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A P P U N T I

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MATERIA : Analisi II
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CANUTO TABACCO "ANALISI MATEMATICA 2" Springer

Ambiente in cui è concentrato il corso.

$$\mathbb{R}^2 = \{(x,y)\} \quad \mathbb{R}^3 = \{(x,y,z)\} \quad \mathbb{R}^n = \{(x_1, x_2, \dots, x_n)\}$$

Topologia in $\mathbb{R}^n$

distanza in X

Def : $d : X \times X \to \mathbb{R}$

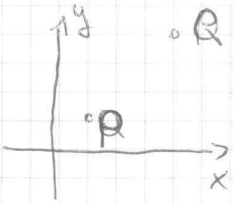
$(x,y) \to d(x,y)$

- $\forall x,y : d(x,y) = d(y,x)$
- $\forall x,x : d(x,y) \geq 0$
- $\forall d(x,y) = 0 \iff x = y$
- disuguaglianza triangolare

$\forall x,y,z \quad d(x,y) \leq d(x,z) + d(z,y)$

$\mathbb{R} : d(x,y) = |x-y|$

$\mathbb{R}^2 : \{(x,y) \mid x,y \in \mathbb{R}\}$



$d(P,Q) = \overline{PQ} \qquad P(x_P, y_P)$

$\qquad Q(x_Q, y_Q)$

$$\overline{PQ} = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$$

$P, Q \in \mathbb{R}^3$

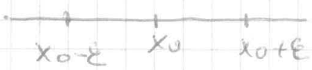
$$d(P,Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2}$$

$\mathbb{R}^n : P(x_1, \dots, x_n) \in \mathbb{R}^n$

$\qquad Q(y_1, \dots, y_n) \in \mathbb{R}^n$

$$d(P,Q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots (x_n - y_n)^2}$$

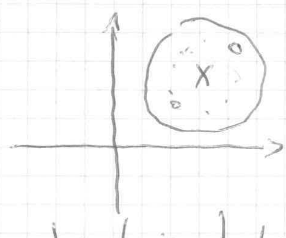
Norma : Insieme dei punti contenuti nell'intervallo



$\{x, d(x_0, x) < \varepsilon\}$

$x^0 = \{x_1^0, x_2^0\}$

$\{x = \{(x_1, x_2) / d(x, x^0) < \varepsilon\}$



cerchio (pieno) di centro x^0 e raggio esclusa la circonferenza

Si dice chiusura di A: $\bar{A} = A \cup \partial A$, $\bar{A}$ insieme chiuso.

Insieme limitato:

Def: $A \subseteq \mathbb{R}^n$ è limitato se:

$\exists M>0 \quad \forall x \in A : d(x,0) < M$

. se $\exists B_M(0) : A \subseteq B_M(0)$

se A non è limitato si dice che è illimitato

$\forall M>0, \exists x \in A \quad d(x,0) \geq M$.

Def: A è compatto se è chiuso e limitato

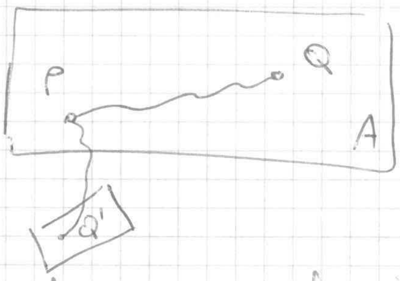
$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ se f è continua e A è compatto

. $\Rightarrow$ f ha max e min

$\exists \bar{x}, \exists \underline{x}$ in A, $\forall x \in A : f(\underline{x}) \leq f(x) \leq f(\bar{x})$

Insieme connesso; (per archi):

A è connesso se $\forall P, Q \in A$: $\exists$ un arco di curva che unisce P a Q che è contenuta in A.



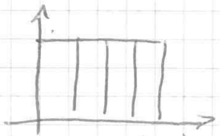
senza Q' A è connesso

con Q' A non è connesso.

Punto di accumulazione:

Def: $A \subseteq \mathbb{R}^n$. $\bar{x} \in \mathbb{R}^n$ è un punto di accumulazione per A se: $\forall B(\bar{x}, \varepsilon) : B(\bar{x}, \varepsilon) \cap A \setminus \{\bar{x}\} \neq \emptyset$

① I punti interni sono di accumulazione per A

②  tutti i punti del grafico sono di accumulazione

Punto isolato:

Def: $\bar{x} \in A$ si dice ISOLATO se non è di accumulazione per A.

$\exists B(\bar{x}, \varepsilon) : B(\bar{x}, \varepsilon) \cap A \setminus \{\bar{x}\} = \emptyset$

$B(\bar{x}, \varepsilon) \cap A = \{\bar{x}\}$

. $f: \mathbb{R}^n \to \mathbb{R}$

$(x_1, \dots, x_n) \to f((x_1, \dots, x_n)) \in \mathbb{R}$

. $f: \mathbb{R}^n \to \mathbb{R}^m \quad n, m > 1$

$\text{dom } G \circ F = \{ x \in \text{dom } F \mid F(x) \in \text{dom } G \}$

Teorema. se F è continua in $\bar{x}$ e G è continua in $F(\bar{x}) \Rightarrow$
$G \circ F$ è continua.

$\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^3 \xrightarrow{F} \mathbb{R}$

$t \longrightarrow \gamma(t) \longrightarrow F(\gamma(t))$

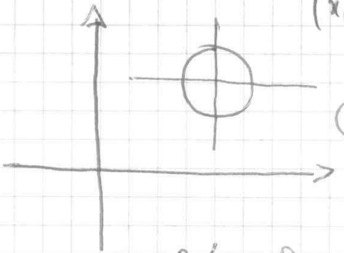
$t \longrightarrow (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \longrightarrow F(\gamma(t))$

$\mathbb{R}^3 \xrightarrow{F} \mathbb{R} \xrightarrow{\gamma} \mathbb{R}^3$

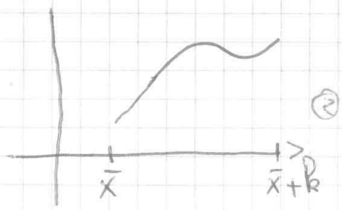
$(t_1, t_2, t_3) \longrightarrow F(t_1, t_2, t_3) \longrightarrow (\gamma(F(t_1)), \gamma(F(t_2)), \gamma(F(t_3)))$

Calcolo differenziale:

$f: \mathbb{R}^2 \to \mathbb{R}$



$A: \mathbb{R}^4 \to \mathbb{R}$



① $\lim_{k \to 0} \dfrac{f(\bar{x}+k, \bar{y}) - f(\bar{x}, \bar{y})}{k}$ se $\exists$ limite $= \dfrac{\partial f}{\partial x}(\bar{x}, \bar{y})$

② $\lim_{k \to 0} \dfrac{f(\bar{x}+k) - f(\bar{x})}{k}$ se $\exists$ limite $f'(\bar{x}) = \dfrac{df}{dx}(\bar{x})$

$f(\bar{x}, \bar{y}) = g(x)$ $\quad (\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$ $\exists$ retta tangente alla curva $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$

$\lim_{k \to 0} \dfrac{f(\bar{x}, \bar{y}+k) - f(\bar{x}, \bar{y})}{k}$ se $\exists$ limite finito $\dfrac{\partial f}{\partial y}(\bar{x}, \bar{y})$

Es: $f(x,y) = \begin{cases} 0 & \text{se } x=0,\ y=0 \\ 1 & \text{se altrimenti} \end{cases}$

$\dfrac{\partial f}{\partial x}(0,0) = 0 \Rightarrow \exists$ la derivata parziale di f in $(0,0)$ ma f non è continua.

$\bar{v} = (v_1, v_2)$ $\quad \|\bar{v}\| = 1$ $\quad \begin{cases} x = \bar{x} + tv_1 \\ y = \bar{y} + tv_2 \end{cases}$

$\lim_{t \to 0} \dfrac{f(\bar{x}+tv_1, \bar{y}+tv_2) - f(\bar{x}, \bar{y})}{t} = \dfrac{\partial f}{\partial v}(\bar{x}, \bar{y})$ se il limite $\exists$ finito

$f: \mathbb{R} \to \mathbb{R}$ $\quad \exists\, \ell \in \mathbb{R}$ $\quad \bar{y}$ interno al dom f

$f(x) = f(\bar{x}) + \ell(x - \bar{x}) + o(x - \bar{x})$ per $x \to \bar{x}$

$\Longleftrightarrow \exists f'(\bar{x})$ $\quad (f'(\bar{x}) = \ell)$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(\bar{x},\bar{y}) = \frac{\partial^2 f}{\partial y \partial x^2}(\bar{x},\bar{y})$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(\bar{x},\bar{y}) = \frac{\partial^2 f}{\partial x \partial y}(\bar{x},\bar{y})$$

=

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(\bar{x},\bar{y}) = \frac{\partial^2 f}{\partial y^2}(\bar{x},\bar{y})$$

Teorema di Schwarz: se f ammette derivata seconda continua in A aperto $\Rightarrow \forall x \in A$: $\frac{\partial^2 f}{\partial y \partial x}(x) = \frac{\partial^2 f}{\partial x \partial y}(x)$

$F : \mathbb{R}^n \to \mathbb{R}^n \quad m > 1$

$(x_1, \dots, x_n) \to (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$

$x = (x_1, \dots, x_n)$

$f_1 : \mathbb{R}^n \to \mathbb{R} \to \nabla f_1(x)$ $\quad \exists \frac{\partial f_i}{\partial x_j} \quad \forall i = 1, \dots, m$

$f_n : \mathbb{R}^n \to \mathbb{R} \to \nabla f_m(x)$ $\quad \forall j = 1, \dots, n$

$J_x F$ Matrice Jacobiana di F in x.

$$\begin{pmatrix} \nabla f_1(x) \\ \vdots \\ \nabla f_m(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_m}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

matrice $m \times n$

$F : \mathbb{R}^n \to \mathbb{R}^{m \times n}$

$x \to J_x F$

Def. $F : \mathbb{R}^n \to \mathbb{R}^m$ è differenziabile in $\bar{x}$ p.to interno al dom di F se $\exists$ un applicazione lineare $d_{\bar{x}} F : \mathbb{R}^n \to \mathbb{R}^m$

$$\underset{\uparrow \mathbb{R}^m}{F(x)} = \underset{\uparrow \mathbb{R}^m}{F(\bar{x})} + \underbrace{(d_{\bar{x}} F)(x - \bar{x})}_{\uparrow \mathbb{R}^n} + o(\|x - \bar{x}\|) \text{ per } x \to \bar{x}$$

$\Rightarrow d_{\bar{x}} F = J_{\bar{x}} F$ differenziabile

* Esercizio: Data un'applicazione lineare.

$L : \mathbb{R}^n \to \mathbb{R}^m$

$\Updownarrow \quad x \to L(x)$

A (matrice) $x \to A(x) \quad \mathbb{R}^2$

$$\underset{(x_1, x_2)}{x} \to \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "g"] \arrow[rr, bend right] & \mathbb{R} \arrow[r, "\gamma"] & \mathbb{R}^m \end{tikzcd} \qquad \gamma(g(x)) = \mathbb{R}^m \to \mathbb{R}^m$$

$\mathbb{R}^n \xrightarrow{F} \mathbb{R}^n \xrightarrow{F^{-1}} \mathbb{R}^m$

$F^{-1}: \mathbb{R}^n \to \mathbb{R}^n$

$y \to x : F(x) = y$

F iniettiva

$\mathbb{R}^n \to \mathrm{Im}F \to \mathbb{R}^m$

$\mathbb{R}^n \xrightarrow{F} \mathbb{R}^n \xrightarrow{F^{-1}} \mathbb{R}^n$

$F^{-1} \circ F(x) = F^{-1}(F(x)) = x$

$F^{-1} \circ F(x) = \text{identità}(x) = Ix = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} (x)$

$F^{-1} \circ F(x) = Ix$

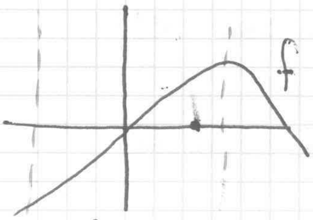
$F: \mathbb{R}^n \to \mathbb{R}^n$ "lineare"

$\downarrow$

A è invertibile $\iff \det A \neq 0$ $\quad \exists A^{-1}$

$F: \mathbb{R}^n \to \mathbb{R}^n$ campi vettoriali

$f: \mathbb{R} \to \mathbb{R}$



f è invertibile $\iff$ stretta in alcuni intervalli

f è strettamente monotona $\Rightarrow$ f è invertibile nel dom f

f monotona crescente stre.

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

La monotonia è una prop. delle funzioni reali di variabili reali

$f: \mathbb{R} \to \mathbb{R}$ $\quad f$ di classe $C^1(A)$ Aperto di $\mathbb{R}$

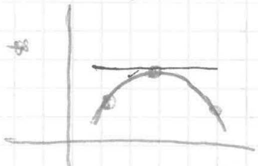
$\exists x_0 \in A$, se $f'(x_0) \neq 0$ $\Rightarrow$ f è localmente invertibile $\Rightarrow \exists B_\delta(x_0), \exists B_\varepsilon(f(x_0))$.

$\exists f^{-1}: B_\varepsilon(f(x_0)) \to B_\delta(x_0)$

: $f^{-1} \circ f(x) = x \quad \forall x \in B_\delta(x_0)$

$f \circ f^{-1}(y) = y \quad \forall y \in B_\varepsilon(f(x_0))$

f^{-1} è di classe C^1, $(f^{-1})'(f(x_0)) = \dfrac{1}{f'(x_0)}$



Teorema di invertibilità locale (o della funz. inversa)

$F: \mathbb{R}^n \to \mathbb{R}^n$

- $A \subseteq \mathbb{R}^n$ aperto, F sia di classe C^1 su A. $\quad F \in C^1(A)$
- $\exists X_0 \in A$, $J_{x_0}F$ invertibile $\left(\det J_{x_0}F \neq 0\right)$

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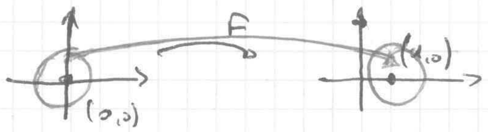
$(0,0) \qquad F(0,0) = (1,0)$



$$\int_{F(0,0)} F^{-1} = \int_{(1,0)} F^{-1} = \left(\int_{(0,0)} F \right)^{-1} = I^{-1} = I$$

$$\int_{(0,0)} F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = I^{-1}$$

Approssimazione lineare della $F(x)$

$$F(x) = F(\bar{x}) + \int_{\bar{x}} F \cdot (x - \bar{x}) + o(\|x - \bar{x}\|) \text{ per } x \to x_0$$

$F(x_0) = y_0$

$$F^{-1}(y) = F^{-1}(y_0) + \int_{y_0} F^{-1} (y - y_0) + o(\|y - y_0\|) \quad \text{per } y \to y_0$$

Es: $x_0 = (0, \pi/2) \longrightarrow F(x_0)$

$$\int_{F(x_0)} F^{-1}$$

$F : \mathbb{R}^n \to \mathbb{R}^n$

$\boxed{F(x) = y}$ cercare le soluzioni di questa equazione.

$\iff x = F^{-1}(y)$

Se trovo $F(\bar{x}) = \bar{y}$ e $\int_{\bar{x}} F$ è invertibile (F di classe C^1)

$(\bar{x}) \qquad \exists B(\bar{x})$

$\qquad \exists B(\bar{y}) \qquad F(x) = y$

$(\bar{y})$

$$F(x,y) = (a,b) \Rightarrow F(x, y + 2k\pi) = (a,b)$$

$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$

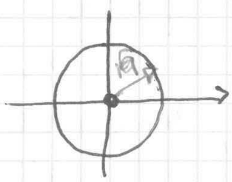
$\{(x,y) : f(x,y) = a\}$ insieme di livello.

$f(x,y) = x^2 + y^2 = a$.

$\{(x,y) : f(x,y) = a < 0\} = \emptyset$

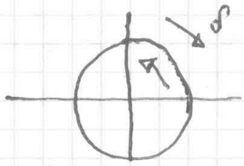
$\{(x,y) : f(x,y) = 0\} = \{(0,0)\}$

$\{(x,y) : f(x,y) = a > 0\}$



$f(x,y) = 3 \qquad$ Piano $z = 3$

$x^2 + y^2 = 1$



$(x, k(x))$ $(k(y), y)$

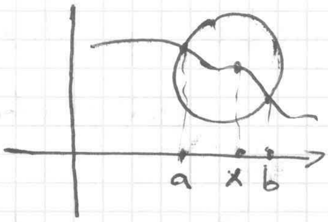
k sia def. per $x \in (a,b)$

$\gamma : (a,b) \longmapsto \mathbb{R}^2$

$t \longmapsto (t, h(t))$

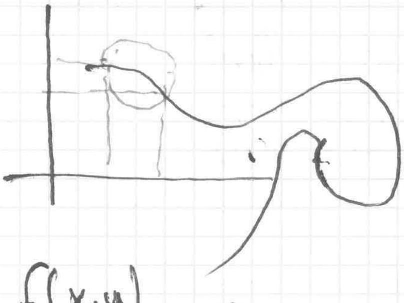
$\gamma'(t) = (1, k'(t)) \neq (0,0)$

Def. $A \subseteq \mathbb{R}^2$ si dice varietà uni-dimensionale (di dim 1) se $\forall P \in A$ $\exists B_\varepsilon(P)$ $B_\varepsilon(P) \cap A$ è il grafico di una funzione di 1 variabile, di classe C^1



$\{ y = h(x), \ x_0 \in (a,b) \}$

$= B_\varepsilon(P) \cap A$



$f(x,y) = a$

Teorema del Dini (o della funzione implicita) in due variabili ...

$f(x,y) = a$

$x = k(y)$

$y = h(x)$

$A \subseteq \mathbb{R}^2$ aperto $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$; $f \in C^1(A)$.

$(x_0, y_0) \in A : f(x_0, y_0) = a \in \mathbb{R}$

$\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$

$\{ e^{xy} + x - y - 1 = 0 \} = L \qquad (0,0) \in L$

Verificare che si applica il teorema del Dini e calcolare $h'(x)$ in $x=0$.

$\frac{\partial f}{\partial y}(0,0) \neq 0 \Rightarrow y = h(x)$

$\frac{\partial f}{\partial y} = x e^{xy} - 1 \qquad \frac{\partial f}{\partial y}(0,0) = -1 \neq 0$

$\frac{\partial f}{\partial x} = y e^{xy} + 1 \qquad \frac{\partial f}{\partial x}(0,0) = 1 \neq 0$

$\cancel{y = h(x)}$. $y = h(x)$

$$h'(x) = -\frac{\frac{\partial f}{\partial x}(x,h(x))}{\frac{\partial f}{\partial y}(x,h(x))} = -\frac{x e^{xh(x)} - 1}{y e^{xh(x)} + 1}$$

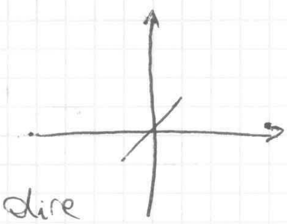
$h(0) = 0$

$h'(0) = -\frac{1}{-1} = 1$

Sviluppo di Taylor:

$h(x) = h(0) + h'(0) + o(x)$ per $x \to 0$

$h(x) = 0 + x + o(x)$ per $x \to 0$



se la funzione è di classe C^2 riesco a dire di più.

* * * * *

$h(x)$ se esiste, soddisfa l'equazione $f(x,h(x)) = a$
$\forall x \in (x_0 - \delta, x_0 + \delta)$

$(x_0 - \delta, x_0 + \delta) \xrightarrow{\gamma} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

$x \longmapsto (x, h(x)) \longrightarrow f(x, h(x))$

$f \circ \gamma(x) = \varphi(x)$

$\parallel$

$f(x, h(x))$

φ è di classe C^1 perché f è di classe C^1 per ipotesi,

4

Teo del Dini

Mer $\begin{cases} 13 - 14.30 & 10A \\ 14:30 - 16:00 & 5S \end{cases}$

Ven. 13.00 - 16:00 8C

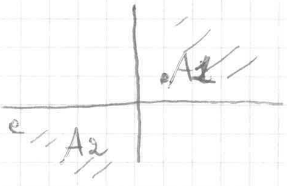
- $f(x,y) = a$
- $f \in C^1(A)$ A aperto di $\mathbb{R}^2$
- $\exists (x_0, y_0) \in A : f(x_0,y_0) = a$
- $\frac{\partial f}{\partial y}(x_0,y_0) \neq 0$ $\left(o \ \frac{\partial f}{\partial x}(x_0,y_0) \neq 0 \right)$

$\Rightarrow \exists B(x_0), \exists B(y_0), \exists h. \ B(x_0) \to B(y_0)$.

t.c.
- $h(x_0) = y_0$ *
- $f(x, h(x)) = a \quad \forall x \in B(x_0)$
- $h \in C^1(B(x_0))$.
- $h'(x_0) = - \dfrac{\partial f / \partial x \,(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)}$.

$f(x,y) = xy + \log xy - 1$

1. dom f
2. $(1,1) \in \{(x,y) \,/\, xy + \log xy - 1 = 0\} = L$
3. Verificare che si può applicare il teo. del Dini loc. in $(1,1)$ e calcolare $h'(1)$ o $k'(1)$ se $x = k(y)$
4. Se L è una varietà unidimensionale.



$\text{dom} f = \{(x,y) \,/\, xy > 0\} =$

$= \{(x,y) \,/\, x>0 \text{ e } y>0\} \cup \{(x,y) \,/\, x<0 \text{ e } y<0\}$.

$f(1,1) = 1 + \log 1 - 1 = 1 - 1 = 0 \Rightarrow (1,1) \in L$

Osserviamo che $f \in C^1(A_1)$, $(1,1) \in L$

$\frac{\partial f}{\partial x} = y + \frac{1}{xy} y = y + \frac{1}{x} \qquad \frac{\partial f}{\partial x}(1,1) = 2 \neq 0$

posso scrivere x in funzione di y

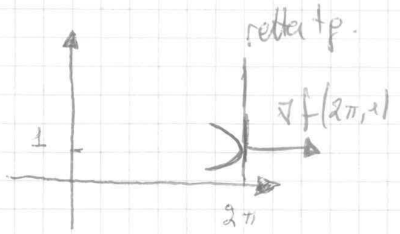
$\frac{\partial f}{\partial y} = x + \frac{1}{xy} x = x + \frac{1}{y} \Rightarrow \frac{\partial f}{\partial y}(1,1) = 2 \neq 0$

Esplicito rispetto ad una delle variabili.

$\exists B(1, \varepsilon), \ \exists B(1, \delta)$

$y = h(x) \qquad h: B(1,\varepsilon) \to B(1,\varepsilon)$

$x \longmapsto h(x)$.



$\vec{\nabla} f(2\pi,1) = (8(\pi-1), 0) \Rightarrow$ gradiente orizzontale.

Eq retta tangente a L in $(2\pi,1)$. $\Longrightarrow x = 2\pi$

Es: $x^3+y^3-4x^2y+2=0$. $\qquad x^5+3x^2y-2y^4-1=0$.

$f(1,0)=0$.

1) $(1,1) \in L$

2)

$f(x,y,z)$ dom $f \subseteq \mathbb{R}^3$

$f(x,y,z) = a$

$f(x,y,z) = x^2+y^2+z^2 = 1$ sup. sferica.

$x^2+y^2+z^2 = a$
$\to \phi$ se $a<0$
$\to \{(0,0,0)$ se $a=0\}$
$\to$ superfi. sferica se $a>0$.

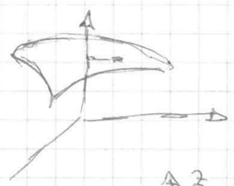
$z = h(x,y)$
$y = k(x,z)$
$x = g(y,z)$

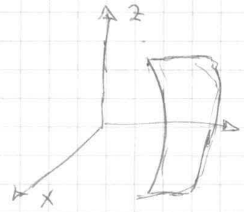
Grafico di due variabili

1) $\{(x,y), h(x,y)\}$

2) $\{(x, k(x,z), z)\}$

3) $\{(g(y,z), y, z)\}$



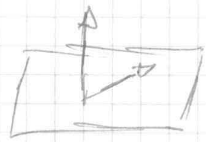


$\sigma : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ di classe C^1

ap., connesso

$(u,v) \longmapsto (\sigma_1(u,v), \sigma_2(u,v), \sigma_3(u,v))$

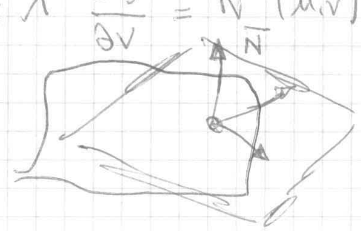
$\quad\quad\quad\quad\quad\quad\quad\quad x \quad\quad\quad\quad y \quad\quad\quad\quad z$



$\frac{\partial \sigma}{\partial u}$ $\quad$ $\frac{\partial \sigma}{\partial v}$

$$\begin{pmatrix} \frac{\partial \sigma_1}{\partial u} & \frac{\partial \sigma_1}{\partial v} \\ \frac{\partial \sigma_2}{\partial u} & \frac{\partial \sigma_2}{\partial v} \\ \frac{\partial \sigma_3}{\partial u} & \frac{\partial \sigma_3}{\partial v} \end{pmatrix} = J\sigma_{(u,v)}$$

$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \vec{N}(u,v)$



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- $f \in \mathcal{C}^k \Rightarrow h$ è di classe $\mathcal{C}^k$.

- Corollario 1:

$f(x,y,z) = a$, $f \in \mathcal{C}^1(A)$ $\qquad P_0(x_0,y_0)$

$f(x_0,y_0,z_0) = a$

e se $\nabla f(x,y,z) \neq (0,0,0)$ $\forall (x,y,z) \in L \Rightarrow$

L è una varietà bidimensionale ($\Rightarrow$ una superficie)

- Corollario 2:

Nelle ipotesi del teorema del Dini

$\Rightarrow \nabla f(x_0,y_0,z_0) \perp$ al piano tangente a L in (x_0,y_0,z_0)

cioè il $\nabla f(x_0,y_0,z_0)$ è un vett. normale.

Dim: L è localmente il grafico di $z = h(x,y)$

$\Rightarrow$ $\boxed{\vec{N} = \left(\frac{\partial h}{\partial x}(x_0,y_0), \frac{\partial h}{\partial y}(x_0,y_0), -1\right) \text{ è un vettore normale}}$

$$\cancel{\frac{\partial f}{\partial z}}\left(-\frac{\partial f/\partial x(P_0)}{\cancel{\partial f/\partial z(P_0)}}, -\frac{\partial f/\partial y(P_0)}{\cancel{\partial f/\partial z(P_0)}}, -1\right) \qquad \frac{\partial f}{\partial z}(x_0,y_0) \neq 0.$$

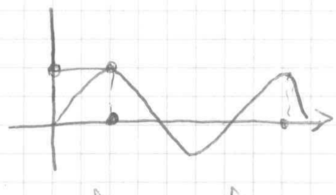
$$= \left(-\frac{\partial f}{\partial x}(P_0), -\frac{\partial f}{\partial y}(P_0), -\frac{\partial f}{\partial z}(P_0)\right) = -\nabla f(x_0,y_0,z_0)$$

Massimi e minimi di $f: \mathbb{R}^n \to \mathbb{R}$.

passare da Max e Min di un sistema aperto ad uno chiuso.

$A \subseteq \mathbb{R}^n$, $A \subseteq \text{dom} f$.

1. $\bar{x} \in A$ è un punto di max (assoluto)
   dif se $\forall x \in A$, $f(x) \leq f(\bar{x})$
   , $f(\bar{x})$ il massimo di f in A



2. $\underline{x} \in A$ è un punto di min (assoluto) di f in A se
   $\forall x \in A$, $f(x) \geq f(\underline{x})$.
   $f(\underline{x})$ è il minimo di f in A.

Teorema di Weierstrass:

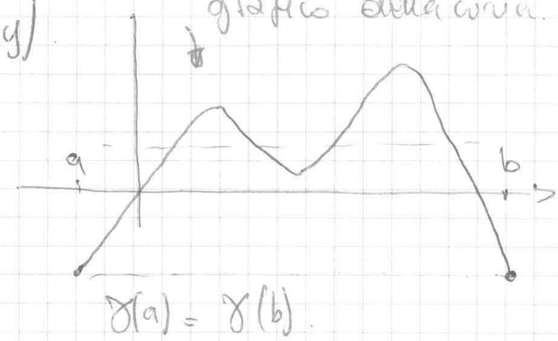
Se $K \subseteq \mathbb{R}^n$ è compatto (chiuso e limitato) f è continua in K
$\Rightarrow$ f ha un max e min assoluto in K.

1° Strategia: descrivere il bordo come una curva in forma parametrica.

$\partial K \longrightarrow \{(x,y) \in \partial K\}$ = sostegno di una curva in forma parametrica.

$\begin{cases} x = \gamma_1(t) \\ y = \gamma_2(t) \end{cases}$ $t \in [a,b]$. $f(x,y)$

grafico della curva.



$f(\gamma_1(t), \gamma_2(t)) = h(t)$

$\gamma(a) = \gamma(b)$.

I punti candidati sono:
- pt. in cui la derivata è nulla
- " " " " " non è nulla
- punti di estremo.

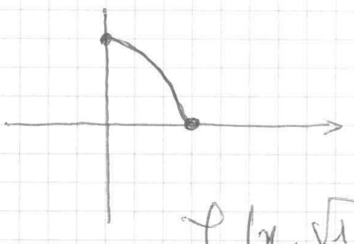
* punti candidati => Se f è continua su K.

=> Per Weierstrass f ha max e min assoluto su K.

$f(P_1)$, $f(P_2)$ .... $f(P_n)$.

$f(x,y) = xy$

i max e i min locali di f sul quarto di circonf $x^2+y^2=1$ contenuta nel primo quadrante.



$\begin{cases} x^2+y^2=1 \\ x \geq 0 \\ y \geq 0 \end{cases}$ ① $\begin{cases} y = \sqrt{1-x^2} \\ 0 \leq x \leq 1 \end{cases}$

$f(x, \sqrt{1-x^2}) = x \cdot \sqrt{1-x^2}$, $x \in [0,1]$

② $\begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases}$ $0 \leq t \leq \pi/2$

$f(x,y) \rightarrow f(\cos t, \operatorname{sen} t) = \cos t \cdot \operatorname{sen} t$, $0 \leq t \leq \pi/2$.

② $h(t) = \cos t \operatorname{sen} t$

$h'(t) = -\operatorname{sen}^2 t + \cos^2 t = -(1-\cos^2 t) + \cos^2 t = -1 + 2\cos^2 t = 0$

$\cos^2 t = \frac{1}{2}$ <=> funz cost è iniettiva => $\exists !$ valore per cui si annulla.

$$\left.\begin{array}{l} f(\bar{x},\bar{y}) = g(h(\bar{x},\bar{y})) \\ f(x,y) = g(h(x,y)) \end{array}\right\} \quad h(\bar{x},\bar{y}) \geq h(x,y) \quad \text{applico } g$$

$$g(h(\bar{x},\bar{y})) \geq g(h(x,y))$$

$\sqrt{x^2+y^2}$ $\qquad g(t) = \sqrt{t}$

$$h(x,y) = x^2+y^2$$

$f(x,y) = \sqrt{x^2+y^2} = g(h(x,y))$ NB i ~~pdti~~ non cambiano ma cambiano i valori.

h e g decrescenti o crescenti non camb.

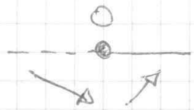
$f(x,y) = x^2+y^2$ nei punti $x = y^2+1$

$\Downarrow$

$$f(x,y) = (y^2+1)^2+y^2 = h(x),\ y \in \mathbb{R}$$

$$h'(y) = 2(y^2+1)\cdot 2y + 2y = 2y\,(\underbrace{2(y^2+1)+1}_{>0}) = 0$$

$\exists$! pto per cui $h'(y) = 0 \Rightarrow y = 0$.



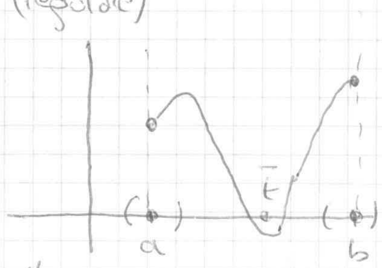
$f(x,y)$ funzione ~~def~~ in un aperto di $\mathbb{R}^2$ di classe C^1
trovare i punti di max e di min di f ma su un "vincolo"

<u>Vincolo</u>: il sostegno di una curva in forma parametrica (regolare)

$\gamma : [a,b] \to \mathbb{R}^2$

$f(\gamma_1(t), \gamma_2(t)) = k(t)$ def in $[a,b]$

$\Rightarrow k$ è di classe C^1



$t = b \Rightarrow \gamma(b)$ c'è il max.

Se $\bar{t} \in (a,b)$ ed è un punto a derivata nulla di k. $k'(\bar{t}) = 0$

$$k'(t) = \frac{d}{dt} f(\gamma_1(t), \gamma_2(t)) = \frac{\partial f}{\partial x}(\gamma(\bar{t}))\cdot \gamma_1'(\bar{t}) + \frac{\partial f}{\partial y}(\gamma_2(\bar{t}))\cdot \gamma_2'(\bar{t}) = 0$$

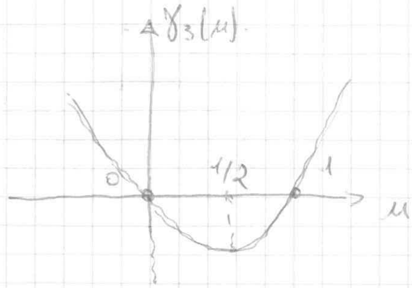
$$= \nabla f(\gamma(\bar{t}))\cdot \underbrace{(\gamma_1'(\bar{t}), \gamma_2'(\bar{t}))}_{\gamma'(\bar{t})} = 0 \quad \text{Vettore tangente}$$

AB : $\gamma_3 : \begin{cases} x = u \\ y = 1-u \end{cases}$ $\quad f(x,y) = x^2+y-1$.

$u = 0 \Rightarrow (0,1) = B$

$u = 1 \Rightarrow (1,0) = A$.

$f|_{\gamma_3} = u^2 + 1 - u - 1 = 0 \Rightarrow u^2 - u = k(u)$



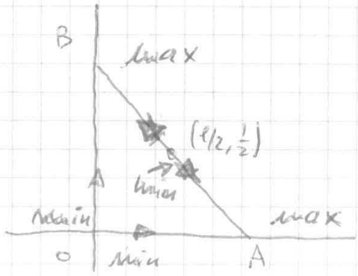
$f|_{\gamma_3} = u^2 - u = u(u-1)$

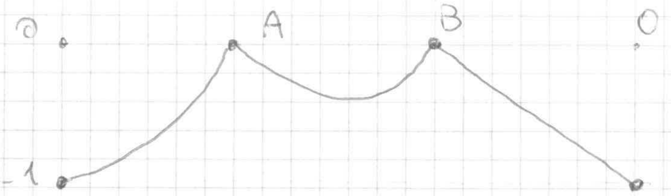
$k'(u) = 2u - 1 = 0 \Rightarrow u = 1/2$.

$u = 0 \Rightarrow (0,1) =$ pt max

$u = 1 \Rightarrow (1,0) =$ pt. max

$u = 1/2 \Rightarrow (1/2, 1/2) =$ pt di min





① $(0,0)$ $\left(\frac{1}{2}, \frac{1}{2}\right)$ sono pti di min locale.

② A, B sono pti di max locale.

$f(0,0) = -1$

$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{2} - 1 = \frac{1+2-4}{4} = -\frac{1}{4} > -1$.

$f(A) = f(B) = 0$

$\Rightarrow$ O $(0,0)$ è punto di minimo assoluto.

$\Rightarrow -1$ è il minimo assoluto.

A $(1,0)$ e B$(0,1)$ sono i pti di max assoluto.

0 è il max assoluto.

Esercizio 8

$f(x,y) = 4x^2 + 3y^2$ $\quad A(1,0)$ $\quad B(2,0)$ $\quad C(0,2)$

$f\left(\frac{6}{7}, \frac{8}{7}\right) = 4 \cdot \frac{36}{49} + 3\,\frac{64}{49} = \frac{336}{49} > 7$

$\Rightarrow$ Conclusione:

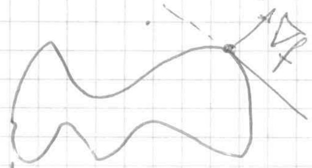
$f\left(\frac{3}{4}, \frac{1}{2}\right) = 3$ ~~punto~~ di min

$f(2,0) = 16$ ~~punto~~ ~~di~~ max.

Moltiplicazione di Lagrange:

$g(x,y) = a$

$\{(x,y) \in \mathbb{R}^2 / g(x,y) = a\}$



$\nabla f(x,y)$ è $\nabla f(\gamma(t))$, $\frac{d}{dt} f(\gamma(t)) = 0$

Dal teorema del Dini, $\cdot \nabla g \perp L$

- $\nabla f(P) \perp L \iff P \in L$ ed è punto stazionario vincolato di f ad L

$P \in L$ è un pto staz. vincolato $\iff \nabla g(P) \parallel \nabla f(P)$

Trovare i $\lambda \neq 0$:

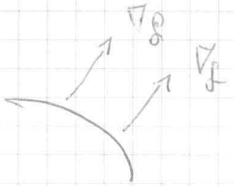
$$\begin{cases} \nabla g(P) = \lambda \nabla f(P) \\ P \in L \end{cases}$$ (vettori paralleli).

Teorema:

- $f(x,y)$ di classe C^1 in un aperto $A \subseteq \mathbb{R}^2$.
- $g(x,y)$ ~~di~~ $\in C^1(A)$
- $\{g(x,y) = a\} = L \neq \emptyset$
- $\nabla g(x,y) \neq (0,0)$, $\forall (x,y) \in L$

$\Rightarrow$ I pti stazionari di f vincolati su L sono i pti di L in cui $\nabla f \parallel \nabla g$ cioè sono le soluzioni del sistema.

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = \lambda \frac{\partial g}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) = \lambda \frac{\partial g}{\partial y}(x,y) \\ g(x,y) = a \end{cases} \quad \lambda \neq 0$$



λ = moltiplicatore di Lagrange.

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2 - Moltiplicatore di Lagrange:

$g(x,y) = x^2+y^2-1$ è di classe C^1

. $\nabla g(x,y) = (2x, 2y) = (0,0) \iff (x,y) = (0,0)$ ma $(0,0) \notin$ dom di g $\notin L$

. f è di classe C^1

$\nabla g(x,y) = \cancel{\lambda}(2x, 2y) \parallel \nabla f(x,y) = \cancel{\lambda}(y,x)$ $\quad \nabla f = \lambda \nabla g$ $\quad \parallel$ $\quad \nabla g = \lambda \nabla f$.

$$\begin{cases} 2x = \lambda y \\ 2y = \lambda x \qquad \lambda \neq 0 \\ x^2+y^2-1=0 \end{cases}$$

OSS: nei punti della circonferenza se

1) 1° eq: $x=0$, $\lambda \neq 0 \Rightarrow y=0$ $\quad (0,0) \notin L$, non interessa

2) 2° eq: $y=0$, $\lambda \neq 0 \Rightarrow x=0$ $\quad$ " $\quad$ " $\quad$ "

$\Rightarrow$ dalla 1), 2) $\Rightarrow x, y \neq 0$ posso dividere per y.

1° eq

$$\begin{cases} \lambda = \dfrac{2x}{y} \\ \lambda = \dfrac{2y}{x} \\ x^2+y^2-1=0 \end{cases} \Rightarrow \frac{2x}{y} = \frac{2y}{x}$$

$2x^2 = 2y^2$

$x^2 = y^2$

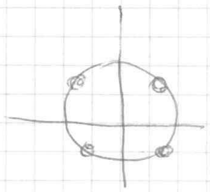
$$\begin{cases} x^2 = y^2 \\ x^2+y^2-1=0 \end{cases} \Rightarrow \begin{cases} x^2 = y^2 \\ 2x^2-1=0 \end{cases}$$

$$\begin{cases} x^2 = \dfrac{1}{2} \Rightarrow x = \pm \dfrac{1}{\sqrt{2}} = \pm \dfrac{\sqrt{2}}{2} \\ x^2 = y^2 \quad (x=y \lor x=-y) \end{cases}$$

$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\qquad P_2 = \left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$P_3 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ $\qquad P_4 \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$P_1, \dots P_4$ punti simmetrici sulla circonferenza



$f(x,y) = xy$

$f(P_1) = f(P_3) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$ max

$f(P_2) = f(P_4) = -\frac{1}{2}$ min

$P_1, P_3 \Rightarrow$ pti di max

$P_2, P_4 \Rightarrow$ pti di min

$$\begin{cases} x(\lambda-1)=0 \\ -2y+6=\lambda(y-2) \\ x^2+(y-2)^2=4 \end{cases}$$

① $\begin{cases} x=0 \\ -2y+6=\lambda(y-2) \\ (y-2)^2=4 \end{cases}$ ② $\begin{cases} \lambda=1 \\ -2y+6=y-2 \\ x^2+(y-2)^2=4 \end{cases}$

①

$$\begin{cases} x=0 \\ y-2=\pm 2 \; {\scriptstyle +2} \end{cases} \Rightarrow \begin{cases} x=0 \\ y=4 \end{cases} \vee \begin{cases} x=0 \\ y=0 \end{cases}$$

$$\Rightarrow \boxed{(0,4),\ (0,0)}$$

②

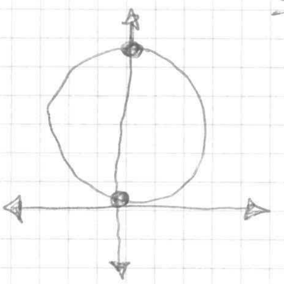
$$\begin{cases} \lambda=1 \\ -2y+6=y-2 \\ x^2+(y-2)^2=4 \end{cases}$$

$\sqrt{32}=\sqrt{16\cdot 2}=\sqrt{16}\cdot\sqrt{2}=4\sqrt{2}$

$6=3y-2 \Rightarrow 3y=8 \Rightarrow y=\frac{8}{3}$

$$\begin{cases} \lambda=1 \\ y=\frac{8}{3} \\ x^2+\left(\frac{4}{9}\right)=4 \Rightarrow x^2=\frac{32}{9} \end{cases}$$

$$x=\pm\frac{\sqrt{32}}{3}=\pm\frac{4\sqrt{2}}{3}$$



$P_1(0,4)$ $P_2(0,0)$ $P_{3,4}\left(\pm\frac{4\sqrt{2}}{3},\frac{8}{3}\right)$

$P(0,3) \Rightarrow$ candidati ad essere punti di max e min assoluto.

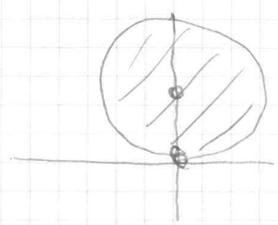
$f(P_1)=-2\cdot 16+12\cdot 4=-32+48=16$

$f(P_2)=0$, $f(P_{3,4})=\frac{32}{9}-\frac{64}{9}+12\cdot\frac{8}{3}=32-\frac{32}{9}=$

$\frac{64}{3}>16$

$f(0,3)=-2\cdot 9+12\cdot 3=-18+36=18=\frac{54}{3}$

$$\begin{cases} P_{3,4}\left(\pm\frac{4\sqrt{2}}{3},\frac{8}{3}\right) \text{ sono i punti di max assoluto} \\ P_2(0,0) \text{ punto di min assoluto} \end{cases}$$



Determinare max e min assoluti di

$f(x,y)=x|x|-2y^2$

$M=\{(x,y): x^2+y^2\le 4,\ y\ge 0\}$

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Semicirconferenza: $C: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t \le \pi$

$$f|_C = 4(\cos t - |\cos t|) - 8\sin^2 t =$$

$$= \begin{cases} 4\cos^2 t - 8\sin^2 t & 0 \le t \le \pi/2 \\ -4\cos^2 t - 8\sin^2 t & \frac{\pi}{2} \le t \le \pi \end{cases}$$

$$\left.\begin{array}{ll} 0 \le t \le \frac{\pi}{2}: & 4(\cos^2 t - 2\sin^2 t) \\ \frac{\pi}{2} \le t \le \pi: & -4(\cos^2 t + 2\sin^2 t) \end{array}\right\} k(t)$$

$$k'(t) = \begin{cases} 4(-2\cos t \sin t - 4\cos t \sin t) & 0 < t < \pi/2 \\ -4(-2\cos t \sin t + 4\cos t \sin t) & \pi/2 < t < \pi \end{cases}$$

$$= \begin{cases} -24 \cos t \sin t & 0 < t < \pi/2 \\ -8 \cos t \sin t & \pi/2 < t < \pi \end{cases}$$

$0, \frac{\pi}{2}, \pi$ sono i punti della circonf in cui la $k'(t)$ si annulla.

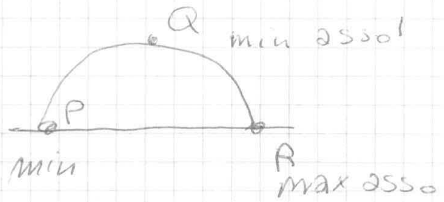
Ho trovato 3 punti candidati: $\quad f = x|x| - 2y^2$

$P(-2,0)$, $Q(0,2)$, $R(2,0)$

$f(P) = -4$ minimo

$f(Q) = -8$ minimo assoluto

$f(R) = 4$ max assoluto



$(0,2)$ = punto di min.

$f(x,y) = xy$

$C: \{ (x,y) : x^2 + y^2 + xy - 1 = 0 \}$

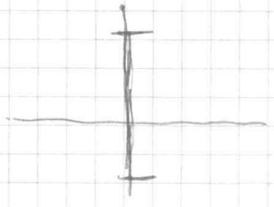
Sostituisco: $y - x^2 = 0$

1) C è compatto?

2) f è continua.

$x^2 + xy + y^2 - 1 = 0$ $\quad$ y come parametro

$$x = \frac{-y \pm \sqrt{y^2 - 4(y^2-1)}}{2} = \frac{-y \pm \sqrt{-3y^2+4}}{2} \qquad -3y^2 + 4 \ge 0$$



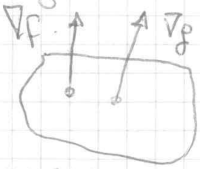
f è ristretta ad una superficie parametrica

$$f(x,y,\varphi(x,y)) = h(x,y)$$

candidati ad essere pti di max e min sono i pti in cui
$\nabla h(x,y) = (0,0)$

$\Rightarrow$ Se P è un punto staz. vincolato di f su $L \Rightarrow$

$\nabla f(P) \perp L$



$P \in L$ è un punto stazionario vin. di f su $L \Leftrightarrow$

$\nabla f(P) \,//\, \nabla g(P)$

Teorema dei moltiplicatori di Lagrange (in 3 variabili)

- $f : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$, Ω aperto.
  $f \in \mathcal{C}^1(\Omega)$
- $L = \{(x,y,z) \,/\, g(x,y,z) = a\} \neq \emptyset$, $L \subseteq \Omega$
  $g \in \mathcal{C}^1(\Omega)$
- $\nabla g(P) \neq \vec{0}$, $\forall P \in L$

$\Rightarrow$ superficie "buona", ha tangente in ogni punto.

$P \in L$ è un punto stazionario di f su L

$\Rightarrow \nabla f(P) \,//\, \nabla g(P)$

$\exists \lambda \neq 0 \quad \nabla f(P) = \lambda \nabla g(P)$

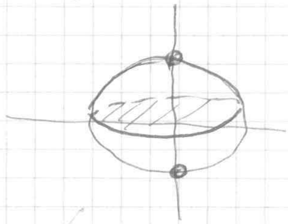
$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y,z) = \lambda \dfrac{\partial g}{\partial x}(x,y,z) \\ \dfrac{\partial f}{\partial y}(x,y,z) = \lambda \dfrac{\partial g}{\partial y}(x,y,z) \\ \dfrac{\partial f}{\partial z}(x,y,z) = \lambda \dfrac{\partial g}{\partial z}(x,y,z) \\ g(x,y,z) = a \end{cases}$$

$\cancel{\lambda}(yz^2-2x)=0 \qquad \lambda\neq 0$

$$\begin{cases} \lambda(xz^2-2y)=0 \\ e^{xy}=\lambda \\ x^2+y^2+z^2=1 \end{cases} \Rightarrow \quad \begin{aligned} & yz^2-xz^2=2x-2y \\ & z^2(y-x)=2(x-y)=-2(y-x) \end{aligned}$$

$$\begin{cases} (y-x)(z^2+2)=0 \\ xz^2=2y \\ x^2+y^2+z^2=1 \end{cases}$$

$z^2+2>0 \quad \forall z$



$y-x=0 \rightarrow y=x$

$$\begin{cases} xz^2=2x \\ 2x^2+z^2=1 \end{cases}$$

$y=x$

$$\begin{cases} x(z^2-2)=0 \\ 2x^2+z^2=1 \end{cases} \Longleftrightarrow \begin{cases} x=0 \\ y=0 \\ z^2=1 \end{cases} \vee \cancel{\begin{cases} z=\pm\sqrt{2} \\ y=x \\ 2x^2+2=1 \Rightarrow 2x^2=-1 \text{ mai} \end{cases}}$$

Conclusione:

l'insieme dei punti $\{(x,y,z) \,/\, x^2+y^2\leq 1,\ z=0\}$ è l'insieme dei punti di min ass

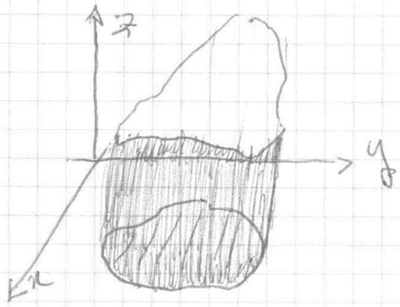
$P_{1,2}\ (0,0,\pm 1)$ sono punti stazionari vincolati.

$f(0,0,\pm 1) = z^2e^{xy}\Big|_{(0,0,\pm 1)} = 1.$

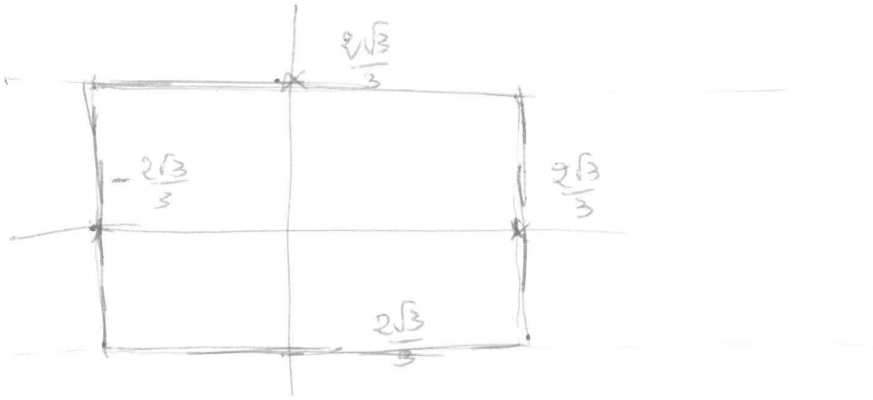
$\Rightarrow P_{1,2}\ (0,0,\pm 1)$ sono i punti di max assoluto di f su L.

Integrali doppi

$f(x,y) \qquad f(x,y,z)$



$f(x,y)=1$



x come parametro ricavo y:

$y^2+xy+x^2-1=0 \Rightarrow y_{1,2}=\frac{-x\pm\sqrt{x^2-4(x^2-1)}}{2}$

$4-3x^2\geq 0 \Rightarrow -\frac{2\sqrt{3}}{3}\leq x\leq +\frac{2\sqrt{3}}{3}$

C. compatto e chiuso.

$f(\vec{x}) = z^2 e^{xy}$ $\qquad L=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2+z^2\leq 1\}$.

sfera.

moltiplicatore di Lagrange:

sul bordo ∂L.

$$\begin{cases} yz^2e^{xy} = 2\lambda x \\ xz^2e^{xy} = 2\lambda y \\ 2ze^{xy} = 2\lambda z \\ x^2+y^2+z^2=1 \end{cases}$$

$$\lambda = \frac{y z^2 e^{xy}}{2x} \qquad x \neq 0.$$

$$x z^2 e^{xy} = 2y \cdot \frac{y z^2 e^{xy}}{2x}$$

$$\cancel{x^2 z^2 e^{xy} = y^2 z^2 e^{xy}}$$

$$2z\left(\lambda - e^{xy}\right) = 0 \quad \begin{cases} z = 0 \\ e^{xy} = \lambda. \end{cases}$$

$$\boxed{z = 0 \;;\; x = 0 \;;\; y = 0}$$

$$\boxed{\lambda = e^{xy}} \Rightarrow$$

$$y z^2 \cancel{e^{xy}} = 2 \cancel{e^{xy}} x. \qquad \boxed{z^2 = \cancel{\frac{2x}{y}} = \frac{2y}{x}}$$

$$x z^2 e^{xy} = 2 e^{xy} y. \qquad \text{sostituisco...}$$

$$\cancel{x^2 + y^2 + \frac{2x}{y} = 1.}$$

$$\boxed{z^4 y^2 + y^2 + z^2 = 1}$$

$$x = z^2 y.$$

$$z^4 y^2 + z^2 + y^2 - 1 = 0. \qquad z^2 = u.$$

$$u^2 y^2 + u + y^2 - 1 = 0 \Rightarrow u_{1,2} = \frac{-1 \pm \sqrt{1 - 4y^2 + 4}}{2} = \frac{-1 \pm \sqrt{5 - 4y^2}}{2}$$

$$5 - 4y^2 \geq 0 \qquad y^2 \leq \frac{5}{4}$$

$$\sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} c_{ij} (x_{i+1} - x_i)(y_{j+1} - y_j) \overset{def}{\Longrightarrow} \iint_R f$$

Data una funzione f limitata su R

- g "maggioranti a scala" di f $\quad (g(x,y) \geq f(x,y))$

$\inf \left\{ \iint_R g , \right\} = \overline{\iint_R} f$

- h "minoranti a scala di f : $h(x,y) \leq f(x,y)$ su R

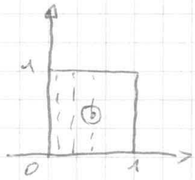
$\sup \left\{ \iint_R h , h \text{ minorante a scala di } f \right\} = \underline{\iint_R} f$

Definizione 3 f è Riemann-integrabile su R se
$$\underline{\iint_R} f = \overline{\iint_R} f$$

Ci sono funzioni che non sono Riemann-integrabili

Dirichlet:
$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

$$f(x,y) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$



$$0 = \underline{\iint_R} f \neq \overline{\iint_R} f = 1$$

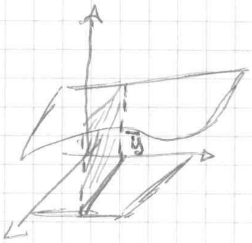
Metodo di RIDUZIONE:

"integrazione secondo Riemann su $R = [a,b] \times [c,d]$

a) Se $\forall y \in [c,d]$, $\exists g(y) = \int_a^b f(x,y)\,dx$ (sopravvive y) ed è integrabile

$$\Rightarrow \iint_R f = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy \qquad \bar{y} = \text{fissato (fermo)}$$

$$f(x,\bar{y}) \Rightarrow \int_a^b f(x,\bar{y})\,dx$$



b) $\forall x \in [a,b], \exists h(x) = \int_c^d f(x,y)\,dy$ (sopravvive x) ed Riemann-integrab. su $[a,b]$

$$\Rightarrow \iint_R f = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

se $f(x,y)$ è continua $\Rightarrow$ integ rispetto a y o a x $\Rightarrow$ ottengo una funzione $h(x) \vee g(y)$ continua

$$\int_1^2 \operatorname{sen}\pi x\, dx = -\frac{1}{\pi}\cos\pi x\Big]_1^2 = -\frac{1}{\pi}\left(\cos 2\pi - \cos\pi\right) =$$
$$= -\frac{1}{\pi}\left(1-(-1)\right) = -\frac{2}{\pi}$$

$$R = [a,b]\times[c,d]$$

$$f(x,y) = g(x)\cdot h(y) \Rightarrow \iint_R f(x,y) = \left(\int_a^b g(x)\,dx\right)\cdot\left(\int_c^d h(y)\,dy\right)$$

Dimostrazione

$$\iint_R g(x)\,h(y) = \int_a^b \Big[\int_c^d \underbrace{g(x)}_{cost}\cdot h(y)\,dy\Big]\,dx = \int_a^b g(x)\cdot\underbrace{\left(\int_c^d h(y)\,dy\right)}_{\in\mathbb{R}}dx$$
$$= \left(\int_c^d h(y)\,dy\right)\cdot\left(\int_a^b g(x)\,dx\right)$$

Esempio: $\iint_R y^2 \operatorname{sen} x \qquad [0,1]\times[0,2] = R$

$$\iint_R y^2 \operatorname{sen} x = \left(\int_0^2 y^2\,dy\right)\cdot\left(\int_0^1 \operatorname{sen} x\,dx\right) = \left(\frac{y^3}{3}\Big]_0^2\right)\cdot\left(-\cos x\Big]_0^1\right)$$

$(+\cos 1 \quad + \quad 1)$

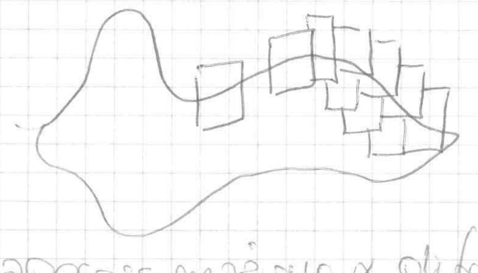
$$\frac{8}{3}\cdot\left(-\cos 1 + \cos 0\right) = \frac{8}{3}\left(1-\cos 1\right)$$

$f(x,y) = 3x^2y + e^{xy} \qquad [0,1]\times[0,2]$

$g(x,y) = x\operatorname{sen}^2 y \qquad [0,2]\times[0,\pi]$

$h(x,y) = 5x^3y^2 - 2xy^2 \qquad [0,1]\times[-1,0]$

Insiemi che vanno bene



$A \subseteq \mathbb{R}^2$ limitato, $\mathring{A} \neq \emptyset$

U finito di rettangoli (U = unione)

$= \{B \supseteq A\} \quad \{\text{area}(B)\}$

$\{C \subseteq A\} \quad \{\text{area}(C)\}$

per approssimazione x difetto prendo il sup

$$\sup\{\text{area}(C),\ C\subseteq A\} \leq \inf\{\text{area}(B),\ B\supseteq A\}$$

misura interna $\qquad$ misura esterna.

$m(\mathring{A})$ = misura di Peano Jordan se la misura interna = misura esterna.

Integrali doppi:

$\Omega \subseteq \mathbb{R}^2$ è misurabile se dato limitato
B rettangolo : $\Omega \subseteq B$

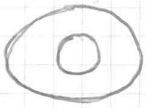
$$f(x,y) = \chi_\Omega = \begin{cases} 1, & (x,y) \in \Omega \\ 0 & (x,y) \notin \Omega \end{cases}$$

Ω è misurabile se f è integrabile su B $\quad |\Omega| = \iint_B f$

Ω è misurabile $\Longleftrightarrow$ $\partial\Omega$ ha misura nulla.

1. sottoinsiemi di ins di mis nulla.
2. $\cup$ finito di ins di mis nulla
3. num finito di punti
4. segmenti
5. $\{ (x, f(x)),\ x \in [a,b],\ f \text{ integr.} \}$
   $\{ (y, g(y)),\ y \in [c,d],\ g \text{ int} \}$
6. Sostegni di archi di curve regolari a tratti:

Proprietà : Ω_1, Ω_2 misurabili



1. $\Omega_1 \subseteq \Omega_2 \Rightarrow |\Omega_1| \leq |\Omega_2|$
2. $\Omega_1 \cup \Omega_2$ è misurabile
   $\Omega_1 \cap \Omega_2$ è misurabile
   $|\Omega_1 \cup \Omega_2| = |\Omega_1| + |\Omega_2| - |\Omega_1 \cap \Omega_2|$

Proposizione :

Ω misurabile $\Rightarrow$

1. $\mathring{\Omega} = \{ P \in \Omega \mid P \text{ è interno ad } \Omega \}$
   $\bar{\Omega} = \Omega \cup \partial\Omega = \mathring{\Omega} \cup \partial\Omega$

   sono misurabili
2. $\forall\, \Omega' \quad \mathring{\Omega} \subseteq \Omega' \subseteq \bar{\Omega}$

Ω'

$\Rightarrow$ Ω' è misurabile e $|\Omega| = |\Omega'| = |\mathring{\Omega}| = |\bar{\Omega}|$

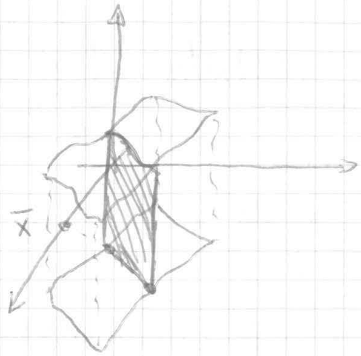
dim: poichè Ω è misurabile $\Rightarrow |\partial\Omega| = 0$

$\partial\mathring{\Omega} = \partial\Omega = \partial\bar{\Omega} \Rightarrow |\partial\mathring{\Omega}| = |\partial\bar{\Omega}| = 0$

$\Rightarrow \mathring{\Omega}$ e $\bar{\Omega}$ sono misurabili

$\bar{\Omega} = \mathring{\Omega} \cup \partial\Omega \qquad |\bar{\Omega}| = |\mathring{\Omega} \cup \partial\Omega| = |\mathring{\Omega}| + \underbrace{|\partial\Omega|}_{=0 \text{ per ipot}} - \underbrace{|\mathring{\Omega} \cap \partial\Omega|}_{=0}$

$$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$



$$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(\bar{x},y)\,dy \right) dx$$

Teorema: (riduzione)

① Ω verticalmente convesso
$f: \Omega \to \mathbb{R}$ continua (q.o.) in Ω.

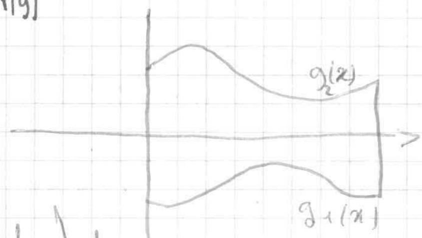
$$\Rightarrow \iint_\Omega f\,dx\,dy = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

② Ω orizzontalmente convesso
$f: \Omega \to \mathbb{R}$ continua (q.o.) in Ω

$$\Rightarrow \iint_\Omega f\,dx\,dy^{**} = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

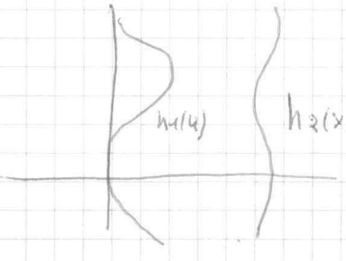
** si può anche ommettere.

Ω verticalmente convessa.



$$\iint_\Omega f = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

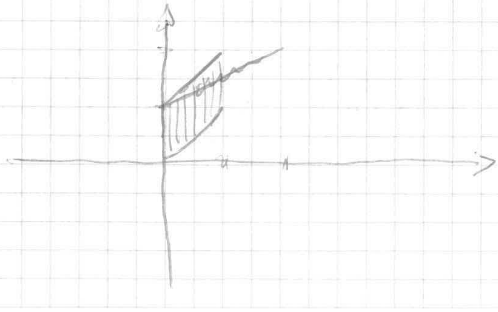
Ω orizz. convessa.



$$\iint_\Omega f = \int_c^d \left(\int_{h_1(x)}^{h_2(x)} f(x,y)\,dx \right) dy$$

Esercizio:

$f(x,y) = xy \qquad \Omega = \left\{ (x,y) \,/\, \begin{array}{l} 0 \le x \le 1 \\ x^2 \le y \le x+1 \end{array} \right\}$

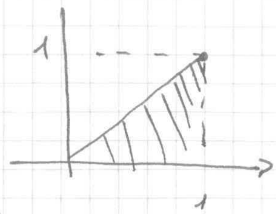


$$\iint_\Omega xy\,dx\,dy$$

$$y^2 dy = \frac{1}{3} dt.$$

$$\int_0^1 \frac{1}{3} \operatorname{sen} t \, dt = -\frac{1}{3} \cos t \Big]_0^1 = -\frac{1}{3}(\cos 1 - 1) = \frac{1}{3}(1 - \underbrace{\cos 1}_{<1}) > 0$$

$$f(x) = \begin{cases} \frac{\operatorname{sen} x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \Big\} \text{cont}$$



$$\Omega = \{(x,y) : 0 \le y \le 1 ; \quad \pi y \le x \le \pi\}$$

$$\int_0^1 \left(\int_{\pi y}^{\pi} \frac{\operatorname{sen} x}{x} dx \right) dy \qquad \int_0^{\pi} \left(\int_0^{1/\pi x} \frac{\operatorname{sen} x}{x} dy \right) dx \qquad \begin{matrix} 0 \le x \le \pi \\ 0 \le y \le \frac{1}{\pi x} \end{matrix}$$

$$\int_0^{1/\pi x} \frac{\operatorname{sen} x}{x} dy = \frac{\operatorname{sen} x}{x} \int_0^{1/\pi x} dy = \frac{\operatorname{sen} x}{x} \cdot \frac{1}{\pi} x = \frac{\operatorname{sen} x}{\pi}$$

$$\int_0^{\pi} \frac{1}{\pi} \operatorname{sen} x \, dx = \frac{1}{\pi} \int_0^{\pi} \operatorname{sen} x \, dx = -\frac{1}{\pi} \cos x \Big]_0^{\pi} =$$

$$\frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

Proprietà degli integrali doppi:

f, g integrali su Ω

$\Omega \subseteq \mathbb{R}^2$ misurabile

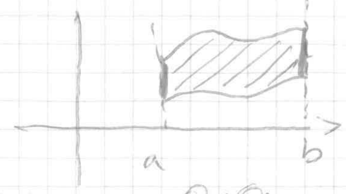
① linearità: $\forall a, b \in \mathbb{R}$

$$\iint_{\Omega} (af + bg) = a \iint_{\Omega} f + b \iint_{\Omega} g$$

(9) $\Omega^\circ \subseteq \Omega' \subseteq \bar{\Omega}$

$$\Rightarrow \iint_{\Omega^\circ} f = \iint_{\Omega'} f = \iint_{\bar{\Omega}} f$$

posso integrare anche su insiemi aperti

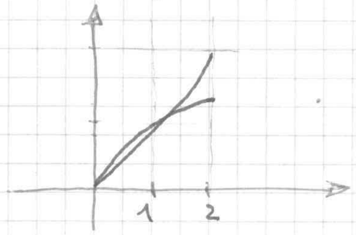


$a < x < b$

$g_1(x) \leq y \leq g_2(x)$

$f(x,y) = x + 2y$

Ω parte di piano compresa tra $y = x^2$ e $y = \sqrt{x}$, con $0 \leq x \leq 2$.



$\Omega_1 = \{ (x,y) ;\ 0 \leq x \leq 1 ;\ x^2 \leq y \leq \sqrt{x} \}$

$\Omega_2 = \{ (x,y) ;\ 1 \leq x \leq 2 ;\ \sqrt{x} \leq y \leq x^2 \}$

$\Omega = \Omega_1 \cup \Omega_2 \qquad \Omega_1 \cup \Omega_2 = \{ (1,1) \}$

$$(*) \quad \iint_{\Omega} f = \iint_{\Omega_1} f + \iint_{\Omega_2} f - \cancel{\iint f} \to 0$$

$\Omega_1 \cap \Omega_2 \to$ mis 0

$$(*) \int_0^1 \left(\int_{x^2}^{\sqrt{x}} (x+2y)\, dy \right) dx + \int_1^2 \left(\int_{\sqrt{x}}^{x^2} (x+2y)\, dy \right) dx =$$

$$\int_{x^2}^{\sqrt{x}} (x+2y)\, dy = xy + \left[\frac{2y^2}{2} \right]_{x^2 = y}^{\sqrt{x}} = x\sqrt{x} + (\sqrt{x})^2 - \left[x\, x^2 + (x^2)^2 \right] = x\sqrt{x} + x - x^3 - x^4$$

$$\int_0^1 \left(x^{3/2} + x - x^3 - x^4 \right) dx - \int_0^2 \left(x^{3/2} + x - x^3 - x^4 \right) dx =$$

[= Se G è una primitiva

$$G(1) - G(0) - (G(2) - G(1)) = 2G(1) - G(0) - G(2)$$

$$G(x) = \frac{2}{5} x^{5/2} + \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^5}{5}$$

$$G(1) = \frac{2}{5} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} =$$

4 10 5 4

$$G(0) = 0$$

$$G(2) = \frac{2}{5} \cdot 2^{5/2} + \frac{4}{2} - \frac{2^4}{4} - \frac{2^5}{5})$$

$$\iint_{D2} f = \int_{-1}^{0} \left(\int_{y+1}^{3y+3} xy \, dx \right) dy \quad (**)$$

$$\int_{y+1}^{3y+3} xy \, dx = y \frac{x^2}{2} \Big]_{x=y+1}^{3y+2} = \frac{1}{2} y \left[(3y+3)^2 - (y+1)^2 \right]$$

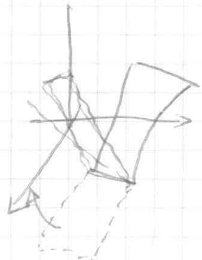
$$(**) \quad \frac{1}{2} \int_{-1}^{0} y \left[(3y+3)^2 - (y+1)^2 \right] dy$$

$$= \frac{1}{2} \int_{-1}^{0} y \left[9(y+1)^2 - (y+1)^2 \right] dy = \cancel{\frac{1}{2}} \int_{-1}^{0} \cancel{8}^{\,4} y \, (y+1)^2 \, dy =$$

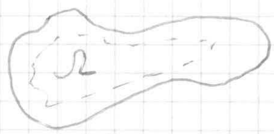
$$4 \int_{-1}^{0} y (y+1)^2 \, dy$$ continua.

① Ω misurabile. $|\Omega| = \iint_\Omega 1 =$ area di Ω

② f integrabile su Ω misurabile



vol $= \iint_\Omega |f|$



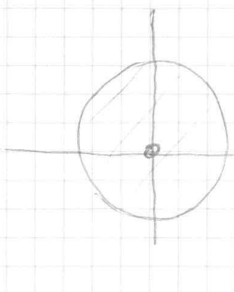
$\rho(x,y)$

Massa della lamina $= \iint_\Omega \rho$

$\rho(x,y) = a \in \mathbb{R}$ $\quad \iint_\Omega a \, dx dy = a \iint 1 \, dx dy$

$$x_G = \frac{1}{M} \iint_\Omega x \, \rho(x,y) \qquad y_G = \frac{1}{M} \iint_\Omega y \, \rho(x,y)$$

(x_G, y_G) = Baricentro

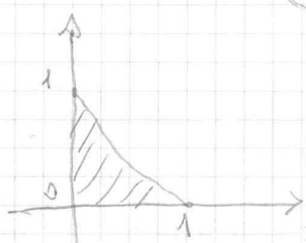


T vertici $(0,0), (1,0), (0,1)$

baricentro di T in due casi

$\rho(x,y) = k$

$\rho(x,y) = 4(x+1)$



area triangolo $= \frac{1}{2}$.

$$4\iint x\,dxdy + \frac{1}{2} = 4\cdot\frac{1}{6} + 4\cdot\frac{1}{2} = \frac{2}{3} + 2 = \frac{2+6}{3} = \frac{8}{3} = M.$$

$$X_G = \frac{1}{M}\iint_T x4(x+1)\,dxdy = ? \quad (1)$$

$$Y_G = \frac{1}{M}\iint_T 4x(x+1)\,dxdy = ? \quad (2)$$

(1) $x_G = \frac{3}{8}\cdot 4\iint_T (x^2+x)\,dxdy = \frac{3}{2}\left[\iint_T x^2 dxdy + \iint_T \underset{1/6}{x\,dxdy}\right]$

...

$y_G = \frac{3}{8}\cdot 4\iint_T y(x+1)\,dxdy =$ 1/6 per simmetria

$= \frac{3}{2}\left[\iint_T yx\,dxdy + \iint_T y\,dxdy\right]$

Cambiamenti di variabili

$\int_a^b f(x)dx$ $\qquad \phi : [c,d] \to [a,b]$ di classe C^1

$y \longmapsto \phi(y)$

$\phi^{-1}(a)\ ,\ \phi^{-1}(b)$

$$\int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(y))\,\phi'(y)dy$$

$x = \phi(y)$

$dx = \phi'(y)\,dy$

$$\parallel$$

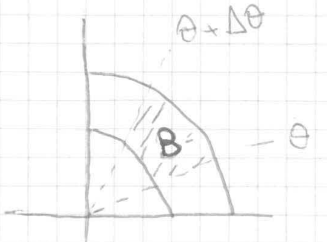
$$\int_c^d f(\phi(y))\,|\phi'(y)|dy$$

R regione: $\mathring{R} \neq \emptyset$, connesso

R'

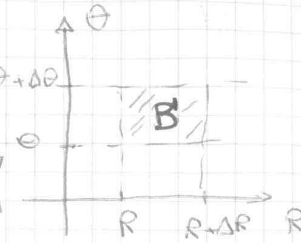
$\phi : R' \longrightarrow R \qquad R, R' \subseteq \mathbb{R}^n$

1. ϕ biettiva
2. ϕ di classe C^1 in un aperto $A \supseteq R'$
3. $J_P\phi$ invertibile $\forall P \in A'$



Area cerchio $= \pi R^2$.

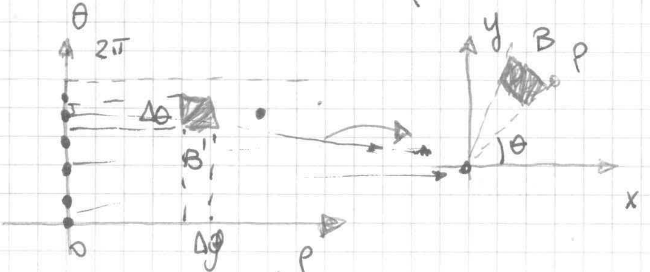
$$\frac{\Delta\theta}{2}\left[(R+\Delta R)^2 - R^2\right]$$
$$\frac{\Delta\theta}{2}\left[R^2 + 2R\Delta R + \underbrace{\Delta R^2}_{\simeq 0} - R^2\right]$$



Area $\simeq R \cdot \Delta R \, d\theta$.

B in coordinate cartesiane
B' in coordinate polari

$$\iint_B dx\,dy = (\text{area } B') \cdot R =$$
$$= \iint$$



det $\phi : [0, +\infty] \times [0, 2\pi) \longrightarrow \mathbb{R}^2$

$$(\rho, \theta) \longrightarrow \begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \end{cases}$$

area B $= \iint_B 1\,dx\,dy \sim$ area B' $= \Delta\theta \cdot \Delta\rho$

$\sim \rho \cdot \Delta\rho \cdot \Delta\theta \longrightarrow \rho\, d\rho\, d\theta$.

$$\iint_B 1\,dx\,dy = \iint_{B'} \rho\, d\rho\, d\theta. \qquad \rho = |\det J\phi|$$

Teorema 8 (cambiamento di variabili negli integrali doppi)

(*) Proprietà: Se $\Omega \subseteq \mathbb{R}^2$, $\Omega' \subseteq \mathbb{R}^2$

$\phi : \Omega' \to \Omega$ è un cambiamento di variabile

$\Rightarrow$ ① $\forall A' \subseteq \Omega'$ misurabile,
$\phi(A')$ è misurabile

② $A \subseteq \Omega$ è misurabile $\Rightarrow \phi^{-1}(A)$ è misurabile.

$\phi : \underset{(u,v)}{\Omega'} \longrightarrow \Omega$ cambiam. variabili

$\Omega \subseteq \mathbb{R}^2$ misurabile ($\Omega' = \phi^{-1}(\Omega)$ è misu.)

f cont. e limitata in $\Omega \Rightarrow \iint_\Omega f^{(x,y)}\,dx\,dy = \iint f(\phi_1(u,v), \phi_2(u,v)) \cdot$

$\cdot \, |\det J\phi(u,v)|\,du\,dv$

$$\begin{cases} x = \phi_1(u,v) \\ y = \phi_2(u,v) \end{cases} \quad (u,v) \in \Omega' \Rightarrow \phi(x,y) \in \Omega$$

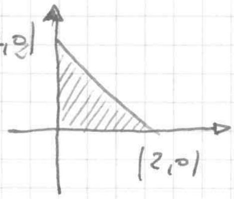
$$\iint_D \frac{xy^2}{x^2+y^2}\,dx\,dy = \iint \frac{\rho\cos\theta\,\cancel{\rho^2}\,\text{sen}^2\theta}{\cancel{\rho^2}}\,\rho\,d\rho\,d\theta$$

$$\iint_{D'} (\rho^3)(\text{sen}^2\theta\,\cos\theta)\,d\rho\,d\theta = \left(\int_1^2 \rho^2\,d\rho\right)\cdot\left(\int_0^{\pi/2} \text{sen}^2\theta\,\underbrace{\cos\theta\,d\theta}\right) =$$

$$\left[\frac{\rho^3}{3}\right]_1^2 \left[\frac{\text{sen}^3\theta}{3}\right]_0^{\pi/2} = \left(\frac{8}{3}-\frac{1}{3}\right)\cdot\left(\frac{1}{3}-0\right) = \frac{7}{3}\cdot\frac{1}{3} = \frac{7}{9}$$

$$f(x,y) = (x^2-y^2)\log\left(1+(x+y)^4\right)$$

$$\Omega = \{(x,y)\,/\,x>0,\ 0<y<2-x\}$$



$$f(x,y) = (x-y)(x+y)\log\left(1+(x+y)^4\right)$$

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \qquad \begin{cases} x = \dfrac{u+v}{2} = \dfrac{1}{2}(u+v) \\ y = \dfrac{u-v}{2} = \dfrac{1}{2}(u-v) \end{cases}$$ Applicazione lineare

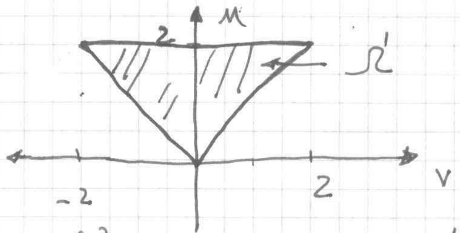
$$f = v\cdot u\cdot\log(1+u^4)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{A}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} \qquad \det A \neq 0$$

$$\det A = -\frac{1}{2} \qquad |\det A| = \frac{1}{2} \neq 0$$

Cambio di coordinate $(u,v) \Leftarrow (x,y)$

$0<u<2 \quad -u<v<u.$



$$\iint_{\Omega'} v\,u\log(1+u^4)\,\frac{1}{2}\,du\,dv.$$

Integro rispetto a v.

$$* \quad \left[\frac{v^2}{2}\right]_{-u}^{u} = 0$$

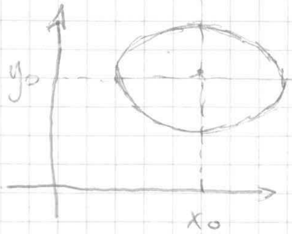
$$\int_0^2 \left(\frac{1}{2}u\log(1+u^4)\int_{-u}^{u*} v\,dv\right)du$$

$$= \int_0^2 u\log(1+u^4)\cdot 0\,du = 0$$

* * * *

$$\iint_D e^{-(x^2+y^2)}\,dx\,dy \qquad \Omega = \{(\rho,\theta)\ :\ \rho<R,\ 0\le\theta\le 2\pi\}$$

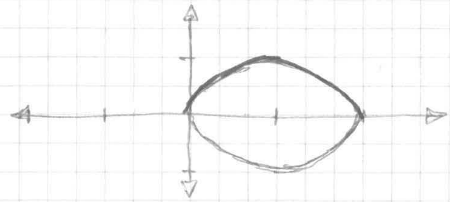
det $J\phi(\rho,\phi) = ab\rho\cos^2\theta + ab\rho\,\text{sen}^2\theta = ab\rho \geq 0$ ok.



$$\iint_B y^2\,dx\,dy$$

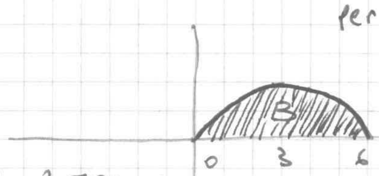
$$B = \{(x,y) /\ 4(x-3)^2 + 9y^2 \leq 36,\ y \geq 0\}$$

$$\frac{(x-3)^2}{3^2} + \frac{y^2}{2^2} \leq 1 \Longrightarrow$$



$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

(x_0,y_0) = centro.



per $y \geq 0$ prendo solo la parte superiore.

$$\begin{cases} x = x_0 + \rho a\cos\theta \\ y = y_0 + \rho b\,\text{sen}\theta \end{cases} \Rightarrow \begin{cases} x = 3 + 3\rho\cos\theta \\ y = 2\rho\,\text{sen}\theta \end{cases} \Rightarrow \left.\begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq \pi \end{cases}\right\} = B'$$

$$\iint_B y^2\,dx\,dy = \iint_{B'} 4\rho^2\,\text{sen}^2\theta\ (3\rho\cdot 2)\,d\rho\,d\theta$$

$$= 24 \iint_{B'} \rho^3\,\text{sen}^2\theta\,d\rho\,d\theta \quad \text{(dominio rettangolare)}$$

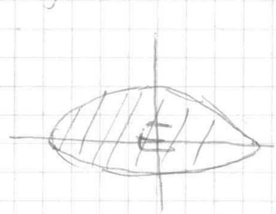
$$24\left(\int_0^1 \rho^3\,d\rho\right)\cdot\left(\int_0^\pi \text{sen}^2\theta\,d\theta\right) \ (*)$$

$$\int \text{sen}^2\theta\,d\theta = \frac{\theta - \text{sen}\theta\cos\theta}{2}\ ;\quad \int \cos^2\theta = \frac{\theta + \text{sen}\theta\cos\theta}{2}$$

$$(*) = 24\left.\frac{\rho^4}{4}\right]_0^1 \cdot \left.\frac{\theta - \text{sen}\theta\cos\theta}{2}\right]_0^\pi = \frac{3}{2}\cdot\frac{1}{4}\cdot\frac{\pi}{2} = 3\pi.$$

Calcolare l'area dell'ellisse di semiassi a, b.

$$\iint_E 1\,dx\,dy \Rightarrow \iint_{B'} ab\rho\,d\rho\,d\theta.$$



$$\frac{(x-a)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1.$$

$$\begin{cases} x = a\rho\cos\theta \\ y = b\rho\,\text{sen}\theta \end{cases} \qquad \left.\begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}\right\} E'$$

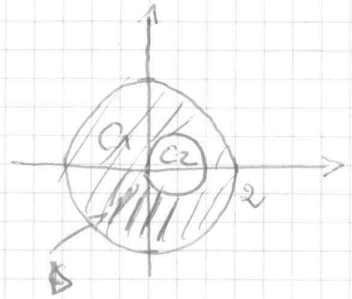
$$a\cdot b\left(\int_0^1 \rho\,d\rho\right)\cdot\left(\int_0^{2\pi} 1\,d\theta\right) \Rightarrow ab\left.\frac{\rho^2}{2}\right]_0^1 2\pi =$$

$$\frac{2^3 \cdot 3^2}{2} \cos^3\theta \Rightarrow \int_0^{\pi/2} 72 \cos^3\theta \, d\theta$$

$$72 \int_0^{\pi/2} \cos^2\theta \cdot \cos\theta \, d\theta = 72 \int_0^{\pi/2} (1 - \operatorname{sen}^2\theta) \cdot \cos\theta \, d\theta$$

$$72 \int_0^{\pi/2} (\cos\theta - \operatorname{sen}^2\theta \cos\theta) \, d\theta =$$

$$72 \left[\operatorname{sen}\theta - \frac{\operatorname{sen}^2\theta}{2} \right]_0^{\pi/2} = 72 \left[1 - \frac{1}{2} \right] = 72 \cdot \frac{1}{2} = 36$$



$$\begin{cases} x^2+y^2 \leq 4 \\ x^2+y^2-x \geq 0 \end{cases}$$

$$f(x,y) = (x^2+y^2)^{3/2}$$

$D = C_1 \setminus C_2$.

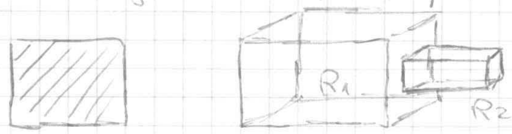
$C_1 = D \cup C_2 \qquad |D \cap C_2| = 0$

$$\iint_{D \cup C_2} f = \iint_D f + \iint_{C_2} f.$$

$$\iint_{D \cup C_2} f - \iint_{C_2} f = \iint_D f = \boxed{\iint_{C_1} f - \iint_{C_2} f = \iint_D f} \text{ ok}$$

Integrali tripli:

Cioè integrali di $f(x,y,z)$ su un dominio in $\mathbb{R}^3$.



$$\int_R f = \sum_{i=1}^{n} C_i \text{ volume}(R_i) \qquad f(x) = C_i$$

$\forall x \in R_i, \ R = R_1 \cup \dots \cup R_n$.

f è limitata su R

$$\underline{\int_R} f = \sup \left\{ \int_R g, \ g \text{ a scala}, \ g \leq f \right\}$$

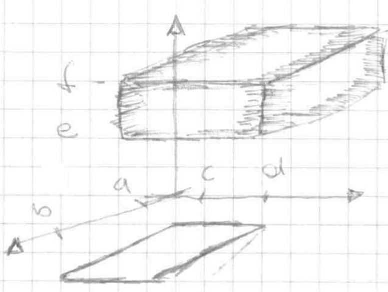
$$\overline{\int_R} f = \inf \left\{ \int_R h, \ h \text{ a scala}, \ h \geq f \right\}$$

sup e inf esistono perché la funzione g, h sono a scala limitate.

$$\underline{\int_R} f \leq \overline{\int_R} f$$

Se $\underline{\int_R} f = \overline{\int_R} f = \int f$ si dice Riemann-integrabile

f continua su un parallelepipedo $[a,b] \times [c,d] \times [e,f]$

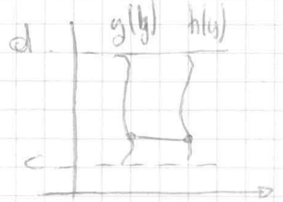


$$\iiint_B f(x,y,z) = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z) \, dx \right) dy \right) dz$$

Esercizio:

$$f(x,y,z) = 2xy^2 - yz^2$$

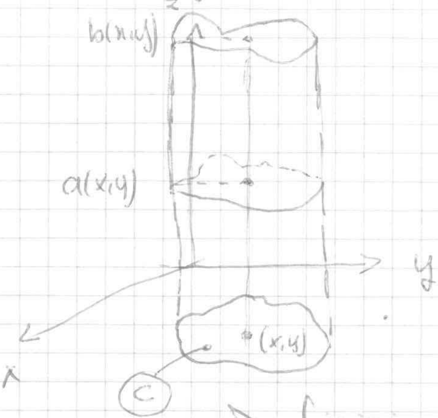
$$B = [-1,0] \times [0,1] \times [0,2]$$



$$\int_c^d \left(\int_{h(y)}^{g(y)} f(x,y)\,dy \right) dx$$

Insieme orizzontalmente connesso.

_ Definizione per fili (seguenti)



$a(x,y) \le z \le b(x,y)$

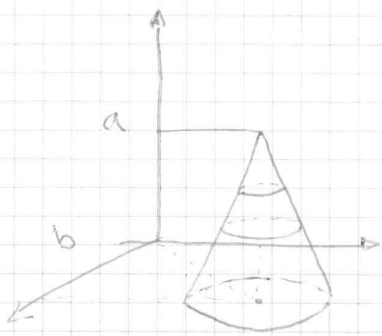
$(x,y) \in C \subseteq \mathbb{R}^2$ misurabile in $\mathbb{R}^2$.

1) C misurabile

2) $a(x,y)$ e $b(x,y)$ continue...

$\Rightarrow$ Se ① e ② sono soddisfatte l'insieme è misurabile

_ Definizione per strati:



$a \le z \le b \qquad \forall z_0 \in [a,b]$

$\{z = z_0\} \cap \Omega = \Omega_{z_0}$ misurabile.

* Esempio di integrazione per fili:

$$\Omega = \left\{ \begin{array}{l} (x,y) \in C \subseteq \mathbb{R}^2 \text{ misurabile} \\ a(x,y) \le z \le b(x,y) \\ a \text{ e } b \text{ continua in } C \end{array} \right\}$$ f continua in Ω per fili

$$\iiint_\Omega f = \iint_C \left(\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz \right) dx\,dy$$ (fili paralleli rispetto all'asse z)

* Integrazione per strati:

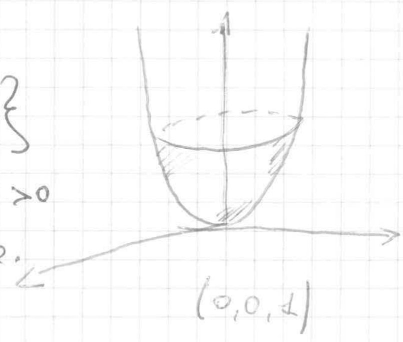
f continua in Ω per strati.

$$\iiint_\Omega f = \int_a^b \left(\iint_{\Omega_z} f(x,y,z)\,dx\,dy \right) dz$$

$\Omega : \{ (x,y,z) : 0 \le z \le 2, \ x^2+y^2 \le z \}$

$x^2+y^2-z=0 \qquad$ se $f(x,y,z): x^2+y^2-z>0$

fuori dalla paraboloide.



z è vincolato $\in [0,2]$

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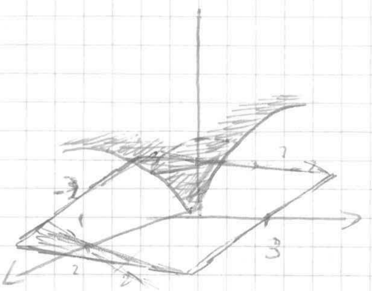
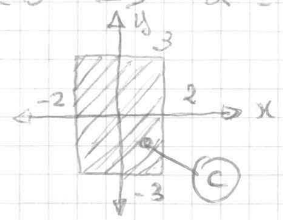
$f(x,y,z) = z \qquad \Omega = \{ |x| \leq 2,\ |y| \leq 3,\ 0 \leq z \leq \sqrt{x^2+y^2} \}$ (*)

(*) Superficie di rotazione.

$y = 0 \Rightarrow z = \sqrt{x^2} = x$ per $x > 0$





$-3 \leq y \leq 3$
$-2 \leq x \leq +2$

$$\iiint_\Omega z = \iint_C \left(\int_0^{\sqrt{x^2+y^2}} z\,dz \right) dx\,dy = \iint_C \left(\frac{1}{2} z^2 \right]_0^{\sqrt{x^2+y^2}}) dx\,dy =$$

$$\int_{-3}^{3} \int_{-2}^{2} \frac{1}{2}(x^2+y^2)\,dx\,dy \Rightarrow \frac{1}{2}\left[\frac{x^3}{3} + xy^2 \right]_{x=-2}^{2} = \frac{1}{2}\left[\frac{8}{3} + 2y^2 - \left(-\frac{8}{3} - 2y^2\right)\right]$$

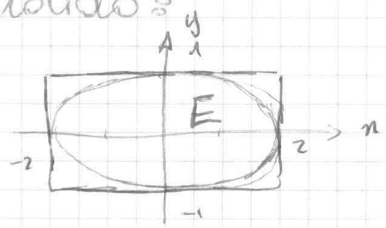
$$= \frac{8}{3} + 2y^2 \Rightarrow \int_{-3}^{3} \left(\frac{8}{3} + 2y^2 \right) dy = \frac{8}{3}y + \frac{2}{3}y^3 \Big]_{-3}^{3} =$$

$$\frac{8}{3}\cdot 3 + \frac{54}{3} - \frac{8}{3}(-3) + \frac{54}{3} = 16 + \frac{2}{3}54 = 16 + \frac{4\cdot 3^{\cancel{3}}}{\cancel{3}} = 16+36$$

$$= 52.$$

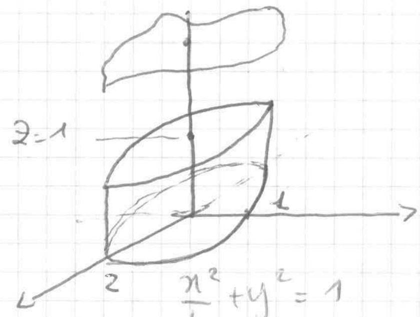
Calcolare il volume del solido:

$$\begin{cases} \frac{x^2}{4} + y^2 = 1 \\ z = 1 \\ z = 12 - xy \end{cases}$$



$(x,y) \in [-2,+2], [-1,1] = E$

$$\begin{cases} z = 1 \\ z = 12 - xy \end{cases} \Rightarrow z = 12 - xy \geq 1. \quad \forall (x,y) \in \text{Ellisse}$$



$\frac{x^2}{4} + y^2 = 1$

$-1 \leq x \leq 1$

$-2 \leq y \leq 2$

$-2 \leq xy \leq 2.$

$2 \leq -xy \leq 2$

$10 \leq 12 - xy \leq 14$

$\leftarrow \mathrm{Vol}(\Omega) = |\Omega| = \iiint_\Omega 1$

$\Omega:\ 1 \leq z \leq 12 - xy$

$(x,y) \in [-2,2] \times [-1,1]$

$\in$ Ellisse.

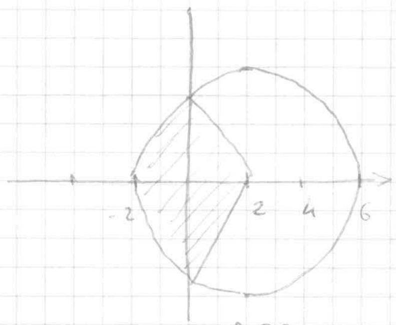
$1 \leq 10 \leq 12 - xy \leq 14$

$$|\Omega| = \iint_E \left(\int_1^{12-xy} 1\,dz \right) dx\,dy = \iint_E (11 - xy)\,dx\,dy =$$

$$11 \iint_E 1\,dx\,dy - \iint_E xy\,dx\,dy \Rightarrow 11 \iint_E 1\,dx\,dy = 11 \cdot 2 \cdot 1 \cdot \pi$$

$\underset{\text{area}(E)}{\shortparallel}$

$$= 22\pi = 11|E|$$



Calcolare: $\iiint_D x\,dx\,dy\,dz = 4\pi$. $\quad D = A \setminus B$ (A eccetto B)

$A = \{ y^2+z^2 \leq 4,\ x \geq 0 \}$

$B = \{ x \geq \sqrt{y^2+z^2} \}$

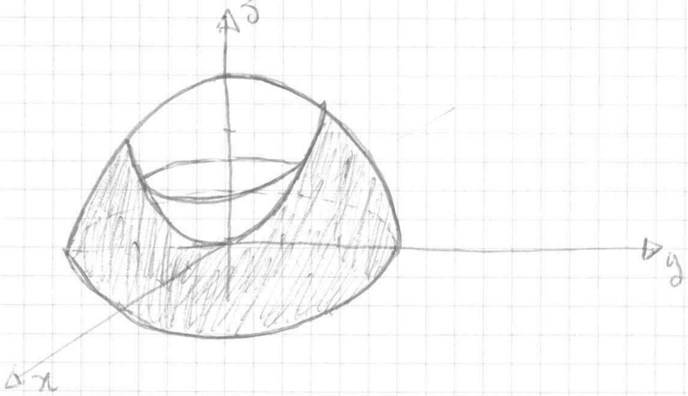
$\iiint_\Omega (1-z^2)\,dx\,dy\,dz$

$$\Omega = \left\{ \begin{array}{l} z \geq 0 \\ x^2+y^2+z^2 \leq 1 \\ z \leq x^2+y^2 \end{array} \right\} \quad \begin{array}{l} \text{semipiano superiore} \\ \text{punti intermedi della sfera} \\ \text{punti sotto la paraboloide} \end{array}$$

- **Per fili:** È complesso.

$0 \leq z \leq x^2+y^2$, $\quad (x,y) \in C\bar{z}$

$0 \leq z \leq 1-(x^2+y^2)$, $\quad (x,y) \in$ corona circolare.



Per strati: $\qquad z \leq x^2+y^2 \leq 1-z^2$

$$\begin{cases} z = (x^2+y^2) \\ 1 = (x^2+y^2)+z^2 \end{cases}$$

$1-z = z^2 \implies z^2+z-1 = 0 \wedge z \geq 0 \qquad \bar{z} = \dfrac{-1 \pm \sqrt{5}}{2} \begin{array}{l} \nearrow \dfrac{-1-\sqrt{5}}{2} < 0 \text{ NO} \\ \searrow \dfrac{-1+\sqrt{5}}{2} > 0 \text{ OK} \end{array}$

$z \leq x^2+y^2 \leq 1-z^2$

$0 \leq z \leq \dfrac{-1+\sqrt{5}}{2}$

Integrali tripli con cambiamento di variabili

$\phi : A' \to A$

- ϕ biunivoca
- ϕ di classe C^1
- $\det(J_P \phi) \neq 0 \quad \forall P \in A'$

} cambiamento di variabili

A' è misurabile $\Rightarrow$ A è mis.

A è misurabile $\Rightarrow$ $\phi^{-1}(A) = A'$ è mis

A misurabile in $\mathbb{R}^3$.

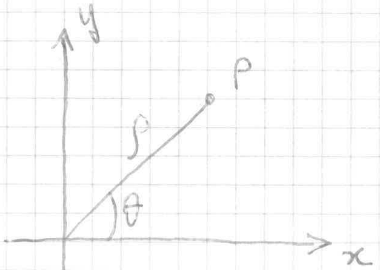
f integrabile su A.

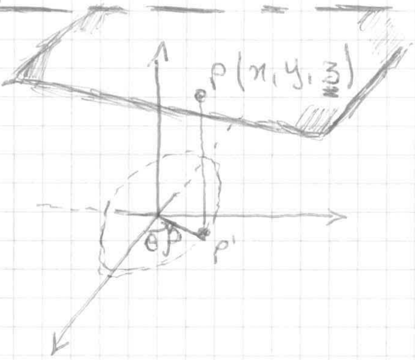
$\phi : A' \to A$ camb. di variabili

$\Rightarrow \iiint_A f(x,y,z)\, dx\,dy\,dz = \iiint_{A'} f(\phi(u,v,w)) |\det J\phi(u,v,w)| \cdot$

$du\,dv\,dw \Rightarrow \iiint_{A'} f(\phi(u,v,w)) |\det J\phi(u,v,w)|\, du\,dv\,dw.$

Coordinate cilindriche





(*)
$$\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = t \end{cases}$$

$\rho \geq 0 \qquad 0 \leq \theta \leq 2\pi$

$t \in \mathbb{R}$

(*) = Coordinate cilindriche

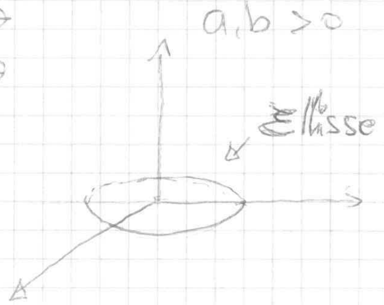
$$J\phi = \begin{pmatrix} \cos\theta & -\rho\operatorname{sen}\theta & 0 \\ \operatorname{sen}\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{dx}{d\rho} & \frac{dx}{d\theta} & \frac{dx}{dt} \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$\det J\phi = 1 \cdot (\rho\cos^2\theta + \rho\operatorname{sen}^2\theta) = \rho \qquad$ per $\rho = 0$ non va bene...

Coordinate ellittiche:

$$\begin{cases} x = a\rho\cos\theta \\ y = b\rho\operatorname{sen}\theta \\ z = t \end{cases}$$

$a, b > 0 \qquad \theta \in [0, 2\pi) \qquad t \in \mathbb{R}$



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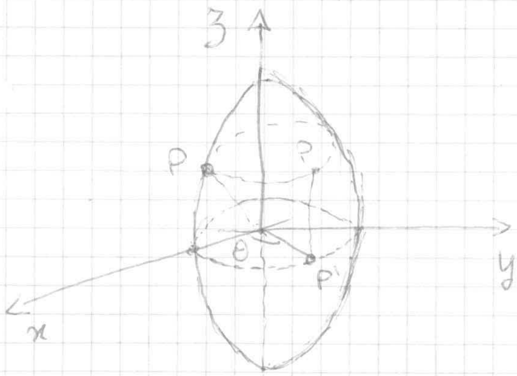
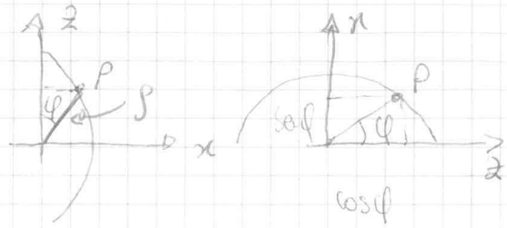
Coordinate sferiche:

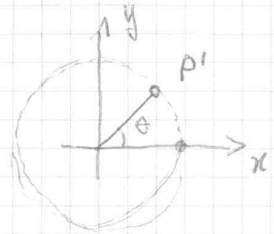




$x = \operatorname{sen}\varphi$

$$\begin{cases} x = (\rho \operatorname{sen}\varphi)\cos\theta \\ y = (\rho \operatorname{sen}\varphi)\operatorname{sen}\theta \\ z = \rho\cos\varphi \end{cases}$$

rotazione intorno a "z"



$y = \operatorname{sen}\theta$

$x = \cos\theta$

$$\rho = d(P, O) = \sqrt{x^2+y^2+z^2}$$

$$\begin{cases} x = \rho \operatorname{sen}\varphi\cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho\cos\varphi \end{cases} \qquad \begin{cases} 0 \le \varphi \le \pi \\ \rho = \sqrt{x^2+y^2+z^2} \ge 0 \\ 0 \le \theta \le 2\pi \end{cases}$$

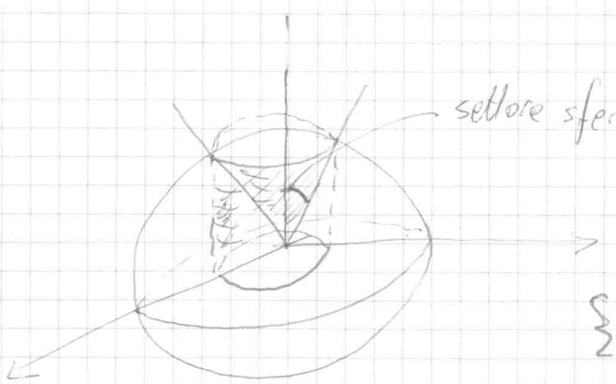
$|\det J\Phi| = |\rho^2 \operatorname{sen}\varphi| = \rho^2 \operatorname{sen}\varphi$ | sempre maggiore di zero perché $\rho \ge 0$, $\operatorname{sen}\varphi \ge 0$ per $0 \le \varphi \le \pi$

$|\det J\Phi| \ne 0 \iff \rho > 0 \wedge \operatorname{sen}\varphi > 0 \Rightarrow \rho > 0 \wedge 0 < \varphi < \pi$.

$$\iiint_\Omega z \, dx\, dy\, dz \qquad \begin{cases} x^2+y^2+z^2 \le 1 \\ z > \sqrt{x^2+y^2} \end{cases}$$

$z = \sqrt{x^2+y^2}$ caso positivo

$z = x > 0$



Soluzione (coord. Sferiche)

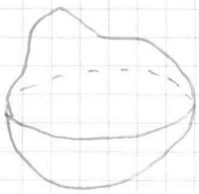
$$\begin{cases} x^2+y^2+z^2 = 1 \\ z = \sqrt{x^2+y^2} \end{cases} \Rightarrow$$

$$\begin{cases} x^2+y^2+z^2 = 1 \\ x^2+y^2 = z^2 \end{cases}$$

$z > 0 \Rightarrow z = 1 - z^2 \Rightarrow$

$2z^2 - 1 = 0 \quad z^2 = 1/2$

$\to z = \pm \frac{1}{\sqrt{2}} = + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$



$$x_G = \frac{1}{M} \iiint_\Omega x\rho(x,y,z)\,dx\,dy\,dz$$

$$y_G = \frac{1}{M} \iiint_\Omega y\,\rho(x,y,z)\,dx\,dy\,dz$$

$$z_G = \frac{1}{M} \iiint_\Omega z\,\rho(x,y,z)\,dx\,dy\,dz$$

$\rho = K$

$$x_G = \frac{1}{\cancel{K}\cdot|\Omega|} \cdot \iiint_\Omega x \cdot K\,dx\,dy\,dz =$$

Volume di un solido che in coordinate cilindriche

$$\begin{cases} x = R\cos\theta \\ y = R\,\text{sen}\theta \\ z = t \end{cases} \qquad \sqrt{1-t^2} \le R \le \sqrt{4-t^2}$$

$R^2 \le t^2$

$0 \le \theta \le 2\pi$

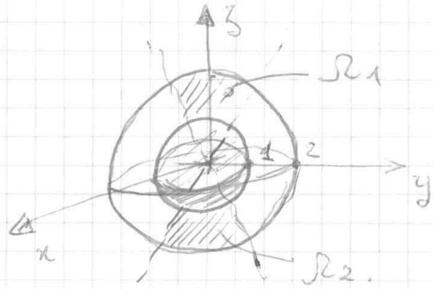
$R^2 = x^2+y^2 = R^2\cos^2\theta + R^2\,\text{sen}^2\theta = R^2$. ok.

$1-t^2 \le R^2 \le 4-t^2$

$1-z^2 \le x^2+y^2 \le 4-z^2$

$1 \le x^2+y^2+z^2 \le 4$.

$x^2+y^2 \le z^2$



Coord. Sferiche:

$$\begin{cases} x = \rho\,\text{sen}\varphi\cos\theta \\ y = \rho\,\text{sen}\varphi\,\text{sen}\theta \\ z = \rho\cos\varphi \end{cases} \qquad \rho = \sqrt{x^2+y^2+z^2}$$

$z > 0$

$$\text{Vol}(\Omega) = 2\,\text{vol}(\Omega_1) \qquad \Omega_1' \begin{cases} 1 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi/4 \end{cases}$$

$$\iiint_{\Omega_1} 1\,dx\,dy\,dz = \iiint_{\Omega_1'} \rho^2\,\text{sen}\varphi\,d\rho\,d\varphi\,d\theta =$$

$$\left(\int_1^2 \rho^2 d\rho\right)\cdot\left(\int_0^{\pi/4} \text{sen}\varphi\,d\varphi\right)\left(\int_0^{2\pi} 1\,d\theta\right) = 2\pi\left[\frac{\rho^3}{3}\right]_1^2\left[-\cos\varphi\right]_0^{\pi/4} =$$

$$2\pi\left[\frac{8}{3}-\frac{1}{3}\right]\left[1-\frac{\sqrt{2}}{2}\right] = 2\pi\cdot\frac{7}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

$$2\pi \iint_{S'} \rho \, d\rho \, dt = 2\pi \iint_{S'} x \, dx \, dz$$

Solido di rotazione Ω ottenuto ruotando $S' \subseteq$ piano (x,z) $(x \geq 0)$ intorno all'asse z ha volume

$$2\pi \iint_{S'} x \, dx \, dz = 2\pi \iint_{S'} y \, dy \, dz$$

$S' \subseteq (y,z)$ rotazione intorno a z

Baricentro di S' $\rho(x,z) = 1$

$$x_G = \frac{1}{|S'|} \cdot \iint_{S'} x \, dx \, dz$$

$$y_G = \frac{1}{|S'|} \cdot \iint_{S'} z \, dx \, dz$$

lunghezza di una circonferenza

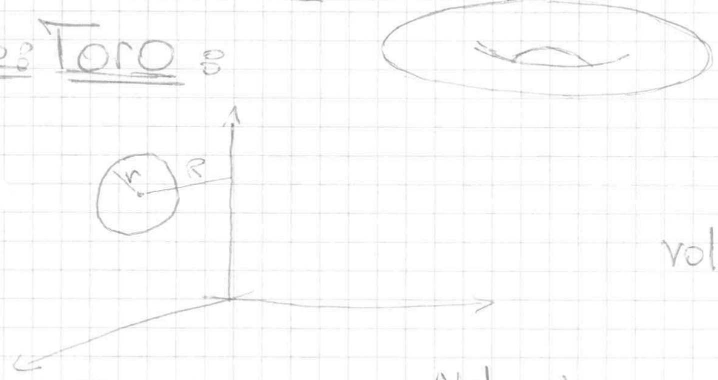
$$Vol(\Omega) = 2\pi \cdot \iint_{S'} x \, dx \, dz = 2\pi \cdot |S| \cdot x_G = 2\pi x_G \cdot |S|$$

$2\pi x_G$ = Circonferenza descritta dal baricentro G nella rota. intorno all'asse z...

x_G tiene conto del fattore distorsivo (superficie non regolare).

Teorema di Guldino

Es: Toro:



$r < R$

$x_G = R$

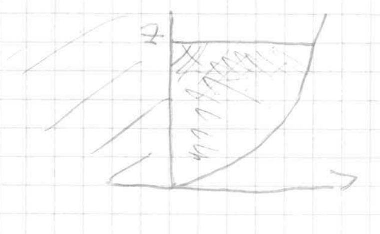
$|S| = \pi r^2$

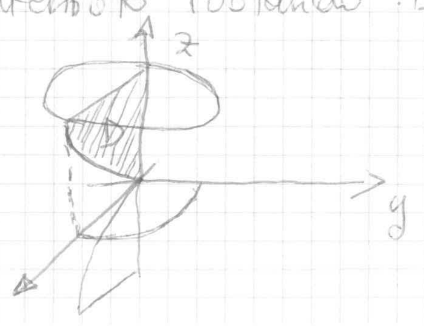
$$vol(Toro) = 2\pi x_G \cdot |S| = 2\pi R \pi r^2 = 2\pi^2 r^2 R$$

Esercizio: Nel piano $y = 0$ è data la superficie

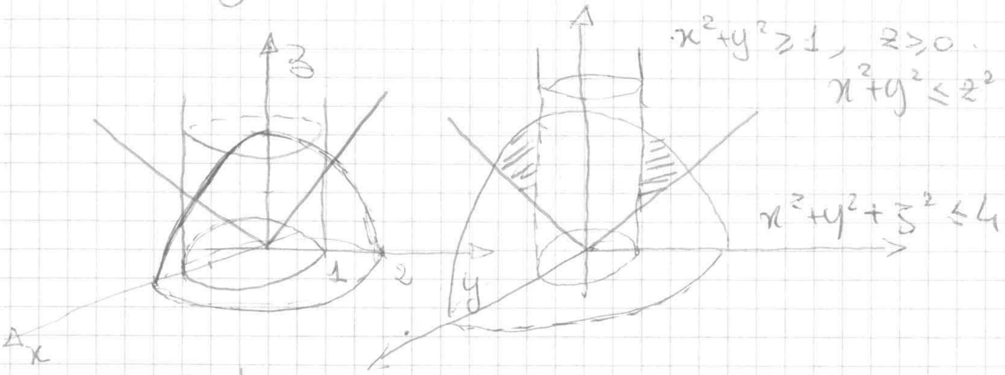
$$D = \{(x,y,z) : y = 0, \ z \geq 2x^2, \ x \geq 0, \ z \leq h\}$$

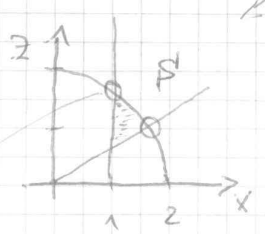
Determinare h in modo che il volume del solido D_z ottenuto ruotando D intorno all'asse z sia uguale al volume del solido D_x ottenuto ruotando D intorno all'asse x.





$$\begin{cases} x^2+y^2 \geq 1 \\ x^2+y^2 \leq z^2 \\ x^2+y^2+z^2 \leq 4 \\ z \geq 0 \end{cases} \qquad x^2+y^2 \leq$$





$$\text{vol}(\Omega) = |\Omega| = 2\pi \iint_{S'} x \, dx \, dz$$

$$y = 0$$

Parametrizzo:

$$y=0 \longrightarrow \begin{cases} x^2 = 1 \Rightarrow x = 1 & x > 0 \\ x^2 \leq z^2 \Rightarrow x < z \\ x^2+z^2 \leq 4 \Rightarrow \end{cases}$$

Intersezione

Se $x = 1 \longrightarrow 1+z^2 = 4 \Rightarrow z^2 = 3 \Rightarrow z = \sqrt{3}$

$x = z, \rightarrow x^2+z^2 = 4 \Rightarrow 2x^2 = 4 \Rightarrow x = \sqrt{2}$.

$$\boxed{\begin{array}{l} 1 \leq x \leq \sqrt{2} \\ x \leq z \leq \sqrt{4-x^2} \end{array}}$$

$$|\Omega| = 2\pi \int_1^{\sqrt{2}} \left(\int_x^{\sqrt{4-x^2}} x \, dz \right) dx = 2\pi \int_1^{\sqrt{2}} x\left(\sqrt{4-x^2}-x\right) dx$$

$$= 2\pi \int_1^{\sqrt{2}} \left[x\sqrt{4-x^2} - x^2 \right] dx$$

$$\int x\sqrt{4-x^2}\, dx = \qquad 4-x^2 = t \qquad dt = -2x\,dx$$

$$1 \leq x \leq \sqrt{2} \Rightarrow 3 \leq t \leq 2 \qquad x\,dx = -\frac{1}{2}\,dt$$

$$-\frac{1}{2}\int_3^2 \sqrt{t}\, dt = -\frac{1}{2}\,\frac{2}{3}\, t^{3/2} = -\frac{1}{3}\, t^{3/2} \Big]_3^2 = \frac{1}{3}\, t^{3/2} \Big]_2^3$$

$$= \frac{2\sqrt{2}}{3} - \frac{1}{3}$$

- Se $\alpha'(s) > 0 \Rightarrow \alpha$ è ~~iniettiva~~ strettamente crescente..
  $\Rightarrow \alpha(c) = a, \quad \alpha(d) = b$
- Se $\alpha'(s) < 0 \Rightarrow \alpha$ è strettamente decrescente
  $\alpha(c) = b, \quad \alpha(d) = a$

$$\begin{tikzcd} t = \alpha(s)\ [a,b] \arrow[r, "\gamma"] & \mathbb{R}^n \\ s\ [c,d] \arrow[u, "\alpha"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

γ regolare...

$\delta(s) = \gamma(\alpha(s)) = \gamma(t)$

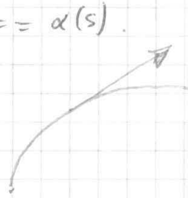
~~Se~~ $\delta(s)$ e $\gamma(t)$ differiscono per un cambiamento di paramet.
se $\exists\, \alpha$ camb. di param. t.c. $\delta(s) = \gamma(\alpha(s)) \quad \forall s \in [c,d]$

1. Sostegno è lo stesso.
2. δ e γ sono di classe C^1.
3. $\delta(s)$ è iniettiva $\Leftrightarrow \gamma$ è iniettiva.
4. δ è chiusa $\Leftrightarrow \gamma$ è chiusa.

$\delta(s) = \gamma(\alpha(s))$

$\delta'(s) = (\delta_1'(s), \dots, \delta_n'(s))$ vett tg a δ

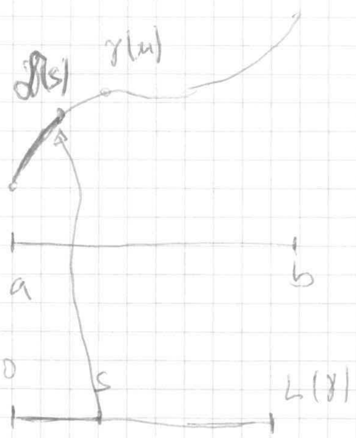
$$\frac{d\gamma}{ds}(\delta(s)) = \frac{d}{ds}\big(\gamma_1(\alpha(s)), \dots, \gamma_n(\alpha(s))\big)$$
$$= \big(\gamma_1'(\alpha(s))\,\alpha'(s), \dots, \gamma_n'(\alpha(s))\,\alpha'(s)\big)$$
$$= \alpha'(s)\,.\,\underbrace{\big(\gamma_1'(\alpha(s)), \dots, \gamma_n'(\alpha(s))\big)}_{\text{vett. tg a } \gamma \text{ nel punto } t = \alpha(s)} = \delta'(s)$$
$$=$$



OSS: (5) γ reg $\Leftrightarrow \delta$ è regolare...

direzione: $\delta'(s) \parallel \gamma'(\alpha(s))$
verso dipende dal segno di $\alpha'(s)$ (scalare)
se $\alpha'(s) > 0 \Rightarrow \delta'(s)$ e $\gamma'(\alpha(s))$ hanno stesso verso $\forall s$
se $\alpha'(s) < 0 \Rightarrow \delta'(s)$ e $\gamma'(\alpha(s))$ hanno verso opposto $\forall s$.

δ e γ sono equivalenti se $\exists\, \alpha : [c,d] \to [a,b]$



$\gamma : [a,b] \longrightarrow \mathbb{R}^n$

$u \longrightarrow \gamma(u)$

$s = L(\gamma(u)) = \int_a^u \|\gamma'(t)\| \, dt$

$s = 0$ se $u = a$. $L(\gamma(a)) = 0$

$s = L(\gamma)$. $u = b$ $L(\gamma(b)) = L(\gamma)$

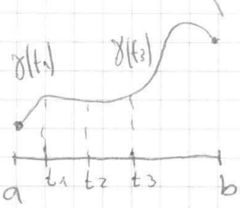
$L(\delta(s)) = s$

$\gamma : [a,b] \to \mathbb{R}^n$ arco di curva regolare $(n \geq 2)$

$f : \mathbb{R}^n \to \mathbb{R}$ funz. continua...

$\gamma([a,b]) \subseteq \operatorname{dom} f$

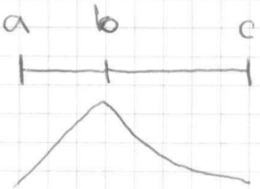
Integrale curvilineo di f lungo γ



$f(\gamma(t))$

$$\int_\gamma f \, ds = \int_a^b f(\gamma(t)) \cdot |\gamma'(t)| \, dt$$

$$\left.\begin{array}{l} f, g \text{ continue} \Rightarrow \int_\gamma (f+g)\, ds = \int_\gamma f + \int_\gamma g \\ \forall \beta \in \mathbb{R}, \ \int_\gamma (\beta f)\, ds = \beta \int_\gamma f \, ds \end{array}\right\} \text{linearità}$$



$\gamma = \gamma_1 + \gamma_2$

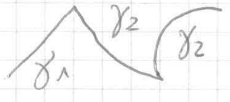
$\gamma_1 : [a,b] \to \mathbb{R}^n$

$\gamma_2 : [b,d] \to \mathbb{R}^n$

$\gamma_1(b) = \gamma_2(c)$ $\qquad \gamma : [\alpha, \beta] \to \mathbb{R}^n$

$$\int_{\gamma = \gamma_1 + \gamma_2} f \, ds = \int_{\gamma_1} f \, ds + \int_{\gamma_2} f \, ds$$

Curva a tratti (se ha punti di cuspide) $\begin{cases} \to \text{vettore tg} = \vec{0} \\ \to \text{vettore tg} \nexists \end{cases}$



$\gamma = \gamma_1 + \gamma_2 + \gamma_3$

$$\int_\gamma f \, ds = \int_{\gamma_1} f \, ds + \int_{\gamma_2} f \, ds + \int_{\gamma_3} f \, ds$$

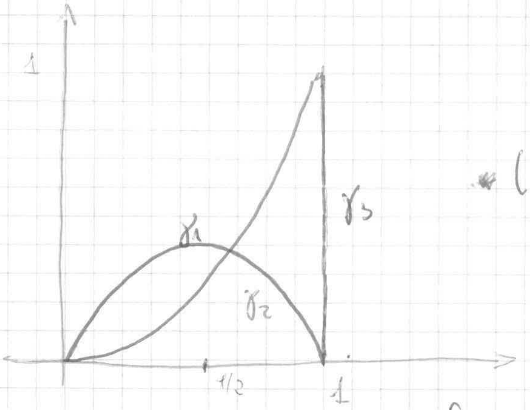
$$\left[2\left(4\cos^4 t + 4\cos^2 t + 1\right)\right]^{1/2} = \sqrt{2}\left(2\cos^2 t + 1\right)^{2/2}$$

$$L(\gamma) = \int_0^{\pi/2} \sqrt{2}\left(\cos^2 t + 1\right) dt = \sqrt{2}\left[2\,\frac{t + \operatorname{sen} t \cos t}{2} + t\right]_0^{\pi/2}$$

$$= \sqrt{2}\; 2\,\frac{\pi}{2} = \sqrt{2}\,\pi.$$

Esercizio $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ nel piano..

$$\begin{cases} \gamma_1 : y \geq 0 & y^2 + x^2 - x = 0 \quad \text{da } (1,0) \xrightarrow{a} (0,0) \\ \gamma_2 : y = x^2 & 0 \leq x \leq 1 \\ \gamma_3 : x = 1 & 0 \leq y \leq 1 \end{cases}$$



$$y^2 + x^2 - 2\,\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} = 0.$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

$$L(\gamma) = \underbrace{\int_{\gamma_1} 1\,ds}_{\pi\left(\frac{1}{2}\right) \text{ raggio}} + \int_{\gamma_2} ds + \underbrace{\int_{\gamma_3} ds}_{=1} =$$

parametrizzo la seconda curva:

$$\gamma_2 \Rightarrow \gamma_2 : \begin{cases} x = t \\ y = t^2 \end{cases} \qquad \gamma_2'(t) = (1, 2t) \qquad 0 \leq t \leq 1.$$

$$|\gamma_2'(t)| = \sqrt{1 + 4t^2}$$

$$\int_0^1 \sqrt{1 + 4t^2}\, dt.$$

~~senht =~~
~~senht = 2t~~

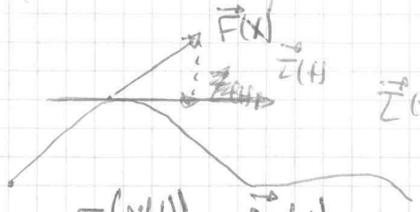
$$\int_0^1 \sqrt{1 + \operatorname{sh}^2 s}\, dt = \frac{1}{2}\int_0^{\alpha} \operatorname{ch} s \operatorname{ch} s\, ds \qquad \operatorname{sh} s = 2t.$$

$$\operatorname{ch} s\, ds = 2\,dt.$$

$$\frac{1}{2}\int \operatorname{ch}^2 s\, ds. \qquad dt = \frac{1}{2}\operatorname{ch} s\, ds.$$

Integrali di linea:

$F: \mathbb{R}^n \to \mathbb{R}^n \qquad X = (x_1, \dots, x_n) \longmapsto (f_1(x), \dots, f_n(x))$



$F(x)$

$\vec{\tau}(t)$ ~~vettore~~ tangente versore

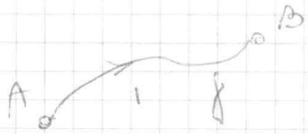
$$\int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$F(\gamma(t)) \cdot \vec{\tau}(t) = f(\gamma(t))$$

$$\int_\gamma f(\gamma(t))\, ds = \int_a^b F(\gamma(t)) \cdot \frac{\gamma'(t)}{\|\gamma'(t)\|} \|\gamma'(t)\| dt \qquad \vec{\tau}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

$$\int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt.$$

Integrale di linea si ottiene calcolando l'integrale curvilineo del vettore $F(x)$ e la sua proiezione sul vettore tangente, tenendo conto del verso di percorrenza (segno di $\vec{\tau}(t)$).



$$\int_\gamma F \cdot dP = -\int_{-\gamma} F\, dP$$ (curva percorsa al contrario)

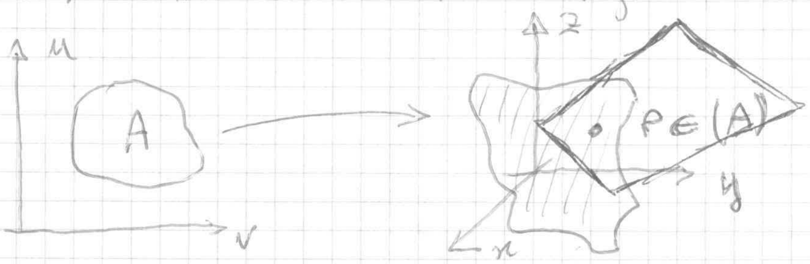
Integrali superficiali:

Se
- A aperto connesso di $\mathbb{R}^2$
- $\alpha: A \to \mathbb{R}^3 \quad (\mathbb{R}^n) \qquad \alpha: A \to \mathbb{R}^n$, $(u,v) \to \alpha(u,v)$
- α di classe C^1
- $J_{(u,v)}^{\alpha} = J\,\alpha(u,v)$ ha rango max. $\forall (u,v) \in A$.

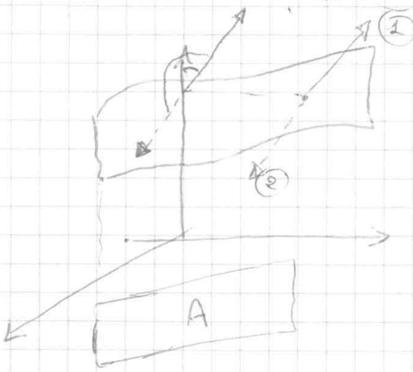
$\Longrightarrow$ Superficie regolare...

$\alpha: (u,v) = (\alpha_1(u,v), \alpha_2(u,v), \alpha_3(u,v))$

$\alpha(A) \subseteq \mathbb{R}^3$ è detto sostegno della superficie



$$\vec{N} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right) \vee \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, +1\right)$$



orientamento

① la proiezione sull'asse z è positiva diretto verso l'alto.

② viceversa ...

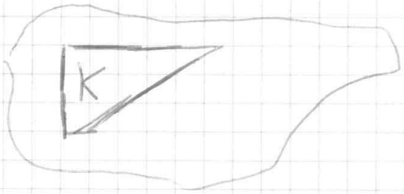
* * * * * * * * * * * *

$z = x^2+y^2$. archi

Calotte superficiali:

$\alpha: A \to \mathbb{R}^3$, $A \subseteq \mathbb{R}^3$ aperto connesso di $\mathbb{R}^2$.

α di classe C^1, $J\alpha$ di rango max in A



~~compatto~~ connesso: (unico insieme, non due pezzi separati)

compatto: (chiuso e limitato)

$\alpha|_K : K \to \mathbb{R}^3$ K compatto, connesso $\mathring{K} \neq \emptyset$

$K = \mathring{K} \cup \partial K$

K misurabile in $\mathbb{R}^2$.

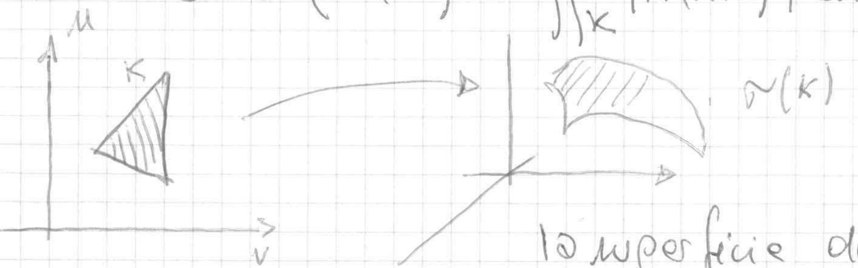
Definizione:

- K compatto, connesso, $\mathring{K} \neq \emptyset$, $K = \mathring{K} \cup \partial K$
- K misurabile.

$\Rightarrow \alpha|_K$: si chiama calotta superficiale regolare.

Se α è una calotta sup. regolare.

$$\Rightarrow \text{area}(\alpha(K)) = \iint_K |N(u,v)|\, du\, dv$$



la superficie della calotta è calcolabile perché K è misurabile