(\tau)\}$

$\phi(\tau) = x(t) \circledast x^*(-t) = x(t) * x(t)$

perchè anche se la ribalto è sempre uguale

$X(f) = AT \frac{\sin(\pi f T)}{\pi f T}$

$\phi(0) = \mathcal{E}_h = A^2 T$

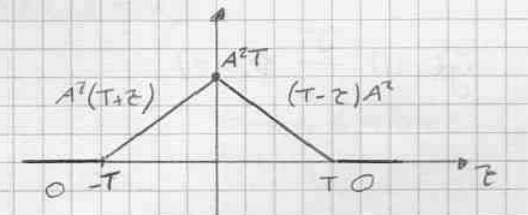


1) $\tau < -T \longrightarrow 0$

2) $\left(\left(\tau + \frac{T}{2}\right) + \frac{\tau}{2}\right) A^2$

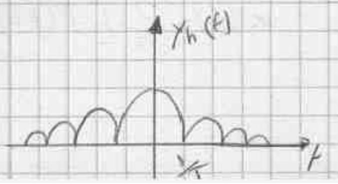
3) $\tau = 0$ parte sovrapposte TA^2

4) $\tau < T \longrightarrow \left(\left(\frac{T}{2} - \tau\right) + \frac{T}{2}\right) A^2$



Devo fare la TF di qst segnale

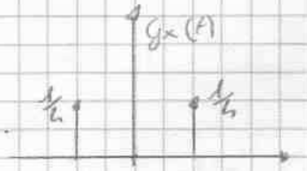
$\mathcal{G}_Y(f) = \mathcal{F}\{\phi(\tau)\} = A^2 T \frac{\sin^2(\pi f T)}{(\pi f T)^2}$



$x(f) = \sum_n \cancel{\mu} \delta(f - nf_0)$

$G_x(f) = \sum_n \mu_n^2 \, \delta(f - f_0)$

$G_Y(f) = \frac{1}{4}\delta(f-f_0) + \frac{1}{4}\delta(f+f_0)$

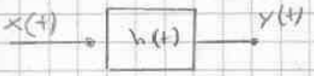


il cos ha un'autocorrelazione costante!

l'antitrasformata è

$\phi(\tau) = \frac{1}{4}\overset{\cos}{e^{j2\pi f_0 t} + \frac{1}{4}e^{-j2\pi f_0 t}}$

l'autoc. di un coseno è ancora un coseno



$G_Y(f) = G_x(f)\, H^2(f)$ l'autoc. in uscita è uguale all'autoc. convoluta con $H(f)$

$\delta_Y(f) = d_x(f)\, H^2(f)$

③ $x(t)$

$\overset{\cos(\,)\cdot\cos(\,)}{x(t) = \cos^2\left(2\pi \frac{t}{T}\right)}$

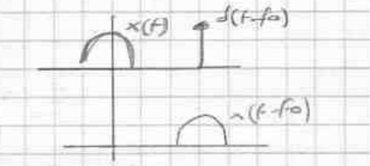
$h(t) = P_T(t)$ calcolare lo spettro di potenza $G_Y(f)$

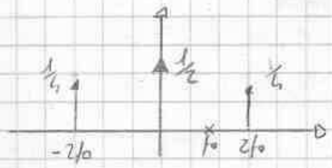
$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$ 1° metodo

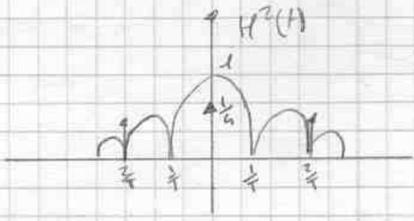
$x(f) = \left[\cancel{\cos}\frac{1}{2}\delta(f-f_0) + \frac{1}{2}(f+f_0)\right] * \left[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right]$ 2° metodo

$x(f) * \delta(f-f_0) = x(f-f_0)$

$x(f) = \frac{1}{2}\delta(f) + \frac{1}{2}\cdot\frac{1}{2}\left(\delta(f-f2) + \delta(f+f2)\right)$

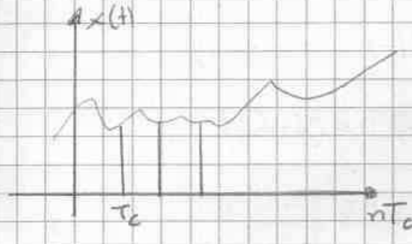






spettro dell' esponenziale bilatera

$$R(f) = \frac{2a}{a^2+4\pi^2 f^2} \qquad \Sigma R\left(f-\frac{n}{T}\right) = \sum_{n=-\infty}^{+\infty} \frac{2a}{a^2+4\pi^2 \frac{n^2}{T^2}}$$



nT_c → campionamento, ogni tot rileva la x(t)
non ho più una scala del tempo

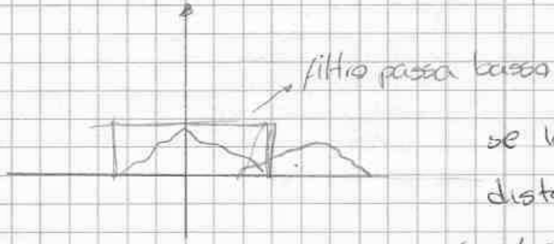
lo spettro di un segnale a righe è un treno segnale periodico.

Un treno di delta trasformato è sempre un treno di δ

$$x(t) \cdot \Sigma \delta(t-nT) \longrightarrow x(f) * \sum_{n=-\infty}^{+\infty} \delta\left(f-\frac{n}{T_c}\right) \qquad f_c = \frac{1}{T_c}$$

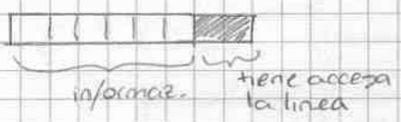
frequenza di campionamento

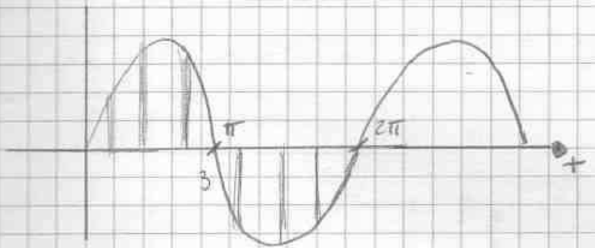
↓ sempre un segnale periodico



se la replica si sovrappone il segnale viene distorto - Fenomeno di interferenza aliasing.
La frequenza di campionamento minima deve essere $\boxed{f_c \geq 2B}$ Th di Nyquist

Perdo campioni fino a quando i segnali si sovrappongono
Questo che abbiamo visto è il Th del campionamento

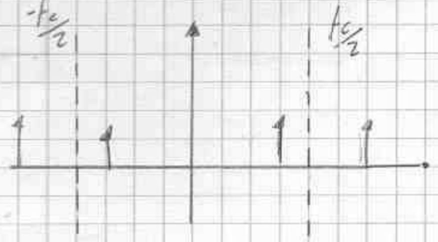




mi bastano 3 campioni per definire la sinusoide

$\sin 2\pi f_0 t \quad f_0 = 1$

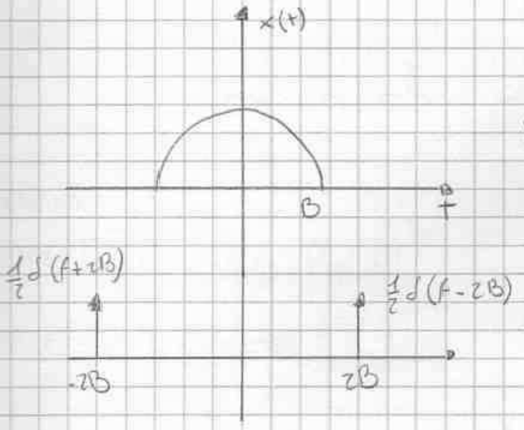
$B = 1Hz \quad f_c = 2Hz$



03-04

x(t) banda limitata

①

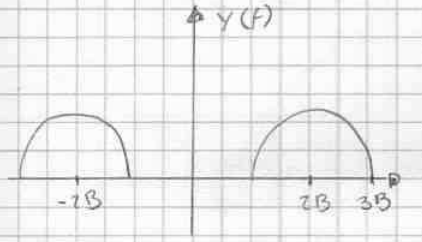


$y(t) = x(t)\cos 2\pi 2Bt$

$Y(f) = x(f) * \frac{1}{2}\left[\delta(f-2B) + \delta(f+2B)\right] =$

$= \frac{1}{2}\left(x(f-2B) + x(f+2B)\right)$

$f_c \geq 6B$



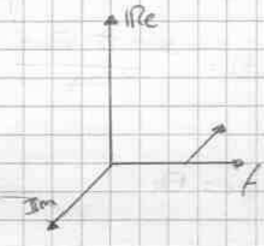
②

$y(t) = x(t)\cos 2\pi 2Bt + x(t)\sin 2\pi 2Bt$

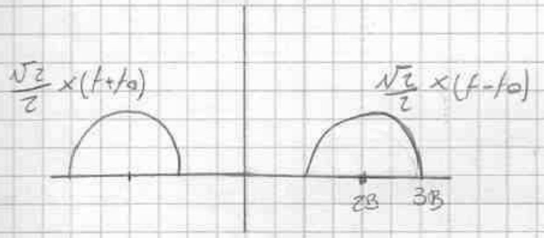
$\hookrightarrow \frac{1}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$

$f_c \geq 6B$

$x(f) * \frac{1}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$

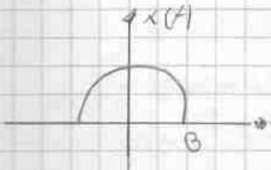


$\frac{\sqrt{2}}{2} \rightarrow$ modulo di $\left(\frac{1}{2} + \frac{1}{2j}\right)$



③

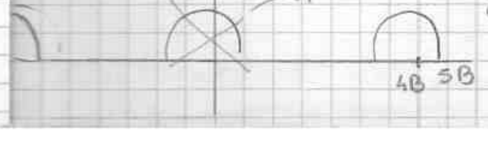
$x(t) \mid x(f) = 0$ $\equiv$ banda limitata $\rightarrow$ spettro nullo per $x > B$
$f > B$



$y(t) = x(t)\cos 2\pi 2Bt \cdot \sin 2\pi 2Bt$

$f_c \geq 10B$

perché sin e cos sono in opposizione di fase. se fosse cos · cos si ha una somma



⑦ $x(t) = \cos 2\pi f_0 t \cdot \sin 2\pi f_0 t$

moltiplicate per il seno

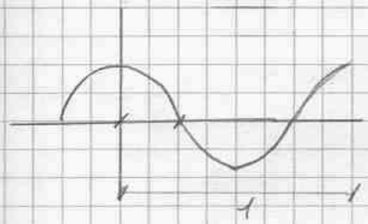
il segnale ha banda A+B

Esercizi:

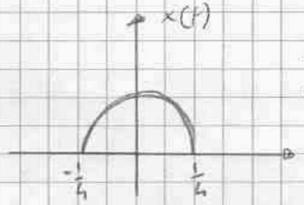
06-04

① $x(f) = \cos 2\pi f \cdot p_{\frac{1}{2}}(f)$

$y(t) = x(t)\left[1 - e^{-j\pi t}\right]$ calcolare la f_c del segnale $y(t)$



$2\pi \cdot x = \frac{\pi}{2}$ $x = \frac{1}{4}$



risultato del prodotto tra cos e $P_{\frac{1}{2}}$

$f_c(x) = \frac{1}{4} \cdot 2 = \frac{1}{2}$

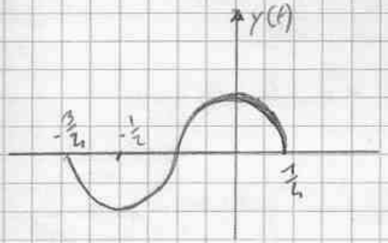
$x(t) - x(t)e^{\overset{f_0 = \frac{1}{2}}{j\pi t}}$

$x(f) - x(f+f_0)$

$x(f)e^{-2\pi f t_0} \longrightarrow x(t-t_0)$

$x(t)e^{2\pi f_0 t_0} \longrightarrow x(f+f_0)$

$f_0 = \frac{1}{2}$ $x(f) - x(f+\frac{1}{2})$



$B = \frac{3}{4} \longrightarrow f_c = \frac{3}{2}$

oppure potevano ragionare tenendo che la banda complessiva del prodotto è la somma delle bande

$$B_{x(f)} + B_{A(f)} = \frac{1}{4} + B\Big[\underbrace{d(f)}_{B=0} + \underbrace{d(f+\tfrac{1}{2})}_{B=\frac{1}{2}}\Big] = \frac{3}{4}$$

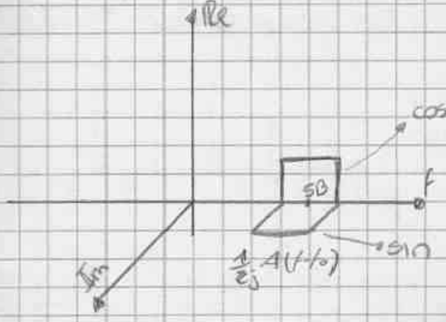
②

$x(t) = \frac{\sin \pi F t}{\pi t}$ $y(t) = \frac{\sin \pi F_4 t}{\pi t}$ $F = 1\,KHz$ $F_4 = 3\,KHz$

$z(t) = x(t)\,y(t)$ $B_{x(t)} = \frac{F}{2}$ $B_{y(t)} = \frac{F_4}{2}$

$B_{z(t)} = 2$ $f_c = 4$

analiticamente viene fuori un trapezio



$$\int_{-\infty}^{+\infty} \left|\frac{1}{2}A(f-f_0)\right|^2 df +$$ parte dovuta al cos

$$+ \int_{-\infty}^{+\infty} \left|\frac{1}{2j}A(f-f_0)\right|^2 df =$$ parte dovuta al sin

$$= \frac{1}{4}\mathcal{E}_A + \frac{1}{4}\mathcal{E}_A = \frac{1}{2}\mathcal{E}_A$$ parte solo di dx

perchè $|A(f-f_0)|^2 = \mathcal{E}_A$

$$\mathcal{E}_X = \mathcal{E}_A = B \qquad f_c = 11 B \qquad B = \frac{11}{2} B$$

②

$$y(t) = a(t)\cos^2 2\pi f_0 t$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2} \qquad a(t) = \frac{\sin \pi B t}{\pi t} \qquad a(t)\left(\frac{1}{2}+\frac{\cos 2\alpha}{2}\right)$$

$$y(f) = \frac{1}{2}a(t) + \frac{1}{2}a(t)\cos 4\pi f_0 t$$

$$= \frac{1}{2}A(f) + \frac{1}{2}A(f) * \left[\frac{1}{2}\delta(f-2f_0) + \frac{1}{2}\delta(f+2f_0)\right] = \frac{1}{2}A(f) + \frac{1}{4}A(f-2f_0) + \frac{1}{4}A(f+2f_0)\Big]$$

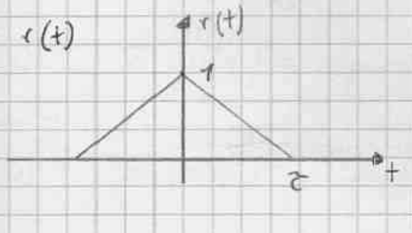
$$\int_{-\infty}^{+\infty} \left|\frac{1}{2}A(f)\right|^2 df = \frac{1}{4}\int_{-\infty}^{+\infty} |A(f)|^2 df =$$

$$= \frac{1}{4}\mathcal{E}_A$$

$$\int_{-\infty}^{+\infty} \left|\frac{1}{4}A(f-2f_0)\right|^2 df = \frac{1}{16}\int_{-\infty}^{+\infty} |A(f-2f_0)| = \frac{1}{16}\mathcal{E}_A + \frac{1}{16}\mathcal{E}_A$$

$$\mathcal{E}_x = \frac{1}{4}\mathcal{E}_A + \frac{1}{16}\mathcal{E}_A + \frac{1}{16}\mathcal{E}_A = \frac{3}{8}B$$

③



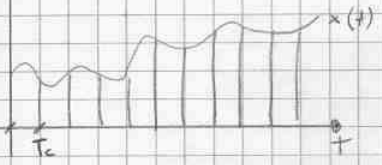
$$r(t) = \mathrm{tri}\left(\frac{t}{\tau}\right) \begin{cases} 1 - \frac{|t|}{\tau} & |t| = \tau \\ 0 & \text{altrove} \end{cases}$$

$$x(t) = A\sum_{n=-\infty}^{+\infty} r(t-nT)$$ calcolare $x(f)$

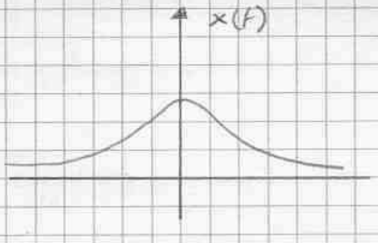
la trasformata del tri è $\tau \dfrac{\sin^2 \pi f \tau}{(\pi f \tau)^2} = R(f)$

Cap. 15 non ci sarà all'esame DFT 17-04

buffer del PC



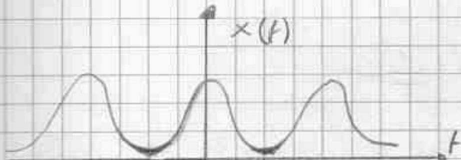
il segnale viene campionato



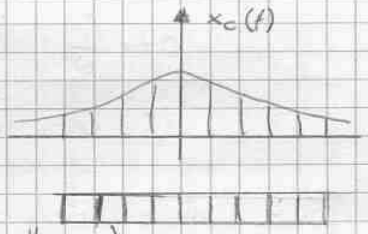
x(t) ha un supporto limitato

↓

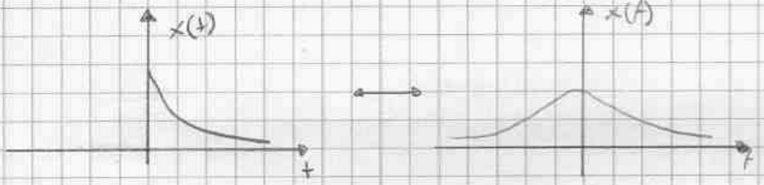
X(f) ha sicuramente supporto illimitato



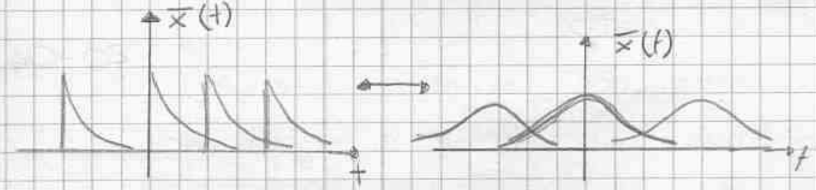
sovrapposizione (effetto elliasing)

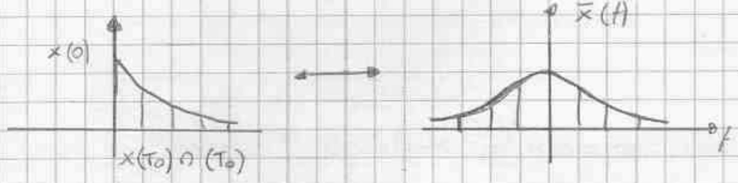


→ qst è qll che legge il PC



il PC vede solo delle robe a righe e quindi il segnale x(t) deve essere periodicizzato





DFT $\bar{X}(nf_0) = \frac{1}{T}\sum_{m=0}^{N-1} \bar{x}(mT_0)\, e^{-j\frac{2\pi}{N}mn}$

IDFT $\bar{x}(mT_0) = \frac{T}{N}\sum_{n=0}^{N-1} \bar{X}(nf_0)\, e^{j\frac{2\pi}{N}mn}$

$$x(nT_c) \longrightarrow \boxed{\text{filtro ricostruttore}} \longrightarrow x(t)$$

sono i campioni di $x(t)$

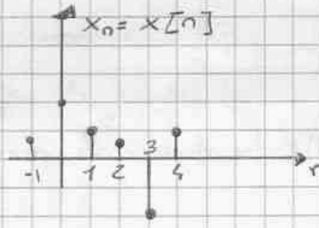
si può manipolare i campioni di $x(t)$ e ricavare subito i campioni del segnale $y(t)$

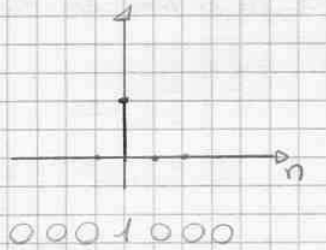
$$x(nT_c) \longrightarrow \boxed{?} \longrightarrow y(nT_c)$$ (riproduzione del digitale)

PC

Il segnale a tempo discreto è una sequenza di num, reali o complessi, indicizzati da un valore intero.

$x[n] = x_n$





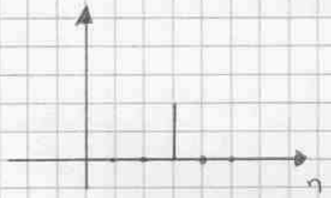
$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

è una sequenza che vive propria nel tempo discreto.

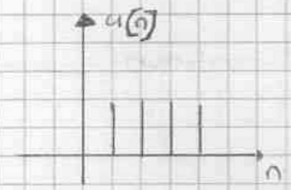
↓

delta di Dirac / delta di Kronecker

Il tempo discreto vive di vita propria al di là dell'esistenza del tempo continuo.



$$\delta[n-k] = \begin{cases} 1 & n-k=0 \quad n=k \\ 0 & n-k\neq 0 \end{cases}$$



$$u[n] = \begin{cases} 1 & n\geq 0 \\ 0 & n<0 \end{cases}$$

prendiamo un segnale (SINUSOIDE NUMERICA)

$x(t) = \sin 2\pi f_0 t \qquad t_n = nT_c$

$x(nT_c) = \sin 2\pi f_0 T_c n$

$T_c = \frac{1}{f_c}$

$x[n] = \sin 2\pi \frac{f_0}{f_c} n$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

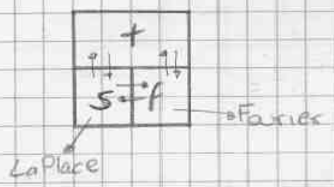
$$y[n] = \mathcal{L}_{n}\left\{\sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]\right\} = \sum_{k=-\infty}^{+\infty} x[k]\,\mathcal{L}\left\{\delta[n-k]\right\}$$

$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]} \rightarrow \text{convoluzione numerica}$$

$$x[n] * h[n]$$

passo dagli integrali alle sommatorie.

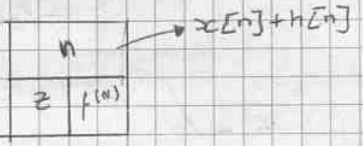
Domini in cui si studiano i segnali a tempo continuo:



nel dominio del tempo $\overbrace{\text{il filtraggio}}^{\text{le operazioni LTI}}$ è una convoluzione

$x(t) * h(t)$ LTI $\quad H(s)\cdot X(s)$ Laplace $\quad H(f)\cdot X(f)$ Fourier

Nel tempo discreto:



ESERCITAZIONE

23-04

① $x(t) \quad t \in \mathbb{R}$

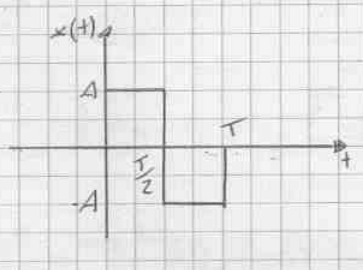
$$\mathcal{E}_x = \int_{\mathbb{R}} |x(t)|^2\,dt \qquad P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt$$

①

$$x(t) = \sum_{n=-\infty}^{-\infty} r(t-nT)$$

$\mathcal{E}_x = ?$

$P_x = ?$



$$\mathcal{E}_x = \int_{\mathbb{R}} |x(t)|^2\,dt$$

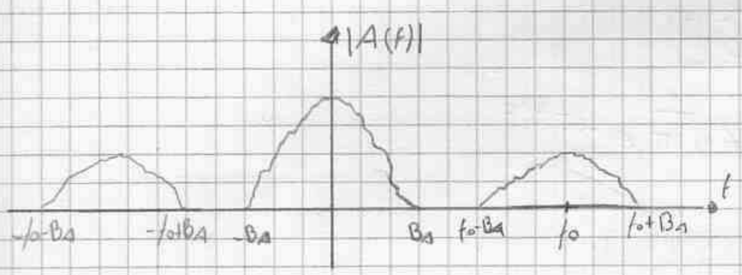
$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T} A^2 T = A^2$$

$$x(f) = \mathfrak{F}\{a(t)\cos(2\pi f_0 t)\} + \mathfrak{F}\{a(t)\sin(2\pi f_0 t)\} =$$

$$= A(f) * \left[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right] + A(f) * \left[\frac{1}{2j}\delta(f-f_0) - \frac{1}{2j}\delta(f+f_0)\right] =$$

$$= \frac{1}{2}A(f-f_0) + \frac{1}{2}A(f+f_0) + \frac{1}{2j}A(f-f_0) - \frac{1}{2j}A(f+f_0)$$



$$x^*(f) = \frac{1}{2}A^*(f-f_0) + \frac{1}{2}A^*(f+f_0) - \frac{1}{2j}A^*(f+f_0) + \frac{1}{2j}A^*(f-f_0)$$

$$|x(f)|^2 = x(f)\,x^*(f) = \frac{1}{4}|A(f-f_0)|^2 - \cancel{\frac{1}{4j}|A(f-f_0)|^2} + \frac{1}{4}|A(f+f_0)|^2 +$$

$$+ \frac{1}{4j}|A(f+f_0)|^2 + \cancel{\frac{1}{4j}|A(f-f_0)|^2} + \cancel{\frac{1}{4}|A(f-f_0)|^2} +$$

$$- \cancel{\frac{1}{4j}|A(f+f_0)|^2} + \frac{1}{4}|A(f+f_0)|^2$$

$$|x(f)|^2 = \frac{1}{2}|A(f-f_0)|^2 + \frac{1}{2}|A(f+f_0)|^2$$

$$\mathcal{E}_x = \int_{\mathbb{R}} |x(f)|^2 df = \frac{1}{2}\int_{\mathbb{R}} |A(f-f_0)|^2 df + \frac{1}{2}\int_{\mathbb{R}} |A(f+f_0)|^2 df =$$

$$= \frac{1}{2}\mathcal{E}_x + \frac{1}{2}\mathcal{E}_x = \mathcal{E}_x$$

④

$$x(t) = \sum_{n=-\infty}^{+\infty} e^{-\frac{\pi}{2}(t-nT)^2}$$

$$x(f) = ?$$

$$y(t) = \sum_{n=-\infty}^{+\infty} x(t-nT) = \sum_{n=-\infty}^{+\infty} x(t) * \delta(t-nT) =$$

$$= x(t) * \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} e^{-\frac{\pi}{2}t^2} * \delta(t-nT) =$$

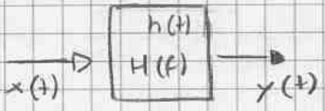
$$= e^{-\frac{\pi}{2}t^2} * \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

$$\mathfrak{F}\{e^{-\alpha t^2}\} = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\pi}{\alpha}}$$

$$x(f) = \sqrt{\frac{\pi}{\frac{\pi}{2}}}\, e^{\frac{\pi^2}{\frac{\pi}{2}} f^2} \cdot \frac{1}{T}\sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T}\right) =$$

$$= \sqrt{2} \cdot \frac{1}{T}\, e^{-2\pi f^2} \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T}\right)$$

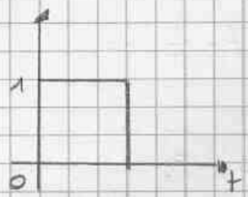
②



$$y(t) = h(t) \ast x(t)$$

$$y(f) = H(f) \cdot x(f)$$

h(t)



$$x(t) = A h(t) \quad A>0$$

$$y(f) = \int_{\mathbb{R}} x(\tau) h(t-\tau) d\tau = \int_{\mathbb{R}} h(\tau) x(t-\tau) d\tau$$

$t<0 \quad y(t)=0$

$0<t<T \quad y(t) = tA$

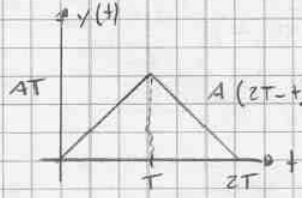
$t=T \quad y(t) = A \cdot T$

$0<t-T<T$

$T<t<2T \quad y(t) = [T-(t-T)]A = A[T-t+T] =$

$= A[2T-t]$

$t>2T \quad y(t)=0$

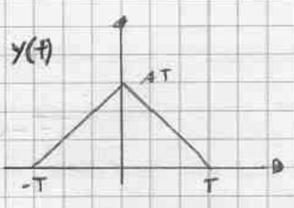


$$y(f) = H(f)\, x(f)$$

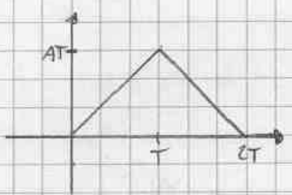
$$H(f) = \frac{\sin(\pi f t)}{\pi f} e^{-j2\pi f \frac{T}{2}}$$

$$x(f) = A H(f) = A \frac{\sin(\pi f t)}{\pi f} e^{-j2\pi f \frac{T}{2}}$$

$$x(f) H(f) = A \underbrace{\frac{\sin^2(\pi f t)}{(\pi f)^2}}_{y(f)} e^{-j2\pi f T}$$

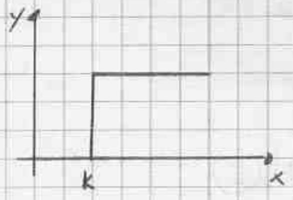


$$y(t) = y'(t-T)$$

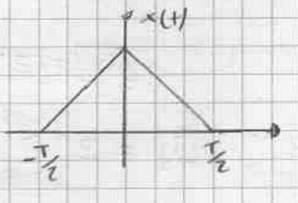


③

$$x(t) \rightarrow \boxed{i/o} \rightarrow y(t)$$

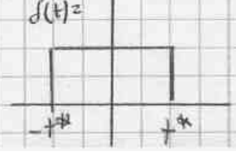


$$K = \frac{A}{2}$$



$y(f) = ?$

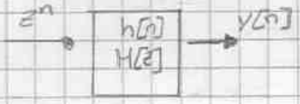
$$y(t) = \sum_{-\infty}^{+\infty} \delta(t-nT) \qquad Y(f) = \delta(f) \frac{1}{T} \sum_{-\infty}^{+\infty} \delta\left(f - \frac{n}{T}\right)$$



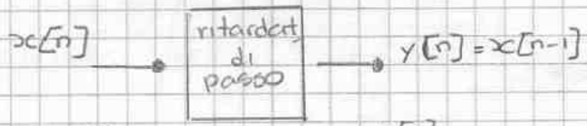
$$H(z) = \sum_{i=-\infty}^{+\infty} h[i]\, z^{-i}$$ trasformata zeta

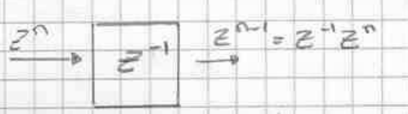
la trasformata zeta di una sequenza $x[n]$ si scriverà

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$$



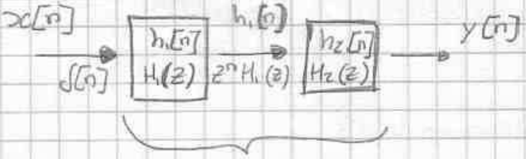
RITARDATORE di PASSO

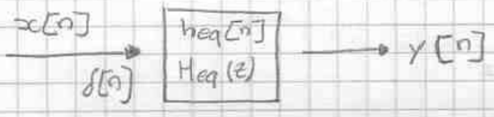




z^{-1} indica un ritardatore di passo

se abbiamo 2 sistemi a cascata:





voglio sapere quanto vale $h_{eq}[n]$

se uso una $\delta[n]$ all'ingresso all'uscita del primo sistema ho $h_1[n]$ e l'uscita dal secondo è $h_1[n] * h_2[n]$

La stessa cosa l'avrei anche sotto introducendo un $\delta[n]$ ottenendo $h_{eq}[n] = h_1[n] * h_2[n]$

Per calcolarmi $H_{eq}(z)$

all'ingresso metto z^n da cui esce $z^n H_1(z)$ (del tipo $z^n \cdot \overset{\text{num. complesso}}{\alpha}$). Il 2° sistema essendo lineare ottengo all'uscita $\alpha z^n H_2(z) = H_1(z) H_2(z) z^n$

$$H_{eq}(z)\, z^n = H_1(z)$$

La cascata di 2 sistemi nel dominio z è il prodotto delle funzioni di trasferimento

$$h_{eq}[n] = h_1[n] * h_2[n]$$
$$\text{z trasf.}\downarrow \qquad \downarrow \text{z trasf.}$$
$$H_{eq}(z) = H_1(z)\, H_2(z)$$

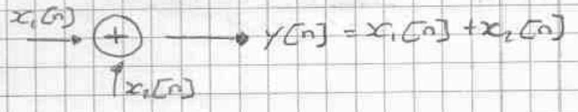
usare le trasformate z per evitare la convoluzione e passare ai prodotti

PROPRIETÀ della CONVOLUZIONE

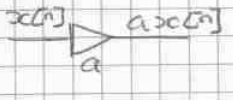
$$\begin{matrix} x_1[n] \longleftrightarrow X_1(z) \\ x_2[n] \longleftrightarrow X_2(z) \end{matrix} \Longrightarrow x_1[n] * x_2[n] \longleftrightarrow X_1(z)\, X_2(z)$$

SISTEMI LTI

nodo di somma



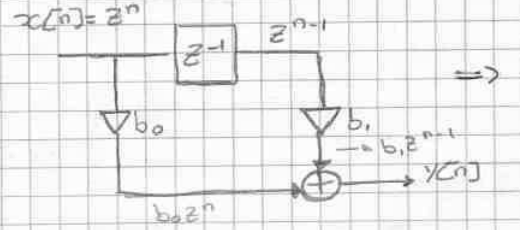
moltiplicatore



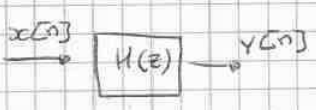
ritardatore

ESERCIZIO:

①



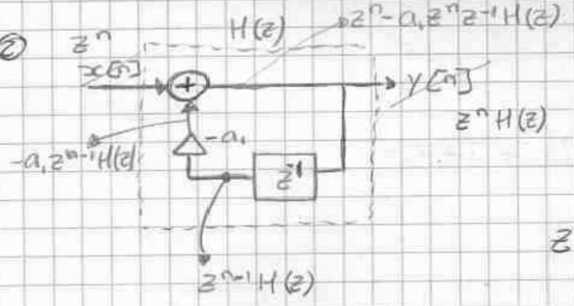
$\Longrightarrow$



$$y[n] = b_0 z^n + b_1 z^{n-1} = z^n \left(b_0 + b_1 z^{-1} \right)$$

$$y[n] = z^n H(z) \qquad H(z) = b_0 + b_1 z^{-1}$$

②



scrivo così perché l'uscita ritorna indietro

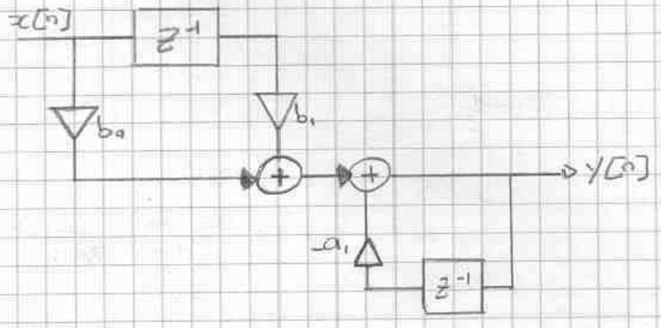
sistema reazionato

$$z^n - a_1 z^n z^{-1} H(z) = z^n H(z)$$

$$1 - a z^{-1} H(z) = H(z)$$

$$H(z)\left[1 + a_1 z^{-1}\right] = 1 \qquad H(z) = \frac{1}{1 + a_1 z^{-1}}$$

③



$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$

sistema a cascata

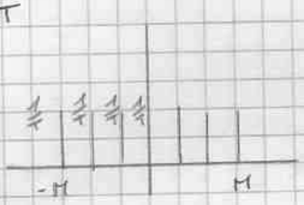
$\Downarrow$

prodotto delle funzioni di trasferimento dei 2 sistemi

$$P = \{-u[n-3] + u[n]\} = \delta[n] + \delta[n-1] + \delta[n-2]$$

qst è un segnale ad energia

④ $x[n] = \frac{1}{T}$

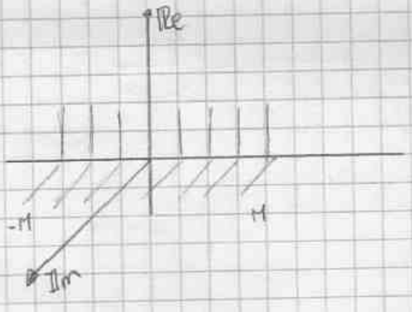


$$P = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{-M}^{M} |x[n]|^2 =$$

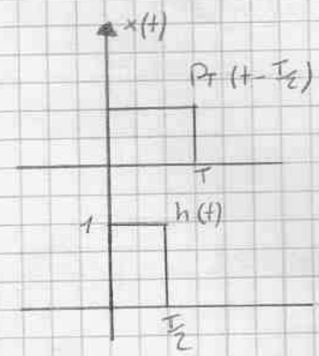
$$= \lim_{M\to\infty} \frac{1}{\cancel{2M+1}} \underbrace{(\cancel{2M+1})}_{\downarrow \text{ qnt campioni ho}} \frac{1}{T^2} = \frac{1}{T^2}$$

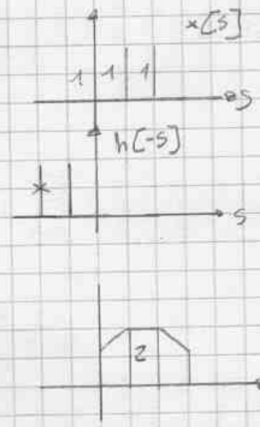
⑤ $x[n] = 3+2j$

$$P = \lim_{M\to\infty} \frac{1}{\cancel{2M+1}} (\cancel{2M+1}) |\cancel{\sqrt{a+b}}|^{\cancel{2}} = 13$$



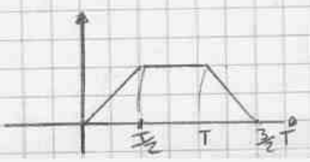
⑥





$$\sum_{-\infty}^{+\infty} x[s]\,h[n-s]$$

$y(t) = x(t) * h(t)$



serie geometrica

$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad |a|<1$

$\sum_{n=0}^{\infty} a^{-n} = \frac{1}{1-a^{-1}} \quad |a|>1$

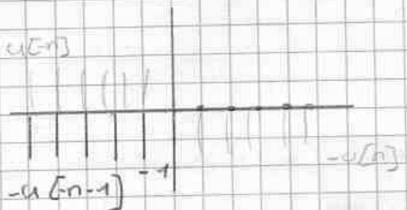
④

$\left(\frac{1}{2}\right)^{n-2} u[n] =$

$= 4 \left(\frac{1}{2}\right)^n u[n] \longrightarrow 4 \frac{1}{1-\frac{1}{2}z^{-1}} \left(\frac{1}{2}\right)^{-1} \quad |z|>\frac{1}{2}$

⑤

$-u[-n-1]$



$\sum_{k=-\infty}^{+\infty} x[k]\, z^{-k} = -\sum_{k=-\infty}^{-1} z^{-k} = -\sum_{m=1}^{+\infty} z^m$

$m=-k$

$-\left(\sum_0^{\infty} z^m - \underset{z^0}{1}\right) = 1 - \sum_0^{\infty} z^m = 1 - \frac{1}{1-z} =$

converge se $|z|<1$

$= \frac{1-z-1}{1-z} = -\frac{z}{1-z} =$

$= \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

una trasf. z è caratterizzata dall'espressione analitica e dalla zona di convergenza. In qst otteniamo lo stesso risultato della serie geo ma ho un'insieme di conv. diverso $|z|<1$

04-05

⑥

$x[n] = a^n u[n] \quad |a|<1$

$\sum_{k=-\infty}^{+\infty} x[k]\, z^{-k} = \sum_{k=-\infty}^{+\infty} a^k z^{-k} = \sum_{k=0}^{+\infty} (az^{-1})^k = \frac{1}{1-az^{-1}} \quad |az^{-1}|<1$

$|z|>|a|$

⑦

$x[n] = \left(\frac{1}{2}\right)^n u[n]$

$\sum_{k=-\infty}^{+\infty} x[k]\, z^{-k} = \sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^k z^{-k} = \sum_{k=0}^{+\infty} (2z)^{-k} = \frac{1}{1-(2z)^{-1}} \quad |z|>\frac{1}{2}$

Individuare linearità e tempo invariante

① $y[n] = \cos(\pi n)\, x[n]$

$a_1(\cos\pi n\, x_1[n]) + a_2(\cos\pi n\, x_2[n])$ SI lineare

$=$

$\cos\pi n(a_1 x_1[n] + a_2 x_2[n])$

$\cos\pi n\, x[n-T]$ NO tempo inv.

$=$

$\cos\pi n\, x[n] \rightarrow \cos\pi[n-T]\, x[n-T]$

②

$y[n] = x[n]\, x[n-2]$

$a_1(x[n]\, x[n-2]) + a_2(x[n]\, x[n-2])$ NO lineare

$=$

$(a_1 x_1[n] + a_2 x_2[n])(a_1 x_1[n-2] + a_2 x_2[n-2])$ $a_1 x_1 + a_2 x_2$

$x[n-T] \rightarrow x[n-T]\, x[n-T-2]$ SI sist. tempo inv.

$=$

$x[n]\, x[n-2] \rightarrow x[n-T]\, x[n-2-T]$

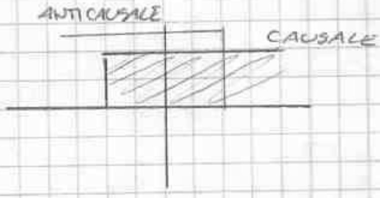
Stabilità del sistema $|h(t)| < \infty$

$\Sigma|h(n)| < \infty$

Memoria del sistema

$x[n] + x[n- \ ]$

Causalità



Calcolare l'antitrasformata METODO DIRETTO

$$x(z) = 1 + z^{-2} \cdot 1$$

$$\delta[n] + \delta[n-2]$$

$$y(z) = 1 + z^{-3} + 2z^{-4}$$

$$y[n] = \delta[n] + \delta[n-3] + 2\delta[n-4]$$

$$x(z) = \frac{1}{1-z^{-1}} \longrightarrow u[n]$$

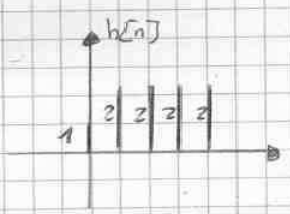
$$y(z) = \frac{z^{-1}}{1-z^{-1}} \longrightarrow u[n-1]$$

ES

$$x(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \longrightarrow \left(\frac{1}{2}\right)^n u[n]$$

$$y(z) = \frac{z^{-1}}{1-\frac{1}{2}z^{-1}} \longrightarrow \left(\frac{1}{2}\right)^{n-1} u[n-1] \qquad |z| > \frac{1}{2}$$

$$H(z) = \frac{(z+1)}{z-1} = \underbrace{\frac{z}{z-1}}_{\downarrow\; u[n]} + \overset{u[n-1]}{\left(\frac{1}{z-1} \cdot \overset{\text{ritardo}}{z^{-1}} \cdot z\right)}$$



i poli indicano il grado del filtro.

$$\frac{1}{(z-\frac{1}{2})(z-\frac{1}{3})}$$

$$\downarrow$$

$$\frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{3}}$$ METODO dei FRATTI SEMPLICI

$$A = \left.\frac{x(z)}{(z-\frac{1}{2})^{-1}}\right|_{z=\frac{1}{2}} = \left.\frac{1}{z-\frac{1}{3}}\right|_{z=\frac{1}{2}} = \frac{1}{1/6} = 6$$

$$B = \left.\frac{1}{z-\frac{1}{2}}\right|_{z=\frac{1}{3}} = \frac{1}{\frac{1}{3}-\frac{1}{2}} = \frac{1}{-\frac{1}{6}} = -6$$

$$6z^{-1}\frac{z}{z-\frac{1}{2}} - 6z^{-1}\frac{z}{z-\frac{1}{3}}$$

$$6\left(\frac{1}{2}\right)^{n-1} u[n-1] - 6\left(\frac{1}{3}\right)^{n-1} u[n-1]$$

$x[n] = \cos\left[\frac{\pi}{8} n\right]$

$x[n+N] = \cos\left[\frac{\pi}{8}(n+N)\right] = \cos\left[\frac{\pi}{8} n + \frac{\pi}{8} N\right] \qquad \frac{\pi}{8} N = 2k\pi \quad N = 16k$

$$x[n] = \mathrm{Re}\left\{e^{j\frac{\pi}{12}n}\right\} + \mathrm{Im}\left\{e^{j\frac{\pi}{18}n}\right\} = \underbrace{\cos\left(\frac{\pi}{12}n\right)}_{N_1=24} + \underbrace{\sin\left\{\frac{\pi}{18}n\right\}}_{N_2=36}$$

$$N = \mathrm{mcm}\{N_1; N_2\} = 72$$

$x[n] = \sin\left(\frac{n}{5} + \pi\right)$

$x[n+N] = \sin\left(\frac{n+N}{5} + \pi\right) = \sin\left(\frac{n}{5} + \pi + \frac{N}{5}\right)$

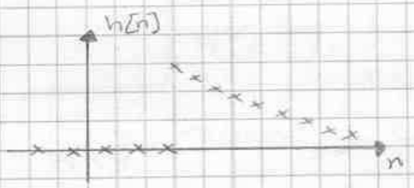
$\frac{N}{5} = 2k\pi \Rightarrow N = 10k\pi$ segnale non periodico nel dominio discreto

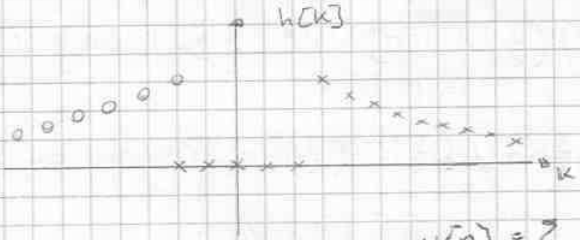
③
$x[n] = e^{j\frac{\pi}{16}n} \cos\left(n\frac{\pi}{17}\right)$

Fare la convoluzione in tempo discreto

①
$x[n] = \left(\frac{1}{6}\right)^{n-6} u[n]$

$h[n] = \left(\frac{1}{3}\right)^{n} u[n-3]$





$$y[n] = \sum_k h[k]\, x[n-k]$$

$n < 3 \quad y[n] = 0$

$$y[n] = \sum_{k=3}^{n} \left(\frac{1}{3}\right)^{k} \left(\frac{1}{6}\right)^{n-k-6} = \sum_{k=3}^{n} \left(\frac{1}{3}\right)^{k} \left(\frac{1}{6}\right)^{n} \left(\frac{1}{6}\right)^{-k} \left(\frac{1}{6}\right)^{-6} = \qquad \sum_{k=k_0}^{k_1} z^k = \frac{z^{k_0} - z^{k_1+1}}{1-z}$$

$$= \left(\frac{1}{6}\right)^{n} 6^6 \sum_{k=3}^{n} \left(\frac{1}{3}\right)^{k} 6^k = \left(\frac{1}{6}\right)^{n} 6^6 \sum_{k=3}^{n} 2^k = \left(\frac{1}{6}\right)^{n} 6^6 \frac{2^3 - 2^{n+1}}{1-2}$$

c'è un errore → risultato corretto è $y[n] = (n+1)\left(\frac{1}{2}\right)^n$

14-05

SINUSOIDE in T.D.

$$x[n] = \sin 2\pi f n = \cos 2\pi f n = e^{j2\pi f n}$$

$$x[n] = x(nT_c) = \sin 2\pi f_0 n T_c = \sin 2\pi \left(\frac{f_0}{f_c}\right) n \qquad f^{(n)} = \frac{f^{(a)}}{f_c^{(a)}}$$

$x(nT_c)$ → segnale analogico campionato

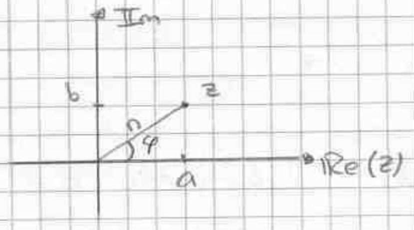
T_c → $f_c = \frac{1}{T_c}$

f → frequenza numerica

f_c → frequenza campionata

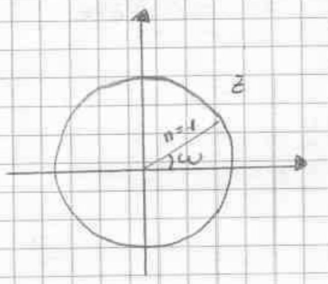
$x(t) = \sin 2\pi f_0 t$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n}$$

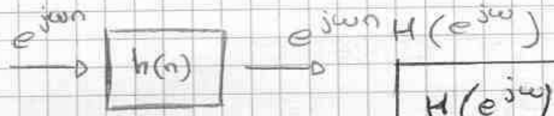


$z = a + jb$

$z = n e^{j\varphi}$



$z = e^{j\omega}$ $\quad z^n = e^{j\omega n}$



$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n}$$

trasformata di Fourier a tempo discreto

solo tempo discreto

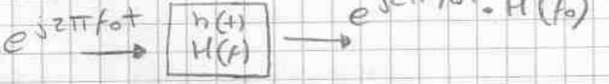
$$\text{DTFT}\,\{h[n]\} = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n}$$

è valida per qualsiasi sequenza numerica

è un caso particolare della formula precedente

data una sinusoide in ingresso:

Modo continuo



entra una sinusoide ed esce una sinusoide modificata di amplez. e fase.

Modo discreto

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_m z^{-m}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_m z^{-m}}$$

$H(z) = 0 \Rightarrow Y(z) = 0$ filtro passa niente

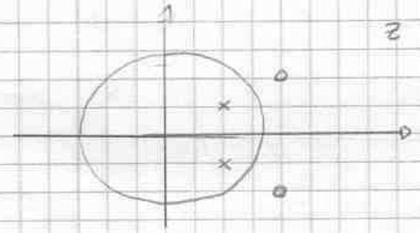
$b_i = 0$

un sistema è stabile se dato all'ingresso un segnale di ampiezza limitata ho un'uscita d ampiezza limitata

Non è stabile qnd invece l'uscita cresce ovvero diverge.

SISTEMA STABILE $\rightarrow \sum_{n=-\infty}^{+\infty} |h[n]| < \infty$

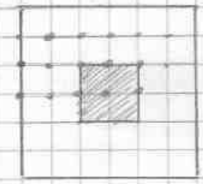
$$H(z) = \frac{b_0 z^m + b_1 z^{m-1} + \dots}{z^m + a_1 z^{m-1} + \dots} = \frac{(z - z_1)(z - z_1^*)\dots}{(z - p_1)(z - p_1^*)\dots}$$

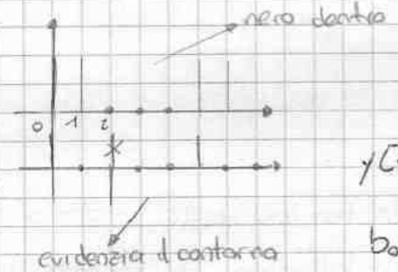


per vedere se è stabile basta verificare che i poli siano all'interno del cerchio unitario

La fase può essere qualsiasi, a noi interessa il modulo.

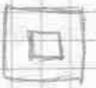
Es.





$y[n] = x[\overset{2}{n}] - x[\overset{2}{n-1}] \quad = -1$

$b_0 = 1 \quad b_1 = 1 \quad b_i = 0 \quad i > 1$



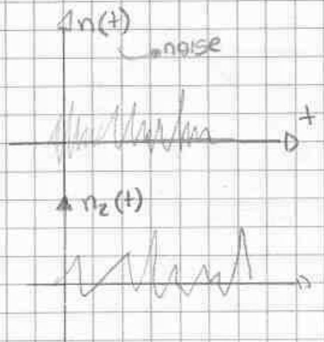
FIR → filtri a risposta all'impulso finita

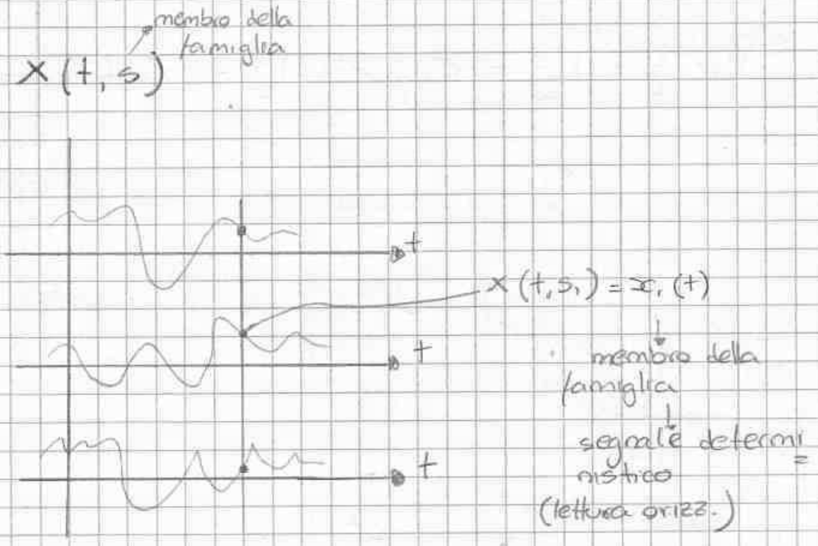
IIR → " " " infinita

$a_i \neq 0$ almeno 1 diverso da zero

$$H(z) = \frac{(z - z_1)(z - z_1)(z - p_1)(z - p^*)}{(z - p_1)(z - p^*)}$$

i poli non devono $\overset{\text{essere}}{\urcorner}$ annullati da degli zeri → poli apparenti





leggendo per verticale trovo una variabile casuale $x(t_1; s) = x_1$

$X_1 \rightarrow f_{x_1}(x_1)$

$f_x(x_1, t_1)$

$$f_{x_1}(x_1) = \frac{d}{dx_1} F_{x_1}(x_1)$$

$$F_{x_1}(x_1) \triangleq \underset{\downarrow \atop \text{probabilità}}{P}(x_1 \leq x_1)$$

$x_1 \quad x_2$

$$f_{x_1 x_2}(x_1, x_2, t_1, t_2)$$

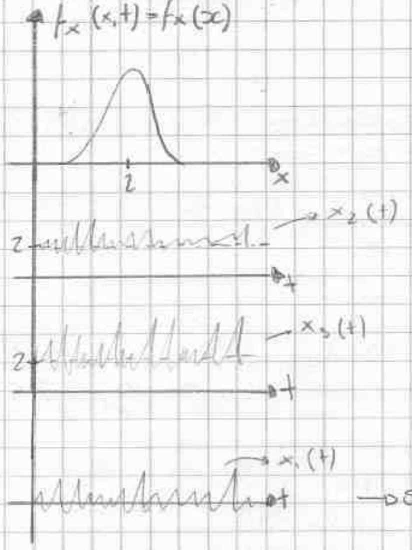
$$F_{x_1 x_2}(x_1, x_2, t_1, t_2) = P(x(t_1) \leq x_1, \; x(t_2) \leq x_2)$$

se x_1, x_2 sono indipendenti

$$f_{x_1 x_2}(x_1, x_2, t_1, t_2) = f_{x_1}(x_1, t_1) f_{x_2}(x_2, t_2)$$

teoria statistica del secondo ordine → grado 2

famiglia di segnali del I° ordine



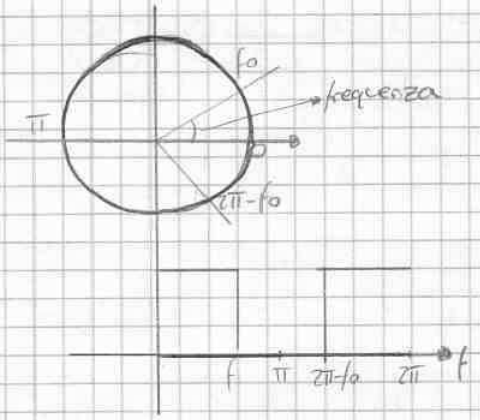
$$E\{x_1\} \Rightarrow E\{x(t)\} =$$

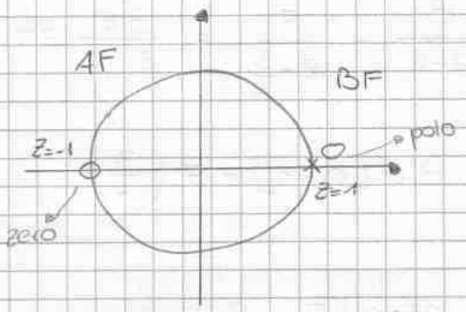
$$= m_x(t) = \int_{-\infty}^{+\infty} x f_x(x, t)\, dx$$

$$E\{x^2(t)\} = \int_{-\infty}^{+\infty} x^2 f_x(x, t)\, dx$$

→ SBAGLIATA non posso avere un valore negativo

Z-05





filtro PASSA BASSO 1° ORDINE vuol dire che ho solo un polo

$$H(z) = \frac{z+1}{z-1} = \frac{z}{z-1} + \frac{1}{z+1}$$

in continua il mio filtro ha un guadagno unitario

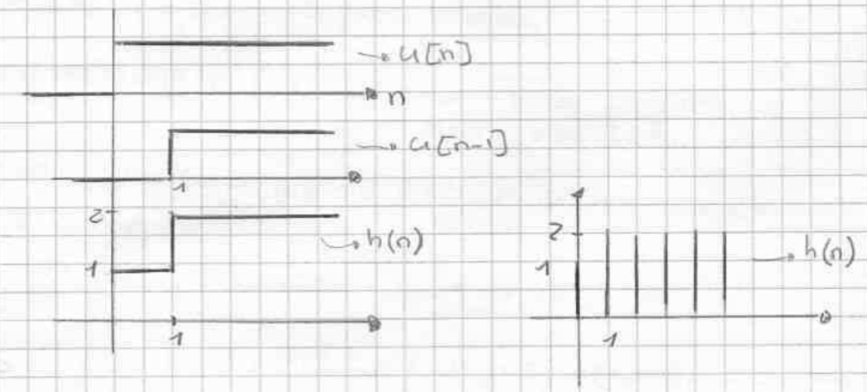
$$z = e^{j2\pi f}$$

Al filtro devo dare coefficienti nel dominio del tempo

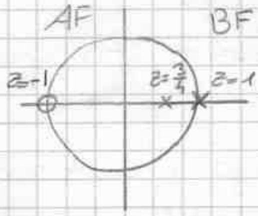
antitrasformata di $\frac{z}{z-1} \longrightarrow u[n]$

" di $\frac{1}{z+1} = \frac{z}{z-1} z^{-1} \longrightarrow u[n-1]$

$h(n) = u[n] + u[n-1]$



Voglio fare un filtro di II° ordine



$$H(z) = \frac{z+1}{(z-1)(z-\frac{3}{4})} = \frac{A}{z-1} + \frac{B}{z-\frac{3}{4}}$$

$$A\left(z-\frac{3}{4}\right) + B(z-1) = z+1$$

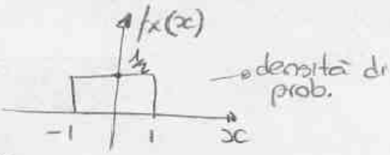
$$A = \frac{z+1}{z-\frac{3}{4}}\Bigg|_{z=1} = 8 \qquad B = \frac{z+1}{z-1}\Bigg|_{z=\frac{3}{4}} = -7$$

Ad esempio se considero un oscillatore

$\sin 2\pi f_0 t$

Processo stazionario del I° ORDINE se $\overline{\text{la densità di prob.}}$ non dipende dal tempo

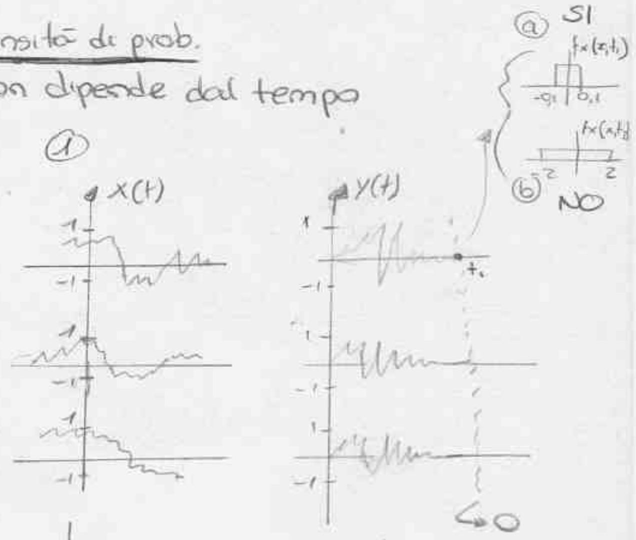
$f_x(x,t) = f_x(x)$



$y = g(x)$

$E\{y\} = \int_{-\infty}^{+\infty} g(x) f_x(x) dx$

①



compatibile con la $f_x(x)$

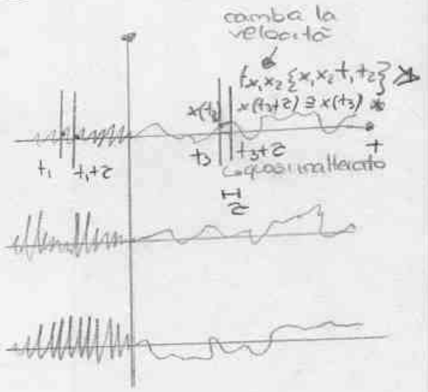
NO xk tende a smorzarsi e per $T<0$ è nullo

✱ densità di prob. del II° ORDINE se $f_{x_1x_2}\{x_1 x_2 \underbrace{t_1 t_2}_{\tau = t_1 - t_2}\}$ dipende solo da τ

Nell'es non è stazionario per il II ordine perché non dipende solo da τ

$\tau \rightarrow$ distanza temporale

②



è stazionario del I° ordine, però è velox per $T<0$ e lento per $T>0$

WSS: processo stazionario in senso lato se

$E\{x(t)\} = \mu_x \rightarrow$ se la media è costante

$E\{x(t)x(t+\tau)\} = r_x(\tau) \rightarrow$ se la media congiunta dipende solo da τ

La media congiunta prende il nome di autocorrelazione $\rightarrow$ legame di parentale tra 2 campioni

L'es ② non è stazionario in senso lato

✱ $x(t_3+\tau) \cong x(t_3)$

$$E\{x(t_1)x(t_1+\tau)\} \cong E\{x(t_1)\} \cdot E\{x(t_1+\tau)\} = 0$$ $\rightarrow$ tenendo conto della densità di prob. di prima

$$E\{x(t_3)x(t_3+\tau)\} \cong E\{x^2(t_3)\} = E\{x_3^2\} =$$

$$= \int_{-\infty}^{+\infty} x_3^2 \underbrace{f_x(x_3)}_{\frac{1}{2}} dx_3 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{2} \frac{x^3}{3}\Big|_{-1}^{+1} = \frac{1}{3}$$

per essere stazionario dovevano venire uguali