

Motivation

Our main motivation is the recent application of integrability in understanding the relation between gauge and string theories. First, we will try to intuitively understand:

- * What are gauge and string theories, and why study them?
- * How are they related, and what are the implications?
- * What is integrability, and how it helps in establishing gauge/string relation?

Gauge theories

Let's start with classical electromagnetism: We can completely describe it using vector-potential A_μ , and derive the Maxwell equations using the Lorentz-invariant action:

$$S = \int \left[-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + j A_\mu \right] d^4x$$

where electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and g is the coupling constant. The action is invariant under gauge transformations:

$$A_\mu \rightarrow A_\mu + \partial_\mu \phi(x) \Rightarrow F'_{\mu\nu} = F_{\mu\nu}; \partial_\mu j^\mu = 0 \text{ (current conservation)}$$

We say that electromagnetism (photons) are described by $U(1)$ (abelian) gauge theory.

We can also construct non-abelian gauge theories, e.g. the standard model is described by $SU(3) \otimes SU(2) \otimes U(1)$ gauge theory (Yang-Mills theory).

This means that the fields A_μ^i (gluons) are now matrices ($i=1 \dots N_c$), and the gauge theory is now described by the action:

$$S_{YM} = \int \left[-\frac{1}{4g_{YM}^2} F_{\mu\nu}^i F^{\mu\nu i} \right] d^4x + (\text{matter fields, interaction etc})$$

where

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + [A_\mu, A_\nu]^i$$

The gauge symmetry becomes: $A_\mu \rightarrow A_\mu^0(x) = \omega(x) A_\mu \omega^\dagger(x) + \partial_\mu \omega(x) \cdot \omega^\dagger(x)$,
where $\omega(x) \in G$ (symmetry group).

- Why we need non-abelian theories?

In quantum theory the coupling constant 'g' is not a constant anymore, but depends on the energy scale μ at which physical process occurs: $g = g(\mu)$. The beta-function:

$$\beta(g) \equiv \frac{dg(\mu)}{d \ln \mu}$$

characterizes the dependence $g(\mu)$.

For electrodynamics (QED):

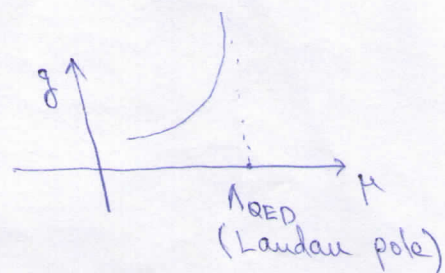
$$\beta(g) = C_{QED} \cdot g^3 \quad ; \quad C_{QED} > 0$$

which means that $g(\mu)$ grows with μ :

This means that at small distances

$g \rightarrow \infty$, and QED cannot be a fundamental theory.

On the other hand, for non-abelian theories:



$$\beta(g) = C_{QED} g^3 \quad ; \quad \text{For } su(3): C_{QED} = - \left(11 - \frac{2n_f}{3}\right) \frac{g^3}{16\pi^2} < 0 \quad \text{for } n_f < 33/2$$

(where n_f is the number of flavours).

So, β -function can become negative, there is no Landau pole,

and the field-theory is well-defined at small distances:

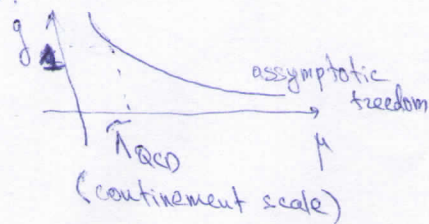
Ideally, one would like QED to be a part

of some larger non-abelian theory.

The integration constant Λ_{QCD} is not determined

by the theory. We can choose Λ_{QCD} corresponding $g(\Lambda_{QCD}) = \infty$, then:

$$g^2(\mu) = \frac{16\pi^2}{\left(11 - \frac{2}{3}n_f\right) \ln \mu^2 / \Lambda_{QCD}^2} \quad ; \quad (su(3))$$



For $su(N)$:

$$g^2(\mu) = \frac{48\pi^2}{\left(11N - 2n_f\right) \ln \mu^2 / \Lambda_{QCD}^2} \quad ; \quad (su(N))$$

We will use general $\beta(\alpha)$ β -function when we do $1/\lambda$ expansion. Let's also consider the case $\beta=0$. This means that the coupling constant does not depend on scale, and the classical scale invariance $x_\mu \rightarrow \lambda x_\mu$ (dilatations) is also a symmetry in the quantum theory. The classical dimension Δ_0 of fields $\Phi(x_\mu)$ is defined as follows:

$$\Phi(x_\mu) \rightarrow \lambda^{\Delta_0} \Phi(\lambda x_\mu) \quad \text{when} \quad x_\mu \rightarrow \lambda x_\mu$$

Classically both QED and QCD are scale invariant, but not in the quantum theories ($\beta \neq 0$). This is called an anomaly. Almost always the scale transformations are a part of a larger conformal group:

$$g_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \quad \text{when} \quad x \rightarrow x'$$

Example of conformal invariant theory: $N=4$ SUSY (supersymmetric) Yang-Mills.

Strings

Now we explain how strings appear from non-abelian gauge theories. This is based on 't Hooft's $1/N$ (large N) expansion.

Remember the path integral formulation in quantum mechanics. We have the evolution operator $U(t, t') = e^{-iH(t-t')}$ which describes the evolution of the wave-function $|i, t'\rangle$ from initial state 'i' at time t' to the state:

$$|f, t\rangle = U(t, t') |i, t'\rangle$$

$\Rightarrow \langle f | f, t \rangle$ is the probability to find the system in the state 'f' at time 't'.

$\Rightarrow$

$$\underbrace{\langle f | f, t \rangle}_{\equiv \psi(f, t)} = \int d^3 q' \underbrace{\langle f | e^{-iH(t-t')} | q' \rangle}_{\equiv K(q, t, q', t')} \underbrace{\langle q' | i, t' \rangle}_{\equiv \psi_i(q', t')}$$

$\Rightarrow$

$$\psi(f, t) = \int d^3 q' K(q, t, q', t') \psi_i(q', t')$$

where $K(q, t, q', t')$ is the amplitude of $q', t' \rightarrow q, t$ transition (propagator).

The path integral formulation gives:

$$K(q, t, q', t') = \sum_{\text{sum over all paths}} e^{iS(q, t, q', t')}$$

$$; \text{ the action: } S(q, t, q', t') = \int_{t'}^t L dt$$

The S-matrix:

$$S \equiv \lim_{\substack{t \rightarrow \infty \\ t' \rightarrow -\infty}} \langle f | e^{-iH(t-t')} | i \rangle$$

Writing $L = L_0 - V$, where L_0 is the free Lagrangian and V is the perturbation, and passing to interaction representation:

$$\psi_I(t) = e^{iH_0 t} \psi(t); \quad A_I(t) = e^{iH_0 t} A e^{-iH_0 t} \quad (\text{where 'A' is an arbitrary operator})$$

(set $\hbar = 1$)

$\Rightarrow$ (DROP THE 'I' INDEX FROM NOW ON)

$$\frac{dA(t)}{dt} = i[H_0, A(t)]; \quad U(t, t') = e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'}$$

where $U(t, t')$ satisfies the differential equation:

$$i \frac{\partial U(t, t')}{\partial t} = V(t) U(t, t'); \quad U(t, t) = 1$$

The integral form:

$$U(t, t') = 1 - \frac{i}{\hbar} \int_{t'}^t V(t'') U(t'', t') dt''$$

$$\Rightarrow U(t, t') = 1 + \left(\frac{i}{\hbar}\right) \int_{t'}^t V(t_1) dt_1 + \left(\frac{i}{\hbar}\right)^2 \int_{t'}^t dt_1 \int_{t'}^{t_1} V(t_1) V(t_2) dt_2 + \dots$$

This is used time-dependent perturbation series from quantum mechanics. In path integral formulation we decompose:

$$K(z, t; z', t') = \sum e^{iS} = \sum e^{iS_0 + iS_I} = \sum e^{iS_0} \left(1 + \frac{i}{\hbar} S_I + \frac{1}{2!} \left(\frac{i}{\hbar}\right)^2 (S_I)^2 + \dots \right)$$

where $S_I = \int V(t) dt \equiv \int \mathcal{L}_I(x) d^4x$

Each term in $U(t, t')$ or $K(z, t; z', t')$ is represented by Feynman diagram.

The S-matrix (operator):

$$S = U(+\infty, -\infty) = 1 - \frac{i}{\hbar} \int_{-\infty}^{+\infty} e^{\frac{-\varepsilon|t'|}{\hbar}} V(t') U(t', -\infty) dt'$$

insert, to make
the integral convergent

$$\Rightarrow \langle n | S | m \rangle = \langle n | S^{(0)} | m \rangle + \langle n | S^{(1)} | m \rangle + \dots$$

Easy to show:

$$*) \langle n | S^{(0)} | m \rangle = \delta_{nm}$$

$$*) \langle n | S^{(1)} | m \rangle = -2\pi i \delta(E_n - E_m) \langle n | V | m \rangle$$

$$*) \langle n | S^{(2)} | m \rangle = -2\pi i \delta(E_n - E_m) \sum_k \frac{\langle n | V | k \rangle \langle k | V | m \rangle}{E_m - E_k + i\epsilon}$$

etc.

$$\Rightarrow \langle n | S | m \rangle = \delta_{nm} - 2\pi i \delta(E_n - E_m) \langle n | T | m \rangle$$

where

$$\langle n | T | m \rangle = \langle n | V | m \rangle + \sum_k \frac{\langle n | V | k \rangle \langle k | V | m \rangle}{E_m - E_k + i\epsilon} + \dots$$

We can also write:

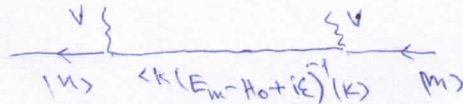
$$\frac{\langle n | V | k \rangle \langle k | V | m \rangle}{E_m - E_k + i\epsilon} = \langle n | V | k \rangle \frac{1}{E_m - E_k + i\epsilon} \langle k | V | m \rangle = \langle n | V | k \rangle \langle k | (E_m - H_0 + i\epsilon)^{-1} | k \rangle \langle k | V | m \rangle$$

Feynman rules:

$$\langle n | V | m \rangle :$$



$$\langle n | S^{(2)} | m \rangle :$$



Let's consider some non-abelian gauge theory. We have gluons A_{μ}^{ij} ; $i, j = 1 \dots N_c$ ($c = \text{'color'}$), and other matter fields, e.g. quarks

q_a^i , $a = 1 \dots N_f$ ($f = \text{'flavour'}$). For QCD, we have $SU(3) \Rightarrow N_c = 3; N_f = 6$.

The Feynman rules are as follows: ($\mathcal{L} \sim \frac{1}{g_{YM}^2} F_{\mu\nu} F^{\mu\nu}$)

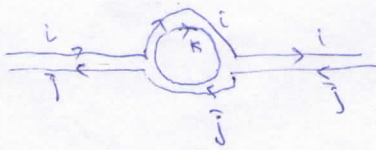
Each gluon is represented by a double line:

$$\begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} : A_{\mu}^{ij}$$

For a quark (anti-quark):

$$\xrightarrow{\quad} : \text{quark} ; \xleftarrow{\quad} : \text{anti-quark}$$

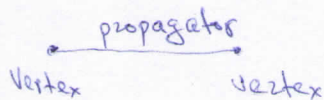
A typical diagram  becomes:



* Each closed loop:  $\sim N_c$

* Each vertex:  $\sim \frac{1}{g_{YM}^2}$

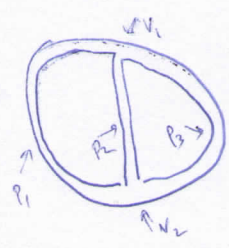
* Each propagator:  $\sim g_{YM}^2$



Let's consider some diagrams.

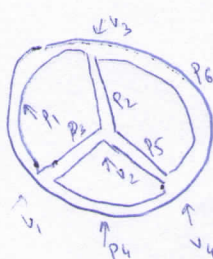


$$\sim N_c^2 \quad (\text{two closed loops, no vertices} \Rightarrow \text{no propagators})$$



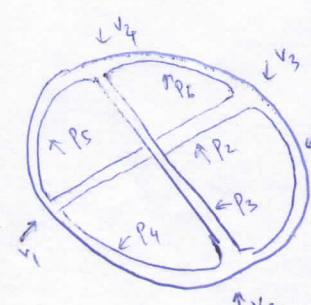
$$\sim N_c^3 \cdot \left(\frac{1}{g_{YM}^2}\right)^2 \cdot (g_{YM}^2)^3 = N_c^2 (g_{YM}^2 N_c)$$

(three closed loops, two vertices, three propagators)



$$\sim (N_c)^4 \left(\frac{1}{g_{YM}^2}\right)^4 (g_{YM}^2)^6 = N_c^2 (g_{YM}^2 N_c)^2$$

(4 closed loops, 4 vertices, 6 propagators)



$$\sim N_c^2 \cdot \left(\frac{1}{g_{YM}^2}\right)^4 (g_{YM}^2)^6 = N_c^2 g_{YM}^4 = N_c^0 (g_{YM}^2 N_c)^2$$

(2 closed loops, 4 vertices, 6 propagators)

In any diagram we have 'edges', 'faces' and 'vertices':



for an arbitrary diagram we have:

$$\sim N_c^F \cdot \left(\frac{1}{g_{YM}^2}\right)^V (g_{YM}^2)^E = N^{V-E+F} \chi^{E-V}$$

where ' $\chi \equiv g_{YM}^2 N_c$ ' is the 't Hooft coupling

Now remember the Euler's formula:

$$\chi = V - E + F$$

where χ is the Euler characteristic, and for two-dimensional surfaces $\chi = 2 - 2g$, where g is the genus of the surface.

For a sphere (or a plane) $\chi=2$; for a torus $\chi=0$ etc.

So, each diagram scales as:

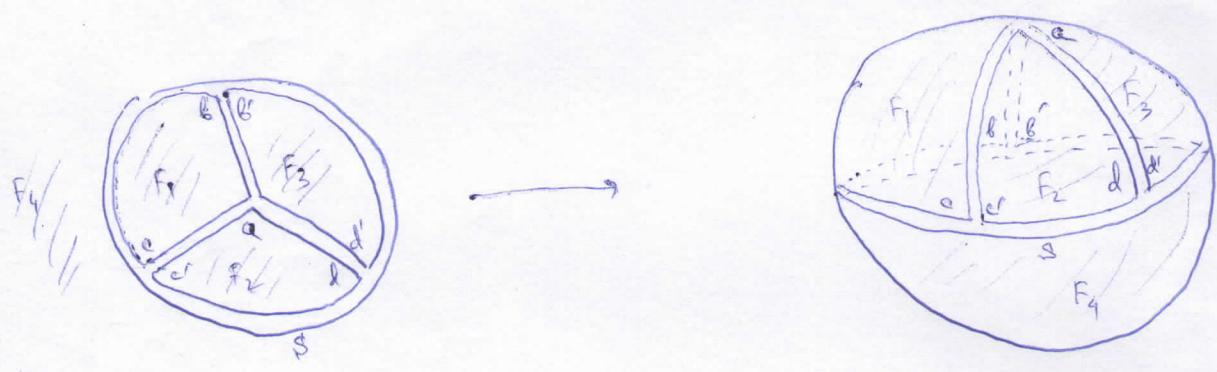
$$\sim N^x \lambda^{F-x}$$

So, we organize the series according to ' χ '.

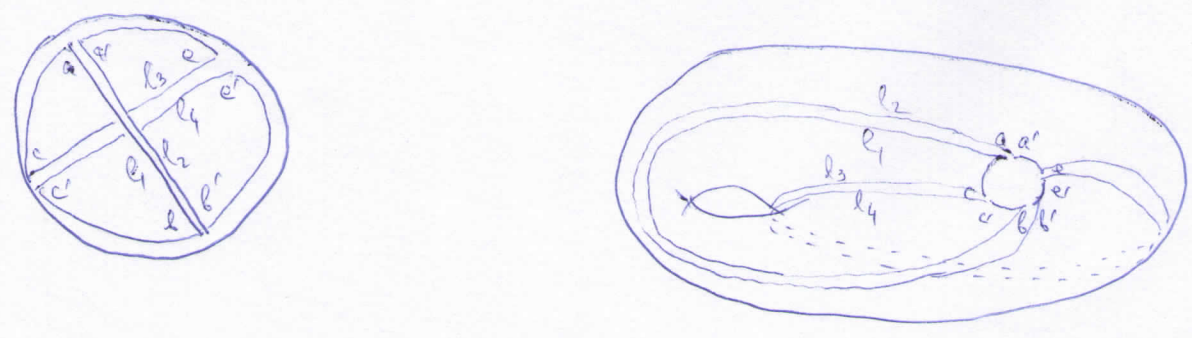
$$\underbrace{N^2 [\tilde{c}_0 \lambda^0 + \tilde{c}_1 \lambda^1 + \dots]}_{\text{planar diagrams}} + \underbrace{N^0 [\tilde{c}_0 \lambda^0 + \tilde{c}_1 \lambda^1 + \dots]}_{\text{non-planar diagrams}} + \dots$$

The powers of $\lambda \sim \lambda^{F-x}$ for a fixed x count the number of loops!

The planar diagrams ($g=0$; $\chi=2$) can be drawn on a sphere:



The non-planar diagrams cannot be drawn on a sphere, but can be drawn for a torus ($g=1$; $\chi=0$) etc. for example:



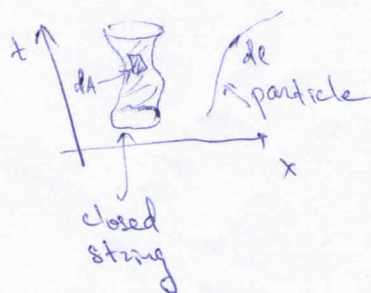
So, we have the following expansion:

$$N_c^2 \cdot \text{[Diagram 1]} + N_c^0 \cdot \text{[Diagram 2]} + \frac{1}{N_c^2} \cdot \text{[Diagram 3]}$$

↑
as we take into account more loops, we get more accurate triangulation of the surface

This is exactly the type of expansion we encounter in closed strings, but with the ~~new~~ string coupling constant $g_s = \frac{1}{N_c}$.

This is because strings are represented by a two-dimensional surface as they move in space-time:



$$S_{\text{particle}} \sim \int dl$$

$$S_{\text{string}} \sim \int dA$$

This is the duality between gauge and string theories.

First concrete realization: AdS/CFT correspondence:

CFT = supersymmetric $su(N)$ Yang-Mills theory

AdS = closed string propagating in $AdS_5 \times S^5$ background: $S_{\text{string}} = \frac{1}{g_s^2} \int d^2\sigma G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$

The parameters are identified as follows:

$$g_s = \frac{g_{YM}^2}{4\pi} ; g_o^2 = \frac{4\pi}{N_c}$$

Let's consider 't Hooft limit: $N_c \rightarrow \infty$; $\lambda = \text{fixed}$. Remembering the β -function, in $N_c \rightarrow \infty$ limit:

$$\lambda(\mu) \sim (\text{const}) \cdot \frac{1}{\ln \mu / \Lambda_{QCD}}$$

we see that the confinement scale Λ_{QCD} is unchanged in 't Hooft's limit!

In 't Hooft limit $N_c \rightarrow \infty \Rightarrow g_s = 1/N_c = 0$, So, the planar limit ($N_c \rightarrow \infty$) corresponds to free string theory ($g_s = 0$). We can do calculations in free string side and this will correspond to all-loop result in the YM theory.

Next, we still have fixed λ . We can consider two limits:

1) $\lambda \gg 1; \Rightarrow g_s \ll 1$ and we can do perturbations on string side

2) $\lambda \ll 1; \Rightarrow g_s \gg 1$, so we can't do much on string side, but we can do perturbation on YM side.

This is the strong/weak duality.

The gauge/string correspondence can be checked due to integrable structures in both theories. By 'integrability' we mean generic term which means that it is possible to solve the theory exactly, without resorting to approximation (perturbation) schemes. Later we will give more precise definitions in classical and quantum theories. Let us see how one such integrable system is discovered from strings.

Strings on $R \times S^2$

Ignoring the fermionic part, the bosonic string action has the form:

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{-g} g^{\alpha\beta} [G_{MN}^{(S^2)} \partial_\alpha X^M \partial_\beta X^N + G_{\mu\nu}^{(AdS_5)} \partial_\alpha \hat{X}^\mu \partial_\beta \hat{X}^\nu]$$

where X^M are S^2 coordinates, and $\hat{X}^\mu$ correspond to AdS_5 :

$$ds_{AdS_5}^2 = dx^2 - \cosh \rho \cdot dt^2 + \sinh \rho (d\varphi^2 + \cos^2 \varphi d\phi_1^2 + \sin^2 \varphi d\phi_2^2)$$

$$ds_{S^2}^2 = dx^2 + \cos^2 \chi d\phi_3^2 + \sin^2 \chi (d\psi^2 + \cos^2 \psi d\phi_1^2 + \sin^2 \psi d\phi_2^2)$$

$$\Rightarrow X^M = (x, \varphi, \phi_1, \phi_2, \phi_3) \leftarrow S^5 \text{ coordinates}; \hat{X}^\mu = (t, \rho, \varphi, \phi_1, \phi_2) \leftarrow AdS_5 \text{ coordinates}$$

Or, we can parametrize S^5 and AdS_5 by embedding coordinates:

$$(\bar{x}_1)^2 + \dots + (\bar{x}_6)^2 = 1 \quad \text{in } \mathbb{R}^6 \quad \leftarrow S^5$$

$$-(\hat{x}_0)^2 + (\hat{x}_1)^2 + (\hat{x}_2)^2 + (\hat{x}_3)^2 + (\hat{x}_4)^2 - (\hat{x}_5)^2 = -1 \quad \text{in } \mathbb{R}^{2,4} \quad \leftarrow AdS_5$$

One has the relations between the sets:

$$\begin{array}{ll} \bar{x}_1 + i\bar{x}_2 = \sin\varphi \cos\varphi e^{i\varphi_1} ; & \hat{x}_1 + i\hat{x}_2 = \sinh\varphi \cos\bar{\varphi} e^{i\varphi_1} \\ \bar{x}_3 + i\bar{x}_4 = \sin\varphi \sin\varphi e^{i\varphi_2} ; & \hat{x}_3 + i\hat{x}_4 = \sinh\varphi \sin\bar{\varphi} e^{i\varphi_2} \\ \bar{x}_5 + i\bar{x}_6 = \cos\varphi \cdot e^{i\varphi_3} ; & \hat{x}_5 + i\hat{x}_0 = \cosh\varphi \cdot e^{i\varphi_3} \end{array}$$

$\underbrace{\hspace{10em}}_{S^5} \qquad \qquad \underbrace{\hspace{10em}}_{AdS_5}$

To obtain $\mathbb{R} \times S^2$ subsector, we take $\mathbb{R}$ to be AdS_5 time 't', thus, we set $g=0$, and the $S^2 \subset S^5$ is obtained by setting $\bar{x}^4 = \bar{x}^5 = \bar{x}^6 = 0$. Then it is easy to show that the action becomes:

$$S = \frac{\sqrt{\lambda}}{4\pi} \int dt d\sigma \sqrt{g} g^{\mu\nu} [\delta_{ij} \partial_\mu \bar{x}^i \partial_\nu \bar{x}^j - \partial_t \bar{x}^i \partial_\sigma \bar{x}^i]$$

where $i, j = 1 \dots 3$ and $\bar{x}^i \bar{x}^i = 1$.

Changing notations: $x^i \equiv \bar{x}^i$; $x^0 \equiv t$

$$\Rightarrow S = \frac{\sqrt{\lambda}}{4\pi} \int dt d\sigma \sqrt{g} g^{\mu\nu} [\delta_{ij} \partial_\mu x^i \partial_\nu x^j - \partial_\mu x^0 \partial_\nu x^0]$$

The equation for $g^{\mu\nu}$:

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \leftarrow \text{Virasoro constraint}$$

$$\Rightarrow \Lambda_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (g^{\mu\nu} \Lambda_{\mu\nu}) = 0$$

where $\Lambda_{\mu\nu} \equiv \delta_{ij} \partial_\mu x^i \partial_\nu x^j - \partial_\mu x^0 \partial_\nu x^0$

In the light-cone coordinates $\sigma_{\pm} = \frac{1}{2}(\tau \pm \sigma)$; $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$
 this becomes:

$$\Lambda_{++} = 0; \Lambda_{--} = 0$$

$$\text{or, } (\partial_+ x^i)^2 = (\partial_+ x^0)^2$$

$$(\partial_- x^i)^2 = (\partial_- x^0)^2$$

Using Virasoro constraints, we can choose a gauge:

$$x^0 = \kappa \tau; \quad \eta_{\mu\nu} g^{\mu\nu} = \eta_{\mu\nu}$$

Then, we obtain the action:

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau [\delta_{ij} \partial_{\alpha} x^i \partial^{\alpha} x^j]; \quad x^i x^i = 1$$

or, introducing Lagrange multiplier λ :

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau [\delta_{ij} \partial_{\alpha} x^i \partial^{\alpha} x^j + \lambda (x^i x^i - 1)]$$

The equations of motion:

$$\partial_+ \partial_- \vec{x} + (\partial_+ \vec{x} \cdot \partial_- \vec{x}) \vec{x} = 0$$

The Virasoro constraint:

$$\partial_+ \vec{x} \cdot \partial_+ \vec{x} = \partial_- \vec{x} \cdot \partial_- \vec{x} = 1$$

$\Rightarrow \partial_+ \vec{x}$ and $\partial_- \vec{x}$ are unit vectors.

$$\Rightarrow \partial_+ \vec{x} \cdot \partial_- \vec{x} = \cos \varphi$$

Then, from the equations of motion:

$$\partial_+ \partial_- \vec{x} + \cos \varphi \cdot \vec{x} = 0 \quad \text{not}$$

Let us find the equation of motion for φ .

First, we will need $\partial_+ \partial_+ \vec{x}$ in terms of $\vec{x}$, $\partial_+ \vec{x}$ and $\partial_- \vec{x}$:

$$\partial_+ \partial_+ \vec{x} = c_1 \vec{x} + c_2 \partial_+ \vec{x} + c_3 \partial_- \vec{x}$$

$$\Rightarrow a) \partial_+ \partial_+ \vec{x} \cdot \vec{x} = c_1 \vec{x}^2 + c_2 \partial_+ \vec{x} \cdot \vec{x} + c_3 \partial_- \vec{x} \cdot \vec{x}$$

$$\text{using: } \vec{x}^2 = 1 \Rightarrow \partial_+ \vec{x} \cdot \vec{x} = 0; \partial_- \vec{x} \cdot \vec{x} = 0; \Rightarrow \partial_+ \vec{x} \cdot \partial_+ \vec{x} + \partial_+ \partial_+ \vec{x} \cdot \vec{x} = 0 \\ \Rightarrow \partial_+ \partial_+ \vec{x} \cdot \vec{x} = -1$$

Therefore, $c_1 = -1$

$$b) \underbrace{\partial_+ \partial_+ \vec{x} \cdot \partial_+ \vec{x}}_{=0, \text{ since } \partial_+ \vec{x} \cdot \partial_+ \vec{x} = 1} = \underbrace{c_1 \vec{x} \cdot \partial_+ \vec{x}}_{=0} + \underbrace{c_2 \partial_+ \vec{x} \cdot \partial_+ \vec{x}}_{=1} + \underbrace{c_3 \partial_- \vec{x} \cdot \partial_+ \vec{x}}_{=\cos \varphi}$$

$$\partial_+ \vec{x} \cdot \partial_+ \vec{x} = 1$$

$$\Rightarrow c_2 + c_3 \cos \varphi = 0$$

$$c) \partial_+ \partial_+ \vec{x} \cdot \partial_- \vec{x} = c_2 \cos \varphi + c_3$$

$$\text{using: } \partial_+ \vec{x} \cdot \partial_- \vec{x} = \cos \varphi \Rightarrow \partial_+ \partial_+ \vec{x} \cdot \partial_- \vec{x} + \underbrace{\partial_+ \vec{x} \cdot \partial_+ \partial_- \vec{x}}_{\substack{=\cos \varphi \cdot \vec{x} \\ =0}} = -\partial_+ \varphi \sin \varphi$$

$$\Rightarrow \partial_+ \partial_+ \vec{x} \cdot \partial_- \vec{x} = -\partial_+ \varphi \sin \varphi$$

$$\Rightarrow c_2 \cos \varphi + c_3 = -\partial_+ \varphi \sin \varphi$$

$$\Rightarrow -c_3 \cos^2 \varphi + c_3 = -\partial_+ \varphi \sin \varphi \Rightarrow c_3(1 - \cos^2 \varphi) = -\partial_+ \varphi \sin \varphi \Rightarrow c_3 = -\frac{\partial_+ \varphi}{\sin \varphi}; c_2 = +\frac{\partial_+ \varphi \cos \varphi}{\sin \varphi}$$

So,
$$\partial_+ \partial_+ \vec{x} = -\vec{x} + \frac{\partial_+ \varphi \partial_+ \vec{x} \cdot \cos \varphi}{\sin \varphi} - \frac{\partial_+ \varphi \partial_- \vec{x}}{\sin \varphi}$$

Similarly, we get:

$$\partial_- \partial_- \vec{x} = -\vec{x} - \frac{\partial_- \varphi \partial_- \vec{x}}{\sin \varphi} + \frac{\partial_- \varphi \partial_+ \vec{x} \cos \varphi}{\sin \varphi}$$

Next, we have the obvious identity:

$$\partial_+ \partial_- \varphi = -\partial_- \left[\frac{\partial_+ (\partial_+ \vec{x} \cdot \partial_- \vec{x})}{\sin \varphi} \right] = -\partial_- \left[\frac{\partial_+ \partial_+ \vec{x} \cdot \partial_- \vec{x} + \overbrace{\partial_+ \vec{x} \cdot \partial_+ \partial_- \vec{x}}^{=0, \text{ since } \partial_- (\partial_+ \vec{x} \cdot \partial_+ \vec{x}) = 0}}{\sin \varphi} \right]$$

$\Rightarrow$

$$\partial_+ \partial_- \varphi = - \left[\frac{\partial_- (\partial_+ \vec{x} \cdot \partial_- \vec{x})}{\sin \varphi} - \frac{(\partial_+ \partial_- \vec{x} \cdot \partial_- \vec{x}) \cdot \cos \varphi \cdot \partial_- \varphi}{\sin^2 \varphi} \right]$$

or, using $\partial_+ \partial_- \vec{x} \cdot \partial_- \vec{x} = -\partial_+ \varphi \sin \varphi$ in the second term:

$$= - \left[\frac{\partial_+ (\partial_- \partial_- \vec{x}) \cdot \partial_- \vec{x} + \partial_+ \partial_- \vec{x} \cdot \partial_- \partial_- \vec{x}}{\sin \varphi} + \frac{\partial_+ \varphi \partial_- \varphi \cdot \cos \varphi}{\sin \varphi} \right]$$

$$= - \frac{1}{\sin \varphi} \left[\partial_+ \varphi \partial_- \varphi \cos \varphi + \partial_+ (-\cos \varphi \cdot \vec{x}) \cdot \partial_- \vec{x} + \partial_+ \partial_- \vec{x} \cdot \partial_- \partial_- \vec{x} \right]$$

$$= - \frac{1}{\sin \varphi} \left[\partial_+ \varphi \partial_- \varphi \cos \varphi - \cos^2 \varphi + \partial_+ \partial_- \vec{x} \cdot \partial_- \partial_- \vec{x} \right]$$

Using expressions for $\partial_+ \partial_- \vec{x}$ and $\partial_- \partial_- \vec{x}$ (p. 14), we obtain:

$$\partial_+ \partial_- \vec{x} \cdot \partial_- \partial_- \vec{x} = 1 - \partial_+ \varphi \partial_- \varphi \cos \varphi$$

$\Rightarrow$

$$\partial_+ \partial_- \varphi = - \frac{1}{\sin \varphi} \left[\cancel{\partial_+ \varphi \partial_- \varphi \cos \varphi} - \cos^2 \varphi + 1 - \cancel{\partial_+ \varphi \partial_- \varphi \cos \varphi} \right] = -\sin \varphi$$

So,

$$\boxed{\partial_+ \partial_- \varphi + \sin \varphi = 0}$$

Thus, we obtain the sine-Gordon equation - a classic example of integrable system! We can write corresponding action:

$$S_{SG} = (\text{const}) \int dx dt \left[\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + (\cos \varphi - 1) \right]$$

The exact classical solutions are known and the quantum theory is also integrable, in the sense which will be defined more precisely in the rest of lectures.

Classical integrability: Arnold-Liouville theorem

Motivation

Consider an-dimensional phase-space: (p_i, q_i) ; $i, j = 1 \dots n$. The Hamiltonian equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

where $H = H(p_i, q_i)$ is the Hamiltonian. There exists a special subclass of systems, called Liouville integrable systems, for which the explicit solutions of the Hamilton equations can be constructed by quadratures:

- a) Solving a finite number of algebraic equations
- b) Evaluating a finite number of integrals

The general idea is to find a change of variables

$$(p_i, q_i) \rightarrow (F_i, \theta_i); \quad H(p_i, q_i) \rightarrow \tilde{H}(F_i)$$

such that the equations of motion become trivial:

$$F_i = \text{const}; \quad \theta_i = \omega_i t + (\theta_i)_0$$

The variables (F_i, θ_i) are called the action-angle variables.

The Liouville theorem provides the conditions and the concrete procedure to solve the system in quadratures. The main requirement is the existence of n commuting and independent functions

$$F_i: \{F_i, F_j\} = 0 \quad (\text{integrals of motion in involution})$$

To make it more precise, let us write the Hamilton equations in the form:

$$\dot{x} = J \cdot \nabla H$$

where

$$x \equiv \begin{pmatrix} p_i \\ q_j \end{pmatrix}; \quad \nabla H \equiv \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial q_j} \end{pmatrix}; \quad J \equiv \begin{pmatrix} 0 & -\mathbb{1}_{n \times n} \\ \mathbb{1}_{n \times n} & 0 \end{pmatrix} \Rightarrow J^2 = -\mathbb{1}_{2n \times 2n}$$

The phase-space M now are the vectors $x = (x^1 \dots x^{2n}) \in M$.

The Poisson brackets are defined on the space of differentiable functions on M : $\mathcal{F}(M)$:

$$\{F, G\}(x) \equiv \sum_{j=1}^n \left(\frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} \right) = J^{ij} \partial_i F \partial_j G = (\nabla F, J \nabla G)$$

where (x, y) is the usual scalar product in $\mathbb{R}^{2n}$: $(x, y) \equiv \sum_{i=1}^{2n} x^i y^i$

The Poisson bracket satisfies several rules:

- 1) $\{F, G\} = -\{G, F\}$ (anti-symmetry)
- 2) $\{F, \alpha G + \alpha' G'\} = \alpha \{F, G\} + \alpha' \{F, G'\}$ (linearity)
- 3) $\{F, G H\} = G \{F, H\} + \{F, G\} H$ (Leibnitz rule)
- 4) $\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0$ (Jacobi identity)

Using the Leibnitz rule, any bracket $\{F, G\}$ can be reduced to calculating:

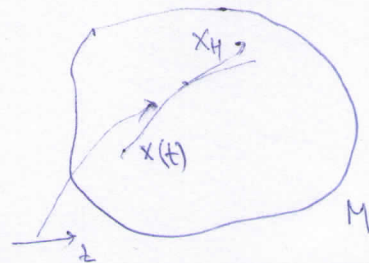
$$\{x^i, x^j\} = J^{ij} \quad \Leftrightarrow \quad \{p_i, p_j\} = 0; \quad \{q_i, q_j\} = 0; \quad \{p_i, q_j\} = \delta_{ij}$$

We can now write the Hamilton equations in the form:

$$\dot{x}^K = \{H, x^K\}$$

We can also think about this as the flow on M determined by the vector field $X_H \equiv \{H, \ast\}$:

$$\dot{x} = X_H(x)$$



For an arbitrary function $F = F(p_i, q_i)$:

$$\dot{F} = \{H, F\}$$

$$\Rightarrow \dot{H} = \{H, H\} = 0$$

Thus, the Hamiltonian is an integral of motion.

Let's introduce the following bi-linear skew-symmetric form:

$$\omega(x, y) \equiv (x, J^{-1}y) ; \quad J^{-1} = -J$$

Consider change of variables:

$$y^j = f^j(x^k)$$

$$\Rightarrow \dot{y}^j = \underbrace{\frac{\partial y^j}{\partial x^k}}_{\equiv A^j_k} \dot{x}^k = A^j_k J^{km} \nabla_m^x H = A^j_k J^{km} \frac{\partial y^p}{\partial x^m} \nabla_p^y H = A^j_k J^{km} A^p_m \nabla_p^y H$$

or, in matrix form:

$$\dot{y} = A J A^t \cdot \nabla_y H$$

In order to obtain Hamilton equations for y : $\dot{y} = J \cdot \nabla \tilde{H}$
we identify:

$$A J A^t = J ; \quad \tilde{H}(y) = H(x(y))$$

The transformations that satisfy this condition are called canonical.

If 'A' does not depend on x , then $A J A^t = J$ is the real symplectic group $Sp(2n, \mathbb{R})$. Under canonical transformations:

$$\omega(Ax, Ay) = (Ax, J^{-1}Ay) = -(Ax, JAy) = -(x, A^t J A y) = -(x, Jy) = \omega(x, y)$$

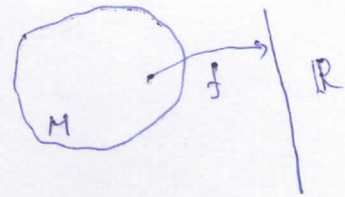
Thus, ω is invariant under canonical transformations.

The ω is known as symplectic form.

When the phase-space is not $\mathbb{R}^{2n}$, and is some manifold, we need to use the language of differential geometry to formulate the Hamiltonian formalism. Together with symplectic form this gives rise to symplectic geometry. Here we give most basic definitions.

Symplectic geometry

Let M be the phase-space manifold; and $F(M)$ is set of all differentiable functions $f: M \rightarrow \mathbb{R}$. Define the Poisson bracket: $\{ \cdot, \cdot \}: F(M) \otimes F(M) \rightarrow F(M)$, so that it satisfies the conditions:



1. $\{f_1, f_2\} = -\{f_2, f_1\}$ (anti-symmetry)
2. $\{f_1, \alpha f_2 + \beta f_3\} = \alpha \{f_1, f_2\} + \beta \{f_1, f_3\}$ (linearity)
3. $\{f_1, f_2 f_3\} = f_2 \{f_1, f_3\} + \{f_1, f_2\} f_3$ (Leibnitz rule)
4. $\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$ (Jacobi identity)

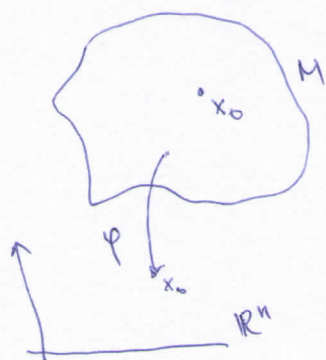
A manifold on which a Poisson bracket is defined is called Poisson manifold.

Tangent vector

Given a smooth manifold M , a tangent vector v_x in the point x is a linear function from $F(M)$ which satisfies the Leibnitz rule:

1. $v_x(\alpha f + \beta g) = \alpha v_x(f) + \beta v_x(g)$
2. $v_x(fg) = f(x) v_x(g) + g(x) v_x(f)$

In local coordinates:



$v_x(\varphi^i) = v^i$ (components of the tangent vector in coordinates)

where $\varphi^i : \varphi(x_0) = (x^1, \dots, x^n)$; $\varphi^i : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\varphi^i(x^1, \dots, x^n) = x^i$.

Then, writing: $f(x) = f(\varphi^i(x_0)) + [\varphi^i(x) - \varphi^i(x_0)] \frac{\partial f(\varphi^i(x))}{\partial x^i} \Big|_x$
where $\bar{x} = \varphi(x_0) + [\varphi(x) - \varphi(x_0)] \cdot s$; $0 < s < 1$

$$\Rightarrow v_{x_0}(f) = 0 + v_{x_0}(\varphi^i) \frac{\partial f(\varphi^i(x))}{\partial x^i} \Big|_{\varphi(x_0)} = v^i \frac{\partial f}{\partial x^i}$$

So, the coordinate representative of v_x is $v^i \frac{\partial}{\partial x^i}$.

The space $T_x M$ of tangent vectors at point x with addition and scalar multiplication:

$$(\alpha u_x + \beta v_x)(f) = \alpha u_x(f) + \beta v_x(f)$$

is turned into a vector space, known as tangent space.

The basis in the local coordinates in $T_x M$ is:

$$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n} \right); \neq 17 \text{ ~~scribbles~~}$$

$\Rightarrow$ The dimension of the space $T_x M$: $\dim T_x M = n$

Assigning to each point $x \in M$ a tangent vector: $x \rightarrow v_x$

gives rise to a vector field on M .

The Poisson bracket $\{*,*\}$, due to the Leibnitz rule

and linearity, gives a tangent vector at each point.

This is clear if we take some fixed function $H \in \mathcal{F}(M)$ and consider:

$$v_H(f) = \{H, f\}$$

This defines a vector field.

A Hamiltonian system is a triple: $(M, \{\cdot, \cdot\}, H)$, where M is the phase-space manifold, $\{\cdot, \cdot\}$ is the Poisson bracket, and H is the Hamiltonian. Define the time evolution:

$$\dot{f} = v_H(f) = \{H, f\}$$

$\Rightarrow$

- 1.) $\{H, f\} = 0$ implies $\dot{f} = 0$, thus f is a conserved quantity
- 2.) If $\{H, f_1\} = 0$; $\{H, f_2\} = 0$, then the Jacobi identity implies that $\{f_1, f_2\}$ is conserved.

$$3.) \text{ If } X_{H_1} = \{H_1, \cdot\}; X_{H_2} = \{H_2, \cdot\}; \Rightarrow [X_{H_1}, X_{H_2}] = -X_{\{H_1, H_2\}}$$

Remembering that in local coordinates: $v_x = v^i \frac{\partial}{\partial x^i}$ and the Poisson bracket defines a tangent vector in each variable, one can represent:

$$\{f_1, f_2\}(x) = \sum_{i,j} J^{ij}(x) \frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial x^j}$$

The properties 1-4. are satisfied if $J^{ij} = -J^{ji}$ and:

$$J^{is} \partial_s J^{jk} + J^{ks} \partial_s J^{ij} + J^{js} \partial_s J^{ki} = 0; \quad (\partial_s \equiv \partial_{x^s})$$

From the definition it follows that:

$$\{x^i, x^j\} = J^{ij} \quad (\text{compare with p. 17})$$

We also need to distinguish degenerate and non-degenerate Poisson brackets. The degenerate Poisson bracket is the one for which there are functions f_q such that $\{f_q, g\} = 0$ for all g . This is equivalent to: $\det(J^{ik}) = 0$.

The set of f_q functions is called the center of the Poisson algebra. To make the Poisson bracket non-degenerate we set:

$$f_{q_1} = \text{const}; f_{q_2} = \text{const}, \text{ etc}$$

This defines a sub-manifold on which the Poisson bracket is non-degenerate, and where $\det(J^{ik}) \neq 0$.

For non-degenerate Poisson bracket we can consider the inverse matrix:

$$(\bar{J}^{-1})_{ij} (J)^{jk} = \delta_j^k \Rightarrow \partial_s (\bar{J}^{-1})_{ij} J^{jk} + (\bar{J}^{-1})_{ij} \partial_s (J^{jk}) = 0$$

$$\Rightarrow \partial_s J^{ij} = -J^{ik} \partial_s (\bar{J}^{-1})_{km} J^{mj}$$

Using the second condition on J^{km} (page 21), we obtain:

$$\partial_i (\bar{J}^{-1})_{km} + \partial_m (\bar{J}^{-1})_{ik} + \partial_k (\bar{J}^{-1})_{mi} = 0$$

This is equivalent to the condition: $d\omega = 0$, where ω is the symplectic form:

$$\omega = - \sum_{i < j} (\bar{J}^{-1})_{ij} dx^i dx^j$$

A symplectic manifold (M, ω) is a manifold M together with a non-degenerate closed 2-form ω : $d\omega = 0$.

Differential forms

Given a vector space V , consider the set of linear functions $f: V \rightarrow \mathbb{R}$. This set forms a vector space with the addition:

$$(\alpha_1 f_1 + \alpha_2 f_2)(v) = \alpha_1 f_1(v) + \alpha_2 f_2(v)$$

This is the dual vector space V^* .

If e_j is basis in V , and e^i is the basis in V^* , we can choose: $e^i(e_j) = \delta_j^i$

$\Rightarrow$

$$f = f_i e^i; v = v^i e_i; \Rightarrow f(v) = f_i v^j e^i(e_j) = f_i v^i \equiv \langle f, v \rangle$$

Take the vector space to be the tangent space $T_x M$.

The dual space is denoted as $T_x^* M$. Thus, the element $\omega \in T_x^* M$ maps $T_x M \xrightarrow{\omega} \mathbb{R}$. The element ω is called 1-form.

$$\Rightarrow \omega = \omega_j e^j; v = v^i e_i \Rightarrow \omega(v) = \omega_i v^i$$

Let's define 1-form $\omega = df$, which in local coordinates takes the form:

$$\omega = df = \frac{\partial f}{\partial x^i} dx^i \Rightarrow \langle df, v \rangle \equiv df(v) \equiv v(f) = v^i \frac{\partial f}{\partial x^i} = \omega_i v^i$$

$\uparrow$
(see p.20)

Thus, $\omega = \omega_i dx^i$; $\omega_i = \frac{\partial f}{\partial x^i}$ are the components of ω , and dx^i is the basis (in the local coordinates). Generally, we write:

$$\omega = \omega_i dx^i \Rightarrow \text{taking } f = x^i; v = \frac{\partial}{\partial x^j} :$$

$$\langle df, v \rangle = v(f) = \frac{\partial x^i}{\partial x^j} = \delta_j^i = df(v) = dx^i \left(\frac{\partial}{\partial x^j} \right)$$

Tensor of the type (p, q) is a map: $\otimes^p T_x^* M \otimes^q T_x M \rightarrow \mathbb{R}$

A differential form of order 'r'

$\uparrow$ p-times $\uparrow$ q-times

is a totally anti-symmetric tensor

of the type $(0, r)$. The basis in local coordinates:

$$dx^{i_1} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_r}$$

($\otimes$ means anti-symmetrization over $i_1, \dots, i_r$)

This is denoted as:

$$dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_r}$$

Any differential form is written as: $\omega = \omega_{i_1, \dots, i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$

The exterior derivative is defined as:

$$d\omega = \frac{\partial}{\partial x^i} \omega_{i_1, \dots, i_r} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_r}$$

For $M = \mathbb{R}^{2n}$ we have $J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \Rightarrow J^T = -J$, and the symplectic form:

$$\omega = -\sum_{i,j} (J^{-1})_{ij} dx^i \wedge dx^j = \sum_{i,j} (J)_{ij} dx^i \wedge dx^j = \sum_k dp_k \wedge dq^k$$

The inverse theorem is known as Darboux theorem:

On any symplectic manifold (M, ω) there exist local (p_i, q_i) coordinates around any point x_0 , so that $\omega = \sum_i dp_i \wedge dq^i$.

Let us also note, that

$$X_H \equiv \{H, \cdot\} = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

Decompose:

$$X_H = X_H^i \frac{\partial}{\partial q^i} \Rightarrow X_H^i \frac{\partial}{\partial q^i} f = X_H(f) = \{H, f\} = \sum_{km} J^{km} \frac{\partial H}{\partial q^k} \frac{\partial f}{\partial q^m}$$

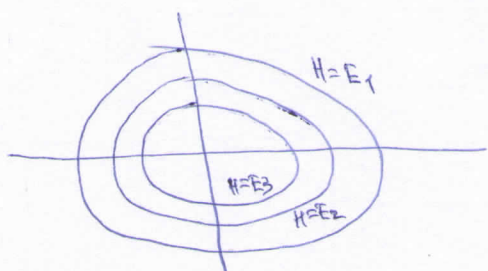
$$\Rightarrow X_H^i = \sum_k J^{ki} \frac{\partial H}{\partial q^k} \Rightarrow \frac{\partial H}{\partial q^m} = X_H^i J_{im}^{-1} \Rightarrow dH = -\omega(X_H, \cdot)$$

Arnold-Liouville theorem

The following simplest example demonstrates the essence of the general theorem. Consider 1-dimensional harmonic oscillator:

$$H = \frac{1}{2}(p^2 + \omega^2 q^2); \{p, q\} = 1$$

Since the energy is conserved: $\dot{H} = 0$, the entire phase-space $\mathbb{R}^2$ is foliated by the ellipses $H = E$:



Parametrize:

$$p = g \cos \theta; \quad q = \frac{g}{\omega} \sin \theta$$

$$\begin{aligned} \Rightarrow \{p, q\} &= 1 = \{g \cos \theta, \frac{g}{\omega} \sin \theta\} = g \{ \cos \theta, \frac{g}{\omega} \sin \theta \} + \cos \theta \{ g, \sin \theta \} \frac{g}{\omega} \\ &= -g \sin \theta \{ \theta, g \} \frac{g}{\omega} \sin \theta + \cos^2 \theta \{ g, \theta \} \frac{g}{\omega} = \frac{g}{\omega} \{ g, \theta \} \end{aligned}$$

$$\Rightarrow \{g, \theta\} = \frac{\omega}{g}$$

The Hamiltonian becomes: $H = \frac{1}{2} g^2$

The equations of motion take the form:

$$\dot{g} = \{H, g\}; \quad \dot{\theta} = \{H, \theta\} = \omega$$

$$\Rightarrow g = \text{const}; \quad \theta = \omega t + \theta_0$$

So, the dynamics is restricted to the ellipse $g = \text{const}$, and (g, θ) are the action-angle variables.

More generally, consider n harmonic oscillators:

$$H = \sum_{i=1}^n \frac{1}{2} (p_i^2 + \omega_i^2 q_i^2)$$

There are n conserved commuting quantities:

$$F_i = \frac{1}{2} (p_i^2 + \omega_i^2 q_i^2); \quad \{F_i, F_j\} = 0$$

The level manifold: $M_f = \{x \in M; F_i = f_i, i=1 \dots n\}$
 $\uparrow$ constants

By the implicit function theorem M_f is a n -dimensional manifold, isomorphic to n -dimensional torus: $\underbrace{S^1 \times \dots \times S^1}_n$. Each

torus is parametrized by (g_i, θ_i) . The motion is conditionally periodic: if for all i, j : $\omega_i/\omega_j = \text{rational number}$, then the trajectory is closed; otherwise it is dense on torus.

Arnold-Liouville theorem:

Definition: The system is Liouville integrable if there exist n independent conserved quantities $F_i, i=1, \dots, n: \{H, F_i\}=0$, which are in involution $\{F_i, F_j\}=0$.

"Independent" means that dF_i are linearly independent, therefore, by the implicit function theorem the level manifold

$M_f: \{F_i = f_i; i=1, \dots, n\}$ is an n -dimensional manifold.

Note, that the Hamiltonian H should be a function of $F_1, \dots, F_n$. The vector fields:

$$X_{F_i} = \{F_i, * \}$$

commute, because: $[X_{F_i}, X_{F_j}] = -X_{\{F_i, F_j\}} = 0$

Theorem: If there are n independent functions F_i in involution on a symplectic $2n$ -dimensional manifold, then:

1. The equations of motion can be obtained by quadratures.
2. If M_f is compact and connected, then it is diffeomorphic to a torus $T^n = S^1 \times \dots \times S^1$
3. The trajectory in the phase-space is conditionally periodic: there are angle-action variables on $M_f (F_i, \phi_i)$ such that

$$\frac{d\phi_i}{dt} = \omega_i; \quad \omega_i = \omega_i(F_j)$$

Proof

Let's consider the 1-form: $\alpha = \sum_i p_i dq_i$ so that the symplectic form: $\omega = d\alpha = \sum_i dp_i \wedge dq_i$. Note that we use here the Darboux theorem, which is not valid globally. Consider a canonical transformation:

$$(p_i, q_i) \rightarrow (F_i, \psi_i)$$

This means that ω is invariant:

$$\omega = \sum_i dq_i \wedge dp_i = \sum_i dF_i \wedge d\psi_i$$

If such transformation exists then:

$$\dot{F}_j = \langle H, F_j \rangle = 0; \quad \dot{\psi}_j = \langle H, \psi_j \rangle = \frac{\partial H}{\partial \psi_j} \equiv \omega_j(F_i)$$

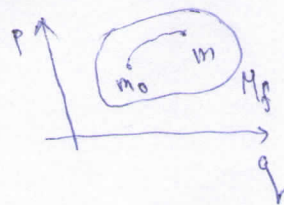
Therefore, $\omega_j(F_i)$ are constants, and we have a solution:

$$F_j(t) = F_j(0); \quad \psi_j(t) = \omega_j \cdot t + \psi_j(0)$$

Now we show how to construct such transformation. Consider

$M_f: F_i(p, q) = f_i$ and solve for $p_i: p_i = p_i(f, q)$. This is possible by the implicit function theorem. Introduce the generating function S :

$$\cancel{S(f, q)} \quad S(f, q) = \int_{m_0}^m \alpha = \int_{q_0}^q \sum_i p_i(f, q_i) dq_i$$



Here $m_0, m \in M_f$.

If S does not depend on the $m_0 \rightarrow m$ path then

$$p_i = \frac{\partial S}{\partial q_i};$$

Define:

$$\psi_j = \frac{\partial S}{\partial f_j}$$

Therefore, we can write:

$$ds = \sum_j \left[\frac{\partial S}{\partial q_j} dq_j + \frac{\partial S}{\partial f_j} df_j \right] = \sum_j [p_j dq_j + \psi_j df_j]$$

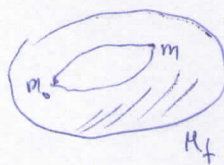
Since $d^2S=0$, $\Rightarrow$

$$\omega = \sum_j dp_j \wedge dq_j = \sum_j df_j \wedge d\psi_j$$

This means that S generates a canonical transformation.

Finally, we need to show that S does not depend on the path $m_0 \rightarrow m$. If M_f does not have any non-trivial cycles (i.e., curves can be continuously deformed one into another); then, using Stokes theorem:

$$\Delta S = \int_{m_0}^{m_0} \alpha = \int_{M_f} d\alpha = \int_{M_f} \omega$$



Let us show that $\omega=0$ on M_f .

The vector fields $x_{F_i} = \{F_i, *\}$ are tangent to M_f : consider an arbitrary point $x \in M$, then $x_{F_i}(f) \rightarrow \mathbb{R}$. But,

$$x_{F_i}(F_j) = \{F_i, F_j\} = 0 \quad ; \quad \text{for } x \text{ is orthogonal to normal vectors of } M_f$$

$\Rightarrow F_i = \text{const}$, which defines the level manifold M_f . Therefore, x_{F_i} are tangent to M_f . Since F_i are independent, x_{F_i} form a basis. Then, for any two vectors $v, v' \in M_f$:

$$\omega(v, v') = v^i v'^j \omega(x_{F_i}, x_{F_j}) = -v^i v'^j dF_i(x_{F_j}) = -v^i v'^j x_{F_j}(F_i) = 0$$

$\uparrow$ use relation on p. 24 $\uparrow$ see p. 23

Therefore $\omega=0$ on M_f and $\Delta S=0$.

However, in this case the level manifold M_f is diffeomorphic to n -dimensional torus $T^n = S^1 \times \dots \times S^1$, which has n cycles. For every cycle the change in S has the form:

$$\Delta S = \int_{\text{cycle}} \alpha$$

and is a function of F_i only.

For each fundamental cycle of the torus:

Define the action variables:

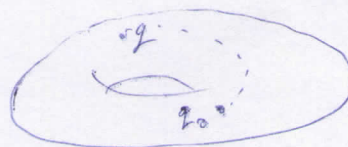


$$I_j = \frac{1}{2\pi} \oint_{C_j} p_i(q, I) dq_i = \frac{1}{2\pi} \oint_{C_j} \alpha$$

Since I_j depend only on f_i , they are conserved in time. The angle variables are constructed as follows:

Consider a generating function $S = S(I_i, q_i)$:

$$S(I, q) = \int_{q_0}^q \alpha = \int_{q_0}^q p_i(q', I) dq'_i$$



where q_0 is some fixed point.

Then,

$$p_j = \frac{\partial S}{\partial q_j} \Rightarrow p = p(q, I)$$

Define:

$$\theta_j = \frac{\partial S}{\partial I_j} \Rightarrow \theta = \theta(q, I) \quad ; \quad \Rightarrow ds = \sum_k \left[\frac{\partial S}{\partial q_k} dq_k + \frac{\partial S}{\partial I_k} dI_k \right]$$

Then consider

$$\frac{1}{2\pi} \oint_{C_j} d\theta_i = \frac{1}{2\pi} \oint_{C_j} d\left(\frac{\partial S}{\partial I_i}\right) = \frac{1}{2\pi} \frac{\partial}{\partial I_i} \oint_{C_j} ds = \frac{1}{2\pi} \frac{\partial}{\partial I_i} \oint_{C_j} \sum_k \left[\frac{\partial S}{\partial q_k} dq_k + \frac{\partial S}{\partial I_k} dI_k \right]$$

$\underbrace{\hspace{10em}}_{=0 \text{ on } M_f}$

$$= \frac{\partial}{\partial I_i} \left(\frac{1}{2\pi} \oint_{C_j} \alpha \right) = \frac{\partial I_j}{\partial I_i} = \delta_{ij}$$

Thus,

$$\frac{1}{2\pi} \oint \frac{d\theta_i}{c_i} = \delta_{ij}$$

which shows that θ_i are angle variables.

Finally, let's show that transformation $(p_i, q_i) \rightarrow (I_i, \theta_i)$ is canonical:

$$\omega = \sum_i dp_i \wedge dq_i = d\left(\sum_i p_i dq_i\right) = d\left(\sum_i \left[ds - \frac{\partial s}{\partial I_i} dI_i\right]\right)$$

$$= -d\left(\sum_i \frac{\partial s}{\partial I_i} dI_i\right) = -d\left(\sum_i \theta_i dI_i\right) = \sum_i dI_i \wedge d\theta_i$$

Let us outline the proof of the fact that M_f is diffeomorphic to a n -torus. We will need a few concepts:

1. An action of a group G on a set S is a map

$$G \times S \rightarrow S$$

2. The orbit of a point $s \in S$ is the set of points:

$$\mathcal{O}_s = \{gs \mid g \in G\}$$

3. Stabilizer of s is:

$$\text{stab}(s) = \{g \in G \mid gs = s\}$$

4. Orbit-stabilizer theorem:

$$\mathcal{O}_p \cong G/\text{stab}(p)$$

We have n commuting vectors X_i on M_f , which generates abelian $\mathbb{R}^n$ group action on M_f . Denote Γ the lattice of all $\mathbb{R}^n$ vectors that leave a point $p \in M_f$ invariant. Γ is a discrete sub-group of $\mathbb{R}^n$ and physical phase-space M_f corresponds to the orbit:

$$M_f = \mathbb{R}^n / \Gamma$$

which gives the n -dimensional torus $T^n = S^1 \times \dots \times S^1$, if M_f is compact

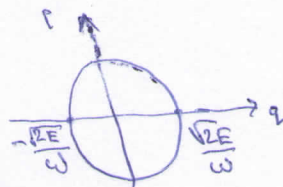
Applications of the Liouville Theorem

Example #1

For harmonic oscillator:

$$E = \frac{1}{2}(p^2 + \omega^2 q^2) \Rightarrow p(E, q) = \pm \sqrt{2E - \omega^2 q^2}$$

$$\Rightarrow I = \frac{1}{2\pi} \oint_E dq \sqrt{2E - \omega^2 q^2} = \frac{2}{2\pi} \int_{-\sqrt{2E}/\omega}^{\sqrt{2E}/\omega} dq \sqrt{2E - \omega^2 q^2} = \frac{E}{\omega}$$



Construct the generating functional:

$$S(I, q) = \int_{q_0}^q p(q, I) dq = \int_{q_0}^q \sqrt{2I\omega - \omega^2 q^2} dq$$

(we chose + sign in p)
(Note, that $\pm$ correspond to periodicity of the θ coordinate).

$$\Rightarrow \theta = \frac{\partial S}{\partial I} = \int \frac{\omega dq}{\sqrt{2I\omega - \omega^2 q^2}} = \arcsin\left(q \sqrt{\frac{\omega}{2I}}\right) - \theta_0$$

This is the angle variable

$$\Rightarrow q = \sqrt{\frac{2I}{\omega}} \sin(\theta + \theta_0) ; \Rightarrow p = \sqrt{2E} \cos(\theta + \theta_0)$$

where $\dot{\theta} = \frac{\partial H(I)}{\partial I} = \omega \Rightarrow \theta(t) = \omega t + \theta_0$

The area of the ellipse:

$$A = \pi \cdot \left(\frac{q}{\omega}\right) \cdot \left(\frac{p}{\omega}\right) ; q = \sqrt{2E}$$

or, $A = \frac{2\pi}{\omega} E$

$$\Rightarrow \frac{dA}{dE} = \frac{2\pi}{\omega} = T \text{ (period of the motion)}$$

Example #2: The Kepler two-body problem

The equations of motion in the center of mass frame:

$$\frac{d^2 x_i}{dt^2} = - \frac{\partial V(r)}{\partial x_i}; \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

where $V(r)$ is any centrally-symmetric potential

The Hamiltonian:

$$H = \frac{1}{2} \sum_{i=1}^3 p_i^2 + V(r); \quad \{p_i, x_j\} = \delta_{ij}$$

The angular momentum $\vec{J} = (J_1, J_2, J_3)$; $J_{ij} \equiv \epsilon_{ijk} J_k = x_i p_j - x_j p_i$ is conserved, but J_i 's do not commute:

$$\{J_i, J_j\} = -\epsilon_{ijk} J_k$$

The independent commuting quantities are:

$$H; \quad J_3 = J_{12}; \quad J^2 = J_1^2 + J_2^2 + J_3^2$$

Introduce spherical coordinates:

$$x_1 = r \sin \theta \cos \phi; \quad x_2 = r \sin \theta \sin \phi; \quad x_3 = r \cos \theta$$

and find the 1-form

$$\alpha = \sum_i p_i dx_i = p_r dr + p_\theta d\theta + p_\phi d\phi.$$

Write:

$$\mathcal{L} = \frac{1}{2} (\dot{r}^2 + r^2 \sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2) - V(r) \quad (\text{Lagrangian})$$

$$\Rightarrow H = \frac{1}{2} (p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{r^2 \sin^2 \theta} p_\phi^2) + V(r) \quad (\text{Hamiltonian})$$

Writing:

$$\alpha = p_1 dx^1 + p_2 dx^2 + p_3 dx^3 = p_r dr + p_\theta d\theta + p_\phi d\phi$$

and using:

$$dx_1 = dr \sin\theta \cos\phi + r \sin\theta \cos\phi d\theta - r \sin\theta \sin\phi d\phi$$

etc

one finds:

$$\begin{pmatrix} p_r \\ p_\theta \\ p_\phi \end{pmatrix} = \underbrace{\begin{pmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ r \cos\theta \cos\phi & r \cos\theta \sin\phi & -r \sin\theta \\ -r \sin\theta \sin\phi & r \sin\theta \cos\phi & 0 \end{pmatrix}}_M \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}; \quad M^{-1} \begin{pmatrix} p_r \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$\Rightarrow$

$$p_1 = \frac{1}{r} \left[r p_r \cos\theta \cos\phi + p_\theta \cos\theta \cos\phi - p_\phi \frac{\sin\phi}{\sin\theta} \right]$$

$$p_2 = \frac{1}{r} \left[r p_r \sin\theta \sin\phi + p_\theta \sin\theta \sin\phi + p_\phi \frac{\cos\phi}{\sin\theta} \right]$$

$$p_3 = p_r \cos\theta - \frac{1}{r} p_\theta \sin\theta$$

Then, one finds:

$$J^2 = p_\theta^2 + \frac{1}{\sin^2\theta} p_\phi^2; \quad J_3 = p_\phi$$

Using the Liouville theorem, on M_4 we have:

$$p_r = \sqrt{2(H - V) - J^2/r^2}; \quad p_\theta = \sqrt{J^2 - J_3^2/\sin^2\theta}; \quad p_\phi = J_3,$$

thus, we have separated variables: $p_r = p_r(r); p_\theta = p_\theta(\theta); p_\phi = p_\phi(\phi)$

The generating functional:

$$S = \int p_r dr + \int p_\theta d\theta + \int p_\phi d\phi$$

$\Rightarrow$ The angle variables:

$$\psi_H = \frac{\partial S}{\partial H}; \quad \psi_{J^2} = \frac{\partial S}{\partial J^2}; \quad \psi_{J_3} = \frac{\partial S}{\partial J_3}$$

From the equations of motion:

$$\dot{\psi}_i = \langle H, \psi_i \rangle = \frac{\partial H}{\partial F_i}$$

$$\Rightarrow \dot{\psi}_H = 1; \quad \dot{\psi}_{J^2} = 0; \quad \dot{\psi}_{J_3} = 0$$

$$\Rightarrow \psi_H = t - t_0; \quad \psi_{J^2} = \text{const}; \quad \psi_{J_3} = \text{const}$$

and we obtain the Kepler's formula:

$$t - t_0 = \int \frac{dr}{\sqrt{2(H - V) - J^2/r^2}}$$

From $\dot{\psi}_{J_3} = 0$

$$\Rightarrow \dot{\phi} = \frac{J_3}{\sin^2 \theta \sqrt{J^2 - J_3^2/\sin^2 \theta}} \cdot \dot{\theta}$$

If we fix $\vec{J} = (0, 0, J)$, we can also find:

$$\psi_{J_3} = 0 = - \int \frac{J dr}{r^2 \sqrt{2(H - V) - J^2/r^2}} + \phi$$

$$\Rightarrow \phi = \int \frac{J dr}{r^2 \sqrt{2(H - V) - J^2/r^2}}$$

Classical integrability: inverse scattering method

We would like to construct action-angle variables for infinite-dimensional systems. This will require developing the inverse scattering method.

Before considering infinite-dimensional systems, let us comment on two important point, regarding finite-dimensional systems, which unlike the Arnold-Liouville theorem, have counterparts in infinite-dimensional systems.

1. Lax pair

The Liouville theorem does not provide means to find the conserved quantities F_i . This can be done by recasting the Hamilton equations in the Lax pair form.

Let $L = L(p, q)$ and $M = M(p, q)$ be two matrices of some dimension, such that the Lax equation:

$$\frac{dL}{dt} = [M, L]$$

is equivalent to Hamilton equations. Then the conserved quantities are:

$$F_k = \text{tr } L^k$$

This is easy to check:

$$\dot{F}_k = k \text{tr}(L^{k-1} \dot{L}) = k \text{tr}([M, L] L^{k-1}) = k \text{tr}([M, L^k]) = 0$$

In practice, one needs to check independence of F_k quantities.

Example: Harmonic oscillator

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}; \quad M = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix}$$

$$\Rightarrow F_1 = H = \text{tr}(L^2/4)$$

2. Classical r-matrix

The existence of the r-matrix is equivalent to the requirement that the quantities $\text{tr}(L^k)$ are in involution. More precisely, the following theorem can be easily proven:

Theorem: The involution property of the quantities $\text{tr}(L^k)$ is equivalent to the existence of the r-matrix, such that:

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]$$

where $L_1 \equiv L \otimes 1$; $L_2 \equiv 1 \otimes L$; and r_{12}, r_{21} are matrices constructed from L and u, Λ matrices, where u diagonalizes L : $L = u \Lambda u^{-1}$. The Jacobi identity puts an additional constraint on r-matrix, which in the case r is independent of dynamical variables, takes the form:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0$$

When $r_{12} = -r_{21}$ this is called classical Yang-Baxter equation

Inverse scattering method

Let us consider the sine-Gordon model (see lecture 1); the action has the form:

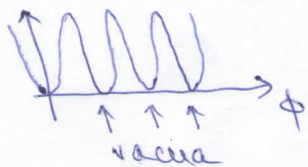
$$S = \int dx \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{\mu^2}{g^2} \cos g\phi(x) \right]$$

The potential:

$$V(\phi) = \frac{\mu^2}{g^2} [1 - \cos(g\phi)]$$

where the constant term is added for convenience.

The minima correspond to the values:



$$\phi = 2\pi n/g; \quad (n=0, \pm 1, \pm 2, \dots)$$

If we expand: $V(\phi) \approx \frac{\mu^2}{2} \phi^2 - \frac{\lambda \phi^4}{4!} + \dots$; $\lambda \equiv (\mu g)^2$

this gives usual ϕ^4 -theory with mass of the particle, corresponding to ϕ : $m = \mu^2$. There are however non-trivial, non-perturbative excitations.

Let's first introduce the topological charge. Since the current $J^\mu \equiv \frac{1}{2\pi g} \epsilon^{\mu\nu} \partial_\nu \phi$ is conserved ($\partial_\mu J^\mu = 0$), we can construct

the corresponding conserved charge:

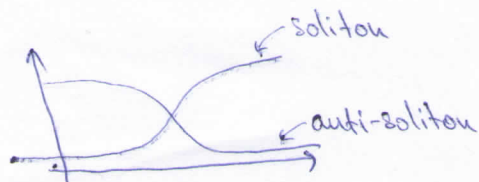
$$Q \equiv \int_{-\infty}^{\infty} J^0 dx = \frac{1}{2\pi g} \int_{-\infty}^{\infty} dx \frac{\partial \phi}{\partial x} \equiv n_1 - n_2$$

where we take $\phi(x)$ at $x \rightarrow \pm\infty$ to correspond the vacua $2\pi n_1/g$ and $2\pi n_2/g$.

It is easy to check that

$$\bar{\phi}(x) = \pm \frac{4}{g} \arctan [e^{m(x-x_0)}]$$

is a static solution (independent of time) of the equations of motion. The $\oplus$ sign corresponds to $Q=1$, and $\ominus$ sign corresponds to $Q=-1$: these are the soliton and anti-soliton:

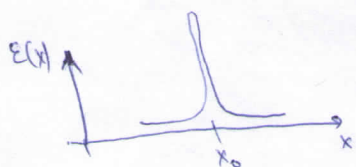


One can also find the energy of a configuration $\phi(x,t)$:

$$E[\phi] = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 + V(\phi) \right] dx$$

For the soliton and anti-soliton, we have:

$$E[\bar{\phi}] = \int_{-\infty}^{\infty} dx \mathcal{E}(x); \quad \mathcal{E}(x) = \frac{4\mu^2}{g^2} \frac{1}{\cosh^2 m(x-x_0)}$$



So, the energy density has a wave-packet shape, localized at x_0 . So, the soliton and anti-soliton can be interpreted as particle excitations.

We can find the mass $E = M_s = 2\mu^2/g^2$. The mass is inversely proportional to the coupling constant, and, therefore, solitons are indeed of non-perturbative nature.

The time-dependent solution is obtained by doing a Lorentz transformation

$$\bar{\phi}(x) \rightarrow \bar{\phi} \left[\frac{m(x-x_0) - vt}{\sqrt{1-v^2}} \right]$$

which gives the relativistic relation between energy and mass:

$$E[\bar{\phi}(x,t)] = \frac{M_s}{\sqrt{1-v^2}}$$

The more complicated solution, corresponding to $Q=0$, has the form:

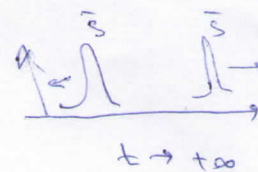
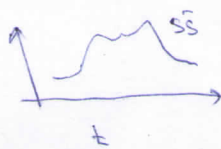
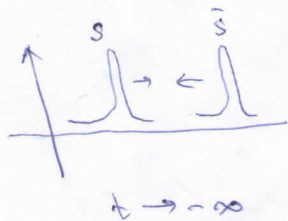
$$\bar{\Phi}_{s\bar{s}}(x,t) = \frac{4}{g} \arctan \left(\frac{\sinh[mvt/\sqrt{1-v^2}]}{v \cosh[mx/\sqrt{1-v^2}]} \right)$$

The most interesting property is the asymptotic behavior:

$$\bar{\Phi}_{s\bar{s}}(x,t) \rightarrow \bar{\Phi}_s \left(\frac{x + v(t \pm \Delta s \bar{s})/2}{\sqrt{1-v^2}} \right) + \bar{\Phi}_{\bar{s}} \left(\frac{x - v(t \pm \Delta s \bar{s})/2}{\sqrt{1-v^2}} \right) ; \quad t \rightarrow \pm \infty ;$$

$$\Delta s \bar{s} \equiv (1-v^2) v \log v$$

Thus, the double soliton solution $\bar{\Phi}_{s\bar{s}}$ can be interpreted as an elastic scattering process: soliton-anti-soliton



This situation remains true for other topological sectors, and multi-soliton solutions describe elastic scattering processes.

To find these multi-soliton solutions, one needs to develop the inverse scattering method.

Zero-curvature representation

Let us consider the equation of motion for the sine-Gordon model:

$$\partial_t^2 \varphi - \partial_x^2 \varphi = -\frac{2m^2}{\beta} \sin(2\beta\varphi); \quad (\text{we replaced } g \rightarrow 2\beta; \mu \rightarrow 4m)$$

This non-linear equation can be replaced by the consistency condition of two linear equations:

$$(\partial_x - A_x)\psi = 0; \quad (\partial_t - A_t)\psi = 0$$

where

$$A_x = i \begin{pmatrix} \frac{\beta}{2} \partial_t \varphi & m(\lambda e^{i\beta\varphi} - \lambda^{-1} e^{-i\beta\varphi}) \\ m(\lambda e^{-i\beta\varphi} - \lambda^{-1} e^{i\beta\varphi}) & -\frac{\beta}{2} \partial_t \varphi \end{pmatrix};$$

$$A_t = i \begin{pmatrix} \frac{\beta}{2} \partial_x \varphi & -m(\lambda e^{i\beta\varphi} + \lambda^{-1} e^{-i\beta\varphi}) \\ -m(\lambda e^{-i\beta\varphi} + \lambda^{-1} e^{i\beta\varphi}) & -\frac{\beta}{2} \partial_x \varphi \end{pmatrix}.$$

Let's denote: $u \equiv A_x$; $v \equiv A_t$. Then, we have:

$$\frac{\partial \varphi}{\partial x} = u\psi; \quad \frac{\partial \varphi}{\partial t} = v\psi$$

$$\Rightarrow \frac{\partial^2 \varphi}{\partial t \partial x} = \partial_t u \psi + u \partial_t \psi; \quad \frac{\partial^2 \varphi}{\partial x \partial t} = \partial_x v \psi + v \partial_x \psi$$

$$\text{or,} \quad \frac{\partial^2 \varphi}{\partial t \partial x} = (\partial_t u + u \cdot v) \psi; \quad \frac{\partial^2 \varphi}{\partial x \partial t} = (\partial_x v + v \cdot u) \psi.$$

The consistency condition means: $\frac{\partial^2 \varphi}{\partial t \partial x} = \frac{\partial^2 \varphi}{\partial x \partial t}$

$$\Rightarrow \partial_t u - \partial_x v + [u, v] = 0$$

Introducing a gauge field $\tilde{a}_a$: $\tilde{a}_x \equiv u$; $\tilde{a}_t \equiv v$, we obtain that the curvature $F_{ab} = 0$:

$$F_{ab} = \partial_a \tilde{a}_b - \partial_b \tilde{a}_a - [\tilde{a}_a, \tilde{a}_b] = 0$$

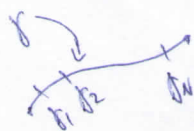
It is easy to check that the sine-Gordon equation is equivalent to the zero-curvature condition with matrices

$\mathcal{L}_x = A_x$; $\mathcal{L}_t = A_t$. The parameter λ is called spectral parameter.

The zero-curvature condition allows us to define the monodromy matrix:

$$T(\lambda) = P e^{\int_0^{2\pi} dx u(x, \lambda)}$$

where P stands for the path ordered exponential. Namely, we divide the curve γ on $N-1$ parts:



Then, the path-ordered exponential is defined as the limit:

$$P e^{\int \mathcal{L} dx} = \lim_{N \rightarrow \infty} \Omega_N$$

where $\Omega_N = L_N \cdot L_{N-1} \dots \cdot L_1$; $L_n = 1 + \int_{x_n} \mathcal{L} dx$

Let's find:

$$\partial_t T(\lambda) = \int_0^{2\pi} dx P e^{\int_x^{2\pi} dy u(y, \lambda)} (\partial_t u) \cdot P e^{\int_0^x dz u(z, \lambda)}$$

This is obvious if we consider:

$$\partial_t \Omega_N = \partial_t L_N \cdot L_{N-1} \dots L_1 + \dots + L_N \cdot L_{N-1} \dots \partial_t L_N \dots L_1 + \dots$$

$$= \dots L_N \cdot L_{N-1} \dots \left(\int_{x_n} \partial_t u \cdot dx \right) \cdot L_{N-1} \dots L_1 + \dots$$

$$\Rightarrow \partial_t T(\lambda) = \int_0^{2\pi} dx P e^{\int_x^{2\pi} dy u(y, \lambda)} \left[\partial_x v + [v, u] \right] P e^{\int_0^x dz u(z, \lambda)}$$

using:

$$F(x) = \int_{a(x)}^{b(x)} f(x, \xi) d\xi; \quad \frac{dF(x)}{dx} = f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x, \xi)}{\partial x} d\xi$$

we get:

$$\partial_t T(\lambda) = \int_0^{2\pi} dx \partial_x \left(P e^{\int_x^{2\pi} dy u(y, \lambda)} \cdot v(x, \lambda) \cdot P e^{\int_0^x dz u(z, \lambda)} \right)$$

$$\Rightarrow \partial_t T(\lambda) = P e^{\int_x^{2\pi} u(y, \lambda) dy} \cdot V(x, \lambda) \cdot P e^{\int_0^x u(z, \lambda) dz} \Big|_0^{2\pi}$$

$$= V(2\pi, t, \lambda) \cdot T(\lambda) - T(\lambda) \cdot V(0, t, \lambda)$$

If $V(x)$ is periodic: $V(2\pi, t, \lambda) = V(0, t, \lambda)$, we get:

$$\partial_t T(\lambda) = [V(2\pi, t, \lambda), T(\lambda)]$$

This has the Lax pair form, therefore $\text{tr}[T(\lambda)]$ generates conserved quantities Q_n :

$$\text{tr}[T(\lambda)] = \sum_n \lambda^n Q_n$$

Finally, let us check that $T(\lambda, t)$ takes $\psi(0, t) \rightarrow \psi(2\pi, t)$:

$$\psi(2\pi, t) = T(\lambda, t) \cdot \psi(0, t); \Rightarrow \underbrace{\partial_t \psi(2\pi, t)}_{= \partial_t(2\pi) \cdot \psi(2\pi)} = \partial_t T \cdot \psi(0, t) + T \cdot \underbrace{\partial_t \psi(0, t)}_{= \partial_t(0) \cdot \psi(0, t)}$$

Indeed, starting from:

$$V(2\pi, t, \lambda) \cdot T(\lambda) = \partial_t T(\lambda) + T(\lambda) \cdot V(0, t, \lambda)$$

or,

$$\partial_t(2\pi, t, \lambda) \cdot T(\lambda) \cdot \psi(0, t) = \partial_t T(\lambda) \cdot \psi(0, t) + T(\lambda) \cdot \partial_t(0, t, \lambda) \cdot \psi(0, t)$$

Comparing with the above equation for ψ , we get:

$$\psi(2\pi, t) = T(\lambda, t) \psi(0, t)$$

Note: Instead of points 0 and 2π , we could have chosen the interval (x, y) and arrive at:

$$\partial_t T(x, y, t, \lambda) = V(y, t, \lambda) \cdot T(x, y, t, \lambda) - T(x, y, t, \lambda) \cdot V(x, t, \lambda)$$

r-matrix for field theories

As in finite-dimensional case, one has the concept of the r-matrix for infinite-dimensional integrable systems.

There are two cases:

1) ultra-local systems

For such systems one has:

$$\{u(x, \lambda) \otimes u(y, \mu)\} = [r(\lambda, \mu), u(x, \lambda) \otimes 1 + 1 \otimes u(y, \mu)] \delta(x-y)$$

ultra-locality means that in the right hand side one has only the delta-function, and not the derivatives $\partial_x \delta(x-y)$ etc.

This leads to:

$$a) \{L_n(\lambda) \otimes L_n(\mu)\} = \left\{ \int_{\Delta_n} u(x, \lambda) dx \otimes \int_{\Delta_n} u(y, \mu) dy \right\} = [r(\lambda, \mu), L_n(\lambda) \otimes L_n(\mu)];$$

$$\{L_n(\lambda) \otimes L_m(\mu)\} = 0 \quad \text{for } n \neq m \quad (\text{because of ultra-locality})$$

$$b) \{T(\lambda) \otimes T(\mu)\} = [r(\lambda, \mu), T(\lambda) \otimes T(\mu)]$$

This immediately leads to involution:

$$\{tr(T(\lambda)), tr(T(\mu))\} = 0$$

The classical Yang-Baxter equation:

$$[r_2(\lambda-\mu), r_3(\lambda) + r_3(\mu)] + [r_3(\lambda), r_3(\mu)] = 0$$

(Compare for finite-dimensional case, p. 36).

There is also the inverse theorem: Any matrix r , satisfying the Yang-Baxter equation and such that $r(\lambda-\mu) = -P(r(\mu-\lambda))P$ where the permutation operator: $P(C \otimes D) = (D \otimes C)P$; $P^2 = 1$, gives an integrable model.

2) Non-ultralocal systems

For such systems $\{u(x, \lambda) \otimes u(y, \mu)\} \sim \partial_x \delta(x-y)$, which leads to not well-defined expression for $\{T(\lambda) \otimes T(\mu)\}$. We won't consider such systems here.

Inverse Scattering Method

Let us consider the sine-Gordon model, and fix the following boundary conditions:

$$\lim_{x \rightarrow -\infty} \varphi(x) = 0; \quad \lim_{x \rightarrow +\infty} \varphi(x) = \frac{Q\pi}{\beta}$$

where Q is the topological charge.

The idea of the inverse scattering method is to replace the original non-linear problem by several equivalent linear problems:

1.) Solve the direct scattering problem:

$$[\partial_x - A_x(\lambda, \varphi(x, 0))] \psi = 0 \quad (\text{see p. 40-})$$

and find the scattering data $a(\lambda, 0); b(\lambda, 0)$ [to be defined below precisely].

2.) Determine the time evolution of the scattering data, using the equation:

$$[\partial_t - A_t(\lambda, \varphi)] \psi = 0$$

This is easy to do, since we know the asymptotics $x \rightarrow \pm\infty$. Thus, we can find $a(\lambda, t); b(\lambda, t)$

3.) From the scattering data $a(\lambda, t); b(\lambda, t)$ one reconstructs $\varphi(x, t)$ by using the Gelfand-Levitan-Marchenko equation (a linear integral equation).

The last step is equivalent to a matrix Riemann-Hilbert problem. Let us first show how it arises.

Consider two solutions:

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}; \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and define the Wronskian:

$$W(\xi, \psi) = \xi_1 \psi_2 - \xi_2 \psi_1$$

$$\text{From } T_2(Ax) = 0 \Rightarrow \frac{dW(\xi, \psi)}{dx} = 0$$

Thus, the Wronskian is independent of 'x', and we can find it using the asymptotic solutions.

Using the boundary conditions: $\psi \rightarrow 0$ at $x \rightarrow -\infty$, and $\psi \rightarrow \frac{iq}{p}$ at $x \rightarrow +\infty$, the linear equation takes the form:

$$*) \left[\frac{\partial}{\partial x} - iK(x)\sigma_1 \right] \psi = 0; \quad x \rightarrow -\infty$$

$$*) \left[\frac{\partial}{\partial x} - i e^{i\pi Q} K(x)\sigma_1 \right] \psi = 0; \quad x \rightarrow +\infty$$

$$\text{where } K(x) \equiv m(1 - \lambda^2)$$

So, at $x \rightarrow \mp\infty$ we have general solutions:

$$\psi_{x \rightarrow -\infty} \sim c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{iKx} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-iKx};$$

$$\psi_{x \rightarrow +\infty} \sim c'_1 \begin{pmatrix} 1 \\ e^{i\pi Q} \end{pmatrix} e^{iKx} + c'_2 \begin{pmatrix} 1 \\ -e^{i\pi Q} \end{pmatrix} e^{-iKx}$$

Define: The Jost solutions are the ones with the following asymptotic behavior:

$$f_1 \sim \begin{pmatrix} 1 \\ e^{i\pi Q} \end{pmatrix} e^{iKx}; \quad x \rightarrow +\infty$$

$$f_2 \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-iKx}; \quad x \rightarrow -\infty$$

Note: Since $A_x^* = G_2 A_x G_2$, then if ψ is a solution, $\bar{\psi} \equiv G_2 \psi^*$ is also a solution.

For Jost solutions f_1, f_2 we have in general:

$$f_1 \sim c_{11}(\lambda) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{ikx} + c_{12}(\lambda) \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-ikx}, \quad x \rightarrow -\infty$$

$$f_1 \sim \begin{pmatrix} 1 \\ e^{iqa} \end{pmatrix} e^{ikx}; \quad x \rightarrow +\infty$$

$$f_2 \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-ikx}; \quad x \rightarrow -\infty$$

$$f_2 \sim c_{21}(\lambda) \begin{pmatrix} 1 \\ e^{iqa} \end{pmatrix} e^{ikx} + c_{22}(\lambda) \begin{pmatrix} 1 \\ -e^{iqa} \end{pmatrix} e^{-ikx}; \quad x \rightarrow +\infty$$

Now we can easily find Wronskians for different solutions, and use the property that the Wronskian does not depend on 'x'.

We have:

	$x = -\infty$	$x = +\infty$
$W(f_1, f_2)$	$-2c_{11}$	$-2c_{22} e^{iqa}$
$W(f_1, \bar{f}_2)$	$2ic_{12}$	$2ic_{21}^*$
$W(f_1, \bar{f}_1)$	$2i(c_{11} ^2 + c_{12} ^2)$	$2i$

Define: $c_{11} \equiv a(\lambda)$; $c_{12} \equiv -b(\lambda)$. These are the scattering data.

Thus, we have:

$$W(f_1, f_2) = -2a(\lambda);$$

$$|a(\lambda)|^2 + |b(\lambda)|^2 = 1$$

The main property is the following theorem: The Jost solutions $f_1(x, \lambda); f_2(x, \lambda)$ are analytic functions in $\text{Im}(\lambda) > 0$. Since $a(\lambda) = -\frac{1}{2}W(f_1, f_2)$, then $a(\lambda)$ is also analytic in the upper half-plane.

Using the time-evolution equation:

$$[\partial_t - A_\pm(\lambda, t)]\psi = 0$$

one can easily determine the time evolution of the scattering data:

$$a(\lambda, t) = a(\lambda, 0); \quad b(\lambda, t) = e^{2im(\lambda + \lambda^{-1})t} \cdot b(\lambda, 0)$$

(Here we ignore, for simplicity, the zeroes of $a(\lambda)$ function: $a(\lambda_n) = 0$)

Let's introduce another Jost solution pair:

$$f_3 \sim \begin{pmatrix} 1 \\ i\sigma \\ -e \end{pmatrix} e^{-ikx}; \quad x \rightarrow +\infty$$

$$f_4 \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{ikx}; \quad x \rightarrow -\infty$$

Since there are only two independent solutions, we can decompose:

$$f_3 = \Lambda_1 f_1 + \Lambda_2 f_2; \quad f_4 = \Lambda_3 f_1 + \Lambda_4 f_2$$

Using various asymptotics one can determine $\Lambda_1, \dots, \Lambda_4$:

$$\Lambda_1 = \frac{b^*(\lambda)}{a(\lambda)} e^{-i\sigma\pi}; \quad \Lambda_2 = \frac{e^{-i\sigma\pi}}{a(\lambda)}$$

$$\Lambda_3 = \frac{1}{a(\lambda)}; \quad \Lambda_4 = \frac{b(\lambda)}{a(\lambda)}$$

The main theorem for f_3, f_4 is that they are analytic functions for $\text{Im}(\lambda) < 0$.

Now, the relation between (f_3, f_4) and (f_1, f_2) can be written in the following form:

$$G_-^{-1} G_+ = \begin{pmatrix} 1 & -b(\lambda) e^{-2ikx} \\ -b^*(\lambda) e^{2ikx} & 1 \end{pmatrix} \equiv G(\lambda)$$

where :

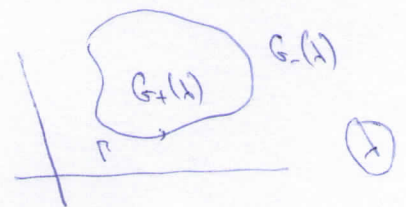
$$(f_2, f_1) \equiv G_+^{(1)} e^{-ik\phi_2 x} ; \quad \left(\frac{e^{i\theta}}{a^*(\lambda)} f_3, \frac{1}{a^*(\lambda)} f_4 \right) \equiv G_-(\lambda) e^{-ik\phi_2 x}$$

The main theorem means that $G_+(\lambda)$ is analytic in $\text{Im}(\lambda) > 0$ and $G_-(\lambda)$ is analytic in $\text{Im}(\lambda) < 0$.

Thus, we reduced the problem to the Riemann-Hilbert problem:

For a given contour Γ , and a function $G(\lambda)$ on the contour, find the functions $G_+(\lambda)$ and $G_-(\lambda)$ analytic inside and outside contour, such that on the contour the following condition is ~~satisfy~~ satisfied:

$$G_-^{-1}(\lambda) G_+(\lambda) = G(\lambda)$$



Since we know $G(\lambda)$ (in terms of $b(\lambda)$),

by solving the Riemann-Hilbert problem we will determine f_1, f_2 ; and, therefore, determine $\psi(t, x)$.

The RH problem is a well-studied mathematical branch, and we know how to solve it. (See the references).

From RH problem one can obtain the Gelfand-Leviton-Marchenko linear integral equation. The detailed derivation is presented in Appendix 1.

Introduction to the Bethe Ansatz

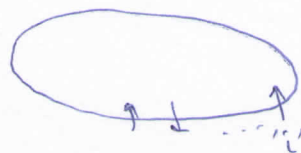
$S = 1/2$ xxz spin chain

The Hamiltonian:

$$H = J_1 \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_2 \sum_i S_i^z S_{i+1}^z$$

The spin operator at each site:

$$\vec{S}_i = \frac{1}{2} \vec{\sigma}_i; \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



Spin states: $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The spin algebra: $[S_i^\alpha, S_j^\beta] = i\delta_{ij} \epsilon^{\alpha\beta\gamma} S_i^\gamma$

When $J_1 = J_2 \Rightarrow H$ is the $S=1/2$ xxz Heisenberg model.

First, consider the unitary transformation $U_i(\vec{n}, \theta) = e^{i\theta \vec{n} \cdot \vec{S}_i}$ and show that it rotates the spin around $\vec{n}$ on angle θ .

Take: $\vec{n} = \hat{e}_z = (0, 0, 1)$

$$\Rightarrow e^{i\theta S_i^z} S_i^\alpha e^{-i\theta S_i^z} = \tilde{S}_i^\alpha(\theta)$$

$$\Rightarrow \frac{d\tilde{S}_i^\alpha(\theta)}{d\theta} = i e^{i\theta S_i^z} [S_i^z, S_i^\alpha] e^{-i\theta S_i^z} = i e^{i\theta S_i^z} S_i^\alpha S_i^z e^{-i\theta S_i^z} - i e^{i\theta S_i^z} S_i^z S_i^\alpha e^{-i\theta S_i^z} = i e^{i\theta S_i^z} [S_i^z, S_i^\alpha] e^{-i\theta S_i^z}$$

$$\Rightarrow \frac{d\tilde{S}_i^x(\theta)}{d\theta} = i e^{i\theta S_i^z} (i S_i^y) e^{-i\theta S_i^z} = -U_i S_i^y U_i^\dagger = -S_i^y(\theta)$$

$$\text{and } \frac{d\tilde{S}_i^y(\theta)}{d\theta} = S_i^z(\theta); \quad \frac{d\tilde{S}_i^z(\theta)}{d\theta} = 0$$

The solution has the form:

$$\left. \begin{aligned} \tilde{S}_i^x(\theta) &= S_i^x \cos \theta - S_i^y \sin \theta \\ \tilde{S}_i^y(\theta) &= S_i^x \sin \theta + S_i^y \cos \theta \\ \tilde{S}_i^z(\theta) &= S_i^z \end{aligned} \right\} \begin{aligned} &*) \text{ consistent with the condition,} \\ &\quad \tilde{S}^x \rightarrow S^x \text{ when } \theta \rightarrow 0 \\ &*) \text{ Describes the rotation around} \\ &\quad \text{the } z\text{-axis on angle } \theta. \end{aligned}$$

Let's take: $\vec{n} = e_z; \theta = \pi; u_i = e^{i\pi S_i^z}$

Consider the operator:

$$T = \prod_{i=1}^N u_i = \prod_{i=1}^N e^{i\pi S_i^z} = e^{i\pi S_1^z} e^{i\pi S_3^z} \dots e^{i\pi S_{2N+1}^z}$$

$$\Rightarrow T S_i^x T^\dagger = \left(e^{i\pi S_1^z} \dots e^{i\pi S_i^z} \dots e^{i\pi S_{2N+1}^z} \right) S_i^x \left(e^{-i\pi S_{2N+1}^z} \dots e^{-i\pi S_i^z} \dots e^{-i\pi S_1^z} \right)$$

commute

$$= e^{i\pi S_1^z} \dots \underbrace{e^{i\pi S_i^z} S_i^x e^{-i\pi S_i^z}}_{\text{rotation on angle } \pi} \dots e^{-i\pi S_1^z} = -S_i^x \quad \text{for odd 'i'}$$

For 'i' even: $\nabla T S_i^x T^\dagger = S_i^x$

Thus:

$$\boxed{\begin{aligned} T S_m^x T^\dagger &= (-i)^m S_m^x \\ T S_m^y T^\dagger &= (-i)^m S_m^y \\ T S_m^z T^\dagger &= S_m^z \end{aligned}}$$

Using this, we find:

$$THT^\dagger = J_1 \sum_i T (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) T^\dagger + J_2 \sum_i T (S_i^z S_{i+1}^z) T^\dagger$$

$$= J_1 \sum_i (-i)^i (-i)^{i+1} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_2 \sum_i S_i^z S_{i+1}^z$$

$$= (-J_1) \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_2 \sum_i S_i^z S_{i+1}^z$$

This shows that under this transformation: $J_1 \rightarrow -J_1$; $J_2 \rightarrow J_2$ and, therefore, only the sign of J_1 is relevant.

Introduce:

$$S^\pm = S^x \pm iS^y$$

$$\Rightarrow S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{so that: } S^+ |\downarrow\rangle = \uparrow; S^- |\uparrow\rangle = |\downarrow\rangle$$

The Hamiltonian becomes:

$$H = \frac{J_1}{2} \sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + J_2 \sum_i S_i^z S_{i+1}^z$$

1. $J_1 = 0$: Ising model
2. $J_2 < 0$: Ferromagnet
3. $J_2 > 0$: Anti-ferromagnet

Jordan-Wigner transformation

Consider transformation:

$$S_i^+ = S_i^x + iS_i^y = S_i^+ K(i) = K(i) S_i^+$$

$$S_i^- = S_i^x - iS_i^y = K(i) S_i^- = S_i^- K(i)$$

$$S_i^z = S_i^+ S_i^- - 1/2$$

where $K(i) = e^{i\pi \sum_{j=1}^{i-1} S_j^+ S_j^-} = e^{i\pi \sum_{j=1}^{i-1} (S_j^z + 1/2)}$

Easy to check:

$$*) K(i) \cdot K(i)^\dagger = 1$$

$$*) [K(i), S_j] = 0 \quad \text{if } i \neq j$$

$$*) [K(i), K(j)] = 0$$

The idea of fermionization comes from the observation that at the site:

$$\{S_j^+, S_j^-\} = \frac{1}{4} \{\sigma_j^+ \sigma_j^-\} = \frac{1}{4} \{\sigma_j^x + i\sigma_j^y, \sigma_j^x - i\sigma_j^y\} = 1$$

Therefore, S_j^+ and S_j^- behave like fermions!

$$\Rightarrow S_j^+ \equiv \psi_j^\dagger; S_j^- \equiv \psi_j; S_j^z = \psi_j^\dagger \psi_j - 1/2$$

The form of S_j^z follows from the commutation relations:

$$[S_j^+, S_j^+] = -S_j^+; \Rightarrow [\psi_j^\dagger, \psi_j^\dagger \psi_j] = -\psi_j^\dagger = -S_j^+$$

$$1/2 \text{ factor comes from } (\vec{S})^2 = \frac{3}{4}$$

The fermionic analogy fails for different sites. Thus, one needs to consider more general transformation with $K(i)$ being a non-local functional of $\psi \equiv f$.

Start from:

$$f_i^\dagger = S_i^\dagger K(i); \quad f_i = S_i^- K(i)$$

$$K(i) = e^{i\pi \sum_{j=1}^{i-1} (S_j^z + 1/2)} = e^{i\pi S_1^z} \dots e^{i\pi S_{i-1}^z} \cdot e^{i\frac{\pi}{2}(i-1)} = \prod_{j=1}^{i-1} U_j(e_z, \pi) \cdot \underbrace{e^{i\frac{\pi}{2}(i-1)}}_{= (i)^{i-1}}$$

Thus, $K(i)$ represents the rotation of the spins from $j=1$ to $i-1$ by π around the z -axis.

$$\Rightarrow S^x \rightarrow -S^x; \quad S^y \rightarrow -S^y$$

$$\Rightarrow \text{for } i < j: \quad K^\dagger(j) S_i^- K(j) = -S_i^- \quad (\text{or } S_i^- K(j) = -K(j) S_i^-)$$

This is clear because among j 's there is ' i ' component, and the rotation is an angle ' π '.

Then:

* For $i < j$:

$$f_i f_j = S_i^- K(i) S_j^- K(j)$$

└───┘
commute (see p. 52)

$$= S_i^- S_j^- K(i) K(j) = S_j^- S_i^- K(i) K(j) = S_j^- S_i^- K(j) K(i) = -S_j^- K(j) S_i^- K(i) = -f_j f_i$$

$$\Rightarrow \boxed{\{f_i, f_j\} = 0}$$

Also, for $i < j$:

$$\begin{aligned} f_i f_j^\dagger + f_j^\dagger f_i &= S_i^- K(i) S_j^\dagger K(j) + S_j^\dagger K(j) S_i^- K(i) \\ &= S_i^- S_j^\dagger K(j) K(i) - S_j^\dagger S_i^- K(j) K(i) \\ &= [S_i^-, S_j^\dagger] K(i) K(j) = 0 \end{aligned}$$

$$\Rightarrow \{f_i, f_j^\dagger\} = \delta_{ij}$$

So, we now have fermionic variables for all sites!

Since $S_j^z = f_j^\dagger f_j - 1/2$ the fermionic number $n_j \equiv f_j^\dagger f_j$ can be either 1 or 0, which corresponds to $S_j^z = \pm 1/2$. Thus, we have the Pauli principle.

Using:

$$S_i^+ S_{i+1}^- = f_i^\dagger K^+(i) K(i+1) f_{i+1} = f_i^\dagger \underbrace{e^{i\pi f_i^\dagger f_i}}_{=1 \ 02 -1} f_{i+1} = f_i^\dagger \underbrace{(1 - 2f_i^\dagger f_i)}_{=1 \ 02 -1} f_{i+1} = f_i^\dagger f_{i+1}$$

Similarly,

$$\begin{aligned} S_i^- S_{j+1}^+ &= f_j e^{i\pi f_j^\dagger f_j} f_{j+1}^\dagger = f_j (1 - 2f_j^\dagger f_j) f_{j+1}^\dagger = f_j f_{j+1}^\dagger - 2f_j f_j^\dagger f_j f_{j+1}^\dagger \\ &= f_j f_{j+1}^\dagger - 2f_j f_{j+1}^\dagger - 2f_j^\dagger f_j f_j f_{j+1}^\dagger = -f_j f_{j+1}^\dagger = f_{j+1}^\dagger f_j \end{aligned}$$

$\Rightarrow$ The Hamiltonian:

$$H = - \frac{J_\pm}{2} \sum_{i=1}^N (f_i^\dagger f_{i+1} + f_{i+1}^\dagger f_i) + J_\pm \sum_{i=1}^N (f_i^\dagger f_i - 1/2) (f_{i+1}^\dagger f_{i+1} - 1/2)$$

$\uparrow$

The sign can be chosen freely.

The periodicity (boundary) condition $\vec{S}_{N+1} = \vec{S}_1$

translates into

$$*) f_{N+1} = -f_1 \quad (\text{even } N = \sum_{i=1}^N f_i^\dagger f_i)$$

$$*) f_{N+1} = f_1 \quad (\text{odd } N)$$

we can immediately obtain the spectrum of the XY-model ($J_z=0$):

Introducing:

$$\tilde{f}_k = \frac{1}{\sqrt{N}} \sum_i f_i e^{ikx_i}; \quad \tilde{f}_k^\dagger = \frac{1}{\sqrt{N}} \sum_i f_i^\dagger e^{-ikx_i}$$

$$\Rightarrow H_{xy} = \sum_k \epsilon(k) \tilde{f}_k^\dagger \tilde{f}_k; \quad \epsilon(k) = -J_x \cos k$$

Coordinate Bethe Ansatz

Consider again the $s=1/2$ xxz spin chain:

$$H = J_x \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_z \sum_i S_i^z S_{i+1}^z$$

Remembering the fermionic picture:

$$S_j^z = n_j - 1/2$$

Thus, we can imagine 'spin down' state as no fermion present in site j , and 'spin up' state as one fermion on site j . Since $[S^z, H] = 0$, we can diagonalize the Hamiltonian in each sector of S^z independently.

In the fermionic picture:

$$H = - \frac{J_1}{2} \sum_{i=1}^N (\hat{f}_i^\dagger \hat{f}_{i+1} + \hat{f}_{i+1}^\dagger \hat{f}_i) + J_2 \sum_{i=1}^N (n_i - 1/2)(n_{i+1} - 1/2)$$

The term $\hat{f}_i^\dagger \hat{f}_{i+1}$ means that a fermion is killed at site $i+1$ and created at site i . Thus, J_1 -term represents the kinetic energy, and J_2 -term represents the potential energy.

Ferromagnetic case: $J_2 < 0$

Denote the wave-function of the state with ' r ' spins up as: $\Phi_{n_1, n_2, \dots, n_r}$, where n_i is the site with spin $\uparrow$ at the i -th place:

$$\begin{array}{cccccc} \downarrow & \downarrow & \uparrow & \downarrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} : \Phi_{n_1, n_2, n_3} ; \text{ where } n_1=3; n_2=5; n_3=6$$

Thus, $N-r$ spins are down.

Consider the all spins down state Φ_0 . Since $\sigma_m^z |\uparrow\rangle_m = |\uparrow\rangle_m$;

$\sigma_m^z |\downarrow\rangle_m = -|\downarrow\rangle_m$, we get:

$$H \Phi_0 = \frac{J_1}{4} \sum_i \sigma_i^z \sigma_{i+1}^z |\downarrow \dots \downarrow\rangle = \underbrace{\frac{J_1}{4} N}_{\equiv E_0} \Phi_0 \equiv E_0 \Phi_0$$

Consider now the space of functions with one spin up: Φ_n .

The superposition of these functions:

$$\psi = \sum_{n=1}^N c_n \Phi_n, \quad (c_{n+k} = c_k)$$

$\Rightarrow$

$$H\psi = \sum_{n=1}^N \left(J_z \sum_i S_i^z S_{i+1}^z | \uparrow \dots \uparrow_n \dots \downarrow \rangle c_n \right) + \sum_{n=1}^N \left(\frac{J_\perp}{2} \sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) | \uparrow \dots \uparrow_n \dots \downarrow \rangle c_n \right)$$

$$= \sum_{n=1}^N c_n \frac{J_z}{4} \left(\sigma_1^z \sigma_2^z + \dots + \sigma_{n-1}^z \sigma_n^z + \sigma_n^z \sigma_{n+1}^z + \dots + \sigma_N^z \sigma_1^z \right) | \uparrow \dots \uparrow_n \dots \downarrow \rangle$$

gives (-1) ↓
gives (-1)

$$+ \sum_{n=1}^N c_n \frac{J_\perp}{2} \left(S_1^+ S_2^- + \dots + S_{n-1}^+ S_n^- + S_n^+ S_{n+1}^- + \dots \right) | \uparrow \dots \uparrow_n \dots \downarrow \rangle$$

all terms give zero

$$+ \sum_{n=1}^N c_n \frac{J_\perp}{2} \left(S_1^- S_2^+ + \dots + S_{n-1}^- S_n^+ + S_n^- S_{n+1}^+ + \dots \right) | \uparrow \dots \uparrow_n \dots \downarrow \rangle$$

the only term that is not zero

$$= \frac{J_z}{4} (N-4) \psi + \frac{J_\perp}{2} \sum_{n=1}^N (c_{n+1} + c_{n-1}) \Phi_n \equiv E^{(1)} \psi$$

Thus, the spectrum satisfies the equation:

$$E \cdot c_n = (E_0 - J_z) c_n + \frac{J_\perp}{2} (c_{n+1} + c_{n-1})$$

Looking for a solution in terms of plane waves:

$$c_n = \frac{1}{\sqrt{N}} e^{ikn}$$

$$\Rightarrow E_k = E_0 - J_z + J_x \cos k$$

The boundary condition: $c_{n+N} = c_n \Leftrightarrow k = \frac{2\pi m}{N}$, where

$$m = 0, \pm 1, \dots$$

The excitation energy:

$$\varepsilon_k \equiv E_k - E_0 = |J_z| + J_x \cos k$$

For $|J_z| < |J_x| \Rightarrow \varepsilon_k < 0$. For simplicity, consider $|J_z| \geq |J_x|$

Consider now two spin up states: $\Phi_{n_1 n_2}$. The linear combination:

$$\psi = \sum_{n_1 < n_2} c_{n_1 n_2} \Phi_{n_1 n_2}$$

There are two cases to be considered:

a) $n_2 > n_1 + 1$

$\Rightarrow$

$$H \Phi_{n_1 n_2} = \frac{J_z}{4} \left(\sigma_1^z \sigma_2^z + \underbrace{\sigma_1^z \sigma_{n_1}^z}_{\text{gives } (-1)} + \underbrace{\sigma_{n_1}^z \sigma_{n_1+1}^z}_{(-1)} + \dots + \underbrace{\sigma_{n_2-1}^z \sigma_{n_2}^z}_{(-1)} + \underbrace{\sigma_{n_2}^z \sigma_{n_2+1}^z}_{(-1)} + \dots \right) |\downarrow \dots \uparrow_{n_1} \dots \uparrow_{n_2} \dots \downarrow \rangle$$

$$+ \frac{J_x}{2} (S_1^+ S_2^- + \dots + \underbrace{S_{n_1-1}^+ S_{n_1}^- + S_{n_1}^+ S_{n_1+1}^- + \dots + S_{n_2-1}^+ S_{n_2}^- + S_{n_2}^+ S_{n_2+1}^-}_{\text{survive}} + \dots) |\downarrow \dots \uparrow_{n_1} \dots \uparrow_{n_2} \dots \downarrow \rangle$$

$$+ \frac{J_x}{2} (+ \leftrightarrow -) |\downarrow \dots \uparrow_{n_1} \dots \uparrow_{n_2} \dots \downarrow \rangle$$

$$= \frac{J_2}{4} (N-8) |\downarrow \dots \uparrow_{n_1} \dots \uparrow_{n_2} \dots \downarrow\rangle + \frac{J_1}{2} (\phi_{n_1-1, n_2} + \phi_{n_1, n_2-1} + \phi_{n_1+1, n_2} + \phi_{n_1, n_2+1})$$

$$= (E_0 - 2J_2) \Phi_{n_1, n_2} + \frac{J_1}{2} (\Phi_{n_1-1, n_2} + \Phi_{n_1+1, n_2} + \Phi_{n_1, n_2-1} + \Phi_{n_1, n_2+1})$$

The second case:

b) $n_2 = n_1 + 1$

$$H \phi_{n_1, n_1+1} = \frac{J_2}{4} \left(\dots + \underbrace{\delta_{n_1-1}^2}_{(-1)} \delta_{n_1}^2 + \underbrace{\delta_{n_1}^2}_{(+1)} \delta_{n_1+1}^2 + \underbrace{\delta_{n_1+1}^2}_{(-1)} \delta_{n_1+2}^2 + \dots \right) |\downarrow \dots \uparrow_{n_1} \uparrow_{n_1+1} \dots \downarrow\rangle$$

$$+ \frac{J_1}{2} \left(\dots \underbrace{S_{n_1-1}^+ S_{n_1}^- + S_{n_1}^+ S_{n_1+1}^- + S_{n_1+1}^+ S_{n_1+2}^-}_{\text{the only surviving term}} + \dots \right) |\downarrow \dots \uparrow_{n_1} \uparrow_{n_1+1} \dots \downarrow\rangle$$

$$+ \frac{J_1}{2} (+ \leftrightarrow -) |\downarrow \dots \uparrow_{n_1} \uparrow_{n_1+1} \dots \downarrow\rangle$$

$$= \frac{J_2}{4} (N-4) \Phi_{n_1, n_1+1} + \frac{J_2}{2} (\phi_{n_1-1, n_1+1} + \phi_{n_1, n_1+2})$$

Thus, for $\mathcal{E} = E - E_0$:

a) $n_2 > n_1 + 1$: $\mathcal{E} c_{n_1, n_2} = -2J_2 c_{n_1, n_2} + \frac{J_1}{2} (c_{n_1+1, n_2} + c_{n_1-1, n_2} + c_{n_1, n_2+1} + c_{n_1, n_2-1})$ (*)

b) $n_2 = n_1 + 1$: $\mathcal{E} c_{n_1, n_1+1} = -J_2 c_{n_1, n_1+1} + \frac{J_1}{2} (c_{n_1-1, n_1+1} + c_{n_1, n_1+2})$ (**)

Look for a solution in the form: (for $n_2 > n_1 + 1$)

$$c_{n_1 n_2} = c_1 e^{i(k_1 n_1 + k_2 n_2)} + c_2 e^{i(k_2 n_1 + k_1 n_2)}$$

Let us formally make (*) compatible with (**) (p. 53);
taking $n_2 = n_1 + 1$ in (*) we obtain:

$$-J_2 c_{n_1, n_1+1} + \frac{J_1}{2} (c_{n_1, n_1} + c_{n_1+1, n_1+1}) = 0$$

This defines the coefficients $c_{n,n}$.

Then, we have:

$$\varepsilon c_{n_1 n_2} = -2J_2 c_{n_1 n_2} + \frac{J_1}{2} (c_{n_1+1, n_2} + c_{n_1-1, n_2}) + \frac{J_1}{2} (c_{n_1, n_2-1} + c_{n_1, n_2+1})$$

$$\Rightarrow e^{i(k_1 n_1 + k_2 n_2)} [c_1 e^{i(k_1 n_1 + k_2 n_2)} + c_2 e^{i(k_2 n_1 + k_1 n_2)}]$$

$$= -2J_2 [c_1 e^{i(k_1 n_1 + k_2 n_2)} + c_2 e^{i(k_2 n_1 + k_1 n_2)}] + \frac{J_1}{2} [c_1 e^{i k_1} + c_2 e^{-i k_1}] e^{i(k_1 n_1 + k_2 n_2)}$$

$$+ \frac{J_1}{2} [c_2 e^{i k_2} + c_1 e^{-i k_2}] e^{i(k_2 n_1 + k_1 n_2)} + \frac{J_1}{2} [c_1 e^{i k_2} + c_2 e^{-i k_2}] e^{i(k_1 n_1 + k_2 n_2)}$$

$$+ \frac{J_1}{2} [c_2 e^{i k_1} + c_1 e^{-i k_1}] e^{i(k_2 n_1 + k_1 n_2)}$$

$$= -2J_2 [c_1 e^{i(k_1 n_1 + k_2 n_2)} + c_2 e^{i(k_2 n_1 + k_1 n_2)}] + J_1 (\cos k_1 + \cos k_2) [c_1 e^{i(k_1 n_1 + k_2 n_2)} + c_2 e^{i(k_2 n_1 + k_1 n_2)}]$$

$$\Rightarrow \varepsilon_{k_1, k_2} = [-J_2 + J_1 \cos k_1] + [-J_2 + J_1 \cos k_2] = \varepsilon_{k_1} + \varepsilon_{k_2}$$

So, the energy is additive: $\boxed{\varepsilon_{k_1, k_2} = \varepsilon_{k_1} + \varepsilon_{k_2}}$

using the equation for $c_{n,n}$: $-J_2 c_{n,n+1} + \frac{J_1}{2} (c_{n,n_1} + c_{n+1,n_1}) = 0$

we get:

$$\frac{c_1}{c_2} = - \frac{J_2 e^{i \frac{k_1 - k_2}{2}} - J_1 \cos \frac{k_1 + k_2}{2}}{J_2 e^{-i \frac{k_1 - k_2}{2}} - J_1 \cos \frac{k_1 + k_2}{2}}$$

$$\Rightarrow \left| \frac{c_1}{c_2} \right| = 1 \quad ; \quad \Rightarrow \quad c_1 = e^{i\phi/2} \quad ; \quad c_2 = e^{-i\phi/2} \quad \text{where } \phi = \phi(k_1, k_2)$$

$$\Rightarrow \cot \phi/2 = \frac{J_2 \sin \frac{k_1 - k_2}{2}}{J_1 \cos \frac{k_1 + k_2}{2} - J_2 \cos \frac{k_1 - k_2}{2}}$$

The boundary condition now gives:

$$c_{n_1, n_2} = c_{n_2, n_1 + N}; \quad \Rightarrow \quad k_1 N = \phi + 2\pi m_1; \quad k_2 N = -\phi + 2\pi m_2$$

Writing:

$$c_{n,n_2} = c_1 \left[e^{i(k_1 n_1 + k_2 n_2)} + S e^{i(k_1 n_2 + k_2 n_1)} \right]; \quad S \equiv \frac{c_2}{c_1} = e^{-i\phi}$$

~~we~~ we see that $S = e^{-i\phi}$ has the meaning of the S -matrix.

The total momentum is quantized: ~~$P = p_1 + p_2$~~

$$P = k_1 + k_2 = \frac{2\pi}{N} (m_1 + m_2)$$

The periodicity condition is equivalent to:

$$e^{ip_1 N} = S(p_1, p_2); e^{ip_2 N} = S(p_2, p_1)$$

These are the Bethe equations.

In general, for r spins up state, one can show that the wave-function has the form:

$$\psi = \sum_{n_1, n_2, \dots, n_r} c_{n_1, \dots, n_r} \Phi_{n_1, \dots, n_r}$$

where

$$c_{n_1, \dots, n_r} = \sum_{\substack{\mathbf{p} \\ \text{(permutations)}}} e^{i \left(\sum_{j=1}^r k_{p_j} n_j + \frac{1}{2} \sum_{j < l} \Phi(k_{p_j}, k_{p_l}) \right)}$$

The energy is additive in general; $\mathcal{E}_{k_1, \dots, k_r} = \mathcal{E}_{k_1} + \mathcal{E}_{k_2} + \dots + \mathcal{E}_{k_r}$,

and the Bethe equations have the factorized form:

$$e^{ip_i N} = \prod_{i \neq j} S(p_i, p_j)$$

In the next lecture we re-obtain these formulas using the integrable properties of the model and applying the Algebraic Bethe Ansatz.

Algebraic Bethe Ansatz

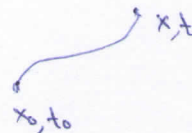
Lattice models

In lattice models the spatial coordinate x is replaced by discrete variable n . In addition, lattice models provide regularization to divergent continuous models, where the lattice spacing $\Delta \sim \frac{1}{\Lambda}$; and Λ is the ultra-violet cut-off.

Let us construct first the zero-curvature condition for lattice models. For continuous models the zero-curvature condition arises as the compatibility condition of the linear equations:

$$\frac{\partial \psi}{\partial x} = u(x, t, \lambda) \psi; \quad \frac{\partial \psi}{\partial t} = v(x, t, \lambda) \psi$$

The monodromy: $T_\delta = P e^{\int_{\delta} u dx + v dt}$
takes:



$$T_\delta \psi(x_0, t_0) = \psi(x, t)$$

The parallel transport along the spatial dimension ($t = t_0$) is given by:

$$T(\lambda) = P e^{\int_{x_0}^x u dx}; \quad \psi(x) = T(\lambda) \psi(x_0)$$

For lattice models parallel transport $T_n(\lambda) = \prod_{\Delta n} u(x, t, \lambda)$ is replaced by the matrix $L_n(t, \lambda)$ which defines

the parallel transport from site n to the site $n+1$:

$$\psi(x + \Delta x) = \psi_{n+1} = T_n(\lambda) \psi(x_0) + \psi(x_0) \approx L_n \psi(x_0)$$

Then, the linear system is replaced with:

$$\psi_{n+1} = L_n(t, \lambda) \psi_n; \quad \frac{d\psi_n}{dt} = V_n(t, \lambda) \psi_n$$

The compatibility condition of these equations is the analogue of the zero-curvature condition for the continuous models. We have:

$$\frac{d\psi_{n+1}}{dt} = \frac{dL_n}{dt} \psi_n + L_n \frac{d\psi_n}{dt}$$

$$\Rightarrow V_{n+1} \cdot \psi_{n+1} = \frac{dL_n}{dt} \psi_n + L_n V_n \psi_n$$

$$\Rightarrow V_{n+1} \cdot L_n \psi_n = \frac{dL_n}{dt} \psi_n + L_n V_n \psi_n$$

Thus, we obtain:

$$\boxed{\frac{dL_n}{dt} + L_n \cdot V_n - V_{n+1} \cdot L_n = 0}$$

This is the zero-curvature condition for the lattice models.

Example: The Toda lattice

Real variables q_n : $1 \leq n \leq N$

The equations of motion: $\frac{d^2 q_n}{dt^2} = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}$

The Hamiltonian:

$$H = \sum_n \frac{1}{2} p_n^2 + V(q); \quad V(q) = \sum_n (e^{q_n - q_{n+1}} - 1)$$

and,

$$\{p_n, p_m\} = 0; \quad \{q_n, q_m\} = 0; \quad \{p_n, q_m\} = \delta_{nm}$$

The L_n, V_n matrices have the form:

$$L_n(\lambda) = \begin{pmatrix} p_n + \lambda & e^{\theta_n} \\ -e^{\theta_n} & 0 \end{pmatrix}; \quad V_n(\lambda) = \begin{pmatrix} 0 & -e^{\theta_n} \\ e^{\theta_{n+1}} & \lambda \end{pmatrix}$$

In continuous theories we have the concept of the r-matrix:

$$\{u(x, \lambda), u(y, \mu)\} = [r(\lambda, \mu), u(x, \lambda) \otimes 1 + 1 \otimes u(y, \mu)] \cdot \delta(x-y)$$

and,

$$\{\tau(\lambda), \tau(\mu)\} = [r(\lambda, \mu), \tau(\lambda) \otimes \tau(\mu)]$$

(see p. 43)

Since in lattice models L_n has the meaning of the transition matrix $T_n = \int_{\Delta_n} u dx$ over the lattice spacing Δ_n , we can immediately write the following commutation relations:

$$[L_n(\lambda), L_m(\mu)] = [r(\lambda, \mu), L_n(\lambda) \otimes L_m(\mu)] \delta_{nm}$$

The full monodromy matrix over the entire lattice:

$$T_N(\lambda) = L_N(\lambda) \cdot L_{N-1}(\lambda) \cdots L_1(\lambda)$$

It satisfies the commutation relation:

$$[T_N(\lambda), T_N(\mu)] = [r(\lambda, \mu), T_N(\lambda) \otimes T_N(\mu)]$$

Therefore, we find that the quantities $F_n(\lambda) = \pm L(T_n(\lambda))$ are in involution:

$$\{F_n(\lambda), F_m(\mu)\} = 0$$

Decomposing: $F_n(\lambda) = \sum_j \lambda_j Q_j$, we obtain commuting quantities: $\{Q_i, Q_j\} = 0$.

S=1/2 xxx model

Let's show how the algebraic Bethe ansatz works for the simplest model - the Heisenberg model.

The Hamiltonian:

$$H = \sum_{n=1}^N \left[\vec{S}_n \cdot \vec{S}_{n+1} - \frac{1}{4} \right]$$



The full Hilbert space:

$$\mathcal{H} = \underbrace{h \otimes h \otimes \dots \otimes h}_N, \quad h \cong \mathbb{C}^2$$

The spin variables:

$$\vec{S}_n = \mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \vec{S} \otimes \dots \otimes \mathbb{1}$$

↑
n-th position

Periodicity: $\vec{S}_{n+N} = \vec{S}_n$

Using the permutation operator:

$$P = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} + \vec{\sigma} \otimes \vec{\sigma})$$

in the basis: $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

we have:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad P(a \otimes b) = b \otimes a; \quad a, b \in \mathbb{C}^2;$$

and the Hamiltonian can be written as

$$H = \frac{1}{2} \sum_{n=1}^N (P_{n,n+1} - 1)$$

Now, we introduce the Lax operator $L_{n,a}$ which acts in $h_n \otimes V$. The space V is the auxiliary space, and in this case $V = \mathbb{C}^2$. In general, V does not have to coincide with h . The $L_{n,a}$ operator has the form:

$$L_{n,a} = \lambda \cdot (\mathbb{1}_n \otimes \mathbb{1}_a) + i(\vec{S}_n \otimes \vec{S}_a) = \begin{pmatrix} \lambda + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda - iS_n^3 \end{pmatrix}$$

$L_{n,a}$ defines parallel transport: $\phi_{n+1} = \frac{1}{\lambda} L_n \phi_n$.

We easily check this: in the continuum limit:

$$S(x) = \lim_{\Delta \rightarrow 0} \frac{S_n}{\Delta}; \quad x = n \cdot \Delta$$

$$\Rightarrow \frac{1}{i} \phi'(x) = \lim_{\Delta \rightarrow 0} \frac{1}{i} \frac{\phi_{n+1} - \phi_n}{\Delta} = \frac{\vec{S} \cdot \vec{\sigma}}{\lambda} \phi(x)$$

This is the correct u -operator for the continuous Heisenberg model (the Landau-Lifshitz model).

The main relation here is the fundamental commutation relation:

$$R_{a_2}(\lambda - \mu) L_{n,a_1}(\lambda) L_{n,a_2}(\mu) = L_{n,a_2}(\mu) L_{n,a_1}(\lambda) R_{a_1,a_2}(\lambda - \mu)$$

Here $R_{a_1,a_2}(\lambda - \mu)$ is the quantum R-matrix, which acts in the product of two auxiliary spaces: $V_{a_1} \otimes V_{a_2}$

The relation is verified directly, with the R-matrix of the following form:

$$R_{a_1,a_2}(\lambda) = \lambda \mathbb{I}_{a_1,a_2} + i P_{a_1,a_2}$$

If one sets: $R_{12} = \mathbb{I}_{12} + \hbar r_{12} + \dots$, then the above relation goes to the classical Poisson bracket relation for L_n -operators, for the monodromy matrix:

$$T_n(\lambda) = L_{n,n}(\lambda) L_{n,n-1}(\lambda) \dots L_{1,n}(\lambda)$$

it is easy to show the following relation:

$$R_{a_2}(\lambda - \mu) T_a(\lambda) T_a(\mu) = T_a(\mu) T_a(\lambda) R_{a_1,a_2}(\lambda - \mu)$$

Consider now the product: $T(\mu) \otimes T(\nu) \otimes T(\lambda)$. By using the above relation we can bring $T(\mu) \otimes T(\nu) \otimes T(\lambda)$ to $T(\lambda) \otimes T(\nu) \otimes T(\mu)$ in two different ways (by interchanging different T-operators). The consistency of this procedure leads to the quantum Yang-Baxter relation:

$$R_{12}(\lambda, \nu) R_{13}(\lambda, \mu) R_{23}(\mu, \nu) = R_{23}(\mu, \nu) R_{13}(\lambda, \mu) R_{12}(\lambda, \nu)$$

Again, taking $R_{12} = \mathbb{I}_{12} + \hbar r_{12}$ we obtain, from the previous relation, the classical Yang-Baxter relation for the r -matrix (see p. 43).

Introduce: $F(\lambda) = \text{tr}_2 [T_a(\lambda)]$. This acts in the Hilbert space $\mathcal{H}$.
Since:

$$R[T(\lambda) \otimes T(\mu)] R^{-1} = T(\mu) \otimes T(\lambda)$$

taking the trace in (a_1, a_2) -spaces and using $\text{tr}(A \otimes B) = \text{tr} A \cdot \text{tr} B$ we find:

$$[F(\lambda), F(\mu)] = 0$$

Thus, we have commuting quantities, and we can check that the Hamiltonian is amongst the $F(\lambda)$:

$$H = \frac{i}{2} \frac{d}{d\lambda} \ln F(\lambda) \Big|_{\lambda = i/2 - N/2}$$

Let's write the monodromy matrix in the form:

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

We want to diagonalize $\text{tr}_2 T_a(\lambda) = A(\lambda) + D(\lambda)$.

In the basis: $e_1 = \uparrow \uparrow \uparrow$; $e_2 = \uparrow \uparrow \downarrow$; $e_3 = \downarrow \uparrow \uparrow$; $e_4 = \downarrow \uparrow \downarrow$

we have:

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}; \quad a(\lambda) = \lambda + i; \quad b = \lambda; \quad c = i$$

Then, the fundamental commutation relation gives:

$$[B(\lambda), B(\mu)] = 0$$

$$A(\lambda)B(\mu) = f(\lambda-\mu)B(\mu)A(\lambda) + g(\lambda-\mu)B(\lambda)A(\mu)$$

$$D(\lambda)B(\mu) = h(\lambda-\mu)B(\mu)D(\lambda) + k(\lambda-\mu)B(\lambda)D(\mu)$$

$$\text{where: } f(\lambda) = \frac{\lambda-i}{\lambda}; \quad g(\lambda) = \frac{i}{\lambda}; \quad h(\lambda) = \frac{\lambda+i}{\lambda}; \quad k(\lambda) = -\frac{i}{\lambda}$$

Define the state Ω :

$$c(\lambda)\Omega = 0$$

Then, using:

$$L_{n,2} = \begin{pmatrix} \lambda + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda - iS_n^3 \end{pmatrix}$$

$$\Rightarrow L_n |\uparrow\rangle = \begin{pmatrix} \lambda + i/2 & * \\ 0 & \lambda - i/2 \end{pmatrix} |\uparrow\rangle$$

Then, for $\Omega = \uparrow \uparrow \uparrow \dots \uparrow$:

$$T(\lambda)\Omega = \begin{pmatrix} \alpha(\lambda) & * \\ 0 & \beta(\lambda) \end{pmatrix} \Omega; \quad \alpha(\lambda) = \lambda + i/2; \quad \beta(\lambda) = \lambda - i/2$$

So, Ω is the eigenstate of $A(\lambda)$ and $D(\lambda)$:

$$C(\lambda)\Omega = 0; A(\lambda)\Omega = \alpha(\lambda)\Omega; D(\lambda)\Omega = \beta(\lambda)\Omega$$

Now, look for the eigenvalues in the form:

$$|\lambda_1 \lambda_2 \dots \lambda_M\rangle = B(\lambda_1) \dots B(\lambda_M) \Omega$$

Then, find:

$$A(\lambda) |\lambda_1 \dots \lambda_M\rangle = \dots$$

$$D(\lambda) |\lambda_1 \dots \lambda_M\rangle = \dots$$

Then, $|\lambda_1 \dots \lambda_M\rangle$ is an eigenvalue of $t_2 T_2(\lambda) = A(\lambda) + D(\lambda)$,

provided:

$$\left(\frac{\lambda_n + i/2}{\lambda_n - i/2} \right)^L = \prod_{j=1}^M \frac{\lambda_n - \lambda_j + i}{\lambda_n - \lambda_j - i} \quad (*)$$

This gives: $t_2 [T_2(\lambda)] |\lambda_1 \dots \lambda_M\rangle = \Lambda |\lambda_1 \dots \lambda_M\rangle$, where

$$\Lambda = (\lambda + i/2)^L \prod_{n=1}^M \frac{\lambda - \lambda_n - i}{\lambda - \lambda_n} + (\lambda - i/2)^L \prod_{n=1}^M \frac{\lambda - \lambda_n + i}{\lambda - \lambda_n} \quad (**)$$

Taking $\lambda_j = \frac{1}{2} \cot p_j/2$ the equation (*) gives:

$$e^{i p_i L} = \prod_{j=1}^M S(p_i, p_j)$$

This is exactly the Bethe equations obtained in the previous lecture via direct coordinate Bethe ansatz.

Moreover, using the expression for the Hamiltonian (p. 69) it is easy to check that the expression (**) yields

the correct spectrum, with the energy: $\mathcal{E}_{\lambda_1 \dots \lambda_M} = \mathcal{E}_{\lambda_1} + \mathcal{E}_{\lambda_2} + \dots + \mathcal{E}_{\lambda_M}$.

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