Relatório - Pibic 2019/2020

Abstract

In this project, we aim to review some aspects of Gauge Field Theory, including its path integral description and the quantization of the Yang-Mills field, furthermore we wish to discuss some problems concerning the non-perturbative description of Non-Abelian Gauge Theories. The motivation for this is the known fact that the large coupling constant in the Standard Model's QCD is an impeding factor for the use of perturbative methods, and thus for the understanding of phenomena like asymptotic freedom these non-perturbative methods become useful. The interesting fact we shall explore is that the principal chiral model exhibits properties of the four dimensional Non-Abelian Yang-Mills theories. The Problem of study then is that there haven't yet been discovered any multi-instanton solutions for the exact description of the bi-dimensional model which may be used in the representation of the theory. Furthermore, we describe many other problems involving instantons and we mention related advances in experimental physics.

1 Theoretical Introduction to Gauge Field Theory

1.1 Non-Abelian Gauge Theories

We begin with general description of the Yang-Mills fields (the simplest case corresponding to electromagnetic interactions is given in Appendix 3). Let Ω be a compact semisimple Lie group. For us it is essential that the generators T^a of the Lie algebra of this group can be normalized by the condition $tr(T^aT^b) = -2\delta^{ab}$. In this case, the structure constants are completely antisymmetric.

The Yang-Mills field can be associated with any compact semisimple Lie group. It is given by the vector field $\mathscr{A}_\mu(x)$. In the case of electrodynamics the gauge transformation is: $\mathscr{A}_\mu(x) \rightarrow \mathscr{A}_\mu(x) + i\partial_\mu\lambda(x)$. The transformation of the fields $\psi(x)$ analogous to the phase transformation in electrodynamics is

$$\psi(x) \rightarrow \psi^\omega = \Gamma[\omega(x)]\psi(x) \tag{1}$$

where $\omega(x) \in \Omega$. Then the derivative $\nabla_\mu = \partial_\mu - \mathscr{A}_\mu$ will be covariant with respect to the rule

$$\mathscr{A}_\mu(x) \rightarrow \mathscr{A}^\omega_\mu(x) = \omega(x)\mathscr{A}_\mu(x)\omega^{-1}(x) - \omega(x)\partial_\mu\omega^{-1}(x) \tag{2}$$

and the change of $\mathscr{A}_\mu$ and $\psi(x)$ under such a transformation will be

$$\delta\mathscr{A}_\mu = \partial_\mu\alpha - [\mathscr{A}_\mu, \alpha] = \nabla_\mu\alpha, \quad \delta\psi = \Gamma(\alpha)\psi \tag{3}$$

where $\omega(x) = 1 + \alpha(x)$. Now we need to consider what is needed to make the Lagrangian invariant under the gauge transformations of the fields. For this we consider the comutator of two covariant derivatives acting on a matter field: $[\nabla_\mu, \nabla_\nu]\psi = -iT^a\mathscr{F}^a_{\mu\nu}\psi, \quad a = 1, ..., n$; where $\mathscr{F}^\gamma_{\mu\nu} = \partial_\nu\mathscr{A}_\mu - \partial_\mu\mathscr{A}_\nu + [\mathscr{A}_\mu, \mathscr{A}_\nu]$, then it follows that $\mathscr{F}_{\mu\nu}(x)$ transforms according to the law

$$\mathscr{F}_{\mu\nu}(x) \rightarrow \omega(x)\mathscr{F}_{\mu\nu}(x)\omega^{-1}(x). \tag{4}$$

From the Jacobi identity we get that $\mathscr{F}_{\mu\nu}(x)$ obeys the Bianchi identity of the theory of gravity. Then our gauge invariant Lagrange function in the case of non-Abelian gauge groups is the following

$$\mathscr{L} = \frac{1}{8g^2}\mathrm{tr}[\mathscr{F}_{\mu\nu}\mathscr{F}_{\mu\nu}] + \mathscr{L}_M(\psi, \nabla_\mu\psi) \tag{5}$$

where $g = det(g_{\mu\nu})$ is the determinant of the metric of the space.

1.2 Quantum Theory in terms of Path Integrals

Now we shall deal with the general formalism of the path integral.

Let p and q be the canonical momentum and coordinate of a particle, and let P and Q be the corresponding momentum and coordinate operators. The dynamics of the system is described with the help of the Hamiltonian function $h(p,q)$ and the corresponding Hamiltonian operator $H = h(P,Q)$. One of the most important concepts here is the evolution operator $U(t'',t') = exp\{-iH(t''-t')\}$, which will be given by the theorem:

Theorem: the matrix elements of the elvolution operator are

$$< q'', t''|q', t' >= \int \exp\{i\int_{t'}^{t''}(p\dot{q} - h(p,q))dt\}\prod_t \frac{dpdq}{2\pi}. \tag{6}$$

Now we shall be interested in the evolution operator for an infinite time interval, since it is precisely this operator which is needed for construction of the scattering matrix, defined by the formula

$$S = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} e^{iH_0t''}e^{-iH(t''-t')}e^{-iH_0t'}, \tag{7}$$

Here H_0 is the energy operator for free motion.

In practice, it is often more convenient to deal with the external source $\eta(x)$ dependent functional

$$Z(\eta) = \exp\big\{-i\int V\big(\frac{1}{i}\frac{\delta}{\delta\eta(x)}dx\big)\big\}\exp\big\{\frac{i}{2}\int \eta(x)D_c(x-y)\eta(y)dxdy\big\}, \tag{8}$$

$$D_c(x) = -\big(\frac{1}{2\pi}\big)^4\int e^{-ikx}\frac{1}{k^2 - m^2 + i0}d^4k,$$

Since from this functional we may derive the S-matrix through a simple procedure: we calculate its variational derivatives to find the Green functions $G_n(x_1, x_2, ..., x_n)$, apply to these functions the differential operator $\prod_{i=1}^n(\Box_{x_i} + m^2)$, then we multiply the result by the product $\frac{1}{n!}\prod_i \varphi_0(x_i)$, and integrate over all x, and sum over n.

1.3 Quantization of The Yang-Mills Field

Now let's turn our attention to constructing a consistent quantization procedure for the Yang-Mills field. For this we must find the true dynamical variables for the Yang-Mills field and verify that they change with time according to the laws of Hamiltonian dynamics. After this, we shall be able, in constructing the evolution operator, to use the path integral formalism developed so far. Let us consider in greater detail the structure of the Lagrangian in the first order formalism:

$$\mathscr{L} = \frac{1}{4}tr\{(\partial_\nu\mathscr{A}_\mu - \partial_\mu\mathscr{A}_\nu + g[\mathscr{A}_\mu, \mathscr{A}_\nu] - \frac{1}{2}\mathscr{F}_{\mu\nu})\mathscr{F}_{\mu\nu}\}, \tag{9}$$

In the 3-D notation ($\mu = 0, k; \nu = 0, l; k, l = 1, 2, 3$) we may rewrite the Lagrangian in the form

$$\mathscr{L} = E_k^a\partial_0A_k^a - h(E_k, A_k) + A_0^aC^a, h = \frac{1}{2}\{(E_k^a)^2 + (G_k^a)^2\}, \tag{10}$$

where $E_k = F_{k0}$, $G_k = \frac{1}{2}\epsilon^{ijk}F_{ji}$, $C = \partial_k E_k - g[A_k, E_k]$. It is clear that the pairs (E_k^a, A_k^a) are canonical variables; h is the Hamiltonian, A_0^a is the Lagrangian multiplier, and C^a is the constraint on the canonical variables. By introducing Poisson brackets $\{E_k^a(x), A_i^b(y)\} = \delta_{ki}\delta^{ab}\delta(x-y)$, we verify that

$$\{C^a(x), C^b(y)\} = gt^{abc}C^c(x)\delta(x-y), \quad \{\int d^3x h(E_k, A_k), C^b(y)\} = 0 \tag{11}$$

and the subsidiary condition $\partial_k \mathscr{A}_k = 0$, which is the Coulomb gauge, is admissible since

$$\{\partial_k A_k^a(x), \partial_i A_i^b(y)\} = 0, \text{and} \quad \{C^a(x), \partial_k A_k^b(y)\} = -\partial_k[\partial_k\delta^{ab} - gt^{abc}A_k^c(x)]\delta(x-y). \tag{12}$$

The canonical fields may be decomposed as $A = A^L + A^T$ and $E = E^L + E^T$

$$A^L = \partial_k B(x); \quad B(x) = \frac{1}{4\pi}\int \frac{1}{|x-y|}\partial_k A_k(y)dy \tag{13}$$

$$E^L(x) = \partial_k Q(x) \tag{14}$$

$$A_l^{Tb}(x,t) = \frac{1}{(2\pi)^{3/2}}\sum\int [e^{ikx}a_i^b(k,t)u_l^i(k) + e^{-ikx}a_i^{*b}(k,t)u_l^i(k)]\frac{d^3k}{\sqrt{2\omega}}, \tag{15}$$

$$E_l^{Tb}(x,t) = \frac{i}{(2\pi)^{3/2}}\sum\int [-e^{ikx}a_i^b(k,t)u_l^i(k) + e^{-ikx}a_i^{*b}(k,t)u_l^i(k)]\frac{\sqrt{\omega}d^3k}{\sqrt{2}} \tag{16}$$

and $u_l^i(k), i = 1, 2$ are two polarization vectors, also the $a_i^b(k,t)$, $a_i^{*b}(k,t)$ are holomorphic variables (See Faddeev and Slavnov [1991]).

This treatment of the subjects of Gauge Theories and Quantization of the Yang-Mills fields follows from (Faddeev and Slavnov [1991]), more on these subjects may be found on (Itzykson and Zuber [2012]), (Weinberg et al. [1995]) and (Weinberg [1995]).

2 Topology of Gauge Fields

The topic that we will discuss here is the use of instantons in solving problems related to charge confinement. This method is valid for both Abelian and Non-Abelian systems. We shall analyse topological properties of non-Abelian instantons and solitons in classical physics, for instance in plasma physics, and discuss associated effects.

2.1 Instantons in $\mathscr{D} = 2$, $N = 3$ $\boldsymbol{n}$-Fields

Let us find minima of the classical action for the $\boldsymbol{n}$-field in the case $N = 3$. In order that this action be finite, we have to consider a boundary condition: $\boldsymbol{n}(x) \to_{x\to\infty} \boldsymbol{n}_0$. Therefore, since infinity can be viewed as one point, our x-space is topologically a sphere. Each configuration $\boldsymbol{n}(x)$ defines a map of such a sphere in x-space onto the sphere $\boldsymbol{n}^2 = 1$, which in the case $N = 3$ gives $S^2 \to S^2$. These maps can be classified by their winding number q which define the number of times the second sphere is covered by the first one [See Appendix 4]. In this case we may write

$$q = \frac{1}{8\pi}\int d^2x\ \boldsymbol{n}\cdot[\partial_\mu\boldsymbol{n}\partial_\nu\boldsymbol{n}]\epsilon^{\mu\nu} \tag{17}$$

Here $\epsilon^{\mu\nu}$ is the standard antisymmetric tensor. The classical Action may be written as

$$S = \frac{1}{2e_0^2}\int(\partial_\mu\boldsymbol{n})^2 d^2x = \frac{4\pi q}{e_0^2} + \frac{1}{4e_0^2}\int(\partial_\mu\boldsymbol{n} + \epsilon_{\mu\nu}[\boldsymbol{n}\times\partial_\nu\boldsymbol{n}])^2 d^2x \tag{18}$$

From this relation it follows that in order to find an absolute minimum for the n-fields with the topological charge q one can avoid the problem of solving classical equations of motion which are second order differential equations, and instead consider the first order equations:

$$\partial_\mu \boldsymbol{n} = -\epsilon_{\mu\nu}[\boldsymbol{n} \times \partial_\nu \boldsymbol{n}] \tag{19}$$

A solution of this equation may be found by introducing a complex field w by stereographic projection:

$$n_1 + in_2 = 2w/(1+|w|^2)$$

$$n_3 = (1-|w|^2)/(1+|w|^2) \tag{20}$$

In this new variable, (19) reduces this equation to:

$$\partial_z w \equiv (\partial_1 + i\partial_2)w = 0 \tag{21}$$

Therefore (19) are just Cauchy-Riemann equations for the function w. Since this function must be meromorphic (otherwise n would have brach cuts), the most general solution has the form:

$$w(z) = \prod_{j=1}^{q} \frac{z-a_j}{z-b_j} \quad \text{(Instantons)} \quad w(z) = \prod_{j=1}^{|q|} \frac{\bar{z}-a_j}{\bar{z}-b_j} \quad \text{(Anti}-\text{Instantons)} \tag{22}$$

where q is the topological charge (17) which we may write in terms of the complex function $w(z)$, and is negative for the anti-instanton solution.

In quantum field theory the partition function in the zeroth order approximation of the one instanton contribution is proportional to $e^{-S_{cl}} = e^{-4\pi/e_0^2}$, for $q = 1$, but for higher order perturbations we have to renormalize the coupling e_0^2. The general renormalization formula is given by

$$e^2(p) = e^2(\mu)/(1 + \frac{N-2}{4\pi}e^2(\mu)\log(p^2/\mu^2)) \tag{23}$$

where p and μ are values for the momentum of the particles on the configuration, respectively in the perturbed and non-perturbed treatment. The contribution $Z^{(1)}$ to the partition function of a single instanton solution is then going to have the form:

$$Z^{(1)} \sim \int \frac{d^2a d^2b}{|a-b|^4}\exp(-4\pi/e^2(|a-b|)) = \lambda^2 \int \frac{d^2a d^2b}{|a-b|^2} \simeq \lambda^2 V \int \frac{d\rho}{\rho} \tag{24}$$

where λ is the Debye length of the system, $\rho = |a-b|$ is the instanton's effective size, V is the volume of the system, and we used the fact $e^2(|a-b|) \simeq 2\pi/\log(\lambda|a-b|)^{-1}$ for our $N = 3$ theory, following directly from the general rule (23). For the multi-instanton contribuition we refer to (Polyakov [1987]):

$$Z^{(q)} = \frac{\lambda^{2q}}{(q!)^2}\int d^2a_1...d^2a_q d^2b_1...d^2b_q \prod_{i<j}|a_i - a_j|^2 \prod_{i<j}|b_i - b_j|^2 \prod_{i,j}|a_i - b_j|^{-2} \tag{25}$$

Or, after summing over q:

$$Z_{\text{INST}} = \sum_{q=0}^{\infty} \frac{\lambda^{2q}}{(q!)^2}\int \prod_j d^2a_j d^2b_j \exp\{\sum_{i<j}(\log|a_i - a_j|^2 + log|b_i - b_j|^2) - \sum_{i,j}\log|a_i - b_j|^2\} \tag{26}$$

This result is surprising because we see that each instanton behaves as if it is composed of a pair of opposite Coulomb charges, placed at a_j and b_j. Since the two dimensional Coulomb energy is given by $(1/4\pi)\log|a-b|^2$, the expression (26) is the partition function for the plasma with inverse temperature $\beta = 4\pi$. This plasma has two different phases. For large β the charge form dipoles and the system is neutral, with no mass gap. At some critical β, dissociation of the dipoles occurs and for $\beta < \beta_{cr} = 8\pi$ we have a plasma phase with Debye screening, and therefore a massgap. There is a very useful representation for (26) based on the bosonization procedure. If one considers a free massless Dirac field $\boldsymbol{\psi} = (\psi_L, \psi_R)$ for $\mathscr{D} = 2$ and introduces two operators $\sigma_+(x) = \psi_L^\dagger \psi_R$ and $\sigma_- = \psi_R^\dagger \psi_L$, then it can be shown that

$$< \sigma_+(a_1)...\sigma_+(a_N)\sigma_-(b_1)...\sigma_-(b_M) >= \prod_{i<j} |a_i - a_j|^2 |b_i - b_j|^2 \prod_{i,j} |a_i - b_j|^{-2} \delta_{N,M} \tag{27}$$

From this it follows that

$$Z_{\text{INST}} = \int \mathscr{D}\psi(x)\mathscr{D}\bar{\psi}(x)\exp\{-\int (\bar{\psi} i\gamma^\mu \partial_\mu \psi + \lambda \bar{\psi}\psi)\} \tag{28}$$

where $\bar{\psi}\psi = \sigma_+(x) + \sigma_-(x)$ is a mass term. We see that in this representation, expansion in instantons becomes a mass expansion. It is also obvious that

$$\log Z_{\text{INST}} = V \int \frac{d^2p}{(2\pi)^2} \text{Tr } \log(\gamma^\mu p_\mu + \lambda) \tag{29}$$

Expansion in λ leads to more and more infrared singular terms contaning $\int d^2p/p^n$ but the sum (29) is well behaved.

Before actually considering Non-Abelian gauge theories let us describe what kind of instanton structure is present in some versions of chiral models. First of all, n-fields with the group $O(N)$, $N \geq 4$ do not have any nontrivial topology; that is to say any map $S^2 \to S^{N-1}$ for $N \geq 4$ is contractible. In fact the map of S^2 onto any Non-Abelian Lie group G, described by the principal chiral field $g(x)$ is also contractible. These theories do not have stable instantons. The chiral theories which do have them are described by the coset spaces G/H in which H contains $U(1)$ as a factor. In mathematical notation this means that:

$$\pi_2(G/H) \simeq \pi_1(H) \text{ if } \pi_2(G) = 0 \tag{30}$$

The most familiar example of a chiral theory with instanton structure is the so-called

$$\mathbb{C}P^{N-1} = \frac{SU(N)}{SU(N-1) \otimes U(1)} \tag{31}$$

2.2 Instantons in Non-Abelian Gauge Theories

Now we are ready to discuss the main physical problem of interest to this project. As opposed to the examples in Abelian theories, the problem we shall discuss here is connected to the fact that multi-instanton solutions have not been explicitly parametrized up to now, and consequently even for one loop computations the multi-instanton solutions, for the case of Non-Abelian gauge theories, have not yet been discovered. Let us describe this problem more precisely.

First we require that $F_{\mu\nu}(x) \to o(1/x^2)$ as $x \to \infty$ in order that the Yang-Mills action be finite. From this we deduce that $A_\mu(x) \to g^{-1}(x)\partial_\mu g(x) + o(1/x)$ as $x \to \infty$, where $g(x) \in G$. Bounding our $\mathscr{D} = 4$ Euclidean space by a large three-dimensional sphere S^3, we obtain a map $g(x) : S^3 \to G$. These maps are classified by the integers for any G. The analogue of the formula (17) in the present case has the form:

$$q = \frac{1}{12\pi^2} \int d^3x \epsilon^{abcd} \epsilon_{\mu\nu\lambda} (\partial_\mu n^a \partial_\nu n^c \partial_\lambda n^d) = \frac{1}{24\pi^2} \int d^3x \epsilon_{\mu\nu\lambda} \text{Tr}(L_\mu L_\nu L_\lambda) \tag{32}$$

with $L_\mu(x) = g^{-1}\partial_\mu g(x)$. We may also write this integral in terms of the function

$$\rho(x) = \frac{1}{4}\epsilon^{\mu\nu\lambda\rho} \text{Tr}(F_{\mu\nu}F_{\lambda\rho})d^4x = \frac{1}{2}\text{Tr}(F_{\mu\nu}{}^*F_{\mu\nu}), \quad ({}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}) \tag{33}$$

Our aim is to find an instanton solution with $q = 1$. As in the case of the n-field we can avoid solving the Yang-Mills equations themselves, by considering first order equations instead. Using the identity

$$S = \frac{1}{4e_0^2}\int \mathrm{Tr} F_{\mu\nu}^2 d^4x$$

$$= \frac{1}{8e_0^2}\int \mathrm{Tr}((F_{\mu\nu} - {}^*F_{\mu\nu})^2)d^4x + \frac{1}{4e_0^2}\int \mathrm{Tr}(F_{\mu\nu}{}^*F_{\mu\nu})d^4x$$

$$= \frac{8\pi^2 q}{e_0^2} + \frac{1}{8e_0^2}\int \mathrm{Tr}((F_{\mu\nu} - {}^*F_{\mu\nu})^2)d^4x \tag{34}$$

We see that if we find a solution of the duality equation

$$F_{\mu\nu} = {}^*F_{\mu\nu} \tag{35}$$

then the action for a fixed q will be minimal. In fact, if (35) are satisfied, then so are the Yang-Mills equations

$$\nabla_\mu F^{\mu\nu} = 0. \tag{36}$$

We see that the duality equations (35) are four dimensional analogues of the Cauchy-Riemann equations. Their most surprising property is that they possess multi-instanton solutions.

For a solution with $q = 1$, for example, we consider a group $SU(2) \otimes SU(2) \simeq O(4)$. Then equations (35) will have the symmetry group $O(4)_S \otimes O(4)_I$, where the first factor is space rotations, and the second isotopic rotations. Our solution should break this group to $O(4)_{S,I}$ preserving simultaneous rotations in x-space and isotopic space. The ansatz in our case then gives

$$A_\mu^a(x) = \frac{2\eta_{a\mu\nu}(x_\nu - a_\nu)}{(\boldsymbol{x} - \boldsymbol{a})^2 + \rho^2} \tag{37}$$

$$F_{\mu\nu}^a = -\frac{4\eta_{a\mu\nu}\rho^2}{((\boldsymbol{x} - \boldsymbol{a})^2 + \rho^2)^2} \tag{38}$$

with arbitrary scale parameter ρ and position paramenter a_μ, ($\eta_{abc} = \epsilon_{abc}, \eta_{ab0} = \delta_{ab}$).

This Non-Abelian instanton can be viewed as a magnetic dipole of size ρ. If we consider now the contribuition of one instanton to the partition function Z we find, just as in case of $\boldsymbol{n}$-field, several factors. First of all we have a factor $e^{-S_{cl}} = e^{-8\pi^2/e_0^2}$ which gets replaced, after taking account of the one loop correction by $e^{-8\pi^2/e^2(\rho)}$ where, for $SU(2)$

$$e^2(\rho) = \frac{3}{11N}\frac{8\pi^2}{-\log(\lambda\rho)} \tag{39}$$

is an effective coupling for the size ρ. The contribuition has to be integrated over ρ and $\boldsymbol{a}$. The measure must be both scale and translationally invariant. The only combination with these properties is $d^4R d\rho\rho^{-5}$. We find from this consideration:

$$Z_{\mathrm{INST}}^{(1)} \sim V\int \frac{d\rho}{\rho^5} e^{-8\pi^2/e^2(\rho)} = V\int \frac{d\rho}{\rho^5}\rho^{11N/3} \tag{40}$$

(V being the 4-volume).

This instanton contribution has an infrared divergence. This means that in the multi-instanton picture, individual instantons tend to grow and to overlap. In this case we expect something like dissociation of dipole-like instantons to their elemetary constituents. However, even one loop computations on the multi-instanton background have not yet been performed and nothing similar to the Coulomb plasma has been discovered.

2.3 Physical Consequences of Instantons

Another physical problem involving instantons is their interaction with massless Dirac fermions. Let us consider the action

$$S_\psi = \int d^4x\bar{\psi}(i\gamma^\mu(\partial_\mu + A_\mu))\psi \tag{41}$$

In classical physics, this action conserves the axial current :

$$\partial_\mu(i\bar{\psi}\gamma^\mu\gamma^5\psi) = 0 \tag{42}$$

In the presence of a non-Abelian external field A_μ, this becomes untrue because of the quantum anomaly. This happens because this current which in terms of the partition function of fermions diverges in the singularities of the Green function:

$$\begin{aligned} J_{\mu 5} = Z^{-1}[A] \int \mathscr{D}\psi\mathscr{D}\bar{\psi}\exp(-\int \bar{\psi}i\gamma^\mu(\partial_\mu + A_\mu)\psi d^4x) i\bar{\psi}\gamma_\mu\gamma_5\psi \\ = -i\mathrm{Tr}\gamma_\mu\gamma_5 G(x,x;A) \end{aligned} \tag{43}$$

where

$$Z[A] = \int \mathscr{D}\psi\mathscr{D}\bar{\psi}\exp(-\int \bar{\psi}i\gamma^\mu(\partial_\mu + A_\mu)\psi)d^4x \tag{44}$$

and $G(x,x';A)$ is the Green function for the Dirac operator in the field A_μ. To avoid this problem we express $G(x,x';A)$ in terms of eigenfunctions of the Dirac equation $\psi_n(x)$, the standard formula for this is

$$G(x,x') = \sum_n \frac{\psi_n(x)\bar{\psi}_n(x')}{E_n} \tag{45}$$

where E_n are the eigenvalues of the $\psi_n(x)$. To regularize this expression we insert a factor of $e^{-\epsilon E_n^2}$. Then

$$J_{\mu 5}(x;A) = \sum_n \frac{i\bar{\psi}_n\gamma_\mu\gamma_5\psi_n}{E_n}e^{-\epsilon E_n^2} \tag{46}$$

Through some basic computations we arrive at

$$\partial_\mu J_{\mu 5}(x) = \lim_{\epsilon\to 0}\sum_n \partial_\mu \frac{i\bar{\psi}_n\gamma_\mu\gamma_5\psi_n}{E_n}e^{-\epsilon E_n^2} = \frac{1}{8\pi^2}\mathrm{Tr}(F_{\mu\nu}{}^*F_{\mu\nu}) \tag{47}$$

This implies that the change in the axial charge under the influence of A_μ, which is the 4-integral of (47):

$$\Delta Q_5 = \int \partial_\mu J_{\mu 5} d^4x = 2q, \quad Q_5 = \int J_{05} d^3x = N_L - N_R \tag{48}$$

This result means that there exists compulsory production of fermions and antifermions in topologically nontrivial fields and that the numbers of left- and right-handed particles N_L and N_R necessarily change. In the Minkowskian interpretation this means that the field strengths are such that they lead to compulsory pair-creation. By compulsory we mean that the transition amplitude without pair creation is exactly zero. To justify this we note that the vacuum-vacuum amplitude, given by

$$\begin{aligned} Z[A] = \int \mathscr{D}\psi\mathscr{D}\bar{\psi}\exp\{-\int i\bar{\psi}\gamma_\mu(\partial_\mu + A_\mu)\psi d^4x\} \\ = \mathrm{Det}(i\gamma_\mu(\partial_\mu + A_\mu)) \end{aligned} \tag{49}$$

may be seen to be zero because the Dirac operator, has zero eigenvalues in the topological fields. The proof of this lies in the fact that if the topological charge $q = n_R - n_L$ is non-zero, then we trivially get zero eigenmodes.

If now we consider the Green functions instead of Z:

$$G(x_i, y_i) = Z^{-1} \int e^{-S_\psi} \mathscr{D}\psi \mathscr{D}\bar{\psi}(\psi(x_1)...\psi(x_N)\bar{\psi}(y_1)...\bar{\psi}(y_N)) \tag{50}$$

We find that this quantity is not well defined in the instanton field because $Z = 0$. Thus for this field we have to consider

$$Z \cdot G(x_i, y_i) = \int e^{-S_\psi} \mathscr{D}\psi \mathscr{D}\bar{\psi}(\psi(x_1)...\psi(x_N)\bar{\psi}(y_1)...\bar{\psi}(y_N)) \tag{51}$$

Since the action is quadratic this amplitude can be computed by expansion in normal modes:

$$\psi(x) = \sum_\alpha C_{0\alpha}\psi_{0\alpha}(x) + \sum_{n\neq 0} C_n \psi_n(x) \tag{52}$$

Here $\{\psi_{0\alpha}\}$ are the zero modes of the Dirac operator and the other are the nonzero modes. Then we use the Berezin rule $\int dC = 0$ and $\int CdC = 1$ for anticommuting variables. Since

$$\mathscr{D}\psi\mathscr{D}\bar{\psi} = \prod_\alpha dC_{0\alpha} d\bar{C}_{0\alpha} \prod_{n\neq 0} dC_n d\bar{C}_n \tag{53}$$

and

$$S(\psi, \bar{\psi}) = \sum_{n\neq 0} E_n \bar{C}_n C_n \tag{54}$$

then the only nonzero terms in the integrand of (51) will contain a product of the $C_{0\alpha}\bar{C}_{0\alpha}$. Since each left-handed $\psi_{0\alpha}$ gives a right-handed $\bar{\psi}_{0\alpha}$ the amplitude will be nonzero, only if the rule $\Delta Q_5 = 2q$ is satisfied. In this case the integral will be proportional to the product of the corresponding zero mode eigenfunctions $\psi_{0\alpha}(x)$.

This configuration if integrated over A_μ, including nontrivial topological fields, has the nonconservation of the axial current. This effect shows that massless quarks tend to suppress instanton contribution, because $Z = 0$ in the instanton field. If we consider instanton-anti-instanton configurations then their contribution will be nonzero, due to the fact that the total topological charge is zero. But then the effective action diverges with the increasing distance between the instantons. This means that exchange of a massless fermion pair leads to long-range forces between instantons and anti-instantons. The most probable consequence of this effect is that due to the strong binding force between fermions the chiral symmetry gets spontaneously broken and as a result the fermions acquire a mass. After which the long range force between instantons and anti-instantons disappears. The only remaining effect of anomalous non-conservation will consist of giving a mass to the corresponding Goldstone boson.

Another curious phenomenon arising because of instantons is the loss of time reversal invariance in the the theory of strong interactons if we insert in the Lagrangian a topological term taking account of instantons:

$$\mathscr{L} = -\frac{1}{4e_0^2} TrF_{\mu\nu}^2 + \frac{i\theta}{16\pi^2} TrF_{\mu\nu}{}^*F_{\mu\nu} \tag{55}$$

This loss of time reversal invariance may be seen through the fact that the term $H \cdot E$ in $TrF_{\mu\nu}{}^*F_{\mu\nu} = -2H \cdot E$ is T-odd.

Due to the presence of instantons, physical transitions will depend on this topological term θ. The vacuum to vacuum amplitude will then be

$$Z = \sum_{q=-\infty}^{+\infty} e^{i\theta q} Z_q \tag{56}$$

Because of this loss of invariance, we expect that θ must be zero or extremely small. Though if we allow this strong T-violation, then it can be shown that if massless fermions with broken chiral symmetry are present in the system, due to instanton effects Goldstone's massless particles obtain some mass (because of nonexact conservation of $J_{\mu 5}$) and simultaneously the θ-term gets absorbed after redefinition of Goldstone's field. This consideration predicted a light isoscalar boson. Unfortunately this particle, called axion, has not yet been found. Though efforts are being made towards a discovery of this particle, specifically the XENON1T experiment based at the Gran Sasso National Laboratory in Italy has recently captured a hint of a signal of axions streaming out of our sun. This experiment is further discussed on Letzter [2020].

3 Appendix: Notations and Introductory Concepts

To start our main discussion of Gauge field theory in the context of quantum field theories and their path integral formalism, we have to start off constructing an action for the classical theory, which will then be generalized.

We will assume that for a free particle

$$S = -mc \int_a^b ds \tag{57}$$

here $ds = \sqrt{dx_\mu dx^\mu}$ ($\mu = 0, 1, 2, 3$), where we are putting $dx_\mu dx^\mu = c^2 dt^2 - dx^2 - dy^2 - dz^2$.

For a charged particle moving in an electromagnetic field, the action function will have the form

$$S = \int (-mcds - \frac{e}{c} A_\mu dx^\mu), \quad \mu = 0, 1, 2, 3. \tag{58}$$

Now from the least action principle we deduce the equations of motion:

$$mc \frac{du_\mu}{ds} = \frac{e}{c} F^{\mu\nu} u_\nu, \quad \mu, \nu = 0, 1, 2, 3 \tag{59}$$

where $F^{\mu\nu} = \partial A_\nu / \partial x^\mu - \partial A_\mu / \partial x^\nu$ is the **electromagnetic field tensor**, and $u_\mu = dx_\mu / ds$.

It is obvious that the quantity $F_{\mu\nu} F^{\mu\nu}$ is an invariant quantity. This will be the last term in the action so that the field equations are linear with respect to the fields. Thus the action now looks like

$$S = -\sum \int mcds - \sum \int \frac{e}{c} A_\mu dx^\mu - \frac{1}{16\pi c} \int F_{\mu\nu} F^{\mu\nu} d\Omega, \quad \mu, \nu = 0, 1, 2, 3 \tag{60}$$

where the sums are meant to take into account systems of many particles, the factor $d\Omega = c\ dt\ dx\ dy\ dz$, and the term $1/16\pi c$ is convenient for experimental agreement.

More on this topic is discussed on (Landau and Lifshitz [1980]).

4 Appendix: The Topology of Soliton Solutions

In field theory we have many classical solutions to the field configurations that obey our given Lagrangians, these we call soliton solutions and one of the classical examples of them in one spatial dimension arises in the theory with a single scalar field ϕ and Lagrangian density:

$$\mathscr{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi), \quad V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2. \tag{61}$$

Here m^2 and λ are both positive and $v = \sqrt{m^2/\lambda}$. The classical Euler-Lagrange equation of the theory is

$$\frac{d^2\phi}{dt^2} - \frac{d^2\phi}{dx^2} = -\lambda(\phi^2 - v^2)\phi \tag{62}$$

and we are particularly interested in static solutions $d^2\phi/dt^2 = 0$. By straightforward computation we arrive at the kink solution

$$\phi(x) = v \tanh[\frac{m}{\sqrt{2}}(x - x_0)] \tag{63}$$

We notice that this configuration has two vacuum solutions at different parts of spatial infinity $\phi(\infty) = v$, $\phi(-\infty) = -v$ and with these values the configuration could not be continuously deformed into a uniform vacuum solution over all of space. In this case it is fairly easy to see the topological stability of the soliton, however we wish to consider theories in which the space of vacuum solutions is more complicated.

4.1 Vacuum Manifolds

The general situation that we want to consider is a field theory with n scalar fields that can be assembled into an $n-$component column vector ϕ and a scalar potential $V(\phi)$ that has a family of degenerate minima that form a manifold $\mathscr{M}$. Let us assume that this degeneracy is a consequence of a symmetry group G that is spontaneously broken to a subgroup H by the vacuum expectation value of ϕ. Given a value ϕ, the action of an element g of G transforms ϕ to $g\phi$. In particular, if ϕ_0 minimizes V, then so does $g\phi_0$ for any choice of g.

If G is completely broken then there is a one-to-one correspondence between elements of G and minima of V. In the case it is only partially broken, there is an unbroken subgroup H that can be defined by the requirement that it leaves ϕ_0 invariant. We can therefore define equivalence classes of elements of G by defining two elements g_1 and g_2 to be equivalent if $g_2 = g_1 h$ for some $h \in H$. The set of such equivalence classes is the coset space G/H. There is a one-to-one correspondence between these equivalence classes and the minima of V, so $\mathscr{M} = G/H$.

4.2 Homotopy and the fundamental group $\pi_1(\mathscr{M})$

Let us start by considering closed paths, or loops on a manifold $\mathscr{M}$. In particular let us pick a point x_0 on $\mathscr{M}$ and restrict our attention to paths that begin and end at x_0. Any such path can be specified by a continuous function $f(t)$ taking values in $\mathscr{M}$, with $0 \leq t \leq 1$ and $f(0) = f(1) = x_0$. Let $f(t)$ and $g(t)$ be continuous paths beginning and ending at x_0. They can be smoothly deformed into one another if and only if there is a continuous function $k(s,t)$ with $0 \leq s,t \leq 1$ such that

$$k(0,t) = f(t), \quad k(1,t) = g(t), \quad k(s,0) = k(s,1) = x_0 \tag{64}$$

Thus $k(s,t)$ can be viewed as a sequence of loops, labeled by s, that begin and end at x_0 with f being the first in the sequence and g the last. Paths f and g are said to be *homotopic* at x_0, and the family of paths that define the function k is a homotopy.

One can define a product on the space of paths. Given paths f and g, their product is defined in the following way $f \circ g$ is the path obtained by going around f and then going around g. An inverse path f^{-1} can be defined as going around f as going around f in the reverse direction $f^{-1}(t) = f(1-t)$. The next step is to divide the paths on $\mathscr{M}$ into *homotopy classes*, with the homotopy class $[f]$ denoting the set of paths that are homotopic to f. Defining a product in this set of equivalence classes we obtain a group structure and we denote this group $\pi_1(\mathscr{M}, x_0)$ and call it the *fundamental group* of $\mathscr{M}$ at x_0. If the manifold $\mathscr{M}$ is connected then the base point x_0 loses importance since the group will be the same in any point of the manifold.

There is a natural generalization of the fundamental group. The latter classifies closed loops, which are maps from a circle, S^1, to a given manifold. The higher homotopy groups, $\pi_n(\mathscr{M})$, classify maps from an n-sphere, S^n, to the manifold. Let us first consider the second homotopy group $\pi_2(\mathscr{M}, x_0)$. Let $f(s,t)$ and $g(s,t)$ be two maps on S^2, both of which are equal to x_0 everywhere on the perimeter of the square. These are homotopic if there is a function $k(s,t,u)$ with $(s,t,u) \in [0,1]^3$ such that the u-parameter takes k from f to g, and k is also equal to x_0 on the perimeter of the variables s,and t. Now the product of maps is defined similarly as before but only for the t-variable. The inverse map to the map $f(s,t)$ is also similar being defined by $f(s,-t)$, and under the equivalence relation it will define the inverse element of the group by $[f(s,t)]^{-1} = [f(s,-t)]$ where the equivalence class of the map $f(s,t)$ is $[f(s,t)]$, being the set of maps homotopic to $f(s,t)$.

This discussion of π_2 can be carried over, with obvious generalizations, to the π_n with $n \geq 3$. An important property of all these groups (with $n \geq 2$) is that they are always Abelian.

4.3 Vortices and Homotopy

In two dimensions, spatial infinity can be described as a circle at $r = \infty$. As θ varies from 0 to 2π, the values of the field $\phi(r = \infty, \theta)$ on this circle trace out a loop in the vacuum manifold $\mathscr{M}$. In the field theory context, the homotopy equivalence relation between two field configurations occurs when $\phi_1(\infty, 0) = \phi(\infty, 0) = \phi_0$ for some fixed ϕ_0. A nonsingular configuration in which ϕ takes on different vacuum values at different points along the circle $r = \infty$ is a continuous family of vacuum states. With a complex scalar field $\phi(x) = \rho(x)e^{i\alpha(x)}$, governed by the Lagrangian density

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^\dagger(\partial^\mu\phi) - \frac{\lambda}{4}\big(|\phi|^2 - \frac{\mu^2}{\lambda}\big)^2 \tag{65}$$

This theory has a global $U(1)$ symmetry. It is minimized by $|\phi| = v = \sqrt{\mu^2/\lambda}$ and the $U(1)$ symmetry is spontaneously broken. There is a continuous set of vacuum states, given by $\rho = v$ and an arbitrary uniform value for α. In this example again the field cannot be continuously deformed to a vacuum solution, this field solution is known as a vortex. The topological formulation for this situation is the following: for some smooth configuration of ϕ, we can define the line integral

$$N[C] = \frac{1}{2\pi}\oint_C d\boldsymbol{l}\cdot\boldsymbol{\nabla}(\arg\phi) = \frac{1}{2\pi}\oint_C d\boldsymbol{l}\cdot\boldsymbol{\nabla}\alpha, \tag{66}$$

where the countour C is the circle at spatial infinity. If ϕ is nonzero everywhere on this circle, so that its phase is well defined at each point of the curve, $N[C]$ counts the number of full rotations that this phase makes during one clockwise circuit around C. This must be an integer n, which may be termed the vorticity or winding number.

We are particularly interest on some results concerning the homotopy groups of spheres for the analysis of later field configurations. Let us begin with the higher homotopy groups of n-spheres $\pi_n(S^n)$, which classifies maps from one n-sphere to another. On S^1 if θ denotes the angle on the first circle, and $\alpha(\theta)$ the angle to which this is mapped on the second circle, the winding number can be written as

$$N = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{d\alpha}{d\theta}. \tag{67}$$

On S^2 using the standard spherical coordinates θ and ϕ for the first sphere and $\alpha(\theta,\phi)$ and $\beta(\theta,\phi)$ for the sencond, the winding number is defined by

$$N = \frac{1}{4\pi}\int d^2\Omega \frac{\sin\alpha}{\sin\theta}\big(\frac{d\alpha}{d\theta}\frac{d\beta}{d\phi} - \frac{d\beta}{d\theta}\frac{d\alpha}{d\phi}\big). \tag{68}$$

It is useful to rewrite in terms of Cartesian coordinates. Let us define a unit vector

$$\hat{\boldsymbol{e}}(\boldsymbol{r}) = (\sin\alpha\,\cos\beta, \sin\alpha\,\sin\beta, \cos\alpha), \tag{69}$$

Then the winding number of $\hat{\boldsymbol{e}}(\boldsymbol{r})$ on a sphere of fixed radius is

$$N = \frac{1}{8\pi}\epsilon^{ijk}\int dS^i \hat{\boldsymbol{e}} \cdot \partial_j \hat{\boldsymbol{e}} \times \partial_k \hat{\boldsymbol{e}}, \tag{70}$$

where dS^i is the surface element on the sphere. This integral is invariant under smooth variation of $\hat{\boldsymbol{e}}$. Furthermore, N is invariant under perturbations of the integration surface, as long as $\hat{\boldsymbol{e}}$ remains well defined. Hence, if we have a field $\phi(r)$ that transforms as an $SO(3)$ vector and define $\hat{\boldsymbol{e}} = \phi/|\phi|$, the winding number of $\hat{\boldsymbol{e}}$ over a surface is invariant under deformations of the surface that do not take it through a zero of ϕ. Arguments analogous to those for the vortex case then show that N is equal to the total number of zeros of ϕ in the region enclosed by the surface of integration, with each zero being counted with a plus or a minus sign according to the sign of the winding on an infinitesimal sphere enclosing the zero.

This quantity may also be writen in terms of elements the group in discussion

$$N[G] = \frac{1}{24\pi^2}\epsilon^{ijk}\int d^3x \mathrm{tr} G^{-1}\partial_i G G^{-1}\partial_j G G^{-1}\partial_k G. \tag{71}$$

The discussion here presented on instanton solutions is very limited and this rich subject is further explored by (Weinberg [2012]), many related topics, such as a description of these instanton solutions in the contex of Fibre Bundles, treatments of index theorems, homotopy theory and physical applications may also be found on (Nakahara [1990]).

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